Constructing Variants of the Category of Partial Equivalence Relations

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

The thesis aims at providing a categorical model of modified realizability, understood as an interpretation of extensional Heyting arithmetic in all finite types. Two variants of the category of partial equivalence relations are studied, namely, PER^* and PER^{**}. The former is shown to be not regular, thus not suitable for our aim. The latter is regular and has enough projectives; it is a good candidate. Both variants are results of alternating iteration of the cover construction and the co-cover construction. The cover construction resembles but differs from the regular completion.

Contents

1 Introduction

1 The goal

The current work aims at providing a categorical model for modified realizability, understood as an interpretation of extensional Heyting arithmetic in all finite types.

$\mathbf{HA}^\omega$ Heyting arithmetic in all finite types, denoted as $\mathbf{HA}^\omega$, is a many sorted theory, where the sorts are finite types. Finite types are defined from an atomic type o, to be thought of as the set of natural numbers, and closed under product ($\times$) and function space ($\to$). For example, $o \times o \to o$ is the type consisting of all computable function indices e that takes two natural numbers as arguments and produces a natural number as result.

In the language of $\mathbf{HA}^\omega$, there are countably infinitely many variable symbols, each associated with a finite type. There is a constant 0 of type o (abbreviated as $0\colon o$), successor $S\colon o \to o$, pairing $\mathbf{p}_{\sigma,\tau}$ and projections $\mathbf{p}_{0,\sigma,\tau}, \mathbf{p}_{1,\sigma,\tau}$ for all finite types σ, τ, combinators $\mathbf{k}_{\sigma,\tau}, \mathbf{s}_{\rho,\sigma,\tau}$, and recursor $\mathbf{r}_\rho\colon \rho \to (\rho \to o \to \rho) \to o \to \rho$. Terms are variables, constants and applications $st\colon \tau$, where $s\colon \sigma \to \tau$ and $t\colon \sigma$. For each type, there is an equality $=_\tau$. Atomic formulas are $s =_\tau t$ for $s, t\colon \tau$; more complex fomulas are inductively formed by connecting with $\to$, $\wedge$, $\vee$, $\forall x^\tau$ and $\exists x^\tau$.

The axioms and rules of $\mathbf{HA}^\omega$ include axioms and rules of many sorted first-order intuitionistic logic with equality, together with axioms for arithmetic and the combinator constants. Concerning application, the axioms for equality include

$$y =_\sigma z \to xy =_\tau xz, \quad x =_{\sigma\to\tau} y \to xz =_\tau yz.$$

The extensional version, $\mathbf{E\text{-}HA}^\omega$, has an extra axiom scheme about equality:

$$\forall yz(\forall x(yx =_\tau zx) \to y =_{\sigma\to\tau} z),$$

for x, y, z of the suitable types. $\mathbf{E\text{-}HA}^\omega$ has a model HEO, the *hereditarily effective operations*. HEO is inductively defined as a set of pairs $(\mathrm{HEO}_\tau, =_\tau)$, one for each finite type τ:

$$\mathrm{HEO}_o = \mathbb{N}, \quad x =_o y \text{ iff } x = y;$$

$$\mathrm{HEO}_{\sigma\to\tau} = \{x \in \mathbb{N}\,;\, \forall yy'(y =_\sigma y' \to x \bullet y =_\tau x \bullet y')\},$$

$$x =_{\sigma\to\tau} y \text{ iff } x, y \in \mathrm{HEO}_{\sigma\to\tau}, \text{ and for all } z \in \mathrm{HEO}_\sigma,\ x \bullet z =_\tau y \bullet z;$$

$$\mathrm{HEO}_{\sigma\times\tau} = \{\langle x, y\rangle\,;\, x \in \mathrm{HEO}_\sigma \wedge y \in \mathrm{HEO}_\tau\},$$

$$x =_{\sigma\times\tau} y \text{ iff } j_1(x) =_\sigma j_1(y) \text{ and } j_2(x) =_\tau j_2(y).$$

The notatioin $x \bullet y$ means applying computable function with index x to y. The notation $\langle , \rangle$ denotes the pairing function, with j_1, j_2 being the corresponding unpairing functions.

Modified realizability Realizability is a relation between constructions and logical statements. If a statement ϕ can be established by exhibiting some construction a, then *a realizes* ϕ, and ϕ is realizable. Formally, constructions are represented by computations, and statements by formulas.

Modified realizability is an interpretation of $\mathbf{HA}^\omega$ in $\mathbf{HA}^\omega$. To each formula ϕ in $\mathbf{HA}^\omega$ is associated a finite type τ. We have (i) $\phi\colon o$ for atomic formula ϕ, (ii) $\phi^\sigma \wedge \psi^\tau\colon \sigma \times \tau$, for ϕ, ψ associated with σ, τ, (iii) $\phi^\sigma \to \psi^\tau\colon \sigma \to \tau$, (iv) $\exists x^\sigma(\psi^\tau)\colon \sigma \times \tau$, and (v) $\forall x^\sigma(\psi^\tau)\colon \sigma \to \tau$. The essential clause is the existential one, where the type of witnesses is specified. The purpose of other clauses is to keep track of the structure of the formula. Then for each formula ϕ in $\mathbf{HA}^\omega$, define a formula x mr ϕ (*x modified realizes ϕ*), where x is a term of the type associated with formula ϕ.

(i) For atomic ϕ, $x \text{ mr } \phi \equiv \phi$; [atomic formulas do not need extra information to judge; x is redundant.]

(ii) $x \text{ mr } (\phi \wedge \psi) \equiv (\mathbf{p}_0 x \text{ mr } \phi) \wedge (\mathbf{p}_1 x \text{ mr } \psi)$;

(iii) $x \text{ mr } (\phi \to \psi) \equiv \forall y((y \text{ mr } \phi) \to (xy \text{ mr } \psi))$;

(iv) $x \text{ mr } \exists y \phi(y) \equiv \mathbf{p}_1 x \text{ mr } \phi(\mathbf{p}_0 x)$;

(v) $x \text{ mr } \forall y \phi(y) \equiv \forall y(xy \text{ mr } \phi(y))$.

The formula x mr ϕ is considered as a statement completed with the information x needed for the judgement of ϕ. Modified realizability can also be regarded as an interpretation of $\mathbf{E\text{-}HA}^\omega$ in $\mathbf{E\text{-}HA}^\omega$. It is this aspect that we wish to capture in this thesis.

Categorical analysis The goal of this thesis is to find a category $\mathcal{C}$, such that

$$\mathcal{C} \vDash \phi \quad \text{iff} \quad \text{HEO} \vDash \exists x(x \text{ mr } \phi),$$

for all ϕ in $\mathbf{HA}^\omega$. Let $\text{AC} = \bigcup_{\sigma,\tau}\{\text{AC}_{\sigma,\tau}\}$; for all finite types σ, τ,

$$\text{AC}_{\sigma,\tau} \qquad \forall x^\sigma \exists y^\tau \phi(x,y) \to \exists f^{\sigma\to\tau} \forall x^\sigma \phi(x, fx).$$

Let $\text{IP} = \bigcup_\tau\{\text{IP}_\tau\}$; for all finite types τ,

$$\text{IP}_\tau \qquad (\phi \to \exists x^\tau \psi(x)) \to \exists x^\tau(\phi \to \psi(x)),$$

where ϕ is existential free and x does not occur free in ϕ. By Troelstra [23],

$$\mathbf{E\text{-}HA}^\omega + \text{AC} + \text{IP} \vdash \phi \leftrightarrow \exists x(x \text{ mr } \phi).$$

As HEO is a model of $\mathbf{E\text{-}HA}^\omega$, and modified realizability validates AC and IP, it is necessary that $\mathcal{C} \vDash \mathbf{E\text{-}HA}^\omega + \text{AC} + \text{IP}$. For any Heyting category $\mathcal{C}$, $\mathcal{C} \vDash \text{AC}_{\sigma,\tau}$ for all sorts σ if the object interpreting τ is internal projective. Under certain conditions, internal projective objects coincide with projective objects. The current work tries to find a regular category with enough projective objects, aiming at $\mathbf{E\text{-}HA}^\omega + \text{AC}$.

2 Results

Results are

- (i) The cover and co-cover constructions that produce the category of partial equivalence relations (PER) and its variants;
- (ii) A definition of density in minimally initiated limits, the regularity condition for the cover construction, and the alternation condition for the cover and co-cover constructions;
- (iii) Among the variants, a natural candidate for modified realizability, PER^*, is not regular, hence could not serve as a model for modified realizability;
- (iv) A further variant PER^{**} is regular and has enough projectives.

3 Related work

Realizability was initiated by Kleene [11] and modified by Kreisel [13, 14]. It seems that to Kreisel, the importance of his interpretation is the ability to define realizability with different ranges of realizers [14, §10, the last sentence]. This may be the reason that he named it 'generalized realizability'.

Many results in intuitionistic arithmetic and realizability have been obtained by Anne Troelstra, collected in his 1973 monograph [23]. In Chapter III.4, he gives a very detailed analysis of modified realizability, with precise citation to Kreisel's work. There is also a definition in his later book [24, Exercise 9.6.5]. For more recent literature, see Kohlenbach's monograph [12] and Streicher's lecture notes [22].

The category of partial equivalence relations can be traced back to Eršov's *Numerierungen* [4, §3][a]. Dana Scott later suggested the name *modest set* for each 'numbering', which can be viewed as a partial equivalence relation[b]. However, later work on effective topos also used 'modest' for objects not being a subset of $\mathbb{N}$. In view of possible confusion, the current work adopts the name 'partial equivalence relation'. Much of the application of this category is in providing semantics for programming languages, especially for polymorphism [1], where there is no non-trivial set theoretic models that fully embeds into the category of sets, while respecting products and exponentials.

In the paper *Effective Topos* [6], Martin Hyland gave effective topos its name, and showed that the topos is a natural home for constructive mathematics. In that paper, he also pointed out a full sub-category of the effective topos (§7), which is equivalent to the category of partial equivalence relations. Since then, the category has been investigated in many works. Among those, Bauer's thesis [2] and Longley's thesis [15] have been important reference for the writing of this thesis.

[a] An English article on the same topic is available from the *Handbook of Computability Theory* [5]. Morphisms are defined on p.479.

[b] That Scott suggested the name is mentioned in Rosolini's paper [20, §3]. It is 'modest' because every set being numbered (by natural numbers) has to be countable.

But the world of realizability toposes is diverse. While the effective topos generalizes Kleene's realizability, the *modified realizability topos* generalizes (intentional) modified realizability. There are also the *extensional realizability topos* and the *Herbrand topos*. If we take the tripos-to-topos approach to define toposes, then the first step is to fix the collection of possible truth values for a proposition, in order to define triposes. This is a denotational idea of semantics: the meaning of a proposition is considered as the collection of objects that realize it. In the case of the effective topos, each truth value of a proposition is a subset of natural numbers, as realizers of Kleene's realizability are indices of computable functions. For the modified realizability topos, each truth value is a pair (A, P), where A is understood as the *actual realizers* and P as the *potential realizers*, $A \subseteq P \subseteq \mathbb{N}$.[a] The construction then goes on to obtain the modified realizability topos. The study of this modified realizability topos began from Hyland and Grayson.[b] The investigation was joint by Streicher [21] (modelling intensional type theory), Hyland and Ong [7] (generalizing strong normalization proof), van Oosten [28] (about a larger topos that includes the modified realizability topos and the effective topos), and Birkedal and van Oosten [3] (combining modified realizability and relative realizability). For the extensional realizability topos, each truth value is (A, R), where $A \subseteq \mathbb{N}$ and R an equivalence relation on A [thus (A, R) form a partial equivalence relation on $\mathbb{N}$]. This extensional realizability topos is known to Pitts [18],[c] and is studied by van Oosten [27]. For the Herbrand topos, the truth value also consists of actual realizers and potential realizers, but each realizer here is a finite list of natural numbers $(n_1, n_2, \ldots, n_k)$. This topos is studied by van den Berg [25] and Johnstone [9].

We have mentioned that the modified realizability topos generalizes the intentional modified realizability, *i.e.*, one that has the *hereditarily recursive operations* (HRO) as a model. As modified realizability can also be understood extensionally, it is natural to expect an *extensional modified realizability topos*. Although this thesis does not enter the study of topos, it is in the same line of thought — to combine modified realizability and extensionality. It is expected that the category PER^{**} can be placed naturally in this envisioned topos.

As remarked in van Oosten's historical overview on realizability [29], one feature of the effective topos and related toposes is to model non-classical theory, including synthetic domain theory, set up by Rosolini [19]; algebraic set theory, explained in Joyal and Moerdijk [10], with recent work by Moerdijk and van den Berg [26]; and intuitionistic non-standard arithmetic, Moerdijk [17]. The reader is encouraged to read this historical essay, as it is an enjoyable experience in itself.

