A Model Of Type Theory In Cubical Sets With Connections

MSc Thesis *(Afstudeerscriptie)*

written by

Simon Docherty
(born 23rd October, 1989 in Oxford, United Kingdom)

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Date of the public defense:	**Members of the Thesis Committee:**
22nd September, 2014	Prof.dr Benedikt Löwe
	Dr. Benno van den Berg
	Dr Jaap van Oosten
	Dr Luca Spada
	Dr Nick Bezhanishvili

Institute for Logic, Language and Computation

Abstract

In this thesis we construct a new model of intensional type theory in the category of cubical sets with connections. To facilitate this we introduce the notion of a *nice path object category*, a simplification of the path object category axioms of [vdBG12] that nonetheless yields the full path object category structure. By defining *cubical n-paths* and *contraction operators* upon them we exhibit the category of cubical sets with connections as a nice path object category, and are therefore able to utilise a general construction of a *homotopy theoretic model of identity types* from the structure of a path object category in order to give our model of type theory.

Introduction

The problem of constructing sound models of intensional type theory goes back at least 30 years, starting with the model of type theory in locally cartesian closed categories given by Seely in [See84]. This had a major deficiency in that it only modelled an *extensional* type theory. That is, the following rule is satisfied

$$\frac{p \in Id_A(a, b)}{a = b \in A} \text{ Id - Reflection}$$

meaning propositional and definitional equality coincide. This is problematic besides the desire to keep these two notions separate, as the addition of reflection causes desirable computational properties such as strong normalisation [Str93] and decidable type checking [Str91] to fail. It wasn't until Hofmann & Streicher's landmark paper [HS98] that an intensional model was constructed. In their paper identity types were interpreted as hom-sets of groupoids, themselves given a discrete groupoid structure. Witnesses to propositional equality were thus given by isomorphisms between terms, themselves interpreted as objects in groupoids. As the hom-sets could be inhabited by more than one arrow this meant extensionality was no longer satisfied, although the lack of higher dimensional structure meant these witnesses could not be further related by identity terms. In order to obtain such towers of identity types it was required to look at higher dimensional structure, which in turn opened the door to methods from homotopy theory. This idea was independently taken up by Voevodsky [Voe06] and Warren [War08], and paved the way for the field now known as *homotopy type theory* [IAS13].

In recent years research in this area has accelerated, yielding a number of models of intensional type theory in familiar mathematical settings such as simplicial sets [KLV12, Str14], chain complexes [War08], topological spaces [vdBG12] and the effective topos [vOar]. In [vdBG12] van den Berg and Garner were able to give a general framework for producing such models that encompassed all of those listed. The inspiration for the present work comes from the *cubical set* model presented in [BCH14], the existence of which was intimated in [Cis14]. A natural question arises: can this model also be brought into the general framework of [vdBG12]? We answer in the affirmative, as long as we take the category of cubical sets with *connections*. In doing so we are able to present a brand new model of intensional type theory.

Structure Of The Thesis

The thesis is split into two sections: the first contains the vast majority of the original work in this thesis. Here we concern ourselves with the identification of a path object category structure on the category of cubical sets with connections. The second section is then an expansion of the work in [vdBG12] and details how one can construct a model of type theory with the path object category structure we have determined. We give a brief summary of each chapter below:

- *Chapter 1.* We introduce *path object categories*, a natural axiomatic framework satisfied by categories with an *internal notion of path*. For the first time in the literature we present these axioms in full detail by abstracting from the characteristic example in **Top**. Over the course of the chapter we prove that the category **Gpd** is a path object category.
- *Chapter 2.* We introduce a new concept, that of a *nice path object category*. We prove that this simplification of the path object category axioms still yields the path object category structure and thus reduce the task of identifying such a structure on the category of cubical sets with connections.
- *Chapter 3.* We give the main result of the thesis: that the category of cubical sets with connections is a path object category. To do so we exhibit a instance of the nice path object category axioms by introducing *cubical n-paths* and *contraction operators*. We then show that we can give the collection of cubical n-paths through X the structure of a cubical set with connections and use the contraction operators to show we have an internal notion of path contraction for these paths.
- *Chapter 4.* We present the categorical semantics for the models of type theory we construct. Concurrently we give motivation for the framework introduced in Chapter 6 by presenting the fibration interpretation of type theory and explaining the coherency issues such models suffer.
- *Chapter 5.* Following [vdBG12], we give the modifications required to solve the coherency issue in model category interpretations of identity types. In doing so we present the framework by which we construct the model in the title of the thesis: that of a homotopy theoretic model of identity types. We prove that such structures produce sound categorical models of intensional type theory.
- *Chapter 6.* We prove that path object categories are homotopy theoretic models of identity types. To do so we introduce a notion of homotopy internal to path object categories and construct a cloven weak factorisation system based on strong deformation retracts and maps corresponding to certain homotopy lifting properties. As an immediate corollary we obtain a model of intensional type theory in the category of cubical sets with connections.
- *Chapter 7.* In this concluding chapter we summarise the work of the thesis and identify some open questions and potential future research with the tools we have developed.
- *Appendix A.* We give the category theoretic background required to read the thesis.
- *Appendix B.* We present the rules of the fragment of type theory we model in this thesis.

Original Contribution

The original contributions of this thesis are thus:

1. The first complete exposition of the path object category axioms in the literature as well as an expansion of the material detailed in [vdBG12].

2. The introduction of *nice path object categories*. We prove that this refinement of the path object category axioms yields the necessary structure to construct categorical models of type theory, and has the benefit of side-stepping the introduction of tensorial strengths. We believe this framework can provide simplified and/or new proofs of known path object category structures as well as providing a simple set of axioms to identify new models of type theory with.

3. A new model of intensional type theory given in cubical sets with connections. This can be distinguished from the model in cubical sets given in [BCH14] in one key respect: in our model any cubical set may be interpreted as a context. In order to do so, we introduce *cubical n-paths* as well as *contraction operators* upon them and then prove that the category of cubical sets with connections carries a nice path object category structure.

Acknowledgements

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Contents

Part I

Path Object Category Structure On Cubical Sets With Connections

Chapter 1

Path Object Categories

We begin by introducing the central notion of this section of the thesis: that of a *path object category* [vdBG12]. By abstracting from the characteristic case in **Top** we can build up to the full definition gradually, articulating the motivation for each axiom and developing the precise details concurrently. To do so we will intermittently require additional category theoretic concepts: for the sake of a self contained exposition we give a full introduction to each as they are needed.

1.1 Axiom 1: Path Objects

Appropriately the first criteria we demand of a path object category is the existence of what we will call *path objects*. The idea is to be able to assign to each object X an object MX in the category that we interpret as containing all of the "paths" between "points" in X. That these collections of paths have the structure of an object of the category is the stringent condition that makes path object categories special. The principal example is given by the Moore path space of a topological space X

$$\{(r, \phi) \in \mathbb{R}_+ \times X^{\mathbb{R}_+} \mid \forall s \geq r(\phi(s) = \phi(r))\}$$

together with the subspace topology inherited from the usual topology on $\mathbb{R}_+ \times X^{\mathbb{R}_+}$. The path object category structure can be seen as an abstraction from this particular case.

What is particularly special in this case is that these Moore paths give the morphisms of a category based in X: We have $dom(r, \phi) = \phi(0)$, $cod(r, \phi) = \phi(r)$, whilst the composition of paths (r, ϕ) and (s, ψ) with $\phi(r) = \psi(0)$ is given by $(r + s, \theta)$ where

$$\theta(t) = \begin{cases} \phi(t) & \text{if } t \leq r \\ \psi(t - r) & \text{if } r \leq t \end{cases}$$

It is then immediate that $id_x = (0, t \mapsto x)$. We note that each of these maps is a continuous function - a morphism in **Top** - so this construction is entirely contained within the category. As we wish to replicate this we must first introduce an *internal* notion of category.

1.1.1 Internal Categories

To help understand the motivation behind this definition, as well as introduce the notation for the general case, we give a category theoretic procedure for specifying a small category within **Set**. We first specify the objects and morphisms of our category by choosing sets C_0 and C_1. These would need to satisfy certain coherence properties of course, and we thus require "source" and "target" morphisms $s, t : C_1 \to C_0$ specifying domain and codomain for our arrows. Each object in C_0 has an identity arrow, and we give this by specifying a morphism $e : C_0 \to C_1$ assigning identity arrows in such a way that

$$se = id_{C_0} = te$$

By taking a pullback with the source and target maps obtain the set of all pairs of composable arrows:

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{p_0} & C_1 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle t} \\ C_1 & \xrightarrow[s]{} & C_0 \end{array} \tag{1.1}$$

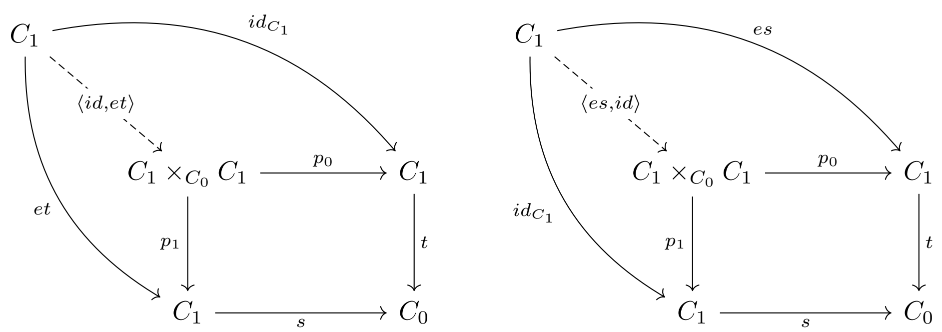
In **Set** the vertex of this pullback is of course given by $\{(f, g) \mid t(f) = s(g)\}$. We thus give composition by specifying a morphism

$$c : C_1 \times_{C_0} C_1 \to C_1$$

with uniqueness following from the fact c is a function. When the context is clear we will interchangeably denote $c(f, g)$ by $g \circ f$. We also require the commutativity of the following diagrams, ensuring that the domain and codomain of $g \circ f$ are $dom(f)$ and $cod(g)$ respectively

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{c} & C_1 \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle s} \\ C_1 & \xrightarrow[s]{} & C_0 \end{array} \qquad \begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{c} & C_1 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle t} \\ C_1 & \xrightarrow[t]{} & C_0 \end{array}$$

We now need to specify some conditions on the data we have thus far to satisfy the usual category axioms. First we ensure that identity arrows work as usual, with $f \circ id_{dom(f)} = f$ and $id_{cod(f)} \circ f = f$. Once again we take pullbacks: from the source and target pullback we obtain the following cones using the fact e is a retract of s and t.

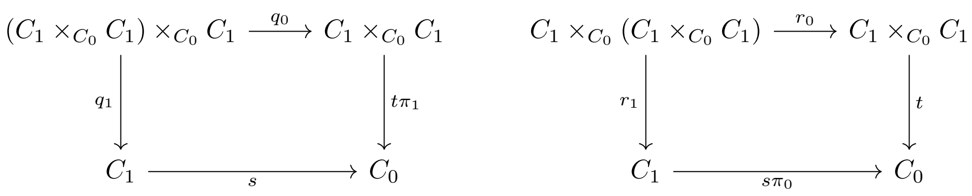


(1.2)

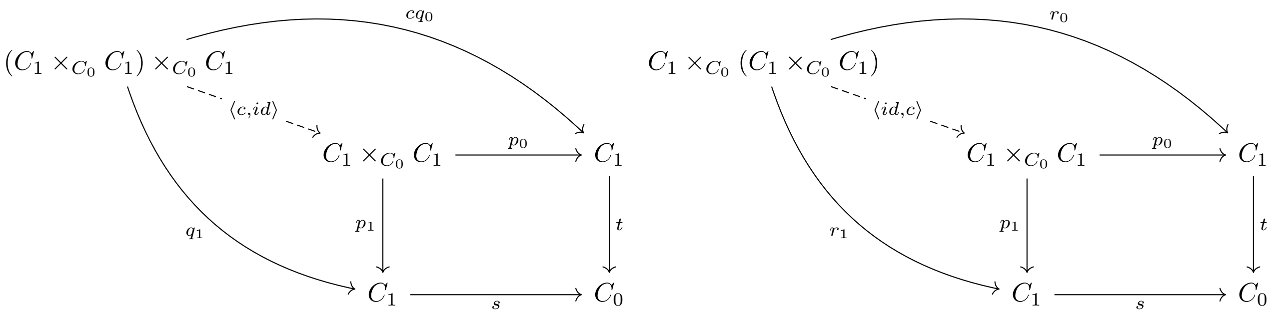
By computation we can see that the function $\langle id, et\rangle$ is given by $f \mapsto (f, id_{cod(f)})$, and similarly the function $\langle es, id\rangle$ is given by $f \mapsto (id_{dom(f)}, f)$. Hence the identity axiom is satisfied iff

$$c \circ \langle id, et\rangle = id_{C_1} = c \circ \langle es, id\rangle$$

Finally we ensure associativity is satisfied. We'd first give a set comprised of triples of composable morphisms by taking a pullback, and as one might expect there are two ways we can do that:



It is clear that these vertexes are identical up to deletion of brackets and hence isomorphic. However these diagrams both induce different cones on the source/target pullback:



(1.3)

Working out the details we see that these induced arrows are given by

$$\langle id, c\rangle(f, (g, h)) = (f, h \circ g)$$

$$\langle c, id\rangle((f, g), h) = (g \circ f, h)$$

Hence associativity reduces to the condition

$$c \circ \langle id, c\rangle = c \circ \langle c, id\rangle$$

With all of these properties satisfied we have specified a small category. By abstracting away from **Set** we obtain a general procedure. We note that we only require the existence of the pullbacks given in the preceding discussion, but we strengthen our definition to include the requirement of finite completeness, as this holds in all the cases of interest to us.

Definition 1.1 (Internal Category)**.** [Bor94a] Given a finitely complete $\mathcal{C}$, a *category internal to* $\mathcal{C}$

$$C_0 \underset{t}{\overset{s}{\leftleftarrows}} \xrightarrow{e} C_1 \xleftarrow{c} C_1 \times_{C_0} C_1$$

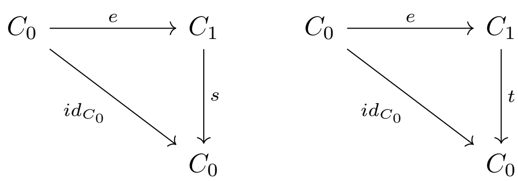
consists of the following data:

- **Objects**: An object C_0 in $\mathcal{C}$;

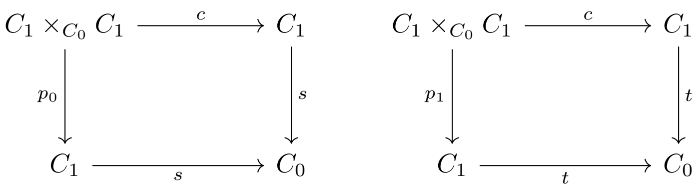
- **Arrows**: An object C_1 in $\mathcal{C}$;
- **Source/Target**: Morphisms $s, t : C_1 \to C_0$;
- **Identity**: A morphism $e : C_0 \to C_1$;
- **Composition**: A morphism $c : C_1 \times_{C_0} C_1 \to C_1$, where $C_1 \times_{C_0} C_1$ is given by (1.1);

such that the following diagrams commute:

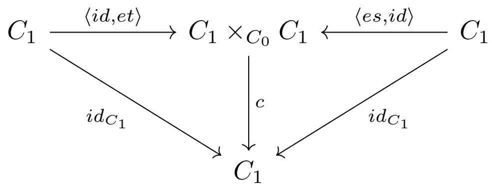
- **Source/Target of identities**:



- **Source/Target of compositions**:

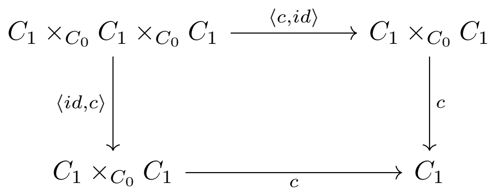


- **Left and right identity laws**:



  Where $\langle id, et \rangle$ and $\langle es, id \rangle$ are as given in (1.2).

- **Associativity**:



  Where $\langle c, id \rangle$ and $\langle id, c \rangle$ are as given in (1.3).

Remark 1.2. Although for our purposes we only require the definition of an internal category, one can also internalise the notion of functor, natural transformation, limits and more. For an introduction to this rich theory we direct the interested reader to [Bor94a] from which our presentation is taken.

1.1.2 Path Objects

With the idea of an internal category under our belts we can give the first half of our definition. Let $\mathcal{E}$ be a finitely complete category. We wish to assign to every object X an object of *paths through* X MX such that there is an internal category

$$X \underset{t_X}{\overset{s_X}{\leftleftarrows}} \xrightarrow{e_X} MX \xleftarrow{c_X} MX \times_X MX$$

We interpret the source map s_X as giving the start point of paths; similarly t_X gives end-points. Then the pullback $MX \times_X MX$ gives the object of concatenable paths and composition c_X performs that concatenation of paths. Finally the map e_X assigns trivial paths.

Notation 1.1.1. We will make frequent reference to the projection maps in (1.1). To prevent any ambiguity we denote the object they are associated with in the superscript:

$$\begin{array}{ccc} MX \times_X MX & \xrightarrow{p_0^X} & MX \\ {\scriptstyle p_1^X}\downarrow & & \downarrow{\scriptstyle t_X} \\ MX & \xrightarrow[s_X]{} & X \end{array}$$

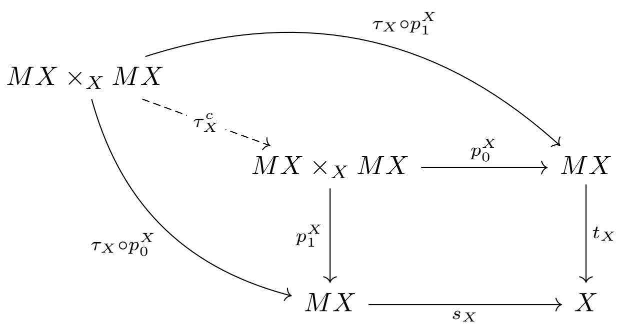
Another feature of the path structure on **Top** is *reversal* of paths. Given a Moore path (r, ϕ), the reversal can be given by (r, ϕ°) where

$$\phi^\circ(t) = \begin{cases} \phi(r-t) & \text{if } t \leq r \\ \phi(0) & \text{if } r \leq t \end{cases}$$

This induces an identity-on-objects involution on the Moore path category structure. We thus require such a morphism $\tau_X : MX \to MX$. This means that the following identities will be satisfied

$$\tau_X \circ \tau_X = id_{MX} \qquad \tau_X \circ e_X = e_X$$
$$s_X \circ \tau_X = t_X \qquad t_X \circ \tau_X = s_X$$

as well as *internal functorality*. We can define the map $\tau_X^c : MX \times_X MX \to MX \times_X MX$ making use of the identities just given:



We thus require $\tau_X \circ c_X = c_X \circ \tau^c_X$. This enforces that the reversal of a composition of paths is the composition of the reversals of the original paths.

To complete the idea we note that in **Top** any continuous function $f : X \to Y$ induces a map between Moore path spaces by the assignment $(r, \phi) \mapsto (r, f \circ \phi)$. It is immediate that this is functorial, and extends the assignment of Moore path spaces to a pullback preserving functor. Not only this, but this extension establishes the categorical structure maps as the components of natural transformations. We thus demand that the assignment M can be extended to a pullback preserving functor making s, t, e, c, τ natural transformations. That is

$$s : M \Rightarrow id \qquad t : M \Rightarrow id$$
$$e : id \Rightarrow M \qquad \tau : M \Rightarrow M$$

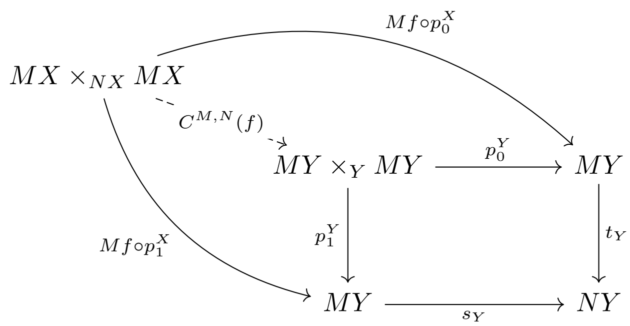
In order to state the case for c we need to confirm the assignment of pullback vertices $X \mapsto MX \times_X MX$ is functorial. To do so we prove the following general proposition:

Proposition 1.3. *Given endofunctors $M, N : \mathcal{C} \to \mathcal{C}$ and natural transformations $s, t : M \Rightarrow N$ there exists a canonical endofunctor $C^{M,N} : \mathcal{C} \to \mathcal{C}$ extending the assignment*

$$X \mapsto MX \times_{NX} MX$$

where $MX \times_X MX$ is the pullback along s_X and t_X.

Proof. By the naturality of s and t it follows for every $f : X \to Y$ in $\mathcal{C}$ we can give an assignment $f \mapsto C^{M,N}(f)$ where $C^{M,N}(f)$ is obtained



(1.4)

Since

$$\begin{aligned} t_Y \circ Mf \circ \pi_0^X &= Nf \circ t_X \circ \pi_0^X \\ &= Nf \circ s_X \circ \pi_1^X \\ &= s_Y \circ Mf \circ \pi_1^X \end{aligned}$$

Functorality of this assignment follows straightforwardly from functorality of M and N.

□

In our particular case we set $N = id$. Hence we demand that

$$c : C^M \Rightarrow M$$

We capture the discussion of this section in a definition:

Definition 1.4 (Has Path Objects)**.** A finitely complete category $\mathcal{E}$ *has path objects* if there exists a pullback preserving endofunctor $M : \mathcal{E} \to \mathcal{E}$ and natural transformations

$$\begin{aligned} s &: M \Rightarrow id & t &: M \Rightarrow id \\ e &: id \Rightarrow M & c &: C \Rightarrow M \\ \tau &: M \Rightarrow M & & \end{aligned}$$

such that, for all X in $\mathcal{E}$

i) $X \underset{t_X}{\overset{s_X}{\leftleftarrows}} \xrightarrow{e_X} MX \xleftarrow{c_X} MX \times_X MX$ is an internal category of $\mathcal{E}$.

ii) τ_X constitutes an identity-on-objects involution on the internal category MX with

$$\begin{aligned} \tau_X \circ \tau_X &= id_X & \tau_X \circ e_X &= e_X \\ s_X \circ \tau_X &= t_X & t_X \circ \tau &= s_X \\ \tau_X \circ c_X &= c_X \circ \tau_X^c & & \end{aligned}$$

Thus we can give the first path object category axiom:

Axiom 1: $\mathcal{E}$ has path objects

As motivation, we give a straightforward example of a category with path objects:

Example 1.1. *We show the category **Gpd** has path objects. Define $\mathcal{I}$ to be the groupoid with two objects 0 and 1 and two non identity arrows $0 \to 1$ and $1 \to 0$ which are each others' inverses. Given Γ in **Gpd** we define $M\Gamma = \Gamma^{\mathcal{I}}$. We can equivalently consider $\Gamma^{\mathcal{I}}$ as the groupoid with arrows $p : \gamma \to \gamma'$ of Γ as objects and commutative squares*

$$\begin{array}{ccc} \gamma & \xrightarrow{p} & \gamma' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ \delta & \xrightarrow[q]{} & \delta' \end{array}$$

as morphisms $(h, k) : p \to q$. *We define* $s_\Gamma, t_\Gamma : \Gamma^{\mathcal{I}} \to \Gamma$ *by the domain and codomain functors respectively:*

$$s_\Gamma(p : \gamma \to \gamma') = \gamma \qquad s_\Gamma((h,k)) = h$$
$$t_\Gamma(p : \gamma \to \gamma') = \gamma' \qquad t_\Gamma((h,k)) = k$$

We then give $e_\Gamma : \Gamma \to \Gamma^{\mathcal{I}}$ *as the functor assigning identity arrows:*

$$e_\Gamma(\gamma) = id_\gamma \qquad e_\Gamma(p : \gamma \to \gamma') = \begin{array}{ccc} \gamma & \xrightarrow{id_\gamma} & \gamma \\ \downarrow p & & \downarrow p \\ \gamma' & \xrightarrow[id_{\gamma'}]{} & \gamma' \end{array}$$

The pullback $\Gamma^{\mathcal{I}} \times_\Gamma \Gamma^{\mathcal{I}}$ *can be computed to be the groupoid comprised of composable pairs of* Γ*-morphisms* (p, q) *with morphisms* $(h, j, k) : (p, q) \to (p', q')$ *given by commutative diagrams*

$$\begin{array}{ccccc} \gamma & \xrightarrow{p} & \gamma' & \xrightarrow{q} & \gamma'' \\ h\downarrow & & j\downarrow & & \downarrow k \\ \delta & \xrightarrow[p']{} & \delta' & \xrightarrow[q']{} & \delta'' \end{array}$$

Hence we define $c_\Gamma : \Gamma^{\mathcal{I}} \times_\Gamma \Gamma^{\mathcal{I}} \to \Gamma^{\mathcal{I}}$ *by*

$$c_\Gamma(p, q) = q \circ p \qquad c_\Gamma(h, j, k) = (h, k)$$

Finally we define the involution $\tau_\Gamma : \Gamma^{\mathcal{I}} \to \Gamma^{\mathcal{I}}$

$$\tau_\Gamma(p : \gamma \to \gamma') = p \qquad \tau_\Gamma(h, k) = (h^{-1}, k^{-1})$$

Now it is straightforward to see that this data equips Γ *with the structure of an internal category (in fact, an internal groupoid) since the required properties are all inherited from* Γ *itself. The assignment* $\Gamma \mapsto \Gamma^{\mathcal{I}}$ *can be extended to the usual exponent functor* $(-)^{\mathcal{I}}$*, which is also pullback preserving. Naturality in the cases* s, t, e, τ *is straightforward, so we focus on* c*. We can compute the assignment of arrows of the functor* $C^{(\cdot)^{\mathcal{I}}} : \textbf{Gpd} \to \textbf{Gpd}$ *to be given by*

$$C^{(\cdot)^{\mathcal{I}}}(f)(p, q) = (f(p), f(q)) \qquad C^{(\cdot)^{\mathcal{I}}}(f)(h, j, k) = (f(h), f(k)) : (f(p), f(q)) \to (f(p'), f(q'))$$

Hence given a functor $f : \Gamma \to \Delta$ *naturality follows immediately by functorality:*

$$f^{\mathcal{I}} \circ c_\Gamma(p, q) = f^{\mathcal{I}}(q \circ p) = (f(q \circ p)) = (f(q) \circ f(p)) = c_\Delta(f(p), f(q)) = c_\Delta \circ C^{(\cdot)^{\mathcal{I}}}(f)(p, q)$$

with the case for morphisms similar.

1.2 Axiom 2: Constant Paths

The next piece of data we require of a path object category is the existence of *constant paths*. Many of the examples we want to fit into this framework come equipped with a notion of *length* for each path. Jumping ahead slightly, if we come to the idea of path contraction without this, problems can arise. For example in the topological case, in order to contract a path p through X to an end point we should demand a path of paths $\eta_X(p)$ given by

$$\begin{array}{ccc} s_X(p) & \xrightarrow{\quad p \quad} & t_X(p) \\ {\scriptstyle p}\downarrow & \overset{\eta_X(p)}{\Longrightarrow} & \downarrow{\scriptstyle e_X(t_X(p))} \\ t_X(p) & \xrightarrow[e_X(t_X(p))]{} & t_X(p) \end{array}$$

This requires that for η_X we have identity upon post-composition with s_{MX} or $M(s_X)$ and the composite $e_X \circ t_X$ upon post-composition with $t_M X$ or $M(t_X)$. However in this case our idea cannot work without constant paths. Hence if the Moore path p is length r, necessarily the corresponding $\eta_X(p)$ must also be length r since $M(s_X)$ preserves path length and $M(s_X)(\eta_X(p)) = p$. However by the same argument, for $r > 0$ we cannot have $M(t_X)(\eta_X(p)) = e_X(p)$ since $e_X(p)$ is length 0 and $M(t_X)$ preserves path length. Hence to obtain a notion of contraction in our structure we need to ensure $M(t_X)(\eta_X(p))$ is a non-trivial path of length r that is constant at $t_X(p)$.

Now we have that the terminal topological space given by $\{\star\}$ with the discrete topology and all Moore paths through this space are of the form $(k, t \mapsto \star)$.: We can then define $Con_x^X : \mathbb{R}^+ \to X$ by $Con_x^X(t) = x$ and give an assignment $M1 \times X \to MX$ by sending $((k, t \to \star), x)$ to (k, Con_x^X). By utilising some properties of the maps assigning this we can use these constant paths to perform path contraction. The idea is thus: interpret $M1$ as the "object of path lengths" and give a map

$$\alpha_{1,X} : M1 \times X \to MX$$

that takes a path length r and an object of x and returns the constant path at x of length r. These maps must interact appropriately with the natural transformations s, t, c, e and τ. To see how to resolve this we need to investigate the structure the product $\times$ enforces on our category.

1.2.1 Strong Functors

We begin with a definition:

Definition 1.5 (Monoidal Category)**.** [Bor94b] A *monoidal structure* $(\otimes, 1, \epsilon, \iota, a)$ on a category $\mathcal{C}$ consists of the following data:

- **Tensor Product**: A functor $\mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$;
- **Unit Object**: An object 1 in $\mathcal{C}$;
- **Unitors**: Natural isomorphisms

$$\epsilon : 1 \otimes - \Rightarrow id_{\mathcal{C}}$$

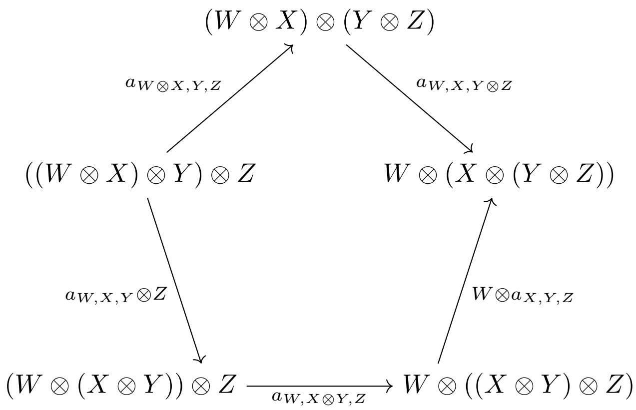
$$\iota : - \otimes 1 \Rightarrow id_{\mathcal{C}}$$

- **Associator**: A natural isomorphism

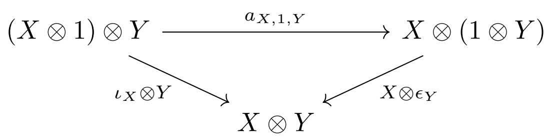
$$a : (- \otimes -) \otimes - \Longrightarrow - \otimes (- \otimes -)$$

Making the following diagrams commute

- **Pentagon Identity**



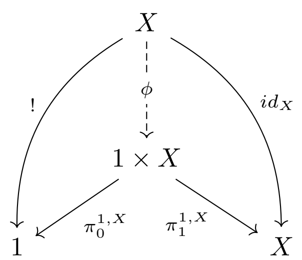
- **Triangle Identity**:



We call a category equipped with a monoidal structure a *monoidal category*. Further, we say a monoidal category is *strict* if the associator and left/right unitors are all identity morphisms.

As the name "tensor product" implies, the inspiration behind this notion comes from the category of vector spaces **Vect**. It is well known that the operation of taking the tensor product of vector spaces extends to linear maps, thus giving the requisite functor. An even simpler example of this phenomenon, however, is given by interpreting $\otimes$ to be $\times$ in a category with finite products:

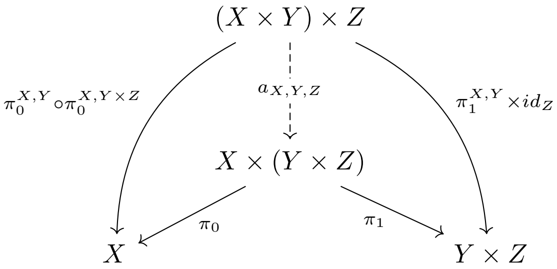
Example 1.2 (Cartesian Monoidal Category)**.** *Let $\mathcal{C}$ be a category with finite products as well as a canonical choice of product $A \times B$ for each pair of objects A, B in $\mathcal{C}$, say by assuming AC. Then there exists a* Cartesian monoidal structure *on $\mathcal{C}$ given by taking the tensor product to be the usual product and the unit object to be the terminal object* 1 *in $\mathcal{C}$.This is functorial because of our choice of products. To obtain the unitors we utilise the universal property of the product. First note we can obtain a morphism $\phi : X \to 1 \times X$ as follows*



We immediately have that $\pi_1^{1,X} \circ \phi = id_X$ *and we obtain* $\phi \circ \pi_1^{1,X} = id_1$ *by the universal property of the product since*

$$\pi_0^{1,X} \circ \phi \circ \pi_1^{1,X} = \pi_0^{1,X} \qquad \text{(1 is terminal)}$$
$$\pi_1^{1,X} \circ \phi \circ \pi_1^{1,X} = \pi_1^{1,X}$$

Hence $\pi_1^{1,X}$ *constitutes an isomorphism, and by exploiting properties of the product we can see that it is natural. Hence we have* $\epsilon_X = \pi_1^{1,X}$*. Similarly we obtain* $\iota_X = \pi_0^{X,1}$*. Finally we obtain the associator* $a_{X,Y,Z}$ *from the diagram*

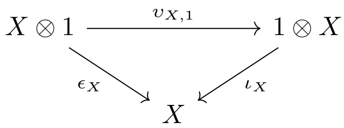


It is a straightforward but tedious exercise to verify the monoidal identities are satisfied by these natural isomorphisms.

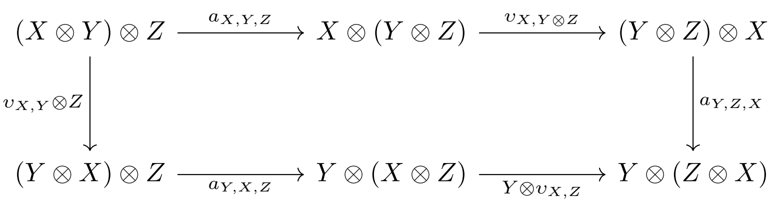
This example highlights a further property a monoidal category may satisfy: symmetry. It is easily verifiable that in a category with finite products there is a natural isomorphism $A \times B \simeq B \times A$. For a monoidal category to be symmetric there are some obvious identities that should hold to ensure the symmetry maps operate coherently with the existing structure:

Definition 1.6 (Symmetric Monoidal Category)**.** A monoidal category $\mathcal{C}$ is *symmetric* if there exists natural isomorphisms $\upsilon_{X,Y} : X \otimes Y \simeq Y \otimes X$ such that the following diagrams commute:

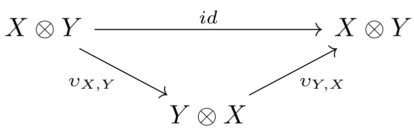
- **Unit Coherence**:



- **Associativity Coherence**:



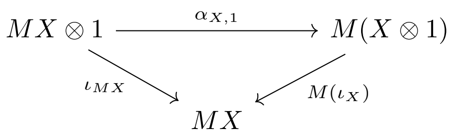
- **Inverse Law**:



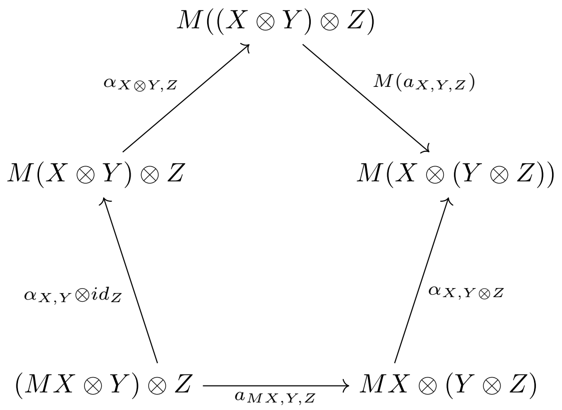
That the Cartesian moinoidal structure satisfies these additional properties follows immediately from basic facts about products. So why is this relevant to our interests? It turns out that, given an endofunctor M on a symmetric monoidal category, there is a notion of functor that allows us to move from tensor products of M-images and objects to the M-image of a tensor product in a coherent way:

Definition 1.7 (Strong Functor)**.** [Koc70] Given a symmetric monoidal category $\mathcal{C}$ a *tensorial strength* on an endofunctor $M : \mathcal{C} \to \mathcal{C}$ is a natural transformation $\alpha : M(-) \otimes (-) \Rightarrow M((-) \otimes (-))$ rendering the following diagrams commutative

- **Unitor**:



- **Associativity**



We call (M, α) a *strong functor.*

A trivial example of a tensorial strength is given by the identity maps $id_{X\otimes Y}$. As one might expect, these turn the identity endofunctor id: $\mathcal{C} \to \mathcal{C}$ into a strong functor. We also have a canonical strength for MM if (M, α) is a strong functor.

Example 1.3. *Suppose (M, α) is a strong functor on a symmetric monoidal category $(\mathcal{C}, \otimes)$. We define $\alpha^{\star}_{X,Y} : MMX \otimes Y \to MM(X \times Y)$ by*

$$\alpha^{\star}_{X,Y} = M(\alpha_{X,Y}) \circ \alpha_{MX,Y}$$

That this is a natural transformation follows from the fact that α is one: let $f : X \to X'$ and $g : Y \to Y'$. Then

$$\begin{aligned} MM(f \otimes g) \circ \alpha^{\star}_{X,Y} &= M(M(f \otimes g) \circ \alpha_{X,Y}) \circ \alpha_{MX,Y} \\ &= M(\alpha_{X',Y'}) \circ M(Mf \otimes g) \circ \alpha_{MX,Y} \\ &= M(\alpha_{X',Y'}) \circ \alpha_{MX',Y'} \circ (MMf \otimes g) \\ &= \alpha^{\star}_{X',Y'} \circ (MMf \otimes g) \end{aligned}$$

$\alpha^\star$ also inherits the unitor axiom:

$$\begin{aligned} MM(\rho_X) \circ \alpha^\star_{X,1} &= M(M(\rho_X) \circ \alpha_{X,1}) \circ \alpha_{MX,1} \\ &= M(\rho_{MX}) \circ \alpha_{MX,1} \\ &= \rho_{MMX} \end{aligned}$$

Showing associativity is a little trickier. We note that by naturality of α we have the commutative square

$$\begin{array}{ccc} M(MX \otimes Y) \otimes Z & \xrightarrow{\alpha_{MX\otimes Y,Z}} & M(MX \otimes Y \otimes Z) \\ \downarrow{\scriptstyle M(\alpha_{X,Y})\otimes id_Z} & & \downarrow{\scriptstyle M(\alpha_{X,Y}\otimes id_Z)} \\ MM(X \otimes Y) \otimes Z & \xrightarrow[\alpha_{M(X\otimes Y),Z}]{} & M(M(X \otimes Y) \otimes Z) \end{array} \tag{1.5}$$

Expanding $M(a_{X,Y,Z}) \circ \alpha^\star_{X\otimes Y,Z} \circ (\alpha^\star_{X,Y} \otimes id_Z)$ we obtain:

$$M(M(a_{X,Y,Z}) \circ \alpha_{X\otimes Y}) \circ \alpha_{M(X\otimes Y),Z} \circ (M(\alpha_{X,Y}) \otimes id_Z) \circ (\alpha_{X,Y} \otimes Z)$$

and then by computing we get

$$\begin{aligned} &M(M(a_{X,Y,Z}) \circ \alpha_{X\otimes Y}) \circ \alpha_{M(X\otimes Y),Z} \circ (M(\alpha_{X,Y}) \otimes id_Z) \circ (\alpha_{X,Y} \otimes Z) \\ &= M(M(a_{X,Y,Z}) \circ \alpha_{X\otimes Y} \circ (\alpha_{X,Y} \otimes id_Z)) \circ \alpha_{MX\otimes Y,Z} \circ (\alpha_{X,Y} \otimes id_Z) && \text{(1.5)} \\ &= M(\alpha_{X,Y\otimes Z} \circ a_{MX,Y,Z}) \circ \alpha_{MX\otimes Y,Z} \circ (\alpha_{MX,Y} \otimes id_Z) && \text{(Associativity of } \alpha) \\ &= M(\alpha_{X,Y\otimes Z}) \circ \alpha_{MX,Y\otimes Z} \circ a_{MMX,Y,Z} && \text{(Associativity of } \alpha) \\ &= \alpha^\star_{X,Y\otimes Z} \circ a_{MMX,Y,Z} \end{aligned}$$

as required: $\alpha^\star$ satisfies associativity and is indeed a tensorial strength for MM.

Appropriately for the task at hand, strong functors come with their own notion of natural transformation, which allows us to ensure the assignment of constant paths interacts coherently with the maps s_X, t_X, c_X, e_X and τ_X:

Definition 1.8 (Strong Natural Transformation)**.** Given a symmetric monoidal category $\mathcal{C}$ and strong functors $(M, \alpha), (N, \beta) : \mathcal{C} \to \mathcal{C}$ a *strong natural transformation* $\sigma : (M, \alpha) \Rightarrow (N, \beta)$ is a collection of maps

$$(\sigma_X : MX \to NX \mid X \text{ in } \mathcal{C}_0)$$

such that for all X, Y in $\mathcal{C}$ the following diagram commutes:

$$\begin{array}{ccc} MX \otimes Y & \xrightarrow{\alpha_{X,Y}} & M(X \otimes Y) \\ \downarrow{\scriptstyle \sigma_X\otimes id_Y} & & \downarrow{\scriptstyle \sigma_{X\otimes Y}} \\ NX \otimes Y & \xrightarrow[\beta_{X,Y}]{} & N(X \otimes Y) \end{array}$$

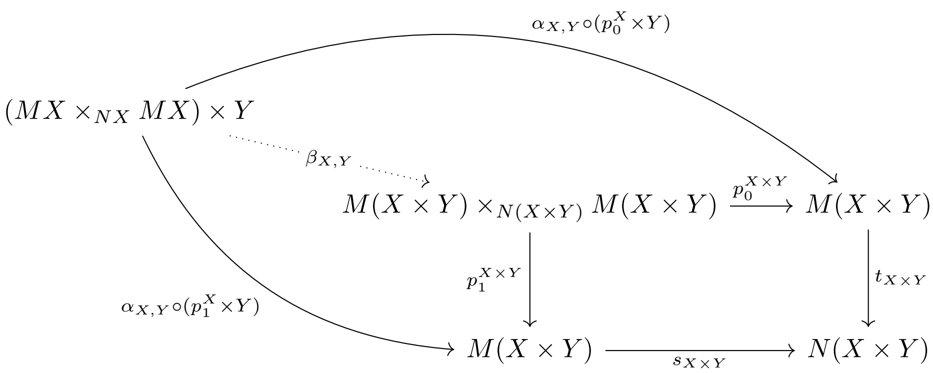
Remark 1.9. Strong functors were introduced by Kock in his and have strong links to the theory of enriched categories. The interested reader can consult for more information about this connection.

1.2.2 Constant Paths

With these new notions we are ready to define the next axiom of our framework. The assignment in **Top** of constant paths can be extended to a strength given by $((k,\phi),y) \mapsto (k,\psi)$ where $\psi(x) = (\phi(x), y)$. This strength has the property that the natural transformations s, t, e, c, τ are *strong* with respect to it. We thus demand for the second path object category the existence of a strength α for the endofunctor M such that the natural transformations s, t, e, c and τ are strong with respect to it. Strength and naturality then exhibit $M1 \times X$ as a retraction of MX, and we can interpret $\alpha_{1,X}$ as representing the subobject of constant paths through X. This idea makes immediate sense in the cases s, t, e and τ since we already know the strengths of all the functors involved, however we must specify the strength C^M is equipped with before we can state the demand that c is a strong natural transformation. Assuming that s and t are already strong natural transformations we once again can give a general result constructing a strength.

