Filtered Order-partial Combinatory Algebras and Classical Realizability

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written by

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Abstract

This thesis is mainly about classical realizability. We study a general construction of abstract Krivine structures from filtered order-partial combinatory algebras. This construction gives interesting models of classical realizability, in the sense that the corresponding Krivine toposes are not Grothendieck. From this construction, we also get a characterization of Krivine toposes among the class of realizability-related toposes. In addition, we generalize some important results about order-partial combinatory algebras to those of filtered order-partial combinatory algebras.

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Contents

Chapter 1

Introduction

The theory of realizability was originated in 1940s, when Stephen Cole Kleene tried to give a precise description of the link between intuitionism and the theory of recursive functions [Kleene, 1945]. The basic idea is to assign a set of natural numbers as *realizers* to each sentence in the language of arithmetic. A number n *realizes* a sentence ϕ is defined inductively as: [1]

(1) n realizes an atomic sentence A if and only if $n = 0$ and A is true;
(2) $n = \langle m, k \rangle$ realizes $\psi_1 \wedge \psi_2$ if and only if m realizes ψ_1 and k realizes ψ_2;
(3) $n = \langle m, k \rangle$ realizes $\psi_1 \vee \psi_2$ if and only if either $m = 0$ and k realises ψ_1 or $m = 1$ and k realizes ψ_2;
(4) n realizes $\psi_1 \to \psi_2$ if and only if n is the Gödel number of a partial recursive function F, such that for each m that realizes ψ_1, $F(m)$ is defined and realizes ψ_2;
(5) $n = \langle m, k \rangle$ realizes $\exists x \psi(x)$ if and only if k realizes $\psi(\overline{m})$;
(6) n realises $\forall x \psi(x)$ if and only if n is the Gödel number of a total recursive function F, such that for all m, $F(m)$ realises $\psi(\overline{m})$,

where $\langle \cdot, \cdot \rangle$ denotes the primitive recursive bijection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$, and $\overline{m}$ is the canonical term in the language of arithmetic that denotes the natural number m.

From then on, a lot of variations and extensions of realizability have been discovered. In particular, recursive functions have been generalized to *partial combinatory algebras* by Feferman [Feferman, 1975]. In 1980, Martin Hyland, Peter Johnstone and Andrew Pitts published the landmark paper *Tripos theory* [Hyland et al., 1980], where they showed that one can construct a topos out of any partial combinatory algebra through the *tripos-to-topos construction*. It brought a new perspective to the research of realizability, sometimes called *topos-theoretic account of realizability*.

[1] This definition is from section 2.2 in [Van Oosten, 2002].

The basic algebras for realizability have been further generalised by Pieter Hofstra and Jaap van Oosten in [Hofstra and van Oosten, 2003], where they defined the notion of *order-partial combanitory algebras* [2]. Later in 2006, Hofstra proposed an even more general notion, *basic combinatorial objects* in [Hofstra, 2006], and distinguished a subclass called *filtered order-partial combanitory algebras*[3].

However, these results were only known for intuitionistic logic for long. Classical logic, due to its non-constructiviness label, was not considered as applicable to realizability. From 2000 on, Jean-Louis Krivine started to develop his theories of *classical realizability* based on extensions of the λ-calculus in a sequence of papers, e.g. [Krivine, 2009]. Thomas Streicher explored this way further, by showing that Krivine's classical realizability can be adapted to the topos-theoretic view [Streicher, 2013]. More precisely, Streicher found that from Krivine classical realizability structures, there is a way of constructing filtered order-partial combinatory algebras which give rise to triposes. Hence, through the tripos-to-topos construction, every Krivine classical realizability structure corresponds to a topos, a *Krivine topos*. These new findings connect two fields: on one side, there are the well-studies theories of realizability-related toposes, and on the other side, there are classical logic, Peano arithmetic, Zermelo-Frænkel set theory and so on.

In this thesis, we aim to firstly develop the theory of filtered order-partial combinatory algebras further, and secondly characterize Krivine toposes by investigating geometric morphisms between the general realizability-related toposes and Krivine toposes.

In Chapter 2, we will introduce the underlying algebras for realizability, i.e., basic combinatorial objects, filtered order-partial combinatory algebras and partial combinatory algebras. Chapter 3 deals with triposes and the tripos-to-topos construction. Classical realizability will be discusses in Chapter 4. The main results of this thesis are also there. We will give a characterization of Krivine toposes in section 4.3, and give a non-trivial model of classical realizability in section 4.4. Chapter 5 contains conclusions and possible directions for future research.

[2] They are called *ordered partial combinatory algebras* in [Hofstra and van Oosten, 2003], the notion here is from [Van Oosten, 2008]

[3] Similarly, the notion in [Hofstra, 2006] is an *ordered partial combinatory algera with filter*.

Chapter 2

Partial Combinatory Algebras

The history of partial combinatory algebras (pcas) can be dated back to early 1920s. In 1920, Moses Schönfinkel gave a talk on *combinatory logic*,[1] where a version of total combinatory algebras was described. The more general partial version was introduced by Solomon Feferman [Feferman, 1975] in 1970s. About one decade before, in 1960s, Eric Wagner proposed an axiomatic framework with the notion of *uniformly reflexive structures*, aiming at developing computability theory in an abstract manner [Wagner, 1963]. These structures turned out to be precisely *decidable* partial combinatory algebras.

Pcas and realizability have been connected with each other ever since realizability was born,[2] although the notion of pcas was not clear then. After pcas were used as the basic building blocks for realizability toposes[3] in the paper *Tripos theory* [Hyland et al., 1980], people started to search for suitable definitions of morphisms between pcas to make a category that correlates well with the category of realizability toposes.

In his PhD thesis, John Longley proposed the notion of *applicative morphisms* of pcas and showed that a particular kind of geometric morphisms of realizability triposes are induced by adjoint pairs of applicative morphisms of pcas [Longley, 1995]. Subsequently, Pieter Hofstra and Jaap van Oosten concentrated on a subclass of applicative morphisms, called *computationally dense* morphisms in [Hofstra and van Oosten, 2003]. They also extended the notion of partial combinatory algebras to that of order-partial combinatory algebras (order-pcas). Pieter Hofstra extended this work further in [Hofstra, 2006] by exploring what is the least structure which gives rise to triposes in the canonical way. He started with a pre-realizability notion, *basic combinatorial objects*, and concluded

[1] The content of this talk was later published in 1924, see [Schönfinkel, 1924].

[2] As we will mention later, the number realizability Kleene defined in [Kleene, 1945] is an example of pca.

[3] Pcas are called *partial applicative structures*, and realizability triposes are called *recursive realizability triposes* in [Hyland et al., 1980].

that at least an order-pca structure with a filter (called *filtered order-pca*) is needed. Interestingly, filtered order-pcas are the structures that Thomas Streicher used to combine Krivine's classical realizabilty and tripos theories [Streicher, 2013].

In this chapter, we will introduce all the basic algebras that are needed to study classical realizabilty. Though, as described above, the history of the development of realizability-related algebras is from special to general, I will treat them in the reverse order. Firstly, the most general notion, basic combinatorial objects, will be introduced in section 2.1, then filtered order-pcas and the special subclass, pcas, will be discusses in section 2.2. The morphisms between filtered order-pcas will be treated in section 2.3.

2.1 Basic Combinatorial Objects

We start with the most general framework. Almost all material in this section is from [Hofstra, 2006].

2.1.1 The category BCO

Definition 2.1. Let $(\Sigma, \leq)$ be a partially ordered set and $\mathcal{F}_\Sigma$ be a set of partial endofunctions on Σ. The tuple $(\Sigma, \leq, \mathcal{F}_\Sigma)$ is called a *basic combinatorial object* (BCO) if it satisfies:

1. $\forall f \in \mathcal{F}_\Sigma\ \forall a \in dom(f)\ \forall b \in \Sigma\ [b \leq a \Rightarrow b \in dom(f) \wedge f(b) \leq f(a)]$.
2. $\exists i \in \mathcal{F}_\Sigma\ \forall a \in \Sigma\ [a \in dom(i) \wedge i(a) \leq a]$.
3. $\forall f, g \in \mathcal{F}_\Sigma\ \exists h \in \mathcal{F}_\Sigma\ \forall a \in dom(f)\ [f(a) \in dom(g) \Rightarrow a \in dom(h) \wedge h(a) \leq g(f(a))]$.

Definition 2.2. A *morphism* $\phi : (\Sigma, \leq, \mathcal{F}_\Sigma) \to (\Theta, \leq, \mathcal{F}_\Theta)$ between two BCOs is a function $\phi : \Sigma \to \Theta$ satisfying:

1. There exists $u \in \mathcal{F}_\Theta$, such that for all $a \leq a'$ in Σ we have $u(\phi(a)) \leq \phi(a')$;
2. For all $f \in \mathcal{F}_\Sigma$, there exists $g \in \mathcal{F}_\Theta$, such that for all $a \in dom(f)$, $\phi(a) \in dom(g)$ and $g(\phi(a)) \leq \phi(f(a))$.

Proposition 2.3. *Basic combinatorial objects and their morphisms form a category, call it* ***BCO***.

The category **BCO** is enriched in preorders. We give the definition of *preorder-enriched category* here.

Definition 2.4. A category $\mathcal{C}$ is called *preorder-enriched* if for any pair of objects A, B in $\mathcal{C}$, there is a preorder structure on the class of morphisms $\mathcal{C}(A, B)$ from A to B, and the composition is functorial in respect to the preorder structures, i.e., for any objects A, B, C of $\mathcal{C}$, the composition map

$$\mathcal{C}(A,B) \times \mathcal{C}(B,C) \to \mathcal{C}(A,C)$$

is order-preserving.

Let $\mathcal{C}$ be a preorder-enriched categor. For any parallel morphisms $f, g : A \to B$ in $\mathcal{C}$, we call f *isomorphic* to g ($f \simeq g$) if $f \leq g$ and $g \leq f$.

For two morphisms $f : A \to B$ and $g : B \to A$, we call g *left adjoint* to f (or f right adjoint to g), write as $g \dashv f$, if $g \circ f \leq id_A$ and $id_B \leq f \circ g$. We call $g \dashv f$ an *adjoint pair of morphisms* from A to B.

Definition 2.5. Let $(\Sigma, \leq, \mathcal{F}_\Sigma)$, $(\Theta, \leq, \mathcal{F}_\Theta)$ be BCOs. For two parallel morphisms $\phi, \psi : \Sigma \to \Theta$, define

$$\phi \leq \psi \quad \Leftrightarrow \quad \exists g \in \mathcal{F}_\Theta, \forall a \in \Sigma, \phi(a) \in dom(g) \wedge g(\phi(a)) \leq \psi(a).$$

It is easy to see that the relation defined above is reflexive and transitive and the requirements on morphisms ensure that composition is functorial.

2.1.2 Internal Finite limits

The category **BCO** has binary products with all structures taken coordinatewise. It also has a terminal object 1, that is the one element poset equipped with the identity function.

With the preorder structure on morphisms, there is a definition of *internal finite limits*, which is different from finite limits in the category **BCO**.

Definition 2.6. A BCO $(\Sigma, \leq, \mathcal{F}_\Sigma)$ is said to have a *top element* $\top$ if the map $\Sigma \to 1$ has a right adjoint $f_\top : 1 \to \Sigma$ where $f_\top(*) = \top$, for the only element $*$ in 1.

When the diagonal morphism $\Delta : \Sigma \to \Sigma \times \Sigma$ has a right adjoint $\wedge : \Sigma \times \Sigma \to \Sigma$, we say that $(\Sigma, \leq, \mathcal{F}_\Sigma)$ has *finite products*[4] and call $\wedge : \Sigma \times \Sigma \to \Sigma$ a *finite-products* map.

$(\Sigma, \leq, \mathcal{F}_\Sigma)$ is said to have *internal finite limits* if it has both top element and finite products.

[4] "Binary products" would be a better name, but it is called "finite products" in [Hofstra, 2006], we follow his notion here.

It is easy to verify that a BCO $(\Sigma, \leq, \mathcal{F}_\Sigma)$ has a top element $\top$ if and only if there is $f \in \mathcal{F}_\Sigma$, such that for all $a \in \Sigma$, $f(a) \leq \top$.

The top element is generally not unique, we call them *designated truth-values*. The formal definition is given in the following.

Definition 2.7. If $(\Sigma, \leq, \mathcal{F}_\Sigma)$ is a BCO with a top element $\top$, then we define

$$TV(\Sigma) \coloneqq \{a \in \Sigma : \exists f \in \mathcal{F}_\Sigma, f(\top) \leq a\},$$

and call elements of $TV(\Sigma)$ the *designated truth-values* of Σ.

From the above definition, it is easy to derive that every $v \in TV(\Sigma)$ has the property that there is $f \in \mathcal{F}_\Sigma$, such that for all $a \in \Sigma$, $f(a) \leq v$, hence, v is also a top element.

As for finite products of BCO $(\Sigma, \leq, \mathcal{F}_\Sigma)$, $\wedge : \Sigma \times \Sigma \to \Sigma$ is a finite-products map if and only if it is a morphism of BCOs from the product $(\Sigma, \leq, \mathcal{F}_\Sigma) \times (\Sigma, \leq, \mathcal{F}_\Sigma)$ to $(\Sigma, \leq, \mathcal{F}_\Sigma)$, and there are $f_0, f_1, g \in \mathcal{F}_\Sigma$, such that for all $a, b \in \Sigma$, $f_0(a \wedge b) \leq a$, $f_1(a \wedge b) \leq b$ and $g(a) \leq (a \wedge a)$.

In the following text, we will mainly concerned about BCOs which have internal finite limits.

We end this part by a definition of *internal-finite-limits-preserving morphisms*.

Definition 2.8. Let Σ, Θ be BCOs which have top elements $\top_\Sigma, \top_\Theta$ and finite products $\wedge_\Sigma$, $\wedge_\Theta$, then we say a morphism $\phi : \Sigma \to \Theta$ *preserves internal finite limits* if

1. there is $g \in \mathcal{F}_\Theta$, such that $g(\top_\Theta) \leq \phi(\top_\Sigma)$;[5]
2. there is $h \in \mathcal{F}_\Theta$, such that for all $a, b \in \Sigma$, $h(\phi(a) \wedge_\Theta \phi(b)) \leq \phi(a \wedge_\Sigma b)$.[6]

2.1.3 Downset Monad on the category BCO

In this section, we will describe a monad on the category **BCO**.

Firstly, definitions of monads and algebras.

Definition 2.9. Let $\mathcal{C}$ be a category, $T : \mathcal{C} \to \mathcal{C}$ an endofunctor, $\mu : T^2 \Rightarrow T$ and $\eta : id_\mathcal{C} \Rightarrow T$ natural transformations. The triple (T, μ, η) is called a *monad* if the

[5]From the definition of top element, it is easy to see that there is a $g' \in \mathcal{F}_\Theta$, such that $g'(\phi(\top_\Sigma)) \leq \top_\Theta$.
[6]Similarly, from the definition of finite-products map, there is an $h' \in \mathcal{F}_\Theta$, such that for all $a, b \in \Sigma$, $h'(\phi(a \wedge_\Sigma b)) \leq \phi(a) \wedge_\Theta \phi(b)$.

following diagrams commute:

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, "id_T"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, "id_T"] \\ & T & \end{tikzcd} .$$

Given a monad (T,μ,η) on a category $\mathcal{C}$, a T-*algebra* is a pair (X,h), where X is an object of $\mathcal{C}$ and $h : T(X) \to X$ is an arrow of $\mathcal{C}$ such that the following diagrams commute:

$$\begin{tikzcd} T^2(X) \arrow[r, "T(h)"] \arrow[d, "\mu_X"'] & T(X) \arrow[d, "h"] \\ T(X) \arrow[r, "h"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "id_X"'] & T(X) \arrow[d, "h"] \\ & X \end{tikzcd} .$$

If $\mathcal{C}$ is an preorder-enriched category, there is a notion of *pseudo-algebras*, where the defining diagrams for algebras only required to commute up to natural isomorphism.

Let $\Sigma = (\Sigma, \leq, \mathcal{F}_\Sigma)$ be a BCO, consider $\mathcal{D}\Sigma := \{\alpha \subseteq \Sigma : \alpha \text{ is downward closed}\}$. Let

$$\mathcal{H} := \{F : \mathcal{D}\Sigma \rightharpoonup \mathcal{D}\Sigma \mid \forall \alpha \in dom(F) \forall \beta \in \mathcal{D}\Sigma\ [\beta \subseteq \alpha \Rightarrow \beta \in dom(F) \wedge F(\beta) \subseteq F(\alpha)]\},$$

in which $F : \mathcal{D}\Sigma \rightharpoonup \mathcal{D}\Sigma$ means F is a partial function from $\mathcal{D}\Sigma$ to $\mathcal{D}\Sigma$. Define $\mathcal{F}_{\mathcal{D}\Sigma} := \{F \in \mathcal{H} \mid \exists f \in \mathcal{F}_\Sigma, \forall \alpha \in dom(F), \forall a \in \alpha,\ (f(a) \in F(\alpha))\}$. Then $(\mathcal{D}\Sigma, \subseteq, \mathcal{F}_{\mathcal{D}\Sigma})$ is a BCO.

In the category **BCO**, we consider the following endofunctor $\mathcal{D} : \mathbf{BCO} \to \mathbf{BCO}$:

On objects, it assigns a BCO Σ to $\mathcal{D}\Sigma = (\mathcal{D}\Sigma, \subseteq, \mathcal{F}_{\mathcal{D}\Sigma})$;
On morphisms, it maps $\phi : \Sigma_1 \to \Sigma_2$ to $\mathcal{D}\phi : \mathcal{D}\Sigma_1 \to \mathcal{D}\Sigma_2$, where

$$\mathcal{D}\phi(\alpha) := \downarrow \phi[\alpha] = \{b \in \Sigma_2 : \exists a \in \alpha, b \leq \phi(a)\},$$

for any $\alpha \in \mathcal{D}\Sigma$.

Let $\downarrow(-) : id_{\mathbf{BCO}} \Rightarrow \mathcal{D}$ defined as: for any Σ, $\downarrow(-) : a \mapsto \downarrow\{a\} = \{a' \in \Sigma : a' \leq a\}$ for any $a \in \Sigma$.

Let $\bigcup : \mathcal{D}\mathcal{D} \Rightarrow \mathcal{D}$ defined as: on any Σ, for any $U \in \mathcal{D}\mathcal{D}\Sigma$, $\bigcup$ sends U to its union $\bigcup U \in \mathcal{D}\Sigma$.

Proposition 2.10. $(\mathcal{D}, \downarrow(-), \bigcup)$ *is a monad on the category* ***BCO****. We call it the downset monad.*

2.2 Filtered Order-Partial Combinatory Algebras

This section is based on section 1.8 in [Van Oosten, 2008], where order-partial combinatory algebras are treated in detail. As we will show in this section, almost all important properties of order-partial combinatory algebras can be generalised to filtered ones.

2.2.1 Filtered Order-pcas

We start with the definition of order-partial applicative structures.

An *order-partial applicative structure* (order-pas) is a structure with a poset $(A, \leq)$ and a partial function $A \times A \to A$ (called *application*), we denote it as $(a, b) \mapsto ab$.

Let V be an infinite set of variables, we define $E(A)$ the set of *terms* over A as the least set containing A, V and closed under application, i.e., for all $t, s \in E(A)$, $(ts) \in E(A)$.

A term is called *closed* if no variable occurs in it. We define a relation $t{\downarrow}\, a$ (term t denotes element a) between the set of closed terms and A, as the least relation satisfying:

for all $a \in A$, $a{\downarrow}\, a$ and
$(ts){\downarrow}\, a$ if and only if there are $b, c \in A$, $t{\downarrow}\, b$, $s{\downarrow}\, c$ and $bc = a$.

We write $t{\downarrow}$ (t denotes) for a closed term, if there is $a \in A$, $t{\downarrow}\, a$. For any two closed terms t and s, we write $t \precsim s$, if $t{\downarrow}$ and $t \leq s$ whenever $s{\downarrow}$.

Definition 2.11. An *order-partial combinatory algebra* (order-pca) is an order-pas which has distinguished elements $\mathsf{k}, \mathsf{s} \in A$ (not necessarily unique), such that for all elements a, b, a', b', c in A,

1. If $a' \leq a$ and $b' \leq b$, then $a'b' \precsim ab$;
2. $\mathsf{k}ab{\downarrow}$ and $\mathsf{k}ab \leq a$;
3. $\mathsf{s}ab{\downarrow}$ and $\mathsf{s}abc \precsim ac(bc)$.

A *partial combinatory algebra* (pca) A is an order-pca of which the order $(A, \leq)$ is discrete.

Any order-pca is weakly combinatory complete, in the sense that any term is represented by an element in it. The following is the definition.

Definition 2.12. An order-pas A is called *weakly combinatory complete*, if for any term $t(x_1, \cdots, x_{n+1})$, there is an element $a \in A$ such that for any $a_1, \cdots, a_{n+1} \in A$, $aa_1 \cdots a_n{\downarrow}$ and $aa_1 \cdots a_{n+1} \precsim t(a_1, \cdots, a_{n+1})$.

Proposition 2.13. *(Theorem 1.8.4 in [Van Oosten, 2008])*
Let A be an order-pas. A carries an order-pca structure if and only if A is weakly combinatory complete.

Definition 2.14. A *filter* in an order-pca A is a subset $A' \subseteq A$, such that A' is closed under application and contains a choice of k and s which satisfy the axioms for A. We write the order-pca A with filter A' as (A, A'), and call it a *filtered order-pca*.

Any order-pca A can be seen as the filtered order-pca (A, A).

We also remark here that A' with the partial order and partial application restricting to it is again an order-pca, hence, it also possesses weakly combinatory completeness. We prove a slightly different version of it here.

Lemma 2.15. *Let (A, A') be a filtered order-pca. For any term $t(x_1, \cdots, x_{n+1})$, where only elements in A' and variables occur, there is an element $\langle x_1 \cdots x_{n+1} \rangle t \in A'$, such that for any $a_1, \cdots, a_{n+1} \in A$, $(\langle x_1 \cdots x_{n+1} \rangle t) a_1 \cdots a_n \downarrow$ and $(\langle x_1 \cdots x_{n+1} \rangle t) a_1 \cdots a_{n+1} \precsim t(a_1, \cdots, a_{n+1})$.*

Proof. Let $\mathsf{k}, \mathsf{s} \in A'$. Following the proof of Theorem 1.1.3 in [Van Oosten, 2008], we define for every variable x and term t, a term $\langle x \rangle t$ inductively as:

1. $\langle x \rangle t = \mathsf{k}t$, if t is a constant $a \in A'$ or variable $y \neq x$;
2. $\langle x \rangle x = \mathsf{skk}$;
3. $\langle x \rangle t_1 t_2 = \mathsf{s}(\langle x \rangle t_1)(\langle x \rangle t_2)$.

Let $\langle x_1 \cdots x_{n+1} \rangle t := \langle x_1 \rangle (\langle x_2 \rangle (\cdots (\langle x_{n+1} \rangle t) \cdots))$. By an easy induction, it is an element in A'. Other properties can also be verified easily by induction. □

2.2.2 Coding of Finite Sequences

The property of weakly combinatory completeness is very powerful. It enables recursion theory inside any filtered order-pca, in the sense that every partial recursive function can be represented by some closed terms in a suitable way. As a consequence, there is a coding of finite sequences of elements in filtered order-pcas. This coding will be used in later chapters. We need some preparations to introduce this coding. First of all, we distinguish some useful closed terms.

i and $\bar{\mathsf{k}}$ are the most commonly used combinators apart from k and s. They are defined as $\mathsf{i} := \mathsf{skk}$ and $\bar{\mathsf{k}} := \mathsf{ki}$ respectively. In any filtered order-pca (A, A'), i and $\bar{\mathsf{k}}$ will denote some elements in the filter A'. It is easy to work out that, for any $a, b \in A$, $\mathsf{i}a\downarrow$, $\bar{\mathsf{k}}ab\downarrow$ and $\mathsf{i}a \leq a$, $\bar{\mathsf{k}}ab \leq b$.