[a] Some detail is omitted. See the following references for a precise definition.

[b] Mentioned in Birkedal and van Oosten's paper on modified and relative realizability [3, §4.2], and van Oosten's book on realizability [30, p.103].

[c] Mentioned in van Oosten's paper [27, §3.1]. The tripos is defined on pp.16–17 in Pitt's thesis.

2 PER variants via alternating constructions

1 PER and two variants

A *partial equivalence relation* on $\mathbb{N}$ is a symmetric and transitive relation on $\mathbb{N}$, abbreviated as per. Such a relation can be written as (A, R), where R is the partial equivalence relation and $A = \{x \in \mathbb{N}\,;\ x\ R\ x\}$.

We are interested in partial computable functions that preserve the relation. Let $(A, R), (B, S)$ be pers, computable function f preserves the relation from (A, R) to (B, S) if (i) for all $x \in A$, $f(x)\!\downarrow$ with $f(x) \in B$, and (ii) $x\ R\ y$ implies $f(x)\ S\ f(y)$. Thus, an equivalence class of A is mapped entirely into an equivalence class of B. We consider two such functions f, g to be the same, if for all $x \in A$, $f(x)\ S\ g(x)$. Then for each pair of pers, we obtain equivalence classes $[f]$ of relation preserving computable functions. The *category of pers* is a category where objects are pers (A, R), and morphisms are equivalence classes of computable functions $[f]\colon (A, R) \to (B, S)$ that preserve equivalence relations on the pers. The composition $[g] \circ [f]$ is defined as $[gf]$ (this is well-defined), which is associative, with $[\mathrm{id}_A]$ as identities, where $\mathrm{id}_A(x) = x$ for all $x \in A$. We denote the category of pers as PER.

PER is a regular category which is finite co-complete, locally cartesian closed with natural number objects.

Mono morphism $[f]\colon (A, R) \to (B, S)$ is a mono iff for all $x, y \in A$, $f(x)\ S\ f(y)$ implies $x\ R\ y$.

Initial object $(\emptyset, \emptyset)$ is the (only) initial object. For any per (A, R), the only morphism $(\emptyset, \emptyset) \to (A, R)$ is the equivalence class of all partial computable functions.

Binary product $(A, R) \times (B, S) = (A \times B, T)$, where $A \times B$ denotes the set $\{\langle x, y\rangle\ ;\ x \in A \text{ and } y \in B\}$, and $\langle x, u\rangle\ T\ \langle y, v\rangle$ iff $x\ R\ y$ and $u\ S\ v$. Projections are $[j_1]$, $[j_2]$, where j_1 and j_2 are the unpairing functions. The pairing morphism $\langle [f], [g]\rangle = [\langle f, g\rangle]$, with $\langle f, g\rangle = \lambda x.\langle f(x), g(x)\rangle$.

Equalizer the equalizer of $[f], [g]\colon (A, R) \to (B, S)$ is $[i]\colon (E, R \cap E \times E) \to (A, R)$, where $E = \{x \in A\,;\ f(x)\ S\ g(x)\}$, and i is the inclusion $A \leftarrow E$. For any $[h]$ with domain (C, U) that equalizes $[f]$ and $[g]$, the unique morphism given by the universal property is $[k]$, where k is set theoretically identical to h, such that $[h] = [i][k]$. (We could have written $[h]_{R,U} = [i]_{R,T} \circ [h]_{T,U}$, with $T = R \cap E \times E$.)

Pullback the pullback of $[f]\colon (A,R) \to (C,T)$ and $[g]\colon (B,S) \to (C,T)$ is (P,U), where $P = \{(f(x), g(y))\ ;\ f(x)\ T\ g(y)\}$, and U is the equivalence relation in $(A,R) \times (B,S)$ restricted to P. The two pulled back morphisms are $[f]^*([g]) = [j_1 i]$ and $[g]^*([f]) = [j_2 i]$, where i is an inclusion.

Epi morphism $[f]\colon (A,R) \to (B,S)$ is an epi iff for all $y \in B$, there is $x \in A$ such that $f(x)\ S\ y$.

Terminal object any $B \subseteq \mathbb{N}$ with the total relation on B is a terminal object. The unique morphism from (A,R) to a terminal object is $[k]$, where k is a constant function on A.

Binary coproduct $(A,R) \amalg (B,S) = (A \amalg B, T)$, where $T = \{\big(\langle 1,x\rangle, \langle 1,y\rangle\big)\ ;\ x\ R\ y\} \cup \{\big(\langle 2,x\rangle, \langle 2,y\rangle\big)\ ;\ x\ S\ y\}$. Coprojections are $[t_1]$, $[t_2]$, where $t_i(x) = \langle i,x\rangle$. The copairing of $[f]$ and $[g]$ is the equivalence class of copairing $[f,g]$.

Coequalizer the coequalizer of $[f],[g]\colon (A,R) \to (B,S)$ is $[\mathrm{id}_B]\colon (B,S) \to (B,T)$, where T is generated from $S \cup \{\big(f(x), g(x)\big)\ ;\ x \in A\}$. For any $[h]$ that coequalizes $[f]$ and $[g]$, the unique morphism given by the universal property is $[k]$, where k is set theoretically identical to h, such that $[h] = [\mathrm{id}_B][k]$.

Pushout The pushout of $[f]\colon (C,T) \to (A,R)$ and $[g]\colon (C,T) \to (B,S)$ is $(A \amalg B, U)$, where U is the equivalence relation generated from $V \cup \{\big(\langle 1, f(z)\rangle, \langle 2, g(z)\rangle\big)\ ;\ z \in C\}$, with V being the equivalence relation in $(A,R) \amalg (B,S)$.

Cover, image Let $[f]\colon (A,R) \to (B,S)$. The cover $\mathrm{cov}[f] = [\mathrm{id}_A]\colon (A,R) \to (A,T)$, where $T = \{(x,y)\ ;\ f(x)\ S\ f(y)\}$. The image $\mathrm{im}[f] = [g]$, where g is set theoretically identical to f, such that $[f] = [g][\mathrm{id}_A]$.

Projective $(A, = \cap A \times A)$ are projectives. If (B,S) is a projective, then it is isomorphic to some $(A, = \cap A \times A)$: there are $[f]$, $[g]$ such that $[g][f] = [\mathrm{id}_A]$ and $[f][g] = [\mathrm{id}_B]$. In other words, for all $x \in A$, $gf(x) = x$ and for all $y \in B$, $fg(y)\ S\ y$.

Natural number object $(\mathbb{N}, =)$ is a natural number object.

A *sub-per* is a triple (P,R,A), where (P,R) is a per with $P \neq \emptyset$, and $A \subseteq P$ is a sub-per: $x \in A$ and $x\ R\ y$ implies $y \in A$. The *category of sub-pers* has sub-pers as objects; morphisms are equivalence classes of partial computable functions $[f]\colon (P,R,A) \to (Q,S,B)$ such that (i) f is defined on P and $f[P] \subseteq Q$; (ii) f preserves the equivalence relation on P; (iii) $f[A] \subseteq B$, and (iv) $f \sim g$ iff they are essentially the same on A: for all $x \in A$, $f(x)\ S\ g(x)$. Composition $[g][f] = [gf]$, and $\mathrm{id}_{(P,R,A)} = [\mathrm{id}_P]$. Denote the category of sub-pers as PER^*. The subcategory of PER with non-empty pers embeds fully into PER^*. For $A \neq \emptyset$, per (A,R) appears as (A,R,A) in PER^*. This is also true with per $(\emptyset,\emptyset)$ included, but we need to make an arbitrary choice.

With a per for potential realizers and a sub-per for actual realizers, PER* is a natural candidate for interpreting modified realiziblity. However, PER* is not regular, which disqualifies it from this very purpose. It is a middle step towards the appropriate category. A *sub-per with quotient* is a quadruple (P, R, A, S), where (P, R, A) is a sub-per, and S a further quotient on A: for all $x, y \in A$, $x\ R\ y$ implies $x\ S\ y$. In the category of sub-pers with quotients, morphisms are equivalence class of computable functions $[f]\colon (P, R, A, S) \to (Q, T, B, U)$ that preserve equivalence relation R, maps A into B and preserve equivalence relation S; functions $f \sim g$ iff for all $x \in A$, $f(x)\ U\ g(x)$. Composition $[g][f] = [gf]$, and $\mathrm{id}_{(P,R,A,S)} = [\mathrm{id}_P]$. The category is denoted as PER**.

2 A pattern in limits and colimits

Notice that a per (A, R) can be described by a surjective function $r\colon A \to A/R$; a class of computable functions $[f]\colon (A, R) \to (B, S)$ can be described by one function $g\colon A/R \to B/S$.

Define a category $\mathcal{S}$. Objects in $\mathcal{S}$ are surjective functions $r\colon A \to X$ with $A \subseteq \mathbb{N}$. Morphisms are $(s, g, r)\colon r \to s$, where g is a function $\mathrm{cod}\, r \to \mathrm{cod}\, s$ such that $gr = sf$ for some computable function f defined on $\mathrm{dom}\, r$; we say f *tracks* g. Composition $(c, h, s)(s, g, r) = (c, hg, r)$ with identities $(r, \mathrm{id}_{\mathrm{cod}\, r}, r)$. The appearance of s, r in (s, g, r) is significant: it is not necessary that $(s, g, r) \simeq (s, g, r_1)$; similarly for s.

Proposition 1 $\mathcal{S} \simeq$ PER.

Proof. Define functors $F\colon \mathrm{PER} \to \mathcal{S}$, and $G\colon \mathcal{S} \to \mathrm{PER}$. On objects,

$$F(A, R) = \{(x, [x]_R)\ ;\ x \in A\},$$

which is a surjective function. On morphisms, suppose $[f]\colon (A, R) \to (B, S)$, $F(A, R) = r$ and $F(B, S) = s$.

$$F([f]) = (s, g, r) \text{ such that } gr = sf.$$

For functor G, suppose $r\colon A \to X$ is a surjective function with $A \subseteq \mathbb{N}$.

$$G(r) = (A, R) \text{ where } x\ R\ y \text{ iff } r(x) = r(y).$$

On morphisms,

$$G(s, g, r) = \{\text{computable functions } f \text{ defined on } \mathrm{dom}\, r \\ \text{such that } f[\mathrm{dom}\, r] \subseteq \mathrm{dom}\, s \text{ and } gr = sf\}.$$

Show GF and FG each gives a natural isomorphism. We have $GF(A, R) = (A, R)$. For naturality, suppose

$$GF([f]) = F(s, g, r) = [f']$$

for some appropriate r, s, g and f'. Then $sf' = gr = sf$, hence $[f] = [f']$. For the other composition, it is easy to see that $FG(r) = r^{-1}r$. As r is surjective, the inverse r^{-1} is a set isomorphism, and $(r^{-1}r,\, r^{-1},\, r)$ in the category $\mathcal{S}$ is tracked by id_A. Thus $FG(r) = r^{-1}r \simeq r$ in $\mathcal{S}$. For naturality, suppose

$$FG(s, g, r) = [f] = (s^{-1}s,\, g',\, r^{-1}r)$$

for some appropriate f and g'. Then

$$g'r^{-1}r = s^{-1}sf = s^{-1}gr.$$

By surjectivity of r, we get $g'r^{-1} = s^{-1}g$, and

$$(s^{-1}s,\, g',\, r^{-1}r)(r^{-1}r,\, r^{-1},\, r) = (s^{-1}s,\, s^{-1},\, s)(s, g, r).$$

The morphism $(r^{-1}r,\, r^{-1},\, r)$ in the equation is thought of as the component σ_r of a natural transformation σ; so for $(s^{-1}s,\, s^{-1},\, s)$. □

Describe limits and colimits in this new representation.

Binary product $r \times_{\mathrm{PER}} s = r \times s$, where $r \times s$ is the unique morphism $\operatorname{dom} r \times \operatorname{dom} s \to \operatorname{cod} r \times \operatorname{cod} s$ in **Set** given by the universal property.

Equalizer The equalizer of (c, f, d) and (c, g, d) is (d, h, e), where h is a equalizer of f and g in **Set**. Let inclusion h_0 be an equalizer of fd and gd, then e is the unique morphism $\operatorname{dom} h_0 \to \operatorname{dom} h$ given by the universal property.

Binary coproduct $r \amalg_{\mathrm{PER}} s = r \amalg s$, where $r \amalg s$ is the unique morphism $\operatorname{dom} r \amalg \operatorname{dom} s \to \operatorname{cod} r \amalg \operatorname{cod} s$ in **Set** given by the universal property.