Proposition 1.10. *Given strong functors $(M,\alpha), (N,\alpha')$ if natural transformations $s,t : M \Rightarrow N$ are also strong then there exists a canonical strength for the functor $C^{M,N}$ of Proposition 1.3.*

Proof. Since s and t are strong natural transformations we have

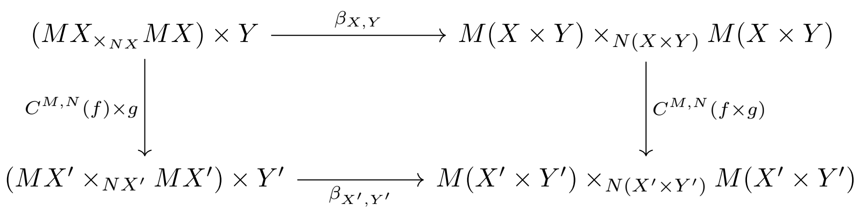


(1.6)

Since

$$\begin{aligned} t_{X\times Y} \circ \alpha_{X,Y} \circ (p_0^X \times id_Y) &= \alpha'_{X,Y} \circ (t_X \times id_Y) \circ (p_0^X \times id_Y) && (t \text{ a strong natural transformation}) \\ &= \alpha'_{X,Y} \circ (t_X \circ p_0^X \times id_Y) \\ &= \alpha'_{X,Y} \circ (s_X \circ p_1^X \times id_Y) \\ &= \alpha'_{X,Y} \circ (s_X \times id_Y) \circ (p_1^X \times id_Y) \\ &= s_{X\times Y} \circ \alpha_{X,Y} \circ (p_1^X \times id_Y) && (s \text{ a strong natural transformation}) \end{aligned}$$

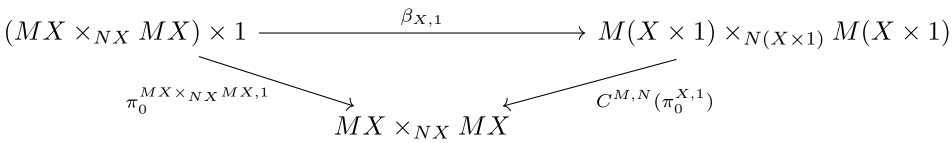
We claim $(\beta_{X,Y} \mid X, Y \text{ in } \mathcal{C}_0)$ is a strength for $C^{M,N}$. To show naturality we require, given $f : X \to X'$ and $g : Y \to Y'$, that the following diagram commutes



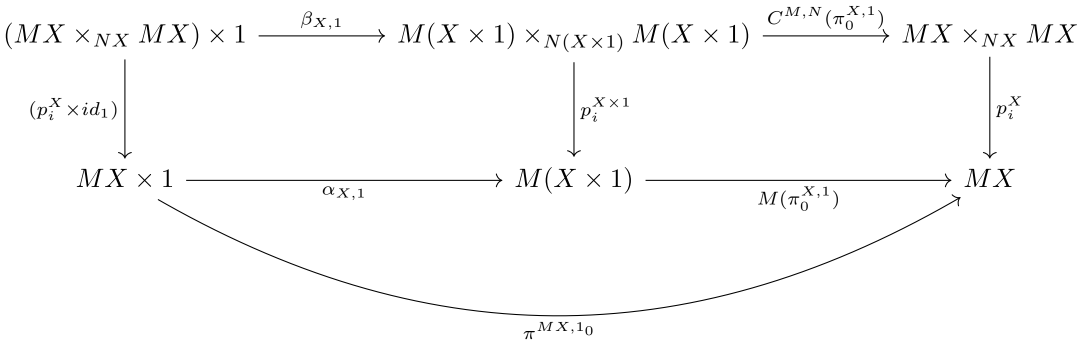
To show this it is sufficient to show equality upon post-composition with $p_0^{X\times Y}$ and $p_1^{X\times Y}$. Let $i \in \{0,1\}$. Then we have:

$$\begin{aligned}
p_i^{X'\times Y'} \circ C^{M,N}(f\times g)\circ \beta_{X,Y} &= M(f\times g)\circ p_i^{X\times Y}\circ \beta_{X,Y} && \text{(Commutativity of (1.4))}\\
&= M(f\times g)\circ \alpha_{X,Y}\circ (p_i^X \times id_Y) && \text{(Commutativity of (1.6))}\\
&= \alpha_{X',Y'}\circ (Mf\times g)\circ (p_i^X\times id_Y) && \text{(Naturality of } \alpha)\\
&= \alpha_{X',Y'}\circ (p_i^{X'}\circ C^{M,N}(f)\times g) && \text{(Commutativity of (1.4))}\\
&= p_i^{X'\times Y'}\circ \beta_{X',Y'}\circ (C^{M,N}(f)\times g) && \text{(Commutativity of (1.6))}
\end{aligned}$$

To prove β satisfies the unitor law we must show that the following diagram commutes:



It suffices to show equality upon post composition by the projections p_0^X and p_1^X. We observe that, for $i \in \{0,1\}$, the following diagram commutes:



The right-hand square commutes by (1.4), the left hand square commutes by (1.6) and the bottom commutes by the unitor law for α. Hence by taking the two possible paths around the perimeter we obtain the identity

$$p_i^X \circ C^{M,N}(\pi_0^{X,1})\circ \beta_{X,1} = \pi_0^{MX,1}\circ (p_i^X\times id_1)$$

We then immediately have

$$\pi_0^{MX,1}\circ (p_0^X\times id_1) = p_i^X \circ \pi_0^{MX\times_{NX}MX,1}$$

and so we obtain the required identities: β satisfies the unitor law. Finally we must show associativity is satisfied. It is sufficient to show equality upon post- composition with the projection maps $p_0^{X\times(Y\times Z)}$ and $p_1^{X\times(Y\times Z)}$. Attending to the route around the left-hand side of the pentagon first, we observe, for $i\in\{0,1\}$, that we have the following commutative diagram:

$$
\begin{array}{ccc}
(MX \times Y) \times Z & \xleftarrow{(p_i^X \times id_Y) \times id_Z} & (C^{M,N}(X) \times Y) \times Z \\
\scriptstyle{\alpha_{X,Y} \times id_Z}\downarrow & & \downarrow\scriptstyle{\beta_{X,Y} \times id_Z} \\
M(X \times Y) \times Z & \xleftarrow[(p_i^{X \times Y} \times id_Z)]{} & C^{M,N}(X \times Y) \times Z \\
\scriptstyle{\alpha_{(X \times Y),Z}}\downarrow & & \downarrow\scriptstyle{\beta_{X \times Y,Z}} \\
M((X \times Y) \times Z) & \xleftarrow{p_i^{(X \times Y) \times Z}} & C^{M,N}((X \times Y) \times Z) \\
\scriptstyle{M(a_{X,Y,Z})}\downarrow & & \downarrow\scriptstyle{C^{M,N}(a_{X,Y,Z})} \\
M(X \times (Y \times Z)) & \xleftarrow[p_i^{X \times (Y \times Z)}]{} & C^{M,N}(X \times (Y \times Z))
\end{array}
$$

The top and middle squares are commutative by (1.6) whilst the bottom square commutes by (1.4). Hence post composition of the left hand side of the β associativity diagram by $p_i^{X \times (Y \times Z)}$ is equal to pre-composition of the left hand side of the α associativity diagram by $(p_i^X \times id_Y) \times id_Z$. Applying α's associativity we obtain

$$\alpha_{X,Y \times Z} \circ a_{MX,Y,Z} \circ ((p_i^X \times id_Y) \times id_Z)$$

To see this is equal to travelling around the right-hand side of the pentagon we observe we have the following commutative diagram

$$
\begin{array}{ccccc}
(C^{M,N}(X) \times Y) \times Z & \xrightarrow{a_{C^{M,N}(X),Y,Z}} & C^{M,N}(X) \times (Y \times Z) & \xrightarrow{\beta_{X,Y \times Z}} & C^{M,N}(X \times (Y \times Z)) \\
\scriptstyle{(p_i^X \times id_Y) \times id_Z}\downarrow & & \downarrow\scriptstyle{p_i^X \times id_{Y \times Z}} & & \downarrow\scriptstyle{p_i^{X \times (Y \times Z)}} \\
(MX \times Y) \times Z & \xrightarrow[a_{MX,Y,Z}]{} & MX \times (Y \times Z) & \xrightarrow[\alpha_{X,Y \times Z}]{} & M(X \times (Y \times Z))
\end{array}
$$

The left hand square commutes by naturality of a, whilst the right commutes by (1.6). It follows that β is associative, and thus a strength for $C^{M,N}$. □

By applying this proposition with $N = id$ we obtain a strength β for C^M. Thus the coherence of composition with respect to the additional structure requires that

$$c : (C^M, \beta) \Rightarrow (M, \alpha)$$

. Once again we collect this discussion in a definition:

Definition 1.11 (Has Constant Paths)**.** Given a category $\mathcal{E}$ with path objects, we say $\mathcal{E}$ *has constant paths* if the endofunctor M is equipped with a strength

$$\alpha_{X,Y} : MX \times Y \to M(X \times Y)$$

with respect to which s, t, e, c and τ are strong natural transformations:

$$
\begin{array}{ll}
s : (M, \alpha) \Rightarrow (id, id) & t : (M, \alpha) \Rightarrow (id, id) \\
e : (id, id) \Rightarrow (M, \alpha) & c : (C^M, \beta) \Rightarrow (M, \alpha) \\
\tau : (M, \alpha) \Rightarrow (M, \alpha) &
\end{array}
$$

Hence we have Axiom 2:

Axiom 2: $\mathcal{E}$ has constant paths

Example 1.4. *We look back to* ***Gpd*** *to give an example of a category satisfying this axiom. We define the strength* $\alpha_{\Gamma,\Delta} : \Gamma^{\mathcal{I}} \times \Delta \to (\Gamma \times \Delta)^{\mathcal{I}}$ *by*

$$\alpha_{\Gamma,\Delta}(p : \gamma \to \gamma', \delta) = (p, id_\delta) : (\gamma, \delta) \to (\gamma', \delta)$$

$$\alpha_{\Gamma,\Delta}((h,k),r) = \begin{array}{ccc} \cdot & \xrightarrow{(p,id_\delta)} & \cdot \\ {\scriptstyle (h,r)}\downarrow & & \downarrow{\scriptstyle (k,r)} \\ \cdot & \xrightarrow[(q,id_{\delta'})]{} & \cdot \end{array}$$

Naturality is straightforward: let $f : \Gamma \to \Gamma'$ *and* $g : \Delta \to \Delta'$*. On objects we have*

$$(f \times g)^{\mathcal{I}} \circ \alpha_{\Gamma,\Delta}(p, \delta) = (f(p), id_{g(\delta)}) = \alpha_{\Gamma',\Delta'}(f(p), g(\delta)) = \alpha_{\Gamma',\Delta'} \circ (f^{\mathcal{I}} \times g)(p, \delta)$$

and naturality on arrows follows immediately by a similar argument: we leave the verification of the unitor and associativity laws to the reader. That s, t, e, τ *are strong natural transformations with respect to this strength is a straightforward argument, so once more we concentrate on* c*. Recall that the functor* $C^{(\cdot)^{\mathcal{I}}}$: ***Gpd*** $\to$ ***Gpd*** *was given by*

$$C^{(\cdot)^{\mathcal{I}}}(f)(p,q) = (f(p), f(q)) \qquad C^{(\cdot)^{\mathcal{I}}}(f)(h,j,k) = (f(h), f(k)) : (f(p), f(q)) \to (f(p'), f(q'))$$

By Proposition 1.10 we can compute the strength β *for this functor as*

$$\beta_{\Gamma,\Delta}((p,q), \delta) = ((p, id_\delta), (q, id_\delta))$$

$$\beta_{\Gamma,\Delta}((h,j,k),r) = \begin{array}{ccccc} \cdot & \xrightarrow{(p,id_\delta)} & \cdot & \xrightarrow{(q,id_\delta)} & \cdot \\ {\scriptstyle (h,r)}\downarrow & & \downarrow{\scriptstyle (j,r)} & & \downarrow{\scriptstyle (k,r)} \\ \cdot & \xrightarrow[(p',id_{\delta'})]{} & \cdot & \xrightarrow[(q',id_{\delta'})]{} & \cdot \end{array}$$

Thus we see that with respect to this strength c *is a strong natural transformation: on objects we have*

$$\alpha_{\Gamma,\Delta} \circ (c_\Gamma \times id_\Delta)((p,q), \delta) = (q \circ p, id_\delta) = c_{\Gamma \times \Delta}((p, id_\delta), (q, id_\delta)) = c_{\Gamma \times \Delta} \circ \beta_{\Gamma,\Delta}((p,q), \delta)$$

with verification on arrows similar.

1.3 Axiom 3: Path Contraction

With the material of the previous sections in place, the final axiom is much easier to state. As alluded to earlier, we ask for a notion of path contraction in the category: that is, the ability to contract paths onto one end point in a uniform and coherent way. In the case of **Top** we can assign to each Moore path (k, ϕ) through X the contraction path $(k, \psi : \mathbb{R}_+ \to \{(r, \phi) \in \mathbb{R}_+ \times X^{\mathbb{R}_+} \mid \forall s \geq r(\phi(s) = \phi(r))\})$ through the Moore path space, where for $t \leq k$ we have $\psi(t) = (l - t, \phi_t)$ where $\phi_t(x) = \phi(x + t)$ and for $k \leq t$ we have $\psi(t) = (0, \phi_k)$, noting that ϕ_k is equal to the constant function at $\phi(k)$. Applying the action of the functor on the codomain map to this path gives the constant path (k, ϕ_k) as discussed in the previous section.

Hence for each path ζ in MX we require a path of paths in MMX that starts at ζ and ends on a constant path at ζ's end point, and the assignment of this contraction path must respect the existing path object category structure. This means the assignment $\eta : M \Rightarrow MM$ must not only be a natural transformation, but also a strong natural transformation. There remains the question of the strength on MM but recall we gave just such a strength $\alpha^\star = M(\alpha_{X,Y}) \circ \alpha_{MX,Y}$. We can formalise this idea in the following definition:

Definition 1.12 (Has Path Contraction)**.** Given a category $\mathcal{E}$ with path objects and constant paths, we say $\mathcal{E}$ *has path contraction* if there exists a strong natural transformation

$$\eta : (M, \alpha) \Rightarrow (MM, \alpha^\star)$$

such that the following equations hold:

$$s_{MX} \circ \eta_X = id_{MX} \tag{1.7}$$

$$t_{MX} \circ \eta_X = e_X \circ t_X \tag{1.8}$$

$$M(s_X) \circ \eta_X = id_{MX} \tag{1.9}$$

$$M(t_X) \circ \eta_X = M(\pi_1^{1,X}) \circ \alpha_{1,X} \circ (M(!), t_X) \tag{1.10}$$

$$\eta_X \circ e_X = e_{MX} \circ e_X \tag{1.11}$$

This gives us the final axiom for a path object category:

Axiom 3: $\mathcal{E}$ has path contraction.

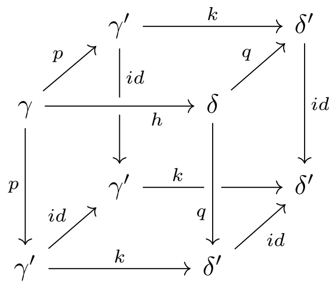
Theorem 1.13. *[vdBG12, Proposition 5.1.1] The category **Top** carries the structure of a path object category.*

Example 1.5. *We continue our case study in **Gpd** and show it satisfies the third and final axiom. We define $\eta : (-)^{\mathcal{I}} \Rightarrow ((-)^{\mathcal{I}})^{\mathcal{I}}$ by*

- ***Objects:***

$$\eta_\Gamma(p : \gamma \to \gamma') = \begin{array}{ccc} \gamma & \xrightarrow{p} & \gamma' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle id_{\gamma'}} \\ \gamma' & \xrightarrow[id_{\gamma'}]{} & \gamma' \end{array}$$

- ***Arrows***: *The morphism* $\begin{array}{ccc} \gamma & \xrightarrow{p} & \gamma' \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} \\ \delta & \xrightarrow{q} & \delta' \end{array}$ *is sent to the commutative cube*



We first verify this is a strong natural transformation. Once again we just show the action on objects and let the reader satisfy herself it works analogously for arrows. We have

$$\eta_{\Gamma\times\Delta} \circ \alpha_{\Gamma,\Delta}(p,\delta) = \eta_{\Gamma\times\Delta}(p, id_\delta)$$

$$= \begin{array}{ccc} & \xrightarrow{(p,id_\delta)} & \\ \downarrow{\scriptstyle (p,id_\delta)} & & \downarrow{\scriptstyle (id,id)} \\ & \xrightarrow{(id,id)} & \end{array}$$

Similarly we obtain

$$(\alpha_{\Gamma,\Delta})^{\mathcal{I}} \circ \alpha_{\Gamma^{\mathcal{I}},\Delta} \circ (\eta_\Gamma \times id_\Delta)(p,\delta) = ((\alpha_{\Gamma,\Delta})^{\mathcal{I}} \circ \alpha_{\Gamma^{\mathcal{I}},\Delta}) \left(\begin{array}{ccc} & \xrightarrow{p} & \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle id} \\ & \xrightarrow{id} & \end{array}, \delta \right)$$

$$= (\alpha_{\Gamma,\Delta})^{\mathcal{I}} \left(\begin{array}{ccc} & \xrightarrow{p} & \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle id} \\ & \xrightarrow{id} & \end{array}, id_\delta \right) = \begin{array}{ccc} & \xrightarrow{(p,id_\delta)} & \\ \downarrow{\scriptstyle (p,id_\delta)} & & \downarrow{\scriptstyle (id,id)} \\ & \xrightarrow{(id,id)} & \end{array}$$

Finally we show each of the equations is satisfied:

(1.7) $s_{\Gamma^{\mathcal{I}}} \circ \eta_\Gamma(p : \gamma \to \gamma') = s_{\Gamma^{\mathcal{I}}}((p, id_{\gamma'}) : p \to id_{\gamma'}) = p = id_{\Gamma^{\mathcal{I}}}(p)$

(1.8) $t_{\Gamma^{\mathcal{I}}} \circ \eta_\Gamma(p : \gamma \to \gamma') = t_{\Gamma^{\mathcal{I}}}((p, id_{\gamma'}) : p \to id_{\gamma'}) = id_{\gamma'} = e(\gamma') = e \circ t(p : \gamma \to \gamma')$

(1.9) $(s_\Gamma)^{\mathcal{I}} \circ \eta_\Gamma(p : \gamma \to \gamma') = (s_\Gamma)^{\mathcal{I}}((p, id_{\gamma'}) : p \to id_{\gamma'}) = (s_\Gamma)(p, id_{\gamma'}) = p = id_{\Gamma^{\mathcal{I}}}(p)$

(1.10) $t_\Gamma^{\mathcal{I}} \circ \eta_\Gamma(p) = t_\Gamma^{\mathcal{I}}(p, id_{\gamma'}) = id_{\gamma'} = M(\pi_1^{1,X}) \circ \alpha_{1,\Gamma}(id_\star, \gamma') = M(\pi_1^{1,X}) \circ \alpha_{1,\Gamma} \circ (!^{\mathcal{I}}, t_\Gamma)(p)$

(1.11 $\eta_\Gamma \circ e_\Gamma(\gamma) = \eta_\Gamma(id_\gamma) = (id_\gamma, id_\gamma) : id_\gamma \to id_\gamma = id_{id_\gamma} = e_{\Gamma^{\mathcal{I}}} \circ e_\Gamma(\gamma)$

Thus we summarise our running example as a theorem:

Theorem 1.14. ***Gpd*** *is a path object category.*

We can now collect the work of this chapter into a single definition for the sake of readability:

Definition 1.15 (Path Object Category)**.** A finitely complete category $\mathcal{E}$ is called a *path object category* if the following three axioms are satisfied:

- **Axiom 1.** $\mathcal{E}$ has path objects:
  There exists a pullback preserving endofunctor $M : \mathcal{E} \to \mathcal{E}$ and natural transformations

$$s : M \Rightarrow id \qquad t : M \Rightarrow id$$
$$e : id \Rightarrow M \qquad c : C^M \Rightarrow M$$
$$\tau : M \Rightarrow M$$

  such that, for all X in $\mathcal{E}$

  i) $X \overset{s_X}{\underset{t_X}{\leftleftarrows}} \xrightarrow{e_X} MX \xleftarrow{c_X} MX \times_X MX$ is an internal category of $\mathcal{E}$.

  ii) τ_X constitutes an identity-on-objects involution on the internal category MX with

$$\tau_X \circ \tau_X = id_X \qquad \tau_X \circ e_X = e_X$$
$$s_X \circ \tau_X = t_X \qquad t_X \circ \tau = s_X$$
$$\tau_X \circ c_X = c_X \circ \tau_X^c$$

- **Axiom 2.** $\mathcal{E}$ has constant paths:
  The endofunctor M comes equipped with a strength

$$\alpha_{X,Y} : MX \times Y \to M(X \times Y)$$

  with respect to which s, t, e, c and τ are strong natural transformations:

$$s : (M, \alpha) \Rightarrow (id, id) \qquad t : (M, \alpha) \Rightarrow (id, id)$$
$$e : (id, id) \Rightarrow (M, \alpha) \qquad c : (C^M, \beta) \Rightarrow (M, \alpha)$$
$$\tau : (M, \alpha) \Rightarrow (M, \alpha)$$

- **Axiom 3.** $\mathcal{E}$ has path contraction:
  There exists a strong natural transformation

$$\eta : (M, \alpha) \Rightarrow (MM, \alpha^\star)$$

such that the following equations hold:

$$\begin{aligned}
s_{MX} \circ \eta_X &= id_{MX} \\
t_{MX} \circ \eta_X &= e_X \circ t_X \\
M(s_X) \circ \eta_X &= id_{MX} \\
M(t_X) \circ \eta_X &= M(\pi_1^{1,X}) \circ \alpha_{1,X} \circ (M(!), t_X) \\
\eta_X \circ e_X &= e_{MX} \circ e_X
\end{aligned}$$

Besides the examples of **Top** and **Gpd** there are a number of other important examples, proofs of which can all be found in van den Berg and Garner's [vdBG12] (in which the definition of path object category was originally given), with the exception of the final item which appears in [vOar].

Theorem 1.16. *The following carry the structure of a path object category:*

- *The category of simplicial sets* ***sSet****; [vdBG12, Section 7]*
- *The category of chain complexes over a ring* $\mathcal{R}$*; [vdBG12, Proposition 5.3.2]*
- *Any interval object category; [vdBG12, Proposition 5.4.3]*
- *The effective topos. [vOar, Proposition 1.6]*

The goal of the first part of this thesis is to add the category of cubical sets with connections to this list. In order to facilitate this we dedicate the next chapter to a refinement of the path object category axioms that we prove is enough to yield the requisite structure.

Chapter 2

Nice Path Object Categories

In this chapter we refine the path object category axioms and introduce a new concept: that of a *nice* path object category. Nice path object categories have two advantages over the regular kind: first, they allow us to ignore all issues of path length; second, they allow us to avoid the introduction of tensorial strengths. Beyond this it seems many examples of path object category structures are already nice path object category structures, or will be after the introduction of a sensible equivalence relation.

The introduction of tensorial strengths appeared to be vital in the case of **Top** as Moore paths come equipped with a notion of length that the path object structure must interact coherently with. Despite this, a close examination of the example of **Gpd** reveals that these concerns do not always apply.

Recall that the endofunctor $M : \mathbf{Gpd} \to \mathbf{Gpd}$ was defined to be exponentiation by the interval groupoid $\mathcal{I}$. In this case the object of path lengths $M1$ is trivial, as

$$M1 = \{\star\}^{\mathcal{I}} \cong \{\star\} = 1$$

We note further that the constant paths assigned by the strength $\alpha_{1,X}$ coincide with the trivial paths assigned by the natural transformation e. Thus it appears that our initial intuition and our work-around coincide in the case of **Gpd**. This begs the question: might we be able to modify the path object category axioms to account for this situation? This motivates the following definition:

Definition 2.1 (Nice Path Object Category)**.** A finitely complete category $\mathcal{E}$ is called a *nice path object category* if the following modified path object category axioms are satisfied:

- **Axiom** 1. $\mathcal{E}$ has *path objects*:
  There exists a pullback preserving endofunctor $M : \mathcal{E} \to \mathcal{E}$ and natural transformations

$$s : M \Rightarrow id \qquad t : M \Rightarrow id$$
$$e : id \Rightarrow M \qquad c : C^M \Rightarrow M$$
$$\tau : M \Rightarrow M$$

  such that, for all X in $\mathcal{E}$

i) $X \underset{t_X}{\overset{s_X}{\leftleftarrows}} \!\!\xrightarrow{e_X}\! MX \xleftarrow{c_X} MX \times_X MX$ is an internal category of $\mathcal{E}$.

ii) τ_X constitutes an identity-on-objects involution on the internal category MX with

$$\begin{aligned} \tau_X \circ \tau_X &= id_X & \qquad \tau_X \circ e_X &= e_X \\ s_X \circ \tau_X &= t_X & \qquad t_X \circ \tau &= s_X \\ \tau_X \circ c_X &= c_X \circ \tau_X^c \end{aligned}$$

- **Axiom 2′.** $\mathcal{E}$ has *nice constant paths*:

$$M1 \cong 1$$

- **Axiom 3′.** $\mathcal{E}$ has *nice path contraction*:
  There exists a natural transformation

$$\eta : M \Rightarrow MM$$

  such that the following equations hold:

$$\begin{aligned} s_{MX} \circ \eta_X &= id_{MX} \\ t_{MX} \circ \eta_X &= e_X \circ t_X \\ M(s_X) \circ \eta_X &= id_{MX} \\ M(t_X) \circ \eta_X &= e_X \circ t_X \\ \eta_X \circ e_X &= e_{MX} \circ e_X \end{aligned}$$

We leave it to the reader to return to the example of Chapter 1 and convince herself of the following theorem:

Theorem 2.2. ***Gpd*** *is a nice path object category.*

We dedicate the remainder of this chapter to proving that this definition is enough to prove an instantiation of the path object category axioms. We begin by constructing a strength:

Proposition 2.3. *Given a nice path object category $\mathcal{E}$ there exists a strength α for the endofunctor M.*

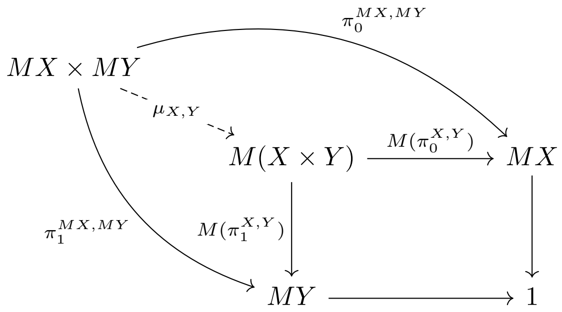
Proof. Recall that given X and Y in a category with pullbacks the product $X \times Y$ can be obtained as the pullback

$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_0^{X,Y}} & X \\ {\scriptstyle \pi_1^{X,Y}} \downarrow & & \downarrow \\ Y & \longrightarrow & 1 \end{array}$$

Since M is pullback preserving we thus obtain a natural isomorphism

$$\mu_{X,Y} : MX \times MY \to M(X \times Y)$$

from the pullback diagram



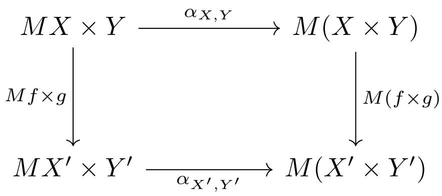
(2.1)

Hence we claim the collection of maps $(\alpha_{X,Y} : MX \times Y \to M(X \times Y) \mid X, Y \text{ in } \mathcal{E}_0)$ defined

$$\alpha_{X,Y} = \mu_{X,Y} \circ (id_{MX} \times e_Y)$$

yields a tensorial strength for the endofunctor M. We must satisfy three properties:

- **Naturality**: Let $f : X \to X'$ and $g : Y \to Y'$. We must show that the following diagram commutes:

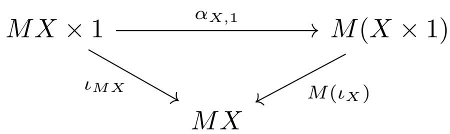


By applying the naturality of μ and e we can compute this directly:

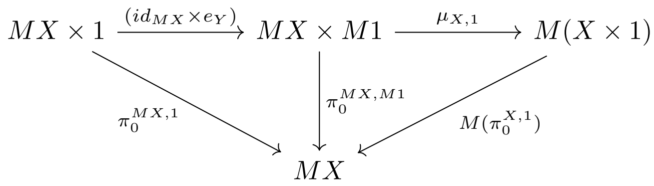
$$\begin{aligned} M(f \times g) \circ \alpha_{X,Y} &= M(f \times g) \circ \mu_{X,Y} \circ (id_{MX} \times e_Y) \\ &= \mu_{X',Y'} \circ (Mf \times Mg) \circ (id_{MX} \times e_Y) && \text{(Naturality of } \mu) \\ &= \mu_{X',Y'} \circ (Mf \times (Mg \circ e_Y)) \\ &= \mu_{X',Y'} \circ (Mf \times (e_{Y'} \circ g)) && \text{(Naturality of } e) \\ &= \mu_{X',Y'} \circ (id_{MX'} \times e_{Y'}) \circ (Mf \times g) \\ &= \alpha_{X',Y'} \circ (Mf \times g) \end{aligned}$$

Hence α is a natural transformation $\alpha : M(-) \times (-) \Rightarrow M((-) \times (-))$ as required.

- **Unitor**: Recall that we must show commutativity of the following diagram:

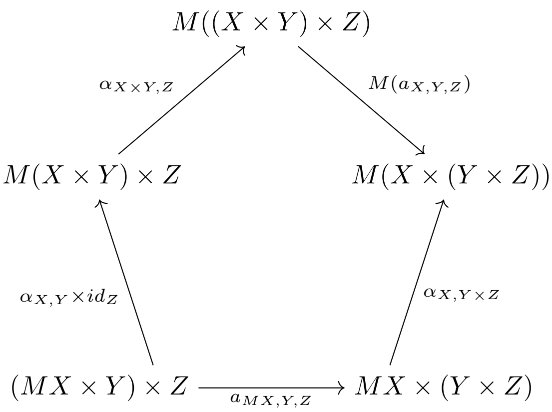


We remind the reader that for the Cartesian monoidal structure the maps ι_X are given by the projections $\pi_0^{X,1}$. Hence commutativity follows from the following diagram

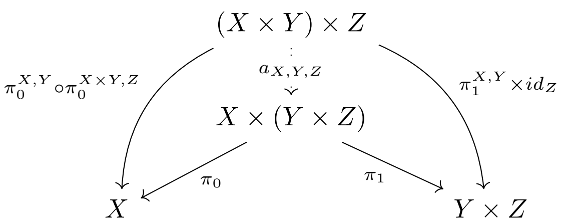


The triangle on the right commutes by (2.1) whilst the left triangle is commutative by the definition of $id_{MX} \times e_Y$.

- **Associativity**: Recall that we must show commutativity of the following diagram



In the Cartesian monoidal structure the natural isomorphism $a_{X,Y,Z}$ is obtained by the universal property of the product as follows:



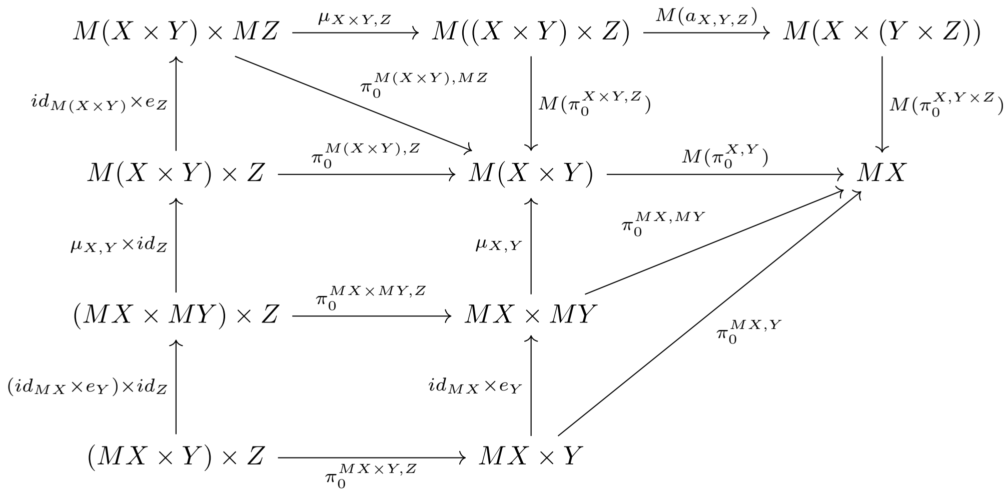
(2.2)

and applying M gives us the identities

$$M(\pi_0^{X,Y\times Z}) \circ M(a_{X,Y,Z}) = M(\pi_0^{X,Y}) \circ M(\pi_0^{X\times Y,Z}) \tag{2.3}$$

$$M(\pi_1^{X,Y\times Z}) \circ M(a_{X,Y,Z}) = M(\pi_1^{X,Y} \times id_Z) \tag{2.4}$$

Now since we have that $M(X \times (Y \times Z))$ is the vertex of a pullback it suffices to prove the two different routes around the associativity diagram are identical upon post-composition with the projection maps $M(\pi_0^{X,Y\times Z})$ and $M(\pi_1^{X,Y\times Z})$. We begin with the first of these cases. First observe that the following diagram is commutative:



In the left-hand column: the top triangle commutes by (2.1), whilst the lower triangle and squares all follow by definition. In the right-hand column: the top square commutes by (2.3), the upper triangle commutes by (2.1) and the lower triangle is an instance of the universal property of the product. Hence by traversing the perimeter of the diagram in both directions, we obtain the identity:

$$M(\pi_0^{X,Y\times Z}) \circ M(a_{X,Y,Z}) \circ \alpha_{X\times Y,Z} \circ (\alpha_{X,Y} \times id_Z) = \pi_0^{MX,Y} \circ \pi_0^{MX\times Y,Z}$$

We also have:

$$\begin{aligned} M(\pi_0^{X,Y\times Z}) \circ \alpha_{X,Y\times Z} \circ a_{MX,Y,Z} &= M(\pi_0^{X,Y\times Z}) \circ \mu_{X,Y\times Z} \circ (id_{MX} \times e_{Y\times Z}) \circ a_{MX,Y,Z} \\ &= \pi_0^{MX,M(Y\times Z)} \circ (id_{MX} \times e_{Y\times Z}) \circ a_{MX,Y,Z} && (2.1) \\ &= \pi_0^{MX,Y\times Z} \circ a_{MX,Y,Z} \\ &= \pi_0^{MX,Y} \circ \pi_0^{MX,Y\times Z} && (2.2) \end{aligned}$$

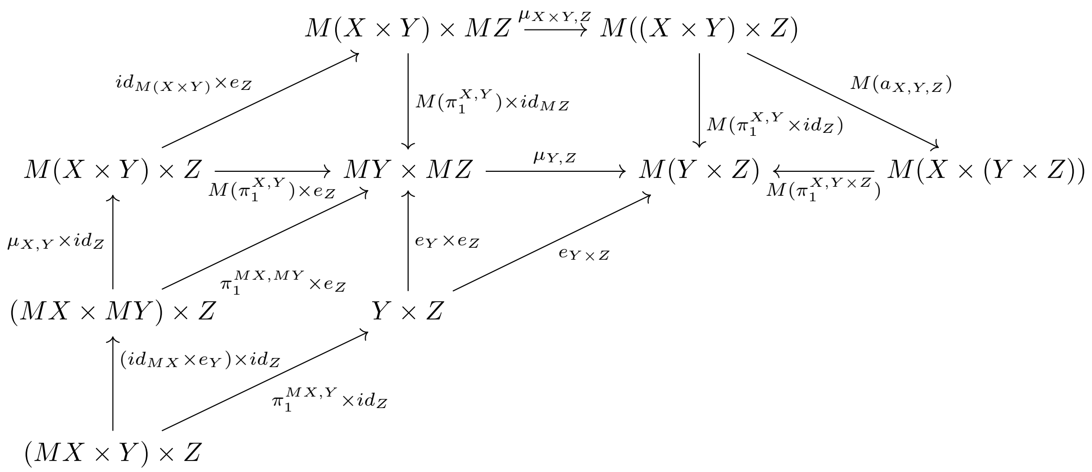
Hence we indeed have equality upon post-composition with $M(\pi_0^{X,Y\times Z})$. Moving to the second case, we first observe that, for all X, Y in $\mathcal{E}$ we have the identity

$$e_{X\times Y} = \mu_{X,Y} \circ (e_X \times e_Y) \tag{2.5}$$

Once again we can verify this by checking we have identity upon post-composition by $M(\pi_0^{X,Y})$ and $M(\pi_1^{X,Y})$. In the case for $M(\pi_0^{X,Y})$ we have:

$$\begin{aligned} M(\pi_0^{X,Y}) \circ \mu_{X,Y} \circ (e_X \times e_Y) &= \pi_0^{MX,MY} \circ (e_X \times e_Y) && 2.1 \\ &= e_X \circ \pi_0^{X,Y} \\ &= M(\pi_0^{X,Y}) \circ e_{X\times Y} && \text{(Naturality of } e\text{)} \end{aligned}$$

The analogous argument gives the latter case. With this we can see that the following diagram commutes:



In the left-most column: that the upper triangle commutes is immediate, the lower triangle commutes by (2.1) and the bottom square commutes by definition. In the middle column: the upper square commutes by naturality of μ and the lower triangle commutes by (2.5). Finally the rightmost triangle commutes by (2.4), By traversing the perimeter of the diagram in the two possible directions we obtain the identity:

$$M(\pi_1^{X,Y\times Z}) \circ M(a_{X,Y,Z}) \circ \alpha_{X\times Y,Z} \circ (\alpha_{X,Y} \times id_Z) = e_{Y\times Z} \circ (\pi_1^{MX,Y} \times id_Z)$$

Conversely we have:

$$\begin{aligned}
M(\pi_1^{X,Y\times Z}) \circ \alpha_{X,Y\times Z} \circ a_{MX,Y,Z} &= M(\pi_1^{X,Y\times Z}) \circ \mu_{X,Y\times Z} \circ (id_{MX} \times e_{Y\times Z}) \circ a_{MX,Y,Z} \\
&= \pi_1^{MX,M(Y\times Z)} \circ (id_{MX} \times e_{Y\times Z}) \circ a_{MX,Y,Z} && (2.1) \\
&= e_{Y\times Z} \circ \pi_1^{MX,Y\times Z} \circ a_{MX,Y,Z} \\
&= e_{Y\times Z} \circ (\pi_1^{MX,Y} \times id_Z) && (2.2)
\end{aligned}$$

So the two morphisms are also equal under post-composition with $M(\pi_1^{X,Y\times Z})$. It follows that the diagram commutes: α satisfies associativity and is thus a strength for M.

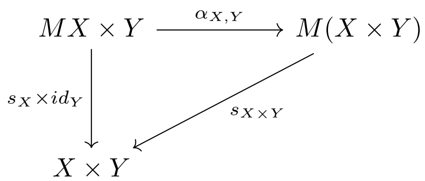
□

The next step is to verify the natural transformations of the nice path object category $\mathcal{E}$ are strong with respect to the strength we have defined. By a result of Kock [Joh97, Proposition 3.1] the strength we have defined is in fact the unique strength for M that renders e strong, but we can show that s, t, c and τ also become strong. We begin with the simpler cases.

Proposition 2.4. *Given a nice path object category $\mathcal{E}$, the natural transformations $s, t : M \Rightarrow id$, $e : id \Rightarrow M$ and $\tau : M \Rightarrow M$ are strong natural transformations with respect to the strength given in Proposition 2.3.*

Proof. We prove each case in turn:

- **s, t**: We only give the proof for s since the argument for t is essentially the same. Recall that we require commutativity of the following diagram:



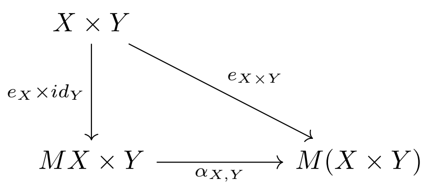
By the universal property of the product it is sufficient to show equality upon post-composition by the projections $\pi_0^{X,Y}$ and $\pi_1^{X,Y}$, We have:

$$\begin{aligned}
\pi_0^{X,Y} \circ s_{X\times Y} \circ \alpha_{X,Y} &= \pi_0^{X,Y} \circ s_{X\times Y} \circ \mu_{X,Y} \circ (id_{MX} \times e_Y) \\
&= s_X \circ M(\pi_0^{X,Y}) \circ \mu_{X,Y} \circ (id_{MX} \times e_Y) && \text{(Naturality of } s\text{)} \\
&= s_X \circ \pi_0^{MX,MY} \circ (id_{MX} \times e_Y) && \text{(Commutativity of (2.1))} \\
&= \pi_0^{X,Y} \circ (s_X \times s_Y) \circ (id_{MX} \times e_Y) \\
&= \pi_0^{X,Y} \circ (s_X \times id_Y) && \text{(Source of Identities law)}
\end{aligned}$$

$$\begin{aligned}
\pi_1^{X,Y} \circ s_{X\times Y} \circ \alpha_{X,Y} &= \pi_1^{X,Y} \circ s_{X\times Y} \circ \mu_{X,Y} \circ (id_{MX} \times e_Y) \\
&= s_Y \circ M(\pi_1^{X,Y}) \circ \mu_{X,Y} \circ (id_{MX} \times e_Y) && \text{(Naturality of } s\text{)} \\
&= s_Y \circ \pi_1^{MX,MY} \circ (id_{MX} \times e_Y) && \text{(Commutativity of (2.1))} \\
&= \pi_1^{X,Y} \circ (s_X \times s_Y) \circ (id_{MX} \times e_Y) \\
&= \pi_1^{X,Y} \circ (s_X \times id_Y) && \text{(Source of Identities law)}
\end{aligned}$$

Hence $s_{X\times Y} \circ \alpha_{X,Y} = s_X \times id_Y$ and we have $s : (M, \alpha) \Rightarrow (id, id)$ as required.

- **e**: We must verify the commutativity of the diagram



Since $M(X \times Y)$ is the vertex of the pullback (2.1) it is suffices to show equality upon post-composition with the morphisms $M(\pi_0^{X,Y})$ and $M(\pi_1^{X,Y})$. We have

$$\begin{aligned}
M(\pi_0^{X,Y}) \circ \alpha_{X,Y} \circ (e_X \times id_Y) &= M(\pi_0^{X,Y}) \circ \mu_{X,Y} \circ (e_X \times e_Y) \\
&= \pi_0^{MX,MY} \circ (e_X \times e_Y) && \text{(Commutativity of 2.1)} \\
&= e_X \circ \pi_0^{MX,Y} \\
&= M(\pi_0^{X,Y}) \circ e_{X \times Y} && \text{(Naturality of } e\text{)}
\end{aligned}$$

$$\begin{aligned}
M(\pi_1^{X,Y}) \circ \alpha_{X,Y} \circ (e_X \times id_Y) &= M(\pi_1^{X,Y}) \circ \mu_{X,Y} \circ (e_X \times e_Y) \\
&= \pi_1^{MX,MY} \circ (e_X \times e_Y) && \text{(Commutativity of 2.1)} \\
&= e_Y \circ \pi_1^{MX,Y} \\
&= M(\pi_1^{X,Y}) \circ e_{X \times Y} && \text{(Naturality of } e\text{)}
\end{aligned}$$

Hence $e_{X \times Y} = \alpha_{X,Y} \circ (e_X \times id_Y)$ and $e : (id, id) \Rightarrow (M, \alpha)$ as required.