Let $\mathsf{p} := \langle xyz\rangle zxy$, $\mathsf{p}_0 := \langle v\rangle v\mathsf{k}$ and $\mathsf{p}_1 := \langle v\rangle v\bar{\mathsf{k}}$. By Lemma 2.15, in any filtered order-pca (A, A'), p, p_0 and p_1 are all in the filter A'. Moreover, for any $a, b \in A$, $\mathsf{p}ab$, $\mathsf{p}_0(\mathsf{p}ab)$ and $\mathsf{p}_1(\mathsf{p}ab)$ denote, and satisfy the following equations

$$\mathsf{p}_0(\mathsf{p}ab) \leq a; \qquad \mathsf{p}_1(\mathsf{p}ab) \leq b.$$

We call p the *pairing operator*, and p_0, p_1 the *projection operators*.

Secondly, we need a representation of natural numbers.

Definition 2.16. Let (A, A') be a filtered order-pca, the *curry numerals* are defined inductively as:

$$\overline{0} := \mathsf{i};$$
$$\overline{n+1} := \mathsf{p}\bar{\mathsf{k}}\overline{n}.$$

Note that all curry numerals are inside the filter A'.

It might be the case that for any $m, n \in \mathbb{N}$, $\overline{m} = \overline{n}$ in A. However, when A is a non-trivial pca, (i.e., A contains more than one element), then for any $m \neq n$, $\overline{m} \neq \overline{n}$. Hence, we have a copy of natural numbers inside any non-trivial pca.

The last tool is the existence of the *primitive recursion operator* inside any filtered order-pca.

Proposition 2.17. *Let (A, A') be a filtered order-pca. There is an element $\mathsf{R} \in A'$, such that for all $a, f \in A$, for all $n \in \mathbb{N}$:*

$$\mathsf{R}af\overline{0} \lesssim a;$$
$$\mathsf{R}af\overline{n+1} \lesssim f\overline{n}(\mathsf{R}af\overline{n}).$$

The proof is totally analogous to that of Proposition 1.3.5 in [Van Oosten, 2008].

With all these preparations, we can code finite sequences of elements in any filtered order-pca (A, A').

Define maps $J^n : A^n \to A$ for $n > 0$ inductively as:

$$J^1(a) := a;$$
$$J^{n+1}(a_1, \cdots, a_{n+1}) := \mathsf{p}a_1 J^n(a_2, \cdots, a_{n+1}).$$

Suppose $u_0, \cdots, u_{n-1}$ is a finite sequence of elements of A, define its code $[u_0, \cdots, u_{n-1}]$ as:

$$\begin{aligned} [\,] &:= \mathsf{p}\overline{0}\overline{0} && (n = 0); \\ [u_0, \cdots, u_{n-1}] &:= \mathsf{p}\overline{n}J^n(u_0, \cdots, u_{n-1}) && (n > 0). \end{aligned}$$

With the primitive recursive operator, we can construct operators $\mathsf{b_i}, \mathsf{c_i} \in A'$, $i \in \mathbb{N}$, such that for any $[u_0, \cdots, u_k]$ with $k \geq i$, $\mathsf{b_i}[u_0, \cdots, u_k]\downarrow$ and $\mathsf{b_i}[u_0, \cdots, u_k] \leq u_i$, $\mathsf{c_i}[u_0, \cdots, u_k]\downarrow$ and $\mathsf{c_i}[u_0, \cdots, u_k] \leq [u_i, \cdots, u_k]$.

Also, there is an operator d in filter A', such that for any $a \in A$, for any $[u_0, \cdots, u_k]$, $\mathsf{d}a[u_0, \cdots, u_k]\downarrow$ and $\mathsf{d}a[u_0, \cdots, u_k] \leq [a, u_0, \cdots, u_k]$. Similarly, there are operators $\mathsf{t}, \mathsf{t}' \in A'$, such that for all $a \in A$, $\mathsf{t}a \leq [a]$ and $\mathsf{t}'[a] \leq a$.

2.2.3 Examples of Filtered Order-pcas

We list some well-studied examples of filtered order-pcas, some of which will be used in the following chapters.

2.2.3.1 Examples of Pcas

The best known pcas are due to Kleene.

Kleene's First Model:
Fix a coding of all Turing machines. We write φ_e for the partial recursive function computed by the Turing machine with code e, and write $\varphi_e(n)$ for the output of the partial recursive function on value n. Note that $\varphi_e(n)$ can be undefined.

Kleene's first model (or the *number realizability*) $\mathcal{K}_1$ is the set $\mathbb{N}$ with partial recursive application $a, b \mapsto \varphi_a(b)$.

Kleene's Second Model:
Kleene's second model $\mathcal{K}_2$ is also called *function realizability*. The carrier set is $\mathbb{N}^{\mathbb{N}}$, the set of all functions from natural numbers to natural numbers. Take a 1-1, surjective coding a finite sequences of natural numbers by natural numbers. We write the coding of the sequence $u_0, \cdots, u_k$ as $\langle u_0, \cdots, u_k \rangle$. Application is defined in the following way: for any $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$, $\alpha\beta\downarrow$ and $\alpha\beta := \gamma$ for some $\gamma \in \mathbb{N}^{\mathbb{N}}$, if and only if

$$\forall n \in \mathbb{N}, \exists k \in \mathbb{N},\ \alpha(\langle n, \beta(0), \cdots, \beta(k)\rangle) = \gamma(n) + 1 \ \wedge\ \forall s < k,\ \alpha(\langle n, \beta(0), \cdots, \beta(s)\rangle) = 0.$$

It can be shown that with this application, there are $\mathsf{k}, \mathsf{s} \in \mathbb{N}^{\mathbb{N}}$ satisfying the requirements for a pca. The set of all recursive functions contains k, s and is closed under application, hence is a filter in $\mathcal{K}_2$.

2.2.3.2 Examples of Order-pcas

Meet-semilattice:

Any meet-semilattice is an order-pca. Suppose $(A, \wedge)$ is a meet-semilattice. If we take $\wedge$ as the application map, and any two elements a, b as k and s respectively, then we get an order-pca.

Downset Monad:

As described in section 2.1.3, the functor $\mathcal{D}$ maps a BCO to another BCO. In particular, if $\Sigma = (A, A')$ with $\mathsf{k}_\Sigma, \mathsf{s}_\Sigma$ is a filtered order-pca, then $\mathcal{D}\Sigma$ is a BCO. We can define application and a filter on $\mathcal{D}\Sigma$ to make it a filtered order-pca, which will still be denoted as $\mathcal{D}\Sigma$.[7]

Application is defined as: for any $\alpha, \beta \in \mathcal{D}(A)$, $\alpha\beta\downarrow$ if and only if for all $a \in \alpha, b \in \beta$, $ab\downarrow$. When $\alpha\beta\downarrow$, define $\alpha\beta :=\downarrow\{ab : a \in \alpha, b \in \beta\}$. With this definition, we have $\mathsf{k}_{\mathcal{D}\Sigma} :=\downarrow\{\mathsf{k}_\Sigma\}$ and $\mathsf{s}_{\mathcal{D}\Sigma} :=\downarrow\{\mathsf{s}_\Sigma\}$ satisfy requirements of order-pcas.

The filter $\Phi_{\mathcal{D}\Sigma}$ is defined as $\Phi_{\mathcal{D}\Sigma} := \{\alpha \in \mathcal{D}\Sigma, \alpha \cap A' \neq \varnothing\}$.

If A is a pca, seen as a filtered order-pca $\Sigma = (A, A)$, then $\mathcal{D}\Sigma$ is the order-pca $(\mathcal{P}(A), \subseteq)$ with the filter $\mathcal{P}(A) \setminus \{\varnothing\}$.

2.3 Applicative Morphisms

In this section, we deal with the morphisms between filtered order-pcas. As mentioned before, Longley is among the first to define morphisms between pcas [Longley, 1995], which are called *applicative morphisms*. Hofstra and van Oosten extended this notion to order-pcas [Hofstra and van Oosten, 2003], and highlighted a subclass of morphisms called *computationally dense morphisms*. Hofstra extended both these notions to the category **BCO** [Hofstra, 2006].

[7] Indeed, as will be discussed in section 3.2, if $[-, \mathcal{D}\Sigma]$ is a tripos, then Σ is a filtered order-pca and $\mathcal{D}\Sigma$ carries a filtered order-pca structure as described here.

2.3.1 Filtered Order-pcas as BCOs

We first clarify that filtered order-pcas are special BCOs and define morphisms between filtered order-pcas as a special subclass of morphisms between BCOs.

A filtered order-pca (A, A') can be seen as a BCO $(A, \leq, \mathcal{F}_{A'})$ by taking $\mathcal{F}_{A'} := \{\phi_a | a \in A'\}$, where $\phi_a : A \to A$ is the function sending any $b \in A$ to $ab \in A$.

Moreover, any filtered order-pca has internal finite limits as a BCO. To see this, firstly note that k is a top element witnessed by $\langle x \rangle \mathsf{k} \in A'$.

Secondly, let p be the paring operator in A'. We claim the map $(a, b) \mapsto \mathsf{p}ab$ is the finite-products map. Let p_0, p_1 be the projection operators in A' and consider $\langle x \rangle \mathsf{p}xx \in A'$, we have for all $a, b \in A$, $\mathsf{p}_0(\mathsf{p}ab) \leq a$, $\mathsf{p}_1(\mathsf{p}ab) \leq b$ and $(\langle x \rangle \mathsf{p}xx)a \leq \mathsf{p}aa$. We only need to check that the map $(a, b) \mapsto \mathsf{p}ab$ is a morphism of BCOs from $(A, \leq, \mathcal{F}_{A'}) \times (A, \leq, \mathcal{F}_{A'})$ to $(A, \leq, \mathcal{F}_{A'})$. For the first condition of Definition 2.2, suppose $(a, b) \leq (c, d)$, then $a \leq c, b \leq d$, and we have $\mathsf{i}(\mathsf{p}ab) \leq \mathsf{p}ab \leq \mathsf{p}cd$. For the second condition, for all $(a', b') \in A' \times A'$, there is $\langle x \rangle \mathsf{p}(a'(\mathsf{p}_0 x))(b'(\mathsf{p}_1 x)) \in A'$, such that for all $(a, b) \in A \times A$, if $a'a\downarrow, b'b\downarrow$, then $((\langle x \rangle \mathsf{p}(a'(\mathsf{p}_0 x))(b'(\mathsf{p}_1 x)))(\mathsf{p}ab) \leq \mathsf{p}(a'a)(b'b)$.

In a filtered order-pca, the set of designated truth-values has a simple description: it is the upward closure of the filter.

Lemma 2.18. *For a filtered order-pca $\Sigma = (A, A')$, $TV(\Sigma) = \{a \in A : \exists a' \in A', a' \leq a\}$.*

Proof. It is clear that $\{a \in A : \exists a' \in A', a' \leq a\} \subseteq TV(\Sigma)$. Suppose $a \in TV(A)$, then there is $a' \in A$, such that for all $b \in A$, $a'b \leq a$. Take $\mathsf{k} \in A'$, then $a'\mathsf{k} \in A'$ and $a'\mathsf{k} \leq a$. Therefore, $a \in \{a \in A : \exists a' \in A', a' \leq a\}$, and we get $TV(\Sigma) \subseteq \{a \in A : \exists a' \in A', a' \leq a\}$. □

2.3.2 Applicative Morphisms

The definition of morphisms between filtered-order pcas is not the direct application of the morphisms between BCOs. Rather, it is a generalization of applicative morphisms between pcas as defined by Longley in [Longley, 1995].

Definition 2.19. An *applicative morphism of filtered order-pca* $(A, A') \to (B, B')$ is a function $f : A \to B$ satisfying:

1. For all $a' \in A'$, there is $b' \in B'$, such that $b' \leq f(a')$.
   (That is, f maps A' into the upward closure of B'.)

2. There is an element $r \in B'$ such that for all $a' \in A'$, $a \in A$, whenever $a'a\downarrow$ in A, $rf(a')f(a)\downarrow$ in B and $rf(a')f(a) \leq f(a'a)$.

3. There is an element $u \in B'$ such that whenever $a \leq a'$ in A, $uf(a)\downarrow$ and $uf(a) \leq f(a')$ in B.

Though applicative morphisms between filtered order-pcas are not exactly morphisms between BCOs, there is a very close relation between them.

The following lemma extends Hofstra's Lemma 5.1 in [Hofstra, 2006] to the filtered case.

Lemma 2.20. *Let (A, A'), (B, B') be filtered order-pcas. The morphism of BCOs $\phi : A \to B$ is an applicative morphism of filtered order-pcas precisely when it preserves internal finite limits.*

Proof. The proof is basically the same as the one for Lemma 5.1 in [Hofstra, 2006].

It is easy to see that applicative morphisms between filtered order-pcas are internal-finite-limits-preserving morphisms between BCOs.

Conversely, suppose ϕ preserves finite limits. To show that it is an applicative morphism between filtered order-pcas, note that the last condition of applicative morphism is exactly the same as the first condition of morphisms between BCOs, hence, we only need to verify condition 1 and 2 of definition 2.3.2.

We prove that if ϕ preserves top elements, then ϕ sends A' into the upward closure of B'. Let $\top_A, \top_B$ be top elements in A and B respectively. For any $a' \in A'$, we want to show there is $b' \in B'$, such that $b' \leq \phi(a')$. Note that $\langle x \rangle a' \in A'$ and $(\langle x \rangle a')\top_A \leq a'$. Hence, by definition of morphism of BCOs, there is $u \in B'$ with $u\phi((\langle x \rangle a')\top_A) \leq \phi(a')$. Again, by definition, there is $g \in B'$ such that $g\phi(\top_A) \leq \phi((\langle x \rangle a')\top_A)$, thus, $u(g\phi(\top_A)) \leq \phi(a')$. Since ϕ preserves top element, there is $v \in B'$ with $v\top_B \leq \phi(\top_A)$. Being a top element, there is $f \in B'$, such that $f\mathsf{k} \leq T_B$. Hence, $u(g(v(f\mathsf{k})))\downarrow$ in B' and $u(g(v(f\mathsf{k}))) \leq \phi(a')$.

For condition 2, firstly note that the morphism $(a_1, a_2) \mapsto \mathsf{p}a_1a_2$ is a finite-products morphism in any filtered order-pca (A, A'). Now consider $\langle x \rangle(\mathsf{p}_0x)(\mathsf{p}_1x) \in A'$, by the second condition of ϕ being a morphism, there is $d \in A'$, such that for all $a' \in A, a \in A$, $d\phi(\mathsf{p}a'a) \leq \phi(\langle x \rangle(\mathsf{p}_0x)(\mathsf{p}_1x)(\mathsf{p}a'a))$. Apply the first condition of ϕ being a morphism, we get a $d' \in B'$, such that $d'\phi(\mathsf{p}a'a) \leq \phi(a'a)$. By the assumption that ϕ preserves finite products, there is a $e' \in B'$, such that for all $a' \in A'$, $a \in A$, $e'(\mathsf{p}\phi(a')\phi(a)) \leq \phi(\mathsf{p}a'a)$.

Take $q := \langle xy \rangle d'(e'(\mathsf{p}xy)) \in B'$, then for any $a' \in A', a \in A$, if $a'a\downarrow$, then $q\phi(a')\phi(a)\downarrow$ and $q\phi(a')\phi(a) \leq \phi(a'a)$. ☐

It is easy to see that the identity morphism preserves internal finite limits and internal-finite-limit-preserving morphisms of BCOs are closed under composition. Therefore, filtered order-pcas with applicative morphisms form a category. It also inherits the enriched preorder from the category **BCO**: for two applicative morphisms $\phi, \psi : A \to B$ between filtered order-pcas (A, A') and (B, B'),

$$f \leq g \quad \Leftrightarrow \quad \exists b' \in B', \forall a \in A \ (b' f(a) \downarrow \ \wedge b' f(a) \leq g(a)).$$

We now introduce Longley's definition of applicative morphisms between pcas. Although the notion of applicative morphisms between filtered order-pcas is a generalization of that between pcas, an applicative morphism from pcas A to B is not an applicative morphism of filtered order-pcas from (A, A) to (B, B), rather, it is an applicative morphism from (A, A) to $(\mathcal{P}(B), \mathcal{P}(B) \setminus \varnothing)$.

Definition 2.21. Let A, B be pcas. An *applicative morphism* of pcas $f : A \to B$ assigns to every element $a \in A$ a non-empty subset $f(a)$ of B, such that there is $r \in B$, for any $a, a' \in A$, $b \in f(a), b' \in f(a')$, if $aa' \downarrow$, then $rbb' \downarrow$ and $rbb' \in f(aa')$.

Pcas with applicative morphisms between them form a category.

2.3.3 Computational Density

Computationally dense morphisms of pcas are defined by Hostra and van Oosten in [Hofstra and van Oosten, 2003]. This notion plays a key role in characterizing geometric morphisms between realizability toposes.

To define computationally density, we first define some abbreviations. Let A be a pca, $a \in A, \alpha \subseteq A$, we write $a\alpha\downarrow$ as abbreviation for the clause: for all $a' \in \alpha$, $aa'\downarrow$. And when $a\alpha\downarrow$, write $a\alpha := \{aa' : a' \in \alpha\}$.

Definition 2.22. Let A, B be pcas. An applicative morphism $f : A \to B$ of pcas is called *computationally dense* if there is an element $m \in B$ such that the following holds:

For any $b \in B$, there is $a \in A$ such that for all $a' \in A$: if $bf(a')\downarrow$, then $aa'\downarrow$ and $mf(aa') \subseteq bf(a')$.

This notion can be generalised to filtered order-pcas.

Definition 2.23. Let $(A, A'), (B, B')$ be filtered order-pcas. An applicative morphism from (A, A') to (B, B') is called *computationally dense* (cd) if there is an element $m \in B'$

satisfying:

$$\text{(cd)} \qquad \forall b' \in B' \exists a' \in A' \forall a \in A \; (b' f(a) \downarrow \Rightarrow a'a \downarrow \wedge m f(a'a) \le b' f(a)).$$

If we assume the Axiom of Choice, then we can rephrase the condition (cd) in the following way: there is a function $s : B' \to A'$ and element $m \in B'$, such that

$$\forall b' \in B' \forall a \in A \; (b' f(a) \downarrow \Rightarrow s(b')a \downarrow \wedge m f(s(b')a) \le b' f(a)).$$

In his recent paper [Johnstone, 2013], Peter Johnstone gave a simpler but equivalent condition of computationally dense morphisms of pcas.

Proposition 2.24. *(Lemma 3.2 in [Johnstone, 2013])*
An applicative morphism of pcas $f : A \to B$ is computationally dense if and only if there exists an element $t \in B$ and a function $h : B \to A$ such that for all $b \in B$, for all $b' \in f(h(b))$, $tb' = b$.

We conclude this chapter with a generalization of this equivalence to filtered order-pcas.

Theorem 2.25. *Let (A, A') and (B, B') be two filtered order-pcas and $f : A \to B$ be an applicative morphism between them, then the following are equivalent:*

1. *f is computationally dense;*
2. *There exists a function $r : B' \to A'$ and an element $t \in B'$, such that for all $b' \in B', b \in B$, if $b \le f(r(b'))$, then $tb \le b'$.*

Proof. Let $\tau, u \in B'$ be the elements that witness the second and third conditions of $f : A \to B$ being an applicative morphism respectively (see Definition 2.3.2).

If 1. holds with function $s : B' \to A'$ and $m \in B'$, for any $\mu \in B'$, define $r(\mu) := s(\mathsf{k}\mu)$, then $r(\mu) \in A'$. Let $v \le f(a')$ for some $a' \in A'$ and $v \in B'$ (such v exists since $f(a')$ belongs to the upward closure of B'), $\mathsf{h} := \langle xyz \rangle x(yz)$, and $\mathsf{e} := \langle xy \rangle yx$, we set $t := \mathsf{h}m(\mathsf{h}(\mathsf{e}v)\tau)$. Note that each subterm of t denotes in B', hence, $t \in B'$. Suppose $b \le f(r(b'))$ for some $b \in B$ and $b' \in B'$. Since $\mathsf{k}b' f(a') \downarrow$, by computational density, $s(\mathsf{k}b')a' \downarrow$, i.e., $r(b')a' \downarrow$ and $mf(r(b')a') \le \mathsf{k}b' f(a')$. Hence, $\tau f(r(b')) f(a') \le f(r(b')a')$ (since $r(b')a' \downarrow$), and $\tau b v \le \tau f(r(b')) f(a') \le f(r(b')a')$. Finally, we have

$$tb \le m(\tau b v) \le m f(r(b')a') \le \mathsf{k}b' f(a') \le b',$$

as desired.

Conversely, if 2. holds, let $\mathsf{p}, \mathsf{p}_0, \mathsf{p}_1 \in A'$ be pairing and projection operators as before, for any $\mu \in B'$, we define $s(\mu) := \mathsf{p}r(\mu) \in A'$. Let $\pi_0 \leq f(\mathsf{p}_0)$ such that $\pi_0 \in B'$, similarly, find $\pi_1 \leq f(\mathsf{p}_1)$ with $\pi_1 \in B'$, then $\tau\pi_0 \downarrow$ in B'. To see this, take any $a, a' \in A'$, then $\mathsf{p}_0(\mathsf{p}aa') \downarrow$, hence $\tau f(\mathsf{p}_0) f(\mathsf{p}aa') \downarrow$, and we get $\tau\pi_0 \leq \tau f(\mathsf{p}_0) \downarrow$. Similarly, $\tau\pi_1 \downarrow$ in B'. Define $m := \mathsf{s}(\mathsf{h}t(\mathsf{h}u(\tau\pi_0)))(\mathsf{h}u(\tau\pi_1))$, where $\mathsf{h} := \langle xyz \rangle x(yz)$, then all subterms of m denotes in B', hence $m \in B'$. Suppose $b' f(a) \downarrow$ for some $b' \in B'$, $a \in A$, then $s(b')a = \mathsf{p}r(b')a \downarrow$ in A. We only need to show that

$$mf(s(b')a) \leq b' f(a).$$

By definition, $mf(s(b')a) \leq t(u(\tau\pi_0 f(s(b')a)))(u(\tau\pi_1 f(s(b')a)))$. Note that

$$u(\tau\pi_0 f(s(b')a)) \leq u(f(\mathsf{p}_0(\mathsf{p}r(b')a))) \leq f(r(b'))$$

(the last inequality by $\mathsf{p}_0(\mathsf{p}r(b')a)) \leq r(b')$ and the property of u), then by 2., $t(u(\tau\pi_0 f(r(b')a))) \leq b'$. Therefore,

$$mf(s(b')a) \leq b'(u(\tau\pi_1 f(s(b')a))) \leq b'(uf(\mathsf{p}_1(\mathsf{p}r(b')a))) \leq b' f(a)$$

(the last inequality is by $\mathsf{p}_1(\mathsf{p}s(b')a)) \leq a$ and the property of u). In conclusion, we get $mf(s(b')a) \leq b' f(a)$. $\square$

Chapter 3

Triposes and Toposes

Due to the paper *Tripos theory* ([Hyland et al., 1980]), where a general method of constructing toposes from pcas was described, and the powerful example, the *effective topos*, discovered by Hyland in [Hyland, 1982], a large amount of attention on realizability has been attracted to the topos-theoretic view since 1980.