Coequalizer The coequalizer of (c, f, d) and (c, g, d) is (hc, h, c), where h is a coequalizer of f, g in **Set**.

Define a category $\mathcal{S}^*$. Every object in $\mathcal{S}^*$ is a pair of functions (m, r), where $r\colon P \to X$ is a surjective function with $P \in \mathbb{N}$, and $m\colon X_1 \to X$ is an injective function with X non-empty. Morphisms are $((n, s), h, (m, r))\colon (m, r) \to (n, s)$, where h is a function $\operatorname{dom} m \to \operatorname{dom} n$ such that for some functions g, f (i) $nh = gm$, $gr = sf$, and (ii) f is a computable function defined on $\operatorname{dom} r$ with $f[\operatorname{dom} r] \subseteq \operatorname{dom} s$. We call h the *essential morphism* of $((n, s), h, (m, r))$, and say g, f *witness* $((n, s), h, (m, r))$.

$$\begin{array}{ccccc} \cdot & \xrightarrow{m} & \cdot & \xleftarrow{r} & \cdot \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f}\ \text{computable} \\ \cdot & \xrightarrow{n} & \cdot & \xleftarrow{s} & \cdot \end{array}$$

Proposition 2 $\mathcal{S}^* \simeq \mathrm{PER}^*$.

Proof. Define $F\colon \mathrm{PER}^* \to \mathcal{S}^*$ and $G\colon \mathcal{S}^* \to \mathrm{PER}^*$.

$$\begin{aligned} F(P,R,A) &= (m,r), \text{ where} \\ r &= \{(x,[x]_R)\ ;\ x \in P\}, \text{ and} \\ m &\text{ is the inclusion } \{[x]_R\ ;\ x \in A\} \to \{[x]_R\ ;\ x \in P\}. \end{aligned}$$

$F([f])$ is defined as $((n,s),h,(m,r))$ such that the commuting requirement is satisfied. Define G.

$$\begin{aligned} G(m,r) &= (\operatorname{dom} r, R, A), \text{ where} \\ R &= \{(x,y)\ ;\ r(x) = r(y)\}, \text{ and} \\ A &= \{x \in \operatorname{dom} r\ ;\ \text{ there is } y \text{ such that } m(y) = r(x)\}. \end{aligned}$$

$G((n,s),h,(m,r))$ is defined as the class of computable functions that satisfy the commuting requirement. $GF(P,R,A) = (P,R,A)$; $FG(m,r) = (m',r') \simeq (m,r)$, where $\operatorname{cod} m' \simeq \operatorname{cod} m$, $\operatorname{dom} m' \simeq \operatorname{dom} m$ and both isomorphisms in $\operatorname{dom} m' \simeq \operatorname{dom} m$ are tracked by $\mathrm{id}_{\operatorname{dom} r}$. ☐

Proposition 3 In PER^*,

(i) $(m,r) \times (n,s) = (m \times n, r \times s)$. Projections have essential morphisms π_1, π_2 from $\operatorname{dom} m \times \operatorname{dom} n$. Pairing $\langle ((m,r),f,(l,q)),((n,s),g,(l,q))\rangle = ((m \times n, r \times s), \langle f,g\rangle,(l,q))$.

(ii) The equalizer of parallel morphisms $((n,c),f,(m,d))$ and $((n,c),g,(m,d))$ is $((m,d),h,(mh,d))$, where h is an equalizer of f and g.

(iii) $(m,r) \amalg (n,s) = (m \amalg n, r \amalg s)$. Coprojections have essential morphisms ρ_1, ρ_2 to $\operatorname{dom} m \amalg \operatorname{dom} n$. Copairing of morphisms with essential morphisms f,g has essential morphism $[f,g]$.

(iv) The coequalizer of parallel morphisms $((n,c),f,(m,d))$ and $((n,c),g,(m,d))$ is $((l,h_1c),h,(n,c))$, where h is an coequalizer of f and g, function h_1 is a coequalizer of nf and ng, and l is the unique morphism $\operatorname{cod} h \to \operatorname{cod} h_1$ given by the universal property of coequalizer.

All limits and colimits for essential morphisms are calculated in **Set**.

Proof. (i) Given monic m,n, morphism $m \times n$ is monic in any category, so $(m \times n, r \times s)$ is an object in PER^*. Projection $\pi_{(m,r)}$ is witnessed by $\pi_{\operatorname{cod} r}$, $\pi_{\operatorname{dom} r}$. Suppose $((m,r),f,(l,q))$ and $((n,s),g,(l,q))$ are witnessed by f_1, f_0 and g_1, g_0, respetively. Then the copairing is witnessed by $\langle f_1, g_1\rangle, \langle f_0, g_0\rangle$. Uniqueness of the copairing comes from uniqueness of $\langle f,g\rangle$.

(ii) Function m in (m,d) is monic in **Set**. As an equalizer, h is monic, so mh is monic and (mh,d) is an object in PER^*. Morphism $((m,d),h,(mh,d))$ is witnessed by $\mathrm{id}_{\operatorname{cod} d}, \mathrm{id}_{\operatorname{dom} d}$. Suppose $((m,d),k,(l,e))$ equalizes $((n,c),f,(m,d))$ and $((n,c),g,(m,d))$. Then k equalizes f and g, and there is i with $k = hi$. Let

k be witnessed by k_1, k_0. We have $((mh, d), i, (l, e))$ witnessed by k_1, k_0, and $((m, d), k, (l, e))$ is the composition of $((m, d), h, (mh, d))$ and $((mh, d), i, (l, e))$.

(iii) Given monic m, n, morphism $m \amalg n$ is monic in **Set**. The rest of the proof is a dual of (i).

(iv) Morphism $((l, h_1 c), h, (n, c))$ is witnessed by $h_1, \mathrm{id}_{\mathrm{dom}\, c}$. Suppose $((p, e), k, (n, c))$ coequalizes $((n, c), f, (m, d))$ and $((n, c), g, (m, d))$, and is witnessed by k_1, k_0. Then there is i such that $((p, e), k, (n, c))$ is a composition of $((p, e), i, (l, h_1 c)$ and $((l, h_1 c), h, (n, c))$, where $((p, e), i, (l, h_1 c)$ is witnessed by i_1, k_0, with i_1 the function in $k_1 = i_1 h_1$. □

In the new form of PER, objects are epis, while in PER*, monos are added. The injective functions part in equalizer of PER* is dual to the surjective functions in coequalizer of PER; similarly for coequalizer of PER* and equalizer of PER. In fact, the two categories can be obtained from dual constructions.

We translate limits and colimits in PER* back to a concrete form for later reference.

Binary product $(P, R, A) \times (Q, S, B) = (P \times Q, T, A \times B)$, where $x \; T \; y$ iff $x \; R \; y$ and $x \; S \; y$. Projections are $[\pi_1]$, $[\pi_2]$. Pairing $\langle [f], [g] \rangle = [\langle f, g \rangle]$.

Equalizer The equalizer of $[f], [g] \colon (P, R, A) \to (Q, S, B)$ is $[\mathrm{id}_P] \colon (P, R, A_1) \to (P, R, A)$, where $A_1 = \{x \in A \; ; \; f(x) \; S \; g(x)\}$.

Binary coproduct $(P, R, A) \amalg (Q, S, B) = (P \amalg Q, T, A \amalg B)$, where $T = \{((1, x), (1, y)) \; ; \; x \; R \; y\} \cup \{((2, x), (2, y)) \; ; \; x \; S \; y\}$. Coprojections are $[\rho_1]$, $[\rho_2]$. The copairing of $[f]$ and $[g]$ is the equivalence class of copairing $[f, g]$.

Coequalizer The coequalizer of $[f], [g] \colon (P, R, A) \to (Q, S, B)$ is $[\mathrm{id}_Q] \colon (Q, S, B) \to (Q, T, B)$, where T is generated from $S \cup \{(f(x), g(x)) \; ; \; x \in A\}$.

3 PER* is not regular

Work in PER*.

Proposition 4 $[f] \colon (P, R, A) \to (Q, S, B)$ and $[g]$ form isomorphisms iff $gf(x) \; R \; x$ on A and $fg(y) \; S \; y$ on B.

Proposition 5 $[f] \colon (P, R, A) \to (Q, S, B)$ is monic iff for all $x, y \in A$, $f(x) \; S \; f(y)$ implies $x \; R \; y$.

Proof. Show sufficiency. Suppose for all $x, y \in A$, $f(x) \; S \; f(y)$ implies $x \; R \; y$. Let $[f][g] = [f][h]$, with domain (U, T, C). Then for any $u \in C$, $fg(u) \; S \; fh(u)$, and $g(u) \; R \; h(u)$. Thus $g \sim h$.

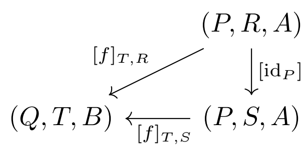
For necessity, suppose $[f]$ is monic. Form a sub-per (X, T, C), where

$$X = \{(x, y) \in P \times P \; ; \; f(x) \; S \; f(y)\},$$
$$(x, y) \; T \; (z, w) \text{ iff } x \; R \; z \text{ and } y \; R \; w, \text{ and}$$
$$C = X \cap (A \times A),$$

with p, q being projections $P \times P \to P$ restricted to X. We have $fp \sim fq$, thus $p \sim q$ by monicity. Let $x, y \in A$ and $f(x)\ S\ f(y)$. Then $p(x, y)\ R\ q(x, y)$, hence $x\ R\ y$. □

Proposition 6 1. Let (P, R, A) and (P, S, A) be sub-pers. If R, S coincide on $P \setminus A$, and S is a quotient of R on A, then $[\mathrm{id}_P]\colon (P, R, A) \to (P, S, A)$ is a cover.

2. Any cover is isomorphic to some $[\mathrm{id}_P]$ as described above. Let $[f]\colon (P, R, A) \to (Q, T, B)$ be a cover, then $[f] \simeq [\mathrm{id}_P]\colon (P, R, A) \to (P, S, A)$, where R, S coincide on $P \setminus A$, and $x\ S\ y$ on A iff $f(x)\ T\ f(y)$ on B. The isomorphism is given by $[f]_{T,S}\colon (P, S, A) \to (Q, T, B)$.



Proof. 1. Suppose $[\mathrm{id}_P] = [g][f]$, with $[g]$ monic. Let $\mathrm{dom}([g]) = (Q, T, B)$, and denote the above $[f]$ explicitly as $[f]_{T,R}$. Then $[g]$ and $[f]_{T,S}\colon (P, S, A) \to (Q, T, B)$ form isomorphisms. Show $[f]_{T,S}$ satisfies the requirement for morphisms in PER*. Suppose $x\ S\ y$ on A, then $gf(x)\ S\ x\ S\ y\ S\ gf(y)$, and $f(x)\ T\ f(y)$ by $[g]$ monic. The situation for $x\ S\ y$ on $P \setminus A$ is trivial. Show $[g]$ and $[f]_{T,S}$ form isomorphisms. Suppose $x \in A$, then $gf(x)\ S\ x$ for $(P, S, A) \to (P, S, A)$ comes from $gf(x)\ S\ x$ for $(P, R, A) \to (P, S, A)$. For $fg(y)\ T\ y$, it follows from $gfg \sim g$ and $[g]$ monic.

2. Define S as described, then it is a quotient of R, and $[\mathrm{id}_P]$ is a morphism $(P, R, A) \to (P, S, A)$. Also by the definition of S, $[f]_{T,S}$ is a monic morphism $(P, S, A) \to (Q, T, B)$, and $[f]_{T,R} = [f]_{T,S}[\mathrm{id}_P]$. Thus $[f]_{T,S}$ is an isomorphism. □

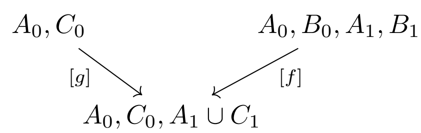
Proposition 7 The pullback of $[f]\colon (P, R, A) \to (U, T, C)$ and $[g]\colon (Q, S, B) \to (U, T, C)$ is $(P \times Q, L, D)$ with morphisms $[p], [q]$, where $(x, y)\ L\ (z, w)$ iff $x\ R\ z$ and $y\ S\ w$, $D = \{(x, y) \in P \times Q\ ;\ f(x)\ T\ g(y)\}$, and p, q are the projections restricted to D.

Fix $a, b, c \in \mathbb{N}$. We write $h_a h_b h_c$ with $h_n \in \{0, 1\}$, for the set of computable function indices e such that the halting of $e \bullet a$, $e \bullet b$ and $e \bullet c$ are as indicated—0 for halting and 1 otherwise. For example, $e \in 001$ means $e \bullet a\downarrow$, $e \bullet b\downarrow$ and $e \bullet c\uparrow$. Call two disjoint sets A and B *computably distinguishable* if there is a computable function f defined on $A \cup B$ such that $f[A]$ and $f[B]$ are computably separable; call them computably indistinguishable otherwise.