- τ: We must verify commutativity of the diagram:

$$\begin{array}{ccc}
MX \times Y & \xrightarrow{\alpha_{X,Y}} & M(X \times Y) \\
{\scriptstyle \tau_X \times id_Y}\downarrow & & \downarrow{\scriptstyle \tau_{X \times Y}} \\
MX \times Y & \xrightarrow[\alpha_{X,Y}]{} & M(X \times Y)
\end{array}$$

Once again it suffices to show equality upon post composition with the morphisms $M(\pi_0^{X,Y})$ and $M(\pi_1^{X,Y})$. Thus we have:

$$\begin{aligned}
M(\pi_0^{X,Y}) \circ \alpha_{X,Y} \circ (\tau_X \times id_Y) &= M(\pi_0^{X,Y}) \circ \mu_{X,Y} \circ (\tau_X \times e_Y) \\
&= \pi_0^{MX,MY} \circ (\tau_X \times e_Y) && \text{(Commutativity of (2.1))} \\
&= \tau_X \circ \pi_0^{MX,Y} \\
&= \tau_X \circ \pi_0^{MX,MY} \circ (id_{MX} \times e_Y) \\
&= \tau_X \circ M(\pi_0^{X,Y}) \circ \mu_{X,Y} \circ (id_{MX} \times e_Y) && \text{(Commutativity of (2.1))} \\
&= M(\pi_0^{X,Y}) \circ \tau_{X \times Y} \circ \alpha_{X,Y} && \text{(Naturality of } \tau\text{)}
\end{aligned}$$

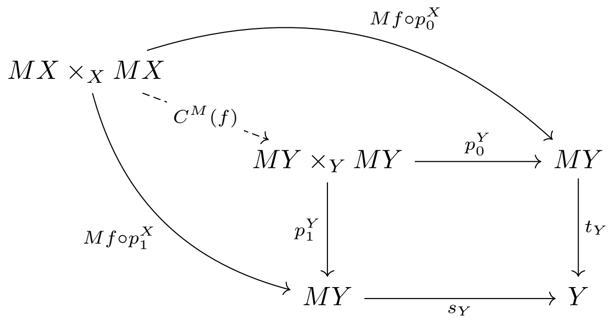
$$\begin{aligned}
M(\pi_1^{X,Y}) \circ \alpha_{X,Y} \circ (\tau_X \times id_Y) &= M(\pi_1^{X,Y}) \circ \mu_{X,Y} \circ (\tau_X \times e_Y) \\
&= \pi_1^{MX,MY} \circ (\tau_X \times e_Y) && \text{(Commutativity of (2.1))} \\
&= e_Y \circ \pi_1^{MX,Y} \\
&= \tau_Y \circ e_Y \circ \pi_1^{MX,Y} && (\tau_Y \circ e_Y = e_Y) \\
&= \tau_Y \circ \pi_1^{MX,MY} \circ (id_{MX} \times e_Y) \\
&= \tau_Y \circ M(\pi_1^{X,Y}) \circ \mu_{X,Y} \circ (id_{MX} \times e_Y) && \text{(Commutativity of (2.1))} \\
&= M(\pi_1^{X,Y}) \circ \tau_{X \times Y} \circ \alpha_{X,Y} && \text{(Naturality of } \tau\text{)}
\end{aligned}$$

Hence $\alpha_{X,Y} \circ (\tau_X \times id_Y) = \tau_{X \times Y} \circ \alpha_{X,Y}$ and $\tau : (M, \alpha) \Rightarrow (M, \alpha)$ as required.

□

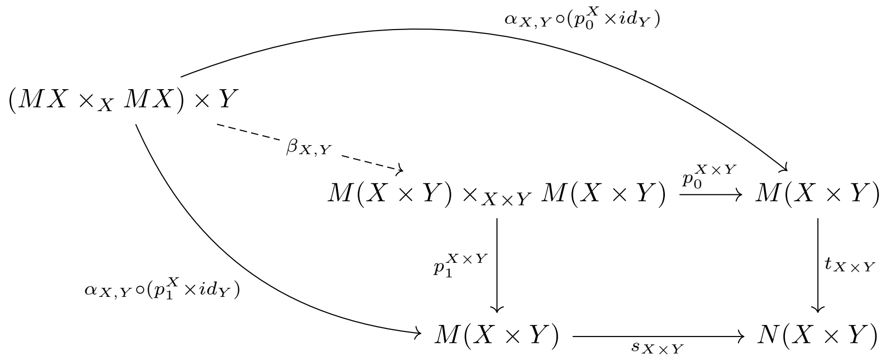
Now in order to show that $\mathcal{E}$ satisfies the full second path object category axiom it remains to show the case for the composition. First we recall some definitions from the previous chapter:

- The functor $C^M : \mathcal{E} \to \mathcal{E}$ is defined
  - **Objects**: $C^M(X) = MX \times_X MX$
  - **Arrows**: Given $f : X \to Y$ we obtain $C^M(f)$ from the pullback diagram:



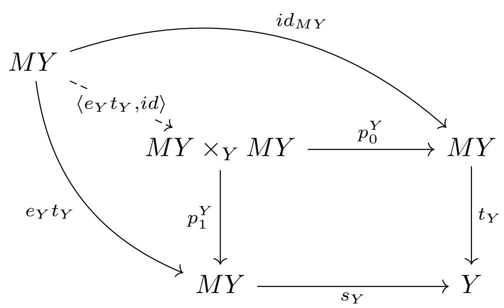
(2.6)

- The components for the strength β for the functor C^M is given by the pullback diagram



(2.7)

- The map $\langle e_Y t_Y, id \rangle$ is defined by the pullback



(2.8)

  and satisfies

$$c_Y \circ \langle e_Y t_Y, id \rangle = id_{MY} \tag{2.9}$$

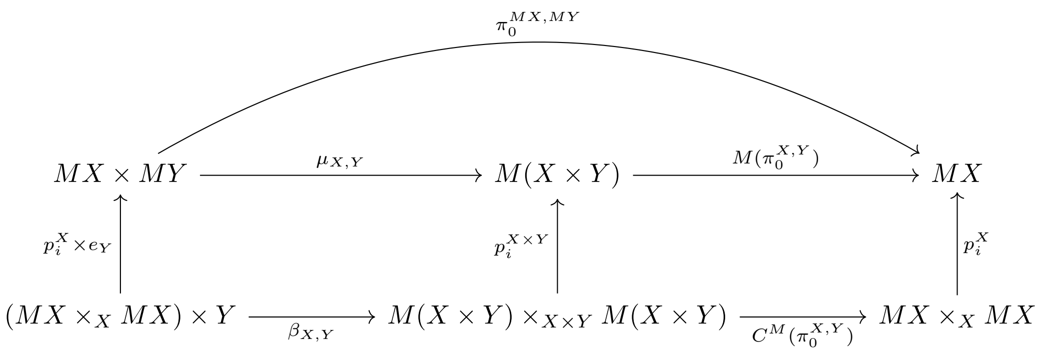
Before attending to the proof we will require the following lemma.

Lemma 2.5. *For X, Y in a nice path object category $\mathcal{E}$ the following equalities hold:*

i) $C^M(\pi_0^{X,Y}) \circ \beta_{X,Y} = \pi_0^{C^M(X),Y}$

ii) $C^M(\pi_1^{X,Y}) \circ \beta_{X,Y} = \langle e_Y t_Y, id \rangle \circ e_Y \circ \pi_1^{C^M(X),Y}$

Proof. i) Since $MX \times_X MX$ is the vertex of a pullback it is sufficient to prove equality upon post-composition with p_0^X and p_1^X. We first observe that for $i \in \{0,1\}$ we can obtain the following commutative diagrams:

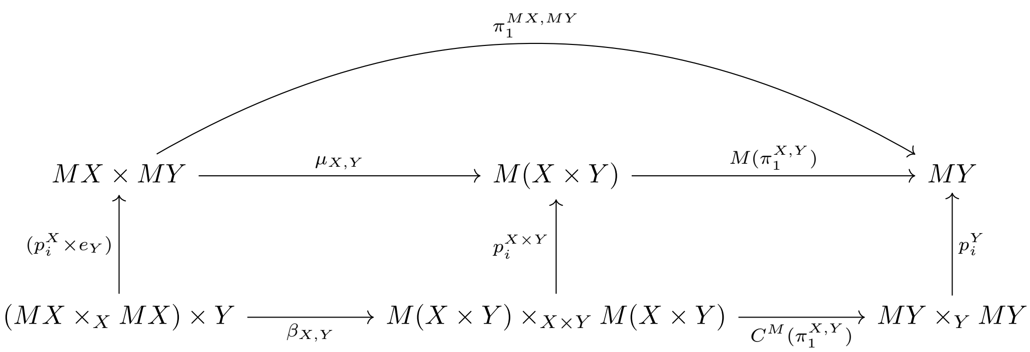


In both cases we have that the left square commutes by (2.7), the right square commutes by (2.6) and the top commutes by (2.1). Hence by diagram chasing we obtain

$$p_i^X \circ C^M(\pi_0^{X,Y}) \circ \beta_{X,Y} = \pi_0^{MX,MY} \circ (p_i^X \times e_Y) = p_i^X \circ \pi_0^{MX \times_X MX, Y}$$

so we obtain the required identities.

ii) First note that since $MY \times_Y MY$ is the vertex of a pullback it suffices to prove equality upon post-composition with the projections p_0^Y and p_1^Y. As in the previous case, letting $i \in \{0,1\}$ we obtain the commutative diagrams



By diagram chasing we obtain the identities:

$$p_i^Y \circ C^M(\pi_1^{X,Y}) \circ \beta_{X,Y} = \pi_1^{MX,MY} \circ (p_i^X \times e_Y) = e_Y \circ \pi_1^{C^M(X),Y}$$

Now for the case $i = 0$ note that by (2.8) we have $id_{MY} = p_0^Y \circ \langle e_Y t_Y, id\rangle$. Hence

$$p_0^Y \circ C^M(\pi_1^{X,Y}) \circ \beta_{X,Y} = p_0^Y \circ \langle e_Y t_Y, id\rangle \circ e_Y \circ \pi_1^{C^M(X),Y}$$

Similarly for, the case $i = 1$ we have by the internal category axioms that $id_Y = t_Y \circ e_Y$. Hence we have

$$\begin{aligned} p_1^Y \circ C^M(\pi_1^{X,Y}) \circ \beta_{X,Y} &= e_Y \circ \pi_1^{C^M(X),Y} \\ &= e_Y \circ t_Y \circ e_Y \circ \pi_1^{C^M(X),Y} \\ &= p_1^Y \circ \langle e_Y t_Y, id\rangle \circ e_Y \circ \pi_1^{C^M(X),Y} \qquad \text{(Commutativity of (2.8))} \end{aligned}$$

It thus follows that

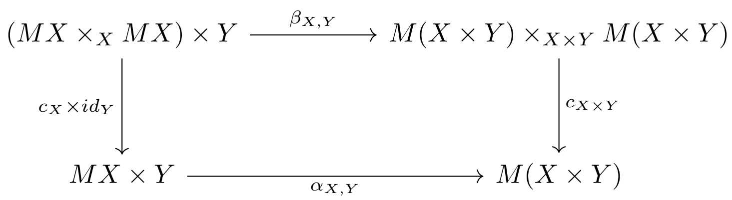
$$C^M(\pi_1^{X,Y}) \circ \beta_{X,Y} = \langle e_Y t_Y, id\rangle \circ e_Y \circ \pi_1^{C^M(X),Y}$$

□

We're now ready to finish the proof that $\mathcal{E}$ satisfies the second path object category axiom.

Proposition 2.6. *Nice path object categories satisfy the second path object category axiom.*

Proof. From our previous propositions it remains to verify that c is a strong natural transformation $c : (C^M, \beta) \Rightarrow (M, \alpha)$. We must verify the commutativity of the following diagram:



Again, since $M(X \times Y)$ is the vertex of a pullback, it suffices to check equality upon post-composition by $M(\pi_0^{X,Y})$ and $M(\pi_1^{X,Y})$. We have

$$
\begin{aligned}
M(\pi_0^{X,Y}) \circ \alpha_{X,Y} \circ (c_X \times id_Y) &= M(\pi_0^{X,Y}) \circ \mu_{X,Y} \circ (c_X \times e_Y) \\
&= \pi_0^{MX,MY} \circ (c_X \times e_Y) && \text{(Commutativity of (2.1))} \\
&= c_X \circ \pi_0^{C^M(X),Y} \\
&= c_X \circ C^M(\pi_0^{X,Y}) \circ \beta_{X,Y} && \text{Lemma 2.5 i))} \\
&= M(\pi_0^{X,Y}) \circ c_{X \times Y} \circ \beta_{X,Y} && \text{(Naturality of } c\text{)}
\end{aligned}
$$

$$
\begin{aligned}
M(\pi_1^{X,Y}) \circ \alpha_{X,Y} \circ (c_X \times id_Y) &= M(\pi_1^{X,Y}) \circ \mu_{X,Y} \circ (c_X \times e_Y) \\
&= \pi_1^{MX,MY} \circ (c_X \times e_Y) && \text{(Commutativity of (2.1))} \\
&= e_Y \circ \pi_1^{C^M(X),Y} \\
&= c_Y \circ \langle e_Y t_Y, id \rangle \circ e_Y \circ \pi_1^{C^M(X),Y} && \text{(2.9)} \\
&= c_Y \circ C^M(\pi_1^{X,Y}) \circ \beta_{X,Y} && \text{(Lemma 2.5 ii))} \\
&= M(\pi_1^{X,Y}) \circ c_{X \times Y} \circ \beta_{X,Y} && \text{(Naturality of } c\text{)}
\end{aligned}
$$

Hence $c_{X \times Y} \circ \beta_{X,Y} = \alpha_{X,Y} \circ (c_X \times id_Y)$ and $c : (C^M, \beta) \Rightarrow (M, \alpha)$ as required. It follows that $\mathcal{E}$ satisfies Axiom 2. $\square$

We are now ready to finish the work of this chapter and prove the third and final path object category axiom is satisfied by a nice path object category $\mathcal{E}$.

Theorem 2.7. *If $\mathcal{E}$ satisfies the nice path category axioms then $\mathcal{E}$ satisfies the path category axioms.*

Proof. Let $\mathcal{E}$ be a nice path object category. We already have Axiom 1 satisfied by definition and we know Axiom 2 is satisfied by Proposition 2.6, hence it remains to give the two missing details of Axiom 3. First we must verify that η is a strong natural transformation $\eta : (M, \alpha) \Rightarrow (MM, \alpha^\star)$, which we recall requires the commutativity of the following diagram:

$$
\begin{array}{ccc}
MX \times Y & \xrightarrow{\alpha_{X,Y}} & M(X \times Y) \\
\Big\downarrow{\scriptstyle \eta_X \times id_Y} & & \Big\downarrow{\scriptstyle \eta_{X \times Y}} \\
MMX \times Y & \xrightarrow[\alpha^\star_{X,Y}]{} & MM(X \times Y)
\end{array}
$$

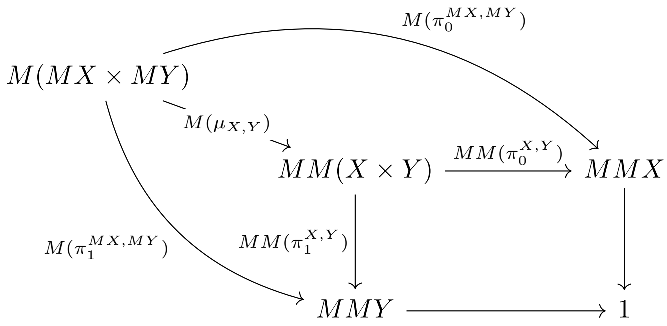
We remind the reader that $\alpha^\star$ was defined

$$\alpha^\star_{X,Y} = M(\alpha_{X,Y}) \circ \alpha_{MX,Y}$$

In our particular case we have

$$\alpha^\star_{X,Y} = M(\mu_{X,Y}) \circ M(id_{MX} \times e_Y) \circ \mu_{MX,Y} \circ (id_{MMX} \times e_Y)$$

Now since M is pullback preserving we have that $MM(X \times Y)$ is the vertex of a pullback. It follows that it suffices to show equality upon post-composition with the maps $MM(\pi_0^{X,Y})$ and $MM(\pi_1^{X,Y})$. Note also that from (2.1) we have the commutative diagram

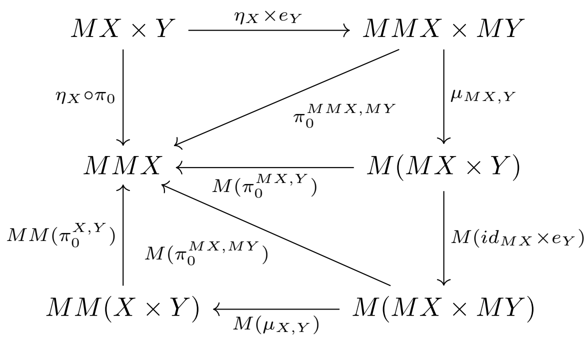


(2.10)

Now we have:

$$\begin{aligned}
MM(\pi_0^{X,Y}) \circ \eta_{X\times Y} \circ \alpha_{X,Y} &= MM(\pi_0^{X,Y}) \circ \eta_{X\times Y} \circ \mu_{X,Y} \circ (id_{MX} \times e_Y) \\
&= \eta_X \circ M(\pi_0^{X,Y}) \circ \mu_{X,Y} \circ (id_{MX} \times e_Y) && \text{(Naturality of } \eta) \\
&= \eta_X \circ \pi_0^{MX,MY} \circ (id_{MX} \times e_Y) && \text{(Commutativity of (2.1))} \\
&= \eta_X \circ \pi_0^{MX,Y}
\end{aligned}$$

Now observe that the following diagram commutes:



In the upper square: the upper triangle commutes by definition whilst the lower triangle commutes by (2.1). In the lower square: the upper triangle commutes once more by definition whilst the lower square commutes by (2.10). Hence by diagram chasing we can see that

$$MM(\pi_0^{X,Y}) \circ \alpha^{\star}_{X,Y} \circ (\eta_X \times id_Y) = \eta_X \circ \pi_0^{MX,Y}$$

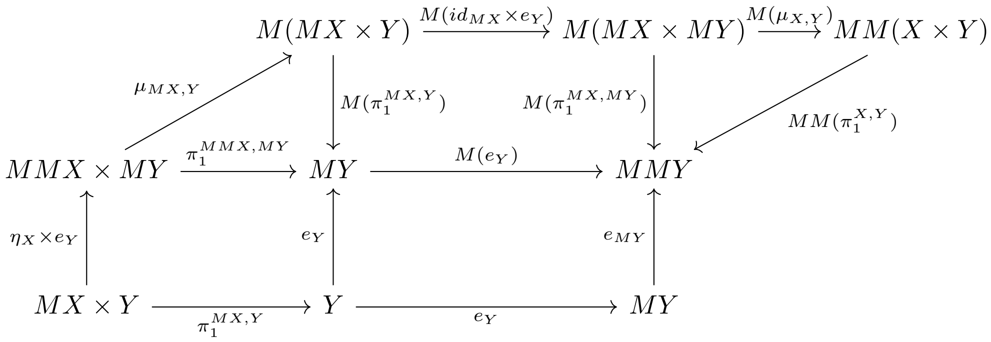
Hence it follows that

$$MM(\pi_0^{X,Y}) \circ \alpha^{\star}_{X,Y} \circ (\eta_X \times id_Y) = MM(\pi_0^{X,Y}) \circ \eta_{X\times Y} \circ \alpha_{X,Y}$$

Similarly we have:

$$
\begin{aligned}
MM(\pi_1^{X,Y}) \circ \eta_{X\times Y} \circ \alpha_{X,Y} &= MM(\pi_1^{X,Y}) \circ \eta_{X,Y} \circ \mu_{X,Y} \circ (id_{MX} \times e_Y) \\
&= \eta_Y \circ M(\pi_1^{X,Y}) \circ \mu_{X,Y} \circ (id_{MX} \times e_Y) && \text{(Naturality of } \eta\text{)} \\
&= \eta_Y \circ \pi_1^{MX,MY} \circ (id_{MX} \times e_Y) && \text{(Commutativity of (2.1))} \\
&= \eta_Y \circ e_Y \circ \pi_1^{MX,Y} \\
&= e_{MY} \circ e_Y \circ \pi_1^{MX,Y} && \text{(Axiom } 3'\text{)}
\end{aligned}
$$

Now observe that we have commutativity of the following diagram:



In the left-most column: the lower square commutes by definition and the upper triangle commutes by (2.1). In the central column: the upper square commutes by definition and the lower square commutes by naturality of e. Finally, the right-most triangle commutes by (2.10).Thus by a diagram chase we see that

$$MM(\pi_1^{X,Y}) \circ \alpha^{\star}_{X,Y} \circ (\eta_X \times id_Y) = e_{MY} \circ e_Y \circ \pi_1^{MX,Y}$$

Hence

$$MM(\pi_1^{X,Y}) \circ \eta_{X\times Y} \circ \alpha_{X,Y} = MM(\pi_1^{X,Y}) \circ \alpha^{\star}_{X,Y} \circ (\eta_X \times id_Y)$$

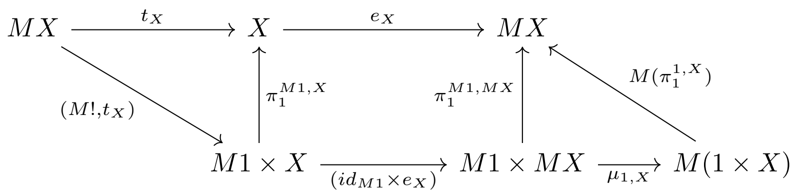
Taken together, we have that $\eta_{X\times Y} \circ \alpha_{X,Y} = \alpha^{\star}_{X,Y} \circ (\eta_X \times id_Y)$ and so η is a strong natural transformation $\eta : (M, \alpha) \Rightarrow (MM, \alpha^{\star})$ as required. The final detail necessary to satisfy the third axiom is the identity

$$M(t_X) \circ \eta_X = M(\pi_1^{1,X}) \circ \alpha_{1,X} \circ (M(!), t_X)$$

We already have by Axiom $3'$ that $M(t_X) \circ \eta_X = e_X \circ t_X$ hence we show

$$M(\pi_1^{1,X}) \circ \alpha_{1,X} \circ (M(!), t_X) = e_X \circ t_X$$

Observe that the following diagram commutes



The left triangle and centre square commute by definition, whilst the triangle on the right hand side commutes by (2.1). Hence we have the required equality: $\mathcal{E}$ satisfies the third path object category axiom. □

Chapter 3

Path Object Category Structure On The Category Of Cubical Sets With Connections

We now prove the key result of the thesis

Theorem 3.1. *The category of cubical sets with connections carries the structure of a path object category.*

In order to do so we take advantage of the work of the previous chapter, exhibiting a nice path object category structure for the category of cubical sets with connections, henceforth $\boldsymbol{cSet^c}$. Before we attend to this proof we give a short introduction to cubical sets with connections: for a more extensive exposition we recommend the lecture notes [Wil12], the presentation of which we follow. A slightly less accessible, but more comprehensive account of the construction of the category $\boldsymbol{cSet^c}$ can be found in [GM03].

3.1 Cubical Sets With Connections

We obtain the category $\boldsymbol{cSet^c}$ as the presheaf category over a category of *cubes with connections* $\square^c$. There are a number of equivalent constructions of $\square^c$ (see [GM03, Theorem 5.2] for five) but we choose the presentation with the most categorical flavour. We first give some definitions

Definition 3.2 (Interval With Contraction Structure and Connections)**.**

1. An *interval* (I^0, I^1, i_0, i_1) in a category $\mathcal{C}$ consists of objects I^0, I^1 in $\mathcal{C}$ together with arrows

$$I^0 \underset{i_1}{\overset{i_0}{\rightrightarrows}} I^1$$

2. Let $\mathcal{C}$ be a category equipped with a monoidal structure $(\otimes, 1, \epsilon, \iota, a)$ and let $\hat{I} = (1, I, i_0, i_1)$ be an interval in $\mathcal{C}$. A *contraction structure* upon $\hat{I}$ is an arrow $p : I^1 \to 1$ in $\mathcal{C}$ such that the following

diagrams commute

$$\begin{array}{ccc} 1 & \xrightarrow{i_0} & I \\ & \searrow^{id_1} & \downarrow{\scriptstyle p} \\ & & 1 \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{i_1} & I \\ & \searrow^{id_1} & \downarrow{\scriptstyle p} \\ & & 1 \end{array}$$

3. Let $\mathcal{C}$ be a category equipped with a monoidal structure $(\otimes, 1, \epsilon, \iota, a)$ and let $\hat{I} = (1, I, i_0, i_1, p)$ be an interval in $\mathcal{C}$ with a contraction structure. An *upper connection structure* upon $\hat{I}$ is an arrow $\Gamma_0 : I \otimes I \to I$ such that the following diagrams commute (where I is identified with $I \otimes 1$ and $1 \otimes I$ via the unitors):

$$\begin{array}{ccc} I & \xrightarrow{id_I\otimes i_0} & I\otimes I \\ & \searrow^{id_I} & \downarrow{\scriptstyle \Gamma_0} \\ & & I \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{i_0\otimes id_I} & I^1\otimes I \\ & \searrow^{id_I} & \downarrow{\scriptstyle \Gamma_0} \\ & & I^1 \end{array}$$

$$\begin{array}{ccc} I & \xrightarrow{id_I\otimes i_1} & I\otimes I \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \Gamma_0} \\ 1 & \xrightarrow[i_1]{} & I \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{i_1\otimes I} & I\otimes I \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \Gamma_0} \\ 1 & \xrightarrow[i_1]{} & I \end{array}$$

It is *compatible* with the contraction structure p if the following diagram commutes:

$$\begin{array}{ccc} I\otimes I & \xrightarrow{\Gamma_0} & I \\ {\scriptstyle id_I\otimes p}\downarrow & & \downarrow{\scriptstyle p} \\ I & \xrightarrow[p]{} & 1 \end{array}$$

4. Let $\mathcal{C}$ be a category equipped with a monoidal structure $(\otimes, 1, \epsilon, \iota, a)$ and let $\hat{I} = (1, I, i_0, i_1, p)$ be an interval in $\mathcal{C}$ with a contraction structure. A *lower connection structure* upon $\hat{I}$ is an arrow $\Gamma_1 : I\otimes I \to I$ such that the following diagrams commute (where once again I is identified with $I \otimes 1$ and $1 \otimes I$):

$$\begin{array}{ccc} I & \xrightarrow{id_I\otimes i_1} & I\otimes I \\ & \searrow^{id_I} & \downarrow{\scriptstyle \Gamma_1} \\ & & I \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{i_1\otimes id_I} & I\otimes I \\ & \searrow^{id_I} & \downarrow{\scriptstyle \Gamma_1} \\ & & I \end{array}$$

$$\begin{array}{ccc} I & \xrightarrow{id_I\otimes i_0} & I\otimes I \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \Gamma_1} \\ 1 & \xrightarrow[i_0]{} & I \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{i_0\otimes id_1} & I\otimes I \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \Gamma_1} \\ 1 & \xrightarrow[i_0]{} & I \end{array}$$

It is *compatible* with the contraction structure p if the following diagram commutes:

$$\begin{array}{ccc} I \otimes I & \xrightarrow{\Gamma_1} & I \\ {\scriptstyle id_I \otimes p}\downarrow & & \downarrow{\scriptstyle p} \\ I & \xrightarrow[p]{} & 1 \end{array}$$

These are natural definitions: when $\mathcal{C}$ has a monoidal structure $(\otimes, 1, \epsilon, \iota, a)$, an interval $\hat{I} = (= 1, I, i_0, i_1)$ suffices to obtain a notion of *homotopy* in the category. Further, a contraction structure for $\hat{I}$ allows one to augment this notion with *constant homotopies*. Finally the presence of compatible upper and lower connection structures is precisely what is required to add *double homotopies*. We direct the interested reader to [Wil12, Sections II.3.3, II.4.2, III.3]. The relevance here is that we can give $\square^c$ as the *free strict monoidal category upon an interval with contraction and connection structures*. We first define the graph $\mathbb{Y}$ as follows:

- For all $n \in \omega$, a vertex I^n;
- For every $1 \leq i \leq n$ and $\delta \in \{-1, 1\}$ a directed edge $f^n_{i,\delta} : I^{n-1} \to I^n$;
- For every $1 \leq i \leq n$, a directed edge $d^n_i : I^n \to I^{n-1}$;
- For every $n \geq 2$, $1 \leq i \leq n-1$ and $\delta \in \{-1, 1\}$ a directed edge $\Gamma^{n-1}_{i,\delta} : I^n \to I^{n-1}$.

We take the free category on this graph and denote it by $F(\mathbb{Y})$. To give this category the strict monoidal structure where $I^n \otimes I^m = I^{n+m}$ we must quotient by the equivalence relation generated by the following *cocubical relations*:

- For any $1 \leq i \leq n$, $1 \leq j \leq n+1$ and $\delta, \epsilon \in \{-1, 1\}$:

$$f^{n+1}_{j,\epsilon} \circ f^n_{i,\delta} \sim \begin{cases} f^{n+1}_{i,\delta} \circ f^n_{j-1,\epsilon} & \text{if } j > i \\ f^{n+1}_{i+1,\delta} \circ f^n_{j,\epsilon} & \text{if } j \leq i \end{cases}$$

- For any $n \geq 2$, $1 \leq i \leq n$ and $1 \leq j \leq n-1$:

$$d^{n-1}_j \circ d^n_i \sim \begin{cases} d^{n-1}_{i-1} \circ d^n_j & \text{if } j < i \\ d^{n-1}_i \circ d^n_{j+1} & \text{if } j \geq i \end{cases}$$

- For any $1 \leq i, j \leq n$ and $\delta \in \{-1, 1\}$:

$$d^n_j \circ f^n_{i,\delta} \sim \begin{cases} id & \text{if } j = i \\ f^{n-1}_{i,\delta} \circ d^{n-1}_{j-1} & \text{if } j > i \\ f^{n-1}_{i-1,\delta} \circ d^{n-1}_j & \text{if } j < i \end{cases}$$

- For any $n \geq 2$, $1 \leq i \leq n$, $1 \leq j \leq n-1$ and $\delta, \epsilon \in \{-1, 1\}$:

$$\Gamma^{n-1}_{j,\epsilon} \circ f^{n}_{i,\delta} \sim \begin{cases} id & \text{if } i = j \text{ and } \delta = \epsilon \textit{ or } i = j+1 \text{ and } \delta = \epsilon \\ f^{n-1}_{j,\delta} \circ d^{n-1}_{j} & \text{if } i = j \text{ and } \delta \neq \epsilon \textit{ or } i = j+1 \text{ and } \delta \neq \epsilon \\ f^{n-1}_{i,\delta} \circ \Gamma^{n-2}_{j-1,\epsilon} & \text{if } n \geq 3 \text{ and } i < j \\ f^{n-1}_{i-1,\delta} \circ \Gamma^{n-2}_{j,\epsilon} & \text{if } n \geq 3 \text{ and } i > j+1 \end{cases}$$

- For any $n \geq 2$, $1 \leq i \leq n-1$, $1 \leq j \leq n-1$ and $\delta \in \{-1, 1\}$:

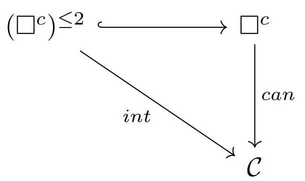
$$d^{n-1}_{j} \circ \Gamma^{n-1}_{i,\delta} \sim \begin{cases} d^{n-1}_{i} \circ d^{i}_{n} & \text{if } i = j \\ \Gamma^{n-2}_{i-1,\delta} \circ d^{n}_{j} & \text{if } n \geq 3 \text{ and } j < i \\ \Gamma^{n-2}_{i,\delta} \circ d^{n}_{j+1} & \text{if } n \geq 3 \text{ and } j > i \end{cases}$$

- For any $n \leq 3$, $1 \leq i \leq n-1$, $1 \leq j \leq n-2$ and $\delta, \epsilon \in \{-1, 1\}$:

$$\Gamma^{n-2}_{j,\epsilon} \circ \Gamma^{n-1}_{i,\delta} \sim \begin{cases} \Gamma^{n-2}_{i-1,\delta} \circ \Gamma^{n-1}_{j,\epsilon} & \text{if } j < i \\ \Gamma^{n-2}_{i,\delta} \circ \Gamma^{n-1}_{j+1,\epsilon} & \text{if } j > i \\ \Gamma^{n-2}_{i,\delta} \circ \Gamma^{n-2}_{i+1,\delta} & \text{if } j = i \text{ and } \epsilon = \delta \end{cases}$$

These turn out to be precisely what is required to give an inductive definition of a bifunctor $\otimes$ on $F(\mathbb{Y})$. We call the quotient category *the category of cubes with connections* $\square^c$. This category contains a "generic" interval with contraction structure and connections. Denote by $(\square^c)^{\leq 2}$ the full subcategory of $\square^c$ containing the objects I^0, I^1 and I^2. We call this *the free- standing interval with a contraction structure and connections*. We obtain the following universal property.

Proposition 3.3. *[Wil12, cf Proposition III.3.4.4] Let $\mathcal{C}$ be a category equipped with a strict monoidal structure $(\otimes, 1)$. For any functor $int : (\square^c)^{\leq 2} \to \mathcal{C}$ there is a unique functor $can : \square^c \to \mathcal{C}$ preserving the strict monoidal structure such that the following diagram commutes*



Definition 3.4 (Category of Cubical Sets With Connections)**.** The *category of cubical sets with connections* $\boldsymbol{cSet^c}$ is given as the presheaf category $\mathbf{Set}^{(\square^c)^{op}}$.

If we were to remove all aspects of the construction involving the connection structures we obtain the *category of cubes* $\square$ and with it cubical sets. We make essential use of the connections in building our nice path object category structure however. We can give an explicit definition of cubical sets with connections that will be easier to work with.

Definition 3.5 (Cubical Set With Connections)**.** A *cubical set with connections* X is comprised of sets $\{X_n\}_{n \in \omega}$ together with morphisms - henceforth *cubical operators* - given by

i) **Face Maps**: For $1 \leq i \leq n$ and $\delta \in \{-1, 1\}$ we have

$$f_n^{i,\delta} : X_n \to X_{n-1}$$

ii) **Degeneracy Maps**: For $1 \leq i \leq n$ we have

$$d_n^i : X_{n-1} \to X_n$$

iii) **Connection Maps**: For $n \geq 2$, $1 \leq i \leq n-1$ and $\delta \in \{-1, 1\}$, we have

$$\Gamma_{n-1}^{i,\delta} : X_{n-1} \to X_n$$

subject to the following *cubical identities*:

- For any $1 \leq i \leq n$, $1 \leq j \leq n+1$ and $\delta, \epsilon \in \{-1, 1\}$:

$$f_n^{i,\delta} \circ f_{n+1}^{j,\epsilon} = \begin{cases} f_n^{j-1,\epsilon} \circ f_{n+1}^{i,\delta} & \text{if } j > i \\ f_n^{j,\epsilon} \circ f_{n+1}^{i+1,\delta} & \text{if } j \leq i \end{cases}$$

- For any $n \geq 2$, $1 \leq i \leq n$ and $1 \leq j \leq n-1$:

$$d_n^i \circ d_{n-1}^j = \begin{cases} d_n^j \circ d_{n-1}^{i-1} & \text{if } j < i \\ d_n^{j+1} \circ d_{n-1}^i & \text{if } j \geq i \end{cases}$$

- For any $1 \leq i, j \leq n$ and $\delta \in \{-1, 1\}$:

$$f_n^{i,\delta} \circ d_n^j = \begin{cases} id & \text{if } j = i \\ d_{n-1}^{j-1} \circ f_{n-1}^{i,\delta} & \text{if } j > i \\ d_{n-1}^j \circ f_{n-1}^{i-1,\delta} & \text{if } j < i \end{cases}$$

- For any $n \geq 2$, $1 \leq i \leq n$, $1 \leq j \leq n-1$ and $\delta, \epsilon \in \{-1, 1\}$:

$$f_n^{i,\delta} \circ \Gamma_{n-1}^{j,\epsilon} = \begin{cases} id & \text{if } i = j \text{ and } \delta = \epsilon \text{ } or \text{ } i = j+1 \text{ and } \delta = \epsilon \\ d_{n-1}^j \circ f_{n-1}^{j,\delta} & \text{if } i = j \text{ and } \delta \neq \epsilon \text{ } or \text{ } i = j+1 \text{ and } \delta \neq \epsilon \\ \Gamma_{n-2}^{j-1,\epsilon} \circ f_{n-1}^{i,\delta} & \text{if } n \geq 3 \text{ and } i < j \\ \Gamma_{n-2}^{j,\epsilon} \circ f_{n-1}^{i-1,\delta} & \text{if } n \geq 3 \text{ and } i > j+1 \end{cases}$$

- For any $n \geq 2$, $1 \leq i \leq n-1$, $1 \leq j \leq n-1$ and $\delta \in \{-1, 1\}$:

$$\Gamma_{n-1}^{i,\delta} \circ d_{n-1}^j = \begin{cases} d_n^i \circ d_{n-1}^i & \text{if } i = j \\ d_n^j \circ \Gamma_{n-2}^{i-1,\delta} & \text{if } n \geq 3 \text{ and } j < i \\ d_n^{j+1} \circ \Gamma_{n-2}^{i,\delta} & \text{if } n \geq 3 \text{ and } j > i \end{cases}$$

- For any $n \geq 3$, $1 \leq i \leq n-1$, $1 \leq j \leq n-2$ and $\delta, \epsilon \in \{-1, 1\}$:

$$\Gamma^{i,\delta}_{n-1} \circ \Gamma^{j,\epsilon}_{n-2} = \begin{cases} \Gamma^{j,\epsilon}_{n-1} \circ \Gamma^{i-1,\delta}_{n-2} & \text{if } j < i \\ \Gamma^{j+1,\epsilon}_{n-1} \circ \Gamma^{i,\delta}_{n-2} & \text{if } j > i \\ \Gamma^{i+1,\delta}_{n-1} \circ \Gamma^{i,\delta}_{n-2} & \text{if } j = i \text{ and } \epsilon = \delta \end{cases}$$

Note the reversal of the subscript and superscript to distinguish the operators from the arrows of the graph $F(\mathbb{Y})$. As is the case in the literature, we will use the same notation for the operators of every cubical set with connections. Although this takes some getting used to, it is the only feasible way to maintain readability. We thus give a similar characterisation of the morphisms of $\boldsymbol{cSet^c}$.

Definition 3.6 (Cubical Morphism)**.** A *cubical morphism* $\mu : X \to Y$ is a collection of maps $\mu_n : X_n \to Y_n$ commuting with the cubical operators.

It will suit us to go back and forth between working with these explicit definitions and utilising the properties $\boldsymbol{cSet^c}$ inherits as a presheaf category. For a cubical set X we call elements ζ of X_n *n-cubes*. If such an element can be obtained as the image of an $(n-1)$-cube under a degeneracy map, we call it a *degenerate n-cube*. We can give some geometric intuitions that explains this nomenclature and will be useful in the work that follows. Each n-cube has $2n$ faces, each of which an $(n-1)$-cube. The action of a face map $f^{i,\epsilon}_n$ collapses an n-cube onto it's (i, ϵ)-th face (where $(i, -1)$ and $(i, 1)$ are a pair of parallel faces) forgetting the rest of its structure. We will often draw a 1-cube ζ as

$$x_0 \xrightarrow{\quad \zeta \quad} x_1$$

Where $f^{1,-1}_1(\zeta) = x_0$ and $f^{1,1}_1(\zeta) = x_1$. Similarly we represent 2-cubes ϕ as

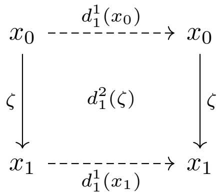
$$\begin{array}{ccc} x_0 & \xrightarrow{\quad \zeta_1 \quad} & x_1 \\ {\scriptstyle \zeta_0}\big\downarrow & \phi & \big\downarrow{\scriptstyle \zeta_2} \\ x_2 & \xrightarrow[\quad \zeta_3 \quad]{} & x_3 \end{array}$$

where $f^{1,-1}_2(\phi) = \zeta_0$, $f^{2,-1}_2(\phi) = \zeta_1$, $f^{1,1}_2(\phi) = \zeta_2$ and $f^{2,1}_2(\phi) = \zeta_3$.The cubical identities thus enforce the intuition that, for example, collapsing onto ζ_1 with $f^{2,1}_2$ and then collapsing onto x_0 with $f^{1,-1}_1$ is the same as collapsing onto ζ_0 with $f^{1,-1}_2$ followed by collapsing onto x_0 with $f^{1,-1}_1$.

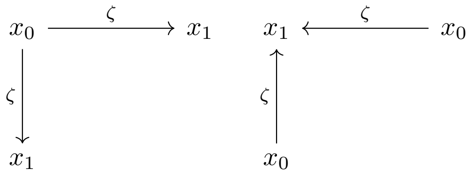
Degeneracies allow us to consider n-cubes to be *thin* $(n+1)$-cubes and the superscript in d^i_n determines the orientation of the degenerate cubes we obtain. Degenerate cubes have a pair of equal opposite faces and all other faces degeneracies. For example, given a 1-cube

$$x_0 \xrightarrow{\quad \zeta \quad} x_1$$

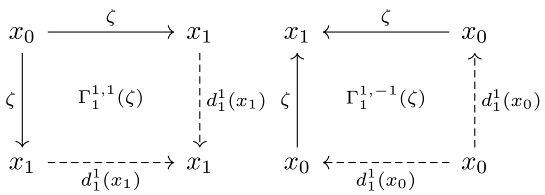
We can apply d_2^1 to obtain the 2-cube



With the identities for compositions of face and degeneracy maps enforcing the idea that the degenerate faces of such thin cubes are themselves degeneracies of vertices. We will typically use dashed arrows to represent degeneracies. Finally we come to the structure distinguishing our setting from regular cubical sets: the connection maps. In essence these are extra degeneracies. The image under a connection map yields a cube with a pair of equal adjacent faces and all over faces either connection cubes or degenerate cubes. For example, given a 1-cube ζ, connections allow us to consider diagrams of the form



as 2-cubes



and the cubical identities involving connections enforce this picture. Of course these intuitions can only take us so far: the trick is to use low dimensional cases to work out how the combinatorics will work in the general case.

3.2 Nice Path Object Category Structure

We now complete this section of the thesis by proving

Theorem 3.7. ***cSet***^{c} *carries the structure of a nice path object category.*

3.2.1 Axiom 1: Path Objects

We begin by defining the notion of path at the centre of the path object category structure we will develop throughout this section; that of a *cubical* n*-path*. In some ways this bears similarity to the simplicial Moore paths given in [vdBG12, Definition 7.1.1] to show that **sSet** is a path object category, however because of the simplified structure we are aiming to obtain, as well as the the flexibility of the connection structure, we are able to give a much more straightforward notion of path. Another source of inspiration is Brown's *Moore hyper rectangles on a space* [Bro09] the collections of which form a cubical ω-category with an underlying cubical set with connections and compositions. Before giving a formal definition we can understand the intuition behind what we will do by way of the case for low dimensions:

First, cubical 0-paths through a cubical set X will be given by a sequence of 0-cubes $x_0, \ldots, x_k$ "connected" as the faces of 1-cubes $\zeta_0, \ldots, \zeta_{k-1}$. A characteristic example would be

$$x_0 \xrightarrow{\zeta_0} x_1 \xleftarrow{\zeta_1} x_2 \xrightarrow{\zeta_2} x_3 \xrightarrow{\zeta_3} x_4 \tag{3.1}$$

By applying the face maps $f_1^{1,-1}$ and $f_1^{1,1}$ in a particular order to the sequence $\zeta_0, \ldots, \zeta_3$ we are able to "travel" from the 0-cube x_0 to the 0-cube x_4. Moving up a dimension, cubical 1-paths will be given by sequences of 1-cubes $\zeta_0, \ldots, \zeta_k$ "connected" as the faces of 2-cubes $\phi_0, \ldots, \phi_{k-1}$. For example

$$\begin{array}{ccccc} x_0 & \longrightarrow & x_1 & \longleftarrow & x_2 \\ {\scriptstyle \zeta_0}\big\downarrow & \phi_0 & \big\downarrow{\scriptstyle \zeta_1} & \phi_1 & \big\downarrow{\scriptstyle \zeta_2} \\ x_0' & \longrightarrow & x_1' & \longleftarrow & x_2' \end{array} \tag{3.2}$$

We note that by applying $f_2^{2,-1}$ to each 2-cube in this 1-path we obtain a 0-path from x_0 to x_2. Similarly, $f_2^{2,1}$ gives us another 0-path from x_0' to x_1'. The idea is that 0-paths will be 0-cubes in a cubical set, and similarly 1-paths will be 1-cubes, with faces given in the way described. A similar argument yields, from the 0-path (3.1), a degenerate 1-path obtained by applying the degeneracy d_2^2 to every 1-cube:

$$\begin{array}{ccccccccc} x_0 & \xrightarrow{\zeta_0} & x_1 & \xleftarrow{\zeta_1} & x_2 & \xrightarrow{\zeta_2} & x_3 & \xrightarrow{\zeta_3} & x_4 \\ {\scriptstyle d_1^1(x_0)}\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle d_1^1(x_4)} \\ x_0 & \xrightarrow{\zeta_0} & x_1 & \xleftarrow{\zeta_1} & x_2 & \xrightarrow{\zeta_2} & x_3 & \xrightarrow{\zeta_3} & x_4 \end{array}$$

By applying the connection map $\Gamma_2^{2,\epsilon}$ to ϕ_0 and ϕ_1 in (3.2) we similarly obtain a 2-path from $\Gamma_1^{1,\epsilon}(\zeta_0)$ to $\Gamma_1^{1,\epsilon}(\zeta_2)$. These intuitions are enforced by the cubical identities, and by generalising this idea we can imbue the collection of cubical n-paths with the structure of a cubical set with connections. We introduce the following definition to formalise our idea and generalise it to all dimensions:

Definition 3.8 (Cubical N-Path)**.** Given a cubical set X, a *cubical n-path through* X from ζ_0 to ζ_k is a tuple

$$\chi = \Big((\zeta_0, \ldots, \zeta_k), (\phi_0, \ldots, \phi_{k-1}), (\epsilon_0, \ldots, \epsilon_{k-1}) \Big)$$

where $\zeta_0, \ldots, \zeta_k$ are n-cubes of X, $\phi_0, \ldots, \phi_{k-1}$ are (n+1)-cubes of X, $\epsilon_0, \ldots, \epsilon_{k-1} \in \{-1, 1\}$ and the following identities hold:

$$\begin{gathered} f_{n+1}^{1,\epsilon_0}(\phi_0) = \zeta_0 \\ f_{n+1}^{1,-\epsilon_0}(\phi_0) = \zeta_1 = f_{n+1}^{1,\epsilon_1}(\phi_1) \\ f_{n+1}^{1,-\epsilon_1}(\phi_1) = \zeta_2 = f_{n+1}^{1,\epsilon_2}(\phi_2) \\ \vdots \\ f_{n+1}^{1,-\epsilon_{k-2}}(\phi_{k-2}) = \zeta_{k-1} = f^{1,\epsilon_{k-1}}(\phi_{k-1}) \\ f_{n+1}^{1,-\epsilon_{k-1}}(\phi_{k-1}) = \zeta_k \end{gathered}$$

We also allow the case $k = 0$ to give "trivial" paths $\Big((\zeta_0), (\,), (\,) \Big)$.