As far as I know, there are two advantages of the topos-theoretic view of realizability. The first one is that in comparison with the algebras for realizability, e.g. pcas, toposes have much richer structures. Toposes share a lot of general categorical properties with the category of sets (Set). Therefore, most constructions in Set also go through in any topos. The other one is that the internal logic of toposes is intuitionistic higher order logic. The truth definitions of realizability can be generalised naturally from first-order logic (or arithmetic) to higher orders.

In [Streicher, 2013], Thomas Streicher brings Krivine's classical realizability to the general topos-theoretic view, by showing that any structure for Krivine's classical realizability gives rise to a Set-tripos. The construction is not based on the notion of pcas, rather, it uses the more general one, filtered order-pcas. Hence, in this chapter, we will not restrict ourselves on the constructions in [Hyland et al., 1980]. Instead, the focus will be on the general setting: constructing triposes from BCOs, which is the main topic in [Hofstra, 2006].

Note that we have already mentioned three levels of categories: categories of BCOs, triposes and toposes. The relation between morphisms of these three categories has been studied since [Longley, 1995]. The motivating questions are: how to define morphisms between triposes, such that they correspond exactly to geometric morphisms between the induced toposes? And, what are the morphisms of BCOs, which correspond to those of the induced triposes? These problems are not completely solved, however, by the collective effort in [Longley, 1995], [Hofstra and van Oosten, 2003],[Hofstra, 2006] and [Johnstone, 2013] during the past two decades, the answers are almost there.

In this chapter, we will, firstly, give the definition of triposes in section 3.1, followed by the construction from BCOs to Set-triposes in section 3.2, with an emphasis on the characterization of the class of BCOs that can generate triposes in the canonical way. In section 3.3, the tripos-to-topos construction is given. Finally, in section 3.4, we deal with the connection between morphisms of BCOs, morphisms of triposes and geometric morphisms of toposes.

3.1 Triposes

The key notion in the construction from pcas to toposes is that of a tripos. It provides a general framework, from which a special class of toposes can be induced. One can associate a tripos to every pca, but it is not the case that every tripos is induced by some pca. This fact makes it possible to generalize theories of realizability. Order-pcas and BCOs are such generalizations. In this section, we will give the definition of triposes.

3.1.1 Preorder-enriched categories

We have defined preorder-enriched categories in the previous chapter. To define triposes, more related terminologies are needed.

Definition 3.1. Let $\mathcal{C}, \mathcal{D}$ be preorder-enriched categories. A *pseudofunctor* $F : \mathcal{C} \to \mathcal{D}$ maps an object X of $\mathcal{C}$ to an object $F(X)$ of $\mathcal{D}$, and maps arrow $f : X \to Y$ of $\mathcal{C}$ to arrow $F(f) : F(X) \to F(Y)$ of $\mathcal{D}$, such that

$$F(id_X) \simeq id_{F(X)};$$
$$F(g \circ f) \simeq F(g) \circ F(f);$$
$$F\restriction_{\mathcal{C}(X,Y)} : \mathcal{C}(X,Y) \to \mathcal{D}(F(X), F(Y)) \text{ is order-preserving.}$$

Similarly, there is a notion of pseudo-natural transformations between pseudofunctors.

Definition 3.2. Suppose $F, G : \mathcal{C} \to \mathcal{D}$ are two pseudofunctors between preorder-enriched categories. A *pseudo-natural transformation* $\mu : F \Rightarrow G$ consists of a family of arrows $(\mu_X : F(X) \to G(X))_{X \in \mathcal{C}}$ of $\mathcal{D}$, such that for any arrow $f : X \to Y$ of $\mathcal{C}$, $G(f) \circ \mu_X \simeq \mu_Y \circ F(f)$.

The following are two example of preorder-enriched category: Preord and Heypre.

The category Preord has preordered sets as objects and order-preserving maps as arrows. The order on arrows is defined point-wise.

The objects of Heypre are *Heyting prealgebras*, which are cartesian closed preorders (seen as categories) with finite coproducts (or equivalenly, preorders whose poset reflection are Heyting algebras). Morphisms between Heyting prealgebras are functors which preserve all these structures. The order on morphisms is also defined point-wise.

3.1.2 Tripos: Definition

Definition 3.3. Let $\mathcal{C}$ be a category with finite products. A *$\mathcal{C}$-tripos* P is a pseudofuntor from $\mathcal{C}^{op}$ to Heypre satisfying:

1. For every morphism $f : X \to Y$ of $\mathcal{C}$, the map $\mathsf{P}(f) : \mathsf{P}(Y) \to \mathsf{P}(X)$ has left adjoint $\exists_f$ and right adjoint $\forall_f$ in Preord, satisfying *Beck-Chevalley condition*: for any pullback diagram in $\mathcal{C}$,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ Z \arrow[r, "k"'] & W \end{tikzcd} ,$$

   the composite maps of preorders $\forall_f \circ \mathsf{P}(g)$ and $\mathsf{P}(h) \circ \forall_k$ are isomorphic.

2. For every object X of $\mathcal{C}$, there is an object $\pi(X)$ of $\mathcal{C}$ and an element $\in_X$ (*membership predicate*) of $\mathsf{P}(X \times \pi(X))$ satisfying: for every object Y of $\mathcal{C}$ and every element ϕ of $\mathsf{P}(X \times Y)$, there is a morphism $\{\phi\} : Y \to \pi(X)$ of $\mathcal{C}$, such that ϕ is isomorphic to $\mathsf{P}(id_X \times \{\phi\})(\in_X)$ in $\mathsf{P}(X \times Y)$.

If $\mathcal{C}$ is cartesian closed, then the second condition can be simplified to: there is a *generic element* in P, i.e., there is an object Σ of $\mathcal{C}$ and an element σ of $\mathsf{P}(\Sigma)$, such that for every object X of $\mathcal{C}$, every $\phi \in \mathsf{P}(X)$, there is $[\phi] : X \to \Sigma$ with $\mathsf{P}([\phi])(\sigma) \simeq \phi$ in $\mathsf{P}(X)$.

Definition 3.4. Let P, Q be two $\mathcal{C}$-triposes. A *transformation* is a pseudo-natural transformation $\mu : \mathsf{P} \Rightarrow \mathsf{Q}$ where P, Q are considered as pseudofunctors from $\mathcal{C}^{op}$ to Preord.

We can define an order on transformations of $\mathcal{C}$-triposes. For a pair of transformations $\mu, \nu : \mathsf{P} \Rightarrow \mathsf{Q}$, $\mu \le \nu$ if and only if for any object X in $\mathcal{C}$, $\mu_X \le \nu_X$ in Preord. Therefore, for a fixed category $\mathcal{C}$ with finite products, $\mathcal{C}$-triposes and transformations between them form a preorder-enriched category $\mathcal{C}$-Trip.

3.2 Triposes Constructed from BCOs

In this section, we deal with one class of triposes, namely triposes constructed from BCOs. This class covers almost all examples of triposes related to realizability, but the reader should keep in mind that it is not the case that any BCO will give rise to a tripos in this way.

3.2.1 $[-,\Sigma]$ and $[-,\mathcal{D}\Sigma]$

Let Set be the category that has sets as objects and functions between sets as arrows.

Definition 3.5. Suppose $\Sigma = (\Sigma, \leq, \mathcal{F}_\Sigma)$ is a BCO, define a functor $[-,\Sigma] : \mathsf{Set}^{op} \to \mathsf{Preord}$ as:

On objects, it sends X to $[X,\Sigma]$ the set of all functions from X to Σ, with the prorder $\leq_X$ on $[X,\Sigma]$: for any $\phi, \psi : X \to \Sigma$,

$$\phi \leq_X \psi \;\Leftrightarrow\; \exists a \in \mathcal{F}_\Sigma \;\forall x \in X \; [\phi(x) \in dom(a) \wedge a(\phi(x)) \leq \psi(x)].$$

On functions, it sends $f : Y \to X$ to $\Sigma(f) : [X,\Sigma] \to [Y,\Sigma]$, defined as $\Sigma(f)(\phi) \coloneqq \phi \circ f$, for any $\phi : X \to \Sigma$.

We remark here that when $\Sigma = (A, A')$ is a filtered order-pca, then the order on $[X,\Sigma]$ is defined as: for any $\tau, \eta : X \to A$,

$$\tau \leq_X \eta \quad \text{iff} \quad \exists a' \in A' \;\forall x \in X \; [a'\tau(x)\downarrow \wedge\, a'\tau(x) \leq \eta(x)].$$

Recall that in section 2.1.3 we have introduced the downset monad $\mathcal{D}$ on the category **BCO**. When Σ is a BCO, $\mathcal{D}\Sigma$ is also a BCO. Similarly, $\mathcal{D}_i(\Sigma)$ with carrier set $\mathcal{D}(\Sigma) \setminus \{\varnothing\}$ and all other structure defined as those in $\mathcal{D}\Sigma$ restrict to it, is also a BCO. Hence, we also have functors $[-,\mathcal{D}\Sigma]$ and $[-,\mathcal{D}_i\Sigma]$ from Set^{op} to Preord.

Generally, $[-,\Sigma]$ or $[-,\mathcal{D}\Sigma]$ are not necessarily Set-triposes. In [Hofstra, 2006], Hofstra characterised when these functors are triposes.

Proposition 3.6. *(Theorem 6.9 in [Hofstra, 2006])*
Let Σ be a BCO. Then the following are equivalent:

1. *$[-,\mathcal{D}\Sigma]$ is a tripos;*
2. *Σ carries a filtered order-pca structure.*

Note that if Σ is a filtered order-pca with the filter Φ, then the preorder on $[X, \mathcal{D}\Sigma]$ for any set X can be given equivalently by:

$$\text{for } \tau, \eta : X \to \mathcal{D}\Sigma,\ \tau \le_X \eta \quad \Leftrightarrow \quad \exists a \in \Phi\ \forall x \in X\ \forall b \in \tau(x)\ (ab{\downarrow} \wedge ab \in \eta(x)).$$

Proposition 3.7. *(Theorem 6.13 in [Hofstra, 2006])*
Let Σ be a BCO with internal finite products, then the following are equivalent:

1. *$[-, \Sigma]$ is a tripos;*
2. *Σ carries an $\mathcal{D}$-pseudo-algebra $(\Sigma, \bigvee)$ and a filtered order-pca structure with the filter Φ, such that there exists $v \in \Phi$, for any $\{c_i : i \in I\} \in \mathcal{D}\Sigma$, if $c_i a \le b$ for all $i \in I$, then $v(\bigvee\{c_i : i \in I\})a \le b$.*

3.2.2 Examples of Triposes

In this section, we introduce some important triposes constructed from BCOs.

Realizability Triposes

The fundamental examples are those triposes come from pcas. Let A be a pca, considered as an filtered order-pca (A, A), applying functor $\mathcal{D}$, we get the filtered order-pca $\mathcal{P}(A)$ with the filter $\mathcal{P}(A) \smallsetminus \{\varnothing\}$. The tripos $[-, \mathcal{P}(A)]$ is called the *realizability tripos* on A.

Relative Realizability Triposes

Let Σ be a pca A with filter A', then the tripos $[-, \mathcal{D}\Sigma]$ is called the *relative realizability tripos*. Like realizability triposes, it assigns a set X to $[X, \mathcal{P}(A)]$ the set of functions from X to $\mathcal{P}(A)$. But the order on $[X, \mathcal{P}(A)]$ is slightly different from that of realizability triposes.

Generalised Relative Realizability Triposes

Let Σ be a filtered order-pca, we call the tripos $[-, \mathcal{D}\Sigma]$ a *generalised relative realizability tripos*.

Triposes from Locales

Let Σ be a locale, i.e., regarded as a poset, Σ is a complete Heyting algebra. Σ is a filtered order-pca with $\wedge$ (meet) as application, $\mathsf{k} = \mathsf{s} = \top$ where $\top$ is the supremum of all elements in Σ, and $\{\top\}$ as the filter. It carries a $\mathcal{D}$-algebra $(\Sigma, \bigvee)$ with $\bigvee$ the supremum map such that $\wedge$ preserves $\bigvee$. Therefore, by Proposition 3.7, $[-, \Sigma]$ is a tripos. We call these triposes *localic triposes* .

3.2.3 Implicative Ordered Combinatory Algebras

For a filtered order-pca Σ, $[-, \Sigma]$ is not necessarily a tripos. Though Proposirion 3.7 gives a characterization of when $[-, \Sigma]$ is a tripos, the conditions are not straightforward to verify. There are some structures with natural requirements that always give rise to triposes. One example is the *implicative ordered combinatory algebras* (${}^{\mathcal{I}}\mathcal{OCA}$) introduced in [Ferrer Santos et al., 2014].

Definition 3.8. An *implicative ordered combinatory algebra* (${}^{\mathcal{I}}\mathcal{OCA}$) is a filtered order-pca (A, Φ) satisfying:

1. the application map is total, i.e., for all $a, b \in A$, $ab\downarrow$;
2. the partially ordered set $(A, \leq)$ is inf-complete;
3. there is a binary operation $imp : A \times A \to A$, $(a, b) \mapsto a \to b$, called implication, antitonic in the first augument and monotone in the second, and for all $a, b, c \in A$, $a \leq b \to c \quad \Rightarrow \quad ab \leq c$.
4. there is a distinguished element e in the filter Φ, such that for all $a, b, c \in A$, $ab \leq c \quad \Rightarrow \quad \mathsf{e}a \leq b \to c$.

For a filtered order-pca $\Sigma = (A, \Phi)$, being an ${}^{\mathcal{I}}\mathcal{OCA}$ is sufficient for $[-, \Sigma]$ to be a tripos.

Proposition 3.9. *(Theorem 5.8 in [Ferrer Santos et al., 2014])*
If Σ is an ${}^{\mathcal{I}}\mathcal{OCA}$, then $[-, \Sigma]$ is a tripos.

The requirements for ${}^{\mathcal{I}}\mathcal{OCA}$ are natural, but not necessary for $[-, \Sigma]$ to be a tripos. With Proposition 3.7, we can find suitable conditions to extend the notion of ${}^{\mathcal{I}}\mathcal{OCA}$s, such that the conditions are both necessary and sufficient for $[-, \Sigma]$ being to become a tripos. Basically, every requirement for ${}^{\mathcal{I}}\mathcal{OCA}$ can be weakened to the one that only requires (in)equality "up to a realizer". We formulate them in the following:

Definition 3.10. A *pre-implicative ordered combinatory algebra* (p–${}^{\mathcal{I}}\mathcal{OCA}$) is a filtered order-pca (A, Φ), satisfying:

1. there is an operator $\bigwedge : \mathcal{P}(A) \to A$, and constants $\mathsf{i}, \mathsf{i}' \in \Phi$, such that for all $\alpha \in \mathcal{P}(A), a \in \alpha$, $\mathsf{i}(\bigwedge \alpha) \leq a$, and

$$\forall \alpha \in \mathcal{P}(A)\ \forall b \in A\ (\forall a \in A(a \in \alpha \to b \leq a) \to \mathsf{i}' b \leq \bigwedge \alpha);$$

2. there is a binary operator $imp : A \times A \to A$, $(a, b) \mapsto a \to b$, constants $\mathsf{e}, \mathsf{e}' \in \Phi$, such that for all $a, b, c \in A$,

   $ab \le c \quad \Rightarrow \quad \mathsf{e}a \le b \to c$;

   $a \le b \to c \quad \Rightarrow \quad \mathsf{e}'ab \le c$.

Theorem 3.11. *For a filtered order-pca $\Sigma = (A, \Phi)$, $[-, \Sigma]$ is a tripos if and only if Σ carries a $p -^{\mathcal{I}} \mathcal{OCA}$ structure.*

To prove this theorem, we need some results in [Hofstra, 2006], as listed below.

Proposition 3.12. *(Lemma 6.12 in [Hofstra, 2006])*
Let Σ be a filtered order-pca, giving rise in the canonical way to a tripos $[-, \mathcal{D}\Sigma]$. Write $\Rightarrow$ for the implication on $\mathcal{D}\Sigma$. Assume also that $(\Sigma, \bigvee)$ is a $\mathcal{D}$-pseudo-algebra. Then the following are equivalent.

1. *$[-, \Sigma]$ has implication, given by a map $a \to b := \bigvee(\downarrow(a) \Rightarrow \downarrow(b))$;*

2. *There exists $v \in \Phi$, such that for any $\{c_i : i \in I\} \in \mathcal{D}\Sigma$, if $c_i a \le b$ for all $i \in I$, then $v(\bigvee\{c_i : i \in I\})a \le b$.*

Proposition 3.13. *(Proposition 4.2 in [Hofstra, 2006])*
For a BCO Σ, pseudo-algebra structures on Σ are in one-to-one correspondence with left adjoints to the unit $\downarrow(-) : \Sigma \to \mathcal{D}\Sigma$.

Now we turn to the proof of Theorem 3.11. For simplicity, in the proof, we will write, for any set X, the preorder $\le_X$ on $[X, \Sigma]$ as $\le$.

Proof. We first show that if Σ is a $p -^{\mathcal{I}} \mathcal{OCA}$, then $[-, \Sigma]$ is a tripos.

Since Σ is a filtered order-pca, $[-, \Sigma]$ is an indexed meet-semilattice, with the meet $\wedge$ defined as: for any set X, $\phi, \psi : X \to \Sigma$, for any $x \in X$, $(\phi \wedge \psi)(x) := \mathsf{p}\phi(x)\psi(x)$ with p the paring operator in the filter. We only need to show that it has implication, universal quantification which satisfies Beck-Chevalley condition and a generic predicate. Assume $\Sigma = (A, \le, \Phi, \to, \bigwedge, \mathsf{i}, \mathsf{i}', \mathsf{e}, \mathsf{e}')$.

1. For any set X, $\phi, \psi : X \to A$, define $\phi \to \psi$ as: $(\phi \to \psi)(x) := \phi(x) \to \psi(x)$, for any $x \in X$. We need to show that for any $\xi : X \to A$, $\xi \wedge \phi \le \psi$ iff $\xi \le \phi \to \psi$.

   Suppose $\xi \wedge \phi \le \psi$, then there is $a \in \Phi$, such that for all $x \in X$, $a(\mathsf{p}\xi(x)\phi(x)) \le \psi(x)$. Let $b := \langle xy \rangle a(\mathsf{p}xy)$, then $b \in \Phi$ and $b\xi(x)\phi(x) \le a(\mathsf{p}\xi(x)\phi(x)) \le \psi(x)$. Therefore, $\mathsf{e}(b\xi(x)) \le \phi(x) \to \psi(x)$. Let $c := \langle x \rangle \mathsf{e}(bx) \in \Phi$, then for all $x \in X$, $c\xi(x) \le \mathsf{e}(b\xi(x)) \le \phi(x) \to \psi(x) = (\phi \to \psi)(x)$. Hence, $\xi \le \phi \to \psi$.

Suppose $\xi \leq \phi \to \psi$, then there is $a \in \Phi$, such that for all $x \in X$, $a\xi(x) \leq \phi(x) \to \psi(x)$. Hence, $\mathsf{e}'(a\xi(x))\phi(x) \leq \psi(x)$. Let $b := \langle x \rangle \mathsf{e}'(a(\mathsf{p}_0 x))(\mathsf{p}_1 x) \in \Phi$, then for all $x \in X$, $b(\mathsf{p}\xi(x)\phi(x)) \leq \mathsf{e}'(a\xi(x))\phi(x) \leq \psi(x)$, therefore, $\xi \wedge \phi \leq \psi$.

2. Let $f : Y \to X$ be a function, then the universal quantification for $\psi : Y \to A$ is defined by:

$$\forall_f(\psi)(x) := \bigwedge\{\psi(y) : f(y) = x\}.$$

To show that $\forall_f$ is right adjoint to $\Sigma(f)$, we need to show that for any $\phi : X \to A, \psi : Y \to A$, $\Sigma(f)(\phi) = \phi \circ f \leq \psi$ iff $\phi \leq \forall_f(\psi)$. Suppose $\phi \circ f \leq \psi$, then there is $a \in \Phi$, such that for all $y \in Y$, $a\phi(f(y)) \leq \psi(y)$. Then for any $x \in X$, $a\phi(x) \leq \psi(y)$ with $x = f(y)$, hence, $\mathsf{i}'(a\phi(x)) \leq \bigwedge\{\psi(y) : f(y) = x\} = \forall_f(\psi)(x)$. Let $b := \langle x \rangle \mathsf{i}'(ax) \in \Phi$, then b witnesses $\phi \leq \forall_f(\psi)$. On the other hand, if $\phi \leq \forall_f(\psi)$, then there is $a' \in \Phi$, such that for all $x \in X$, $a'\phi(x) \leq \bigwedge\{\psi(y) : f(y) = x\}$. Also note that for all $y \in Y$, $\mathsf{i}(\bigwedge\{\psi(y') : f(y') = f(y)\}) \leq \psi(y)$, hence, $\mathsf{i}(a'\phi(f(y))) \leq \psi(y)$. Let $b' := \langle x \rangle \mathsf{i}(a'x)$, then for all $y \in Y$, $b'\phi(f(y)) \leq \psi(y)$, therefore, $\phi \circ f \leq \psi$.

For the Beck-Chevalley condition, suppose

$$(*) \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ Z \arrow[r, "k"'] & W \end{tikzcd}$$

is a pullback diagram in Set, we need to show that for all $\phi : Z \to A$, $\forall_f(\phi \circ g)$ is isomorphic to $\forall_k(\phi) \circ h$. For any $y \in Y$,

$$\forall_f(\phi \circ g)(y) = \bigwedge\{\phi(g(x)) : f(x) = y\} = \bigwedge\{\phi(z) : \exists x, z = g(x), f(x) = y\},$$
$$\forall_k(\phi) \circ h(y) = \bigwedge\{\phi(z) : k(z) = h(y)\}.$$

Since $(*)$ is a pullback in Set, we have for any $y \in Y$, $\{z : k(z) = h(y)\} = \{z : \exists x, z = g(x), f(x) = y\}$. Therefore, $\forall_f(\phi \circ g) = \forall_k(\phi) \circ h$.

3. We pick A in Set and $id_A \in A^A$ as the generic element. For any X and $\phi : X \to A$, it is clear that $\Sigma(\phi)(id_A) = id_A \circ \phi = \phi$.