Let $A_0 = 010$, $A_1 = 011$, $B_0 = 100$, $B_1 = 101$. Among the sets, A_0 and A_1 are indistinguishable, B_0 and B_1 are indistinguishable; other pairs of sets are distinguishable. Let (P, R, P) be a sub-per formed by those four sets, each being an equivalence class. Let f be the computable functional such that given an index e, value $e' = f(e)$ is the index of the following function

$$e' \bullet x \simeq \begin{cases} \downarrow & \text{if } x = a, \\ e \bullet x & \text{otherwise.} \end{cases}$$

Function f maps A_0, A_1, B_0, B_1 into A_0, A_1, $C_0 = 000$, $C_1 = 001$ respectively. Let (Z, T, Z) be a sub-per formed by equivalence classes A_0, C_0, $A_1 \cup C_1$. Then $[f]\colon (P, R, P) \to (Z, T, Z)$ is a morphism $P \to Z$ in PER*, and dom(im$[f]$) has equivalence classes $\{A_0, B_0, A_1 \cup B_1\}$, all actual.[a] Notice that $A_1 \cup B_1$ is indistinguishable from either A_0 or B_0. Let (Q, S, Q) be the sub-per formed by equivalence classes A_0, C_0, indistinguishable from each other. Let g be the identity function. Then $[g]\colon (Q, S, Q) \to (Z, T, Z)$ is a morphism in PER*. Pullback im$[f]$, $[f]$ along $[g]$.



The domain of $[g]^*([f])$ has equivalence classes

$$\{U \times V\ ;\ U \in \{A_0, C_0\}, V \in \{A_0, B_0, A_1, B_1\}\},$$

with $A_0 \times A_0$ and $C_0 \times B_0$ being actual. Morphism im$[g]^*([f]) = [g]^*([f])$. The domain of $[g]^*(\text{im}[f])$ has equivalence classes

$$\{U \times V\ ;\ U \in \{A_0, C_0\}, V \in \{A_0, B_0, A_1 \cup B_1\}\},$$

also with $A_0 \times A_0$ and $C_0 \times B_0$ being actual. There is no isomorphism in PER* between im$[g]^*([f])$ and $[g]^*(\text{im}[f])$. More specifically, there is no computable function k that maps any of $(-, A_1 \cup B_1)$ into an equivalence class in dom im$[g]^*([f])$ while preserving $A_0 \times A_0$ and $C_0 \times B_0$ (so that $[k]\colon [g]^*(\text{im}[f]) \to \text{im}[g]^*([f])$ is a morphism in PER*).

Let $[k]$ be a morphism $[g]^*(\text{im}[f]) \to \text{im}[g]^*([f])$, preserving $A_0 \times A_0$ and $C_0 \times B_0$. The set $A_0 \times (A_1 \cup B_1)$ cannot be mapped into any of $- \times B_0$, otherwise A_0 and $A_1 \cup B_1$ would be distinguishable: take $\pi_2 k(a_0, e)$, where $a_0 \in A_0$ and $e \in A_0 \cup (A_1 \cup B_1)$. If $e \in A_0$, then $\pi_2 k(a_0, e) \in A_0$, as $A_0 \times A_0$ is preserved by k; if $e \in A_1 \cup B_1$, then $\pi_2 k(a_0, e) \in B_0$; but sets A_0 and B_0 are distinguishable. For the same reason, $A_0 \times (A_1 \cup B_1)$ cannot be mapped into any of $- \times B_1$. In other words, were k to be as wished, $A_0 \times (A_1 \cup B_1)$ can only be mapped into $- \times A_0$ or $- \times A_1$; similarly, $C_0 \times (A_1 \cup B_1)$ can only be mapped into $- \times B_0$ or $- \times B_1$. However, $A_0 \cup A_1$ and $B_0 \cup B_1$ are distinguishable, it would mean A_0 and C_0 were distinguishable under such k: take $e \in A_0 \cup C_0$ and $b \in A_1 \cup B_1$, then $\pi_2 k(e, b)$ would show whether $e \in A_0$ or $e \in C_0$.

Theorem 1 PER* is not regular.

[a] An equivalence class $[x]_R$ is actual in a sub-per (P, R, A) if $[x]_R \in A/R$.

4 The cover and cocover constructions

Proposition 8 In a regular category, covers are closed under composition, pullback, product, pushout, and coproduct.

Proposition 9 In a regular category, projectives are closed under binary coproduct.

Definition 1 (The cover construction) Let $\mathcal{C}$ be a category, $\mathcal{C}_1$ be a subcategory of $\mathcal{C}$. The *category of covers* on $(\mathcal{C}_1, \mathcal{C})$ is a category where objects are covers c in $\mathcal{C}$ with $\operatorname{dom} c$ in $\mathcal{C}_1$ and projective in $\mathcal{C}$. Morphisms are $(c, f, d)\colon d \to c$ where (i) $f\colon \operatorname{cod} d \to \operatorname{cod} c$ is a morphism in $\mathcal{C}$, and (ii) f is tracked: there is a morphism g in $\mathcal{C}_1$ with $fd = cg$. Composition $(b, g, c)(c, f, d) = (b, gf, d)$; identities are $(d, \mathrm{id}_{\operatorname{dom} d}, d)$. Denote the category as $\mathrm{k}(\mathcal{C}_1, \mathcal{C})$.

Given object d, c in $\mathrm{k}(\mathcal{C}_1, \mathcal{C})$, there is a bijection between morphisms (c, f, d) in $\mathrm{k}(\mathcal{C}_1, \mathcal{C})$, and partial equivalence classes $[g]$ defined on the set of morphisms $\operatorname{dom} d \to \operatorname{dom} c$ in $\mathcal{C}_1$, where $g \sim h$ iff $cg = ch$.

Dually, we have the *category of co-covers* on $(\mathcal{C}_1, \mathcal{C})$ defined with co-covers and injectives in $\mathcal{C}$.

Definition 2 (Density) Let $\mathcal{C}_1$ be a subcategory of $\mathcal{C}$. Category $\mathcal{C}_1$ is *dense in binary products* of $\mathcal{C}$, if for any objects a, b in $\mathcal{C}_1$, a, b have a product in $\mathcal{C}$ implies they have a product in $\mathcal{C}_1$, and the product cone in $\mathcal{C}_1$ is also a product cone in $\mathcal{C}$. Category $\mathcal{C}_1$ is *dense in regular monos* of $\mathcal{C}$, if for any regular mono m in $\mathcal{C}$ with $\operatorname{cod} m$ in $\mathcal{C}_1$, there is m_1 in $\mathcal{C}_1$ with $\operatorname{cod} m_1 = \operatorname{cod} m$, such that (i) $m_1 \simeq m$ in $\mathcal{C}$, and (ii) for any l in $\mathcal{C}_1$ with $l = m_1 n$ in $\mathcal{C}$, the unique morphism n is in $\mathcal{C}_1$.

Dualize the definition to obtain density in coproducts and regular epis.

Definition 3 (Regularity condition) Categories $(\mathcal{C}_1, \mathcal{C})$ satisfy the *regularity condition* for the cover construction, if (i) $\mathcal{C}$ is regular, (ii) projectives in $\mathcal{C}$ are closed under binary products and regular subobjects, and (iii) $\mathcal{C}_1$ is dense in binary products and regular mono of $\mathcal{C}$.

Proposition 10 (Limit) Let $(\mathcal{C}_1, \mathcal{C})$ satisfy the regularity condition.

(i) The product of a and b in $\mathrm{k}(\mathcal{C}_1, \mathcal{C})$ is $a \times b$, with $\operatorname{dom} a \times \operatorname{dom} b$ taken according to density;

(ii) Let (c, f, d), (c, g, d) be morphisms in $\mathrm{k}(\mathcal{C}_1, \mathcal{C})$. Let h be an equalizer of f, g in $\mathcal{C}$, h_0 be an equalizer of fd and gd taken according to density, and e be the unique morphism $\operatorname{dom} h_0 \to \operatorname{dom} h$ given by the universal property of equalizer h. The equalizer of (c, f, d) and (c, g, d) is (d, h, e).

Proof. (i) Morphism $a \times b$ is a cover with projective domain. The projections are $(d, \pi_{\operatorname{cod} d}, d \times c)$ and $(c, \pi_{\operatorname{cod} c}, d \times c)$, tracked by $\pi_{\operatorname{dom} d}$ and $\pi_{\operatorname{dom} c}$ respectively. Pairing $\langle (d, f, e), (c, g, e) \rangle = (d \times c, \langle f, g \rangle, e)$, where $\operatorname{cod}\langle f, g \rangle = \operatorname{cod} d \times \operatorname{cod} c$. By density, $\pi_{\operatorname{dom} d}$ and $\pi_{\operatorname{dom} c}$ are in $\mathcal{C}_1$. Suppose f, g are tracked by f_1, g_1, then $\langle f_1, g_1 \rangle$ is in $\mathcal{C}_1$ due to density, thus $\langle f, g \rangle$ is tracked.

(ii) Morphisms h, h_0, d, e form a pullback square, so e as a pullback of d is a cover. By density, h_0 is in $\mathcal{C}_1$. Any unique morphism with codomain $\operatorname{dom} h_0$ given by the universal property of equalizer h_0 is in $\mathcal{C}_1$, due to density. □

Proposition 11 (Colimit) Let $\mathcal{C}$ be a regular category, $\mathcal{C}_1$ be a subcategory of $\mathcal{C}$ dense in coproducts.

(i) The coproduct of a and b in $\mathrm{k}(\mathcal{C}_1, \mathcal{C})$ is $a \amalg b$, with $\operatorname{dom} a \amalg \operatorname{dom} b$ taken according to density.

(ii) The coequalizer of (c, f, d) and (c, g, d) in $\mathrm{k}(\mathcal{C}_1, \mathcal{C})$ is (hc, h, c), where h is a coequalizer of f and g in $\mathcal{C}$.

Proof. (i) Morphism $a \amalg b$ is a cover with a projective domain. The coprojections are $(a \amalg b, \rho_1, a)$ and $(a \amalg b, \rho_2, b)$, tracked by σ_1, σ_2 respectively. Copairing of (c, f, a) and (c, g, b) is $(c, [f, g], a \amalg b)$. Tracking morphisms and morphisms given by universal property are in $\mathcal{C}_1$ due to density.

(ii) The coequalizer h is a cover, so hc is a cover with a projective domain. Morphism h is tracked by $\mathrm{id}_{\operatorname{dom} c}$. For any (b, k, d) coequalizing (c, f, d) and (c, g, d), let k be tracked by k_0, then the unique morphism given by the equalizer h is also tracked by k_0. □

Proposition 12 (Monicity) Let $(\mathcal{C}_1, \mathcal{C})$ satisfy the regularity condition. Then (c, f, d) is monic in $\mathrm{k}(\mathcal{C}_1, \mathcal{C})$ iff f is monic in $\mathcal{C}$.

Proof. Show necessity. The forgetful functor $F^-\colon \mathrm{k}(\mathcal{C}_1, \mathcal{C}) \to \mathcal{C}$, mapping (c, f, d) to f, preserves binary products and equalizers, thus preserves pullbacks and monos. □

Proposition 13 (Cover) Let $(\mathcal{C}_1, \mathcal{C})$ satisfy the regularity condition.

(i) If k is a cover in $\mathcal{C}$, then (kd, k, d) are covers in $\mathrm{k}(\mathcal{C}_1, \mathcal{C})$.

(ii) Morphism (c, h, d) in $\mathrm{k}(\mathcal{C}_1, \mathcal{C})$ is a cover only if it is isomorphic to some (kd, k, d) where k is a cover in $\mathcal{C}$.

Proof. (i) Suppose k is a cover, then it is a coequalizer of its kernel pair a_1, a_2. Pullback $d \times d$ along $\langle a_1, a_2 \rangle$ to obtain cover e, then (kd, k, d) is a coequalizer of (d, a_i, e) [k is tracked by $\mathrm{id}_{\operatorname{dom} d}$]. Suppose (kd, k, d) can be factored as $(kd, j, b)(b, h, d)$ with (kd, j, b) monic. Then (b, h, d) coequalizes (d, a_i, e) and the monic (kd, j, b) has a section, thus an isomorphism.