Notation 3.2.1. For readability, we denote by $\Big((\vec{\zeta}), (\vec{\phi}), (\vec{\epsilon})\Big)$ the arbitrary cubical n-path $\Big((\zeta_0, \ldots, \zeta_k), (\phi_0, \ldots, \phi_{k-1}), (\epsilon_0, \ldots, \epsilon_{k-1})\Big)$. Abusing this notation slightly, if we wish to highlight a particular section of an n-path - for example, the $n+1$-cube ϕ_i connecting ζ_i and ζ_{i+1} - we write

$$\Big((\vec{\zeta}, \zeta_i, \zeta_{i+1}, \vec{\zeta}), (\vec{\phi}, \phi_i, \vec{\phi}), (\vec{\epsilon}, \epsilon_i, \vec{\epsilon})\Big)$$

Given maps $f : X_n \to X_m$ and $g : X_{n+1} \to X_l$ we write

$$\Big((\overrightarrow{f(\zeta)}), (\overrightarrow{g(\phi)}), (\vec{\epsilon})\Big) = \Big((f(\zeta_1), \ldots, f(\zeta_k)), (g(\phi_1), \ldots, g(\phi_{k-1})), (\epsilon_0, \ldots, \epsilon_{k-1})\Big)$$

We begin the work of identifying the path objects of $\boldsymbol{cSet^c}$. The obvious candidate is the collections of cubical n-paths, but this entails giving them the structure of a cubical set with connections. First we quotient the paths under a useful equivalence relation.

Definition 3.9 ($(MX)_n$)**.** We define $(MX)_n$ to be the collection of cubical n-paths through X, quotiented by the equivalence relation $\sim$ generated by

$$\Big((\vec{\zeta}, \xi, \xi, \vec{\zeta}), (\vec{\phi}, d^1_{n+1}(\xi), \vec{\phi}), (\vec{\epsilon}, \delta, \vec{\epsilon})\Big) \sim \Big((\vec{\zeta}), (\vec{\phi}), (\vec{\epsilon})\Big)$$

The idea here is that the degenerate $(n+1)$-cubes $d^1_{n+1}(\zeta)$ are"thin", being that they are n-cubes artificially considered to be $(n+1)$-cubes. As such they don't genuinely represent any distance travelled along the paths. A cubical n-path with a string of 20 $(n+1)$-cubes $d^1_{n+1}(\zeta)$ in a row is for all intents and purposes identical to the one with the 20 degenerate cubes removed. A similar trick is used in van Oosten's [vOar]. The question of whether the unquotiented case yields a regular path object category structure is an interesting one but we reserve further remarks on this to the final chapter "Conclusions and Further Work".

To avoid overloading the notation we will generally make no distinction between cubical n- paths and their equivalence classes. We of course have a canonical representative for each class: the cubical n-path with each degenerate $(n+1)$-cube $d^1_{n+1}(\zeta)$ removed. In most cases it does no harm to suppose we are working with these. Nevertheless we will show that the structure we define throughout this section is independent of the choice of representative: we will first work with representatives and then pass through to the equivalence classes to show the structure is as we require.

We must now imbue the collection $\{(MX)_n\}_{n\in\omega}$ with the structure of a cubical set with connections. In order to do so, we define face, degeneracy and connection maps and verify they satisfy the cubical identities.

Definition 3.10 (Face Maps)**.** Let $1 \le i \le n$, $\delta \in \{-1, 1\}$ and $\chi = \Big((\vec{\zeta}), (\vec{\phi}), (\vec{\epsilon})\Big) \in (MX)_n$. The face map $f^{i,\delta}_n : (MX)_n \to (MX)_{n-1}$ is defined:

$$f^{i,\delta}_n(\chi) = \Big((\overrightarrow{f^{i,\delta}_n(\zeta)}), (\overrightarrow{f^{i+1,\delta}_{n+1}(\phi)})), (\vec{\epsilon})\Big)$$

We show that this is well defined: that is, given an n-path χ we obtain an $(n-1)$-path $f^{i,\delta}_n(\chi)$. Clearly each $f^{i,\delta}_n(\zeta_j)$ is an $(n-1)$-cube of X, and each $f^{i+1,\delta}_{n+1}((\phi_j)$ is similarly an n-cube. That the tuple satisfies the $(n-1)$-path conditions follows from the cubical identities. For all $1 \le i \le n$ we have $1 < i+1$, so the cubical identity

$$f^{1,\epsilon_j}_n \circ f^{i+1,\delta}_{n+1} = f^{i,\delta}_n \circ f^{1,\epsilon_j}_{n+1}$$

holds. Thus it follows:

$$f_n^{1,\epsilon_0}(f_{n+1}^{i+1,\delta}(\phi_0)) = f_n^{i,\delta}(f_{n+1}^{1,\epsilon_0}(\phi_0)) = f_n^{i,\delta}(\zeta_0)$$
$$f_n^{1,-\epsilon_0}(f_{n+1}^{i+1,\delta}(\phi_0)) = f_n^{i,\delta}(f_{n+1}^{1,-\epsilon_0}(\phi_0)) = f_n^{i,\delta}(\zeta_1) = f_n^{1,\epsilon_1}(f_{n+1}^{i+1,\delta}(\phi_1))$$
$$\vdots$$
$$f_n^{1,-\epsilon_{k-2}}(f_{n+1}^{i+1,\delta}(\phi_{k-2})) = f_n^{i,\delta}(f_{n+1}^{1,-\epsilon_{k-2}}(\phi_{k-2})) = f_n^{i,\delta}(\zeta_{k-1}) = f_n^{1,\epsilon_{k-1}}(f_{n+1}^{i+1,\delta}(\phi_{k-1}))$$
$$f_n^{1,-\epsilon_{k-1}}(f_{n+1}^{i+1,\delta}(\phi_{k-1})) = f_n^{i,\delta}(f_{n+1}^{1,-\epsilon_{k-1}}(\phi_k)) = f_n^{i,\delta}(\zeta_k)$$

We must also show the face maps interact appropriately with the equivalence relation. That is, if $\chi \sim \chi'$ then $f_n^{i,\delta}(\chi) \sim f_n^{i,\delta}(\chi')$. It is easy to see that this follows from the cubical identity

$$f_{n+1}^{i+1,\delta} \circ d_{n+1}^1 = d_n^1 \circ f_n^{i,\delta}$$

We then have

$$\begin{aligned} & f_n^{i,\delta}\Big((\overrightarrow{\zeta}, \zeta_i, \zeta_i, \overrightarrow{\zeta}), (\overrightarrow{\phi}, d_{n+1}^1(\zeta_i), \overrightarrow{\phi}), (\overrightarrow{\epsilon}, \epsilon, \overrightarrow{\epsilon})\Big) \\ &= \Big((\overrightarrow{f_n^{i,\delta}(\zeta)}, f_n^{i,\delta}(\zeta_i), f_n^{i,\delta}(\zeta_i), \overrightarrow{f_n^{i,\delta}(\zeta)}), (\overrightarrow{f_{n+1}^{i+1,\delta}(\phi)}, f_{n+1}^{i+1,\delta} \circ d_{n+1}^1(\zeta_i), \overrightarrow{f_{n+1}^{i+1,\delta}(\phi)}), (\overrightarrow{\epsilon}, \epsilon, \overrightarrow{\epsilon})\Big) \\ &= \Big((\overrightarrow{f_n^{i,\delta}(\zeta)}, f_n^{i,\delta}(\zeta_i), f_n^{i,\delta}(\zeta_i), \overrightarrow{f_n^{i,\delta}(\zeta)}), (\overrightarrow{f_{n+1}^{i+1,\delta}(\phi)}, d_{n+1}^1(f_n^{i,\delta}(\zeta_i)), \overrightarrow{f_{n+1}^{i+1,\delta}(\phi)}), (\overrightarrow{\epsilon}, \epsilon, \overrightarrow{\epsilon})\Big) \\ &\sim f_n^{i,\delta}\Big((\overrightarrow{\zeta}), (\overrightarrow{\phi}), (\overrightarrow{\epsilon})\Big) \end{aligned}$$

Next we define the degeneracy maps.

Definition 3.11 (Degeneracy Maps)**.** Let $1 \le i \le n$ and $\chi = \Big((\vec{\zeta}), (\vec{\phi}), (\vec{\epsilon})\Big) \in (MX)_{n-1}$. The degeneracy map $d_n^i : (MX)_{n-1} \to (MX)_n$ is defined

$$d_n^i(\chi) = \Big((d_n^i(\vec{\zeta})), (d_{n+1}^{i+1}(\vec{\phi})), (\vec{\epsilon})\Big)$$

Once again we must show that this is well defined, and this argument is similar to the previous case. First we verify the image under d_n^i is an (n)-path. Clearly each $d_n^i(\zeta_j)$ is an n-cube of X and similarly each $d_{n+1}^{i+1}(\phi_j)$ is an $(n+1)$-cube, so we check that the tuple satisfies the (n)-path conditions. Once again, this follows from the cubical identities: for $1 \le i \le n+1$ we have $1 < i+1$, hence $f_{n+1}^{1,\epsilon_j} \circ d_{n+1}^{i+1} = d_n^i \circ f_n^{1,\epsilon_j}$. Thus we have:

$$f_{n+1}^{1,\epsilon_0}(d_{n+1}^{i+1}(\phi_0) = d_n^i(f_n^{1,\epsilon_0}(\phi_0)) = d_{n+1}^i(\zeta_0)$$
$$f_{n+1}^{1,-\epsilon_0}(d_{n+1}^{i+1}(\phi_0)) = d_n^i(f_n^{1,-\epsilon_0}(\phi_0)) = d_n^i(\zeta_1) = f_{n+1}^{1,\epsilon_1}(d_{n+1}^{i+1}(\phi_1))$$
$$\vdots$$
$$f_{n+1}^{1,-\epsilon_{k-2}}(d_{n+1}^{i+1}(\phi_{k-2})) = d_n^i(f_n^{1,-\epsilon_{k-2}}(\phi_{k-2})) = d_n^i(\zeta_{k-1}) = f_{n+1}^{1,\epsilon_{k-1}}(d_{n+1}^{i+1}(\phi_{k-1}))$$
$$f_{n+1}^{1,-\epsilon_{k-1}}(d_{n+1}^{i+1}(\phi_{k-1}) = d_n^i(f_n^{1,-\epsilon_{k-1}}(\phi_{k-1})) = d_n^i(\zeta_k)$$

We must also show that $\chi \sim \chi'$ implies $d_n^i(\chi) \sim d_n^i(\chi')$, but this follows immediately from the cubical identity

$$d_{n+1}^{i+1} \circ d_n^1 = d_{n+1}^1 \circ d_n^i$$

with an identical argument to the case for face maps:

$$\begin{aligned}
&d_n^i\Big((\overrightarrow{\zeta},\zeta_i,\zeta_i,\overrightarrow{\zeta}),(\overrightarrow{\phi},d_n^1(\zeta_i),\overrightarrow{\phi}),(\overrightarrow{\epsilon},\epsilon,\overrightarrow{\epsilon})\Big)\\
&\quad=\Big((\overrightarrow{d_n^i(\zeta)},d_n^i(\zeta_i),d_n^i(\zeta_i),\overrightarrow{d_n^i(\zeta)}),(\overrightarrow{d_{n+1}^{i+1}(\phi)},d_{n+1}^{i+1}\circ d_n^1(\zeta_i),\overrightarrow{d_{n+1}^{i+1}(\phi)}),(\overrightarrow{\epsilon},\epsilon,\overrightarrow{\epsilon})\Big)\\
&\quad=\Big((\overrightarrow{d_n^i(\zeta)},d_n^i(\zeta_i),d_n^i(\zeta_i),\overrightarrow{d_n^i(\zeta)}),(\overrightarrow{d_{n+1}^{i+1}(\phi)},d_{n+1}^{1}(d_n^i(\zeta_i)),\overrightarrow{d_{n+1}^{i+1}(\phi)}),(\overrightarrow{\epsilon},\epsilon,\overrightarrow{\epsilon})\Big)\\
&\quad\sim d_n^i\Big((\overrightarrow{\zeta}),(\overrightarrow{\phi}),(\overrightarrow{\epsilon})\Big)
\end{aligned}$$

Finally we come to the connection maps.

Definition 3.12 (Connection Maps)**.** Let $n\geq 2$, $1\leq i\leq n-1$, $\delta\in\{-1,1\}$ and $\chi=\Big((\vec{\zeta}),(\vec{\phi}),(\vec{\epsilon})\Big)\in(MX)_{n-1}$. The connection map $\Gamma_{n-1}^{i,\delta}:(MX)_{n-1}\to(MX)_n$ is defined

$$\Gamma_{n-1}^{i,\delta}(\chi)=\Big((\Gamma_{n-1}^{i,\delta}(\vec{\zeta})),(\Gamma_n^{i+1,\delta}(\vec{\phi})),(\vec{\epsilon})\Big)$$

For a final time we verify that this is well defined. First we must check the image under $\Gamma_{n-1}^{i,\delta}$ is indeed a cubical n-path. We know immediately that the $\Gamma_{n-1}^{i,\delta}(\zeta_j)$ are n-cubes, and similarly that the $\Gamma_n^{i+1,\delta}(\phi_j)$ are $(n+1)$-cubes as required. To prove that the path conditions are satisfied we note that $n+1\geq 3$ in all cases, and so we can make use of the cubical identity $f_{n+1}^{1,\epsilon_j}\circ\Gamma_n^{i+1,\delta}=\Gamma_{n-1}^{i}\circ f_n^{1,\epsilon_j}$, to obtain:

$$\begin{gathered}
f_{n+1}^{1,\epsilon_0}(\Gamma_n^{i+1,\delta}(\phi_0))=\Gamma_{n-1}^{i,\delta}(f_n^{1,\epsilon_0}(\phi_0))=\Gamma_{n-1}^{i,\delta}(\zeta_0)\\
f_{n+1}^{1,-\epsilon_0}(\Gamma_n^{i+1,\delta}(\phi_0))=\Gamma_{n-1}^{i,\delta}(f_n^{1,-\epsilon_0}(\phi_0))=\Gamma_{n-1}^{i,\delta}(\zeta_1)=f_{n+1}^{1,\epsilon_1}(\Gamma_n^{i+1,\delta}(\phi_1))\\
\vdots\\
f_{n+1}^{1,-\epsilon_{k-2}}(\Gamma_n^{i+1,\delta}(\phi_{k-2}))=\Gamma_{n-1}^{i,\delta}(f_n^{1,-\epsilon_{k-2}}(\phi_{k-2}))=\Gamma_{n-1}^{i,\delta}(\zeta_{k-1})=f_{n+1}^{1,\epsilon_{k-1}}(\Gamma_n^{i+1,\delta}(\phi_{k-1}))\\
f_{n+1}^{1,-\epsilon_{k-1}}(\Gamma_n^{i+1,\delta}(\phi_{k-1}))=\Gamma_{n-1}^{i,\delta}(f_n^{1,-\epsilon_{k-1}}(\phi_{k-1}))=\Gamma_{n-1}^{i,\delta}(\zeta_k)
\end{gathered}$$

In order to show this definition is compatible with the equivalence relation, we note that we have the cubical identity

$$\Gamma_n^{i+1,\delta}\circ d_n^1=d_n^1\circ\Gamma_{n-1}^{i,\delta}$$

Then, in much the same way as the cases for the face and degeneracy maps we have:

$$\begin{aligned}
&\Gamma_{n-1}^{i,\delta}\Big((\overrightarrow{\zeta},\zeta_i,\zeta_i,\overrightarrow{\zeta}),(\overrightarrow{\phi},d_n^1(\zeta_i),\overrightarrow{\phi}),(\overrightarrow{\epsilon},\epsilon,\overrightarrow{\epsilon})\Big)\\
&\quad=\Big((\overrightarrow{\Gamma_{n-1}^{i,\delta}(\zeta)},\Gamma_{n-1}^{i,\delta}(\zeta_i),\Gamma_{n-1}^{i,\delta}(\zeta_i),\overrightarrow{\Gamma_{n-1}^{i,\delta}(\zeta)}),(\overrightarrow{\Gamma_n^{i+1,\delta}(\phi)},\Gamma_n^{i+1,\delta}\circ d_n^1(\zeta_i),\overrightarrow{\Gamma_n^{i+1,\delta}(\phi)}),(\overrightarrow{\epsilon},\epsilon,\overrightarrow{\epsilon})\Big)\\
&\quad=\Big((\overrightarrow{\Gamma_{n-1}^{i,\delta}(\zeta)},\Gamma_{n-1}^{i,\delta}(\zeta_i),\Gamma_{n-1}^{i,\delta}(\zeta_i),\overrightarrow{\Gamma_{n-1}^{i,\delta}(\zeta)}),(\overrightarrow{\Gamma_n^{i+1,\delta}(\phi)},d_n^1(\Gamma_{n-1}^{i+1,\delta}(\zeta_i)),\overrightarrow{\Gamma_n^{i+1,\delta}(\phi)}),(\overrightarrow{\epsilon},\epsilon,\overrightarrow{\epsilon})\Big)\\
&\quad\sim\Gamma_{n-1}^{i,\delta}\Big((\overrightarrow{\zeta}),(\overrightarrow{\phi}),(\overrightarrow{\epsilon})\Big)
\end{aligned}$$

With all of the cubical operators defined we have

Proposition 3.13. *$MX=\{(MX)_n\}_{n\in\omega}$ together with the face, degeneracy and connection maps defined above is a cubical set with connections.*

Proof. It is sufficient to verify the cubical identities hold for the maps defined above - but this follows immediately from the definitions and the structure on X. □

From this we obtain the path objects MX. We extend this assignment to an endofunctor M. Let $\mu : X \to Y$ be a cubical morphism. For $n \in \omega$ and $\chi = \Big((\vec{\zeta}), (\vec{\phi}), (\vec{\epsilon})\Big) \in (MX)_n$ we define $M(\mu)_n : (MX)_n \to (MY)_n$ by

$$M(\mu)_n(\chi) = \Big((\overrightarrow{\mu_n(\zeta)}), (\overrightarrow{\mu_{n+1}(\phi)}), (\vec{\epsilon})\Big)$$

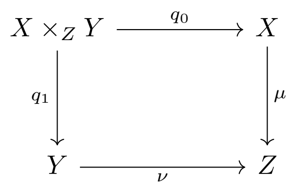
To see this maps cubical n-paths of X to cubical n-paths of Y we note that by definition the components of μ commute with the cubical operators. It then follows that

$$\begin{gathered}
f^{1,\epsilon_0}_{n+1}(\mu_{n+1}(\phi_0)) = \mu_n(f^{1,\epsilon_0}_{n+1}(\phi_0)) = \mu_n(\zeta_0) \\
f^{1,-\epsilon_0}_{n+1}(\mu_{n+1}(\phi_0)) = \mu_n(f^{1,-\epsilon_0}_{n+1}(\phi_0)) = \mu_n(\zeta_1) = f^{1,\epsilon_1}_{n+1}(\mu_{n+1}(\phi_1)) \\
\vdots \\
f^{1,-\epsilon_{k-2}}_{n+1}(\mu_{n+1}(\phi_{k-2})) = \mu_n(f^{1,-\epsilon_{k-2}}_{n+1}(\phi_{k-2})) = \mu_n(\zeta_{k-1}) = f^{1,\epsilon_{k-1}}_{n+1}(\mu_{n+1}(\phi_{k-1})) \\
f^{1,-\epsilon_{k-1}}_{n+1}(\mu_{n+1}(\phi_{k-1})) = \mu_n(f^{1,-\epsilon_{k-1}}_{n+1}(\phi_{k-1})) = \mu_n(\zeta_k)
\end{gathered}$$

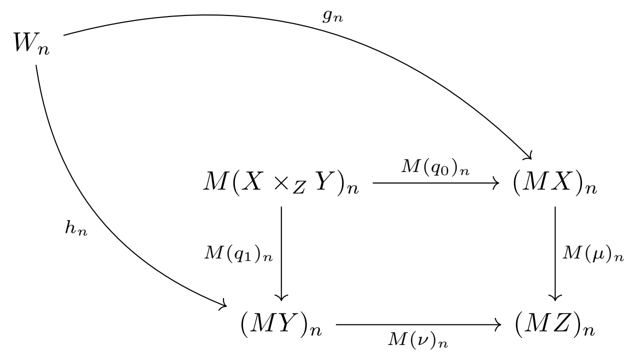
For much the same reason, the maps $(M(\mu))_n$ respect the equivalence relations. Hence $M(\mu)$ is well defined. That this assignment is functorial is immediate. With this in place we can begin to prove that $\boldsymbol{cSet^C}$ satisfies the first nice path object category axiom.

Proposition 3.14. *The endofunctor M is pullback preserving.*

Proof. Suppose we have a pullback



in $\boldsymbol{cSet^c}$. We wish to show the image under M is also a pullback. Since $\boldsymbol{cSet^c}$ is a presheaf category it suffices to verify this pointwise, so we suppose we have the following commutative diagram.



Given $w \in W_n$ we denote the cubical n-paths $g_n(w)$ and $h_n(w)$ by

$$g_n(w) = ((\overrightarrow{g_n(w)^0}), (\overrightarrow{g_n(w)^1}), (\overrightarrow{g_n(w)^2})) \qquad h_n(w) = ((\overrightarrow{h_n(w)^0}), (\overrightarrow{h_n(w)^1}), (\overrightarrow{h_n(w)^2}))$$

With the length of the sequence $(\overrightarrow{g_n(w)^0})$ given by k and the length of the sequence $(\overrightarrow{h_n(w)^0})$ given by k'. Now commutativity of this pullback tells us that $M(\mu)_n(g_n(w)) \sim M(\nu)_n(h_n(w))$. This means that we can add degenerate cubes to one path - without loss of generality suppose $M(\mu)_n(g_n(w))$ and call the modified path $(M(\mu)_n(g_n(w))^+$ - to obtain equality on the cubical n-paths: $(M(\mu)_n(g_n(w))^+ = M(\nu)_n(h_n(w))$. In turn, by the commutativity of the cubical operators we may add degenerate cubes to the path $g_n(w)$ to obtain a unique $g_n(w)^+$ such that the length of the sequence $\overrightarrow{g_n(w)^{+,0}}$ is k' and $M(\mu)_n((g_n(w)^+) = (M(\mu)_n(g_n(w))^+ = M(\nu)_n(h_n(w))$. Thus by the definition of M, for $1 \leq j \leq k'$, $1 \leq l \leq k'-1$ we have

$$\mu_n(g_n(w)_j^{+,0}) = \nu_n(h_n(w)_j^0) \qquad\qquad \mu_{n+1}(g_n(w)_l^{+,1}) = \nu_{n+1}(h_n(w)_l^1)$$

This means that the pairs $((g_n(w)_j^{+,0}), h_n(w)_j^0)$ are n-cubes of $X \times_Z Y$ and the pairs $(g_n(w)_l^{+,1}, h_n(w)_l^1)$ are $n+1$-cubes. We also have the identity, for $1 \leq l \leq k-1$,

$$g_n(w)_l^{+,2} = h_n(w)_l^2$$

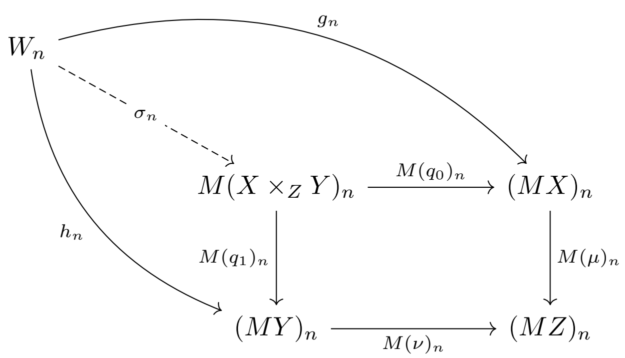
We thus define $\epsilon(w)_l = g_n(w)_l^{+,2} = h_n(w)_l^2$ and claim that

$$\sigma_n(w) = ((\overrightarrow{(g_n(w)^{+,0}, h_n(w)^0)}), (\overrightarrow{(g_n(w)^{+,1}, h_n(w)^1)}), (\overrightarrow{\epsilon(w)}))$$

yields a cubical n-path through $X \times_Z Y$. We have:

$$f_{n+1}^{1,\epsilon(w)_l}(g_n(w)_l^{+,1}, h_n(w)_l^1) = (f_{n+1}^{1,g_n(w)_l^{+,2}}(g_n(w)_l^{+,1}), f_{n+1}^{1,h_n(w)_l^2}(h_n(w)_l^1)) = (g_n(w)_l^{+,0}, h_n(w)_l^0)$$
$$f_{n+1}^{1,-\epsilon(w)_l}(g_n(w)_l^{+,1}, h_n(w)_l^1) = (f_{n+1}^{1,-g_n(w)_l^{+,2}}(g_n(w)_l^{+,1}), f_{n+1}^{1,-h_n(w)_l^2}(h_n(w)_l^1)) = (g_n(w)_{l+1}^{+,0}, h_n(w)_{l+1}^0)$$

as required. Hence we obtain



Commutativity follows straightforwardly from the definition of the projection maps q_0, q_1 and M's action upon them together with the equivalence relation. Similarly, by direct computation we can see this map must be unique when we pass to equivalence classes: for any other σ'_n such that the diagram commutes, for all w we necessarily have $\sigma_n(w) \sim \sigma'_n(w)$ so the functions are identical on the equivalence classes. It follows that M is pullback preserving. $\square$

Next we give the natural transformations $s, t : M \Rightarrow id$ and $e : id \to M$. Given a cubical set X, $n \in \omega$, an n-cube ζ and a cubical n-path $\chi = \Big((\vec{\zeta}), (\vec{\phi}), (\vec{\epsilon})\Big)$ we define

$$(s_X)_n(\chi) = \zeta_0 \qquad\qquad (t_X)_n(\chi) = \zeta_k \qquad\qquad (e_X)_n(\zeta) = \Big((\zeta), (\ ), (\ )\Big)$$

To show that, for each cubical set X, s_X, t_X and e_X are cubical morphisms it suffices to show they commute with the cubical operators: but this is immediate from our definitions. To show naturality, we assume $\mu : X \Rightarrow Y$ is a cubical morphism. It suffices to check commutativity at n, and once again this is straightforward. We have

$$\begin{aligned}
(s_Y)_n \circ M(\mu)_n(\chi) &= (s_Y)_n\Big((\overrightarrow{\mu_n(\zeta)}), (\overrightarrow{\mu_{n+1}(\phi)}), (\vec{\epsilon})\Big) = \mu_n(\zeta_0) = (s_X)_n \circ \mu_n(\chi) \\
(t_Y)_n \circ M(\mu)_n(\chi) &= (t_Y)_n\Big((\overrightarrow{\mu_n(\zeta)}), (\overrightarrow{\mu_{n+1}(\phi)}), (\vec{\epsilon})\Big) = \mu_n(\zeta_k) = (t_X)_n \circ \mu_n(\chi) \\
(e_Y)_n \circ \mu_n(\zeta) &= \Big((\mu_n(\zeta)), (\ ), (\ )\Big) = M(\mu)_n \circ (e_X)_n(\zeta)
\end{aligned}$$

So s, t, e are natural transformations as required. With s and t defined we are able to determine the pullback

$$\begin{array}{ccc}
MX \times_X MX & \xrightarrow{\ p_0^X\ } & MX \\
{\scriptstyle p_1^X}\big\downarrow & & \big\downarrow{\scriptstyle t_X} \\
MX & \xrightarrow[\ s_X\ ]{} & X
\end{array}$$

Computing directly in **Set** we have

$$(MX \times_X MX)_n = \{(\chi, \chi') \in (MX)_n \times (MX)_n \mid (t_X)_n(\chi) = (s_X)_n(\chi')\}$$

Thus an n-cube of $MX \times MX$ is given by a pair of cubical n-paths through X

$$\begin{aligned}
\chi &= \Big((\zeta_0, \ldots, \zeta_k), (\vec{\phi}), (\vec{\epsilon})\Big) \\
\chi' &= \Big((\zeta_k, \zeta'_1, \ldots, \zeta'_{k'}), (\vec{\phi'}), (\vec{\epsilon'})\Big)
\end{aligned}$$

As short-hand for this situation we say χ and χ' are *compatible.* Since

$$f_{n+1}^{1,-\epsilon_{k-1}}(\phi_{k-1}) = \zeta_k = f_{n+1}^{1,\epsilon'_0}(\phi'_0)$$

we can construct a new n-path through X by concatenation of the $(n+1)$-cubes, pasting at the shared n-cube:

$$(c_X)_n(\chi, \chi') = \Big((\zeta_0, \ldots, \zeta_k, \zeta'_1, \ldots, \zeta'_{k'}), (\vec{\phi}, \vec{\phi'}), (\vec{\epsilon}, \vec{\epsilon'})\Big)$$

We introduce the notation $\chi \cdot \chi'$ for $(c_X)_n(\chi, \chi')$. Keeping X fixed, we claim the collection of maps $(c_X)_n : (MX \times_X MX)_n \to (MX)_n$ defined in this way constitutes a cubical morphism $c_X : MX \times_X MX \to MX$. Once again it is sufficient to show commutativity with the cubical operators: we attend only to the face maps as the argument is identical for degeneracy and connection maps:

$$\begin{aligned}
f_n^{i,\delta} \circ (c_X)_n(\chi, \chi') &= \Big(\big(f_n^{i,\delta}(\zeta_0), \ldots, f_n^{i,\delta}(\zeta_k), f_n^{i,\delta}(\zeta'_1), \ldots, f_n^{i,\delta}(\zeta'_{k'})\big), \big(\overrightarrow{f_{n+1}^{i+1,\delta}(\phi)}, \overrightarrow{f_{n+1}^{i+1,\delta}(\phi')}\big), \big(\vec{\epsilon}, \vec{\epsilon'}\big) \Big) \\
&= (c_X)_{n-1}(f_n^{i,\delta}(\chi), f_n^{i,\delta}(\chi')) = (c_X)_{n-1} \circ f_n^{i,\delta}(\chi, \chi')
\end{aligned}$$

We thus claim that the collection of cubical morphisms

$$c = (c_X : MX \times_X MX \to MX \mid X \text{ in } \boldsymbol{cSet^c})$$

constitutes a natural transformation $c : C^M \Rightarrow M$. Let $\mu : X \Rightarrow Y$ be a cubical morphism. We wish to show commutativity of the diagram:

$$\begin{array}{ccc}
(MX \times MX)_n & \xrightarrow{C^M(\mu)_n} & (MY \times_Y MY)_n \\
\Big\downarrow {\scriptstyle (c_X)_n} & & \Big\downarrow {\scriptstyle (c_Y)_n} \\
(MX)_n & \xrightarrow[M(\mu)_n]{} & (MY)_n
\end{array}$$

Recall that the components $C^M(\mu)_n$ are defined via the pullback

$$\begin{array}{ccccc}
(MX \times_X MX)_n & & \xrightarrow{M(\mu)_n \circ (p_0^X)_n} & & \\
& \overset{C^M(\mu)_n}{\searrow} & & & \\
& & (MY \times_Y MY)_n & \xrightarrow{(p_0^Y)_n} & (MY)_n \\
{\scriptstyle M(\mu)_n \circ (p_1^X)_n} \searrow & & \Big\downarrow {\scriptstyle (p_1^Y)_n} & & \Big\downarrow {\scriptstyle (t_Y)_n} \\
& & (MY)_n & \xrightarrow[(s_Y)_n]{} & Y_n
\end{array}$$

Computing directly we have that

$$\begin{aligned}
C^M(\mu)_n(\chi, \chi') &= (M(\mu)_n(\chi), M(\mu)_n(\chi')) \\
&= \Big(\big((\overrightarrow{\mu_n(\zeta)}), (\overrightarrow{\mu_{n+1}(\phi)}), (\vec{\epsilon}) \big), \big((\mu_n(\zeta_k), \overrightarrow{\mu_n(\zeta')}), (\overrightarrow{\mu_{n+1}(\phi')}), (\vec{\epsilon'}) \big) \Big)
\end{aligned}$$

Hence it follows that

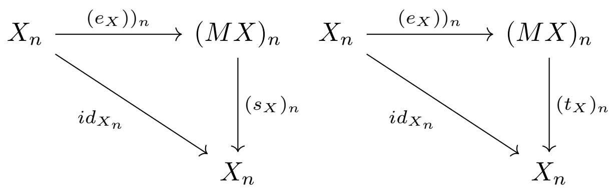
$$\begin{aligned}
(c_Y)_n \circ C^M(\mu)_n(\chi, \chi') &= \Big(\big(\mu_n(\zeta_0), \ldots, \mu_n(\zeta_k), \mu_n(\zeta'_1), \ldots, \mu_n(\zeta'_{k'})\big), \big(\overrightarrow{\mu_{n+1}(\phi)}, \overrightarrow{\mu_{n+1}(\phi')}\big), \big(\vec{\epsilon}, \vec{\epsilon'}\big) \Big) \\
&= (c_X)_n \circ M(\mu)_n(\chi, \chi')
\end{aligned}$$

as required: c is a natural transformation. It is also immediate from the definitions that all of these natural transformations respect the equivalence relation. With this we have everything in place to verify the internal category properties.

Proposition 3.15. *The morphisms s_X, t_X, e_X, c_X equip MX with the structure of an internal category.*

Proof. We must verify each of the internal category axioms. As usual, it is sufficient to check commutativity of the diagrams at n.

- **Source/Target of Identities**: We must show that the diagrams

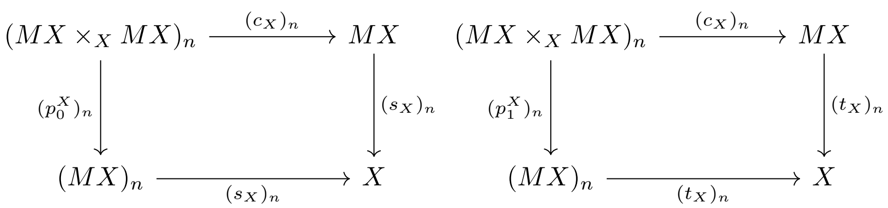


commute. We have

$$(s_X)_n \circ (e_X)_n(\zeta) = (s_X)_n\Big((\zeta), (\ ), (\ )\Big) = \zeta$$

$$(t_X)_n \circ (e_X)_n(\zeta) = (t_X)_n\Big((\zeta), (\ ), (\ )\Big) = \zeta$$

- **Source/Target of Composition**: We must show that the diagrams

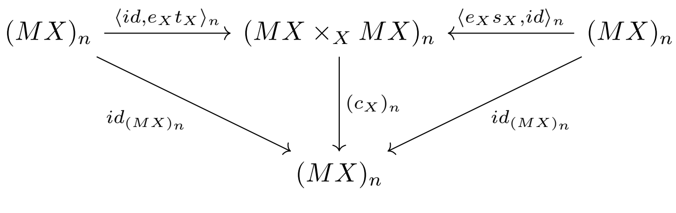


commute. Given compatible cubical n-paths χ and χ' we have

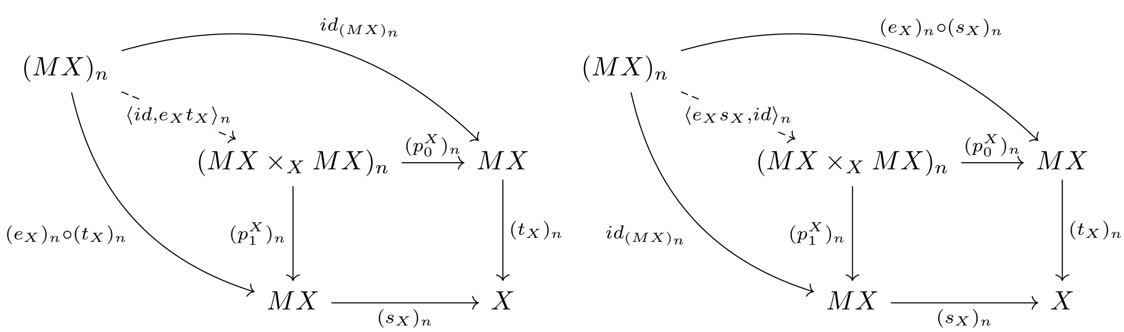
$$(s_X)_n \circ (c_X)_n(\chi, \chi') = \zeta_0 = (s_X)_n \circ (p_0^X)_n(\chi, \chi')$$

$$(t_X)_n \circ (c_X)_n(\chi, \chi') = \zeta'_{k'} = (t_X)_n \circ (p_1^X)_n(\chi, \chi')$$

- **Left/Right Identity Laws**: The next diagram we require to commute is given by



Recall that we obtain the morphisms $\langle id, e_X t_X\rangle$ and $\langle e_X s_X, id\rangle$ are obtained as follows:



Computing directly we see that

$$\langle id, e_X t_X\rangle_n(\chi) = \Big(\chi, \Big((\zeta_k), (\ ), (\ )\Big)\Big)$$

$$\langle e_X s_X, id\rangle_n(\chi) = \Big(\Big((\zeta_0), (\ ), (\ )\Big), \chi\Big)$$

So we have

$$\begin{aligned}(c_X)_n \circ \langle id, e_X t_x\rangle_n(\chi) &= \Big(\big(\zeta_0, \ldots, \zeta_k\big), \big(\phi_0, \ldots, \phi_{k-1}\big), \big(\epsilon_0, \ldots, \epsilon_{k-1}\big)\Big) \\ &= \chi\end{aligned}$$

$$\begin{aligned}(c_X)_n \circ \langle e_X s_X, id\rangle_n(\chi) &= \Big(\big(\zeta_0, \ldots, \zeta_k\big), \big(\phi_0, \ldots, \phi_{k-1}\big), \big(\epsilon_0, \ldots, \epsilon_{k-1}\big)\Big) \\ &= \chi\end{aligned}$$

- **Associativity**: Finally we require commutativity of

$$\begin{array}{ccc}(MX \times_X MX \times_X MX)_n & \xrightarrow{\langle c_X, id\rangle_n} & MX \times_{C_0} MX \\ \Big\downarrow{\scriptstyle \langle id, c_n\rangle_n} & & \Big\downarrow{\scriptstyle (c_X)_n} \\ MX \times_X MX & \xrightarrow[(c_X)_n]{} & MX\end{array}$$

It is straightforward to verify, given a compatible triple $(\chi, \chi', \chi'') \in (MX \times_X MX \times_X MX)_n$, we have

$$\langle c_X, id\rangle_n(\chi, \chi', \chi'') = (\chi \cdot \chi', \chi'') \qquad \langle id, c_X\rangle_n(\chi, \chi', \chi'') = (\chi, \chi' \cdot \chi'')$$

and so commutativity follows from the obvious fact that concatenation is associative

It follows that we have an internal category $X \underset{t_X}{\overset{s_X}{\leftleftarrows}} \xrightarrow{e_X} MX \xleftarrow{c_X} MX \times_X MX$

□

The final piece of data we require in order to satisfy the first axiom is the natural transformation $\tau : M \Rightarrow M$ such that, for every X in $\boldsymbol{cSet^c}$, τ_X is an identity-on-objects involution of the internal category MX. The maps $(\tau_X)_n : (MX)_n \to (MX)_n$ are defined:

$$(\tau_X)_n(\chi) = \Big(\big(\zeta_k, \ldots, \zeta_0\big), \big(\phi_{k-1}, \ldots, \phi_0\big), \big(-\epsilon_{k-1}, \ldots, -\epsilon_0\big)\Big)$$

It follows immediately from the definitions that τ_X commutes with the cubical operators, and is thus a cubical morphism. That the collection of maps τ_X constitutes a natural transformation follows analogously: due to the similarity to previous arguments we leave these simple details to the reader. We require:

$$\tau_X \circ \tau_X = id_X \qquad \tau_X \circ e_X = e_X$$
$$s_X \circ \tau_X = t_X \qquad t_X \circ \tau = s_X$$
$$\tau_X \circ c_X = c_X \circ \tau_X^c$$

That $\tau_X \circ \tau_X = id_X$ holds is immediate from the definition. We have

$$(\tau_X)_n \circ (s_X)_n(\chi) = \zeta_k = (t_X)_n(\chi) \qquad (\tau_X)_n \circ (t_X)_n(\chi) = \zeta_0 = (s_X)_n(\chi)$$

For a compatible pair (χ, χ') it is straightforward to compute that

$$(\tau_X^c)_n(\chi, \chi') = ((\tau_X)_n(\chi'), (\tau_X)_n(\chi))$$

so we obtain

$$(\tau_X)_n \circ (c_X)_n(\chi, \chi') = (\tau_X)_n(\chi \cdot \chi') = (\tau_X)_n(\chi') \cdot (\tau_X)_n(\chi) = c_X \circ (\tau_X^c)_n(\chi, \chi')$$

Finally we obtain the remaining identity:

$$(\tau_X)_n \circ (e_X)_n(\zeta) = \tau_X\Big((\zeta), (\,), (\,)\Big) = \Big((\zeta), (\,), (\,)\Big) = (e_X)_n(\zeta)$$

It follows that $\boldsymbol{cSet^c}$ satisfies the first nice path object category axiom.

3.2.2 Axiom 2: Nice Constant Paths

Next we must show that $M1 \cong 1$, and in order to do so we investigate what the terminal object in $\boldsymbol{cSet^c}$ looks like. Making use of the fact $\boldsymbol{cSet}$ is a presheaf category we obtain the following lemma as a particular instance of the pointwise computation of limits in presheaf categories.