For the other direction, suppose $[-, \Sigma]$ is a tripos, we need to establish operators $\bigwedge$ and $\to$. By Proposition 3.7, if $[-, \Sigma]$ is a tripos, then Σ is a $\mathcal{D}$-algebra, there is a morphism of BCOs $\bigvee : \mathcal{D}\Sigma \to \Sigma$, with $\bigvee \circ \downarrow(-)$ isomorphic to id_Σ, and by Proposition 3.13, $\bigvee$ is left adjoint to $\downarrow(-)$. Suppose the underlying set of Σ is A, define $\bigwedge : \mathcal{P}(A) \to A$ as: for any $\alpha \subseteq A$, $\bigwedge \alpha := \bigvee\{b \in A : \text{ for all } a \in \alpha, b \leq a\}$. Since $\bigvee \circ \downarrow(-)$ is isomorphic to id_Σ, there is $a_0 \in \Phi$, such that for all $a \in A$, $a_0(\bigvee \circ \downarrow(a)) \leq a$. Moreover, $\bigvee$ is a morphism of BCOs, hence, there is $a_1 \in \Phi$, such that if $\alpha \subseteq \beta \in \mathcal{D}\Sigma$, then $a_1(\bigvee \alpha) \leq \bigvee \beta$. For any $a \in \alpha$, $\{b \in A : \text{ for all } a \in \alpha, b \leq a\} \subseteq \downarrow(a)$, therefore,

$$a_1(\bigwedge \alpha) = a_1(\bigvee \{b \in A : \text{ for all } a \in \alpha, b \leq a\}) \leq \bigvee \circ \downarrow(a),$$

and we get $a_0(a_1 \wedge \alpha) \leq a_0(\vee \circ \downarrow(a)) \leq a$. Take $\mathsf{i} := \langle x \rangle a_0(a_1 x) \in \Phi$, then we get $\mathsf{i}(\wedge \alpha) \leq a$, for any $a \in \alpha$. On the other hand, since $\vee$ is left adjoint to $\downarrow(-)$, there is $\mathsf{i}' \in \Phi$, such that for all $\alpha \in \mathcal{D}\Sigma$, for all $a \in \alpha$, $\mathsf{i}'a \leq \vee \alpha$. For any $b \in A$, if for all $a \in \alpha$, $b \leq a$, then $b \in \{b \in A : \text{ for all } a \in \alpha, b \leq a\}$, hence, $\mathsf{i}'b \leq \vee \{b \in A : b \leq a, \text{ for all } a \in \alpha\} = \wedge \alpha$.

By Proposition 3.12, $[-, \Sigma]$ has implication, given by $a \to b := \vee(\downarrow(a) \Rightarrow \downarrow(b))$, where $\downarrow(a) \Rightarrow \downarrow(b) := \{a' \in A : a'a\downarrow, a'a \leq b\}$. For any $a, b, c \in A$, suppose $ab \leq c$, then $a \in \downarrow(b) \Rightarrow \downarrow(c) = \{a' \in A : a'b\downarrow, a'b \leq c\}$, hence $\mathsf{i}'a \leq \vee(\downarrow(b) \Rightarrow \downarrow(c)) = b \to c$. Suppose $a \leq b \to c = \vee\{a' \in A : a'b\downarrow, a'b \leq c\}$. Since for any $a' \in \{a' \in A : a'b\downarrow, a'b \leq c\}$, $a'b \leq c$, by Proposition 3.7, there is $v \in \Phi$, such that $v(\vee\{a' \in A : a'b\downarrow, a'b \leq c\})b \leq c$, hence $vab \leq v(\vee\{a' \in A : a'b\downarrow, a'b \leq c\})b \leq c$, which is the desired result.

$\square$

3.3 The Tripos-to-topos Construction

We will describe the tripos-to-topos construction in this section, following the treatment of chapter 2 in [Van Oosten, 2008].

3.3.1 Internal Logic of Triposes

This part is a preparation of the construction. We will define an interpretation of typed relational languages in triposes, and give the soundness theorem. This interpretation shows that triposes are contexts for intuitionistic logic without equality. In a sense, we can see the tripos-to-topos construction as a way of extending this interpretation to languages with equality.

Let P be a $\mathcal{C}$-tripos. A *$\mathcal{C}$-typed relational language* is a set of relational symbols each with a *type*, i.e., a sequence $(X_1, \cdots, X_n)$, $n \geq 0$, each X_i an object of $\mathcal{C}$.

Given a $\mathcal{C}$-typed relational language $\mathcal{L}$, the set of *$\mathcal{L}$-terms* contains for each object X in $\mathcal{C}$, an infinite set of variables $x_1^X, x_2^X, \cdots$ of type X and for each morphism $f : X_1 \times \cdots \times X_n \to X$, if $t_1, \cdots, t_n$ are $\mathcal{L}$-terms of type $X_1, \cdots, X_n$ respectively, then $f(t_1, \cdots, t_n)$ is a $\mathcal{L}$-term of type X.

For any term t of type X with variable $x_1^{X_1}, \cdots, x_n^{X_n}$, we define a morphism

$$[t] : X_1 \times \cdots \times X_n \to X$$

inductively by letting $[x^X]$ be the identity arrow on X and $[f(t_1, \cdots, t_n)]$ be the composition of f with $[t_i]$.

$\mathcal{L}$-formulas are defined as:
(i) $\top$ and $\bot$ are $\mathcal{L}$-formulas;
(ii) If R is a relational symbol of type $(X_1, \cdots, X_n)$ and $t_1, \cdots, t_n$ are $\mathcal{L}$-terms of type $X_1, \cdots, X_n$ respectively, then $R(t_1, \cdots, t_n)$ is an $\mathcal{L}$-formula;
(iii) If ϕ and ψ are $\mathcal{L}$-formulas, then so are $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \to \psi$ and $\neg\psi$;
(iv) If ϕ is an $\mathcal{L}$-formula and x^X is a variable, then $\forall x^X \phi$, $\exists x^X \phi$ are $\mathcal{L}$-formulas.

An *interpretation* $[\cdot]$ of $\mathcal{L}$ in P assigns to every relation symbol R of type $(X_1, \cdots, X_n)$ an element $[R]$ of $\mathsf{P}(X_1 \times \cdots \times X_n)$.

Fix an interpretation $[\cdot]$, we define for each formula ϕ with free variable $x_1^{X_1}, \cdots, x_n^{X_n}$ an element $[\phi]$ of $\mathsf{P}(X_1 \times \cdots \times X_n)$ ($[\phi]$ will be an element of $\mathsf{P}(1)$ if ϕ is a sentence, where 1 is the terminal object in $\mathcal{C}$) inductively as:
(i) $[\top]$ and $[\bot]$ are the top and bottom element of $\mathsf{P}(1)$;
(ii) If R is of type $(X_1, \cdots, X_n)$ and $y_1^{Y_1}, \cdots, y_m^{Y_m}$ are the variables appearing in $R(t_1, \cdots, t_n)$, then there is a morphism

$$Y_1 \times \cdots \times Y_m \xrightarrow{[t]:=([t_1], \cdots, [t_n])} X_1 \times \cdots X_n \;,$$

let $[R(t_1, \cdots, t_n)] := [t]^*([R]) \in \mathsf{P}(Y_1 \times \cdots \times Y_m)$, where $[t]^*$ is the abbreviation for $\mathsf{P}([t])$.
(iii) $[\phi \wedge \psi]$, $[\phi \vee \psi]$, $[\phi \to \psi]$ and $[\neg\psi]$ are defined by the corresponding operators in Heyting prealgebras;
(iv) Finally $[\forall x^X \phi]$, $[\exists x^X \phi]$ are defined by $\forall_\pi([\phi]), \exists_\pi([\phi])$, where

$$\pi : X \times X_1 \times \cdots \times X_n \to X_1 \times \cdots \times X_n$$

is the projection. (For simplicity, we assume the free variables in ϕ are $x^X, x_1^{X_n}, \cdots, x_n^{X_n}$.)

Definition 3.14. Let ϕ be a sentence in $\mathcal{L}$ and $[\cdot]$ be an interpretation. Then we say that ϕ is *true* in P or $\mathsf{P} \vDash \phi$ relative to $[\cdot]$, if $[\phi]$ is the top element of $\mathsf{P}(1)$.

Proposition 3.15. *(Soundness Theorem, Theorem 2.1.6 in [Van Oosten, 2008])* *Suppose ϕ is a sentence in a $\mathcal{C}$-typed relational language $\mathcal{L}$. If ϕ is provable in intuitionistic logic without equality, then $\mathsf{P} \vDash \phi$ for every $\mathcal{C}$-tripos and every interpretation $[\cdot]$ of $\mathcal{L}$ in P.*

3.3.2 The Construction

We now turn to the tripos-to-topos construction, which has already been mentioned a few times.

Firstly, we give the definitions of elementary toposes and geometric morphisms of toposes.

Definition 3.16. A category $\mathcal{C}$ is called an *elementary topos* if it:

1. has finite limits,
2. is cartesian closed and
3. has a subobject classifier.

Let $\mathcal{E}$ and $\mathcal{F}$ be two toposes, a *geometric morphism* $f : \mathcal{E} \to \mathcal{F}$ consists of an adjoint pair of functors $f^* \dashv f_*$ where $f_* : \mathcal{E} \to \mathcal{F}$ and $f^* : \mathcal{F} \to \mathcal{E}$, such that f^* preserves finite limits. f_* is called the *direct image* and f^* is the *inverse image* of f. We write $f = (f_*, f^*)$.

Let $\mathcal{C}$ be a category with finite products and P be a $\mathcal{C}$-tripos. Define a category $\mathcal{C}[\mathsf{P}]$ of *partial equivalence relation over* P as follows.

An Object in $\mathcal{C}[\mathsf{P}]$ is a pair $(X, \sim)$ where X is an object of $\mathcal{C}$ and $\sim$ is an element of $\mathsf{P}(X \times X)$, satisfying:

$$\mathsf{P} \vDash \forall x^X y^X (x^X \sim y^X \to y^X \sim x^X);$$
$$\mathsf{P} \vDash \forall x^X y^X z^X (x^X \sim y^X \wedge y^X \sim z^X \to x^X \sim z^X),$$

with the obvious interpretation.

A morphism $(X, \sim_X) \to (Y, \sim_Y)$ of $\mathcal{C}[\mathsf{P}]$ is an isomorphism class of elements F of $\mathsf{P}(X \times Y)$ satisfying:

$$\mathsf{P} \vDash \forall x^X y^Y (F(x^X, y^Y) \to x^X \sim_X x^X \wedge y^Y \sim_Y y^Y);$$
$$\mathsf{P} \vDash \forall x^X x'^X y^Y y'^Y (F(x^X, y^Y) \wedge x^X \sim_X x'^X \wedge y^Y \sim_Y y'^Y \to F(x'^X, y'^Y));$$
$$\mathsf{P} \vDash \forall x^X y^Y y'^Y (F(x^X, y^Y) \wedge F(x^X, y'^Y) \to y^Y \sim_Y y'^Y);$$
$$\mathsf{P} \vDash \forall x^X (x^X \sim_X x^X \to \exists y^Y F(x^X, y^Y)),$$

with the obvious interpretation.

Proposition 3.17. *(Theorem 2.2.1 in [Van Oosten, 2008])*
Let $\mathcal{C}$ be a category with finite products and P be a $\mathcal{C}$-tripos, then $\mathcal{C}[\mathsf{P}]$ is an elementary topos.

Let Σ be a filtered order-pca, if $[-,\Sigma]$ is a tripos, then it gives rise to a topos $\mathsf{Set}[-,\Sigma]$. In particular, if A is a pca, then we have the topos $\mathsf{Set}[-,\mathcal{P}(A)]$, called the *realizability topos over A*. Similarly we have *relative realizability topos* $\mathsf{Set}[-,\mathcal{D}\Sigma]$ for $\Sigma = (A, A')$ filtered pca (pca A with filter A'), and *generalized relative realizability topos* $\mathsf{Set}[-,\mathcal{D}\Sigma']$ when Σ' is a filtered order-pca.

3.4 Geometric Morphisms

This section deals with the answers to the question which class of morphisms between filtered order-pcas (or BCOs) corresponds to geometric morphisms between the generated toposes. As the structure of filtered order-pcas (or BCOs) is simpler than that of the induced toposes, the morphisms between filtered-order pcas are much easier to describe and check than geometric morphisms of toposes. Hence, any answer to the question above will bring much convenience for the study of realizability-related toposes.

Since the construction from BCOs to toposes contains two steps: first from BCOs to Set-triposes, then from triposes to toposes, the answer to the question above also has two parts. The first one is from morphisms of triposes to geometric morphisms of the corresponding toposes. And the second one is from morphisms of BCOs to that of the generated Set-triposes.

We will treat a special class of geometric morphisms, the *inclusions*, at the end of this section. Inclusions are related to the notions of subtriposes and subtoposes, which will play an important role in the characterization of Krivine toposes in the next chapter.

3.4.1 Geometric Morphisms of Triposes

Let P and Q be $\mathcal{C}$-triposes. A *geometric morphism* $\mathsf{P} \to \mathsf{Q}$ is an adjoint pair $\Phi^+ \dashv \Phi_+$ of transformations, with $\Phi_+ : \mathsf{P} \Rightarrow \mathsf{Q}$, $\Phi^+ : \mathsf{Q} \Rightarrow \mathsf{P}$, such that for any object X in $\mathcal{C}$, the preorder-map $\Phi^+_X : \mathsf{Q}(X) \to \mathsf{P}(X)$ preserves finite meets. We write it $\Phi = (\Phi_+, \Phi^+)$.

Let $\mathcal{C}$ be a category with finite products and P be a $\mathcal{C}$-tripos. For any object X of $\mathcal{C}$, $\mathsf{P}(X)$ is an object of Heypre, there is a top element $\top_X$ in $\mathsf{P}(X)$. Let $\delta : X \to X \times X$ be the diagonal arrow, then $\exists_\delta(\top_X)$ is an element of $\mathsf{P}(X \times X)$. It can be shown that $[X, \exists_\delta(\top_X)]$ is an object of $\mathcal{C}[\mathsf{P}]$. For any arrow $f : X \to Y$ of $\mathcal{C}$, there is an element $\exists_{\langle id_X, f\rangle}(\top_X)$ of $\mathsf{P}(X \times Y)$. The isomorphism class of $\exists_{\langle id_X, f\rangle}(\top_X)$ is an arrow of $\mathcal{C}[\mathsf{P}]$ from $(X, \exists_\delta(\top_X))$ to $(Y, \exists_\delta(\top_Y))$. So, we can define a functor $\nabla_{\mathsf{P}} : \mathcal{C} \to \mathcal{C}[\mathsf{P}]$ as:

for any object X in $\mathcal{C}$, $\nabla_{\mathsf{P}}(X) \coloneqq (X, \exists_\delta(\top_X))$;

for a morphism $f : X \to Y$ in $\mathcal{C}$, ∇_{P} sends f to the isomorphism class of $\exists_{\langle id_X, f\rangle}(\top_X)$.

Definition 3.18. Let P, Q be $\mathcal{C}$-triposes, the inverse image Φ^* of a geometric morphism $(\Phi_*, \Phi^*) : \mathcal{C}[\mathsf{P}] \to \mathcal{C}[Q]$ is said to *preserve constant objects* if the following diagram commutes up to natural isomorphism:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\nabla_{\mathsf{Q}}"] \arrow[dr, "\nabla_{\mathsf{P}}"'] & \mathcal{C}[\mathsf{Q}] \arrow[d, "\Phi^*"] \\ & \mathcal{C}[\mathsf{P}] \end{tikzcd} .$$

Proposition 3.19. *(Theorem 2.5.8 in [Van Oosten, 2008])*
Let $\Phi = (\Phi_+, \Phi^+) : \mathsf{P} \to \mathsf{Q}$ *be a geometric morphism of triposes, then it induces a geometric morphism of toposes* $(\Phi_*, \Phi^*) : \mathcal{C}[\mathsf{P}] \to \mathcal{C}[\mathsf{Q}]$ *whose inverse image part preserves constant objects.*

Conversely, every geometric morphism $\mathcal{C}[\mathsf{P}] \to \mathcal{C}[\mathsf{Q}]$ *whose inverse image preserves constant objects is induced by an essentially unique geometric morphism of triposes.*

3.4.2 Geometric Morphisms of BCOs

In this section, we study which class of morphisms between BCOs corresponds to geometric morphisms between triposes. Notions in this part mainly come from [Hofstra, 2006].

Definition 3.20. Let Σ and Θ be BCOs with internal finite limits. A *geometric morphism* from Σ to Θ consists of an adjoint pair of morphisms $\phi^\circ \dashv \phi_\circ$ with $\phi_\circ : \Sigma \to \Theta$, $\phi^\circ : \Theta \to \Sigma$, such that the inverse image part ϕ° preserves internal finite limits.

Definition 3.21. Let $\phi : \Sigma \to \Theta$ be a morphism of BCOs. Then ϕ is called *computationally dense* if there exists an $h \in \mathcal{F}_\Theta$, such that for all $g \in \mathcal{F}_\Theta$ there exists $f \in \mathcal{F}_\Sigma$ such that for all a with $\phi(a) \in dom(g)$, we have $h\phi(f(a)) \le g(\phi(a))$.

Note that when Σ and Θ are filtered order-pcas, then this definition is exactly the same as Definition 2.23.

Proposition 3.22. *(Lemma 7.5 in [Hofstra, 2006])*
Let Σ, Θ *be BCOs with finite limits. Any geometric morphism* $\psi : \mathcal{D}\Theta \to \mathcal{D}\Sigma$ *of BCOs is induced by a unique computationally dense map* $\phi : \Sigma \to \mathcal{D}\Theta$ *up to natural isomorphism.*

Before Krivine triposes were introduced, almost all triposes related to realisibility are generalized relative realizability triposes, i.e., triposes of the form $[-, \mathcal{D}\Sigma]$ for a filtered order-pca Σ. As a consequence, more attention has been paid to the geometric morphisms between this class of triposes. The following result is a characterization of geometric morphisms between triposes of the form $[-, \Sigma]$.

Theorem 3.23. *Let $\Sigma_1 = (A_1, A_1')$ and $\Sigma_2 = (A_2, A_2')$ be filtered order-pcas, such that both $[-, \Sigma_1]$ and $[-, \Sigma_2]$ are triposes, then the following are equivalent:*

1. *a geometric morphism (f_+, f^+) of triposes from $[-, \Sigma_1]$ to $[-, \Sigma_2]$;*

2. *a geometric morphism $f^\circ \dashv f_\circ : \Sigma_1 \underset{f_\circ}{\overset{f^\circ}{\rightleftarrows}} \Sigma_2$ of BCOs;*

3. *a geometric morphism $\mathsf{Set}[-, \Sigma_1] \longrightarrow \mathsf{Set}[-, \Sigma_2]$ whose inverse image preserves constant objects.*

Proof. The equivalence between 1 and 3 is directly from Proposition 3.19.

We only need to establish the equivalence between 1 and 2.

Claim 3.24. Every geometric morphism (f_+, f^+) from $[-, \Sigma_1]$ to $[-, \Sigma_2]$ induces an adjoint pair $I(f^+) \dashv I(f_+) : \Sigma_1 \underset{I(f_+)}{\overset{I(f^+)}{\rightleftarrows}} \Sigma_2$ of BCO-morphisms such that $I(f^+)$ preserves finite limits. Conversely, every geomtric morphism $f^\circ \dashv f_\circ : \Sigma_1 \underset{f_\circ}{\overset{f^\circ}{\rightleftarrows}} \Sigma_2$ of BCOs induces a geometric morphism $(V(f_\circ), V(f^\circ))$ of triposes from $[-, \Sigma_1]$ to $[-, \Sigma_2]$. Moreover, we have $IV(f^\circ) = f^\circ$, $IV(f_\circ) = f_\circ$ and for any set X, any $\tau \in [X, \Sigma_1]$, $\eta \in [X, \Sigma_2]$, $VI(f^+)(X)(\eta) \simeq f^+(X)(\eta)$, $VI(f_+)(X)(\tau) \simeq f_+(X)(\tau)$.

Proof. Let (f_+, f^+) be a geometric morphism from $[-, \Sigma_1]$ to $[-, \Sigma_2]$. Consider $f_+(A_1) : [A_1, \Sigma_1] \to [A_1, \Sigma_2]$. Let $id_{\Sigma_1} : A_1 \to A_1$ be the identity function, then $id_{\Sigma_1} \in [A_1, \Sigma_1]$, define $I(f_+) \coloneqq f_+(A_1)(id_{\Sigma_1}) : A_1 \to A_2$. Similarly, we define $I(f^+) \coloneqq f^+(A_2)(id_{\Sigma_2}) : A_2 \to A_1$. We show that $id_{\Sigma_2} \leq I(f_+) \circ I(f^+)$ and $I(f^+) \circ I(f_+) \leq id_{\Sigma_1}$. Since $f^+ \dashv f_+$, we have $id_{\Sigma_2} \leq f_+(A_2)(f^+(A_2)(id_{\Sigma_2}))$ and $f^+(A_1)(f_+(A_1)(id_{\Sigma_1})) \leq id_{\Sigma_1}$, hence, $id_{\Sigma_2} \leq f_+(A_2)(I(f^+))$ and $f^+(A_1)(I(f_+)) \leq id_{\Sigma_1}$. Consider the following square:

$$
\begin{tikzcd}
{[A_1, \Sigma_1]} \arrow[r, "f_+(A_1)"] \arrow[d, "\Sigma_1(I(f^+))"'] & {[A_1, \Sigma_2]} \arrow[d, "\Sigma_2(I(f^+))"] \\
{[A_2, \Sigma_1]} \arrow[r, "f_+(A_2)"'] & {[A_2, \Sigma_2]}
\end{tikzcd}
,
$$

by f_+ being a pseudo-natural transformation, we have

$$f_+(A_2)(I(f^+)) = f_+(A_2)(\Sigma_1(I(f^+))(id_{\Sigma_1})) \simeq \Sigma_2(I(f^+))(f_+(A_1)(id_{\Sigma_1})) = I(f_+) \circ I(f^+),$$

hence $id_{\Sigma_2} \leq I(f_+) \circ I(f^+)$. Similarly, we can show that

$$f^+(A_1)(I(f_+)) \simeq I(f^+) \circ I(f_+) \leq id_{\Sigma_1}.$$

$I(f^+)$ preserves internal finite limits: let $\top_1$, $\top_2$ be the top elements of Σ_1 and Σ_2 respectively, clearly, the function $\tau : A_2 \to A_2$ which sends any $a \in A_2$ to $\top_2$ is a top element in $[A_2, \Sigma_2]$, since $f^+(A_2)$ preserves finite meets, in particular the top element, hence, we have $f^+(A_2)(\tau) \simeq \eta$ where $\eta : A_2 \to A_1$ is defined by $\eta(a) = \top_1$, for all $a \in A_2$. We use the pseudo-natural transformation f^+ again here:

$$\begin{tikzcd}
{[A_2, \Sigma_2]} \arrow[r, "f^+(A_2)"] \arrow[d, "\Sigma_2(\tau)"'] & {[A_2, \Sigma_1]} \arrow[d, "\Sigma_2(\tau)"] \\
{[A_2, \Sigma_2]} \arrow[r, "f^+(A_2)"'] & {[A_2, \Sigma_1]}
\end{tikzcd},$$

then $I(f^+) \circ \tau = \Sigma_2(\tau)(f^+(A_2)(id_{\Sigma_2})) \simeq f^+(A_2)(\Sigma_2(\tau)(id_{\Sigma_1})) = f^+(A_2)(\tau) \simeq \eta$. Hence, there is $u \in A_1'$, such that $u\top_1 \leq I(f^+)(\top_2)$. We conclude that $I(f^+)$ preserves the top element.

Let p, $\mathsf{p}_0, \mathsf{p}_1$ be the pairing and projection operators. We need to show that $I(f^+)$ preserves finite products. Let $\pi_0, \pi_1 : \Sigma_2 \times \Sigma_2 \to \Sigma_2$ be the projections to the first and second coordinates respectively. By the assumption that $f^+(A_2)$ preserves finite meets,

$$f^+(A_2)(\pi_0 \wedge \pi_1) \simeq f^+(A_2)(\pi_0) \wedge f^+(A_2)(\pi_1),$$

where $(\pi_0 \wedge \pi_1)(a, b) := \mathsf{p}ab$, and

$$(f^+(A_2)(\pi_0) \wedge f^+(A_2)(\pi_1))(a, b) := \mathsf{p}(f^+(A_2)(\pi_0)(a, b))(f^+(A_2)(\pi_1)(a, b)),$$

for all $a, b \in A_2$. Use the naturality of f^+, we have $f^+(A_2)(\pi_0 \wedge \pi_1) \simeq I(f^+) \circ (\pi_0 \wedge \pi_1)$, $f^+(A_2)(\pi_0) \simeq I(f^+) \circ \pi_0$ and $f^+(A_2)(\pi_1) \simeq I(f^+) \circ \pi_1$. Therefore,

$$(I(f^+) \circ \pi_0) \wedge (I(f^+) \circ \pi_1) \simeq f^+(A_2)(\pi_0) \wedge f^+(A_2)(\pi_1) \simeq f^+(A_2)(\pi_0 \wedge \pi_1) \simeq I(f^+) \circ (\pi_0 \wedge \pi_1).$$

Hence, there is $v \in A_1'$, such that $v\mathsf{p}I(f^+)(a)I(f^+)(b) \leq I(f^+)(\mathsf{p}ab)$.