(ii) Factor h as $\operatorname{im} h \circ \operatorname{cov} h$, then $(c, \operatorname{im} h, \operatorname{cov} h \circ d)$ is a morphism in $\mathrm{k}(\mathcal{C}_1, \mathcal{C})$, in which $\operatorname{im} h$ is tracked by any morphism that tracks h. $\operatorname{im} h$ is monic, so $(c, \operatorname{im} h, \operatorname{cov} h \circ d)$ is monic, thus an isomorphism. Take $k = \operatorname{cov} h$. □

Proposition 14 (Cover-image factorization) Let $(\mathcal{C}_1, \mathcal{C})$ satisfy the regularity condition. Let (c, f, d) be a morphism in $\mathrm{k}(\mathcal{C}_1, \mathcal{C})$, then the cover of (c, f, d) is $(\operatorname{cov} f \circ d, \operatorname{cov} f, d)$, and the image is $(c, \operatorname{im} f, \operatorname{cov} f \circ d)$.

Proof. Let f be tracked by f_0. Then $\operatorname{cov} f$ in $(\operatorname{cov} f \circ d, \operatorname{cov} f, d)$ is tracked by $\mathrm{id}_{\operatorname{dom} d}$, and $\operatorname{im} f$ in $(c, \operatorname{im} f, \operatorname{cov} f \circ d)$ is tracked by f_0. □

We can characterize projective objects now. Let $\mathcal{C}_1$ be a subcategory of $\mathcal{C}$. Category $\mathcal{C}_1$ is *dense in co-covers*, if for any co-cover k in $\mathcal{C}$ with $\operatorname{cod} k$ in $\mathcal{C}_1$, there is k_1 in $\mathcal{C}_1$ with $\operatorname{cod} k_1 = \operatorname{cod} k$, such that (i) $k_1 \simeq k$ in $\mathcal{C}$, and (ii) for any l in $\mathcal{C}_1$ with $l = k_1 j$ in $\mathcal{C}$, morphism j is in $\mathcal{C}_1$.

Theorem 2 (Projective object, Bauer) Let $(\mathcal{C}_1, \mathcal{C})$ satisfy the regularity condition.

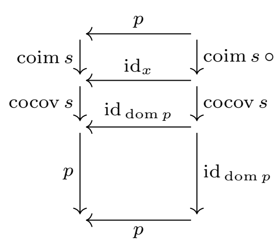
(i) Identities in $\mathcal{C}_1$ are projective in $\mathrm{k}(\mathcal{C}_1, \mathcal{C})$.

(ii) If additionally, $\mathcal{C}$ has co-images with projectives closed under coimage-cocover factorization, and $\mathcal{C}_1$ is dense in co-covers. then p is a projective in $\mathrm{k}(\mathcal{C}_1, \mathcal{C})$ only if $p \simeq i$ for some identity i.

(iii) Every object in $\mathrm{k}(\mathcal{C}_1, \mathcal{C})$ is covered by a projective object.

Proof. (i) Let i be an identity in $\mathcal{C}_1$, (kd, k, d) be a cover, and (kd, f, i) be a morphism where f is tracked by f_0. Then $(kd, f, i) = (kd, k, d)(d, df_0, i)$, and df_0 in (d, df_0, i) is tracked by f_0.

(ii) Morphism $(p, p, \mathrm{id}_{\operatorname{dom} p})$ is a cover in $\mathrm{k}(\mathcal{C}_1, \mathcal{C})$, and p is projective, so $(p, p, \mathrm{id}_{\operatorname{dom} p})$ has a section $(\mathrm{id}_{\operatorname{dom} p}, s, p)$, in which s is tracked by some s_0. Take the coimage-cocover factorization of s so that the co-cover is in $\mathcal{C}_1$, and let x be the domain of $\operatorname{cocov} s$. We have $(p, p \circ \operatorname{cocov} s, \mathrm{id}_x)$, $(\mathrm{id}_x, \operatorname{coim} s, p)$ being a pair of isomorphisms: (i) $\operatorname{coim} s$ is tracked by $\operatorname{coim} s \circ p$, given by $\operatorname{cocov} s \circ \operatorname{coim} s \circ p = s_0$ and density; (ii) $p \circ \operatorname{cocov} s$, $\operatorname{coim} s$ form isomorphisms, because $\operatorname{coim} s$ is an epi that has a retraction. Take $i = \mathrm{id}_x$.



(iii) Every object c is covered by $(c, c, \mathrm{id}_{\operatorname{dom} c})$. □

See Bauer's thesis [2]*Theorem 1.3.4 for the original proof.

Proposition 15 (Pullback) The pullback of (c, f, a) and (c, g, b) in $\mathrm{k}(\mathcal{C}_1, \mathcal{C})$ is $(a, \pi_{\operatorname{cod} a} \circ i, e)$ and $(b, \pi_{\operatorname{cod} b} \circ i, e)$, given by a product followed by an equalizer. Morphisms f, g, $\pi_{\operatorname{cod} a} \circ i$, $\pi_{\operatorname{cod} b} \circ i$ form a pullback square in $\mathcal{C}$. Let f be tracked by f_0, morphism g by g_0, and i by i_0 [in $(a \times b, i, e)$], then fa, gb, $\pi_{\operatorname{dom} a} \circ i_0$, $\pi_{\operatorname{dom} b} \circ i_0$ form a pullback square, so for cf_0, cg_0, $\pi_{\operatorname{dom} a} \circ i_0$ and $\pi_{\operatorname{dom} b} \circ i_0$.

Theorem 3 Let $(\mathcal{C}_1, \mathcal{C})$ satisfy the regularity condition. Then $\mathrm{k}(\mathcal{C}_1, \mathcal{C})$ is regular.

Proof. Show covers are stable under pullback. Let (kb, k, b) be a cover. Pull it back along (kb, f, a), obtaining (a, l, e), then l is a cover, tracked by some l_0. We show $a \simeq al_0 = le$, so that $(a, l, e) \simeq (le, l, e)$ in $\mathrm{k}(\mathcal{C}_1, \mathcal{C})$, hence a cover. The non-obvious direction is, there is l_1 with $al_0l_1 = a$. Let f_0 tracks f. We have

$$(kb)\mathrm{id}_{\mathrm{dom}\, b} \circ f_0 = (kb)f_0 \circ \mathrm{id}_{\mathrm{dom}\, a},$$

and l_0 is in the pullback square given by $(kb)\mathrm{id}_{\mathrm{dom}\, b}$ and $(kb)f_0$, so there is l_1 such that $l_0l_1 = \mathrm{id}_{\mathrm{dom}\, a}$. This l_1 is in $\mathcal{C}_1$ due to density. Thus $al_0l_1 = a$ and $a \simeq al_0 = le$. Therefore $(a, l, e) \simeq (le, l, e)$ is a cover. (In the diagram, f/f_0 means f is tracked by f_0.) ☐

$$\begin{array}{ccc} b & \longleftarrow & e \\ {\scriptstyle k/\mathrm{id}_{\mathrm{dom}\, b}}\downarrow & & \downarrow{\scriptstyle l/l_0} \\ kb & \underset{f/f_0}{\longleftarrow} & a \end{array}$$

The cover construction looks similar to regular completion: both constructions take equivalence classes of morphisms and produce regular categories. However, the cover construction $\mathrm{k}(\mathcal{C}_1, \mathcal{C})$ is not necessarily a regular completion of $\mathcal{C}_1$. A regular category is the result of a regular completion exactly when every object is covered by a projective and embeds into a projective, and the full subcategory of projectives are closed under finite limits. Let $\mathcal{N}_1$ be the category of subsets of $\mathbb{N}$, with morphisms $f\colon X \to Y$, where f is a computable function defined on X and $f[X] \subseteq Y$. The category $\mathrm{PER} \simeq \mathrm{k}(\mathcal{N}_1, \mathbf{Set})$, but not every object embeds into a projective. In PER, a morphism (c, m, d) is monic iff m is monic in **Set**. Embedding every object into projectives would mean for any cover d in **Set**, there were a monic m such that $(\mathrm{id}_{\mathrm{cod}\, m}, m, d)$ were in PER — there were a computable function n with $md = n$. This is not the case, as we can take d to be the characterising function for the the halting problem.

5 Generalization to functors

For any category $\mathcal{C}$, define $\mathrm{q}(\mathcal{C})$ to be the full subcategory of $\mathcal{C}$ containing all objects covered by some projective object; define $\mathrm{p}(\mathcal{C})$ to be the full subcategory of projectives. For regular completion, we have $\mathrm{reg}(\mathcal{C}) \simeq \mathrm{reg}\,\mathrm{p}\,\mathrm{reg}(\mathcal{C})$. Assume $(\mathcal{C}_1, \mathcal{C})$ satisfy the regularity condition for the following proposition and its corollaries.

Proposition 16 $\mathrm{k}(\mathcal{C}, \mathcal{C}) \simeq \mathrm{q}\,\mathcal{C}$.

Proof. Define $F^+\colon \mathrm{q}\,\mathcal{C} \to \mathrm{k}(\mathcal{C}, \mathcal{C})$. On objects,

$$F^+(x) = \begin{cases} \mathrm{id}_x, & \text{if } x \text{ is projective in } \mathcal{C}, \\ \text{any cover } c\colon p \to x \text{ with projective } p, & \text{otherwise.} \end{cases}$$

On morphisms, $F^+(f) = (F^+(\operatorname{cod} f), f, F^+(\operatorname{dom} f))$. Functor $F^-\colon \mathrm{k}(\mathcal{C},\mathcal{C}) \to \mathrm{q}\mathcal{C}$ is forgetful. On objects, $F^-(c) = \operatorname{cod} c$; on morphisms, $F^-(c,f,d) = f$. Composition F^-F^+ is an identity. $F^+F^-(c) = c' \simeq c$, because $\operatorname{dom} c' = \operatorname{dom} c$, and both c', c are covers with projective domains. □

Corollary 1 $\mathrm{q}\mathcal{C}$ is regular, with projectives closed under finite limits.

Generalize the cover construction with functor.

Definition 4 (The cover construction) For any faithful functor I, define category $\mathrm{k}(I)$: objects are (c, c_1) where c is a cover in $\operatorname{cod} I$ with projective domain, and c_1 is an object in $\operatorname{dom} I$ with $I(c_1) = \operatorname{dom} c$; morphisms are $((c,c_1), f, (d,d_1))\colon (d,d_1) \to (c,c_1)$ where f is a morphism in $\operatorname{cod} I$, such that for some $g\colon d_1 \to c_1$ in $\operatorname{dom} I$, $fd = c \circ I(g)$. Define $\mathbf{k}(I)$ as a functor $\mathrm{k}(I) \to \mathrm{k}(\mathrm{id}_{\operatorname{cod} I})$, mapping morphisms $((c,c_1), f, (d,d_1))$ to $((c, I(c_1)), f, (d, I(d_1)))$.

The previous notation $\mathrm{k}(\mathcal{C}_1,\mathcal{C})$ is a special case when I is an inclusion functor. The objects c_1, d_1 in the generalized definition did not appear explicitly in this special case, because they are the domains of c, d.

Definition 5 (Minimal object) An object i is *minimal* if $\operatorname{cod} f = i$ implies $f = \mathrm{id}_i$. A category is *minimally initiated* if for any object j, there is a morphism $f\colon i \to j$ with i minimal. A diagram F is minimally initiated if $\operatorname{dom} F$ is minimally initiated.

For example, $\{*,*\}$ and $* \to * \leftarrow *$ are minimally initiated, but $* \longleftrightarrow *$ is not.

Definition 6 (Density) Let I be a functor, F be a finite diagram where $\operatorname{dom} F$ is minimally initiated and $\operatorname{cod} F = \operatorname{cod} I$. Let n be the set of minimal objects in $\operatorname{dom} F$, and let $\{z_i\}_{i\in n}$ be a set of $\operatorname{dom} I$ objects with $I(z_i) = F(i)$ for $i \in n$. Functor I is *dense in the limit of F* at $\{z_i\}_i$, if there are $\operatorname{dom} I$ morphisms $\{a_i\}_{i\in n}$ with $\operatorname{dom} a_i = x$ and $\operatorname{cod} a_i = z_i$, such that (i) $\{F(f) \circ I(a_i)\}_j$ and $I(x)$ form a limiting cone in $\operatorname{cod} I$, where $j \in \operatorname{ob}(\operatorname{dom} F)$ and $f\colon i \to j$ is any morphism with i minimal, and (ii) for any cone $\{F(f) \circ I(b_i)\}_j$ with vertex $I(y)$ on the same diagram, where $\operatorname{dom} b_i = y$ and $\operatorname{cod} b_i = z_i$, there is $c\colon y \to x$ in $\operatorname{dom} I$, such that $a_i c = b_i$ for $i \in n$.