Lemma 3.16. *In a presheaf category $Set^{\mathcal{C}^{op}}$, a presheaf X is terminal iff for every C in $\mathcal{C}_0$ $X(C)$ is a singleton.*

We thus define 1 from scratch

Definition 3.17 (Terminal Cubical Set)**.** The terminal cubical set 1 is defined

- For all $n \in \omega$: $1_n = \{n\}$
- For all $1 \leq i \leq n$ and $\delta \in \{-1, 1\}$: $f_n^{i,\delta}(n) = n - 1$
- For all $1 \leq i \leq n$: $d_n^i(n-1) = n$
- For all $n \geq 2$, $1 \leq i \leq n-1$ and $\delta \in \{-1, 1\}$: $\Gamma_{n-1}^{i,\delta}(n-1) = n$

That this definition satisfies the cubical identities follows trivially; that it is terminal follows from Lemma 3.16. Another way of looking at 1 is as the degenerate cubical set generated from a single vertex. It follows that every cubical n-path of 1 is of the form

$$\Big((n,\ldots,n),(n+1,\ldots,n+1),(\vec{\epsilon})\Big)$$

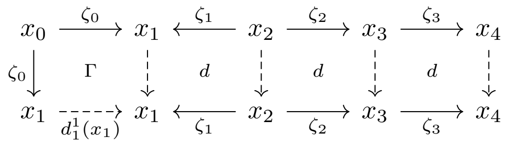
Under the equivalence relation we have that for each n there exists a single equivalence class containing the path $\Big((n),(\,),(\,)\Big)$. Hence applying Lemma 3.16 again gives us that $M1$ is also terminal, and thus isomorphic to 1: $\boldsymbol{cSet^c}$ satisfies the second nice path category axiom.

3.2.3 Axiom 3: Nice Path Contraction

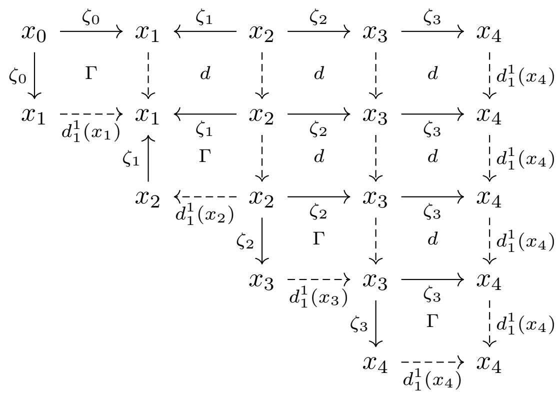
For the final axiom we must construct the natural transformation that contracts paths. It is here that we make essential use of the connection structure on our cubical sets. Before we delve into the combinatorics we illustrate the intuition behind our argument by looking at the case in low dimensions. We look back to the example (3.1) given at the start of the section:

$$x_0 \xrightarrow{\zeta_0} x_1 \xleftarrow{\zeta_1} x_2 \xrightarrow{\zeta_2} x_3 \xrightarrow{\zeta_3} x_4$$

The idea is to give a 1-path which has as one of its faces (3.1) and as the other the same path with ζ_0 removed. By applying a connection map to ζ_0 and degeneracy maps elsewhere we can achieve this:



By iterating this idea we obtain the following picture



where each row of squares is a 2-path. This gives us a 0-path through MX, and by applying the equivalence relation it has our original path as source and the trivial path at x_4 as its target. Not only this, but by the definition of M on cubical morphisms we have that the image under $M(s_X)_1$ is the original path and the image under $M(t_X)_1$ is the trivial path at x_4 (again by the equivalence relation) and this is precisely what we require. We now tackle the problem in full generality and make this idea precise. First we introduce two *contraction operators* on cubical n-paths.

Definition 3.18 (Contraction Operators). Let $\chi = \Big((\zeta_0,\ldots,\zeta_k),(\phi_0,\ldots,\phi_{k-1}),(\epsilon_0,\ldots,\epsilon_{k-1})\Big)$ be a cubical n-path through a cubical set X with $k \geq 1$. We define

- χ^- to be the cubical n-path given by $\Big((\zeta_1, \dots, \zeta_k), (\phi_1, \dots, \phi_{k-1}), (\epsilon_1, \dots, \epsilon_{k-1})\Big)$
- $\Theta(\chi)$ to be the cubical $(n+1)$-path given by

$$\Big(\big(\phi_0, d^1_{n+1}(\zeta_1), \dots, d^1_{n+1}(\zeta_k)\big), \big(\Gamma^{1,\epsilon_0}_{n+1}(\phi_0), d^2_{n+2}(\phi_1), \dots, d^2_{n+2}(\phi_{k-1})\big), \big(\epsilon_0, \dots, \epsilon_{k-1}\big)\Big)$$

We first verify that this definition makes sense. It's obvious that for a cubical n-path χ, χ^- is also an n-path. That $\Theta(-)$ yields $(n+1)$-paths is a little less immediate however, and so we compute directly. First recall that the following cubical identities hold

$$f^{1,\delta}_{n+2} \circ \Gamma^{1,\delta}_{n+1} = id \qquad f^{1,-\delta}_{n+2} \circ \Gamma^{1,\delta}_{n+1} = d^1_{n+1} \circ f^{1,-\delta}_{n+1} \qquad f^{1,\delta}_{n+2} \circ d^2_{n+2} = d^1_{n+1} \circ f^{1,\delta}_{n+1}$$

We then have

$$\begin{gathered}
f^{1,\epsilon_0}_{n+2}(\Gamma^{1,\epsilon_0}_{n+1}(\phi_0)) = \phi_0 \\
f^{1,-\epsilon_0}_{n+2}(\Gamma^{1,\epsilon_0}_{n+1}(\phi_0)) = d^1_{n+1}(f^{1,-\epsilon_0}_{n+1}(\phi_0)) = d^1_{n+1}(\zeta_1) = d^1_{n+1}(f^{1,\epsilon_1}_{n+1}(\phi_1)) = f^{1,\epsilon_1}_{n+2}(d^2_{n+2}(\phi_1)) \\
f^{1,-\epsilon_1}_{n+2}(d^2_{n+2}(\phi_1)) = d^1_{n+1}(f^{1,-\epsilon_1}_{n+1}(\phi_1)) = d^1_{n+1}(\zeta_2) = d^1_{n+1}(f^{1,\epsilon_2}_{n+1}(\phi_2)) = f^{1,\epsilon_2}_{n+2}(d^2_{n+2}(\phi_2)) \\
\vdots \\
f^{1,-\epsilon_{k-1}}_{n+2}(d^2_{n+2}(\phi_{k-1})) = d^1_{n+1}(f^{1,-\epsilon_{k-1}}(\phi_{k-1})) = d^1_{n+1}(\zeta_k)
\end{gathered}$$

And so $\Theta(\chi)$ is indeed an $(n+1)$-path. With that out of the way we now justify our operators' suggestive name by way of a lemma.

Lemma 3.19. *Let* $\chi = \Big((\zeta_0, \dots, \zeta_k), (\phi_0, \dots, \phi_{k-1}), (\epsilon_0, \dots, \epsilon_{k-1})\Big)$ *be a cubical n-path through X with $k \geq 1$. Then*

i)

$$f^{1,\epsilon_0}_{n+1}(\Theta(\chi)) = \chi$$

ii)

$$f^{1,-\epsilon_0}_{n+1}(\Theta(\chi)) \sim \chi^-$$

Proof. We compute directly, noting that we have the cubical identities

$$f^{i,\delta}_{n+1} \circ d^i_{n+1} = id \qquad f^{i+1,\delta}_{n+2} \circ \Gamma^{i,\delta}_{n+1} = id \qquad f^{i+1,-\delta}_{n+2} \circ \Gamma^{i,\delta}_{n+1} = d^i_{n+1} \circ f^{i,-\delta}_{n+1}$$

i) $f^{1,\epsilon_0}_{n+1}(\Theta(\chi)) = \chi$:

$$\begin{aligned}
f^{1,\epsilon_0}_{n+1}(\Theta(\chi)) &= \Big(\big(f^{1,\epsilon_0}_{n+1}(\phi_0), \overrightarrow{f^{1,\epsilon_0}_{n+1} \circ d^1_{n+1}(\zeta)}\big), \big(f^{2,\epsilon_0}_{n+2} \circ \Gamma^{1,\epsilon_0}_{n+1}(\phi_0), \overrightarrow{f^{2,\epsilon_0}_{n+2} \circ d^2_{n+2}(\phi)}\big), \big(\vec{\epsilon}\big)\Big) \\
&= \Big(\big(\zeta_0, \dots, \zeta_k\big), \big(\phi_0, \dots, \phi_{k-1}\big), \big(\vec{\epsilon}\big)\Big) \\
&= \chi
\end{aligned}$$

ii) $f^{1,-\epsilon_0}_{n+1}(\Theta(\chi)) \sim \chi^-$:

$$\begin{aligned} f^{1,-\epsilon_0}_{n+1}(\Theta(\chi)) &= \Big(f^{1,-\epsilon_0}_{n+1}(\phi_0), \overrightarrow{f^{1,-\epsilon_0}_{n+1} \circ d^1_{n+1}(\zeta)}\big), \big(f^{2,-\epsilon_0}_{n+2} \circ \Gamma^{1,\epsilon_0}_{n+1}(\phi_0), \overrightarrow{f^{2,-\epsilon_0}_{n+2} \circ d^2_{n+2}(\phi)}\big), (\vec{\epsilon})\Big) \\ &= \Big((\zeta_1, \zeta_1, \ldots, \zeta_k), (d^1_{n+1} \circ f^{1,-\epsilon_0}_{n+1}(\phi_0), \phi_1, \ldots, \phi_{k-1}), (\vec{\epsilon})\Big) \\ &= \Big((\zeta_1, \zeta_1, \ldots, \zeta_k), (d^1_{n+1}(\zeta_1), \phi_1, \ldots, \phi_{k-1}), (\vec{\epsilon})\Big) \\ &\sim \chi^- \end{aligned}$$

□

Notation 3.2.2. Given a cubical n-path χ through X, we define χ^{-^j} for $0 \leq j \leq k$ by

i) $\chi^{-^0} = \chi$ ii) $\chi^{-^1} = \chi^-$ iii) $\chi^{-^{i+1}} = (\chi^{-^i})^-$

Note that $\chi^{-^k} = (e_X)_n(t_X)_n(\chi)$ - precisely what we wish to contract our paths to. We thus set

$$(\eta_X)_n(\chi) = \Big((\chi^{-^0}, \ldots, \chi^{-^k}), (\Theta(\chi^{-^0}), \ldots, \Theta(\chi^{-^{k-1}})), (\vec{\epsilon})\Big)$$

By Lemma 3.19, this indeed gives us an assignment $(\eta_X)_n : (MX)_n \to (MMX)_n$. There are a number of things left to check before we can see if the Axiom 3 identities are satisfied. Firstly, that this assignment is independent of choice of representative; secondly, that each η_X is a cubical morphism and finally that η constitutes a natural transformation. The following lemma shows that the contraction operators themselves don't respect the equivalence relation, but we can use it to show the map $(\eta_X)_n$ does.

Lemma 3.20.

1. *Let $\chi \sim \chi'$ with $\chi = \Big((\vec{\zeta}), (\vec{\phi}), (\vec{\epsilon})\Big)$ and $\chi' = \Big((\zeta_0, \zeta_0, \vec{\zeta}), (d^1_{n+1}(\zeta_0), \vec{\phi}), (\epsilon, \vec{\epsilon})\Big)$. Then $\Theta(\chi') \sim d^1_{n+1}(\chi)$.*
2. *Let $\chi \sim \chi'$ with $\chi = \Big((\vec{\zeta}), (\vec{\phi}), (\vec{\epsilon})\Big)$, $\chi' = \Big((\vec{\zeta}, \zeta_i, \zeta_i, \vec{\zeta}), (\vec{\phi}, d^1_{n+1}(\zeta_i), \vec{\phi}), (\vec{\epsilon}, \epsilon, \vec{\epsilon})\Big)$ and $i > 0$. Then $\Theta(\chi) \sim \Theta(\chi')$.*

Proof. We compute both cases directly:

1. First note that we have the cubical identity $\Gamma^{1,\epsilon}_{n+1} \circ d^1_{n+1} = d^1_{n+2} \circ d^1_{n+1}$. It thus follows that

$$\begin{aligned} \Theta(\chi') &= \Big((d^1_{n+1}(\zeta_0), d^1_{n+1}(\zeta_0), \ldots, d^1_{n+1}(\zeta_k)), (\Gamma^{1,\epsilon}_{n+1}(d^1_{n+1}(\zeta_0)), \overrightarrow{d^2_{n+2}(\phi)}), (\epsilon, \vec{\epsilon})\Big) \\ &= \Big((d^1_{n+1}(\zeta_0), d^1_{n+1}(\zeta_0), \ldots, d^1_{n+1}(\zeta_k)), (d^1_{n+2}(d^1_{n+1}(\zeta_0)), \overrightarrow{d^2_{n+2}(\phi)}), (\epsilon, \vec{\epsilon})\Big) \\ &\sim d^1_{n+1}(\chi) \end{aligned}$$

2. We note that we have the cubical identity $d^2_{n+2} \circ d^1_{n+1} = d^1_{n+2} \circ d^1_{n+1}$. That $\Theta(\chi) \sim \Theta(\chi')$ then follows from a similar computation to the previous case:

$$\begin{aligned}
\Theta(\chi') = \quad & \Big(\big(\phi_0, \overrightarrow{d^1_{n+1}(\zeta)}, d^1_{n+1}(\zeta_i), d^1_{n+1}(\zeta_i), \overrightarrow{d^1_{n+1}(\zeta)}\big), \\
& \big(\Gamma^{1,\epsilon}_{n+1}(\phi_0), \overrightarrow{d^2_{n+2}(\phi)}, d^2_{n+2}(d^1_{n+1}(\zeta_i)), \overrightarrow{d^2_{n+2}(\phi)}\big), \big(\overrightarrow{\epsilon}, \epsilon, \overrightarrow{\epsilon}\big) \Big) \\
= \quad & \Big(\big(\phi_0, \overrightarrow{d^1_{n+1}(\zeta)}, d^1_{n+1}(\zeta_i), d^1_{n+1}(\zeta_i), \overrightarrow{d^1_{n+1}(\zeta)}\big), \\
& \big(\Gamma^{1,\epsilon}_{n+1}(\phi_0), \overrightarrow{d^2_{n+2}(\phi)}, d^1_{n+2}(d^1_{n+1}(\zeta_i)), \overrightarrow{d^2_{n+2}(\phi)}\big), \big(\overrightarrow{\epsilon}, \epsilon, \overrightarrow{\epsilon}\big) \Big) \\
\sim \quad & \Theta(\chi)
\end{aligned}$$

□

Corollary 3.21. *For cubical n-paths χ, χ' through X, if $\chi \sim \chi'$ then $(\eta_X)_n(\chi) \sim (\eta_X)_n(\chi')$.*

Proof. Without loss of generality we assume $\chi = \Big((\overrightarrow{\zeta}), (\overrightarrow{\phi}), (\overrightarrow{\epsilon}) \Big)$ and $\chi' = \Big((\overrightarrow{\zeta}, \zeta_i, \zeta_i, \overrightarrow{\zeta}), (\overrightarrow{\phi}, d^1_{n+1}(\zeta_i), \overrightarrow{\phi}), (\overrightarrow{\epsilon}, \epsilon, \overrightarrow{\epsilon}) \Big)$ with no other degenerate $(n+1)$-cubes occurring. We can make the following observations immediately:

i) $\chi^{-j} \sim (\chi')^{-j}$ for $0 \leq j \leq i$

ii) $\chi^{-j} = (\chi')^{-j+1}$ for $i \leq j \leq k-1$

Applying the previous lemma and our observations we have that

i) $\Theta((\chi')^{-j}) \sim \Theta(\chi^{-j})$ for $0 \leq j < i$

ii) $\Theta((\chi')^{-i}) \sim d^1_{n+1}(\chi^i)$

iii) $\Theta(\chi')^{-j+1} = \Theta(\chi^{-j})$ for $i \leq j \leq k-1$

Hence we have

$$\begin{aligned}
(\eta_X)_n(\chi') &= \Big(\big(\overrightarrow{(\chi')^{-(\cdot)}}\big), \big(\overrightarrow{\Theta(\chi^{-(\cdot)})}\big), \big(\overrightarrow{\epsilon}, \epsilon, \overrightarrow{\epsilon}\big) \Big) \\
&= \Big(\big(\overrightarrow{\chi^{-(\cdot)}}, \chi^{-i}, \chi^{-i}, \overrightarrow{\chi^{-(\cdot)}}\big), \big(\overrightarrow{\Theta(\chi^{-(\cdot)})}, d^1_{n+1}(\chi^{-i}), \overrightarrow{\Theta(\chi^{-(\cdot)})}\big), \big(\overrightarrow{\epsilon}, \epsilon, \overrightarrow{\epsilon}\big) \Big) \\
&\sim \Big(\big(\overrightarrow{\chi^{-(\cdot)}}\big), \big(\overrightarrow{\Theta(\chi^{-(\cdot)})}\big), \big(\overrightarrow{\epsilon}\big) \Big) \\
&= (\eta_X)_n(\chi)
\end{aligned}$$

It follows that the maps $(\eta_X)_n$ are well defined. □

Next we must show the components of η_X commute with the cubical operators. To prove this it suffices to show they commute with the contraction operators in the appropriate way. That this is the case with $(-)^-$ is obvious, but we require slightly more argument for $\Theta(-)$.

Lemma 3.22.

1. *Let* $1 \leq i \leq n$, $\delta \in \{-1,1\}$ *and* $\chi \in (MX)_n$. *Then* $f_{n+1}^{i+1,\delta}(\Theta(\chi)) = \Theta(f_n^{i,\delta}(\chi))$.
2. *Let* $1 \leq i \leq n$ *and* $\chi \in (MX)_{n-1}$. *Then* $d_{n+1}^{i+1}(\Theta(\chi)) = \Theta(d_n^i(\chi))$.
3. *Let* $n \geq 2$, $1 \leq i \leq n-1$, $\delta \in \{-1,1\}$ *and* $\chi \in (MX)_{n-1}$. *Then*

$$\Gamma_n^{i+1,\delta}(\Theta(\chi)) = \Theta(\Gamma_{n-1}^{i,\delta}(\chi))$$

.

Proof.

1. We first note that we have the following cubical identities:
   - Since $i \geq 1$, we have $i+1 > 1$ and $i+2 > 2$. Hence

$$f_{n+1}^{i+1,\delta} \circ d_{n+1}^1 = d_n^1 \circ f_n^{i,\delta} \qquad\qquad f_{n+2}^{i+2,\delta} \circ d_{n+2}^2 = d_{n+1}^2 \circ f_{n+1}^{i+1,\delta}$$

   - Since $1 \leq i \leq n$ we have $n+2 \geq 3$ and $i+2 \geq 1+1$. Hence

$$f_{n+2}^{i+2,\delta} \circ \Gamma_{n+1}^{1,\epsilon_0} = \Gamma_n^{1,\epsilon_0} \circ f_{n+1}^{i+1}$$

   Hence we have

$$\begin{aligned} f_{n+1}^{i+1,\delta}(\Theta(\chi)) &= \Big(\big(f_{n+1}^{i+1,\delta}(\phi_0), \overrightarrow{f_{n+1}^{i+1,\delta} \circ d_{n+1}^1(\zeta)}\big), \big(f_{n+2}^{i+2,\delta} \circ \Gamma_{n+1}^{1,\epsilon_0}(\phi_0), \overrightarrow{f_{n+2}^{i+2,\delta} \circ d_{n+2}^2(\phi)}\big), \big(\overrightarrow{\epsilon}\big)\Big) \\ &= \Big(\big(f_{n+1}^{i+1,\delta}(\phi_0), \overrightarrow{d_n^1(f_n^{i,\delta}(\zeta))}\big), \big(\Gamma_n^{1,\epsilon_0}(f_{n+1}^{i+1}(\phi_0)), \overrightarrow{d_{n+1}^2(f_{n+1}^{i+1}(\phi))}\big), \big(\overrightarrow{\epsilon}\big)\Big) \\ &= \Theta(f_n^{i,\delta}(\chi)) \end{aligned}$$

2. Next we verify the degeneracy maps. Once again we note that some relevant cubical identities hold
   - Since $i \geq 1$ we have $i+1 > 1$ and $i+2 > 2$. Hence

$$d_{n+1}^{i+1} \circ d_n^1 = d_{n+1}^1 \circ d_n^i \qquad\qquad d_{n+2}^{i+2} \circ d_{n+1}^2 = d_{n+2}^2 \circ d_{n+1}^{i+1}$$

   - Since $1 \leq i \leq n$ we have $n+2 \geq 3$ and $i+1 > 1$. Hence

$$\Gamma_{n+1}^{1,\epsilon_0} \circ d_{n+1}^{i+1} = d_{n+2}^{i+2} \circ \Gamma_n^{1,\epsilon_0}$$

   It then follows that

$$\begin{aligned} d_{n+1}^{i+1}(\Theta(\chi)) &= \Big(\big(d_{n+1}^{i+1}(\phi_0), \overrightarrow{d_{n+1}^{i+1} \circ d_n^1(\zeta)}\big), \big(d_{n+2}^{i+2} \circ \Gamma_n^{1,\epsilon_0}(\phi_0), \overrightarrow{d_{n+2}^{i+2} \circ d_{n+1}^2(\phi)}\big), \big(\overrightarrow{\epsilon}\big)\Big) \\ &= \Big(\big(d_{n+1}^{i+1}(\phi_0), \overrightarrow{d_{n+1}^1(d_n^i(\zeta))}\big), \big(\Gamma_{n+1}^{1,\epsilon_0}(d_{n+1}^{i+1}(\phi_0)), \overrightarrow{d_{n+2}^2(d_{n+1}^{i+1}((\phi)))}\big), \big(\overrightarrow{\epsilon}\big)\Big) \\ &= \Theta(d_n^i(\chi)) \end{aligned}$$

3. Finally we attend to the connection maps. In this case the pertinent cubical identities are as follows:

   - Since $i \geq 1$ we have $i+1 > 1$ and $i+2 > 2$. Hence

$$\Gamma_n^{i+1,\delta} \circ d_n^1 = d_n^1 \circ \Gamma_{n-1}^{i,\delta} \qquad\qquad \Gamma_{n+1}^{i+2,\delta} \circ d_{n+1}^2 = d_{n+1}^2 \circ \Gamma_n^{i+1,\delta}$$

   - Since $i+2 > 1$ we have

$$\Gamma_{n+1}^{i+2,\delta} \circ \Gamma_n^{1,\epsilon_0} = \Gamma_{n+1}^{1,\epsilon_0} \circ \Gamma_n^{i+1,\delta}$$

   Hence

$$\begin{aligned}
\Gamma_n^{i+1,\delta}(\Theta(\chi)) &= \Big(\big(\Gamma_n^{i+1,\delta}(\phi_0), \overrightarrow{\Gamma_n^{i+1,\delta} \circ d_n^1(\zeta)}\big), \big(\Gamma_{n+1}^{i+2,\delta} \circ \Gamma_n^{1,\epsilon_0}(\phi_0), \overrightarrow{\Gamma_{n+1}^{i+2,\delta} \circ d_{n+1}^2(\phi)}\big), \big(\vec{\epsilon}\big) \Big) \\
&= \Big(\big(\Gamma_n^{i+1,\delta}(\phi_0), \overrightarrow{d_n^1(\Gamma_{n-1}^{i,\delta}(\zeta))}\big), \big(\Gamma_{n+1}^{1,\epsilon_0}(\Gamma_n^{i+1,\delta}(\phi_0)), \overrightarrow{d_{n+1}^2(\Gamma_n^{i+1,\delta}(\phi))}\big), \big(\vec{\epsilon}\big) \Big) \\
&= \Theta(\Gamma_{n-1}^{i,\delta}(\chi))
\end{aligned}$$

□

Thus the fact that each η_X is a cubical morphism follows immediately:

Corollary 3.23. *For each cubical set X, η_X is a cubical morphism.*

Proof. We must show the components of η_X commute with the cubical operators. This is straightforward by the previous lemma and we restrict attention to the face map case since all three follow from an identical argument:

$$\begin{aligned}
f_n^{i,\delta} \circ (\eta_X)_n(\chi) &= f_n^{i,\delta}\Big(\big(\chi^{-0}, \dots, \chi^{-k}\big), \big(\Theta(\chi^{-0}), \dots, \Theta(\chi^{-k-1})\big), \big(\vec{\epsilon}\big) \Big) \\
&= \Big(\big(\overrightarrow{f_n^{i,\delta}(\chi^{-(\cdot)})}\big), \big(\overrightarrow{f_{n+1}^{i+1,\delta}(\Theta(\chi^{-(\cdot)}))}\big), \big(\vec{\epsilon}\big) \Big) \\
&= \Big(\big(\overrightarrow{(f_n^{i,\delta}(\chi))^{-(\cdot)}}\big), \big(\overrightarrow{(\Theta(f_n^{i,\delta}(\chi^{-(\cdot)}))}\big), \big(\vec{\epsilon}\big) \Big) \\
&= (\eta_X)_{n-1} \circ f_n^{i,\delta}(\chi)
\end{aligned}$$

□

We're left with the final verification:

Proposition 3.24. *The collection of cubical morphisms $\eta = (\eta_X : MX \to MMX \mid X$ in $\boldsymbol{cSet^c})$ constitutes a natural transformation $\eta : M \Rightarrow MM$.*

Proof. Let $\mu : X \to Y$ be a cubical morphism. We require the commutativity of the following diagram

$$\begin{array}{ccc}
(MX)_n & \xrightarrow{(\eta_X)_n} & (MMX)_n \\
{\scriptstyle M(\mu)_n}\downarrow & & \downarrow{\scriptstyle MM(\mu)_n} \\
(MX)_n & \xrightarrow[(\eta_Y)_n]{} & (MMY)_n
\end{array}$$

It follows immediately from the definition of $M(\mu)_n$ that

$$M(\mu)_n(\chi^-) = (M(\mu)_n(\chi))^- \qquad\qquad M(\mu)_{n+1}(\Theta(\chi)) = \Theta(M(\mu)_n(\chi))$$

Hence by these identities we have

$$\begin{aligned} MM(\mu)_n \circ (\eta_X)_n(\chi) &= MM(\mu)_n\Big(\big(\chi^{-_0}, \ldots, \chi^{-_k}\big), \big(\Theta(\chi^{-_0}), \ldots, \Theta(\chi^{-_{k-1}})\big), \big(\vec{\epsilon}\big)\Big) \\ &= \Big(\big(\overrightarrow{M(\mu)_n(\chi^{-_{(\cdot)}})}\big), \big(\overrightarrow{M(\mu)_{n+1}(\Theta(\chi^{-_{(\cdot)}}))}\big), \big(\vec{\epsilon}\big)\Big) \\ &= \Big(\big(\overrightarrow{(M(\mu)_n(\chi))^{-_{(\cdot)}}}\big), \big(\overrightarrow{\Theta(M(\mu)_n(\chi^{-_{(\cdot)}}))}\big), (\epsilon)\Big) \\ &= (\eta_Y)_n \circ M(\mu)_n(\chi) \end{aligned}$$

It follows that η is a natural transformation. □

With this in place all that remains is to verify the requisite identities for Axiom 3 and thus finish our proof that $\boldsymbol{cSet^c}$ is a nice path object category.

Proposition 3.25. *The natural transformation $\eta : M \Rightarrow MM$ satisfies the nice path object category contraction identities. Hence $\boldsymbol{cSet^c}$ satisfies the third nice path object category axiom.*

Proof. We compute each in turn, once again noting that it is sufficient to compute each identity at n. We obtain the identities

$$s_{MX} \circ \eta_X = id_{MX} \qquad\qquad t_{MX} \circ \eta_X = e_X \circ t_X$$

immediately, since we have

$$(s_{MX})_n \circ (\eta_X)_n(\chi) = \chi \qquad\qquad (t_{MX})_n \circ (\eta_X)_n(\chi) = (e_X)_n \circ (t_X)_n(\chi)$$

Next we must show $M(s_X) \circ \eta_X = id_{MX}$. Computing directly, we obtain

$$\begin{aligned} M(s_X)_n \circ (\eta_X)_n(\chi) &= \Big(\big(\overrightarrow{(s_X)_n(\chi^{-_{(\cdot)}})}\big), \big(\overrightarrow{(s_X)_{n+1}(\Theta(\chi^{-_{(\cdot)}}))}\big), \big(\vec{\epsilon}\big)\Big) \\ &= \Big(\big(\zeta_0, \ldots, \zeta_k\big), \big(\phi_0, \ldots, \phi_{k-1}\big), \big(\vec{\epsilon}\big)\Big) \\ &= \chi \end{aligned}$$

Similarly we attend to $M(t_X) \circ \eta_X = e_X \circ t_X$:

$$\begin{aligned} M(t_X)_n \circ (\eta_X)_n(\chi) &= \Big(\big(\overrightarrow{(t_X)_n(\chi^{-_{(\cdot)}})}\big), \big(\overrightarrow{(t_X)_{n+1}(\Theta(\chi^{-_{(\cdot)}}))}\big), \big(\vec{\epsilon}\big)\Big) \\ &= \Big(\big((t_X)_n(\chi), \ldots, (t_X)_n(\chi)\big), \big(d^1_{n+1}((t_X)_n(\chi)), \ldots, d^1_{n+1}((t_X)_n(\chi))\big), \big(\vec{\epsilon}\big)\Big) \\ &\sim \Big(\big((t_X)_n(\chi)\big), (\,), (\,)\Big) \\ &= (e_X)_n \circ (t_X)_n(\chi) \end{aligned}$$

Finally we require $\eta_X \circ e_X = e_{MX} \circ e_X$ - but this is immediate:

$$(\eta_X)_n \circ (e_X)_n(\zeta) = \Big(\big(e_X(\zeta)\big), (\,), (\,)\Big) = (e_{MX})_n \circ (e_X)_n(\zeta)$$

Hence the nice contraction identities hold: $\boldsymbol{cSet^c}$ satisfies the third nice path object category axiom. □

As an immediate corollary we obtain Theorem 3.1, and this concludes the first part of the thesis. We dedicate the second half to constructing a model of type theory using this structure.

Part II

Constructing A Model Of Type Theory

Chapter 4

Categorical Semantics

We begin the second half of the thesis by giving the categorical semantics we use to model type theory. There are many options: Dybjer's *categories with families* [Dyb95], Cartmell's *contextual categories* [Car86], Awodey's *natural models* [Awo14] and Pitts' *type categories* [Pit00] to name a few. For essentially aesthetic reasons we choose to go with the latter in this thesis, though this state of affairs should not alarm the reader: these approaches are all essentially equivalent - for example, see [Awo14] for the equivalence between natural models and categories with families. Though the focus of the thesis is away from syntax - and for that reason we leave discussion of this area to Appendix B - the reader may desire a suitable reference to help contextualise the work we do. For this we recommend [NPS90].

4.1 Type Categories

Definition 4.1 (Type Category)**.** A *type category* consists of the following data

- A category $\mathcal{C}$ of *contexts* and *context morphisms* with a terminal object 1.
- For each $\Gamma \in \mathcal{C}_0$, a collection $Ty_{\mathcal{C}}(\Gamma)$ of *types* in context Γ.
- For each $A \in Ty_{\mathcal{C}}(\Gamma)$ an *extended context* $\Gamma.A \in \mathcal{C}_0$ and a dependent projection $\pi_A : \Gamma.A \to \Gamma$.
- For each $f : \Delta \to \Gamma$ in $\mathcal{C}$ and $A \in Ty_{\mathcal{C}}(\Gamma)$, a type $A[f] \in Ty(\Delta)$ and a morphism

  $$f^+ : \Delta.A[f] \to \Gamma.A$$

  making the following square into a pullback

  $$\begin{array}{ccc} \Delta.A[f] & \xrightarrow{f^+} & \Gamma.A \\ {\scriptstyle \pi_{A[f]}}\downarrow & & \downarrow{\scriptstyle \pi_A} \\ \Delta & \xrightarrow[f]{} & \Gamma \end{array}$$

A type category is said to be *split* if it satisfies the coherence axioms

$$\begin{array}{ll} A[id_\Gamma] = A & (id_\Gamma)^+ = id_{\Gamma.A} \\ A[fg] = A[f][g] & (fg)^+ = f^+g^+ \end{array}$$

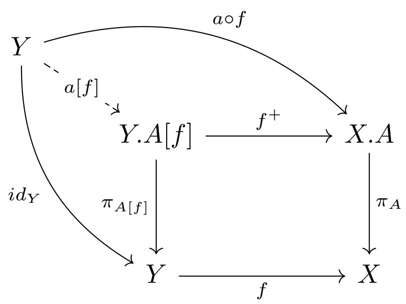
The only thing missing is an interpretation of terms. We obtain these as sections of the projection maps π_A.

Definition 4.2 (Global Section)**.** Given a type category $\mathcal{C}$, a context X in $\mathcal{C}$ and a type A in $Type_{\mathcal{C}}(X)$ a *global section* of A is a morphism $a : X \to X.A$ satisfying $\pi_A \circ a = id_X$. We denote this by $a \in_X A$.

Note that we are able to extend the definition of $\_[f]$ to global sections. Suppose $f : Y \to X$ with A in $Type_{\mathcal{C}}(X)$ and $a \in_X A$. Then we have

$$\pi_A \circ (a \circ f) = id_X \circ f = f = f \circ id_Y$$

Hence by the universal property of the pullback we have a unique $a[f] : Y \to Y.A[f]$ such that the diagram



commutes. This gives that $\pi_{A[f]} \circ a[f] = id_Y$ so $a[f] \in_Y A[f]$ and so substitution of terms is also modelled. With this data satisfied it is straightforward to give an inductive construction that models dependent types, taking care of substitution with the pullback condition. Interpreting the terminal object as the empty context, we can use context extension to construct well formed contexts in $\mathcal{C}$. Terms are given as global sections of types, and substitution of types and terms is taken care of by the substitution pullbacks described above. Definitional equality is then actual equality in $\mathcal{C}$. We have:

Theorem 4.3 ([Pit00])**.** *A split type category has the necessary structure to soundly model dependent types.*

The exact details of the model construction are given in Chapter 6 of [Pit00]. The type categories we develop in the course of this thesis will not be split, given that substitution in them will be given by pullback and hence the coherence axioms will hold up to natural isomorphism instead. This coherence issue can be happily swept under the rug though: by [Hof95] we have that any type category may be replaced by a split type category 'equivalent' to the original in a suitable way.

What is missing from our exposition so far are the type constructors that give dependent type theories their characteristic expressiveness. In this thesis we work towards modelling a fragment of intuitionistic type theory containing the basic inference rules and the axioms governing identity types. A brief introduction to this system can be found in Appendix B. Identity types are the novel feature of such type theories that allow us to express propositional equality in the logic and the aspect of particular interest when it comes to modelling type theory. For reference we give the rules governing identity types for the system:

Identity Formation

$$\frac{a, b : A}{Id_A(a, b)} \text{ Id} - \text{Form;}$$

Identity Introduction

$$\frac{a : A}{r_A(a) : Id_A(a, a)} \text{ Id} - \text{Intro;}$$

Identity Elimination

$$\frac{\begin{array}{c} x, y : A, z : Id_A(x, y), \Delta(x, y, p) \vdash C(x, y, z) \text{ type} \\ a : A, \Delta(a, a, r_A(a)) \vdash d(a) : C(a, a, r_A(a)) \\ a, b : A \vdash p : Id_A(a, b) \end{array}}{\Delta(a, b, p) \vdash J_d(a, b, p) : C(a, b, p)} \text{ Id-Elim}$$

Identity Coherence

$$\frac{\Delta, x : C \vdash a(x), b(x) : A(x) \qquad c : C}{\Delta \vdash Id_{A(x)}(a(x), b(x))[c/x] = Id_{A[c]}(a[c/x], b[c/x])} \text{ Id Coherence}$$

$$\frac{\Delta, x : C \vdash a(x) : A(x) \qquad c : C}{\Delta \vdash r_{A(x)}(a(x))[c/x] = r_{A[c/x]}(a[c/x]) : Id_{A[c/x]}(a[c/x], a[c/x])} \text{ r coherence}$$

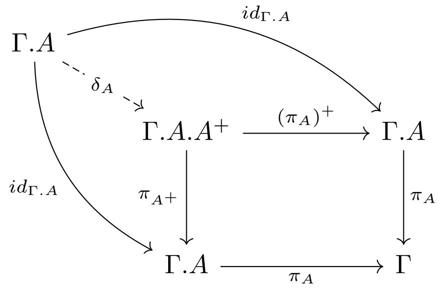
$$\frac{\begin{array}{c} x, y : A, z : Id_A(x, y), \Delta(x, y, p) \vdash C(x, y, z) \text{ type} \\ x : A, \Delta(x, x, r(x)) \vdash d(x) : C(x, x, r(x)) \qquad a : A \end{array}}{\Delta(a, a, r(a)) \vdash J_d(a, a, r(a)) = d(a) : C(a, a, r(a))} \text{ J Coherence}$$

We note the addition of the extra contextual parameter Δ in the elimination rules. Supposing we included the rules governing product types in our fragment this would be equivalent to the usual rules: we can take take the product type of the types in context Δ and C to obtain a new dependent type C' and hence obtain the usual formulation of the rule. However without product types we require this strictly stronger formulation in order to derive the non-trivial properties of the identity types. Now for a type category to model type constructors it requires additional operations that reflect the constructor's derivation rules. Before specifying precisely the conditions necessary to model these rules we introduce some notation:

Given $A, B \in Ty(\Gamma)$ we denote $B[\pi_A] \in Ty(\Gamma.A)$ by the abbreviation B^+. This gives us the pullback square:

$$\begin{array}{ccc} \Gamma.A.B^+ & \xrightarrow{(\pi_A)^+} & \Gamma.B \\ \downarrow{\scriptstyle \pi_{B^+}} & & \downarrow{\scriptstyle \pi_B} \\ \Gamma.A & \xrightarrow[\pi_A]{} & \Gamma \end{array}$$

Letting $A = B$, the identity arrow $id_{\Gamma.A}$ gives a pullback cone inducing the *diagonal morphism* $\delta_A : \Gamma.A \to \Gamma.A.A^+$:



which by commutativity satisfies

$$\pi_{A^+} \circ \delta_A = (\pi_A)^+ \circ \delta_A = id_{\Gamma.A}$$

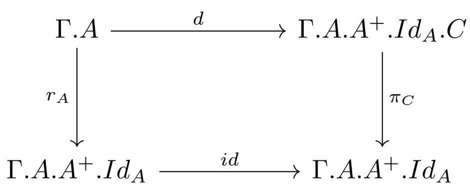
Thus to interpret identity types in a type category we require the satisfaction of the conditions in the following definition:

Definition 4.4 ($\mathcal{C}$ Has Identity Types)**.** [vdBG12] A type category $\mathcal{C}$ *has identity types* if the following conditions are satisfied:

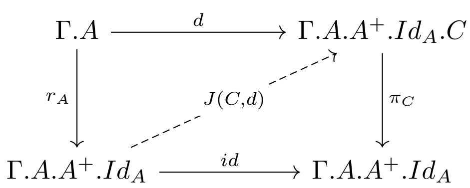
- Formation: For each $A \in Ty(\Gamma)$ a type $Id_A \in Ty(\Gamma.A.A^+)$
- Introduction: For each $A \in Ty(\Gamma)$, an arrow

  $$r_A : \Gamma.A \to \Gamma.A.A^+.Id_A$$

  whose composite with π_{Id_A} is δ_A.
- Elimination/Computation: For each $C \in Ty(\Gamma.A.A^+.Id_A)$ and commutative diagram

  a diagonal filler $J(C, d)$ rendering the following diagram commutative:

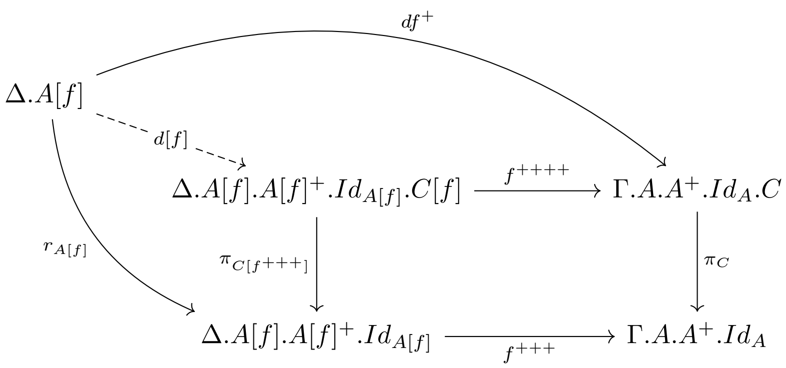
  

- Substitution: For every morphism $f : \Delta \to \Gamma$ in $\mathcal{C}$ we have $Id_A[f^{++}] = Id_{A[f]}$ together with the commutativity of the following squares:

$$\begin{array}{ccc} \Delta.A[f] & \xrightarrow{r_{A[f]}} & \Delta.A[f].A[f]^+.Id_{A[f]} \\ \downarrow{\scriptstyle f^+} & & \downarrow{\scriptstyle f^{+++}} \\ \Gamma.A & \xrightarrow[r_A]{} & \Gamma.A.A^+.Id_A \end{array}$$

$$\begin{array}{ccc} \Delta.A[f].A[f]^+.Id_{A[f]} & \xrightarrow{J(C[f],d[f])} & \Delta.A[f].A[f]^+.Id_{A[f]}.C[f^{+++}] \\ \downarrow{\scriptstyle f^{+++}} & & \downarrow{\scriptstyle f^{+++}} \\ \Gamma.A.A^+.Id_A & \xrightarrow[J(C,d)]{} & \Gamma.A.A^+.Id_A.C \end{array}$$

where $d[f]$ is the morphism defined by the pullback



We can strengthen the definition to account for contextual parameters:

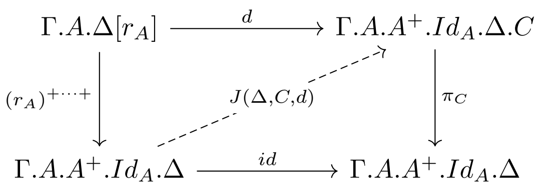
Definition 4.5 ($\mathcal{C}$ Has Strong Identity Types)**.** A type category $\mathcal{C}$ *has strong identity types* if for every $A \in Ty(\Gamma)$ there are given Id_A and r_A as above, but now for every

$$\begin{gathered} B_1 \in Ty(\Gamma.A.A^+.Id_A) \\ \vdots \\ B_n \in Ty(\Gamma.A.A^+.Id_A.B_1 \dots B_{n-1}) \\ C \in Ty(\Gamma.A.A^+.Id_A.B_1 \dots B_{n-1}.B_n) \end{gathered}$$

and commutative diagram

$$\begin{array}{ccc} \Gamma.A.\Delta[r_A] & \xrightarrow{d} & \Gamma.A.A^+.Id_A.\Delta.C \\ \downarrow{\scriptstyle (r_A)^{+\cdots+}} & & \downarrow{\scriptstyle \pi_C} \\ \Gamma.A.A^+.Id_A.\Delta & \xrightarrow{id} & \Gamma.A.A^+.Id_A.\Delta \end{array}$$

with Δ abbreviating the context $B_1, \ldots, B_n$, we have a diagonal filler $J(\Delta, C, d)$ rendering the diagram



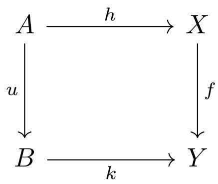
commutative, with the structure stable under substitution as in the previous definition.

We call such a type category a *categorical model of identity types* or a *categorical model of intensional type theory*. By a modification of [War08, Theorem 2.48] we can see that the process producing the corresponding split type category from a non-split one also yields an identity type structure whenever the original type category had one, so once again we are able to ignore issues of coherence.

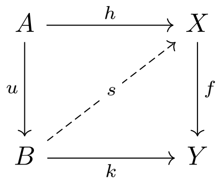
4.2 Model Categories

Introduced by Quillen in [Qui67], model categories abstract away topological properties to provide a general framework within which homotopy theory can be developed. An accessible introduction can be found in [Hov00]. In recent years interpretations of type theory in model categories have motivated the composite field *homotopy type theory* as well as the *univalent foundations* research programme [IAS13]. Although we will not make explicit use of model categories in the remainder of the thesis, the model category interpretation of type theory is the key motivation behind the construction of *homotopy theoretic models* that we will embark upon in the next chapter, as well as providing a non-trivial instantiation of a type category. To state the definition we first require some additional concepts.

Definition 4.6 (Lifting Properties)**.** Let $u : A \to B$ and $f : X \to Y$. We say u has the *left lifting property* with respect to f (respectively f has the *right lifting property* with respect to u) if for every commutative square



there exists a diagonal filler $s : B \to X$ rendering the following diagram commutative:



We denote this by $u \pitchfork f$.