$I(f^+), I(f_+)$ are BCO-morphisms: we need to verify conditions 1 and 2 in definition 2.2 for $I(f^+)$ and $I(f_+)$.

Condition 1 for $I(f_+)$: for any $a' \in A_1'$, let $B := \{a \in A_1 : a'a \downarrow\}$, let $\tau_{a'}, id_B : B \to A_1$ defined by for any $a \in B$, $\tau_{a'}(a) := a'a$ and $id_B(a) := a$. Then $\tau_{a'}, id_B \in [B, \Sigma_1]$ and $id_B \leq \tau_{a'}$ witnessed by $a' \in A_1'$. Therefore, $f_+(B)(id_B) \leq f_+(B)(\tau_{a'})$. By naturality of

f_+, $f_+(B)(\tau_{a'}) \simeq I(f_+) \circ \tau_{a'}$, similarly, $f_+(B)(id_B) \simeq I(f_+) \circ id_B$. Therefore, $I(f_+) \circ id_B \leq I(f_+) \circ \tau_{a'}$, which is exactly there is an $b' \in A_2'$, such that for all $a \in A_1$, if $a'a\downarrow$, then $b'I(f_+)(a)\downarrow$ and $b'I(f_+)(a) \leq I(f_+)(aa')$.

Condition 2 for $I(f_+)$: let $C = \{(a, a') : a \leq a', a, a' \in A_1\}$, $\pi_0, \pi_1 : C \to A_1$ be the projections on the first and second coordinates. Then $\pi_0 \leq \pi_1$ in $[C, \Sigma_1]$, hence, $f_+(C)(\pi_0) \leq f_+(C)(\pi_1)$. Use the naturality again, we have $I(f_+) \circ \pi_0 \simeq f_+(B)(\pi_0) \leq f_+(C)(\pi_1) \simeq I(f_+) \circ \pi_1$, and we get the desired result. Use the same arguments for $I(f^+)$, we see that condition 1 and 2 also hold for $I(f^+)$.

Conversely, suppose $f^\circ \dashv f_\circ : \Sigma_1 \underset{f_\circ}{\overset{f^\circ}{\rightleftarrows}} \Sigma_2$ is an adjoint pair of BCO-morphisms, and f° preserves finite limits. Define $V(f_\circ)$ as: for any set X, $\tau : X \to A_1$, $V(f_\circ)(X)(\tau) := f_\circ \circ \tau$. We define $V(f^\circ)$ in the same way. The naturality of $V(f_\circ)$ and $V(f^\circ)$ is clear. Also, for any X, any $\tau : X \to A_1$, $V(f^\circ)(X) \circ V(f_\circ)(X)(\tau) = f^\circ \circ f_\circ \circ \tau \leq \tau$ since $f^\circ \circ f_\circ \leq id_{\Sigma_1}$. Similarly, for any X, any $\tau : X \to A_2$, $V(f_\circ)(X) \circ V(f^\circ)(X)(\tau) = f_\circ \circ f^\circ \circ \tau \geq \tau$ since $f_\circ \circ f^\circ \geq id_{\Sigma_2}$.

If $\tau \leq \eta : X \to A_1$ in $[X, \Sigma_1]$, then there is $v \in A_1'$, such that for all $x \in X$, $v\tau(x)\downarrow$ and $v\tau(x) \leq \eta(x)$. By condition 1 of BCO-morphisms (Definition 2.2), there is $b' \in A_2'$, such that for all $x \in X$, $b'f_\circ(\tau(x))\downarrow$ and $b'f_\circ(\tau(x)) \leq f_\circ(v\tau(x))$. By condition 2 of BCO-morphisms (Definition 2.2), there is $u \in A_2'$, such that for all $x \in X$, $uf_\circ(v\tau(x))\downarrow$ and $uf_\circ(v\tau(x)) \leq f_\circ(\eta(x))$. Take $b := \langle x \rangle u(b'x)$, then for all $x \in X$, $bf_\circ(\tau(x))\downarrow$ and $bf_\circ(\tau(x)) \leq f_\circ(\eta(x))$, hence, $V(f_\circ)(X)$ preserves order. Similarly, $V(f^\circ)(X)$ also does.

Let $t_X : X \to A_2$ be defined as $t_X(x) := \top_2$, then by f° preserving top element, for any $x \in X$,

$$V(f^\circ)(X)(t_X)(x) = f^\circ \circ t_X(x) = f^\circ(\top_2) \simeq \top_1,$$

where $f(\top_2) \simeq \top_1$ means there are $a_1, a_2 \in A_1'$, such that $a_1 f(\top_2) \leq \top_1$ and $a_2 \top_1 \leq f(\top_2)$. Similarly, for any $\tau, \eta : X \to A_2$,

$$V(f^\circ)(X)(\tau \wedge \eta) = f^\circ(\tau \wedge \eta) \simeq f^\circ(\tau) \wedge f^\circ(\eta) = V(f^\circ)(X)(\tau) \wedge V(f^\circ)(X)(\eta).$$

Hence, $V(f^\circ)(X)$ preserves internal finite limits.

Therefore, $(V(f_+), V(f^+))$ is a geometric morphism of triposes from $[-, \Sigma_1]$ to $[-, \Sigma_2]$.

It is clear that $IV(f^\circ) = f^\circ$, $IV(f_\circ) = f_\circ$. On the other hand, for any set X, any $\tau : X \to A_1$, $VI(f_+)(X)(\tau) = I(f_+) \circ \tau = f_+(X)(id_{\Sigma_1}) \circ \tau \simeq f_+(X)(\tau)$ where the last step is by naturality of f_+. Similarly, for any $\eta : X \to A_2$, $VI(f^+)(X)(\eta) \simeq f^+(X)(\eta)$. □

□

We remark here that this theorem is not a new result, most of them is an easy corollary of Proposition 3.1 in [Hofstra, 2006], which states that *the assignment $\Sigma \mapsto [-,\Sigma]$ is the object part of an embedding of the preorder-enriched category **BCO** into the Set-index preorders.* The only thing that is not contained in Proposition 3.1 in [Hofstra, 2006] is that internal-finite-limits-preserving morphisms of BCOs correspond to finite-meet-preserving transformations of Set-index preorders of the form $[-,\Sigma]$, where Σ is a BCO.

In the special case of pcas, Theorem 3.23 restricts to a nicer result. The condition that "the inverse image part of the geometric morphism of toposes preserves constant object" can be removed, which is a main result in Jonstone's paper *Geometic Morphisms of Realizability Toposes* [Johnstone, 2013].

Proposition 3.25. *Let A, B be pcas, any geometric morphism*

$$\Phi = (\Phi_*, \Phi^*) : \mathsf{Set}[-,\mathcal{P}(A)] \to \mathsf{Set}[-,\mathcal{P}(A)]$$

satisfies the condition that the inverse image Φ^* preserves constant objects.

For the proof, see section 2 of [Johnstone, 2013].

Therefore, up to equivalence, any geometric morphisms between realizability toposes corresponds to a unique geometric morphism between the realizability triposes. Moreover, by Proposition 3.22, they can be further related to computationally dense morphisms.

Corollary 3.26. *Let A, B be pcas, denote the filtered order-pca $\mathcal{D}(A,A) = (\mathcal{P}(A), \mathcal{P}(A) \setminus \{\varnothing\})$ as $\mathcal{P}(A)$, and $\mathcal{D}(B,B)$ as $\mathcal{P}(B)$, then the following are equivalent:*

1. *a geometric morphism of triposes from $[-,\mathcal{P}(A)]$ to $[-,\mathcal{P}(B)]$;*
2. *a geometric morphism of toposes from $\mathsf{Set}[-,\mathcal{P}(A)]$ to $\mathsf{Set}[-,\mathcal{P}(B)]$;*
3. *an adjoint pair $f^\circ \dashv f_\circ$: $\mathcal{P}(A) \underset{f_\circ}{\overset{f^\circ}{\leftrightarrows}} \mathcal{P}(B)$ of applicative morphisms between filtered order-pcas.*
4. *a computationally dense morphism of pcas from B to A;*

Proof. The equivalence between 1 and 2 is from Theorem 3.23 and Proposition 3.25.

The equivalence of 2 and 3 is from Theorem 3.23 and the fact that in the case of filtered order-pcas, a geometric morphism of BCOs is exactly an adjoint pair of applicative morphisms between filtered order-pcas. To see this, notice that both the direct image and the inverse image of a geometric morphism preserve internal finite limits, and by

Lemma 2.20, applicative morphisms between filtered order-pcas are precisely internal-finite-limit-preserving morphisms of BCOs.

The equivalence between 3 and 4 is from Proposition 3.22 and the fact that a computationally dense morphism of pcas from B to A is a computationally dense morphism of BCOs from B to $\mathcal{P}(A) = \mathcal{D}(A, A)$. □

3.4.3 Inclusions of Triposes and Toposes

In this section, we deal with inclusions of triposes and toposes.

Firstly, we introduce the definition of *geometric inclusions* of toposes.

Definition 3.27. Let $f = (f_*, f^*) : \mathcal{E} \to \mathcal{F}$ be a geometric morphism of toposes. Then f is called an *inclusion* or *geometric inclusion* if f_* is full and faithful.

Let $\mathcal{E}, \mathcal{F}$ be toposes, $\mathcal{E}$ is called a *subtopos* of $\mathcal{F}$, if there exists a geometric inclusion from $\mathcal{E}$ to $\mathcal{F}$.

Similarly, we have the notion of *inclusion of triposes*.

Definition 3.28. Let Q and P be $\mathcal{C}$-triposes. A geometric morphism $\Phi = (\Phi_+, \Phi^+)$: $\mathsf{Q} \to \mathsf{P}$ is an *(geometric) inclusion of triposes* if Φ_+ is full and faithful, equivalently, the transformation $\Phi^+ \circ \Phi_+ : \mathsf{Q} \to \mathsf{Q}$ is isomorphic to the identity.

If there is an inclusion of triposes from Q to P, then we call Q a subtripos of P.

In topos theory, it is well-known that geometric inclusions of toposes correspond to *local operators*. The following definition is from Definition A4.4.1 in [Johnstone, 2002].

Definition 3.29. Let $\mathcal{E}$ be a topos and Ω be the subobject classifier in $\mathcal{E}$ with the classifying map $\top : 1 \to \Omega$, and the binary meet $\wedge : \Omega \times \Omega \to \Omega$. A *local operator* on $\mathcal{E}$ is a morphism $j : \Omega \to \Omega$ such that the following diagrams commute:

$$
\begin{tikzcd}
1 \arrow[r, "\top"] \arrow[dr, "\top"'] & \Omega \arrow[d, "j"] \\
& \Omega
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega \arrow[r, "j"] \arrow[dr, "j"'] & \Omega \arrow[d, "j"] \\
& \Omega
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega \times \Omega \arrow[r, "\wedge"] \arrow[d, "j \times j"'] & \Omega \arrow[d, "j"] \\
\Omega \times \Omega \arrow[r, "\wedge"] & \Omega
\end{tikzcd}
.
$$

We formulate the correspondence of subtoposes and local operators in the following proposition, for the proof see section A4.3 and A4.4 in [Johnstone, 2002].

Proposition 3.30. *Let $\mathcal{E}$ be a topos. Every local operator $j : \Omega \to \Omega$ on $\mathcal{E}$ induces a Geometric inclusion of toposes $i_j : \mathsf{sh}_j(\mathcal{E}) \to \mathcal{E}$.*

Conversely, if $f : \mathcal{F} \to \mathcal{E}$ is a Geometric inclusion of toposes, then there is a local operator $j : \Omega \to \Omega$ and an equivalence of categories $e : \mathcal{F} \to \mathsf{sh}_j(\mathcal{E})$, such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "f"] \arrow[d, "e"'] & \mathcal{E} . \\ \mathsf{sh}_j(\mathcal{E}) \arrow[ur, "i_j"'] & \end{tikzcd}$$

This correspondence extends to the level of triposes. Let P be a $\mathcal{C}$-tripos. Then any local operator j on $\mathcal{C}[\mathsf{P}]$ induces a transformation $\Phi_j : \mathsf{P} \to \mathsf{P}$ of triposes (see section 2.5.4 in [Van Oosten, 2008]), which satisfies conditions for *closure transformations* on P. Conversely, every closure transformation on P induces a local operator on $\mathcal{C}[\mathsf{P}]$. We give the definition of closure transformation of triposes in the following.

Definition 3.31. Let P be a $\mathcal{C}$-tripos, the transformation $\Phi : \mathsf{P} \to \mathsf{P}$ of triposes is said to be a *closure transformation* on P, if for any $\mathcal{C}$-object X, the following holds:

$$\begin{aligned} \text{i) } & \Phi_X(\top_X) \simeq \top_X; \\ \text{ii) } & \Phi_X(\Phi_X(\phi)) \simeq \phi; \\ \text{iii) } & \Phi_X(\phi \wedge \psi) \simeq \Phi_X(\phi) \wedge \Phi_X(\psi), \end{aligned}$$

where $\top_X$ is the top element and $\wedge$ is the meet in the Heyting prealgbra $\mathsf{P}(X)$, and ϕ, ψ are elements of $\mathsf{P}(X)$.

Closure transformations are in one-one correspondence with (equivalent classes of) subtriposes.

Proposition 3.32. *(Theorem 2.5.11 in [Van Oosten, 2008])*
Let P be a $\mathcal{C}$-tripos, then inclusions into P correspond, up to equivalence, to closure transformations on P.

There is a direct connection between inclusions of triposes and inclusions of the corresponding toposes, as stated in the following proposition.

Proposition 3.33. *(Theorem 2.5.11 in [Van Oosten, 2008])*
Let P be a $\mathcal{C}$-tripos, then every inclusion of toposes into $\mathcal{C}[\mathsf{P}]$ is, up to equivalence, of the form $f : \mathcal{C}[\mathsf{Q}] \to \mathcal{C}[\mathsf{P}]$ for a $\mathcal{C}$-tripos Q and f is an inclusion of toposes induced by an inclusion of triposes from Q to P.

We conclude this part with a theorem about the relation between local operators and Boolean subtoposes.

A special class of local operators are associated with Boolean subtoposes. Let $\mathcal{E}$ be a topos, Ω the subobject classifier in $\mathcal{E}$. Fix a subterminal object U of $\mathcal{E}$, define a local operator $q(U) : \Omega \to \Omega$ by the following diagram:

$$\Omega \simeq \Omega \times 1 \xrightarrow{1\times(u,u)} \Omega \times \Omega \times \Omega \xrightarrow{\Rightarrow\times 1} \Omega \times \Omega \xrightarrow{\Rightarrow} \Omega\ ,$$

where u is the classifying map of $U \rightarrowtail 1$ (1 is the terminal object of $\mathcal{E}$) and $\Rightarrow$ is the Heyting implication on Ω.

Proposition 3.34. *(Lemma A4.5.21 in [Johnstone, 2002])*
Let j be a local operator on a topos $\mathcal{E}$. Then the subtopos $\mathsf{sh}_j(\mathcal{E})$ induced by j is Boolean if and only if $j = q(U)$ for some subterminal object U of $\mathcal{E}$.

Chapter 4

Classical Realizability

Realizability had been associated to intuitionistic logic, intuitionitic Peano arithmetic and constructive mathematics ever since it was born. It stayed unclear for quite a long time whether realizability can be extended to classical logic and classical mathematics. Around 1990, due to a paper by Griffin [Griffin, 1990], it became clear that the λ-calculus can be extended to systems which correspond to classical logic. Jean-Louis Krivine is among the first to work on such systems. From 2000 on, he began to develop the theory of *classical realizability* in a sequence of papers. However, it was not clear then how Krivine's account of classical realizability may fit into topos-theoretic approach of realizability.

In [Streicher, 2013], Thomas Streicher showed that from a Krivine classical realizability structure, which is reformulated as an *abstract Krivine structure*, one can construct a tripos out of it. Hence the tripos-to-topos construction is also applicable to classical realizability. While Krivine's interpretation is in classical second-order logic (or arithmetic), Streicher's construction enables a uniform interpretaion in higher order logics. Also, it highlights a subclass of realizability-related toposes, which are called *Krivine toposes*. Here comes a natural question, how to characterize this subclass among all realizability-related toposes? More precisely, Thomas Streicher described a way of generating Krivine toposes. It specifies the particular structures that Krivine toposes have. This can be seen as an intensional definition. Our aim is towards an extensional characterization of Krivine toposes.

Interestingly, the charchterization achieved in this thesis is an unexpected outcome of the study of one class of abstract Krivine structures. These structures come from filtered order-pcas, due to a construction by Jaap van Oosten in [van Oosten, 2012], inspired by the idea that some local operators on a topos will induce Boolean subtoposes. As a consequence, we will show that every Krivine topos is a Boolean subtopos of some generalized relative realizability topos.

With all these in mind, it is not surprising that we managed to find concrete examples of non-Grothendieck Krivine toposes from the construction mentioned above.

In this chapter, we will give a brief recapitulation of Krivine's classical realizability in section 4.1, followed by notions of abstract Krivine structures, Krivine triposes and Krivine toposes in section 4.2. In section 4.3, we will describe the construction of abstract Krivine structures from filtered order-pcas and give the characterization of Krivine toposes. In the last section, section 4.4, a concrete example of non-Grothendieck Krivine toposes will be discussed.

4.1 Krivine's Classical Realizability

As stated by Krivine in [Krivine, 2009], he was motivated to expand the *Curry-Howard correspondence* to the realm of classical logic, especially second order classical Peano arithmetic and Zermelo-Frænkel set theory with axiom of choice. The key idea of assigning proof terms in classical logic is to extend the λ-calculus with control operators which realize classical principles like *Pierce's law*. Similarly, to realize non-logical axioms (e.g. the axiom of countable choice) more operators are needed. In the following, we will only sketch an outline of Krivine's classical realizability for classical second order logic, leaving out those parts concerning Peano arithmetic and Zermelo-Frænkel set theory.

Firstly, in section 4.1.1 we describe the programming frame defined by Krivine which extends the λ-calculus. Then we introduce the logic, namely classical second order logic, and the deduction system in 4.1.2. Finally in section 4.1.3, these two parts will be connected via the realizability interpretation, and a form of soundness theorem will be proved.

4.1.1 Programming Frame for Classical Logic

The frame contains *terms* and *stacks*. Terms are basically the λ-terms built from some set of constants containing cc. And stacks can be seen as finite sequences of terms.

Let cc be a constant and Π_0 be a set. Let Λ_c (the set of *terms*) and Π (the set of *stacks*) be defined by the following grammar:

$$\text{Terms:} \quad t ::= x \mid \lambda x t \mid tt \mid \mathsf{cc} \mid \mathsf{k}_\pi, \qquad x \text{ variable;}$$
$$\text{Stacks:} \quad \pi ::= a \mid t.\pi, \qquad a \in \Pi_0, \quad t \text{ closed.}$$

A *process* is a pair $t \star \pi$ with $t \in \Lambda_c$, $\pi \in \Pi$. The *execution* of process, denoted by $\succ$, is given by:

$$\begin{array}{rl} \text{(push)} & tu \star \pi \ \succ \ t \star u.\pi; \\ \text{(pop)} & \lambda x v \star u.\pi \ \succ \ v[u/x] \star \pi; \\ \text{(store the stack)} & \mathsf{cc} \star t.\pi \ \succ \ t \star \mathsf{k}_\pi.\pi \\ \text{(restore the stack)} & \mathsf{k}_\pi \star t.\rho \ \succ \ t \star \pi. \end{array}$$

We say that the process $t \star \pi$ *reduces* to $t' \star \pi'$ (write as $t \star \pi > t' \star \pi'$) if $t' \star \pi'$ can be obtained from $t \star \pi$ by finitely many (possibly zero) execution steps.

Apart from terms, stacks and execution rules, we distinguish a set of processes and a set of terms which will play important roles in the realizability interpretation.

We fix a set of processes $\perp\!\!\!\perp$ which is cc-*saturated*, i.e.,

$$t \star \pi \in \perp\!\!\!\perp \ \wedge t' \star \pi' > t \star \pi \quad \Rightarrow \quad t' \star \pi' \in \perp\!\!\!\perp .$$

The set $\perp\!\!\!\perp$ is called a *pole*.

Fix a pole $\perp\!\!\!\perp$, let $\mathrm{PL} \subseteq \Lambda_c$ be a set of closed terms which contains cc but does not contain k_π for any $\pi \in \Pi$, and satisfies the condition that for any $t \in \mathrm{PL}$, there is $\pi \in \Pi$ with $t \star \pi \notin \perp\!\!\!\perp$. PL is called the *set of proof-like terms*.

4.1.2 Classical Second Order Logic

The classical second order logic is given in the style of typed λ-calculus. It contains *types*, *contexts*, *terms* and a set of rules, where types are formulas, contexts are declarations of variables' types.

Fix a countable set M of individuals, define *types* as the second order formulas built from logical symbols $\to$ and $\forall$; first order variables $x, y, \cdots$; second order variables $X, Y, \cdots$, each of which has an arity; function symbols of arity $k \geq 0$, which are functions from M^k to M and predicate symbols of arity $k \geq 0$, which are functions from M^k to $\mathcal{P}(\Pi)$.

Let $V = \{x_1, x_2, \cdots\}$ be a countable set of variables, define the set of $\lambda\mathsf{cc}$-*terms* as the least set built from variables, cc and closed under application and λ-abstraction.

A *context* is an expression of the form $x_1 : A_1, \cdots, x_n : A_n$, $n \geq 0$, where $x_1, \cdots, x_n \in V$ and $A_1, \cdots, A_n$ are types, i.e., second order formulas. We define the deduction rules for second order logic, and at the same time, the typing rules for $\lambda\mathsf{cc}$-terms as the following: let $\Gamma = x_1 : A_1, \cdots, x_n : A_n$ be a context,

1. $\Gamma \vdash x_i : A_i \ \ (1 \le i \le n)$;
2. $\Gamma \vdash t : A \to B, \ \ \Gamma \vdash u : A \ \ \Rightarrow \ \ \Gamma \vdash tu : B$;
3. $\Gamma, x : A \vdash t : B \ \ \Rightarrow \ \ \Gamma \vdash \lambda xt : A \to B$;
4. $\Gamma \vdash t : (A \to B) \to A \ \ \Rightarrow \ \ \Gamma \vdash \mathsf{cc}t : A$;
5. $\Gamma \vdash t : A \ \ \Rightarrow \ \ \Gamma \vdash t : \forall x A$, if x is not free in Γ;

5'. $\Gamma \vdash t : A \ \ \Rightarrow \ \ \Gamma \vdash t : \forall X A$, if X is not free in Γ;

6. $\Gamma \vdash t : \forall x A \ \ \Rightarrow \Gamma \vdash t : A[\tau/x]$ for every term τ of $\mathcal{L}$;

6'. $\Gamma \vdash t : \forall X A \ \ \Rightarrow \Gamma \vdash t : A[F/Xx_1 \cdots x_k]$ for every formula F.