Functor I is *dense in minimally initiated finite limits*, if for any finite diagram F with $\operatorname{dom} F$ minimally initiated and $\operatorname{cod} F = \operatorname{cod} I$, and any set of $\operatorname{dom} I$ objects $\{z_i\}_{i\in n}$ with $I(z_i) = F(i)$ for $i \in n$, where n is the set of minimal objects in $\operatorname{dom} F$, the functor I is dense in $\lim F$ at $\{z_i\}_i$.

A cone (μ, x) is a *weak limit*, if for any cone (ν, y) on the same diagram, there is a morphism $a\colon (\nu, y) \to (\mu, x)$. Uniqueness of this morphism is not required.

Proposition 17 Let I be a functor with $\operatorname{cod} I$ finitely complete. (i) If I is dense in minimally initiated finite limits, then $\operatorname{dom} I$ has weak limits for minimally initiated finite diagrams. (ii) If I is faithful and dense in minimally initiated finite limits, then I has limits for minimally initiated finite diagrams and preserves pullbacks.

Proof. (i) Let F be a minimally initiated finite diagram with $\operatorname{cod} F = \operatorname{dom} I$. Then IF is a minimally initiated finite diagram. We get weak limits by density.

(ii) For any cone ν on F, $I\nu$ is a cone on IF. Let c be the morphism $\nu \to \mu$ given by density, where μ is a weak limit, then $I(c)$ is the unique morphism $I\nu \to I\mu$. If I is faithful, then c is unique, thus μ is a limit. By the above argument, if $\lambda = \lim F$, then the cone μ given by density is isomorphic to λ, hence $I\lambda \simeq I\mu = \lim(IF)$. Thus I preserves minimally initiated finite limits. Diagrams for pullback cones are minimally initiated, thus I preserves pullbacks. □

Functor I is dense in finite products if it is dense in nullary, unary and binary products. Density in nullary products gives weak terminal objects. Density in unary products is trivial: for any object z in $\operatorname{dom} I$, object $I(z)$ is the unary product in $\operatorname{cod} I$, and $I(\mathrm{id}_z)$ is the projection; for any f with $\operatorname{cod} f = z$, we have $\mathrm{id}_z \circ f = f$.

Proposition 18 Let I be a functor. I is dense in minimally initiated finite limits iff it is dense in finite products and equalizers.

Proof. Show sufficiency. Suppose I is dense in finite products and equalizers. Let F be a minimally initiated finite diagram, n be the set of minimal objects in $\operatorname{dom} F$, and $\{z_i\}_{i \in n}$ be a set of $\operatorname{dom} I$ objects such that $I(z_i) = F(i)$ for $i \in n$. Construct the limit in $\operatorname{cod} I$ as follows, according to density. (1) Take $\prod_i I(z_i)$ for $i \in n$, obtaining $\{p_i\}_i$ with $I(p_i) = \pi_i$ for $i \in n$. (2) For every object j in $\operatorname{dom} F$, construct the equalizer of

$$\{F(f)\pi_{\operatorname{dom} f}\,;\ \operatorname{cod} f = j \text{ and } \operatorname{dom} f \text{ minimal}\},$$

obtaining g_j with $\operatorname{cod}(I(g_j)) = \prod_i I(z_i)$ for each j. (3) Construct the fibred product of all $I(g_j)$: construct $\prod_j \operatorname{dom} I(g_j)$, obtaining s_j with $I(s_j)$ being the projections σ_j; construct the equalizer of $\{I(g_j) \circ \sigma_j\,;\ j \in \operatorname{ob}(\operatorname{dom} F)\}$, obtaining h. Claim: $\mu_j = F(f)I(p_{\operatorname{dom} f})I(g_j)I(s_j)I(h)$ is a limiting cone component at j, where f is any morphism with $\operatorname{cod} f = j$ and a minimal domain.

Show $\{\mu_j\}_j$ is a cone. Let $f\colon j \to k$ be a morphism in $\operatorname{dom} F$ with no constraint on j, k. Take $e\colon i \to j$ with i minimal,

$$\mu_k = F(fe)\mu_i = F(f)F(e)\mu_i = F(f)\mu_j.$$

Suppose (ν, y) is a cone on F. By universal properties of the limits, we get unique morphisms from y to $\prod_i z_i$, $\operatorname{dom}(I(g_j))$ and $\operatorname{dom}(I(h))$, thus μ is limiting. If there are morphisms $\{b_i\}_i$ in $\operatorname{dom} I$ with $\operatorname{dom} b_i = w$, $\operatorname{cod} b_i = z_i$ and $I(b_i) = \nu_i$ for $i \in n$, then by the same tracing process, we get $c\colon w \to \operatorname{dom} h$ such that $p_i g_j s_j h c = b_i$. □

The usual way of constructing limits by products and equalizers does not work for this proposition.

Functor I is *dense in covers*, if for any cover c in $\operatorname{cod} I$ and any object z with $I(z) = \operatorname{dom} c$, there is c_1 in $\operatorname{dom} I$ with $\operatorname{dom} c_1 = z$, such that (i) $I(c_1) \simeq c$, and (ii) if $I(e_1) = d \circ I(c_1)$, then there is d_1 in $\operatorname{dom} I$ with $I(d_1) = d$ and $e_1 = d_1 c_1$.

Proposition 19 Let I be faithful, pullback preserving, and dense in covers. Then given a cover c in $\operatorname{cod} I$ and an object z with $I(z) = \operatorname{dom} c$, the morphism c_1 taken according to density is a cover in $\operatorname{dom} I$.

Proof. Suppose $c_1 = mb$ with m monic. Functor I preserves pullbacks, so $I(m)$ is monic, hence an isomorphism in $\operatorname{cod} I$. We have $I(b) = I(m)^{-1} I(c_1)$, thus by density, there is n with $b = nc_1$ and $I(n) = I(m)^{-1}$. Then $I(mn) = I(\mathrm{id}_{\operatorname{cod} b})$ and $I(nm) = I(\mathrm{id}_{\operatorname{cod} c})$. By faithfulness, m, n form isomorphisms, thus c_1 is a cover in $\operatorname{dom} I$. □

Recall that $\mathbf{k}(I)$ is a functor $\mathrm{k}(I) \to \mathrm{k}(\mathrm{id}_{\operatorname{cod} I})$, mapping morphisms $((c, c_1), f, (d, d_1))$ to $((c, \operatorname{dom} c), f, (d, \operatorname{dom} d))$. Define $\mathbf{p}(I)$ as the inclusion functor $\mathrm{p}(\operatorname{dom} I) \to \operatorname{cod} I$.

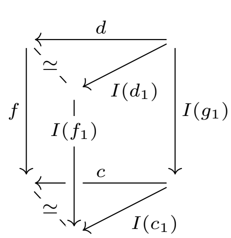
Theorem 4 Let I be a faithful functor where (i) $\operatorname{dom} I$ and $\operatorname{cod} I$ have enough projectives, (ii) I preserves covers and projectives, and (iii) I is dense in covers. Then $I \simeq \mathbf{k}\,\mathbf{p}(I)$.

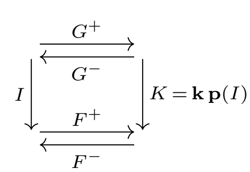
Proof. We have $\operatorname{cod} I \simeq \operatorname{cod} \mathbf{k}\,\mathbf{p}(I) = \mathrm{k}(\mathrm{id}_{\operatorname{cod} I})$, because $\operatorname{cod} I$ has enough projectives so that $\mathrm{q}(\operatorname{cod} I) = \operatorname{cod} I$ [Proposition 16]. Denote the witnessing functors as F^+, F^-. Functor F^+ maps morphism f to $(\mathrm{id}_{\operatorname{cod} f}, f, \mathrm{id}_{\operatorname{dom} f})$, and F^- is forgetful, mapping (c, f, d) to f. [Strictly speaking, we should have written $((c, \operatorname{dom} c), f, (d, \operatorname{dom} d))$ instead of (c, f, d).]

For $\operatorname{dom} I \simeq \operatorname{dom} \mathbf{k}\,\mathbf{p}(I)$, define functor G^+ on $\operatorname{dom} I$, and G^- on $\operatorname{dom} \mathbf{k}\,\mathbf{p}(I)$. On morphisms,

$$G^+(f) = \big((I(c), \operatorname{dom} c),\ I(f),\ (I(d), \operatorname{dom} d)\big),$$

where c covers $\operatorname{cod} f$ by a projective, and d covers $\operatorname{dom} f$ by a projective. The morphism $I(f)$ is tracked by some g in $\mathrm{p}(\operatorname{dom} I)$, because c is a cover, and $\operatorname{dom} d$ is projective. For G^-, suppose $((c, y), f, (d, x))$ is a morphism in $\operatorname{dom} \mathbf{k}\,\mathbf{p}(I)$. Then c, d are covers in $\operatorname{cod} I$. Take c_1, d_1 in $\operatorname{dom} I$ according to density in covers, at y, x respectively, so that $I(c_1) \simeq c$ and $I(d_1) \simeq d$. Let f have tracking morphism g_1 in $\mathrm{p}(\operatorname{dom} I)$. By density in covers, there is f_1 in $\operatorname{dom} I$ with $I(f_1) I(d_1) = I(c_1) I(g_1)$ [morphism $I(c_1) I(g_1)$ factor through $I(d_1)$ via f]. Define $G^-((c, y), f, (d, x)) = f_1$.





$G^- G^+(x) = G^-(I(c_1), p) = \operatorname{cod} c_2$, where c_1 covers x by p, $\operatorname{dom} c_2 = p$, and $I(c_1) \simeq I(c_2)$ with c_2 taken according to density. Faithful functor reflects isomorphisms, so $\operatorname{cod} c_1 \simeq \operatorname{cod} c_2$, thus $G^- G^+(x) \simeq x$. For the other composition,

$G^+G^-(c, y) = G^+(\operatorname{cod} c_1) = (I(c_2), \operatorname{dom} c_2)$, where c_1 is a cover taken according to density at y, and c_2 is a cover given by $\operatorname{dom} I$ having enough projectives. We have $c \simeq I(c_2)$ because $\operatorname{cod} c \simeq \operatorname{cod} I(c_1) = \operatorname{cod} I(c_2)$, and c, c_2 are covers with projective domains in $\operatorname{dom} I$ [so that there are tracking morphisms].

Let $K = \mathbf{k}\,\mathbf{p}(I)$. Show $F^+I \simeq KG^+$ and $IG^- \simeq F^-K$. Let f be a morphism in $\operatorname{dom} I$.

$$\begin{aligned} F^+I(f) &= (\mathrm{id}_{\operatorname{dom} I(f)}, I(f), \mathrm{id}_{\operatorname{cod} I(f)}), \\ KG^+(f) &= K((I(c), \operatorname{dom} c), I(f), (I(d), \operatorname{dom} d)) \\ &= (I(c), I(f), I(d)). \end{aligned}$$

The results are isomorphic in $\operatorname{cod} K$ due to covers and projectives. For $IG^- \simeq F^-K$, let $((c, y), f, (d, x))$ be a morphism in $\operatorname{dom} K$.

$$\begin{aligned} IG^-((c, y), f, (d, x)) &= I(f_1), \\ F^-K((c, y), f, (d, x)) &= F^-(c, f, d) = f. \end{aligned}$$

Let d_1 be the morphism taken according to density to obtain $G^-((c, y), f, (d, x)) = f_1$. We get $f \simeq I(f_1)$ by $d, I(d_1)$ being covers, hence epic. □

Category $\operatorname{dom} I$ in the theorem is not necessarily regular, thus the cover-construction alone does not guarantee regularity. The ambient category is significant.

By dualizing the definition and propositions, we have the co-cover construction for faithful functors, density of maximally terminated finite limits, and the condition for recovering a category by a co-cover construction from its injective objects.

Definition 7 (The co-cover construction) For any faithful functor I, define category $\mathrm{j}(I)$: objects are (c, c_1) where c is a co-cover in $\operatorname{cod} I$ with injective codomain, and c_1 is an object in $\operatorname{dom} I$ with $I(c_1) = \operatorname{cod} c$; morphisms are $((c, c_1), f, (d, d_1))\colon (d, d_1) \to (c, c_1)$ where f is a morphism in $\operatorname{cod} I$, such that for some $g\colon d_1 \to c_1$ in $\operatorname{dom} I$, $cf = I(g) \circ d$. Define $\mathbf{j}(I)$ as a functor $\mathrm{j}(I) \to \mathrm{j}(\mathrm{id}_{\operatorname{cod} I})$, mapping morphisms $((c, c_1), f, (d, d_1))$ to $((c, I(c_1)), f, (d, I(d_1)))$.

6 Alternation

Definition 8 (Regularity condition) A functor I satisfies the regularity condition for the cover construction, if (i) I is faithful, (ii) $\operatorname{cod} I$ is regular, (iii) projectives in $\operatorname{cod} I$ are closed under finite products and regular subobjects, and (iv) I is dense in finite products and regular monos.