If $\mathcal{M}$ is any collection of morphisms in $\mathcal{C}$ we denote by ${}^{\pitchfork}\mathcal{M}$ the collection of morphisms with the left lifting property with respect to all maps in $\mathcal{M}$; analogously $\mathcal{M}^{\pitchfork}$ denotes the collection of morphisms with the right lifting property with respect to all maps in $\mathcal{M}$.

Definition 4.7 (Weak Factorisation System)**.** A weak factorisation system on a category $\mathcal{C}$ is a pair of classes of morphisms $(\mathcal{L}, \mathcal{R})$ closed under retracts, such that the following properties are satisfied:

1. (Factorisation) Every morphism $f : X \to Y$ of $\mathcal{C}$ can be factored as $f = pi$ with $i \in \mathcal{L}$ and $p \in \mathcal{R}$:

$$\begin{array}{ccc} X & \xrightarrow{i} & P \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle p} & \\ & Y & \end{array}$$

2. (Lifting) $\mathcal{L} = {}^{\pitchfork}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\pitchfork}$

We call morphisms in $\mathcal{L}$ *left maps*, whilst morphisms in $\mathcal{R}$ are called *right maps*.

We can derive a useful closure property for any weak factorisation system:

Lemma 4.8. *For any weak factorisation system $(\mathcal{L}, \mathcal{R})$, the class $\mathcal{R}$ is closed under taking pullbacks.*

Proof. Let $r : X \to Z$ be a right map and suppose we have $f : Y \to Z$ such that the pullback

$$\begin{array}{ccc} W & \xrightarrow{\overline{f}} & X \\ {\scriptstyle \overline{r}}\downarrow & & \downarrow{\scriptstyle r} \\ Y & \xrightarrow[f]{} & Z \end{array}$$

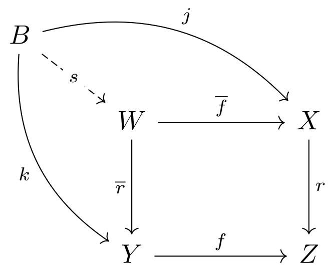
exists. Suppose we have a left map $l : A \to B$ and a commutative square

$$\begin{array}{ccc} A & \xrightarrow{h} & W \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle \overline{r}} \\ B & \xrightarrow[k]{} & Y \end{array}$$

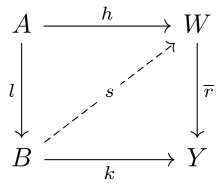
By looking at the composite of this square and the pullback we obtain a diagonal filler $j : B \to X$ rendering the following diagram commutative

$$\begin{array}{ccc} A & \xrightarrow{\overline{f} \circ h} & X \\ {\scriptstyle l}\downarrow & \overset{j}{\nearrow} & \downarrow{\scriptstyle r} \\ B & \xrightarrow[f \circ k]{} & Z \end{array}$$

We thus obtain from the universal property of the pullback the morphism $s : B \to W$ as follows:



We claim this renders the following diagram commutative



The lower triangle is given immediately by commutativity of the diagram defining s, whereas for the upper triangle is suffices to verify identity holds upon post-composition with the projection maps $\overline{f}$ and $\overline{r}$. We have

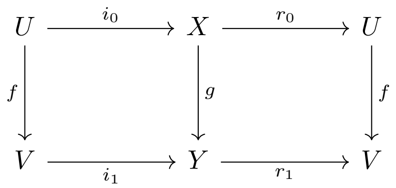
$$\overline{r}sl = kl = \overline{r}h \qquad\qquad \overline{f}sl = jl = \overline{f}h$$

as required. □

The link between weak factorisation systems and intensional type theory has been explored by Garner and Gambino in [GG08]. There they prove that any dependent type theory with identity types admits a non-trivial weak factorisation system. In the other direction, the interpretation of type theory in model categories is based upon one of the weak factorisation systems that comprises part of the model structure: before we get ahead of ourselves we give the full definition.

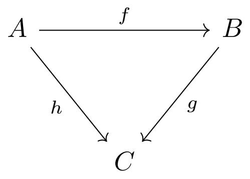
Definition 4.9 (Model Category)**.** A *model structure* on a bicomplete category $\mathcal{C}$ is a triple of classes of morphisms (Cof, Fib, W) - *cofibrations*, *fibrations* and *weak equivalences* respectively - such that:

1. (Retracts): Each distinguished class of arrows is closed under retracts in the arrow category $\mathcal{C}^{\to}$. Specifically, $f : U \to V$ is a retract of $g : X \to Y$ in $\mathcal{C}^{\to}$ if arrows i_0, r_0, i_1, r_1 exist such that the following diagram commutes



with $r_0 \circ i_0 = id_U$ and $r_1 \circ i_1 = id_V$;

2. (2-Out-Of-3) Given a commutative triangle



   if any two out of f, g, h are weak equivalences than so too is the third;

3. (W.F.S) $(Cof, Fib \cap W)$ and $(Cof \cap W, Fib)$ are weak factorisation systems on $\mathcal{C}$.

We call a bicomplete category $\mathcal{C}$ together with a model structure a *model category*. A morphism f is an *acyclic fibration* if it is in $Fib \cap W$; analogously f is an *acyclic cofibration* if it is in $Cof \cap W$. We call an object A *fibrant* if the unique map $!_A : A \to 1$ is a fibration. The dual notion of a cofibrant object is obtained when the unique map $0 \to A$ is a cofibration.

Motivating examples of model categories can be found in the categories of groupoids, topological spaces and simplicial sets. We simply state the model structures of each, as a full exposition of the definitions involved would take us too far off course. The interested reader may consult the cited references for the full details.

Theorem 4.10.

- *[And78] The classical model structure on the category* ***Gpd*** *is given by*
  - *Cof: The class of all functors injective on objects.*
  - *Fib: The class of isofibrations.*
  - *W: The class of equivalences of categories.*
- *[Hov00] The classical model structure on the category* ***Top*** *is given by*
  - *Cof: The smallest saturated class containing the boundary inclusions* $S^{n-1} \hookrightarrow D^n$
  - *Fib: The class of maps that have the right lifting property with respect to all inclusions* $i_0 : D^n \hookrightarrow D^n \times I$ *that include the n-disk as* $D^n \times \{0\}$ *- the* Serre Fibrations.
  - *W: The class of weak homotopy equivalences.*
- *[GM96] The classical model structure on the category* ***sSet*** *is given by*
  - *Cof: The class of all monos.*
  - *Fib: The class of all Kan fibrations - maps* $a : A \to I$ *such that* $i^n_k \pitchfork a$ *for all* $n, k \in \mathbb{N}$.
  - *W: The class of all maps* $f : X \to Y$ *whose geometric realization is a homotopy equivalence.*

We thus construct a type category from the structure of a model category.

Theorem 4.11. *For any model category* $\mathcal{C}$ *there exists a type category with* $\mathcal{C}$ *as its category of contexts.*

Proof. Let $\mathcal{C}$ be a model category. We give the following data to obtain a type category:

- We take $\mathcal{C}$ as the category of contexts and by completeness we have the terminal object 1;
- For each object X in $\mathcal{C}$ we a type A in context X is represented by a *fibration* $f : dom(f) \to X$;
- For an object X and a type A over X represented by $f : dom(f) \to X$ we obtain the extended context $X.A$ as $dom(f)$ and the dependent projection $\pi_A : X.A \to X$ as f.
- Given a type A represented by the fibration $\pi_A : X.A \to X$ and an arrow $f : Y \to X$ we obtain the type $A[f]$ and the substitution map f^+ by taking the pullback

$$\begin{array}{ccc} Y.A[f] & \xrightarrow{f^+} & X.A \\ {\scriptstyle \pi_{A[f]}}\downarrow & & \downarrow{\scriptstyle \pi_A} \\ Y & \xrightarrow[f]{} & X \end{array}$$

noting that $\pi_{A[f]} : Y.A[f] \to Y$ is a fibration by Lemma 4.8.

□

The question is, can this type category model identity types? The answer is sadly no, but it is a qualified no as we can in fact meet half the criteria. In [AW09] Awodey and Warren give the idea behind a proof that requires strengthening in order to coherently model identity types. We can sketch this idea after we have the following definition.

Definition 4.12 ((Very Good) Path Object)**.** In a model category $\mathcal{C}$ a (very good) path object A^I consists of a factorisation

$$\begin{array}{ccc} X & \xrightarrow{r} & X^I \\ & {\scriptstyle \delta}\searrow & \downarrow{\scriptstyle p} \\ & & X \times X \end{array}$$

of the diagonal map $\delta : X \to X \times X$ as a weak equivalence r followed by a fibration p.

In **Gpd** and **Top** path objects are given by exponentiation by a suitable "interval space", whilst in **sSet** the same method works when the object in question is a *Kan complex* - a fibrant object of the model structure. In all cases path objects exist but they may not be uniquely determined or functorial. The idea is to interpret Id_A as the fibration $p : (X.A)^I \to X.A.A^+$ and r_A as r. This clearly satisfies the first two conditions. Moreover, because r_A is a trivial cofibration together with the fact that $(Cof \cap W, Fib)$ is a weak factorisation system we have that the required filler for the elimination/computation square exists.

The problem with this argument, however, is that although the fillers exist there is no guarantee that they are *coherent* - that is, satisfy the substitution squares. To soundly model the type theory we require the assignment to be stable under substitution, which in our case means stable under pullback. There are a number of possible solutions that rectify this situation: one is developed in Warren's PhD thesis [War08], and involving restricting the fibrations we interpret to be identity types to a suitably well behaved subset allowing pullback stability of fillers. Streicher gives an alternative solution in [Str14] by lifting a Grothendieck universe to a type theoretic universe. This gives a "generic" filler that induces a series of pullback stable fillers as required. The solution we use constitutes the next chapter of the thesis; van den Berg and Garner's homotopy theoretic model framework. It is in within this framework that we will give our model of type theory in the category of cubical sets with connections.

Chapter 5

Homotopy Theoretic Models

In this chapter we solve the issue highlighted at the end of the previous chapter by providing the modifications required to strengthen Awodey's idea to a proof. Essentially we need a type category to have enough structure to *specify* choices of diagonal fillers in such way that they are functorial and stable under pullback, thus allowing us to satisfy the substitution squares. Presenting this argument has the dual purpose of introducing the framework within which we will build our model of type theory from the path object category structure.

5.1 Cloven Weak Factorisation Systems

In constructing the type category from an arbitrary model categories we only made use of the weak factorisation system $(Cof \cap W, Fib)$. It thus makes sense that in order to resolve the identity type issue we should strengthen our notion of weak factorisation system. This is achieved by the introduction of *cloven weak factorisation systems*:

Definition 5.1 (Cloven Weak Factorisation System)**.** A *cloven weak factorisation system* on a category $\mathcal{C}$ is specified by the following data:

- For each $f : X \to Y$ in $\mathcal{C}$ a choice of factorisation:

$$\begin{array}{ccc} X & \xrightarrow{\lambda_f} & Pf \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \rho_f} \\ & & Y \end{array}$$

- For each commutative square

$$\begin{array}{ccc} U & \xrightarrow{h} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow[k]{} & Y \end{array}$$

a choice of diagonal filler $P(h,k) : Pf \to Pg$ rendering the following diagram commutative

$$\begin{array}{ccc} U & \xrightarrow{\lambda_g \circ h} & Pg \\ {\scriptstyle \lambda_f}\downarrow & \overset{P(h,k)}{\nearrow} & \downarrow{\scriptstyle \rho_g} \\ Pf & \xrightarrow[k \circ \rho_f]{} & Y \end{array} \tag{5.1}$$

and such that the assignment $(h,k) \mapsto P(h,k)$ is functorial in (h, k);

- For each $f : X \to Y$, choices of diagonal fillers $\sigma_f : Pf \to P\lambda_f$ and $\pi_f : P\rho_f \to Pf$ rendering the following diagrams commutative:

$$\begin{array}{ccc} X & \xrightarrow{\lambda_{\lambda_f}} & P\lambda_f \\ {\scriptstyle \lambda_f}\downarrow & \overset{\sigma_f}{\nearrow} & \downarrow{\scriptstyle \rho_{\lambda_f}} \\ Pf & \xrightarrow[id_{Pf}]{} & Pf \end{array} \qquad \begin{array}{ccc} Pf & \xrightarrow{id_{Pf}} & Pf \\ {\scriptstyle \lambda_{\rho_f}}\downarrow & \overset{\pi_f}{\nearrow} & \downarrow{\scriptstyle \rho_f} \\ P\rho_f & \xrightarrow[\rho_{\rho_f}]{} & Y \end{array}$$

We show that this is a strengthening of the notion of weak factorisation system by giving the w.f.s induced by a cloven W.F.S. First some definitions:

Definition 5.2 (Cloven Maps)**.** Given a cloven weak factorisation system on a category $\mathcal{C}$, a *cloven* $\mathcal{L}$*-map structure* on a morphism $f : X \to Y$ of $\mathcal{C}$ is a map $s : Y \to Pf$ rendering the following diagram commutative

$$\begin{array}{ccc} X & \xrightarrow{\lambda_f} & Pf \\ {\scriptstyle f}\downarrow & \overset{s}{\nearrow} & \downarrow{\scriptstyle \rho_f} \\ Y & \xrightarrow[id_Y]{} & Y \end{array} \tag{5.2}$$

We say $(f,s) : X \to Y$ is a *cloven* $\mathcal{L}$*-map*. Correspondingly, a *cloven* $\mathcal{R}$*-map structure* on f is given by a map $p : Pf \to X$ rendering the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{id_X} & X \\ {\scriptstyle \lambda_f}\downarrow & \overset{p}{\nearrow} & \downarrow{\scriptstyle f} \\ Pf & \xrightarrow[\rho_f]{} & Y \end{array} \tag{5.3}$$

and we call $(f,p) : X \to Y$ a *cloven* $\mathcal{R}$*-map.*

We can thus reformulate the last condition in the definition of cloven weak factorisation: for all $f : X \to Y$ there exists a choice of cloven $\mathcal{L}$-map structure σ_f for λ_f and a choice of cloven $\mathcal{R}$-map structure π_f for ρ_f. That we denote these structured maps to be $\mathcal{L}$-maps and $\mathcal{R}$-maps is no coincidence:

Proposition 5.3. *Every cloven weak factorisation system has an underlying weak factorisation system whose two classes of maps are given by*

$$\mathcal{L} = \{f : A \to B \mid \textit{there is a cloven } \mathcal{L}\textit{-map structure on } f\}$$

$$\mathcal{R} = \{g : C \to D \mid \textit{there is a cloven } \mathcal{R}\textit{-map structure on } g\}$$

Furthermore, this weak factorisation system has a canonical choice of diagonal fillers.

Proof. We show each axiom is satisfied in turn.

- Closed under retracts: We verify the case for $\mathcal{L}$: the case for $\mathcal{R}$ is similar and thus left to the reader. Suppose $g : X \to Y$ is in $\mathcal{L}$ and has a retract $f : U \to V$: thus we have a commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{i_0} & X & \xrightarrow{r_0} & U \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ V & \xrightarrow[i_1]{} & Y & \xrightarrow[r_1]{} & V \end{array}$$

  with $r_0 \circ i_0 = id_U$ and $r_1 \circ i_1 = id_V$. We then have a choice of fillers $P(i_0, i_1) : Pf \to Pg$ and $P(r_0, r_1) : Pg \to Pf$ such that the following diagrams commute:

$$\begin{array}{ccc} U & \xrightarrow{\lambda_g \circ i_0} & Pg \\ {\scriptstyle \lambda_f}\downarrow & \overset{P(i_0,i_1)}{\nearrow} & \downarrow{\scriptstyle \rho_g} \\ Pf & \xrightarrow[i_1 \circ \rho_f]{} & Y \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\lambda_f \circ r_0} & Pf \\ {\scriptstyle \lambda_g}\downarrow & \overset{P(r_0,r_1)}{\nearrow} & \downarrow{\scriptstyle \rho_f} \\ Pg & \xrightarrow[r_1 \circ \rho_g]{} & V \end{array}$$

  Since g is in $\mathcal{L}$ we have a cloven $\mathcal{L}$-map structure on g given by $s : Y \to Pg$. We thus claim that $s' = P(r_0, r_1) \circ s \circ i_1$ gives a cloven $\mathcal{L}$-map structure on f. That this is the case follows from straightforward computations using the above diagrams and functorality of $P(-, -)$, and we leave this to the reader.

- Factorisation: By definition, for every $f : X \to Y$ there is a factorisation $f = \rho_f \circ \lambda_f$ with (ρ_f, π_f) a cloven $\mathcal{R}$-map and (λ_f, σ_f) a cloven $\mathcal{L}$-map.

- Fillers: Suppose we have f in $\mathcal{L}$ and g in $\mathcal{R}$ with a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow[k]{} & Y \end{array}$$

and let s be a cloven $\mathcal{L}$-map structure on f and p a cloven $\mathcal{R}$-map structure on g. We then have a *canonical choice of diagonal filler* $j = p \circ P(h,k) \circ s$ for (f,s) and (g,p):

$$
\begin{aligned}
p \circ P(h,k) \circ s \circ f &= p \circ P(h,k) \circ \lambda_f && \text{(Commutativity of (5.2))} \\
&= p \circ \lambda_g \circ h && \text{(Commutativity of (5.1))} \\
&= h && \text{(Commutativity of (5.3))}
\end{aligned}
$$

$$
\begin{aligned}
g \circ p \circ P(h,k) \circ s &= \rho_g \circ P(h,k) \circ s && \text{(Commutativity of (5.3))} \\
&= k \circ \rho_f \circ s && \text{(Commutativity of (5.1))} \\
&= k && \text{(Commutativity of (5.2))}
\end{aligned}
$$

Hence $f \pitchfork g$ as required.

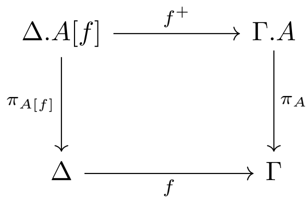
□

With this taken care of we can essentially mimic the type category construction of Theorem 4.11 to give a type category for any complete category $\mathcal{C}$ equipped with a cloven weak factorisation system.

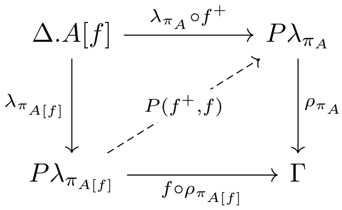
Theorem 5.4. *Let $\mathcal{C}$ be a complete category equipped with a cloven weak factorisation system. Then there is a type category whose category of contexts is $\mathcal{C}$.*

Proof. We list the requisite structure:

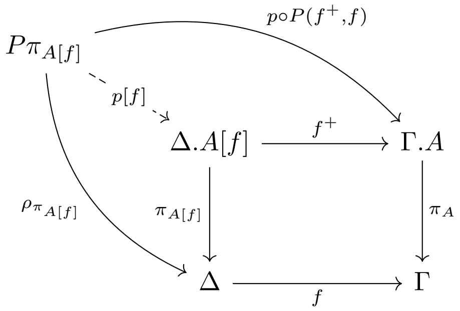
- Empty Context: By completeness we have terminal object 1;
- Types: A type $A \in Ty(\Gamma)$ is given by a cloven $\mathcal{R}$-map $(f,p) : dom(f) \to \Gamma$;
- Context Extension: For a type $A \in Ty(\Gamma)$ represented by a cloven $\mathcal{R}$- map $(f,p) : dom(f) \to \Gamma$ we take $\Gamma.A = dom(f)$ and $\pi_A = f$;
- Substitution: Let $f : \Delta \to \Gamma$ with $A \in Ty(\Gamma)$ given by a cloven $\mathcal{R}$-map $(\pi_A, p) : \Gamma.A \to \Gamma$. We can take the pullback



That $\pi_{A[f]}$ can be equipped with a cloven $\mathcal{R}$-map structure follows from Proposition 5.3 together with Lemma 4.8, but we note that there is a *canonical* choice of cloven $\mathcal{R}$-map structure for $\pi_{A[f]}$. Since p is a cloven $\mathcal{R}$-map structure for π_A we have, in particular, that $\pi_A \circ p = \rho_{\rho_{\pi_A}}$. We also have, by commutativity of the pullback square, a choice of diagonal filler $P(f^+, f)$ rendering the following diagram commutative:

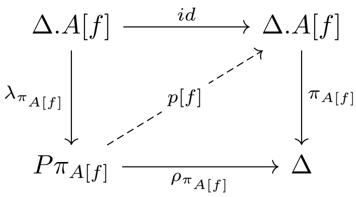


Hence by the universal property of the pullback we obtain a morphism $p[f] : P\pi_{A[f]} \to \Delta.A[f]$ as follows:



(5.4)

We claim $p[f]$ is a cloven $\mathcal{R}$-map structure for $\pi_{A[f]}$. We must verify commutativity of the diagram:



We note that the lower triangle commutes immediately by (5.4). For the upper triangle it suffices to show we have equality upon post-composition by the projection maps f^+ and $\pi_{A[f]}$. We have

$$f^+ \circ p[f] \circ \lambda_{\pi_{A[f]}} = p \circ P(f^+, f) \circ \lambda_{\pi_{A[f]}} = p \circ \lambda_{\pi_A} \circ f^+ = f^+$$

$$\pi_{A[f]} \circ p[f] \circ \lambda_{\pi_{A[f]}} = \rho_{\pi_{A[f]}} \circ \lambda_{\pi_{A[f]}} = \pi_{A[f]}$$

as required. Hence we interpret $A[f]$ as the cloven $\mathcal{R}$-map $(\pi_{A[f]}, p[f]) : \Delta.A[f] \to \Delta$.

□

For this type category to interpret identity types we require our cloven weak factorisation system to satisfy a few more properties.

5.2 Diagonal Factorisations

The next step is to specify a choice of factorisations for the diagonal maps δ_f. With these in tow we can mimic the interpretation of identity types sketched at the end of the previous chapter. We will require certain regularity principles for these choices in order for substitution to be modelled soundly, and to express these we need to fit our cloven $\mathcal{L}$ and $\mathcal{R}$-maps into their own categories.

Definition 5.5 (Morphism Of Cloven Maps)**.** Given cloven $\mathcal{L}$-maps $(f,s) : X \to Y$ and $(g,t) : Z \to W$ a *morphism of cloven $\mathcal{L}$-maps* $(h,k) : (f,s) \to (g,t)$ is given by a commutative square

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[k]{} & W \end{array}$$

with the additional property that the following square commutes

$$\begin{array}{ccc} Y & \xrightarrow{s} & Pf \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle P(h,k)} \\ W & \xrightarrow[t]{} & Pg \end{array}$$

Dually, given cloven $\mathcal{R}$-maps $(f,p) : X \to Y$ and $(g,r) : Z \to W$ a morphism $(h,k) : (f,p) \to (g,r)$ is a commutative square as above, with the additional property that the following square commutes

$$\begin{array}{ccc} Pf & \xrightarrow{P(h,k)} & Pg \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle r} \\ X & \xrightarrow[h]{} & W \end{array}$$

This notion allows us to prove a number of properties that we give in the following lemma.

Lemma 5.6.

1. *Given a cloven weak factorisation system on a category $\mathcal{C}$ there exist a category $\boldsymbol{\mathcal{L}}$**-Map**$_{\boldsymbol{\mathcal{C}}}$ comprised of cloven $\mathcal{L}$-maps and morphisms of cloven $\mathcal{L}$-maps and a category $\boldsymbol{\mathcal{R}}$**-Map**$_{\boldsymbol{\mathcal{C}}}$ comprised of cloven $\mathcal{R}$-maps and morphism of cloven $\mathcal{R}$-maps.*

2. *Precomposition by a morphism of cloven $\mathcal{L}$-maps sends canonical fillers to canonical fillers; postcomposition by a morphism of cloven $\mathcal{R}$-maps sends canonical fillers to canonical fillers.*

3. *The morphism $p[f]$ of (5.4) is the unique such cloven $\mathcal{R}$-map structure on p turning the pullback square into a morphism of cloven $\mathcal{R}$-maps.*

Proof. 1. By defining $id_{(f,p)} = (id_{dom(f)}, id_{cod(f)})$ and $(h',k') \circ (h,k) = (h' \circ h, k' \circ k)$ we obtain categories $\boldsymbol{\mathcal{L}}$**- Map**$_{\boldsymbol{\mathcal{C}}}$ and $\boldsymbol{\mathcal{R}}$**-Map**$_{\boldsymbol{\mathcal{C}}}$. We have the commutative composite:

$$\begin{array}{ccccc} X & \xrightarrow{h} & Z & \xrightarrow{h'} & Z' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ Y & \xrightarrow[k]{} & W & \xrightarrow{k'} & W' \end{array}$$

Then in the case of $\mathcal{L}$-maps the commutative composite

$$\begin{array}{ccc} Y & \xrightarrow{s} & Pf \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle P(h,k)} \\ W & \xrightarrow{t} & Pg \\ {\scriptstyle k'}\downarrow & & \downarrow{\scriptstyle P(h',k')} \\ W' & \xrightarrow[t']{} & Pg' \end{array}$$

and in the case of $\mathcal{R}$-maps the commutative composite

$$\begin{array}{ccccc} Pf & \xrightarrow{P(h,k)} & Pg & \xrightarrow{P(h',k')} & Pg' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle g'} \\ X & \xrightarrow[h]{} & W & \xrightarrow[h']{} & W' \end{array}$$

The functorality of the assignment $P(h, k)$ thus completes the proof that this notion of composition is well defined.

2. Suppose we have a cloven $\mathcal{L}$-map $(f, s) : U \to V$ and an $\mathcal{R}$-map $(g, p) : X \to Y$. As before, given a commutative square

$$\begin{array}{ccc} U & \xrightarrow{h} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow[k]{} & Y \end{array}$$

we have a canonical choice of filler given by $p \circ P(h, k) \circ s$. Supposing we have a morphism of cloven $\mathcal{L}$-maps $(i, j) : (f', s') \to (f, s)$ we obtain the commutative composite:

$$\begin{array}{ccc} U' & \xrightarrow{hi} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g} \\ V' & \xrightarrow[kj]{} & Y \end{array}$$

We then have an obvious choice of filler given by precomposing the canonical choice for the original diagram $p \circ P(h, k) \circ s$ with the map j. By the defining property of the cloven morphism it follows that

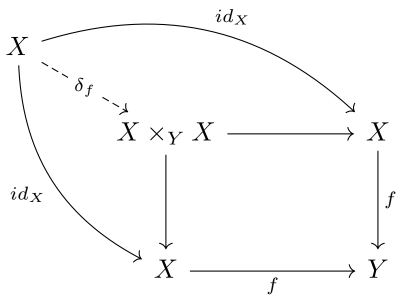
$$\begin{aligned} p \circ P(h, k) \circ s \circ j &= p \circ P(h, k) \circ P(i, j) \circ s' \\ &= p \circ P(hi, kj) \circ s' \end{aligned}$$

That is, the canonical choice of filler for the composite diagram. This works analogously for morphisms of cloven $\mathcal{R}$-maps and post-composition.

3. Finally we note that by commutativity of (5.4) $(f^+, f) : (\pi_{A[f]}, p[f]) \to (\pi_A, p)$ is a morphism of cloven $\mathcal{R}$-maps. Not only this, but by the universal property of the pullback $p[f]$ is the *unique* cloven $\mathcal{R}$-map structure on $\pi_{A[f]}$ such that (f^+, f) is a morphism of $\mathcal{R}$-maps. We collect the preceding discussion in a lemma.

□

Taken together, this shows that this notion of morphism is a good one, and equipped with the data given thus far we have tight control over our canonical choice of fillers. With one more property we will be able to prove that our type category has identity types. Recall that given a morphism $f : X \to Y$ we obtain the diagonal map δ_f from the following pullback



Definition 5.7 (Choice Of Diagonal Factorisations)**.** A cloven weak factorisation system has *a choice of diagonal factorisations* if, for every cloven $\mathcal{R}$-map $(f, p) : X \to Y$ of $\mathcal{C}$ we have an assignment of a factorisation of δ_f:

$$X \xrightarrow{i_f} I(f) \xrightarrow{j_f} X \times_Y X$$

together with an assignment of a cloven $\mathcal{L}$-map structure on i_f and a cloven $\mathcal{R}$-map structure on j_f. This choice is *functorial* if this assignment can be extended to a functor $I : \boldsymbol{\mathcal{R}}\textbf{-Map}_{\boldsymbol{\mathcal{E}}} \to \boldsymbol{\mathcal{R}}\textbf{-Map}_{\boldsymbol{\mathcal{E}}} \times_{\mathcal{E}} \boldsymbol{\mathcal{L}}\textbf{-Map}_{\boldsymbol{\mathcal{E}}}$. That is, for every morphism of cloven $\mathcal{R}$-maps $(h, k) : (f, p) \to (g, r)$ there is given a map $I(h, k)$ functorial in h, k such that the following squares commute:

$$\begin{array}{ccc} X & \xrightarrow{h} & U \\ {\scriptstyle i_f}\downarrow & & \downarrow{\scriptstyle i_g} \\ I(f) & \xrightarrow[I(h,k)]{} & I(g) \end{array} \qquad \begin{array}{ccc} I(f) & \xrightarrow{I(h,k)} & I(g) \\ {\scriptstyle j_f}\downarrow & & \downarrow{\scriptstyle j_g} \\ X \times_Y X & \xrightarrow[h \times_k h]{} & U \times_V U \end{array} \tag{5.5}$$

with the left hand square a morphism of cloven $\mathcal{L}$-maps and the right hand square a morphism of cloven $\mathcal{R}$-maps. Finally the choice is *stable* if the square

$$\begin{array}{ccc} I(f) & \xrightarrow{I(h,k)} & I(g) \\ {\scriptstyle j_f}\downarrow & & \downarrow{\scriptstyle j_g} \\ X \times_Y X & \xrightarrow[h \times_k h]{} & Z \times_W Z \end{array}$$

is a pullback whenever the underlying commutative square of the morphism of $\mathcal{R}$- maps (h, k) is.

Theorem 5.8. *Given a cloven weak factorisation system for a category $\mathcal{C}$, if there exists a functorial and stable choice of diagonal factorisations then the associated type category of Theorem 5.4 has identity types.*

Proof. We show each property can be satisfied case by case.

- Formation/Introduction: Let $A \in Ty(\Gamma)$ be given by a cloven $\mathcal{R}$-map $(\pi_A, p) : \Gamma.A \to \Gamma$. By the substitution property of the type category we have that the following square is a pullback:

$$\begin{array}{ccc} \Gamma.A.A^+ & \xrightarrow{(\pi_A)^+} & \Gamma.A \\ {\scriptstyle \pi_{A^+}}\downarrow & & \downarrow{\scriptstyle \pi_A} \\ \Gamma.A & \xrightarrow[\pi_A]{} & \Gamma \end{array}$$

  We thus obtain the diagonal map $\delta_{\pi_A} : \Gamma.A \to \Gamma.A.A^+$ and use our choice of diagonal factorisation to obtain

$$\Gamma.A \xrightarrow{i_{\pi_A}} I(\pi_A) \xrightarrow{j_{\pi_A}} \Gamma.A.A^+$$

  together with an assigned s and p making (i_{π_A}, s) a cloven $\mathcal{L}$-map and (j_{π_A}, p) a cloven $\mathcal{R}$-map. We thus take (j_{π_A}, p) to be the identity type $Id_A \in Ty(\Gamma.A.A^+)$, making $I(\pi_A) = \Gamma.A.A^+.Id_A$ and $j_{\pi_A} = \pi_{Id_A}$. We can then take i_{π_A} as the introduction morphism $r_A : \Gamma.A \to \Gamma.A.A^+.Id_A$, and clearly $\pi_{Id_A} \circ r_A = \delta_{\pi_A}$ as required.

- Elimination/Computation: Suppose $C \in Ty(\Gamma.A.A^+.Id_A)$ and we have a commutative square

$$\begin{array}{ccc} \Gamma.A & \xrightarrow{d} & \Gamma.A.A^+.Id_A.C \\ {\scriptstyle r_A}\downarrow & & \downarrow{\scriptstyle \pi_C} \\ \Gamma.A.A^+.Id_A & \xrightarrow[id]{} & \Gamma.A.A^+.Id_A \end{array}$$

  Since π_C has a cloven $\mathcal{R}$-map structure p and r_A has a cloven $\mathcal{L}$- map structure s, by Proposition 5.3 we have a canonical choice of filler $J(C, d) = p \circ P(d, id) \circ s : \Gamma.A.A^+.Id_A \to \Gamma.A.A^+.Id_A.C$.

- Substitution:

  1. $Id_A[f^{++}] = Id_{A[f]}$: We first note that Lemma 5.6 iii), the pullback square

$$\begin{array}{ccc} \Delta.A[f] & \xrightarrow{f^+} & \Gamma.A \\ {\scriptstyle \pi_{A[f]}}\downarrow & & \downarrow{\scriptstyle \pi_A} \\ \Delta & \xrightarrow[f]{} & \Gamma \end{array} \qquad (5.6)$$

     is a morphism of cloven $\mathcal{R}$-maps when π_A and $\pi_{A[f]}$ are equipped with their assigned cloven $\mathcal{R}$-map structures. By functorality of the choice of diagonal factorisations we obtain the commutative square

$$\begin{array}{ccc}
\Delta.A[f].A[f]^+.Id_{A[f]} & \xrightarrow{I(f^+,f)} & \Gamma.A.A^+.Id_A \\
\downarrow{\scriptstyle \pi_{Id_{A[f]}}} & & \downarrow{\scriptstyle \pi_{Id_A}} \\
\Delta.A[f].A[f]^+ & \xrightarrow[f^{++}]{} & \Gamma.A.A^+
\end{array}$$

underlying a morphism of $\mathcal{R}$-maps. By stability this square is a pullback and so $I(f^+, f) = f^{+++}$. We then note that since the assigned $\mathcal{R}$-map structure on $\pi_{A[f]}$ is the unique such structure making the square a morphism of $\mathcal{R}$-maps it follows that it is identical to the assigned $\mathcal{R}$-map structure for $\pi_{Id_A[f^{++}]}$ and hence the required identity holds.

2. Commutativity of

$$\begin{array}{ccc}
\Delta.A[f] & \xrightarrow{r_{A[f]}} & \Delta.A[f].A[f]^+.Id_{A[f]} \\
\downarrow{\scriptstyle f^+} & & \downarrow{\scriptstyle f^{+++}} \\
\Gamma.A & \xrightarrow[r_A]{} & \Gamma.A.A^+.Id_A
\end{array}$$

As in the previous case we note that (5.6) is a morphism of $\mathcal{R}$-maps. Thus functorality of the choice of diagonal factorisations establishes the requisite square's commutativity, as it is precisely the left hand commutative square of (5.5).

3. Commutativity of

$$\begin{array}{ccc}
\Delta.A[f].A[f]^+.Id_{A[f]} & \xrightarrow{J(C[f],d[f])} & \Delta.A[f].A[f]^+.Id_{A[f]}.C[f^{+++}] \\
\downarrow{\scriptstyle f^{+++}} & & \downarrow{\scriptstyle f^{++++}} \\
\Gamma.A.A^+.Id_A & \xrightarrow[J(C,d)]{} & \Gamma.A.A^+.Id_A.C
\end{array}$$

First note by functorality the commutative square of 2. is a morphism of $\mathcal{L}$-maps from $r_{A[f]}$ to r_A. By naturality of canonical fillers with respect to morphisms of cloven $\mathcal{L}$-maps we have that $J(C, d) \circ f^{+++}$ is the canonical filler of the square:

$$\begin{array}{ccc}
\Delta.A[f] & \xrightarrow{df^+} & \Gamma.A.A^+.Id_C \\
\downarrow{\scriptstyle r_{A[f]}} & & \downarrow{\scriptstyle \pi_C} \\
\Delta.A[f].A[f]^+.Id_{A[f]} & \xrightarrow[f^{+++}]{} & \Gamma.A.A^+.Id_C
\end{array}$$

Similarly we have that the commutative square

$$\begin{array}{ccc}
\Delta.A[f].A[f]^+.Id_{A[f]}.C[f] & \xrightarrow{f^{++++}} & \Gamma.A.A^+.Id_A.C \\
\downarrow{\scriptstyle \pi_{C[f^{+++}]}} & & \downarrow{\scriptstyle \pi_C} \\
\Delta.A[f].A[f]^+.Id_{A[f]} & \xrightarrow[f^{+++}]{} & \Gamma.A.A^+.Id_A
\end{array}$$

underlies a morphism of cloven $\mathcal{R}$-maps. Using naturality of canonical fillers with respect to cloven $\mathcal{R}$-maps we have that $f^{++++} \circ J(C[f], d[f])$ is the canonical filler of the square.

$$\begin{array}{ccc} \Delta.A[f] & \xrightarrow{f^{++++}d[f]} & \Gamma.A.A^{+}.Id_C \\ \downarrow{\scriptstyle r_{A[f]}} & & \downarrow{\scriptstyle \pi_C} \\ \Delta.A[f].A[f]^{+}.Id_{A[f]} & \xrightarrow[f^{+++}]{} & \Gamma.A.A^{+}.Id_C \end{array}$$

Observing that $df^{+} = f^{++++}d[f]$ thus gives the required commutativity.

□

5.3 The Frobenius Property

We require one further property in order to model the strong identity types, and thus obtain a sound categorical model of identity types. We must be able to replicate these arguments in the presence of contextual parameters, and in order to do this we require a particular stability property of our cloven maps.

Definition 5.9 (Frobenius Property)**.** A cloven weak factorisation system for a category $\mathcal{E}$ has the *Frobenius property* if, given a pullback of a cloven $\mathcal{L}$-map $(i, q) : X \to A$ and a cloven $\mathcal{R}$-map $(f, p) : B \to A$

$$\begin{array}{ccc} f^{\star}(X) & \xrightarrow{\bar{f}} & X \\ \downarrow{\scriptstyle \bar{i}} & & \downarrow{\scriptstyle i} \\ B & \xrightarrow[f]{} & A \end{array}$$

the map $\bar{i}$ can be equipped with a cloven $\mathcal{L}$-map structure. We say this is *functorial* if the assignment extends to a functor $\boldsymbol{\mathcal{R}}$**- Map**$_{\boldsymbol{\mathcal{E}}} \times_{\mathcal{E}} \boldsymbol{\mathcal{L}}$**-Map**$_{\boldsymbol{\mathcal{E}}} \to \boldsymbol{\mathcal{L}}$**-Map**$_{\boldsymbol{\mathcal{E}}}$.

We suggestively compile the definitions given thus far under the name *homotopy theoretic model of identity types*.

Definition 5.10 (Homotopy Theoretic Model Of Identity Types)**.** A category $\mathcal{C}$ is a *homotopy theoretic model of identity types* if it can be equipped with a cloven weak factorisation system satisfying the functorial Frobenius property such that a functorial and stable choice of diagonal factorisations exists.

We thus complete the chapter and justify this terminology by proving that such a category models strong identity types

Theorem 5.11. *The associated type category to a homotopy theoretic model of identity types is a categorical model of identity types.*

Proof. By Theorem 5.8 it suffices to show the following property is satisfied: for every

$$B_1 \in Ty(\Gamma.A.A^{+}.Id_A)$$
$$\vdots$$
$$B_n \in Ty(\Gamma.A.A^{+}.Id_A.B_1 \dots B_{n-1})$$
$$C \in Ty(\Gamma.A.A^{+}.Id_A.B_1 \dots B_{n-1}.B_n)$$

and commutative diagram

$$\begin{array}{ccc} \Gamma.A.\Delta[r_A] & \xrightarrow{\quad d \quad} & \Gamma.A.A^{+}.Id_A.\Delta.C \\ {\scriptstyle (r_A)^{+\cdots+}}\downarrow & & \downarrow{\scriptstyle \pi_C} \\ \Gamma.A.A^{+}.Id_A.\Delta & \xrightarrow[id]{\qquad} & \Gamma.A.A^{+}.Id_A.\Delta \end{array}$$

with Δ abbreviating the context $B_1, \dots, B_n$, we have a diagonal filler $J(\Delta, C, d)$ rendering the diagram

$$\begin{array}{ccc} \Gamma.A.\Delta[r_A] & \xrightarrow{\quad d \quad} & \Gamma.A.A^{+}.Id_A.\Delta.C \\ {\scriptstyle (r_A)^{+\cdots+}}\downarrow & {\scriptstyle J(\Delta,C,d)} \nearrow & \downarrow{\scriptstyle \pi_C} \\ \Gamma.A.A^{+}.Id_A.\Delta & \xrightarrow[id]{\qquad} & \Gamma.A.A^{+}.Id_A.\Delta \end{array}$$

We first note that the Frobenius property exhibits $(r_A)^{+}$ as a cloven $\mathcal{L}$-map structure. We have the pullback

$$\begin{array}{ccc} \Gamma.A.B_1[r_A] & \xrightarrow{(r_A)^{+}} & \Gamma.A.A^{+}.Id_A.B_1 \\ {\scriptstyle \pi_{B_1[r_A]}}\downarrow & & \downarrow{\scriptstyle \pi_{B_1}} \\ \Gamma.A & \xrightarrow[r_A]{\qquad} & \Gamma.A.A^{+}.Id_A \end{array}$$

and we know r_A has an assigned cloven $\mathcal{L}$-map structure and π_{B_1} an assigned cloven $\mathcal{R}$-map structure. Hence $(r_A)^{+}$ is assigned a cloven $\mathcal{L}$- map structure by functorality. Repeatedly applying this argument we obtain an assigned cloven $\mathcal{L}$-map structure of $(r_A)^{+\cdots+}$. Since π_C has an assigned cloven $\mathcal{R}$-map structure we have a canonical choice of diagonal filler $J(\Delta, C, d)$ for the square as required. To see that the analogous substitution conditions hold it suffices to once again use repeated applications of the Frobenius property and apply the arguments of the previous Theorem: functorality guarantees the requisite identities hold. $\square$

We now have a set of conditions that allow us to produce sound models of intensional types. We dedicate the penultimate chapter of the thesis to showing these conditions can be satisfied by constructions on the path object category structure. We thus obtain a model of type theory in cubical sets with connections.

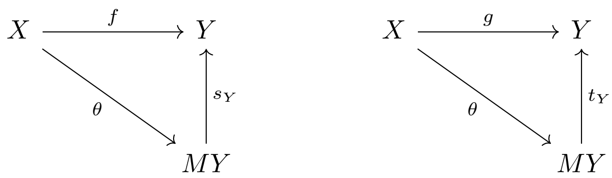
Chapter 6

Constructing A Homotopy Theoretic Model

We now complete the thesis by producing a homotopy theoretic model from a path object category structure. Before we delve into the construction it will help us to use the path object category structure to introduce some homotopy theoretic notions with which to work.

6.1 Homotopy In Path Object Categories

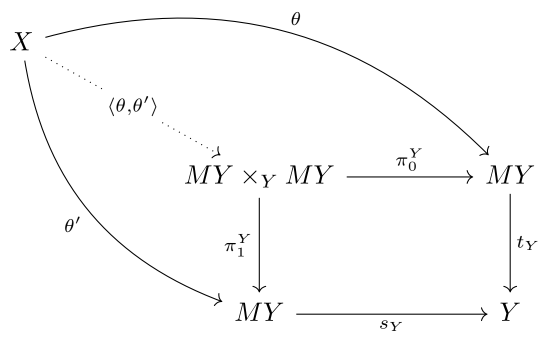
Definition 6.1 (Homotopy)**.** Let $f, g : X \to Y$ be morphisms in a path object category $\mathcal{E}$. A *homotopy* $\theta : f \Rightarrow g$ is a morphism $X \to MY$ such that the following diagrams commute



Proposition 6.2. *For each X, Y in $\mathcal{E}$ there is a category $\underline{\mathcal{E}}(X, Y)$ given by*

- ***Objects*** *Morphisms $f : X \to Y$*
- ***Arrows*** *Homotopies $\theta : f \Rightarrow g$*

Proof. We use the structure of the internal category to define composition of homotopies. Suppose we have $f, g, h : X \to Y$ with $\theta : f \Rightarrow g$ and $\theta' : g \Rightarrow h$. In particular. since $t_Y \circ \theta = s_Y \circ \theta'$ we have the pullback diagram



Hence we define the composition $\theta' \circ_1 \theta = c_Y \circ \langle \theta, \theta' \rangle : X \to MY$. By the internal category identities and the commutativity of the diagram we have

$$s_Y \circ c_Y \circ \langle \theta, \theta' \rangle = s_Y \circ \pi_0^Y \circ \langle \theta, \theta' \rangle = s_Y \circ \theta = f$$

$$t_Y \circ c_Y \circ \langle \theta, \theta' \rangle = t_Y \circ \pi_1^Y \circ \langle \theta, \theta' \rangle = t_Y \circ \theta' = h$$

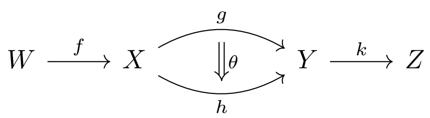
So we indeed have $\theta' \circ_1 \theta : f \Rightarrow h$. To give the identities of the category we take $id_f = e_Y \circ f$. Then by the rules of source and target of identities in an internal category we have

$$s_Y \circ e_Y \circ f = id_Y \circ f = f = id_Y \circ f = t_Y \circ e_Y \circ f$$

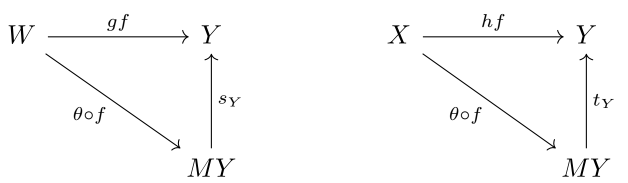
so $Id_f : f \Rightarrow f$ and it is straight forward to see this satisfies the identity axioms. Similarly we inherit associativity of composition from the analogous internal category axiom.