4.1.3 Realizability Interpretation

Every formula will be associated with a set of terms, which are called *realizaers* of the formula.

Firstly, we define the truth values of a formula.

The set $\mathcal{P}(\Pi)$ is called the set of *truth values*. Let A be a closed second order formula, define its *truth value*[1] $||A|| \in \mathcal{P}(\Pi)$ inductively as:
If A is atomic, i.e., $A = R(t_1, \cdots, t_k)$ with $R \in \mathcal{P}(\Pi)^{M^k}$ and $t_1, \cdots, t_k$ closed terms. Let $a_1, \cdots, a_k \in M$ be the values of $t_1, \cdots, t_k$, then we put

$$||R(t_1, \cdots, t_k)|| := R(a_1, \cdots, a_k) \in \mathcal{P}(\Pi);$$

Other steps are:
$||A_1 \to A_2|| := \{t.\pi : t \in |A_1|, \pi \in ||A_2||\}$ where $|A_1| := \{t \in \Lambda_c : \forall \pi \in ||A_1||, t \star \pi \in \perp\!\!\!\perp\}$;
$||\forall x A|| := \bigcap\{||A[a/x]|| : a \in M\}$;
$||\forall X A|| := \bigcup\{||A[R/X]|| : R \in \mathcal{P}(\Pi)^{M^k}\}$ if X is a second order variable with arity k.

Terms are associated to formulas via truth values and the pole $\perp\!\!\!\perp$.

Define $|A| := \{t \in \Lambda_c : \forall \pi \in ||A||, t \star \pi \in \perp\!\!\!\perp\} \in \mathcal{P}(\Lambda_c)$, and write $t \Vdash A$ (t realises A) if $t \in |A|$.

We remark that it can be the case that every formula is realized by some term in Λ_c, to avoid this situation, we only consider realizers inside PL.

The following is the soundness theorem of the classical realizability interpretation.

Proposition 4.1. *(Theorem 2 in [Krivine, 2009])*
Let $A_1, \cdots, A_k, A$ be closed formulas such that $x_1 : A_1, \cdots, x_n : A_n \vdash t : A$ is provable in the deduction system above. If $t_i \Vdash A_i$ for $1 \le i \le k$, then $t[t_1/x_1, \cdots, t_k/x_k] \Vdash A$.

[1] The notion is from [Krivine, 2009]. It might better to call $||A||$ the falsity value of A rather than truth value, since intuitively elements of $||A||$ are witnesses against A.

In particular, if A is a closed formula and $\vdash t : A$ is provable, then $t \Vdash A$ and $t \in PL$.

4.2 Krivine Triposes

In [Streicher, 2013], Streicher reformulated Krivine's programming frames for classical logic as *abstract Krivine structues* based on the idea of combinators. He showed that every abstract Krivine structure gives rise to a filtered order-pca, from which a Set-tripos can be built. Such triposes are called *Krivine triposes*.

Ferrer Santos et al. characterised the class of filtered order-pcas obtained from abstract Krivine structures in [Ferrer Santos et al., 2014]. They proposed a class of structures called *Krivine ordered combinatory algebras*, which are filtered order-pcas with some additional structures, and showed that they give rise to the same class of triposes as Krivine triposes.

4.2.1 Abstract Krivine Structures

We introduce Streicher's notion of abstract Krivine structures.

Definition 4.2. An *abstract Krivine structure* (aks) is given by:

1. A set Λ of terms, and a binary application operation $app : \Lambda \times \Lambda \to \Lambda$ (written as $t_1 t_2$), and distinguished elements $\mathsf{K}, \mathsf{S}, \mathsf{cc} \in \Lambda$;
2. A subset QP (set of quasi-proofs)[2] of Λ which is closed under application and contains $\mathsf{K}, \mathsf{S}, \mathsf{cc}$;
3. A set Π of stacks together with a push operation from $\Lambda \times \Pi$ to Π (written $t.\pi$) and a unary operation $\mathsf{k} : \Pi \to \Lambda$ (written as k_π);
4. A saturated subset $\perp\!\!\!\perp$ of $\Lambda \times \Pi$, where saturated means:
   (S1) $(ts, \pi) \in \perp\!\!\!\perp$ whenever $(t, s.\pi) \in \perp\!\!\!\perp$;
   (S2) $(\mathsf{K}, t.s.\pi) \in \perp\!\!\!\perp$ whenever $(t, \pi) \in \perp\!\!\!\perp$;
   (S3) $(\mathsf{S}, t.s.u.\pi) \in \perp\!\!\!\perp$ whenever $(tu(su), \pi) \in \perp\!\!\!\perp$;
   (S4) $(\mathsf{cc}, t.\pi) \in \perp\!\!\!\perp$ whenever $(t, \mathsf{k}_\pi.\pi) \in \perp\!\!\!\perp$;
   (S5) $(\mathsf{k}_\pi, t.\pi') \in \perp\!\!\!\perp$ whenever $(t, \pi) \in \perp\!\!\!\perp$.

A *strong abstract Krivine structure* (strong aks) is an aks where (S1) can be strengthened to (SS1) $(ts, \pi) \in \perp\!\!\!\perp$ iff $(t, s.\pi) \in \perp\!\!\!\perp$.

[2] "Quasi-proof" is just another name for proof-like terms.

4.2.2 Krivine Triposes

As mentioned before, Streicher showed that from any aks, one can construct a Set-tripos. The key step in this construction is to associate with any aks a filtered-order pca which gives rise to a Set-tripos.

Let $\mathcal{K}$ be an aks. Define an operation $(\cdot)^{\perp\!\!\!\perp} : (\mathcal{P}(\Pi) \to \mathcal{P}(\Lambda)) \cup (\mathcal{P}(\Lambda) \to \mathcal{P}(\Pi))$ as: for any $\alpha \in \mathcal{P}(\Pi)$, $\beta \in \mathcal{P}(\Lambda)$,

$$\alpha^{\perp\!\!\!\perp} := \{t \in \Lambda : \forall \pi \in \alpha, (t, \pi) \in \perp\!\!\!\perp\}; \qquad \beta^{\perp\!\!\!\perp} := \{\pi \in \Pi : \forall t \in \beta, (t, \pi) \in \perp\!\!\!\perp\}.$$

When $\alpha \in \mathcal{P}(\Pi)$, we also write $\alpha^{\perp\!\!\!\perp}$ as $|\alpha|$.

The operator $(\cdot)^{\perp\!\!\!\perp}$ induces a closure operator $(\cdot)^{\perp\!\!\!\perp\perp\!\!\!\perp}$ both on $\mathcal{P}(\Pi)$ and $\mathcal{P}(\Lambda)$. To spell it out, the operator $(\cdot)^{\perp\!\!\!\perp\perp\!\!\!\perp} : \alpha \mapsto \alpha^{\perp\!\!\!\perp\perp\!\!\!\perp}$ on $\mathcal{P}(\Lambda)$ (or $\mathcal{P}(\Pi)$) satisfies: for any α, β

$$\begin{aligned} \alpha &\subseteq \alpha^{\perp\!\!\!\perp\perp\!\!\!\perp}; \\ \alpha^{\perp\!\!\!\perp\perp\!\!\!\perp} &= (\alpha^{\perp\!\!\!\perp\perp\!\!\!\perp})^{\perp\!\!\!\perp\perp\!\!\!\perp}; \\ \alpha \subseteq \beta \quad &\Rightarrow \quad \alpha^{\perp\!\!\!\perp\perp\!\!\!\perp} \subseteq \beta^{\perp\!\!\!\perp\perp\!\!\!\perp}. \end{aligned}$$

Let $\mathcal{P}_{\perp\!\!\!\perp}(\Pi) := \{\alpha \in \mathcal{P}(\Pi) : \alpha^{\perp\!\!\!\perp\perp\!\!\!\perp} = \alpha\}$, with the order $\leq := \supseteq$ on it. Define an application $\bullet$ on $\mathcal{P}_{\perp\!\!\!\perp}(\Pi)$ as: for any $\alpha, \beta \in \mathcal{P}_{\perp\!\!\!\perp}(\Pi)$,

$$\alpha \bullet \beta := \{\pi \in \Pi : \forall t \in |\alpha|, s \in |\beta|, (t, s.\pi) \in \perp\!\!\!\perp\}^{\perp\!\!\!\perp\perp\!\!\!\perp}.$$

Let $\Phi := \{\alpha \in \mathcal{P}_{\perp\!\!\!\perp}(\Pi) : |\alpha| \cap \mathrm{QP} \neq \varnothing\}$.

Proposition 4.3. *$(\mathcal{P}_{\perp\!\!\!\perp}(\Pi), \leq, \bullet)$ is an order-pca and Φ is a filter in it.*

For the proof, see section 5 in [Streicher, 2013].

We write $\Sigma_{\mathcal{K}}$ for the filtered order-pca $((\mathcal{P}_{\perp\!\!\!\perp}(\Pi), \leq, \bullet), \Phi)$ constructed from the aks $\mathcal{K}$. It turned out that as a filtered order-pca $\Sigma_{\mathcal{K}}$ is rich enough to gives rise to a Set-tripos $[-, \Sigma_{\mathcal{K}}]$.

Proposition 4.4. *(Theorem 5.9 in [Streicher, 2013])*
Let $\mathcal{K}$ be an aks, then $[-, \Sigma_{\mathcal{K}}]$ is a Boolean Set-tripos, i.e., it is a tripos such that for all set X, the poset reflection of the preorder on $[X, \Sigma_{\mathcal{K}}]$ is a Boolean algebra.

The tripose $[-, \Sigma_{\mathcal{K}}]$ from an aks $\mathcal{K}$ is called a *Krivine tripos*. The corresponding topos Set$[-, \Sigma_{\mathcal{K}}]$ is called a *Krivine topos*.

4.2.3 Krivine Ordered Combinatory Algebras

Recall that we have defined the structure ${}^{\mathcal{I}}\mathcal{OCA}$ (implicative ordered combinatory algebra) in chapter 2, section 3.2.3. A *Krivine ordered combinatory algebra* (${}^{\mathcal{K}}\mathcal{OCA}$) is based on an ${}^{\mathcal{I}}\mathcal{OCA}$. In section 3.2.3, we also mentioned that from any ${}^{\mathcal{I}}\mathcal{OCA}$ one can obtain a Set-tripos. Similarly, any ${}^{\mathcal{K}}\mathcal{OCA}$ is associated with a Set-tripos. Ferrer Santos et al. showed that the class of triposes obtained from ${}^{\mathcal{K}}\mathcal{OCA}$ are equivalent to the class of Krivine triposes [Ferrer Santos et al., 2014].

Definition 4.5. A *Krivine ordered combinatory algebra* (${}^{\mathcal{K}}\mathcal{OCA}$) consists of an ${}^{\mathcal{I}}\mathcal{OCA}$ equipped with a distinguished element $\mathsf{c} \in \Phi$ such that for all $a, b \in A$,

$$\text{(PC)} \quad \mathsf{c} \leq ((a \to b) \to a) \to a.$$

Let $\mathcal{K}$ be an aks and $\Sigma_{\mathcal{K}} = ((\mathcal{P}_{\perp\!\!\!\perp}(\Pi), \leq, \bullet), \Phi)$ be the filtered order-pca associated to $\mathcal{K}$. $\Sigma_{\mathcal{K}}$ carries more than a filtered order-pca strucute, indeed, it can be extended to a ${}^{\mathcal{K}}\mathcal{OCA}$. The implication $imp : \mathcal{P}_{\perp\!\!\!\perp}(\Pi) \times \mathcal{P}_{\perp\!\!\!\perp}(\Pi) \to \mathcal{P}_{\perp\!\!\!\perp}(\Pi)$ in $\Sigma_{\mathcal{K}}$ is defined as: for any $\alpha, \beta \in \mathcal{P}_{\perp\!\!\!\perp}(\Pi)$,

$$\alpha \to \beta := \{t.\pi : t \in |\alpha|, \pi \in \beta\}^{\perp\!\!\!\perp\perp\!\!\!\perp}.$$

Proposition 4.6. *(Theorem 5.10 in [Ferrer Santos et al., 2014])*
$\Sigma_{\mathcal{K}}$ with imp, $\mathsf{e} := \{\mathsf{EE}\}^{\perp\!\!\!\perp\perp\!\!\!\perp}$, $\mathsf{c} := \{\mathsf{cc}\}^{\perp\!\!\!\perp\perp\!\!\!\perp}$ *is a* ${}^{\mathcal{K}}\mathcal{OCA}$, *where* $\mathsf{E} := \mathsf{S}(\mathsf{KI})$, $\mathsf{I} := \mathsf{SKK}$.

Conversely, one can construct an aks from any ${}^{\mathcal{K}}\mathcal{OCA}$.

Definition 4.7. Let $\mathcal{A} = (A, \leq, app_{\mathcal{A}}, \Phi, imp, \mathsf{k}, \mathsf{s}, \mathsf{c}, \mathsf{e})$ be an ${}^{\mathcal{K}}\mathcal{OCA}$, define the following structure:

$$\mathcal{K}_{\mathcal{A}} := (\Lambda, \Pi, \mathrm{QP}, \mathsf{K}, \mathsf{S}, \mathsf{cc}, \perp\!\!\!\perp, \mathsf{k} : \pi \mapsto \mathsf{k}_\pi, app, push)$$

with $\Lambda = \Pi := A$; $\mathrm{QP} = \Phi$;
$\mathsf{K} := \mathsf{e}(\mathsf{bek})$, $\mathsf{S} := \mathsf{e}(\mathsf{b}(\mathsf{be}(\mathsf{be}))\mathsf{s})$, $\mathsf{cc} := \mathsf{ec}$ where $\mathsf{b} := \langle xyz \rangle z(x(yz))$;
$\perp\!\!\!\perp := \leq$, i.e., $(s, \pi) \in \perp\!\!\!\perp \;\Leftrightarrow\; s \leq \pi$;
$\mathsf{k}_\pi := \pi \to \perp = imp(\pi, \perp)$, where $\perp$ is the least element in the inf-complete lattice $(A, \leq)$;
$app := app_{\mathcal{A}}$; $push := imp$.

Proposition 4.8. *(Theorem 5.12 in [Ferrer Santos et al., 2014])*
The structure $\mathcal{K}_{\mathcal{A}}$ defined from a ${}^{\mathcal{K}}\mathcal{OCA}$ $\mathcal{A}$ is an aks.

Given a ${}^{\mathcal{K}}\mathcal{OCA}$ $\mathcal{A}$, it is also an ${}^{\mathcal{I}}\mathcal{OCA}$, hence $[-, \mathcal{A}]$ is a Set-tripos. From $\mathcal{A}$, there is also the corresponding aks $\mathcal{K}_{\mathcal{A}}$, hence a filtered order-pca $\Sigma_{\mathcal{K}_{\mathcal{A}}}$. Now we have two triposes: $[-, \mathcal{A}]$ and the Krivine tripos $[-, \Sigma_{\mathcal{K}_{\mathcal{A}}}]$.

Proposition 4.9. *(Theorem 5.15 in [Ferrer Santos et al., 2014])*
Let $\mathcal{A}$ be a $^{\mathcal{K}}\mathcal{OCA}$, then the tripos $[-,\mathcal{A}]$ and the Krivine tripos $[-,\Sigma_{\mathcal{K}_{\mathcal{A}}}]$ are equivalent.

4.3 A Characterization of Krivine Toposes

In this section, we want to investigate the relation between Krivine toposes and other realizability-related toposes. In particular, we want to connect Krivine toposes with the more studied toposes, e.g. realizability toposes and relative realizability toposes, via geometric morphisms. We will first establish some results at the level of triposes and then extend them to the level of toposes.

Krivine triposes are of the form $[-,\Sigma]$ for some filtered order-pca Σ, by definition, they are not generalized relative realizability triposes, i.e., triposes of the form $[-,\mathcal{D}\Sigma]$. However, in Streicher's construction of filtered order-pcas out of abstract Krivine structures, the carrier set $\mathcal{P}_{\perp\!\!\!\perp}(\Pi)$ and the definition of application are similar to the downset monad construction $\mathcal{D}\Sigma$. One might wonder if every Krivine tripos is equivalent to some generalized relative realizability tripos. The answer is negative. We will prove this result in section 4.3.1.

Though it is not the case that every Krivine tripos is a generalized relative realizability tripos, it is true that every Krivine tripos is a Boolean subtripos of some generalized relative realizability tripos. This result will also be proved in section 4.3.1. Given this fact, one might ask whether the converse also holds, i.e., if every Boolean subtripos of a generalized relative realizability tripos is equivalent to a Krivine tripos. The answer is positive. This result is established in section 4.3.2.

Together, we get the characterization of Krivine toposes: a topos is a Krivine topos if and only if it is equivalent to some Boolean subtopos of the generalized relative realizability topos.

4.3.1 Krivine Triposes and Genralized Relative Realizability Triposes

We will first give the result that not every Krivine tripos is equivalent to a generalized relative realizibility tripos.

Consider the complete Boolean algebra $B = [A, \wedge, \vee, \neg, \top, \bot]$ with $A = \{\top, \bot, x, \neg x\}$, and the obvious operations on these elements. It is easy to see that $B = [A, \wedge, \vee, \neg, \top, \bot]$ is a $^{\mathcal{K}}\mathcal{OCA}$ with the filter $\Phi' = \{\top\}$, $\mathsf{k} = \mathsf{s} = \mathsf{e} = \mathsf{c} = \top$, $app := \wedge$, $a \to b := \neg a \vee b$, for all $a, b \in A$. By Proposition 4.9, the tripos $[-, B]$ is equivalent to a Krivine tripos.

Theorem 4.10. *There is no filtered order-pca Σ, such that the tripos $[-, B]$ is equivalent to $[-, \mathcal{D}\Sigma]$ or $[-, \mathcal{D}_i\Sigma]$.*

Proof. We first prove that $[-, B]$ is not equivalent to $[-, \mathcal{D}\Sigma]$ for any filtered order-pca Σ. Suppose towards contradiction that there is such Σ, then by Theorem 3.23, the equivalence induces an equivalence (f_*, f^*) of applicative morphisms from the filtered order-pca $\mathcal{D}\Sigma$ to B. Since $f_* \circ f^* \simeq id_B$ and only $\top$ is in the filter, we get $f_* \circ f^* = id_B$. Consider $\varnothing \in \mathcal{D}\Sigma$, since $\varnothing \subseteq f^*(\bot)$, by f_* being an applicative morphism,

$$\top \wedge f_*(\varnothing) \leq f_*(f^*(\bot)) = \bot,$$

thus, $f_*(\varnothing) = \bot$. Since $f^* \circ f_* \leq id_{\mathcal{D}\Sigma}$, there is $u \in \mathcal{D}\Sigma$, with $u \cap \Phi \neq \varnothing$ (Φ is the filter of Σ), such that $uf^*(\bot) = uf^*(f_*(\varnothing)) \subseteq \varnothing$. Since $u \neq \varnothing$, $f^*(\bot) = \varnothing$. Note that f^* is injective, hence, $f^*(x) \neq f^*(\neg x) \neq \varnothing$. By f^* being an applicative morphism, there is $v \in \mathcal{D}\Sigma$, such that $v \cap \Phi \neq \varnothing$, and

$$vf^*(x)f^*(\neg x) \leq f^*(x \wedge \neg x) = f^*(\bot) = \varnothing.$$

But that contradicts $v, f^*(x), f^*(\neg x) \neq \varnothing$.

The proof above uses heavily the property of $\varnothing$ in $\mathcal{D}\Sigma$, thus the proof that $[-, B]$ is not equivalent to $[-, \mathcal{D}_i\Sigma]$ for some filtered order-pca Σ is not the same. Suppose towards contradiction that there is such Σ. Since $[-, B]$ is localic, $[-, \mathcal{D}_i\Sigma]$ must be localic. By Proposition 7.9 in [Hofstra, 2006][3], the poset of designated truth-values of $\mathcal{D}_i\Sigma$ has a least element u. Hence, $u \cap \Phi \neq \varnothing$, for the filter Φ of Σ. Consider any element $a \in u \cap \Phi$, for any $b \in \Phi$, $\downarrow\{b\} := \{c \in \Sigma : c \leq b\}$ is in the set of designated truth-values of $\mathcal{D}_i\Sigma$, hence, $u \subseteq \downarrow\{b\}$, and $a \leq b$. Therefore, the filter Φ of Σ has a least element a. By the assumption that $[-, B]$ is equivalent to $[-, \mathcal{D}_i\Sigma]$, there is an equivalence (f_*, f^*) of applicative morphisms from $\mathcal{D}_i\Sigma$ to B and $f_* \circ f^* = id_B$ (since only $\top$ is in the filter of B). Consider $f^*(x) \neq f^*(\neg x) \in \mathcal{D}_i\Sigma$. Let $\uparrow\Phi$ be the upward closure of Φ, we claim that $f^*(x) \cap \uparrow\Phi = \varnothing$. Otherwise, $\downarrow\{a\} \subseteq (\downarrow\{a\})f^*(x)$ and by f_* being an applicative morphism

$$\top = \top \wedge f_*(\downarrow\{a\}) \leq f_*(\downarrow\{a\}f^*(x)) \leq f_*(\downarrow\{a\}) \wedge f_*(f^*(x)) = \top \wedge x = x,$$

where $f_*(\downarrow\{a\}) = \top$ is by the fact that applicative morphisms preserve filters. Hence, we get a contradiction. Similarly, $f^*(\neg x) \cap \uparrow\Phi = \varnothing$. Consider $\Sigma \setminus \uparrow\Phi$, it is not empty and downward closed, hence in $\mathcal{D}_i\Sigma$, moreover, $f^*(x), f^*(\neg x) \subseteq (\Sigma \setminus \uparrow\Phi)$, hence, $x = \top \wedge f_*(f^*(x)) \leq f_*(\Sigma \setminus \uparrow\Phi)$ and similarly, $\neg x \leq f_*(\Sigma \setminus \uparrow\Phi)$, thus, $f_*(\Sigma \setminus \uparrow\Phi) = \top$. But

[3] The full version of this proposition will be stated in section 4.4, Proposition 4.17.

there is $v \in \mathcal{D}_i\Sigma$, such that $vf^*(f_*(\Sigma \setminus \uparrow\Phi)) \subseteq \Sigma \setminus \uparrow\Phi$, hence $vf^*(\top) \subseteq \Sigma \setminus \uparrow\Phi$. Since $f^*(\top) \cap \Phi \neq \varnothing$, we get $(\Sigma \setminus \uparrow\Phi) \cap \Phi \neq \varnothing$, contradiction. □

The theorem above shows that the class of Krivine triposes is not a subclass of generalized relative realizability triposes. However, it is a subclass of the Boolean subtriposes of generalized relative realizability triposes, as demonstrated by the following theorem.

Theorem 4.11. *Let Σ be a filtered order-pca, if $[-,\Sigma]$ is a tripos, then $[-,\Sigma]$ is a subtripos of $[-,\mathcal{D}\Sigma]$.*

Proof. The proof is essentially contained in [Hofstra, 2006] the characterization of when $[-,\mathcal{D}\Sigma]$ is a tripos.

We establish an inclusion from $[-,\Sigma]$ to $[-,\mathcal{D}\Sigma]$. Define $\downarrow(-) : \Sigma \to \mathcal{D}\Sigma$ as: for any $a \in \Sigma$, $\downarrow(a) := \downarrow\{a\} = \{b \in \Sigma : b \le a\}$. It induces a natural transformation: $f_+ : [-,\Sigma] \to [-,\mathcal{D}\Sigma]$, i.e., for any set X, $\tau \in \Sigma^X$, $f_+(X)(\tau) := \downarrow(-) \circ \tau$. It is easy to see that f_+ is order-preserving.