Proposition 20 Let I satisfy the regularity condition. (i) $\operatorname{dom} \mathbf{k}(I)$ is regular. (ii) $\mathbf{k}(I)$ is dense in finite products, regular monos and regular epis. (iii) If I is dense in finite coproducts, then $\mathbf{k}(I)$ is dense in finite coproducts.

Proof. (i) The argument for an inclusion functor I generalizes. During the argument, we need the functor I to preserve monos. As I is faithful and dense in finite products and regular monos, I preserves pullbacks, hence preserves monos.

(ii) For density in nullary products, let z be a weak terminal object in $\operatorname{dom} I$ given by density. [z is in fact terminal due to faithfulness of I.] Then (id_1, z) is the object in $\operatorname{dom} \mathbf{k}(I)$ needed by density in nullary products, where 1 is the terminal object in $\operatorname{cod} I$. Density in unary products is trivial. For density in binary products, let (a, x), (b, y) be objects in $\operatorname{dom} \mathbf{k}(I)$. By density, there is z with $I(z) = \operatorname{dom} a \times \operatorname{dom} b$, and $(a \times b, z)$ is the object needed by density in binary products. Similarly for regular monos. For density in regular epis, let $((kc, \operatorname{dom} c), k, (c, \operatorname{dom} c))$ be an object in $\operatorname{cod} \mathbf{k}(I)$, and let (c, x) be an object in $\operatorname{dom} \mathbf{k}(I)$ with $I(x) = \operatorname{dom} c$. Then $((kc, x), k, (c, x))$ is the morphism needed by density in regular epis. It is important that k is tracked by an identity id_x.

(iii) This is a similar argument as in the product case. □

Corollary 2 Let I satisfy the regularity condition. Then $\mathbf{k}(I)$ is a faithful regular functor where (i) $\operatorname{dom} \mathbf{k}(I)$ and $\operatorname{cod} \mathbf{k}$ have enough projectives, (ii) $\mathbf{k}(I)$ preserves covers and projectives, and (iii) $\mathbf{k}(I)$ is dense in covers.

Proof. Faithfulness is by construction. Suppose (d, x), (c, y) are objects in $\operatorname{dom} \mathbf{k}(I)$, with morphisms $((c, y), f, (d, x))$ and $((c, y), g, (d, x))$. If the morphisms are mapped to the same morphism by $\mathbf{k}(I)$, namely $((c, \operatorname{dom} c), h, (d, \operatorname{dom} d))$, then $f = g = h$.

Let $J = \mathbf{k}(I)$. Categories $\operatorname{dom} J$ and $\operatorname{cod} J$ are regular, and J preserves finite products and equalizers due to density and faithfulness. Let $((kc, x), k, (c, x))$ be a cover in $\operatorname{dom} J$, then $((kc, \operatorname{dom} c), k, (c, \operatorname{dom} c))$ is a cover in $\operatorname{cod} J$. Thus covers are preserved and J is a regular functor. Projectives in $\operatorname{dom} J$ are (id_u, x) modulo isomorphism, and $J(\mathrm{id}_u, x) = (\mathrm{id}_u, u)$ is projective in $\operatorname{cod} J$. Thus J preserves projectives. For $\operatorname{dom} J$ and $\operatorname{cod} J$ having enough projectives, we can use the argument in the inclusion functor case.

By the previous proposition, J is dense in regular epis, hence dense in covers, as $\operatorname{cod} J$ is regular. □

Corollary 3 Let I satisfy the regularity condition. $\mathbf{k}(I) \simeq \mathbf{k}\,\mathbf{p}\,\mathbf{k}(I)$.

Proof. Functor $\mathbf{k}(I)$ satisfies the condition in Theorem 4, in the previous section. □

This is an analogue of $\mathrm{reg}(\mathcal{C}) \simeq \mathrm{reg}\,\mathrm{p}\,\mathrm{reg}(\mathcal{C})$, where $\mathrm{reg}(\mathcal{C})$ is the regular completion of $\mathcal{C}$.

Definition 9 (Co-regularity condition) A functor I satisfies the co-regularity condition for the co-cover construction, if (i) I is faithful, (ii) $\operatorname{cod} I$ is regular, (iii) injectives in $\operatorname{cod} I$ are closed under finite coproducts and regular quotient objects, and (iv) I is dense in finite coproducts and regular epis.

The condition is dual to the regular condition for the cover construction.

Proposition 21 Let I satisfy the following conditions: (i) the regularity condition for the cover construction, (ii) the co-regularity condition for the co-cover construction, except that density in regular epis is not required, and (iii) $\operatorname{cod} I$ has enough projectives and injectives. Then $\mathbf{k}(I)$ satisfies the regularity and co-regularity condition, and $\operatorname{cod} \mathbf{k}(I)$ has enough projectives and injectives.

Proof. By possessing enough projectives, $\operatorname{cod} I \simeq \operatorname{cod} \mathbf{k}(I)$. The cover construction gives density in regular epis, thus $\mathbf{k}(I)$ satisfies the co-regularity condition. □

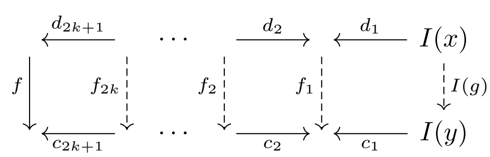
If I has the property described in the proposition, the cover construction and the co-cover construction can alternate. Let $\mathbf{j}(I)$ denote the co-cover construction on functor I. Define

$$I^{(0)} = I, \quad I^{(2k+1)} = \mathbf{k}(I^{(2k)}), \quad I^{(2k+2)} = \mathbf{j}(I^{(2k+1)}).$$

Then we obtain a series of functors $I^{(n)}$, where $I^{(2k+1)}$ is regular with enough projectives, and projectives are exactly objects in $I^{(2k)}$ up to isomorphism; similarly for $I^{(2k+2)}$. Alternatively, define

$$I_0 = I, \quad I_{2k+1} = F^{-} \circ \mathbf{k}(I_{2k}), \quad I_{2k+2} = E^{-} \circ \mathbf{j}(I_{2k+1}),$$

where F^{-} is the isomorphic forgetful functor $\operatorname{cod} \mathbf{k}(I_{2k}) \to \operatorname{cod} I$, and E^{-} is the corresponding functor $\operatorname{cod} \mathbf{j}(I_{2k+1}) \to \operatorname{cod} I$ for the co-cover construction. Then an object in I_n is a pair (c, x), where c is an n-tuple with c_{2k+1} a cover and c_{2k+2} a co-cover, and x an object in $\operatorname{dom} I$ with $I(x) = \operatorname{dom} c_1$. Morphisms in $\operatorname{dom} I_n$ are $((c, y), f, (d, x)) \colon (d, x) \to (c, y)$, where f is a morphism in $\operatorname{cod} I$, and there are $(f_i)_{0 \le i < n}$ in $\operatorname{cod} I$ and $g \colon x \to y$ in $\operatorname{dom} I$, such that (i) $f_{2i+1} d_{2i+1} = c_{2i+1} f_{2i}$ (let $f_n = f$, same below), (ii) $f_{2i+1} d_{2i+2} = c_{2i+2} f_{2i+2}$, and (iii) $I(g) = f_0$.



As each of the functors F^{-} and E^{-} is an isomorphism, $I^{(n)} \simeq I_n$.

Call the conditions in the previous proposition the *alternation condition* for the cover and co-cover constructions.

Theorem 5 Let I be the inclusion functor $\mathcal{N}_1 \to \mathbf{Set}$.

(i) I satisfies the alternation condition.

(ii) $\operatorname{dom} I_1 \simeq \mathrm{PER}$, $\operatorname{dom} I_2 \simeq \mathrm{PER}^*$ and $\operatorname{dom} I_3 \simeq \mathrm{PER}^{**}$.

(iii) PER^{**} is a regular category with enough projectives, where projectives are exactly objects in PER^* up to isomorphism.

Proof. (i) is a straightforward verification. For (ii), cases I_1, I_2 are essentially proved in Section 2; similar proof works for I_3. (iii) follows from the property of the cover construction. □

Proposition 22 Let I satisfy the alternation condition. Functor I_{2k+1} is a regular and co-cartesian functor, and I_{2k+2} is a co-regular and cartesian functor.

Proof. Functor $\mathbf{k}(I_{2k})$ is faithful, dense in finite coproducts and regular epis, thus it is co-cartesian. Functor F^- is isomorphic, so $I_{2k+1} = F^- \circ \mathbf{k}(I_{2k})$ is regular and co-cartesian. Dualize for I_{2k+2}. □

Let I satisfy the alternation condition. Some categorical structures in $\operatorname{dom} I_{2k+1}$ are as follows.

Mono $((c, y), f, (d, x))$ is monic iff f is monic.

Initial object If (c, x) is an initial object, then $\operatorname{cod} c_{2k+1} = 0$.

Binary product $(a, x) \times (b, y) = ((a_i \times b_i)_i, x \times y)$.

Equalizer The equalizer of $((c, y), f, (d, x))$ and $((c, y), g, (d, x))$ is $((d, x), h, (e, w))$, where h is the equalizer of f, g. If $k = 0$, let h_1 be the equalizer of fd_1, gd_1 taken according to density, then e_1 is unique morphism given by the equalizer h. If $k > 0$, let h_1 be any equalizer of fd_1, gd_1, then e_{2k+1} is the unique morphism given by the equalizer h, $e_{2k} = d_{2k} \circ h_1$, and $e_i = d_i$ for $i < 2k$.

Epi $((c, y), f, (d, x))$ is epic iff f is epic.

Terminal object If (c, x) is a terminal object, then $\operatorname{cod} c_{2k+1} = 1$.

Binary coproduct $(a, x) \amalg (b, y) = ((a_i \amalg b_i)_i, x \amalg y)$.

Coequalizer The coequalizer of $((c, y), f, (d, x))$ and $((c, y), g, (d, x))$ is $((b, y), h, (c, y))$, where h is the coequalizer of f, g, morphism $b_{2k+1} = hc_{2k+1}$ and $b_i = c_i$ for $i < 2k + 1$.

mu

3 Conclusion

We have shown,

(i) The category PER^*, where every object is formed by a per and a sub-per, is not regular, hence cannot serve as a model of modified realizability.

(ii) The category PER^{**}, where sub-pers are equipped with a further quotient, is regular and has enough projectives. It is a likely candidate for modelling modified realizability with extensional equality.

(iii) Both PER^* and PER^{**} can be constructed by alternating constructions, consisting of a cover construction and the dual form co-cover construction, where the cover construction resembles but differs from the regular completion.

Future work

Concerning modelling modified realizability with extensional equality, an immediate task is to characterize the locally cartesian closed structure in PER^{**}, and confirm that $\mathrm{PER}^{**} \vDash \mathrm{AC}$. To this end, results in modified assemblies could help. For example, in Streicher's *Investigations into Intensional Type Theory*, it is shown that the category of modified assemblies is locally cartesion closed and regular (Thm 3.2, p.88).[a] After that, we can try to show the equivalence of $\mathrm{PER}^{**} \vDash \phi$ and $\mathrm{HEO} \vDash \exists x(x \text{ mr } \phi)$ for any $\mathbf{E\text{-}HA}^\omega$ formula ϕ.

By the alternating constructions, we have a family of PER variants. A very basic question is: are they different? We could try to answer by asking succesive, more specific questions. Assuming the alternation condition, we know that categories obtained by cover constructions are regular. We could ask, are all categories obtained from the co-cover construction not regular? If this is true, then we separate the cover-constructed categories from the co-cover-constructed ones. To answer this question, it is natural to seek for a generalization of the proof that shows irregularity of PER^*. This could be an interesting computability theory exercise in its own. After this separation, we could ask, within each of the separated families, are all member categories different? One possible strategy could be, show that PER and PER^{**} are different (as we are expecting), then try to show that the co-cover construction followed by a cover construction preserve this difference.

In our investigation, the co-cover construction seems to produce categories with less pleasing properties. However, this could well be caused by the way we

[a]Modified assemblies are called modified realizability sets (mr-sets) there.

choose to observe them: we do pullbacks, ask for regularity, ask for adjoints of pullbacks, ask for quotients, etc.. In general, we are asking for *less information* — we would like to keep the essential and discard detail. The co-cover construction is the opposite: by separating subobjects, more information is added. The smaller the subobject is, the more precise we get. Maybe we can let those categories play their strength? For example, do pushouts, ask for co-regularity, ask for adjoints of pushouts, ask for subobjects, etc., and see do they also capture interesting notions?