□

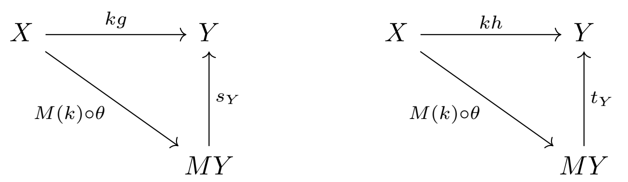
We also have a notion of *whiskering* for our homotopies: suppose we have



Then we can obtain an $\underline{\mathcal{E}}(W, Y)$-morphism $\theta.f : gf \Rightarrow hf$ by the arrow $\theta \circ f$:



Similarly, we obtain an $\underline{\mathcal{E}}(X, Z)$-morphism $k.\theta : kg \Rightarrow kh$ by the arrow $M(k) \circ \theta$: we have commuting diagrams



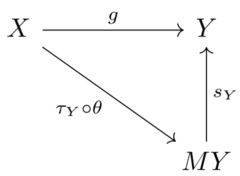
obtained via the naturality of s and t. It is straightforward to verify the following coherence properties of these operations:

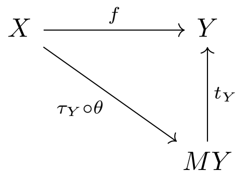
Lemma 6.3. *The operations $\theta.f$ and $k.\theta$ are functorial in θ. Further they satisfy the following coherence equations:*

$$\theta.(f.f') = (\theta.f).f', \qquad \theta.id_X = \theta$$

$$(k'.k).\theta \qquad id_Y.\theta = \theta$$

Finally we have a "reversal" operation $(-)^\circ$ given by the involutions τ_X. Given $\theta : f \Rightarrow Y$ we have $\theta^\circ : g \Rightarrow f$ given by the arrow $\tau_Y \circ \theta$. We obtain the diagrams





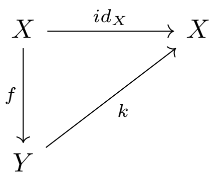
since $s_Y \circ \tau_Y = t_Y$ and $t_Y \circ \tau_Y = s_Y$.

With a notion of homotopy we can also define strong deformation retracts in our path object categories.

Definition 6.4 (Strong Deformation Retract)**.** Given a map $f : X \to Y$ in a path object category $\mathcal{E}$, a *strong deformation retraction* is a retraction $k : Y \to X$ of f together with a homotopy $\theta : id_Y \Rightarrow fk$ which is *trivial on* X: $\theta.f = id_f$. A *strong deformation retract* is a map f equipped with a strong deformation retraction.

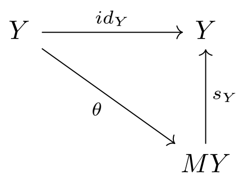
We can unpack this definition to see that $k : Y \to X$ and $\theta : id_Y \Rightarrow fk$ together constitute a strong deformation retract on f iff the following diagrams commute:

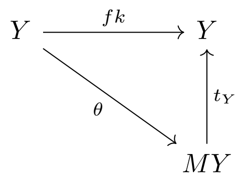
- **Retraction**



(6.1)

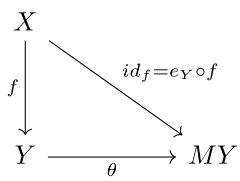
- **Homotopy**





(6.2)

- **Trivial On X**



(6.3)

6.2 Constructing The Model

Equipped with these notions we construct a homotopy theoretic model from the path object category data. For reasons of length, we sketch some of the details, concentrating on the parts of the argument either omitted from the original exposition in [vdBG12] or crucially involving parts of the path object category structure.

6.2.1 The Cloven Weak Factorisation System

The first step in our construction is to equip an arbitrary path object category with a *cloven weak factorisation system*. Recall that a cloven weak factorisation system on a category $\mathcal{E}$ is given by the following data

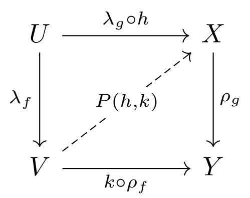
1. For each morphism $f : X \to Y$ in $\mathcal{E}$ a *choice of factorisation*

$$X \xrightarrow{\lambda_f} Pf \xrightarrow{\rho_f} Y$$

2. For every commutative square of the form

$$\begin{CD} U @>{h}>> X \\ @V{f}VV @VV{g}V \\ V @>>{k}> Y \end{CD} \tag{6.4}$$

   a diagonal filler $P(h,k) : Pf \to Pg$ making the diagram

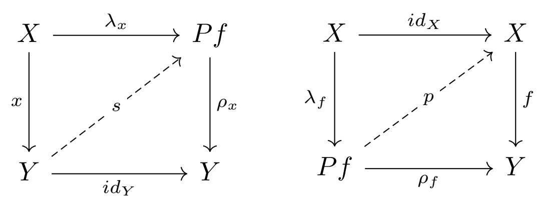


(6.5)

   commute, such that the assignment $(h,k) \mapsto P(h,k)$ is functorial in h, k;

3. For each morphism $f : X \to Y$, a choice of cloven $\mathcal{L}$-map structure σ_f on λ_f and a choice of cloven $\mathcal{R}$-map structure π_f on ρ_f.

Where a cloven $\mathcal{L}$-map structure s and a cloven $\mathcal{R}$-map structure p on a morphism $x : X \to Y$ are given by diagonal fillers rendering the following squares commutative



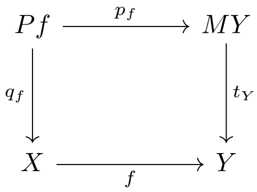
(6.6)

We begin with the choice of factorisations.

Lemma 6.5. *For each morphism $f : X \to Y$ in a path object category $\mathcal{E}$ there exists a choice of factorisation*

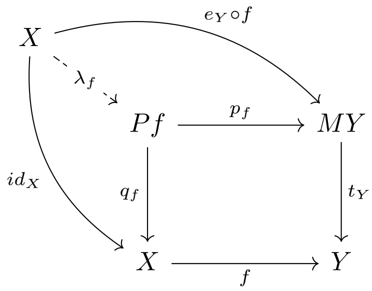
$$X \xrightarrow{\lambda_f} Pf \xrightarrow{\rho_f} Y$$

Proof. : Let $f : X \to Y$ be an arrow in $\mathcal{E}$. We define the required factorisation by first taking a pullback



(6.7)

We have that $t_Y \circ e_Y \circ f = id_Y \circ f = f = f \circ id_X$ so we can obtain λ_f from the universal property of the pullback:



(6.8)

Now defining $\rho_f = s_Y \circ p_f$, by the source/target property of the maps e_Y together with the commutativity of (6.8) we have

$$\rho_f \circ \lambda_f = s_Y \circ p_f \circ \lambda_f = s_Y \circ e_Y \circ f = id_Y \circ f = f$$

as required. $\square$

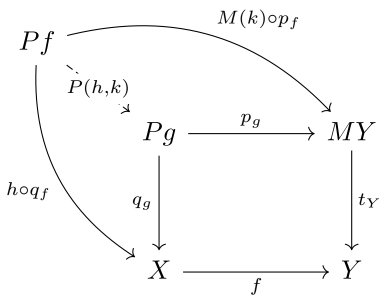
Next we give the diagonal fillers $P(h, k)$.

Lemma 6.6. *For the factorisations of Lemma 6.5, given a commutative square of the form (6.4) there exists a diagonal filler $P(h, k) : Pf \to Pg$ making the diagram (6.5) commute, with the assignment $(h, k) \mapsto P(h, k)$ functorial in h and k.*

Proof. Suppose we have a commutative square of the form (6.4). Using the notation from the pullback of the previous lemma, we have

$$\begin{aligned}
t_Y \circ M(k) \circ p_f &= k \circ t_V \circ p_f && \text{(Naturality of } t \text{)}\\
&= k \circ f \circ q_f && \text{(Commutativity of (6.7))}\\
&= g \circ h \circ q_f && \text{(Commutativity of (6.4))}
\end{aligned}$$

Hence we obtain the map $P(h,k) : Pf \to Pg$ by the universal property of the pullback:



(6.9)

That this assignment is functorial follows from the functorality of M. It remains to verify that the diagram (6.5) commutes. For the upper triangle it suffices to check identity holds upon post-composition with the projection maps p_g and q_g. For q_g we simply use the commutativity of the pullback diagrams:

$$q_g \circ P(h,k) \circ \lambda_f = h \circ q_f \circ \lambda_f = h = q_g \circ \lambda_g \circ h$$

Whilst for p_g we can compute

$$\begin{aligned} p_g \circ \lambda_g \circ h &= e_Y \circ g \circ h && \text{(Commutativity of (6.8))} \\ &= e_Y \circ k \circ f && \text{(Commutativity of (6.4))} \\ &= M(k) \circ e_V \circ f && \text{(Naturality of } e\text{)} \\ &= M(h) \circ p_f \circ \lambda_f && \text{(Commutativity of (6.8))} \\ &= p_g \circ P(h,k) \circ \lambda_f && \text{(Commutativity of (6.9))} \end{aligned}$$

This completes the proof that the upper triangle commutes. For the lower triangle we straightforwardly compute

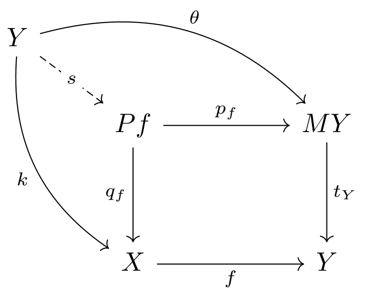
$$\begin{aligned} \rho_g \circ P(h,k) &= s_Y \circ p_g \circ P(h,k) && \\ &= s_Y \circ M(k) \circ p_f && \text{(Commutativity of (6.9))} \\ &= k \circ s_V \circ p_f && \text{(Naturality of } s\text{)} \\ &= k \circ \rho_f && \end{aligned}$$

□

Before we prove the final property required for the cloven weak factorisation system we can give a characterisation of the cloven $\mathcal{L}$-maps in path object categories that utilises the internal notion of homotopy we developed.

Lemma 6.7. *In a path object category, a map $f : X \to Y$ has a cloven $\mathcal{L}$-structure s with respect to the factorisation of Lemma 6.5 iff it has a strong deformation retract (θ, k).*

Proof. First assume we have a strong deformation retraction (θ, k) for f. From the target homotopy diagram 6.2 we obtain $s : Y \to Pf$ from the universal property of the pullback:



(6.10)

We claim s is a cloven $\mathcal{L}$-map structure for f, thus making the left hand diagram of (6.6) commute. For the upper triangle it is sufficient to verify equality upon post-composition with the projection maps p_f and q_f. Since θ is trivial on X we have

$$p_F \circ s \circ f = \theta \circ f = e_Y \circ f = p_f \circ \lambda_f$$

Whilst the fact that k is a retraction of f gives us

$$q_f \circ s \circ f = k \circ f = id_X = q_f \circ \lambda_f$$

For the lower triangle, since $\theta : id_Y \Rightarrow fk$ we have

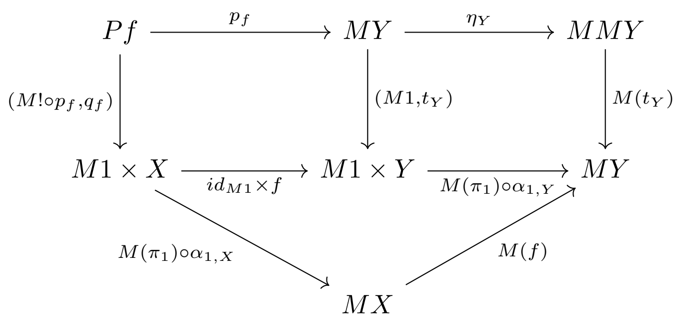
$$\rho_f \circ s = s_Y \circ p_f \circ s = s_Y \circ \theta = id_Y$$

Hence s is a cloven-$\mathcal{L}$-map structure for f as required. Conversely assume (f, s) is a cloven $\mathcal{L}$-map. Defining $\theta = p_f \circ s$ and $k = q_f \circ s$ we claim (θ, k) gives a strong deformation retract on f - we leave the straightforward computations to the reader. □

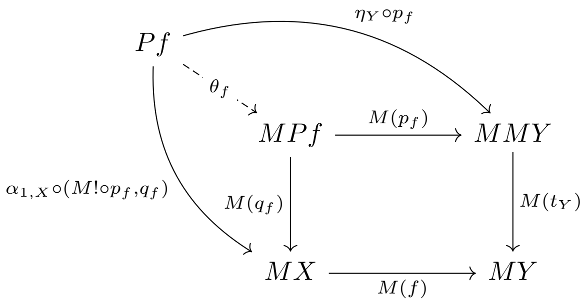
Using this we can prove that the final condition for the cloven weak factorisation system holds:

Lemma 6.8. *For each $f : X \to Y$ in a path object category $\mathcal{E}$ there exists a choice of cloven $\mathcal{L}$-structure σ_f on λ_f and a choice of cloven $\mathcal{R}$-structure π_f on ρ_f with respect to the choice of factorisation given in Lemma 6.5.*

Proof. We first attend to the cloven $\mathcal{L}$-structure on λ_f. By Lemma 6.7 it suffices to give a choice of strong deformation retract. We already have a retraction $q_f : Pf \to X$ of λ_f from the pullback defining it. Hence we seek $\theta_{\lambda_f} : id_{Pf} \Rightarrow \lambda_f \circ q_f$. Observe that the following diagram commutes



The right hand square commutes by the third path object category axiom, the left hand square commutes by 6.8 and the bottom triangle commutes by naturality of α. Recall also that M is a pullback preserving functor. Hence we obtain θ_f from the universal property of the pullback diagram:



(6.11)

It remains to verify that θ satisfies the requisite properties:

- **Homotopy**: We wish to establish that $\theta : id_{Pf} \Rightarrow \lambda_f q_f$. Thus for the source homotopy diagram we require $s_{Pf} \circ \theta_f = id_{Pf}$: equivalently, that we have identity upon post-composition with the projection maps p_f and q_f. We have

$$
\begin{aligned}
p_f \circ s_{Pf} \circ \theta_f &= s_{MY} \circ \eta_{M(p_f)} \circ \theta_f && \text{(Naturality of } s\text{)} \\
&= s_{MY} \circ \eta_Y \circ p_f && \text{(Commutativity of 6.11)} \\
&= id_{MY} \circ p_f && \text{(Path Object Category Axiom 3)} \\
&= p_f
\end{aligned}
$$

$$
\begin{aligned}
q_f \circ s_{Pf} \circ \theta_f &= s_X \circ M(q_f) \circ \theta_f && \text{(Naturality of } s\text{)} \\
&= s_X \circ \alpha_{1,X} \circ (M! \circ p_f, q_f) && \text{(Commutativity of 6.11)} \\
&= (s_1 \times id_X) \circ (M! \circ p_f, q_f) && \text{(} s \text{ is a strong natural transformation)} \\
&= q_f
\end{aligned}
$$

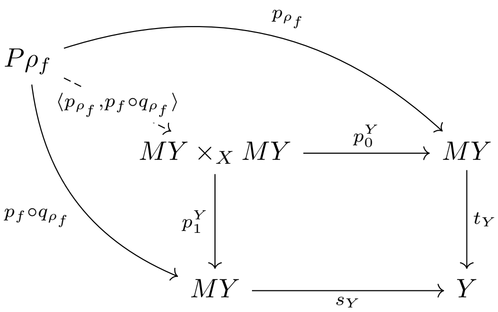
We can establish that the target homotopy diagram commutes in a similar fashion and we leave this detail to the reader.

- **Trivial On X**: Finally we require that $\theta_f \circ \lambda_f = e_{Pf} \circ \lambda_f$ and again it suffices to verify this identity upon post-composition with the projection maps $M(p_f)$ and $M(q_f)$:

$$
\begin{aligned}
M(p_f) \circ \theta_f \circ \lambda_f &= \eta_Y \circ p_f \circ \lambda_f && \text{(Commutativity of 6.11)} \\
&= \eta_Y \circ e_Y \circ f && \text{(Commutativity of 6.8)} \\
&= e_{MY} \circ e_Y \circ f && \text{(Path Object Category Axiom 3)} \\
&= e_{MY} \circ p_f \circ \lambda_f && \text{(Commutativity of 6.8)} \\
&= M(p_f) \circ e_{Pf} \circ \lambda_f && \text{(Naturality of } e\text{)}
\end{aligned}
$$

$$
\begin{aligned}
M(q_f) \circ \theta_f \circ \lambda_f &= \alpha_{1,x} \circ (M!_Y \circ p_f \circ \lambda_f, q_f \circ \lambda_f) && \text{(Commutativity of 6.11)} \\
&= \alpha_{1,X}(M!_Y \circ e_Y \circ f, id_X) && \text{(Commutativity of 6.8)} \\
&= \alpha_{1,X}(e_1 \circ !_Y \circ f, id_X) && \text{(Naturality of e)} \\
&= \alpha_{1,X}(e_1 \circ !_X, id_X) && \text{(1 is terminal)} \\
&= e_X \circ id_{MX} && \text{(e is a strong natural transformation)} \\
&= e_X \circ q_f \circ \lambda_f && (q_f \text{ is a retraction of } \lambda_f) \\
&= M(q_f) \circ e_{Pf} \circ \lambda_f && \text{(Naturality of } e\text{)}
\end{aligned}
$$

This takes care of the choice of $\mathcal{L}$-structure σ_f for λ_f. Next we construct a cloven $\mathcal{R}$-map structure π_f for ρ_f. First note that the factorisation pullback for ρ_f allows us to obtain a map $\langle p_{\rho_f}, p_f \circ q_{\rho_f} \rangle : P\rho_f \to MY \times_Y MY$ from the universal property of the composition pullback diagram of MY:

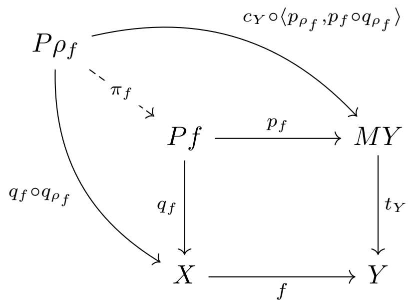


(6.12)

Now we have that

$$
\begin{aligned}
t_Y \circ c_Y \circ \langle p_{\rho_f}, p_f \circ q_{\rho_f} \rangle &= t_Y \circ p_1^Y \circ \langle p_{\rho_f}, p_f \circ q_{\rho_f} \rangle && \text{(Target diagram of composition)} \\
&= t_Y \circ p_f \circ q_{\rho_f} && \text{(Commutativity of 6.12)} \\
&= f \circ q_f \circ q_{\rho_f} && \text{(Commutativity of 6.8)}
\end{aligned}
$$

Hence we obtain π_f:



(6.13)

It remains to show that π_f renders the right-hand diagram of (6.6) commutative for the case $x = \rho_f$, $p = \pi_f$. Once again, for the upper triangle we verify identity upon post-composition by the projection maps p_f and q_f.

$$\begin{aligned}
p_f \circ \pi_f \circ \lambda_{\rho_f} &= c_Y \circ \langle p_{\rho_f}, p_f \circ q_{\rho_f} \rangle \circ \lambda_{\rho_f} && \text{(Commutativity of 6.13)} \\
&= c_Y \circ \langle p_{\rho_f} \circ \lambda_{\rho_f}, p_f \circ q_{\rho_f} \circ \lambda_{\rho_f} \rangle \\
&= c_Y \circ \langle e_Y \circ \rho_f, p_f \rangle && \text{(Commutativity of (6.8))} \\
&= c_Y \circ \langle e_Y \circ s_Y, id_{MY} \rangle \circ p_f \\
&= p_f && \text{(Right Identity Law)}
\end{aligned}$$

$$\begin{aligned}
q_f \circ \pi_f \circ \lambda_{\rho_f} &= q_f \circ q_{\rho_f} \circ \lambda_{\rho_f} && \text{(Commutativity of 6.13)} \\
&= q_f && \text{(Commutativity of (6.8))}
\end{aligned}$$

For the lower triangle we can straightforwardly compute:

$$\begin{aligned}
\rho_f \circ \pi_f &= s_Y \circ p_f \circ \pi_f \\
&= s_Y \circ c_Y \circ \langle p_{\rho_f}, p_f \circ q_{\rho_f} \rangle && \text{(Commutativity of 6.13)} \\
&= s_Y \circ \pi_0 \circ \langle p_{\rho_f}, p_f \circ q_{\rho_f} \rangle && \text{(Source of composition law)} \\
&= s_Y \circ p_{\rho_f} && \text{(Commutativity of 6.12)} \\
&= \rho_{\rho_f}
\end{aligned}$$

Hence π_f is indeed a cloven $\mathcal{R}$-structure for ρ_f as required. □

With all of these lemmas together we have established:

Theorem 6.9. *A path object category $\mathcal{E}$ may be equipped with the structure of a cloven weak factorisation system.*

We will henceforth refer to this as the *associated cloven w.f.s* of the path object category $\mathcal{E}$.

6.2.2 Diagonal Factorisations

Next we must give a stable and functorial choice of diagonal factorisations. Recall that given a morphism $f : X \to Y$ we obtain the diagonal map δ_f from the following pullback

$$\begin{array}{ccccc} X & & \xrightarrow{\quad id_X \quad} & & \\ & \overset{\delta_f}{\searrow} & & & \\ & & X \times_Y X & \longrightarrow & X \\ {\scriptstyle id_X} \searrow & & \downarrow & & \downarrow {\scriptstyle f} \\ & & X & \xrightarrow[f]{} & Y \end{array}$$

Then a *choice of diagonal factorisations* gives, for every cloven $\mathcal{R}$-map $(f, p) : X \to Y$ in $\mathcal{E}$ a factorisation of δ_f:

$$X \xrightarrow{i_f} I(f) \xrightarrow{j_f} X \times_Y X$$

such that i_f is assigned a cloven $\mathcal{L}$-map structure and j_f is assigned a cloven $\mathcal{R}$-map structure. This choice is *functorial* if the assignment can be extended to a functor $I : \boldsymbol{\mathcal{R}}\textbf{-Map}_{\boldsymbol{\mathcal{E}}} \to \boldsymbol{\mathcal{R}}\textbf{-Map}_{\boldsymbol{\mathcal{E}}} \times_{\mathcal{E}} \boldsymbol{\mathcal{L}}\textbf{-Map}_{\boldsymbol{\mathcal{E}}}$. Finally the choice is *stable* if the square

$$\begin{array}{ccc} I(f) & \xrightarrow{I(h,k)} & I(g) \\ {\scriptstyle j_f} \downarrow & & \downarrow {\scriptstyle j_g} \\ X \times_Y X & \xrightarrow[h \times_k h]{} & Z \times_W Z \end{array}$$

is a pullback whenever the underlying commutative square of the morphism of $\mathcal{R}$-maps (h, k) is. Before we can construct this data we require two definitions. The first is that of an *indexed functor*.

Definition 6.10 (Indexed Functor)**.** Given a finitely complete category $\mathcal{E}$, an $\mathcal{E}$*-indexed functor* $M^{(-)}$: $\mathcal{E}/(-) \to \mathcal{E}/(-)$ consists of a family of functors $M^\Gamma : \mathcal{E}/\Gamma \to \mathcal{E}/\Gamma$ indexed by the objects of $\mathcal{E}$ commuting up to coherent isomorphism with the pullback functors $f^\star : \mathcal{E}/\Gamma \to \mathcal{E}/\Delta$ induced by morphisms $f : \Delta \to \Gamma$ of $\mathcal{E}$.

We similarly have an indexed notion of natural transformation.

Definition 6.11 (Indexed Natural Transformation)**.** Given a finitely complete category $\mathcal{E}$ and $\mathcal{E}$-indexed functors $M^{(-)}, N^{(-)} : \mathcal{E} \to \mathcal{E}$, a $\mathcal{E}$*-indexed natural transformation* $\mu^{(-)} : M^{(-)} \Rightarrow N^{(-)}$ consists of a family of natural transformations $\mu^\Gamma : M^\Gamma \Rightarrow N^\Gamma$ indexed by the objects of $\mathcal{E}$ commuting up to coherent isomorphism with the pullback functors $f^\star : \mathcal{E}/\Gamma \to \mathcal{E}/\Delta$ induced by morphisms $f : \Delta \to \Gamma$ of $\mathcal{E}$.

Remark 6.12*.* Readers with a deeper background in category theory may notice that these are far more restricted notions of indexed functors and natural transformations than is usual. In fact taking the slices of a category is a particular instantiation of a more general phenomenon: that of an indexed category. As we only require this particular instance for our present purposes we omit the broader definitions and instead direct the interested reader to [Joh02, Section B1].

With these definitions we are able to state a result of Paré's reported by Johnstone in [Joh97].

Proposition 6.13. *1. Given a finitely complete category $\mathcal{E}$ and a pullback preserving endofunctor M equipped with a strength α there exists a unique (up to canonical isomorphism) extension to an indexed functor $M^{(-)} : \mathcal{E}/(-) \to \mathcal{E}/(-)$ such that each M^{Γ} is pullback preserving.*

2. For any strong natural transformation $\mu : (M, \alpha) \Rightarrow (N, \beta)$ with the underlying functors M and N pullback preserving functors, there exists a unique (up to canonical isomorphism) extension to an indexed natural transformation $\mu^{(-)} : M^{(-)} \Rightarrow N^{(-)}$.

Proof. We omit the full details of the proof, directing the reader instead to [Joh97, Proposition 3.3]. However we give the definitions of the indexed functors and natural transformations for future use.

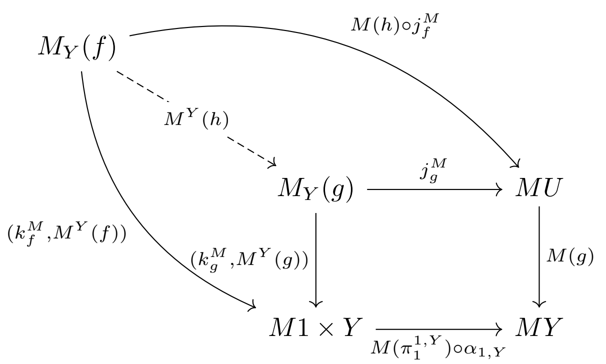
1. Let $(M, \alpha) : \mathcal{E} \to \mathcal{E}$ be a strong endofunctor with M pullback preserving. Given Y in $\mathcal{E}$ we define the functor M^Y as follows

   - **Objects**: Let $f : X \to Y$ be a morphism in $\mathcal{E}$. We define $M^Y(f)$ by taking a pullback as follows

$$\begin{array}{ccc} M_Y(f) & \xrightarrow{\;j_f^M\;} & MX \\ {\scriptstyle (k_f^M, M_Y(f))}\downarrow & & \downarrow{\scriptstyle M(f)} \\ M1 \times Y & \xrightarrow[M(\pi_1^{1,Y})\circ\alpha_{1,Y}]{} & MY \end{array}$$

     Thus yielding an arrow $M_Y(f) : M_Y(f) \to Y$.

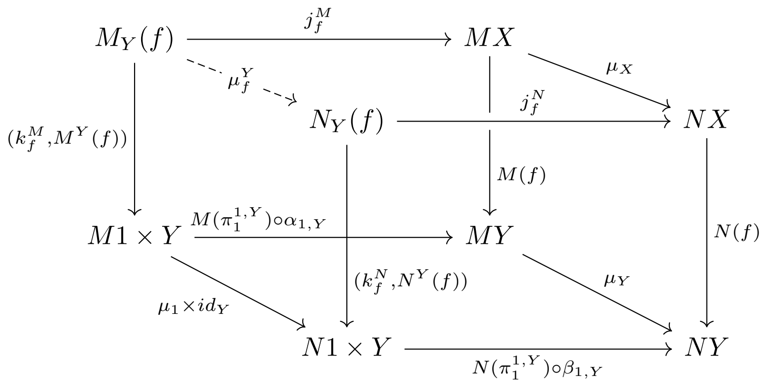
   - **Arrows**: Let $\begin{array}{ccc} X & \xrightarrow{\;h\;} & U \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & Y & \end{array}$ be a commutative triangle in $\mathcal{E}$. By the universal property of the pullback we obtain



     yielding a commutative triangle $\begin{array}{ccc} M_Y(f) & \xrightarrow{\;M^Y(h)\;} & M_Y(g) \\ & {\scriptstyle M_Y(f)}\searrow \quad \swarrow{\scriptstyle M_Y(g)} & \\ & \Gamma & \end{array}$

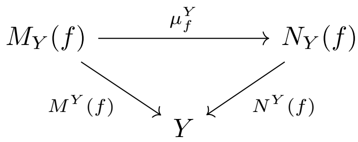
2. Let $\mu : (M, \alpha) \Rightarrow (N, \beta)$ be a strong natural transformation with the underlying functors pullback preserving. Given $f : X \to Y$, we obtain a morphism $\mu_f^Y : M_Y(f) \to N_Y(f)$ by the universal property

of the pullback:



(6.14)

To see this is the case, note that the front and back face commute because they're pullbacks, the right hand face commutes by naturality and the bottom face commutes by strength of μ. Hence from commutativity of the diagram we obtain the commutative triangle



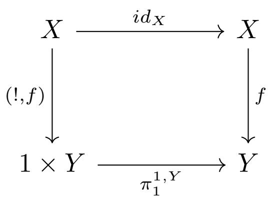
as required.

□

Applying this proposition we see that the data of path object category axiom 1 is indexed. This manifests itself as indexed internal subcategories [vdBG12, Proposition 6.1.2] but we omit these details, instead describing enough of this structure to define the diagonal factorisations.

Proposition 6.14. *The associated cloven weak factorisation system for a path object category $\mathcal{E}$ has a choice of diagonal factorisations.*

Proof. Let $(f, p) : X \to Y$ be a cloven $\mathcal{R}$-map of $\mathcal{E}$. We first observe that the commutative square



is a pullback. Thus plugging the natural transformations s, t and e into the diagram (6.14) for $f : X \to Y$ we obtain components $s_f^Y, t_f^Y : M_Y(f) \to X$ and $e_f^Y : X \to M_Y(f)$ together with the identities

$$s_f^Y = s_X \circ j_f^M \qquad t_f^Y = t_X \circ j_f^Y \qquad e_X = j_f^M \circ e_f^Y$$

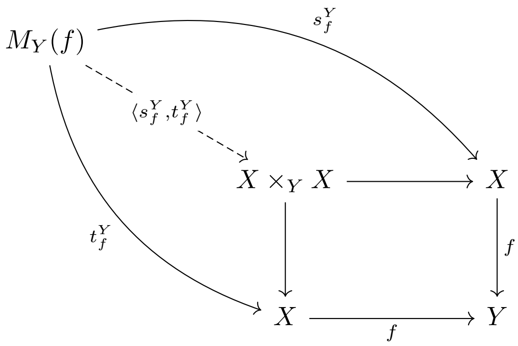
It then follows that

$$s_f^Y \circ e_f^Y = s_X \circ j_f^M \circ e_f^Y = s_X \circ e_X = id_X$$
$$t_f^Y \circ e_f^Y = t_X \circ j_f^M \circ e_f^Y = t_X \circ e_X = id_X$$

We also have, by commutativity of (6.14), the following identity

$$f \circ s_f^Y = M^Y(f) = f \circ t_f^Y$$

inducing a unique map $\langle s_f^Y, t_f^Y \rangle$ from the universal property of the pullback:



It thus follows that the diagonal map $\delta_f : X \to X \times_Y X$ can be factored as

$$X \xrightarrow{e_f^Y} M_Y(f) \xrightarrow{\langle s_f^Y, t_f^Y \rangle} X \times_Y X$$

We omit the (rather lengthy) proof that e_f^Y and $\langle s_f^Y, t_f^Y \rangle$ can be equipped with a cloven $\mathcal{L}$ and $\mathcal{R}$ structure respectively, instead directing the interested reader to [vdBG12, Lemma 6.2.3 - 6.2.4] ☐

To show this choice of diagonal factorisations satisfies the functorality condition we must prove another interesting property of path object categories: functor categories based on path object categories are themselves path object categories. Not only this, but the cloven $\mathcal{L}$ and $\mathcal{R}$-maps of these functor path object categories have a useful characterisation.

Lemma 6.15. *Given a path object category $\mathcal{E}$ and any other category $\mathcal{C}$:*

1. *The functor category $\mathcal{E}^{\mathcal{C}}$ is a path object category.*

2. *The category of $\mathcal{R}$-maps $\boldsymbol{\mathcal{R}}$**- Map**$_{\mathcal{E}^{\mathcal{C}}}$ of the associated cloven w.f.s of $\mathcal{E}^{\mathcal{C}}$ is given by the functor category $\boldsymbol{\mathcal{R}}$**-Map**$_{\mathcal{E}}{}^{\mathcal{C}}$. Similarly, $\boldsymbol{\mathcal{L}}$**-Map**$_{\mathcal{E}^{\mathcal{C}}}$ is given by $\boldsymbol{\mathcal{L}}$**-Map**$_{\mathcal{E}}{}^{\mathcal{C}}$.*

Proof.

1. Let $\mathcal{E}$ be a path object category with pullback preserving endofunctor M, strength α and strong natural transformations s, t, e, c and τ. For an arbitrary category $\mathcal{C}$ we may define the path object structure as follows:

   - Let $F : \mathcal{C} \to \mathcal{E}$ be a functor, $\mu : F \Rightarrow G$ a natural transformation and C an object in $\mathcal{C}$. Then the pullback preserving endofunctor $M^{\to}$ is defined

$$M^{\to}(F) = M \circ F \qquad\qquad M^{\to}(\mu)_C = M(\mu_C)$$

- Let $F : \mathcal{C} \to \mathcal{E}$ be a functor and C an object in $\mathcal{C}$. Then we obtain the internal category

$$F \underset{t_F^{\to}}{\overset{s_F^{\to}}{\leftleftarrows}} \overset{e_F^{\to}}{\to} M^{\to}F \xleftarrow{c_F^{\to}} M^{\to}F \times_F M^{\to}F$$

as follows:

$$(s_F^{\to})_C = s_{F(C)} \qquad (t_F^{\to})_C = t_{F(C)}$$
$$(e_F^{\to})_C = e_{F(C)} \qquad (c_F^{\to})_C = c_{F(C)}$$

whilst we similarly obtain the involution $\tau_F^{\to}$ as $(\tau_F^{\to})_C = \tau_{F(C)}$.

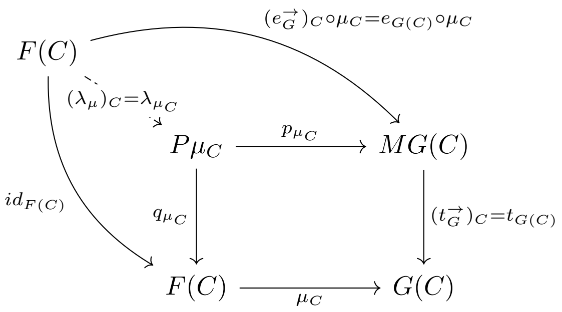
- Let $F, G : \mathcal{C} \to \mathcal{E}$ be functors and C and object in $\mathcal{C}$. Then we obtain the components of the strength $\alpha^{\to}$ as

$$(\alpha_{F,G}^{\to})_C = \alpha_{F(C),G(C)}$$

whilst the contraction natural transformation $\eta^{\to}$ is given by $(\eta_F^{\to})_C = \eta_{F(C)}$.

That this data satisfies the path object category axioms can be computed pointwise, but we immediately see that each property is inherited from $\mathcal{E}$'s path object category structure. We leave this simple verification to the reader.

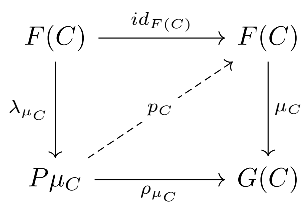
2. We restrict attention to $\mathcal{R}$-maps since the proof for $\mathcal{L}$-maps is dual. First assume $\mu : F \Rightarrow G$ is a natural transformation between functors $F : \mathcal{C} \to \mathcal{E}$. Looking at the factorisation of μ pointwise it is straightforward to verify that $(\lambda_\mu)_C = \lambda_{\mu_C}$ and $(\rho_\mu)_C = \rho_{\mu_C}$: by the definition of the internal category maps for $\mathcal{E}^{\mathcal{C}}$, we define $(\lambda_\mu)_C$ by the pullback



and we then have

$$(\rho_\mu)_C = (s_G^{\to})_C \circ p_{\mu_C} = s_{G(C)} \circ p_{\mu_C} = \rho_{\mu_C}$$

We can give a functor $P : \mathcal{C} \to \mathcal{E}$ by sending objects C to $P\mu_C$ and arrows f to $P(Ff, Gf)$ - that this is functorial follows from the fact that the assignment $(h, k) \mapsto P(h, k)$ is. Thus a cloven $\mathcal{R}$-map structure p on μ is a natural transformation $p : P \Rightarrow F$ comprised of components $p_C : P\mu_C \to F(C)$ making the following diagrams commute



In other words, each p_C gives a cloven $\mathcal{R}$-map structure for μ_C. We thus have an assignment $C \mapsto (\mu_C, p_C)$ for each cloven $\mathcal{R}$-map (μ, p) of $\mathcal{E}^{\mathcal{C}}$. To see that this can be extended to a functor $\mathcal{C} \to \boldsymbol{\mathcal{R}}\text{-}\mathbf{Map}_{\boldsymbol{\mathcal{E}}}$, let $f : C \to C'$ be a morphism in $\mathcal{C}$. By naturality we have the commutative square

$$\begin{array}{ccc} F(C) & \xrightarrow{Ff} & F(C') \\ {\scriptstyle \mu_C}\downarrow & & \downarrow{\scriptstyle \mu_{C'}} \\ G(C) & \xrightarrow[Gf]{} & G(C') \end{array}$$

and for this to give the data for a morphism of $\mathcal{R}$-maps we require commutativity of the diagram

$$\begin{array}{ccc} P\mu_C & \xrightarrow{P(Ff,Gf)} & P\mu_{C'} \\ {\scriptstyle p_C}\downarrow & & \downarrow{\scriptstyle p_{C'}} \\ F(C) & \xrightarrow[Ff]{} & F(C') \end{array}$$

but this is precisely the statement of p's naturality. Similarly we can take any functor $X : C \to \boldsymbol{\mathcal{R}}\text{-}\mathbf{Map}_{\boldsymbol{\mathcal{E}}}$ and induce an $\mathcal{R}$-map of $\mathcal{E}^{\mathcal{C}}$. For C in $\mathcal{C}$ we have $X(C) = (X(C)_0, X(C)_1)$ where $X(C)_0$ is a morphism in $\mathcal{E}$ and $X(C)_1$ a cloven $\mathcal{R}$-map structure upon it. Similarly for $f : C \to C'$ we have $X(f) = (X(f)_0, X(f)_1)$ where

$$\begin{array}{ccc} dom(X(C)_0) & \xrightarrow{X(f)_0} & dom(X(C')_0) \\ {\scriptstyle X(C)_0}\downarrow & & \downarrow{\scriptstyle X(C')_1} \\ cod(X(C)_0) & \xrightarrow[X(f)_1]{} & cod(X(C')_0) \end{array}$$

gives a map of $\mathcal{R}$-maps $X(C) \to X(C')$. Then we can obtain a cloven $\mathcal{R}$-map of $\mathcal{E}^{\mathcal{C}}$ as follows. Define functors $F, G : \mathcal{C} \to \mathcal{E}$ by

$$F(C) = dom(X(C)_0) \qquad F(f) = X(f)_0$$
$$G(C) = cod(X(C)_0) \qquad G(f) = X(f)_1$$

Functorality of these definitions follows immediately from functorality of X. We then have a natural transformations $\mu : F \Rightarrow G$ and $p : P\mu_{(-)} \Rightarrow F$ given by $\mu_C = X(C)_0$ and $p_C = X(C)_1$ respectively. It is easy to see these are well defined by using the fact $X(f)$ is a morphism of $\mathcal{R}$-maps, and we can use the same property to show p is a cloven $\mathcal{R}$-structure for μ. This correspondence forms an equivalence of categories, and we leave the remaining details to the reader.

□

Proposition 6.16. *The choice of diagonal factorisations given in Proposition 6.14 is functorial and stable.*

Proof. By the previous lemma we have that, given a path object category $\mathcal{E}$ and any category $\mathcal{C}$, the functor category $\mathcal{E}^{\mathcal{C}}$ is a path object category with the corresponding category of $\mathcal{R}$-maps given by $\boldsymbol{\mathcal{R}}\text{-}\ \mathbf{Map}_{\boldsymbol{\mathcal{E}}}{}^{\mathcal{C}}$. We thus consider the case for $\mathcal{C} = \boldsymbol{\mathcal{R}}\textbf{-Map}_{\boldsymbol{\mathcal{E}}}$. In particular, we have the "generic" $\mathcal{R}$-map corresponding to the identity functor $\boldsymbol{\mathcal{R}}\textbf{-Map}_{\boldsymbol{\mathcal{E}}} \to \boldsymbol{\mathcal{R}}\textbf{-Map}_{\boldsymbol{\mathcal{E}}}$: a cloven $\mathcal{R}$-map of $\mathcal{E}^{\boldsymbol{\mathcal{R}}\textbf{-Map}_{\boldsymbol{\mathcal{E}}}}$ given by $(\mathfrak{x}, \mathfrak{p})$ where

$$\begin{aligned} \mathbf{dom}(x,p) &= dom(x) & \qquad \mathbf{dom}(h,k) &= h \\ \mathbf{cod}(x,p) &= cod(x) & \qquad \mathbf{cod}(h,k) &= k \\ \mathfrak{x} : \mathbf{dom} &\Rightarrow \mathbf{cod} & \qquad \mathfrak{p} : P\mathfrak{x}_{(-)} &\Rightarrow \mathbf{dom} \end{aligned}$$

With the components of $\mathfrak{x}_{(x,p)} = x$ and $\mathfrak{p}_{(x,p)} = p$. Thus applying the construction of Proposition 6.14 to this cloven $\mathcal{R}$-map yields the required functorality. For stability we once again direct the reader to [vdBG12, Proposition 6.2.5]. □

6.2.3 Functorial Frobenius Structure

The last piece of data we require is the functorial Frobenius structure of the associated cloven weak factorisation system. Recall that a cloven weak factorisation system for a category $\mathcal{E}$ has the *Frobenius property* if, given a pullback of a cloven $\mathcal{L}$-map $(i,q) : X \to A$ and a cloven $\mathcal{R}$-map $(f,p) : B \to A$

$$\begin{array}{ccc} f^{\star}(X) & \xrightarrow{\;\overline{f}\;} & X \\ {\scriptstyle \overline{i}}\downarrow & & \downarrow{\scriptstyle i} \\ B & \xrightarrow[\;f\;]{} & A \end{array}$$

the map $\overline{i}$ can be equipped with a cloven $\mathcal{L}$-map structure. We say this is *functorial* if the assignment extends to a functor $\boldsymbol{\mathcal{R}}\textbf{- Map}_{\boldsymbol{\mathcal{E}}} \times_{\mathcal{E}} \boldsymbol{\mathcal{L}}\textbf{-Map}_{\boldsymbol{\mathcal{E}}} \to \boldsymbol{\mathcal{L}}\textbf{-Map}_{\boldsymbol{\mathcal{E}}}$.