On the other hand, let $M = \{(a,\alpha) : a \in \alpha \in \mathcal{D}\Sigma\}$, let $\pi_1 : M \to \Sigma$, $\pi_2 : M \to \mathcal{D}\Sigma$ be the projections on first and second coordinates respectively, since $[-,\Sigma]$ is tripos, there is $\exists_{\pi_2}(\pi_1) : \mathcal{D}\Sigma \to \Sigma$. Similarly, it induces a natural transformation $f^+ : [-,\mathcal{D}\Sigma] \to [-,\Sigma]$.

To complete the proof, we need to show:
1. f^+ is order-preserving;
2. for any set X, $f^+(X) \circ f_+(X) \simeq id_{\Sigma^X}$;
3. $id_{(\mathcal{D}\Sigma)^X} \le f_+(X) \circ f^+(X)$.

1. Let X be a set, $\tau, \eta : X \to \mathcal{D}\Sigma$, suppose $\tau \le \eta$, i.e., there is $a' \in \Phi_\Sigma$ (the filter of Σ), such that for all $x \in X$, for all $a \in \tau(x)$, $a'a\downarrow$ and $a'a \in \eta(x)$, we need to show that $f^+(X)(\tau) \le f^+(X)(\eta)$. Let

$$P_\tau := \{(x,a) : x \in X, a \in \tau(x)\},$$
$$P_\eta := \{(x,b) : x \in X, b \in \eta(x)\}.$$

Let $t_\tau : P_\tau \to M$ send (x,a) to $(a,\tau(x))$, $t_\eta : P_\eta \to M$ send (x,b) to $(b,\eta(x))$, $\pi_\tau : P_\tau \to X$ and $\pi_\eta : P_\eta \to X$ be the projections. Consider the following diagram:

$$\begin{tikzcd} P_\tau \arrow[r, "\pi_\tau"] \arrow[d, "t_\tau"'] & X \arrow[d, "\tau"] \\ M \arrow[r, "\pi_2"'] & \mathcal{D}\Sigma \end{tikzcd},$$

it is a pullback square, hence, $\exists_{\pi_\tau}(\Sigma(t_\tau)(\pi_1)) \simeq \Sigma(\tau)(\exists_{\pi_2}(\pi_1)) = f^+(X)(\tau)$. Similarly, we have $\exists_{\pi_\eta}(\Sigma(t_\eta)(\pi_1)) \simeq \Sigma(\eta)(\exists_{\pi_2}(\pi_1)) = f^+(X)(\eta)$. Hence, it suffices to show that $\exists_{\pi_\tau}(\pi_1 \circ t_\tau) \leq \exists_{\pi_\eta}(\pi_1 \circ t_\eta)$. Note that there is a morphism $g : P_\tau \to P_\eta$ defined by $g(x,a) = (x, a'a)$. Now, we have $\pi_\tau = \pi_\eta \circ g$, and $\pi_1 \circ t_\tau \leq \pi_1 \circ t_\eta \circ g$ in Σ^{P_τ}. Therefore,

$$\exists_{\pi_\tau}(\pi_1 \circ t_\tau) \simeq \exists_{\pi_\eta}(\exists_g(\pi_1 \circ t_\tau)) \leq \exists_{\pi_\eta}(\exists_g(\pi_1 \circ t_\eta \circ g)) = \exists_{\pi_\eta}(\exists_g(\Sigma(g)(\pi_1 \circ t_\eta)) \leq \exists_{\pi_\eta}(\pi_1 \circ t_\eta),$$

of which the last inequality is from $\exists_g \circ \Sigma(g) \leq id_{\Sigma^{P_\eta}}$.

2. Consider the following diagram:

$$\begin{tikzcd} \Sigma \arrow[r, "id_\Sigma"] \arrow[d, "{(id_\Sigma, \Downarrow(-))}"'] & \Sigma \arrow[d, "\Downarrow(-)"] \\ M \arrow[r, "\pi_2"'] & \mathcal{D}\Sigma \end{tikzcd},$$

it is a pullback square, hence, $\Sigma(\downarrow(-))(\exists_{\pi_2}(\pi_1)) \simeq \exists_{id_\Sigma}(\pi_1 \circ (id_\Sigma, \downarrow(-)))$, note that the left hand side is $\exists_{\pi_2}(\pi_1) \circ \downarrow(-)$ and the right hand side is isomorphic to id_Σ, thus, $\exists_{\pi_2}(\pi_1) \circ \downarrow(-) \simeq id_\Sigma$. From this we can easily get, for every set X, $f^+(X) \circ f_+(X) \simeq id_{\Sigma^X}$.

3. It suffices to show that $id_{\mathcal{D}\Sigma} \leq \downarrow(-) \circ \exists_{\pi_2}(\pi_1)$, which is equivalent to that there is $a' \in \Phi_\Sigma$, such that for all $\alpha \in \mathcal{D}\Sigma$, for all $a \in \alpha$, $a'a \leq \exists_{\pi_2}(\pi_1)(\alpha)$, which is further equivalent to $\pi_1 \leq \exists_{\pi_2}(\pi_1) \circ \pi_2 = \Sigma(\pi_2)(\exists_{\pi_2}(\pi_1))$ in Σ^M. Since $id_{\Sigma^M} \leq \Sigma(\pi_2) \circ \exists_{\pi_2}$, we certainly have $\pi_1 \leq \Sigma(\pi_2)(\exists_{\pi_2}(\pi_1))$ in Σ^M.

□

Corollary 4.12. *Let $\mathcal{K}$ be an abstract Krvivine structure, and $\Sigma_\mathcal{K}$ be the filtered order-pca built from $\mathcal{K}$ as described in section 4.2.2, then the Krivine tripos $[-, \Sigma_\mathcal{K}]$ is a Boolean subtripos of the generalized relative realizability tripos $[-, \mathcal{D}\Sigma_\mathcal{K}]$.*

4.3.2 Krivine Triposes Constructed from Filtered Order-pcas

In this section we establish the result that every Boolean subtripos of a generalized relative realizability tripos is equivalent to a Krivine tripos. It is obtained from a construction proposed by Jaap van Oosten in [van Oosten, 2012].[4] We describe the construction as follows.

[4]There is a similar construction appeared in Wouter Stekelenburg's PhD thesis [Stekelenburg, 2013].

Let (A, A') be a filtered order-pca. Recall that in section 2.2.2 we have mentioned that there is a coding of finite sequences of elements in A. Let

$$\Pi := \{\pi \in A : \pi \text{ is the code of a finite sequence}\}.$$

We denote the code of sequence $u_0, \cdots, u_k$ as $[u_0, \cdots, u_k]$.

Recall that in any filter A', there are operators $\mathsf{b_i}, \mathsf{c_i}$, $i \in \mathbb{N}$, such that for any $[u_0, \cdots, u_k]$ with $k \geq i$, $\mathsf{b_i}[u_0, \cdots, u_k] \downarrow$ and $\mathsf{b_i}[u_0, \cdots, u_k] \leq u_i$, $\mathsf{c_i}[u_0, \cdots, u_k] \downarrow$ and $\mathsf{c_i}[u_0, \cdots, u_k] \leq [u_i, \cdots, u_k]$. Also, there is a combinator d in the filter, such that for any $a, [u_0, \cdots, u_k]$, $\mathsf{d}a[u_0, \cdots, u_k] \downarrow$ and $\mathsf{d}a[u_0, \cdots, u_k] \leq [a, u_0, \cdots, u_k]$. We write $\mathsf{d}a[u_0, \cdots, u_k]$ as $a.[u_0, \cdots, u_k]$ and $a.(b.[u_0, \cdots, u_k])$ as $a.b.[u_0, \cdots, u_k]$.

Fix a downward closed subset U, define the following structure:

- $\Lambda := A$, QP:= A', $\Pi := \{\pi \in A : \pi \text{ is the code of a finite sequence}\}$, push operation $a.\pi$, and a total application $\cdot$ on A by:

$$a \cdot b := \langle \pi \rangle a(b.\pi);$$

- $\perp\!\!\!\perp := \{(a, \pi) : a \in A, \pi \in \Pi, a\pi\downarrow \in U\}$ where $a\pi\downarrow \in U$ means $a\pi\downarrow$ and $a\pi \in U$;
- $\mathsf{K} := \langle \pi \rangle \mathsf{b_0}\pi(\mathsf{c_2}\pi)$;

  $\mathsf{S} := \langle \pi \rangle ((\mathsf{b_0}\pi) \cdot (\mathsf{b_2}\pi) \cdot ((\mathsf{b_1}\pi) \cdot (\mathsf{b_2}\pi)))(\mathsf{c_3}\pi)$

  $\mathsf{k}_\pi := \langle \rho \rangle \mathsf{b_0}\rho\pi$

  $\mathsf{cc} := \langle \pi \rangle \mathsf{b_0}\pi(\mathsf{k}_{\mathsf{c_1}\pi}.(\mathsf{c_1}\pi))$

Lemma 4.13. *The structure defined above is an aks.*

Proof. For any $a, b, c \in A$, $\pi, \pi' \in \Pi$:

(S1) Suppose $(a, b.\pi) \in \perp\!\!\!\perp$, then $a(b.\pi)\downarrow \in U$. Consider $(a \cdot b)\pi := \langle \rho \rangle a(b.\rho)\pi \leq a(b.\pi)\downarrow$, since $a(b.\pi) \in U$ and U is downward closed, we get $(a \cdot b)\pi\downarrow \in U$, hence, $(a \cdot b, \pi) \in \perp\!\!\!\perp$.

(S2) Suppose $(a, \pi) \in \perp\!\!\!\perp$, then $a\pi \downarrow \in U$.

$$\mathsf{K}(a.b.\pi) := \langle \pi' \rangle \mathsf{b_0}\pi'(\mathsf{c_2}\pi')(a.b.\pi) \leq \mathsf{b_0}(a.b.\pi)(\mathsf{c_2}(a.b.\pi)).$$

Suppose $\pi = [u_0, \cdots, u_k]$, note that $a.b.\pi \leq [a, b, u_0, \cdots, u_k]$, hence

$$\mathsf{b_0}(a.b.\pi) \leq \mathsf{b_0}[a, b, u_0, \cdots, u_k] \leq a,$$

and $\mathsf{c}_2(a.b.\pi) \leq [u_0, \cdots, u_k] = \pi$. Therefore, $\mathsf{K}(a.b.\pi) \leq \mathsf{b}_0(a.b.\pi)(\mathsf{c}_2(a.b.\pi)) \leq a\pi\downarrow \in U$, thus, $(\mathsf{K}, a.b.\pi) \in \perp\!\!\!\perp$.

(S3) If $(a{\cdot}c{\cdot}(b{\cdot}c), \pi) \in \perp\!\!\!\perp$, then $(a{\cdot}c{\cdot}(b{\cdot}c))\pi\downarrow \in U$. Note that $\mathsf{S}(a.b.c.\pi) \leq (a{\cdot}c{\cdot}(b{\cdot}c))\pi\downarrow \in U$, hence, $(\mathsf{S}, a.b.c.\pi) \in \perp\!\!\!\perp$.

(S4) If $(a, \mathsf{k}_\pi.\pi) \in \perp\!\!\!\perp$, then $a(\mathsf{k}_\pi.\pi)\downarrow \in U$. $\mathsf{cc}(a.\pi) \leq a(\mathsf{k}_\pi.\pi)\downarrow \in U$, hence we get $(\mathsf{cc}, a.\pi) \in \perp\!\!\!\perp$.

(S5) Suppose $(a, \pi) \in \perp\!\!\!\perp$, then $a\pi\downarrow \in U$. $\mathsf{k}_\pi(a.\pi') \leq a\pi\downarrow \in U$, therefore, $(\mathsf{k}_\pi, a.\pi') \in \perp\!\!\!\perp$. □

We remark here that if the order of (A, A') is discrete, i.e., (A, A') is a filtered pca, then the resulting structure is a strong aks.

Call this aks $\mathcal{K}^U_{A,A'}$. It gives rise to a Krivine tripos $[-, \Sigma_{\mathcal{K}^U_{A,A'}}]$. Also there is a generalized relative realizability tripos $[-, \mathcal{D}(A, A')]$ associated to the filtered order-pca (A, A').

In the following, we will show that the Krivine tripos $[-, \Sigma_{\mathcal{K}^U_{A,A'}}]$ is equivalent to a Boolean subtripos of $[-, \mathcal{D}(A, A')]$.[5]

We need some preparations.

Firstly we introduce a Set-indexed preorder $\mathcal{P}(\mathcal{K}^U_{A,A'})$, which is equivalent to the Krivine tripos $[-, \Sigma_{\mathcal{K}^U_{A,A'}}]$.

For any set X, define a preorder $\vdash$ on $\mathcal{P}(\Pi)^X$: for any $\phi, \psi \in \mathcal{P}(\Pi)^X$,

$$\phi \vdash \psi \quad \Leftrightarrow \quad \exists a \in A', \forall x \in X, \forall t \in \phi(x)^{\perp\!\!\!\perp}, \forall \pi \in \psi(x), a(t.\pi)\downarrow \in U.$$

Let $\mathcal{P}(\mathcal{K}^U_{A,A'}) : \mathrm{Set}^{op} \to \mathsf{HeyPre}$ defined by:
on set X, $\mathcal{P}(\mathcal{K}^U_{A,A'})(X) := (\mathcal{P}(\Pi)^X, \vdash)$;
on functions, defined by the obvious composition.

By Lemma 5.5 in [Ferrer Santos et al., 2014], the Krivine tripos Krivine tripos $[-, \Sigma_{\mathcal{K}^U_{A,A'}}]$ and $\mathcal{P}(\mathcal{K}^U_{A,A'})$ are equivalent.

Secondly, we describe the Boolean subtripos $\mathcal{D}^U_{A,A'}$ of $[-, \mathcal{D}(A, A')]$, which is induced by a closure transformation $j_U : [-, \mathcal{D}(A, A')] \to [-, \mathcal{D}(A, A')]$, for the fixed downward closed set U.

Define a transformation j_U on the tripos $[-, \mathcal{D}(A, A')]$ as: for any set X, $\phi \in \mathcal{D}(A)^X$,

$$j_U(X)(\phi) := (\phi \to U) \to U,$$

[5] The following proof is worked out by Jaap van Oosten.

where for any $\psi \in \mathcal{D}(A)^X$, $\psi \to U$ is defined as: for any $x \in X$,

$$(\psi \to U)(x) := \psi(x) \to U := \{a \in A : \forall \pi \in \psi(x), a\pi \downarrow \in U\}.$$

It is straightforward to verify that j_U is a closure transformation on $[-, \mathcal{D}(A, A')]$. The subtripos it corresponds to is a Boolean subtripos $\mathcal{D}^U_{A,A'}$, which on any set X, has the same elements as $[-, \mathcal{D}(A, A')]$ on X, but a new order $\leq_U$, defined by, for any $\phi, \psi : X \to \mathcal{D}(A)$,

$$\phi \leq_U \psi \text{ iff } \phi \leq j_U(\psi).$$

In the following, we will show that $\mathcal{D}^U_{A,A'}$ and $\mathcal{P}(\mathcal{K}^U_{A,A'})$ are equivalent, hence $[-, \Sigma_{\mathcal{K}^U_{A,A'}}]$ is equivalent to $\mathcal{D}^U_{A,A'}$.

We need one last lemma.

Lemma 4.14. *For any set X, $\phi, \psi \in \mathcal{P}(\Pi)^X$, let $\mathcal{D}(\phi), \mathcal{D}(\psi) \in \mathcal{D}(A)^X$ defined by: for any $x \in X$, $\mathcal{D}(\phi)(x) := \downarrow \phi(x)$ and $\mathcal{D}(\psi)(x) := \downarrow \psi(x) = \{a \in A : \exists \pi \in \psi(x), a \leq \pi\}$[6], then $\phi \vdash \psi$ if and only if $\phi \to U \leq \psi \to U$ in $\mathcal{D}(A)^X$ if and only if $\mathcal{D}(\psi) \leq_U \mathcal{D}(\phi)$.*

Proof. Note that if $\phi(x) \in \mathcal{P}(\Pi)$, then $\phi(x) \to U$ is downward closed in A, hence $\phi \to U$ and $\psi \to U$ are in $\mathcal{D}(A)^X$.

The first if and only if: suppose $\phi \vdash \psi$, then there is $a \in A'$, such that for all $x \in X$, $t \in \phi(x) \to U$, $\pi \in \psi(x)$, $a(t.\pi) \downarrow \in U$. Define $a' := \langle xy \rangle a(x.y) \in A'$, then for all $t \in \phi(x) \to U$,

$$a't := \langle xy \rangle a(x.y)t \leq \langle y \rangle a(t.y) \downarrow.$$

For all $\pi \in \psi(x)$, $a't\pi \leq a(t.\pi) \downarrow \in U$, thus, $a't \in \psi(x) \to U$. We get $\phi \to U \leq \psi \to U$.

Suppose $\phi \to U \leq \psi \to U$, then there is $a \in A'$, such that for all $x \in X$, $t \in \phi(x) \to U$, $\pi' \in \psi(x)$, $at\pi' \downarrow$ and $at\pi' \in U$. Let $a' := \langle \pi \rangle a(\mathsf{b}_0\pi)(\mathsf{c}_1\pi) \in A'$, then for all $t \in \phi(x) \to U$, $\pi' \in \psi(x)$, $a'(t.\pi') \leq at\pi' \downarrow$ and $a'(t.\pi') \in U$.

The second if and only if: suppose $\phi \to U \leq \psi \to U$, then there is $a \in A'$, such that for all $x \in X$, $b \in (\phi(x) \to U)$, $\pi \in \psi(x)$, $ab\pi \downarrow \in U$. Consider $a' := \langle xy \rangle ayx$, then $a' \in A'$, we want to show that $\mathcal{D}(\psi) \leq_U \mathcal{D}(\phi)$ witnessed by a', i.e., for all $\pi' \in \mathcal{D}(\psi)(x)$, for all $b \in (\mathcal{D}(\phi)(x) \to U)$, we need to show $a'\pi'b \downarrow \in U$. Since $\pi' \in \mathcal{D}(\psi)(x)$, there is $\pi \in \psi(x)$, such that $\pi' \leq \pi$. Also note that $\mathcal{D}(\phi)(x) \to U = \phi(x) \to U$, therefore, $b \in \phi(x) \to U$. We have $a'\pi'b \leq a'\pi b \leq ab\pi \downarrow \in U$. The proof of the other direction is similar. □

[6]This is well-defined, since in our setting, $\Pi \subseteq A = \Lambda$.

Theorem 4.15. *The Krivine tripos* $[-, \Sigma_{\mathcal{K}^U_{A,A'}}]$ *is equivalent to the Boolean subtripos* $\mathcal{D}^U_{A,A'}$ *of* $[-, \mathcal{D}(A, A')]$.

Proof. Since $\mathcal{D}^U_{A,A'}$ is Boolean, it is equivalent to $(\mathcal{D}^U_{A,A'})^{op}$, i.e., $\mathcal{D}^U_{A,A'}$ with the opposite order. It suffices to show that $\mathcal{P}(\mathcal{K}^U_{A,A'})$ is equivalent to $(\mathcal{D}^U_{A,A'})^{op}$, since $\mathcal{P}(\mathcal{K}^U_{A,A'})$ is equivalent to $[-, \Sigma_{\mathcal{K}^U_{A,A'}}]$.

Define a Set-indexed preorder $\mathcal{D}^U_{A,A'}(\Pi)$ as, on X, $\mathcal{D}^U_{A,A'}(\Pi)(X) \coloneqq (\mathcal{D}(\downarrow\Pi)^X, \leq_U)$. We claim that $\mathcal{D}^U_{A,A'}(\Pi)$ is equivalent to $\mathcal{D}^U_{A,A'}$. We have an inclusion of indexed preorders: $i : \mathcal{D}^U_{A,A'}(\Pi) \hookrightarrow \mathcal{D}^U_{A,A'}$, it is enough to show that for any set X, i_X is essentially surjective. For any $\phi : X \to \mathcal{D}(A)$, let $\phi' : X \to \mathcal{D}(\downarrow\Pi)$ defined by, for any $x \in X$, $\phi'(x) \coloneqq \downarrow\{[a] : a \in \phi(x)\}$ where $[a]$ is the code of one element sequence a. We need to show that $\phi \leq_U \phi'$ and $\phi' \leq_U \phi$. Recall that there are combinators t, t' in A', such that for any $a \in A$, $\mathsf{t}a \leq [a]$, and $\mathsf{t}'[a] \leq a$. Therefore, $\phi \leq \phi'$ witnessed by t and $\phi' \leq \phi$ witnessed by t'. Note that for any $\alpha \in \mathcal{D}(A)$, $\alpha \subseteq (\alpha \to U) \to U$, hence, $\phi \leq_U \phi'$ and $\phi' \leq_U \phi$ with the same witnesses.

Therefore, it is sufficient to show that $\mathcal{P}(\mathcal{K}^U_{A,A'})$ is equivalent to $(\mathcal{D}^U_{A,A'}(\Pi))^{op}$. Consider the natural transformation $f_+ : (\mathcal{D}^U_{A,A'}(\Pi))^{op} \to \mathcal{P}(\mathcal{K}_{A,A'})$, which sends ϕ to ϕ itself for any set X, any $\phi \in \mathcal{D}(\downarrow\Pi)^X$. Then, for any X, $\phi, \psi \in \mathcal{D}(\downarrow\Pi)^X$, by lemma 4.14,

$$\phi \leq_U \psi \text{ if and only if } \psi \to U \leq \phi \to U \text{ if and only if } f_+(X)(\psi) = \psi \vdash \phi = f_+(X)(\phi).$$

Define the natural transformation $f^+ : \mathcal{P}(\mathcal{K}_{A,A'}) \to (\mathcal{D}^U_{A,A'}(\Pi))^{op}$ as: for any X, $\phi \in \mathcal{P}(\Pi)^X$, $f^+(X)(\phi) \coloneqq \mathcal{D}(\phi)$. Then for any $\phi, \psi \in \mathcal{P}(\Pi)^X$, by lemma 4.14,

$$\phi \vdash \psi \text{ if and only if } f^+(X)(\psi) = \mathcal{D}(\psi) \leq_U \mathcal{D}(\phi) = f^+(X)(\phi).$$

Furthermore, it is clear that for any X, $f^+(X) \circ f_+(X) = id_{\mathcal{D}(\downarrow\Pi)^X}$. And for any $\phi \in \mathcal{P}(\Pi)^X$, since $\mathcal{D}(\phi) \to U = \phi \to U$, by lemma 4.14, we have $\mathcal{D}(\phi) \vdash \phi$ and $\phi \vdash \mathcal{D}(\phi)$, hence, $f_+(X) \circ f^+(X) \simeq id_{\mathcal{P}(\Pi)^X}$. $\square$

Now we can give the characterization of Krivine toposes.

Theorem 4.16. *Every Krivine topos is equivalent to some Boolean subtopos of a generalised relative realizability topos.*

Conversely, every Boolean subtopos of a generalized relative realizability topos is equivalent to some Krivine topos.

Proof. The first statement is from Corollary 4.12. Let $\mathcal{K}$ be an aks, then by Corollary 4.12, there is an inclusion of triposes $i : [-, \Sigma_{\mathcal{K}}] \to [-, \mathcal{D}\Sigma_{\mathcal{K}}]$. It induces an inclusion of toposes $f_i : \mathsf{Set}[-, \Sigma_{\mathcal{K}}] \to \mathsf{Set}[-, \mathcal{D}\Sigma_{\mathcal{K}}]$. Hence, the Krivine topos $\mathsf{Set}[-, \Sigma_{\mathcal{K}}]$ is a Boolean subtopos of the generalized relative realizability topos $\mathsf{Set}[-, \mathcal{D}\Sigma_{\mathcal{K}}]$.