We could also put more attention into the alternating construction itself. The alternating construction presented here seems to be offset by one. Objects in PER are *partial* equivalence relations, but we start the alternating constructions by performing a cover-construction, which corresponds to a total equivalence relation. There could be an initial co-cover construction, producing $(\mathcal{N}_1, \mathbf{Set})$ from some $(\mathcal{C}_1, \mathcal{C})$, where $\mathcal{C}_1$ has only one object $\mathbb{N}$ with minor modification. The minor modification in consideration is $\mathbb{N} \cup \{\bot\}$, *i.e.*, adding a helper element representing divergence. Some effort is needed to getting the definition of this monoidal category right. If case of difficulty, we may find that relations, with its categorical counterpart, allegories, could be of help. There are also notions about partial computable functionals that we could make use of, *e.g.*, call-by-name, call-by-value [16, §4].

In the big picture of the elephant (the world of realizability toposes), we can ask several questions: can we define an extensional modified realizability topos, where PER^{**} fits in naturally? What is the relation of the effective topos, the modified realizability topos, the extensional realizability topos, and the possible extensional modified realizability topos? More specifically, can we lift the cover construction up to the topos level? Speaking of the cover and co-cover constructions, it is hoped that the cover and co-cover constructions can unify some variants of realizability, as the value of a more general theory lies in providing simpler explanation. If that turns out to be successful, then by making the picture a bit more regular, the constructions can show their nontrivial value, rather than merely being another species in the zoo of categorical constructions.

Terminology and facts of basic category theory

A morphism k is a *cover* if whenever $k = mh$ with m monic, m is an isomorphism. In other words, the only subobject of the codomain that k can factor through is an isomorphism. Let $f = ik$. Morphism i is an *image* of f, if whenever $f = jh$ and j monic, $i = jm$ with m monic. In other words, i is the least subobject that f factors through. It follows that k is a cover. An object P is *projective* if for any cover k and morphism $f\colon P \to \operatorname{cod} k$, there is g with $f = kg$. A category is *regular* if it has finite limits and images, and covers are stable under pullback. The dual notions are *co-cover*, *coimage*, *injectives*, and *co-regularity*.

Proposition 23 (i) Covers are epic. (ii) Every mono with a section is an isomorphism. (iii) Regular epis are covers.

Proof. (i) Take equalizer. (ii) Call the section s; consider $ms \circ m$. (iii) If regular epi e factors as mn with m monic, then n equalizes the pair of morphisms defining the regularity of e. By universal property of equalizer e, the monic m has a section, thus an isomorphism. □

Proposition 24 In a regular category, covers are closed under (i) composition, (ii) pullback, and (iii) product.

Proof. (i) Let $e = cd$ with c, d being covers. Factor e as $\operatorname{im} e \circ \operatorname{cov} e$. Let d_1, d_2 be the kernel pair of d, and e_1, e_2 be the kernel pair of e. By the universal property of pullback, there is morphism a with $e_i a = d_i$. Thus $\operatorname{cov} e$ coequalizes d_i, and there is b with $bd = \operatorname{cov} e$ (given by coequalizer d). We have $cd = \operatorname{im} e \circ bd$, so $c = \operatorname{im} e \circ b$ and $\operatorname{im} e$ is an isomorphism. Thus $e \simeq \operatorname{cov} e$ is a cover. (ii) By definition. (iii) Suppose a, b are covers. Decompose $a \times b$ as $(\operatorname{id}_{\operatorname{cod} a} \times b)(a \times \operatorname{id}_{\operatorname{dom} b})$. Morphism $a \times \operatorname{id}_{\operatorname{dom} b}$ is a pullback of a along projection, thus a cover; similarly for $\operatorname{id}_{\operatorname{cod} a} \times b$. So $a \times b$ is a cover. □

Proposition 25 In a regular category, covers are regular epis.

See the Compendium [8], Prop. 1.3.4 for a proof.

Proposition 24 (Continued) In a regular category, covers are closed under (iv) pushout and (v) coproduct.

Proof. (iv) Regular epis are stable under pushout. Take kernel pairs of regular epi e and its push out f, obtaining a commuting square containing $\langle e_1, e_2 \rangle$ and

$\langle f_1, f_2\rangle$ (e_i, f_i are the kernel pairs). Assume a morphism coequalizing f_i; use the pushout diagram to obtain the needed unique morphism. (v) Dualize the proof in the product case. □

Proposition 26 In a regular category, (i) an object x is projective iff any cover with codomain x has a section. (ii) projectives are closed under binary coproduct.

Proof. (i) Pullback the cover. (ii) Let x, y be projectives, k be a cover with $\operatorname{cod} k = x \amalg y$. Then there are a, b with $ak = \rho_x$ and $bk = \rho_y$. The copairing $[a, b]$ is a section of k. □

Proposition 27 (Monicity) (i) In any category, a morphism f is monic iff its kernel pair are identities. (ii) A functor preserving pullbacks preserves monos. (iii) Let $(\mathcal{C}_1, \mathcal{C})$ satisfy the regularity condition for the cover construction. Then f is monic in $\mathcal{C}_1$ iff it is monic in $\mathcal{C}$.

Proof. (i) Show necessity. By monicity, suppose the kernel pair of f are a, a. By $f \circ \mathrm{id}_{\operatorname{dom} f}$ equals itself, we get b with $ab = \mathrm{id}_{\operatorname{dom} f}$. By $faba$ equals itself, we get $ba = \mathrm{id}_{\operatorname{dom} a}$. Thus a is an isomorphism and $\mathrm{id}_{\operatorname{dom} f}$ are the kernel pairs. (ii) By (i). (iii) By density, the inclusion functor $\mathcal{C}_1 \to \mathcal{C}$ preserves products and equalizers, thus preserving pullbacks and monos. □

Let $\mathcal{C}$ be a category with finite limits. The *regular completion* of $\mathcal{C}$ is a category where objects are morphisms in $\mathcal{C}$, and every morphism is an equivalence class $[g]\colon d \to c$ in which (i) $g\colon \operatorname{dom} d \to \operatorname{dom} c$ is a morphism in $\mathcal{C}$ such that cg equalizes the kernel pair of d, and (ii) $g_1 \sim g_2$ iff $cg_1 = cg_2$. The resulting category of $\mathcal{C}$ is denoted as $\operatorname{reg}(\mathcal{C})$. We say a category *has enough projectives* if every object is covered by a projective object: for any object x, there is a cover $p \to x$ with p projective. We say an object x *embeds into a projective object* if there is a mono $x \to p$ with p projective.

Proposition 28 (Carboni, Vitale) Let $\mathcal{C}$ be a category with finite limits. The regular completion $\operatorname{reg}(\mathcal{C})$ has the following properties: (i) it is regular, (ii) it has enough projectives, (iii) projectives in $\operatorname{reg}(\mathcal{C})$ are closed under finite limits, and (iv) every object in $\operatorname{reg}(\mathcal{C})$ embeds into a projective object. Conversely, if a regular category $\mathcal{D}$ has the above properties, then $\mathcal{D} \simeq \operatorname{reg}(\mathcal{P})$, where $\mathcal{P}$ is the full subcategory of projectives in $\mathcal{D}$.[a]

Proof sketch. In $\operatorname{reg}(\mathcal{C})$, (1) $[f]\colon d \to c$ is monic iff the kernel pairs of cf and d coincide, (2) if e is a split epi, then $[e]$ is a cover, and (3) the cover-image factorization of $[f]_{c,d}$ is $[f]_{c,cf}[\mathrm{id}_x]_{cf,d}$, where $[f]_{c,d}$ abbreviates $[f]\colon d \to c$, and $x = \operatorname{dom} d = \operatorname{dom} f$. Then for all object y in $\mathcal{C}$, id_y is projective in $\operatorname{reg}(\mathcal{C}_1)$: for any cover $[k]_{c,d}$, its image $[k]_{c,ck}$ is an isomorphism and k is a split epi; use the section to define the needed factor in the definition of a projective object. Every object f in $\operatorname{reg}(\mathcal{C})$ is covered by $[\mathrm{id}_x]\colon \mathrm{id}_x \to f$, where $x = \operatorname{dom} f$. Every object f is embedded into a projective object by $[f]\colon f \to \mathrm{id}_y$, where $y = \operatorname{cod} f$.

[a]This form is taken from the Compendium [8], Remarks 1.3.10(b).

For finite limits, the functor $I\colon x \mapsto \mathrm{id}_x$ is full and faithful, so it preserves finite limits.

For the converse, define functors $F\colon \mathcal{C} \to \mathrm{reg}(\mathcal{P})$ and $G\colon \mathrm{reg}(\mathcal{P}) \to \mathcal{C}$. For objects x in $\mathcal{C}$, define $F(x) = mc$, where m embeds x into a projective, and c covers x by a projective. For morphisms $f\colon x \to y$, suppose $F(x) = nd$ and $F(y) = mc$. Then $F(f)$ is defined as the class of morphisms $g\colon \mathrm{dom}\, d \to \mathrm{dom}\, c$ such that $fd = cg$. Since d is the coequalizer of the kernel pair of nd, the composition $cg = fd$ coequalizes the kernel pair of nd.

For objects a in $\mathrm{reg}(\mathcal{P})$ (a morphism between projectives in $\mathcal{C}$), define $G(a)$ as the image of a; for morphisms $[g]\colon a \to b$, define $G([g])$ as the unique morphism $\mathrm{dom}\,\mathrm{im}\, a \to \mathrm{dom}\,\mathrm{im}\, b$. It is given by the universal property of coequalizer $\mathrm{cov}\, a$, as $\mathrm{cov}\, b \circ g$ coequalizes the kernel pair of a.

$GF(x) \simeq x$, because the cover-image factorization in a regular category is unique up to isomorphism. $FG(a) = a' \simeq a$, because for $g\colon \mathrm{dom}\, a \to \mathrm{dom}\, a'$ and $h\colon \mathrm{dom}\, a' \to \mathrm{dom}\, a$ given by the projectives, $\mathrm{cov}\, a \circ hg = \mathrm{cov}\, a$, thus hg is an identity, so for gh. □

Proposition 29 Let L be a full functor where every object in $\mathrm{cod}\, L$ is covered by some $L(x)$ and embedded into some $L(y)$. For any functor F with $\mathrm{dom}\, F = \mathrm{dom}\, L$ and $\mathrm{cod}\, F$ regular, if $F = GL$ with G preserving covers and monos, then G is unique.

Proof. Suppose G, G_1 satisfy the property in the conclusion. Let g be a morphism in $\mathrm{cod}\, L$. Cover-embed $\mathrm{dom}\, g$ properly with d, m, cover-embed $\mathrm{cod}\, g$ properly with c, n, then we get $G(d) \simeq G_1(d)$ and $G(n) \simeq G_1(n)$, using fullness of L and regularity of $\mathrm{cod}\, F$. We have $ngc = L(f)$ for some f, and $G(ngc) = G_1(ngc) = F(f)$. Covers are epic, thus $G(ng) \simeq G_1(ng)$ and $G(g) \simeq G_1(g)$. □

Proposition 30 Let L be a full cartesian embedding functor with the following universal property: for any cartesian functor F with $\mathrm{dom}\, F = \mathrm{dom}\, L$ and $\mathrm{cod}\, F$ regular, there is a unique regular functor $\tilde{F}$ with $F = \tilde{F}L$. Then $\mathrm{cod}\, L \simeq \mathrm{reg}(\mathrm{dom}\, L)$. (The Compendium [8], Theorem 1.3.9.)

Proof. Let R be the inclusion $\mathrm{dom}\, L \to \mathrm{reg}(\mathrm{dom}\, L)$. Define $\tilde{L}$ as $\tilde{L}(c, [g], d) = L(\mathrm{cod}\,\mathrm{cov}\, d) \to L(\mathrm{cod}\,\mathrm{cov}\, c)$ given by the universal property of $\mathrm{cov}\, d$ [being coequalizer of its kernel pair]. We can verify that $L = \tilde{L}R$ with $\tilde{L}$ regular. Let $\tilde{R}$ be the unique regular functor with $R = \tilde{R}L$. Then $R = \tilde{R}L = \tilde{R}\tilde{L}R$ with $\tilde{R}\tilde{L}$ regular, thus preserving covers and monos. We also have $R = \mathrm{id}_{\mathrm{cod}\, R} \circ R$, thus $\tilde{R}\tilde{L} \simeq \mathrm{id}_{\mathrm{cod}\, R}$, as R has the property that every object in $\mathrm{cod}\, R$ is covered by some $R(x)$ and embedded into some $R(y)$. Similarly, $\tilde{L}\tilde{R} \simeq \mathrm{id}_{\mathrm{cod}\, L}$. Thus $\mathrm{cod}\, L \simeq \mathrm{cod}\, R = \mathrm{reg}(\mathrm{dom}\, L)$. □

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