Before we attend to this we prove a final useful lemma. Recall that we gave a characterisation of maps equipable with a cloven $\mathcal{L}$-map structure as those that have a strong deformation retract. We can give a similar criterion for cloven $\mathcal{R}$-map structures by way of a *path lifting property*: a cloven $\mathcal{R}$-map structure p on a morphism $f : X \to Y$ corresponds precisely to an operation that lifts V-parametrised paths $\phi : V \to MY$ to paths in X.

Lemma 6.17. *Given a path object category $\mathcal{E}$ and a morphism $f : X \to Y$, cloven $\mathcal{R}$-map structures on f in the associated cloven w.f.s are in bijective correspondence with operations which to every morphism $x : V \to Y$ and homotopy $\phi : y \Rightarrow fx : V \to Y$ assign a homotopy $\overline{\phi} : \phi^{\star}(x) \Rightarrow x$ such that*

i) $f.\overline{\phi} = \phi$;

ii) $\overline{\phi}$ is an identity homotopy whenever ϕ is;

iii) For any map $h : W \to V$: $(\phi h)^{\star}(xh) = \phi^{\star}(x)h$ and $\overline{\phi h} = \overline{\phi}.h$

Proof. We first note that by the embedding given in the Yoneda lemma, equipping $f : X \to Y$ with a cloven $\mathcal{R}$-map structure $p : Pf \to X$ corresponds precisely to giving a natural family of maps $\mathcal{E}(V,p) : \mathcal{E}(V, Pf) \to$

$\mathcal{E}(V, X)$ making all diagrams of the form

$$\begin{array}{ccc} \mathcal{E}(V,X) & \xrightarrow{id} & \mathcal{E}(V,X) \\ {\scriptstyle \mathcal{E}(V,\lambda_f)}\downarrow & \overset{\mathcal{E}(V,p)}{\nearrow} & \downarrow{\scriptstyle \mathcal{E}(V,f)} \\ \mathcal{E}(V,Pf) & \xrightarrow[\mathcal{E}(V,\rho_f)]{} & \mathcal{E}(V,Y) \end{array} \tag{6.15}$$

Now, since Pf is the vertex of a pullback, maps $g : V \to Pf$ are in bijective correspondence to commutative squares of the form

$$\begin{array}{ccc} V & \xrightarrow{\phi} & MY \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle t_Y} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{6.16}$$

Given such a square, we obtain a map $g : V \to Pf$ by the universal property of the pullback. Conversely, if we have $g : V \to Pf$ we can obtain such a square by setting $\phi = p_f \circ g$ and $x = q_f \circ g$. Commutative squares of this sort are themselves in bijective correspondence with homotopies $\phi : y \Rightarrow fx : V \to Y$. Given such a homotopy, the target diagram gives precisely such a commutative square and conversely setting $y = s_Y \circ \phi$ makes $\phi : y \Rightarrow fx$ a homotopy. Hence we claim that giving a natural family of maps $\mathcal{E}(V, p) : \mathcal{E}(V, Pf) \to \mathcal{E}(V, X)$ is equivalent to providing, for every homotopy $\phi : y \Rightarrow fx : V \to Y$, a morphism $\phi^\star(x) : V \to X$ satisfying the following conditions:

1. $f \circ \phi^\star(x) = y$;
2. If ϕ is the identity homotopy $fx \Rightarrow fx$ then $\phi^\star(x) = x$;
3. For any $h : W \to V$ we have $(\phi h)^\star(xh) = \phi^\star(x)h$

First assume we have such a natural family and let $\phi : y \Rightarrow fx$ be a homotopy. From the target diagram we obtain a commutative square (6.16) corresponding to a map $g : V \to Pf$, and we define $\phi^\star(x) = \mathcal{E}(V, p)(g)$. By the commutativity of the lower triangle in (6.15) we have

$$\begin{aligned} f.\phi^\star(x) &= f \circ \mathcal{E}(V, p)(g) = \mathcal{E}(V, f)(\mathcal{E}(V, p)(g)) = \mathcal{E}(V, \rho_f)(g) \\ &= \rho_f \circ g = s_Y \circ p_f \circ g = s_Y \circ \phi = y \end{aligned}$$

hence 1. holds. Assuming ϕ is the identity homotopy $e_Y \circ fx$ with $y = fx$, the morphism induced by the commutative square given in the target diagram is $\lambda_f \circ x$. Thus by commutativity of the upper square in (6.15)

$$\phi^\star(x) = \mathcal{E}(V, p)(\lambda_f \circ x) = \mathcal{E}(V, p)(\mathcal{E}(V, \lambda_f)(x)) = x$$

and so we obtain 2. Finally, letting $h : W \to V$, if g is the morphism induced by the square (6.16), it follows from the universal property of the pullback that the induced map from the commutative square

$$\begin{array}{ccc} W & \xrightarrow{\phi h} & MY \\ {\scriptstyle xh}\downarrow & & \downarrow{\scriptstyle t_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is given by gh. Thus by naturality of the maps $\mathcal{E}(V, p)$ we have

$$(\phi h)^\star(xh) = \mathcal{E}(W, p)(gh) = \mathcal{E}(V, p)(g) \circ h = \phi^\star(x)h$$

and so the third criterion is satisfied. In the other direction we assume we have such an operation. A morphism $g : V \to Pf$ corresponds to the homotopy $p_f g : s_Y p_f g \Rightarrow q_f g$, hence we have a family of maps defined $\mathcal{E}(V, (p_f)^\star(q_f))(g) = (p_f)^\star(q_f)g = (p_f g)^\star(q_f g)$. To see that this renders the requisite diagrams commutative, first consider $x : V \to X$. By condition 1. we have

$$\mathcal{E}(V, (p_f)^\star(q_f)) \circ \mathcal{E}(V, \lambda_f)(x) = (p_f \lambda_f x)^\star(q_f \lambda_f x) = (e_Y \circ fx)^\star(x) = x$$

giving us commutativity of the upper triangle. For the lower triangle, consider $g : V \to Pf$. By condition 2. we have

$$\mathcal{E}(V, f) \circ \mathcal{E}(V, (p_f)^\star(q_f))(g) = f.(p_f g)^\star(q_f g) = s_Y p_f g = \rho_f g = \mathcal{E}(V, \rho_f)(g)$$

hence the diagram commutes. Naturality follows immediately from our definition and condition 3. That these constructions are inverse to each other can be straightforwardly verified.

We can now strengthen this characterisation to the statement of the lemma. One direction is obvious, so suppose we have a cloven $\mathcal{R}$-map structure on $f : X \to Y$. From our previous work this corresponds uniquely to an assignment $\phi^\star(x) : V \to X$ for every homotopy $\phi : y \Rightarrow fx : V \to Y$ that satisfies properties 1-3. So let $\phi : y \Rightarrow fx$ be a homotopy, corresponding uniquely to a morphism $g : V \to Pf$. Recall in Lemma 6.8 that we had a morphism $\theta_f : Pf \to MPf$ defined from the diagram (6.11) to give the strong deformation retraction for λ_f. As it is a homotopy $\theta_f : id_{Pf} \Rightarrow \lambda_f \circ q_f$ we have

$$s_{Pf} \circ \theta_f = id_{Pf} \qquad\qquad t_{Pf} \circ \theta_f = \lambda_f \circ q_f$$

We thus define a morphism $l = M(p) \circ \theta_f$ and claim that $\overline{\phi} = lg$ satisfies the requirements of the lemma. We must first verify that $\overline{\phi}$ is a homotopy $\phi^\star(x) \Rightarrow x$. We have, by naturality of s together with the identity given above:

$$s_X \circ \overline{\phi} = s_X \circ M(p) \circ \theta_f \circ g = p \circ s_{Pf} \circ \theta_f \circ g = pg = \mathcal{E}(V, p)(g) = \phi^\star(x)$$

Similarly, by the naturality of t, the identity given above and the fact p is a cloven $\mathcal{R}$-map structure on f we have:

$$t_X \circ \overline{\phi} = t_X \circ M(p) \circ \theta_f \circ g = p \circ t_{Pf} \circ \theta_f \circ g = p \circ \lambda_f \circ q_f \circ g = q_f \circ g = x$$

We show $i - iii)$ are satisfied. First we require $f.\overline{\phi} = \phi$. That is, $M(f) \circ \overline{\phi} = \phi$. Using the fact p is a cloven $\mathcal{R}$-map structure together with the diagram (6.11) is easy to compute that $M(f) \circ l = p_f$. Hence $M(f) \circ \overline{\phi} = p_f \circ g = \phi$ as required. Next, we assume $\phi = e_Y \circ fx$. Once again this entails that the corresponding morphism $g : V \to Pf$ is given by $\lambda_f \circ x$. Now since θ_f is a strong deformation retraction for λ_f, by triviality on X we have $\theta_f \circ \lambda_f = e_{Pf} \circ \lambda_f$. Hence

$$\overline{\phi} = l\lambda_f x = M(p) \circ \theta_f \circ \lambda_f x = M(p) \circ e_{Pf} \circ \lambda_f x = e_X \circ p \circ \lambda_f x = e_X \circ x$$

That is, $\overline{\phi}$ is the identity homotopy for x. Finally we must show, given $h : V \to W$, that $\overline{\phi h} = \overline{\phi}h$ - but this is clearly true. Note that if g is the morphism corresponding to $\phi : y \Rightarrow fx$ then gh corresponds to $\phi h : yh \Rightarrow fxh$. Hence $\overline{\phi h} = lgh = \overline{\phi}h$, and we're done. □

Note the similarity here with the homotopy lifting property that characterises the fibrations of the model structure on **Top**. Recall that in the type category constructed from a model category, types are interpreted as fibrations, whilst in the homotopy theoretic model types are interpreted as cloven $\mathcal{R}$-maps. In this way we can draw a connection between the two approaches: despite the abstraction we're back in a similar situation to our starting point. We use this lifting property to prove the associated cloven weak factorisation system is functorially Frobenius.

Proposition 6.18. *Given a path object category $\mathcal{E}$, the associated cloven weak factorisation system is functorially Frobenius.*

Proof. Let $(f, p) : B \to A$ be a cloven $\mathcal{R}$-map, $(i, q) : X \to A$ a cloven $\mathcal{L}$-map, and consider the pullback square

$$\begin{array}{ccc} f^\star X & \xrightarrow{\overline{f}} & X \\ {\scriptstyle \overline{i}}\downarrow & & \downarrow{\scriptstyle i} \\ B & \xrightarrow[f]{} & A \end{array}$$

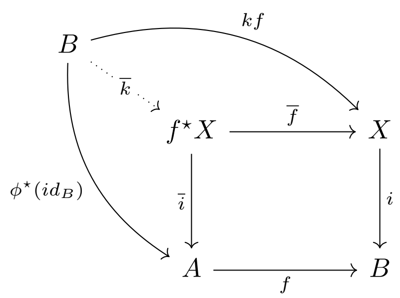
By lemma 6.7 the Frobenius property is satisfied if we can give a strong deformation retraction for $\overline{i}$. This lemma also tells us that we have a strong deformation retraction (θ, k) for i, whilst the previous lemma gives us that we have an operation uniquely corresponding to p assigning to every homotopy $\phi : y \Rightarrow x : V \to A$ a homotopy $\overline{\phi} : \phi(x) \Rightarrow x$ such that

i) $f.\overline{\phi} = \phi$;

ii) If ϕ is an identity homotopy then $\overline{\phi}$ is;

iii) For any $h : W \to V$ we have $(\phi h)^\star(xh) = \phi^\star(x)h$ and $\overline{\phi h} = \overline{\phi}.h$

Hence we consider $\phi = (\theta f)^\circ : ikf \Rightarrow f : B \to A$ and apply the path lifting property. We have a homotopy $\overline{\phi}(id_B) : \phi^\star(id_B) \Rightarrow id_B$ such that $f.\overline{\phi} = \phi$, which in turn necessitates that $f\phi^\star(id_B) = ikf$. Hence by the universal property of the pullback we have a map:



We thus define $\overline{\theta} = (\overline{\phi})^\circ : id_B \Rightarrow \phi^\star(id_B)$. By commutativity of the pullback diagram we have $\overline{i}\overline{k} = \phi^\star(id_B)$ so we have satisfied the homotopy criterion for a strong deformation retract. To see that $\overline{k}$ is a retract of $\overline{i}$ it suffices to show identity upon post composition with the pullback projections $\overline{f}$ and $\overline{i}$, and this is straightforward using the commutativity of the diagram and the conditions for the path lifting property. Finally we require $\overline{\theta} \circ \overline{i} = id_{\overline{i}} = e_A \circ \overline{i}$. By applying the reversal operation $(-)^\circ$ to both sides this is equivalent to showing that $\overline{\phi} \circ \overline{i} = id_{\overline{i}}$. By the third condition of the path lifting property we have $\overline{\phi} \circ \overline{i} = \overline{\phi\overline{i}}$ and utilising the commutativity of the pullback square we have

$$\phi\overline{i} = (\theta f)^\circ \overline{i} = (\theta f \overline{i})^\circ = (\theta i \overline{f})^\circ = (id_{i\overline{f}})^\circ = id_{i\overline{f}}$$

Hence applying the second condition of the path lifting property gives us that $\overline{\phi\overline{i}} = id_{\overline{i}}$ as required: $(\overline{\theta}, \overline{k})$ is a strong deformation retract and so $\overline{i}$ can be equipped with a cloven $\mathcal{L}$-map structure.

To obtain functorality, we simply invoke Lemma 6.15 and use a similar argument to that of Proposition 6.16, this time considering the case where $\mathcal{C} = \boldsymbol{\mathcal{R}}\textbf{-Map}_{\boldsymbol{\mathcal{E}}} \times_{\mathcal{E}} \boldsymbol{\mathcal{L}}\textbf{-Map}_{\boldsymbol{\mathcal{E}}}$. As this is a pullback it comes equipped with two projection functors, one into $\boldsymbol{\mathcal{L}}\textbf{-Map}_{\boldsymbol{\mathcal{E}}}$ and one into $\boldsymbol{\mathcal{R}}\textbf{-Map}_{\boldsymbol{\mathcal{E}}}$. These then correspond to a cloven $\mathcal{L}$ map and a cloven $\mathcal{R}$ map of $\mathcal{E}^{\boldsymbol{\mathcal{R}}\textbf{-Map}_{\boldsymbol{\mathcal{E}}} \times_{\mathcal{E}} \boldsymbol{\mathcal{L}}\textbf{-Map}_{\boldsymbol{\mathcal{E}}}}$ upon which we can apply the preceding argument. This gives functorality and thus completes the proof. □

We can thus summarise the work of this chapter, as well as the thesis.

Theorem 6.19. *Path object categories may be equipped with the structure of a homotopy theoretic model of identity types.*

Corollary 6.20. *For any path object category $\mathcal{E}$ there is a categorical model of identity types with $\mathcal{E}$ as its category of contexts.*

Corollary 6.21. *There is a categorical model of identity types with $\boldsymbol{cSet^c}$ as its category of contexts.*

Chapter 7

Conclusions And Further Work

In this thesis we established a path object category structure on the category of cubical sets $\boldsymbol{cSet^c}$ and utilising the general construction given in Part II from this we were able to give a categorical model of identity types with $\boldsymbol{cSet^c}$ as its category of contexts. In the process we introduced the new notion of a *nice path object category*, a simplification of the original path object category axioms that nonetheless produces the full path object category structure. Our original motivation in pursuing this construction was to bring the cubical set model of [BCH14] into the general framework given in [vdBG12] but our model has some key differences from the one presented there. Most apparent is the presence of connections, additional structure that the authors of [BCH14] do not make use of in their construction. It is still an open question as to whether the category **cSet** of cubical sets has a path object category structure, and the essential use of connections for giving path contraction in our proof shows that a very different notion of path would be required if it is indeed the case. The other distinction is that in our model there is no restriction on which cubical sets may be interpreted as contexts. This mirrors the situation with the model of simplicial sets developed in [vdBG12] which is distinguished from the model given in [Str14] in precisely the same way. We now highlight some possible directions for future research in this area:

1. *Eliminating The Equivalence Relation*: One question we might ask is whether we give a nice path object category structure on $\boldsymbol{cSet^c}$ without using an equivalence relation. If we are to keep the same notion of path then we have $M1 \ncong 1$ and so we will not be able to give an instantiation of a nice path object category structure. We might, however, be able to give a regular path object category structure for $\boldsymbol{cSet^c}$. Recall that the terminal cubical set 1 was defined by

   - For all $n \in \omega$: $1_n = \{n\}$
   - For all $1 \leq i \leq n$ and $\delta \in \{-1, 1\}$: $f_n^{i,\delta}(n) = n - 1$
   - For all $1 \leq i \leq n$: $d_n^i(n-1) = n$
   - For all $n \geq 2$, $1 \leq i \leq n-1$ and $\delta \in \{-1, 1\}$: $\Gamma_{n-1}^{i,\delta}(n-1) = n$

   and thus all cubical n-paths through 1 are of the form

$$\Big((n, \dots, n), (n+1, \dots, n+1)\left(\vec{\epsilon}\right)\Big)$$

We might define a strength that takes such a path in which n occurs k times with orientations $\vec{\epsilon}$ together with an n-cube ζ and returns the n-path through X given by $\Big((\zeta,\ldots,\zeta),(d^1_n+1(\zeta),\ldots,d^1_{n+1}(\zeta)),(\vec{\epsilon})\Big)$. This is the only feasible definition of constant cubical n-path and with this practically all of our original argument goes through with the contraction identity $t_{MX}\circ\eta_X = e_X\circ t_X$ as a notable exception: can a modification be made to solve this issue?

2. *Creating Equivalence Relations*: In the other direction, we might look to existing path object categories and see if their proofs can be simplified - or if new, more economical nice path object category structures can be identified - by quotienting by an appropriate equivalence relation . For one example, recall that the path object category structure on **Top** was given by taking MX to be the Moore path space given by

$$\{(r,\phi)\in\mathbb{R}_+\times X^{\mathbb{R}_+}\mid \forall s\geq r(\phi(s)=\phi(r))\}$$

together with the subspace topology inherited from the usual topology on $\mathbb{R}_+\times X^{\mathbb{R}_+}$ Suppose instead that we take the quotient space given by the following equivalence relation on Moore path spaces: for a topological space X and $x\in X$ define $con^X_x:\mathbb{R}^+\to X$ as $con^X_x(r)=x$ for all r. We thus define the equivalence relation on Moore paths generated by

$$(k,con^X_x)\sim(l,con^X_x)$$

for all $k,l\in\mathbb{R}^+$. It then follows immediately that $M1\cong 1$ and this also solves the issue we highlighted in the development of path object category axiom 2. It only remains to show that the existing structure operates appropriately with the equivalence relation and remains continuous with respect to the quotient topology. Perhaps a more interesting propsect is the case for simplicial sets. Although our proposal here is only a slight improvement over the original proof for topological spaces, there may be scope for greatly simplifying in the case of simplicial sets, as the original proof [vdBG12, Section 7.] is extremely complex. One option would be to use a more restricted notion of path in the same manner as the cubical sets with connections proof, perhaps by only considering paths comprised of simplices joined in a restricted number of orientations.

3. *Identifying Path Object Categories*: Perhaps the most obvious direction for future work is the identification of (nice) path object category structures in other categories. As in this thesis, we can look to existing models of intensional type theory to direct our work. Alternatively, we might look to existing path object category structures as inspiration. We give two suggestions encompassing each of these approaches.

   (a) *The Category of Categories **Cat***: In [Lam14] LaMarche gives a model of intensional Martin-Löf type theory in **Cat**. In order to do so he defines a notion of path together with a *path endofunctor* that associates to each category X a category comprised of *paths through* X. The similarity with the path object category structure should be immediately apparent, and it would be interesting to investigate whether this assignment can be brought into the path object category framework.

   (b) *The Category of Equilogical Spaces*: An *equilogical space* is a T_0 topological space together with an (arbitrary) equivalence relation $\sim$ on its points. It comes equipped with a notion of morphism that yields a category: a morphism of equilogical spaces $f:(X,\sim)\to(Y,\equiv)$ is a continuous function $f:X\to Y$ such that $x\sim y$ implies $f(x)\equiv f(y)$. In [BBS04] an interpretation of type theory was given in equilogical spaces, but this type theory did not include identity types. Might a clever choice of equivalence relation allow us to apply something analogous to the path object category structure construction of **Top** to the category of equilogical spaces?

4. *Expanding The Fragment*: The final direction we give for future work is something we wished to achieve in this thesis but unfortunately time constraints did not allow us to pursue it. Given that we have produced a model of a fragment of Martin-Löf type theory, it would be of interest to expand the construction to model the whole language. In theory this should be straightforward, as the modelling of identity types is the genuinely difficult part of the endeavour and we already obtain sigma types by the closure of right maps under composition.

Appendix A

Category Theory

In this appendix we give the necessary category theoretic preliminaries to read the thesis. This is not intended to be a comprehensive introduction to any of the topics presented but instead an opportunity to fix notation and nomenclature, as well as guide the reader towards the concepts she will require. For a thorough introduction to category theory we recommend [Awo06], [Lan71] and [vO95], upon which this brief presentation is based.

A.1 Basics

A *category* $\mathcal{C}$ is a collection of *objects* $\mathcal{C}_0$ and a collection of *arrows* $\mathcal{C}_1$ together with maps $dom, cod : C_1 \to C_0$ assigning to each arrow f a domain $dom(f)$ and codomain $cod(f)$ in C_0, such that the following axioms are satisfied:

- **Composition**: Given arrows f, g with $cod(f) = dom(g)$ there exists a unique arrow

$$g \circ f : dom(f) \to cod(g)$$

- **Associativity**: Given arrows f, g, h with $cod(f) = dom(g)$ and $cod(g) = dom(h)$ we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

- **Identity**: For every object X there exists an identity arrow id_X such that, given arrows f, g with $dom(f) = X = cod(g)$ we have

$$f \circ id_X = f \qquad\qquad id_X \circ g = g$$

In this thesis we denote arbitrary objects by upper case letters $A, B, C, \ldots, X, Y, Z$ and arbitrary morphisms by f, g, h. We will also refer to arrows interchangeably as morphisms, and use $f : X \to Y$ or $X \xrightarrow{f} Y$ to denote that $dom(f) = X$ and $cod(f) = Y$. For arbitrary categories we use calligraphic capital letters: $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$. For concrete categories of mathematical structures we use boldface: for example we have the category of sets and functions **Set** and the category of topological spaces and continuous functions **Top**. Given any category $\mathcal{C}$ we can obtain the *dual* or *opposite* category $\mathcal{C}^{op}$ by formally reversing the direction of arrows. By passing to the opposite category we can *dualise* many results and definitions.

We can distinguish some special species of arrow that generalise familiar set theoretic properties of functions. We call an arrow $f : X \to Y$ a *monomorphism* if it is *left cancellable*: for any pair of arrows $g, h : Z \to X$ such that $fg = fh$ we have $g = h$. Dually we have the notion of an *epimorphism*; a *right cancellable* morphism. If we have arrows $f : X \to Y$ and $g : Y \to X$ such that $gf = id_X$ we call f a *section* of g and g a *retraction* of f. An isomorphism $f : X \to Y$ is an invertible arrow: that is, there exists an arrow $g : Y \to X$ such that $gf = id_X$ and $fg = id_Y$. We can also give a generalisation of singleton and empty set. A terminal object 1 in $\mathcal{C}$ is an object such that for any other object A of $\mathcal{C}$ there exists a unique arrow $!_A : A \to 1$. Dually we have the notion of *initial object*.

The appropriate notion of morphism between categories is called a *functor*. Given categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ is given by a pair of maps $F_0 : C_0 \to D_0$ and $F_1 : C_1 \to C_1$ that interacts coherently with the category structure:

- For all arrows $f : X \to Y$, $F_1(f) : F_0(X) \to F_1(Y)$
- For all objects X, $F_1(id_X) = id_{F_0(X)}$
- For all arrows $f : X \to Y$, $g : Y \to Z$, $F_1(g \circ f) = F_1(g) \circ F_1(f)$

Where the context is clear we drop the subscript and simply refer to both actions of the functor as F. A functor that is injective on arrows is called *faithful*, whilst a functor that is surjective on arrows is called *full*. If a functor is full and faithful and injective on objects we call it an *embedding*.

Note that the composition of functors is again a functor and there also exists, for every category $\mathcal{C}$, an *identity* functor $id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ that given by taking the identity on both objects and arrows. Thus the collection of categories together with the functors between them forms a category **Cat**.

One important example of a functor is given by the hom-set functors. Given objects X, Y in a category we denote the collection of all arrows $X \to Y$ by $\mathcal{C}(X, Y)$. If this collection is set-sized we call it a *hom-set*, and if all such collections are set-sized we call the category *locally small*. If the collection of objects $\mathcal{C}_0$ is also a set we call the category *small*. In both cases we then have emphhom-set functors for each object C: $\mathcal{C}(-, C) : \mathcal{C}^{op} \to \mathbf{Set}$ and $\mathcal{C}(C, -) : \mathcal{C} \to \mathbf{Set}$. These send an object D to the hom-sets $\mathcal{C}(D, C)$ and $\mathcal{C}(C, D)$ respectively, whilst for arrows we have

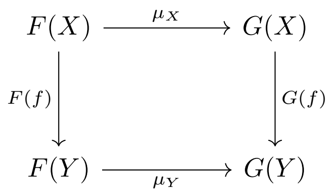
$$\mathcal{C}(f : D \to D', C)(g : D' \to C) = gf : D \to C$$
$$\mathcal{C}(C, f : D \to D')(g : C \to D) = fg : C \to D'$$

Another we make use of is the *constant functor*: for an object D of $\mathcal{D}$, $\Delta_D : \mathcal{C} \to \mathcal{D}$ is defined by sending all objects to D and all arrows to id_D.

Next we give a notion of morphism between functors. Given functors $F, G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\mu : F \Rightarrow G$ is given by a collection of $\mathcal{C}$-indexed maps

$$(\mu_X : F(X) \to G(X) \mid X \text{ in } \mathcal{C}_0)$$

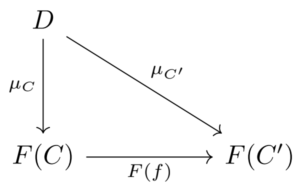
satisfying the following *naturality* condition: given any arrow $f : X \to Y$, the following diagram commutes:



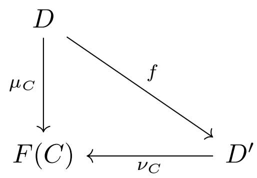
We denote arbitrary natural transformations by greek letters μ, ν, η. A *natural isomorphism* is a natural transformation $\mu : F \Rightarrow G$ such that every component is an isomorphism: we say F and G are naturally isomorphic. With this we can give the correct notion of equivalence for categories: an *equivalence of categories* between $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that there exist natural isomorphisms $F \circ G \simeq id_{\mathcal{D}}$ and $G \circ F \simeq id_{\mathcal{C}}$.

A.2 Limits

A *diagram* in $\mathcal{D}$ of type $\mathcal{C}$ is a functor $F : \mathcal{C} \to \mathcal{D}$. We call $\mathcal{C}$ the *index* of the diagram. Given a diagram $F : \mathcal{C} \to \mathcal{D}$, we define a *cone for* F to be a pair (D, μ), consisting of an object D of $\mathcal{D}$ (the *vertex* of the cone) together with a natural transformation $\mu : \Delta_D \Rightarrow F$. This gives, for every $f : C \to C'$ in $\mathcal{C}$ a commutative triangle



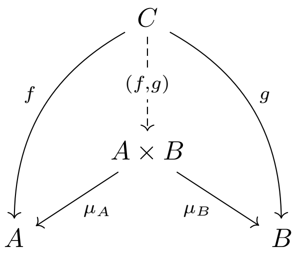
Cones for a diagram F also come with a notion of morphism yielding a category $Cone(F)$. A *map of cones* $f : (D, \mu) \to (D', \nu)$ is given by a morphism $f : D \to D'$ in $\mathcal{D}$ such that, for all C in $\mathcal{C}$ we have a commutative triangle



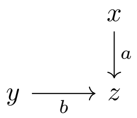
In the category $Cone(F)$ identities and composition of morphisms coincide with that of $\mathcal{D}$ and it is a straightforward exercise to verify that this satisfies the category axioms. We can now give the definition of a *limit* (or *limiting cone*): a limit for a diagram $F : \mathcal{C} \to \mathcal{D}$ is a terminal object in the category $Cone(F)$. We say $\mathcal{D}$ has limits of type $\mathcal{C}$ if limiting cones exist for every diagram with index $\mathcal{C}$. While this definition may seem intimidatingly abstract, it allows us to bring a whole swathe of interesting mathematical phenomena into a single framework. Not only this, using the fact that limits are terminal objects allows us to uniquely define morphisms whenever we have a cone for the relevant diagram. We demonstrate this idea with the following two examples, which we choose for their relevance to this thesis.

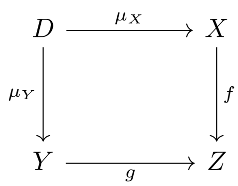
1. *Binary Products*: We first give the category theoretic analogue of the Cartesian product. Denote by $\mathbf{2}$ the discrete category on the set $\{0, 1\}$. A diagram F: $\mathbf{2} \to \mathcal{D}$ picks out two objects of the target category - say A and B - and we thus see that a limit for F is given by a pair $(A \times B, (\mu_A, \mu_B))$ with

the following universal property: for any object C together with arrows $f : C \to A$ and $g : C \to B$ there exists a *unique* morphism $(f, g) : C \to A \times B$ rendering the following diagram commutative:

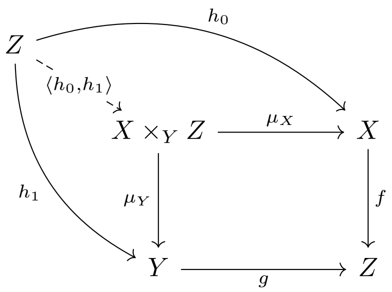


We call $A \times B$ the *binary product* of A and B. Throughout this thesis we use the notation $\mu_A = \pi_0^{A,B}$ and $\mu_B = \pi_1^{A,B}$ and call these morphisms the *projection maps* of the product $A \times B$.

2. *Pullbacks* Next we see a categorical analogue of both set theoretic intersection and inverse image. Denote by **J** the category  . A diagram $F : \mathbf{J} \to \mathcal{D}$ picks out a pair of arrows $f : X \to Z$ and $g : Y \to Z$ and a cone (D, μ) for such a functor gives a commutative square



A limit for this cone is thus given by a pair $(X \times_Z Y, (\mu_X, \mu_Y))$ such that $f \circ \mu_X = g \circ \mu_Y$ with the following universal property: given any other object Z with morphisms $h_0 : Z \to X$ and $h_1 : Z \to Y$ satisfying $fh_0 = gh_1$, there exists a unique morphism $\langle h_0, h_1 \rangle : Z \to X \times_Z Y$ rendering the following diagram commutative:

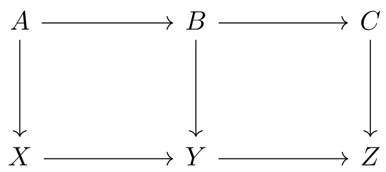


We call μ_X and μ_Y the *projection maps* of the *pullback* $X \times_Y Z$. As we will so frequently use pullbacks to define both objects and morphisms it suits us to keep the notation variable; instead we always make clear from the context that the diagrams in question are pullbacks.

Note that the vertices of limits are defined up to natural isomorphism, as they are obtained via terminal objects. Even so, in many cases we have a canonical choice of vertex. For example in the category **Set** the product of sets A and B is given by the Cartesian product $A \times B = \{(a, b) \mid a \in A, b \in B\}$. Staying in **Set**, the pullback of functions $f : X \to Z$ and $g : Y \to Z$ is given by $\{(x, y) \in X \times Y \mid f(x) = g(y)\}$. If, like **Set**, limits exist for all diagrams of type $\mathcal{C}$ where $\mathcal{C}$ is a small, discrete category we say the category *has products*; analogously we say a category *has pullbacks* if limits exist for all diagrams of type **J**. More generally, if a category $\mathcal{D}$ has all limits of type $\mathcal{C}$ for all small categories $\mathcal{C}$ then we call it *complete*. Letting $G : \mathcal{D} \to \mathcal{E}$ be a functor we say G *preserves limits* of type $\mathcal{C}$ if for every diagram $F : \mathcal{C} \to \mathcal{D}$ with limit (D, μ) we have that $(G(D), G(\mu))$ is a limit for the diagram $GF : \mathcal{C} \to \mathcal{E}$.

There are a few useful lemmas pertaining to products and pullbacks that we will make use of throughout the thesis:

Lemma A.1 (Pasting Lemma)**.** *Let*

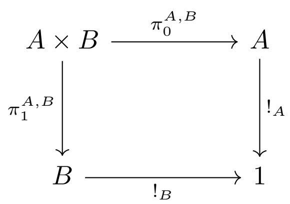


be a commutative diagram. If the right hand square is a pullback then the outer rectangle is a pullback iff the left hand square is a pullback.

Proof. Diagram chase. □

Lemma A.2. *If a category $\mathcal{C}$ has terminal object 1 and pullbacks then it has products.*

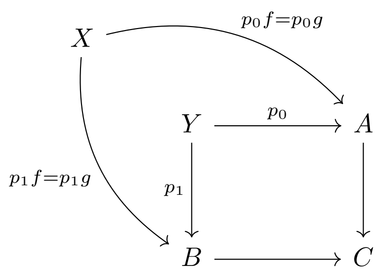
Proof. We obtain the binary product $A \times B$ from the pullback



It is straightforward to verify that this satisfies the universal property of the product. We can then define n-ary products for $n \geq 3$ by a straightforward inductive argument. □

Lemma A.3. *Given $f, g : X \to Y$ where Y is a pullback with projections p_0, p_1 we have $f = g$ iff $p_i f = p_i g$ for $i \in \{0, 1\}$.*

Proof. The left-to-right direction is immediate so we assume $p_i f = p_i g$ for $i \in \{0, 1\}$. Then since both f and g suffice to complete the diagram



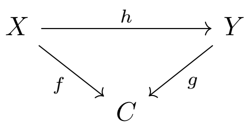
and render it commutative, by the universal property of the pullback necessarily $f = g$. □

The analogous result holds for products. Though we make no use of them in this thesis it is worth noting that by passing to the opposite category we can dualise all of the notions of this section to give *colimits*. A category with all small colimits is called *cocomplete* and a category with all small limits *and* colimits is called *bicomplete*.

A.3 Categorical Constructions

In this final section of our category theory rundown we give some categorical constructions that yield new categories from old. Again we restrict attention to those we make use of in the course of the thesis.

1. *Slice Category*: Given a category $\mathcal{C}$ and an object C in $\mathcal{C}$ we can define the *slice category* $\mathcal{C}/C$ as follows: as objects we take arrows in $\mathcal{C}$ with codomain C; as arrows we take commutative triangles of the form

   Composition and identity are inherited from $\mathcal{C}$.

2. *Quotient Category*: We define a *congruence relation* on a category $\mathcal{C}$ by specifying for each pair of objects X, Y an equivalence relation $\sim_{X,Y}$ on $\mathcal{C}(X, Y)$ such that

   - For $f, g : X \to Y$ and $h : Y \to Z$ if $f \sim_{X,Y} g$ then $hf \sim_{X,Z} hg$
   - For $f : X \to Y$ and $g, h : Y \to Z$ if $g \sim_{Y,Z} h$ then $gf \sim_{X,Z} hf$

   Given such a relation on $\mathcal{C}$ we can form the quotient category $\mathcal{C}/\sim$ by taking the objects to be the same as those of $\mathcal{C}$ and taking equivalence classes of $\mathcal{C}$-morphisms as arrows.

3. *Functor Category* : It is straightforward to verify that the collection of functors $\mathcal{C} \to \mathcal{D}$ together with natural transformations constitute a category: the *functor category* $\mathcal{C}^{\mathcal{D}}$. This has some useful properties. For example, if the category $\mathcal{C}$ has (co)limits of type $\mathcal{E}$ then so too does $\mathcal{C}^{\mathcal{D}}$. Further these limits can be computed *pointwise*: to give a limit for a diagram $F : \mathcal{E} \to \mathcal{C}^{\mathcal{D}}$ it suffices to compute the limit in $\mathcal{C}$ at an arbitrary C. Details of this can be found in [vO95, Section 3.3].

4. *Presheaf Category* We first give one of the most fundamental results in category theory. Given a locally small category $\mathcal{C}$, we can define the *Yoneda Functor* $Y_{(-)} : C \to \mathbf{Set}$. On objects C we have $Y_C = \mathcal{C}(-, C)$, whilst for arrows $f : C \to C'$ we obtain a natural transformation $Y_f : \mathcal{C}(-, C) \Rightarrow \mathcal{C}(-, C')$ given by post-composition by f. We have:

   Lemma A.4. *[Yoneda Lemma] Given a locally small category $\mathcal{C}$, for any object C and any functor $F : \mathcal{C}^{op} \to$ **Set** there is an isomorphism $\mathbf{Set}^{\mathcal{C}^{op}}(Y_C, F) \simeq FC$ natural in F and C.*

   Proof. [Awo06, Lemma 8.2 p 162]

   □

   and from this we obtain:

Corollary A.5. *The Yoneda functor is an embedding.*

Proof. The Yoneda functor is clearly injective on objects, whilst for all objects C, D of $\mathcal{C}$ Lemma A.4 gives

$$\mathcal{C}(C, D) = y_D(C) \simeq \mathbf{Set}^{\mathcal{C}^{op}}(y_C, y_D)$$

□

It follows that the Yoneda embedding gives a representation of any locally small category $\mathcal{C}$ within the category of set valued functors on $\mathcal{C}^{op}$. We call categories of the form $\mathbf{Set}^{\mathcal{C}^{op}}$ *presheaf categories*, and these have some extremely desirable properties. Given that they are functors categories and **Set** is bicomplete we have that presheaf categories are bicomplete. Further, these limits can be computed pointwise. The Yoneda embedding also preserves all limits in $\mathcal{C}$.

Appendix B

Type Theory

We briefly present the fragment of intuitionistic type theory we model in the thesis. For a comprehensive introduction to the full system we recommend [NPS90]. Type theory consists of rules for making judgements of the following six forms:

Judgements Governing Contexts

$$\Gamma \text{ context} \qquad\qquad \Gamma = \Delta \text{ context}$$

Specifying when Γ is a well formed context and when well formed contexts Γ and Δ are definitionally equal.

Judgements Governing Types

$$A \text{ type} \qquad\qquad A = B \text{ type}$$

Specifying when A is a well formed type and when well formed types A and B are definitionally equal.

Judgements Governing Terms

$$a : A \qquad\qquad a = b : A$$

Specifying when a is a well formed term of type A and when well formed terms a and b of type A are definitionally equal.

Contexts are finite lists of variable declarations $x_0 : A_0, \ldots, x_n : A_n$ such that $FV(A_i) \subseteq \{x_0, \ldots, x_{i-1}\}$ when $0 \leq i \leq n$. Judgements may be made in the presence of a context Γ ($\Gamma \vdash \mathcal{J}$) by taking the variable declarations as assumptions. For example, we have the axiom

$$\frac{x : A \vdash B(x) \text{ type} \quad a : A}{B(a) \text{ type}}$$

saying that given a dependent type $B(x)$ over A, for each term $a : A$ we have a type $B(a)$. This is the dependency characteristic of the Martin-Löf formulation of type theory. The axioms governing the construction of well formed contexts are given as follows:

$$\frac{}{(\ )\ \text{context}}\ \text{Empty} \qquad \frac{\Gamma \vdash A\ \text{type}}{(\Gamma, x : A)\ \text{context}}\ \text{Extension}$$

Note that the order the variable declarations of a context appear in is crucial, as each type A_i in the list may depend on declarations earlier in the list to be given as well formed types. The remaining rules can be split into three categories: structural rules, equality rules and logical rules. For readability we omit the ambient context Γ common to both antecedent and consequent. We first give the *structural rules* governing substitution and weakening:

Variable Declaration

$$\frac{A\ \text{type}}{x : A \vdash x : A}\ \text{Var Dec}$$

Weakening

$$\frac{B\ \text{type}}{\Delta \vdash B\ \text{type}}\ \text{Weak}$$

Substitution

$$\frac{a : A \quad x : A, \Delta \vdash b(x) : B(x)}{\Delta[a/x] \vdash B[a/x]\ \text{type}}\ \text{Type Sub} \qquad \frac{a : A \quad x : A, \Delta \vdash b(x) : B(x)}{\Delta[a/x] \vdash b[a/x] : B[a/x]}\ \text{Term Sub}$$

We have the following *equality rules* governing the definitional equality congruence relation:

Equality Between Types

$$\frac{A\ \mathit{type}}{A = A}\ \text{Type Ref} \qquad \frac{A = B}{B = A}\ \text{Type Sym} \qquad \frac{A = B \quad B = C}{A = C}\ \text{Type Trans}$$

Equality Between Terms

$$\frac{a : A}{a = a : A}\ \text{Term Ref} \qquad \frac{a = b : A}{b = a : A}\ \text{Term Sym} \qquad \frac{a = b : A \quad b = c : A}{a = c : A}\ \text{Term Trans}$$

Conversion

$$\frac{a : A \quad A = B}{a : B}\ \text{Term Conv} \qquad \frac{a = b : A \quad A = B}{a = b : B}\ \text{Conv Equality}$$

Substitution

$$\frac{a = a' : A \quad x : A \vdash B(x)\ \text{type}}{B[a/x] = B[a'/x]\ \text{type}} \qquad \frac{a = a' \quad x : A \vdash b(x) : B(x)}{b[a/x] = b[a'/x] : B(a)}$$

$$\frac{a : A \quad x : A \vdash B(x) = C(x)\ \text{type}}{B(a) = C(a)\ \text{type}} \qquad \frac{a : A \quad x : A \vdash b(x) = c(x) : B(x)}{b(a) = c(a) : B(a)}$$

These substitution rules also come in a form for simultaneous substitution of n terms for all n: we omit these from this presentation as they are entirely analogous to those above. Finally we come to the logical rules governing *type constructors* that give the language additional expressivity. These rules have the same structure for each constructor: a formation rule governing the conditions under which the new type can be constructed; an introduction rule governing the introduction of canonical terms of the type; an elimination rule giving an inductive procedure to prove propositions about arbitrary terms with respect to the new type and finally coherence rules governing substitution in the new terms and types. In this thesis we concentrate on the fragment that has only the rules governing *strong identity types* as logical rules:

Identity Formation

$$\frac{a, b : A}{Id_A(a, b)} \text{ Id} - \text{Form;}$$

Identity Introduction

$$\frac{a : A}{r_A(a) : Id_A(a, a)} \text{ Id} - \text{Intro;}$$

Identity Elimination

$$\frac{\begin{array}{c} x, y : A, z : Id_A(x, y), \Delta(x, y, p) \vdash C(x, y, z) \text{ type} \\ a : A, \Delta(a, a, r_A(a)) \vdash d(a) : C(a, a, r_A(a)) \\ a, b : A \vdash p : Id_A(a, b) \end{array}}{\Delta(a, b, p) \vdash J_d(a, b, p) : C(a, b, p)} \text{ Id-Elim}$$

Identity Coherence

$$\frac{\Delta, x : C \vdash a(x), b(x) : A(x) \qquad c : C}{\Delta \vdash Id_{A(x)}(a(x), b(x))[c/x] = Id_{A[c]}(a[c/x], b[c/x])} \text{ Id Coherence}$$

$$\frac{\Delta, x : C \vdash a(x) : A(x) \qquad c : C}{\Delta \vdash r_{A(x)}(a(x))[c/x] = r_{A[c/x]}(a[c/x]) : Id_{A[c/x]}(a[c/x], a[c/x])} \text{ r coherence}$$

$$\frac{\begin{array}{c} x, y : A, z : Id_A(x, y), \Delta(x, y, p) \vdash C(x, y, z) \text{ type} \\ x : A, \Delta(x, x, r(x)) \vdash d(x) : C(x, x, r(x)) \qquad a : A \end{array}}{\Delta(a, a, r(a)) \vdash J_d(a, a, r(a)) = d(a) : C(a, a, r(a))} \text{ J Coherence}$$

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