For the second statement, let $\mathsf{Set}[-, \mathcal{D}(A, A')]$ be a generalized relative realizability topos, with (A, A') a filtered order-pca. Suppose $\mathcal{E}$ is a Boolean subtopos of it, then by Proposition 3.34, $\mathcal{E}$ is induced by a local operator $q(U) : \Omega \to \Omega$ with Ω the subobject classifier in $\mathsf{Set}[-, \mathcal{D}(A, A')]$ and U a subterminal object.

A subterminal object in $\mathsf{Set}[-, \mathcal{D}(A, A')]$ corresponds to (the isomorphism class of) some $\phi : \{*\} \to \mathcal{D}(A)$, hence, specifies some downward closed subset U in $\mathcal{D}(A)$. As mentioned in section 3.4.3, the local operator $q(U)$ induces a closure transformation on $[-, \mathcal{D}(A, A')]$, which in this case is exactly the transformation j_U defined as above. By Theorem 4.15, the Krivine tripos $[-, \Sigma_{\mathcal{K}^U_{A,A'}}]$ is equivalent to the subtripos of $[-, \mathcal{D}(A, A')]$ induced by j_U. Hence, the Krivine topos $\mathsf{Set}[-, \Sigma_{\mathcal{K}^U_{A,A'}}]$ is equivalent to $\mathcal{E}$. □

As a direct consequence of the theorem above, every abstract Krivine structure is, up to equivalence[7], constructed from a filtered order-pca as described in section 4.3.2.

4.4 An Example

We end this chapter with an example of Krivine toposes, given by the construction described in section 4.3.2. We will show that some abstract Krivine structures built from the pca $\mathcal{K}_2$ (Kleene's sencond model) give rise to non-Grothendieck Krivine toposes.

4.4.1 Localic Triposes

Before discussing the example, we first analyse the conditions of localic triposes in this section.

Recall that localic triposes are those from locales (see section 3.2.2 for the definition). In [Hofstra, 2006], Hofstra has a characterization of when a tripos $[-, \Sigma]$, for a filtered order-pca Σ, is localic.

The following proposition is from Proposition 7.9 in [Hofstra, 2006].

[7]The equivalence here means giving rise to equivalent Krivine triposes.

Proposition 4.17. *Suppose Σ is a filtered order-pca, and $[-,\Sigma]$ is a tripos. Then the following are equivalent:*

1. *$[-,\Sigma]$ is localic;*
2. *there is a geometric morphism of toposes $\mathsf{Set}[-,\Sigma] \to \mathsf{Set}$;*
3. *the poset of designated truth-values of Σ has a least element.*

We could apply this proposition to the special case of Krivine triposes.

Let $\mathcal{K}$ be an aks, let $\Sigma_{\mathcal{K}} = (\mathcal{P}_{\perp\!\!\!\perp}(\Pi), \leq, \bullet)$ be the corresponding order-pca with the filter Φ as defined in section 4.2.2. By Proposition 4.17, the Krivine tripos $[-,\Sigma_{\mathcal{K}}]$ is localic iff the poset of designated truth-values of $\Sigma_{\mathcal{K}}$ has a least element. Recall that by Lemma 2.18, the set of designated truth-values of a filtered order-pca is the upward closure of the filter. Since Φ is upward closed, the filter Φ is the poset of designated truth-values of $\Sigma_{\mathcal{K}}$.

Krivine has a relevant characterization in [Krivine, 2012]: a classical realizability model is a forcing model iff there exists a proof-like term $a \in \mathsf{QP}$ realising

$$|\top \to (\bot \to \bot)| \cap |\bot \to (\top \to \bot)|,$$

i.e.,

$$\exists a \in \mathsf{QP}\ \forall \pi \in \Pi\ \forall s \in \Lambda\ \forall t \in\ |\bot|\ \ (\ (a, s.t.\pi) \in \perp\!\!\!\perp\ \wedge\ (a, t.s.\pi) \in \perp\!\!\!\perp),$$

where $|\bot| := \{t \in \Lambda : \forall \pi \in \Pi, (t,\pi) \in \perp\!\!\!\perp\}$.

Lemma 4.18. *The poset of designated truth value of $\Sigma_{\mathcal{K}}$ has a least element if and only if there is $a \in \mathsf{QP}$ realising $|\top \to (\bot \to \bot)| \cap |\bot \to (\top \to \bot)|$.*

Proof. Let $\mathsf{K}' := \mathsf{K}(\mathsf{SKK})$, we first prove that for any $t, s \in \Lambda$, $\pi \in \Pi$, $(\mathsf{K}', s.t.\pi) \in \perp\!\!\!\perp$ whenever $(t,\pi) \in \perp\!\!\!\perp$.

By the closure conditions of $\perp\!\!\!\perp$, we have:
$(t,\pi) \in \perp\!\!\!\perp \quad\Rightarrow\quad (\mathsf{K}, t.(\mathsf{K}t).\pi) \in \perp\!\!\!\perp \quad\Rightarrow\quad (\mathsf{K}t(\mathsf{K}t), \pi) \in \perp\!\!\!\perp \quad\Rightarrow\quad (\mathsf{S}, \mathsf{K}.\mathsf{K}.t.\pi) \in \perp\!\!\!\perp$
$\Rightarrow\quad (\mathsf{SKK}, t.\pi) \in \perp\!\!\!\perp \quad\Rightarrow\quad (\mathsf{K}, (\mathsf{SKK}).s.t.\pi) \in \perp\!\!\!\perp \quad\Rightarrow\quad (\mathsf{K}(\mathsf{SKK}), s.t.\pi) \in \perp\!\!\!\perp$.

Suppose Φ has a least element, then there is $\alpha \in \Phi$, such that for all $\beta \in \Phi$, $\beta \subseteq \alpha$. In particular, for any $b \in \mathsf{QP}$, $\{b\}^{\perp\!\!\!\perp} \subseteq \alpha$. Since $\alpha \in \Phi$, there is $a \in \mathsf{QP}$, with $a \in |\alpha|$, hence, $\alpha = \alpha^{\perp\!\!\!\perp\perp\!\!\!\perp} = |\alpha|^{\perp\!\!\!\perp} \subseteq \{a\}^{\perp\!\!\!\perp}$. Therefore, for all $b \in \mathsf{QP}$, $\{b\}^{\perp\!\!\!\perp} \subseteq \{a\}^{\perp\!\!\!\perp}$. By definition of $|\bot|$, for any $t \in\ |\bot|$, for any $\pi \in \Pi$, $(t,\pi) \in \perp\!\!\!\perp$, hence for all $s \in \Lambda$, $(\mathsf{K}, t.s.\pi) \in \perp\!\!\!\perp$ and $(\mathsf{K}', s.t.\pi) \in \perp\!\!\!\perp$.

Since K and K' are in QP, we have $\{\mathsf{K}\}^{\perp\!\!\perp} \subseteq \{a\}^{\perp\!\!\perp}$ and $\{\mathsf{K}'\}^{\perp\!\!\perp} \subseteq \{a\}^{\perp\!\!\perp}$. Therefore, for all $\pi \in \Pi, s \in \Lambda$, for all $t \in |\bot|$, $(a, t.s.\pi) \in \perp\!\!\perp$ and $(a, s.t.\pi) \in \perp\!\!\perp$.

Conversely, suppose there is $a \in \mathsf{QP}$ realising $|\top \to (\bot \to \bot)| \cap |\bot \to (\top \to \bot)|$, then by the Theorem in page 16 in [Krivine, 2012], there is $t \in \mathsf{QP}$, for all $b \in \mathsf{QP}$, for all $X \subseteq \Pi$, if $b \in |X|$, then $t \in |X|$. Consider $\{t\}^{\perp\!\!\perp} \in \Phi$, for any $\beta \in \Phi$, there is $b \in \mathsf{QP}$ with $b \in |\beta|$, thus $t \in |\beta|$, and we get $\beta = |\beta|^{\perp\!\!\perp} \subseteq \{t\}^{\perp\!\!\perp}$. Therefore, $\{t\}^{\perp\!\!\perp}$ is the least element in the designated truth-values of $\Sigma_{\mathcal{K}}$. $\square$

Therefore, Hofstra's and Krivine's conditions are equivalent. A classical realizability model is a forcing model if and only if the corresponding Krivine tripos is localic.

4.4.2 Krivine Toposes Constructed from $\mathcal{K}_2$

We now discuss an example of Krivine toposes constructed from Kleene's second model $\mathcal{K}_2$.

Let A be Kleene's second model $\mathcal{K}_2$ (the carrier set of A is $\mathbb{N}^{\mathbb{N}}$), A' be the set of recursive functions. Fix a finite subset $U = \{\tau_1, \cdots, \tau_k\}$ with $k \geq 1$ and $U \cap A' = \varnothing$. Then by the construction described in section 4.3.2, we get an abstract Krivine structure $\mathcal{K}^U_{A,A'}$ and a corresponding Krivine tripos $[-, \Sigma_{\mathcal{K}^U_{A,A'}}]$. It turned out that this Krivine tripos is not localic.

Before we state and prove this result, we describe a concrete way of coding finite sequences in $\mathcal{K}_2$. For a finite sequence $\alpha_0, \cdots, \alpha_k$ with $k \geq 0$, for all $0 \leq i \leq k$, $\alpha_i \in \mathbb{N}^{\mathbb{N}}$, we define $\alpha : \mathbb{N} \to \mathbb{N}$ as: $\alpha(0) := k$ and for any $j \geq 0$, $1 \leq i \leq k+1$, $\alpha(j(k+1)+i) := \alpha_{i-1}(j)$. It is easy to see that under this coding, $\Pi = \mathbb{N}^{\mathbb{N}}$.

Let (A, A') be defined as above.

Theorem 4.19. *For any $U \neq \varnothing$ a finite set of nonrecursive functions, the Krivine tripos $[-, \Sigma_{\mathcal{K}^U_{A,A'}}]$ is not localic.*

Proof. Suppose towards contradiction that $[-, \Sigma_{\mathcal{K}^U_{A,A'}}]$ is localic, then by Proposition 4.17, the filter Φ of the filtered order-pca $\Sigma_{\mathcal{K}^U_{A,A'}}$ has a least element, say $\Delta \in \mathcal{P}_{\perp\!\!\perp}(\Pi)$. By definition of the filter, there is a total recursive function α in $|\Delta|$.

For any $\beta \in A'$, consider $\{\beta\}^{\perp\!\!\perp} = \{\pi \in \Pi : \beta\pi\downarrow,\ \beta\pi \in U\} \in \Phi$, then by the fact that Δ is a least element in Φ, we get $\Delta \leq \{\beta\}^{\perp\!\!\perp}$, i.e., $|\Delta| \subseteq |\{\beta\}^{\perp\!\!\perp}|$, hence, $\alpha \in |\{\beta\}^{\perp\!\!\perp}|$. Therefore, for any total recursive function β, $\pi \in \Pi = \mathbb{N}^{\mathbb{N}}$, if $\beta\pi \in U$, then $\alpha\pi \in U$.

Suppose $U = \{\tau_1, \cdots, \tau_k\}$ with $k \geq 1$, $\tau_1, \cdots, \tau_k \in \mathbb{N}^{\mathbb{N}}$ nonrecursive. Take any π, such that $\alpha\pi \in U$ (such π certainly exists). Suppose $\alpha\pi = \tau_i$. Since U is fintie, there is $N \in \mathbb{N}$, such that for all $1 \leq j \neq i \leq k$, $\tau_i \restriction_{\leq N} \neq \tau_j \restriction_{\leq N}$. For all $0 \leq j \leq N$, there is $N_j \in \mathbb{N}$, such that $\alpha(\langle j, \pi(0), \cdots, \pi(N_j)\rangle) = \tau_i(j) + 1$, and for all $0 \leq t < N_j$, $\alpha(\langle j, \pi(0), \cdots, \pi(t)\rangle) = 0$. Take $N' := \max\{N_j : 0 \leq j \leq N\}$. We claim that for all $\pi' \in \Pi$ with $\pi' \restriction_{\leq N'} = \pi \restriction_{\leq N'}$, for all $j \in \mathbb{N}$, either $\alpha\pi'(j) = \tau_i(j)$ or $\alpha\pi'(j)$ is not defined, i.e., for all $t \in \mathbb{N}$, $\alpha(\langle j, \pi'(0), \cdots, \pi'(t)\rangle) = 0$.

The proof of the claim above is the following. Suppose not, then there is $\pi' \in \mathbb{N}^{\mathbb{N}}$ and $j_0 \geq 0$, such that $\pi' \restriction_{\leq N'} = \pi \restriction_{\leq N'}$ and $\alpha\pi'(j_0) \neq \tau_i(j_0)$. Suppose $\alpha\pi'(j_0) = m$, then by definition, there is $t \in \mathbb{N}$, such that $\alpha(\langle j_0, \pi'(0), \cdots, \pi'(t)\rangle) = m + 1$, and for all $l < t$, $\alpha(\langle j_0, \pi'(0), \cdots, \pi'(l)\rangle) = 0$. Let $h := \max\{N', t\}$, define $\pi" \in \mathbb{N}^{\mathbb{N}}$ as: for any $l \leq h$, $\pi"(l) := \pi'(l)$; for any $l > h$, $\pi'(l) := \tau_i(l - h - 1)$. Therefore, $\pi" \restriction_{>h} = \tau_i$. Clearly, there is a total recursive function β, such that $\beta\pi" = \tau_i \in U$. Hence, $\alpha\pi" \in U$. Note that $\pi" \restriction_{\leq N'} = \pi \restriction_{\leq N'}$, thus, for all $0 \leq l \leq N$, $\alpha\pi"(l) = \tau_i(l)$. By the choice of N, $\tau_i \restriction_{\leq N} \neq \tau_j \restriction_{\leq N}$, for all $1 \leq j \neq i \leq k$. Therefore, $\alpha\pi" = \tau_i$. However, since $\pi" \restriction_{\leq t} = \pi' \restriction_{\leq t}$, $\alpha\pi"(j_0) = \alpha\pi'(j_0) = m \neq \tau_i(j_0)$, contradiction.

Now we can construct a Turing machine M as: on input n, M enumerates finite sequences of natural numbers which are initial sequences of $\pi \restriction_{\leq N'}$ or have $\pi \restriction_{\leq N'}$ as a initial sequence, and at the same time, on each such sequence $t_0, \cdots, t_k$, M runs α on input $\langle n, t_0, \cdots, t_k\rangle$, if $\alpha(\langle n, t_0, \cdots, t_k\rangle) \neq 0$, then M finds the least least $k' \leq k$, such that $\alpha(\langle n, t_0, \cdots, t_{k'}\rangle) \neq 0$, then M halts and output $\alpha(\langle n, t_0, \cdots, t_{k'}\rangle) - 1$. Since $\alpha\pi = \tau_i$, for any $n \in \mathbb{N}$, there is a sequence $\pi(0), \cdots, \pi(n_k)$, such that $\alpha(\langle n, \pi(0), \cdots, \pi(n_k)\rangle) = \tau_i(n) + 1 \neq 0$. Therefore, on each input n, M will always halt. Moreover, if M halts on input n and outputs m, then by the claim we proved before, $m = \tau_i(n)$. In conclusion, M is the Turing machine that computes τ_i, hence, τ_i is total recursive, contradiction. $\square$

Therefore, the Krivine topos $\mathsf{Set}[-, \Sigma_{\mathcal{K}^U_{A,A'}}]$ is non-localic when U is a non-empty finite set of nonrecursive functions. Moreover, it is not even a Grothendieck topos.

Corollary 4.20. *For any $U \neq \varnothing$ a finite set of nonrecursive functions, the Krivine topos $\mathsf{Set}[-, \Sigma_{\mathcal{K}^U_{A,A'}}]$ is not a Grothendieck topos.*

Proof. By Theorem 4.19, the tripos $[-, \Sigma_{\mathcal{K}^U_{A,A'}}]$ is not localic. Hence, by Proposition 4.17, there is no geometric morphism of toposes from $\mathsf{Set}[-, \Sigma_{\mathcal{K}^U_{A,A'}}]$ to Set. We know that every Grothendieck topos admits a geometric morphism to Set. Therefore, $\mathsf{Set}[-, \Sigma_{\mathcal{K}^U_{A,A'}}]$ is not a Grothendieck topos. $\square$

We end this section by some discussion on when U induces a localic Krivine tripos.

From the proof of Theorem 4.19, we know that if $[-, \Sigma_{\mathcal{K}^U_{A,A'}}]$ is localic then there is a recursive function α, such that for any recursive function β, $\pi \in \Pi = \mathbb{N}^{\mathbb{N}}$, if $\beta\pi \in U$, then $\alpha\pi \in U$. It is easy to see that the converse also holds, since if such α exists then $\{\alpha\}^{\perp\!\!\perp}$ will be the least element in the filter Φ of $\Sigma_{\mathcal{K}^U_{A,A'}}$. Hence, we can derive sufficient conditions of U for when $[-, \Sigma_{\mathcal{K}^U_{A,A'}}]$ is localic.

For example, when U is an upward closed set with respect to *Turing reducibility*: for α, β in $\mathcal{K}_2$, we call α *Turing reducible* to β (write as $\alpha \leq_T \beta$) if there is a recursive γ, such that $\gamma\beta = \alpha$.

Corollary 4.21. *If $U \subseteq \mathbb{N}^{\mathbb{N}}$ satisfies*

$$\forall \beta \in \mathbb{N}^{\mathbb{N}}((\exists \alpha \in \mathbb{N}^{\mathbb{N}}\ \alpha \in U \wedge \alpha \leq_T \beta) \to \beta \in U),$$

then the tripos $[-, \Sigma_{\mathcal{K}^U_{A,A'}}]$ is localic.

Proof. There is a total recursive function $\alpha \coloneqq \langle x \rangle x$, such that for any recursive function β, $\pi \in \Pi = \mathbb{N}^{\mathbb{N}}$, if $\beta\pi \in U$, then $\beta\pi \leq_T \pi$ and $\pi \in U$. Therefore, $\alpha\pi = \pi \in U$. □

Corollary 4.22. *Let $U_0 = \{\alpha \in \mathbb{N}^{\mathbb{N}} : \alpha$ non-arithmetical$\}$, then the resulting tripos $[-, \Sigma_{\mathcal{K}^{U_0}_{A,A'}}]$ is localic.*

Proof. By Corollary 4.21, we only need to show that U_0 is upward closed with respect to $\leq_T$. For any total recursive function β, for all $\gamma \in \mathbb{N}^{\mathbb{N}}$, if $\beta\gamma$ is defined and non-arithmetical, we need to show that γ is also non-arithmetical. Suppose not, then γ is defined by a formula $\varphi_\gamma(x, y)$. Since β is total recursive, it is also defined by some formula $\varphi_\beta(x, y)$. Let $ln : \mathbb{N} \to \mathbb{N}$ be the total recursive functions that gives the length of a coded sequence, and $p : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with $p(a, i)$ the ith component of the sequence coded by a. Suppose $\phi_{ln}(x, y)$ and $\phi_p(x, y, z)$ defines ln and p respectively. We want to give a formula $\psi(x, y)$ which defines the function $\beta\gamma$. We break it into several subformulas. Let

$$\psi_0(a,k,x,y) \coloneqq \phi_{ln}(a,k+2) \wedge \phi_p(a,0,x) \wedge \forall 0 \leq i \leq k\ \exists t(\phi_p(a,i+1,t) \wedge \varphi_\gamma(i,t)) \wedge \varphi_\beta(a,y+1)$$
$$(a \text{ equals to } \langle x, \gamma(0), \cdots, \gamma(k)\rangle \text{ and } \beta(a) = y+1);$$

$$\psi_1(a',j,x) \coloneqq \phi_{ln}(a',j+2) \wedge \phi_p(a',0,x) \wedge \forall 0 \leq i \leq j\ \exists t(\phi_p(a',i+1,t) \wedge \varphi_\gamma(i,t)) \to \varphi_\beta(a',0)$$
$$(\text{if } a' \text{ equals to } \langle x, \gamma(0), \cdots, \gamma(j)\rangle, \text{ then } \beta(a') = 0).$$

Define

$$\psi(x,y) \coloneqq \exists a \exists k\ (\psi_0(a,k,x,y) \wedge \forall 0 \leq j < k \forall a'\ \psi_1(a',j,x)).$$

Now it is easy to see that $\psi(x, y)$ defines $\beta\gamma$, hence $\beta\gamma$ is arithmetical, contradiction. □

Chapter 5

Conclusions and Future Research

5.1 Conclusions

In this thesis, we have extended the research of classical realizability in several aspects.

Firstly, in Chapter 2 and Chapter 3 we explore the theories of the realizability algebras on which Krivine triposes are based, namely, theories of the filtered order-pcas. We extend properties of pcas and order-pcas to those of filtered ones. We define applicative morphisms of filtered order-pcas based on those of order-pcas. Also, in Theorem 3.23,[1] the correspondence between adjoint pairs of applicative morphisms of filtered order-pcas and geometric morphisms of the corresponding triposes has been made clear. In section 3.2.3, we make suitable adjustments of the conditions for an ${}^{\mathcal{I}}\mathcal{OCA}$, and propose the structure a $p-{}^{\mathcal{I}}\mathcal{OCA}$, which, regarded as a filtered order-pca Σ, is both necessarily and sufficient for the functor $[-, \Sigma]$ to become a Set-tripos.

Secondly, in Chapter 4 section 4.3, we characterise Krivine toposes in terms of generalized relative realizability toposes. We show that a topos is a Krivine topos if and only if it is equivalent to a Boolean subtopos of some generalized relative realizability topos. This result connects classical realizability with the intuitionistic one.

Last but not the least, we describe a construction of abstract Krivine structures from filtered order-pcas in section 4.3.2. With this construction, in section 4.4, we give concrete examples of Krivine toposes which are not Grothendieck.

[1] We remark here again that this theorem is not a new result, it is contained implicitly in Hofstra's paper [Hofstra, 2006].

5.2 Future Research

We list some unanswered questions as well as possible directions for future research in the following. I think all of them are nice problems to work on.

1. As Proposition 3.22 states, any geometric morphism $\psi : \mathcal{D}\Theta \to \mathcal{D}\Sigma$ of BCOs corresponds to a unique computationally dense map $\phi : \Sigma \to \mathcal{D}\Theta$. It would be nice to also characterise the inverse image part f° of an adjoint pair of applicative morphisms $f = (f_{\circ}, f^{\circ}) : (A, A') \to (B, B')$ between filtered order-pcas, i.e., when an applicative morphism $\phi : (B, B') \to (A, A')$ has a right adjoint.

   The closest result I know is Theorem 7.8 in [Hofstra, 2006], which states that when $\phi : \Sigma \to \Theta$ is a map of $\mathcal{D}$-algebras for the downset monad $\mathcal{D}$, then ϕ is computationally dense if and only if it has a right adjoint. However, it is not clear to me that if
   $$f = (f_{+}, f^{+}) : [-, \Sigma_1] \to [-, \Sigma_2]$$
   is a geometric morphism of triposes for some filtered order-pcas Σ_1 and Σ_2, then the induced applicative morphism $I(f^{+}) : \Sigma_2 \to \Sigma_1$ (as described in the proof of Theorem 3.23) is a map of $\mathcal{D}$-algebras.

2. The second problem relates to the correspondence between geometric morphisms of triposes and those of toposes. By Proposition 3.19, the correspondence only holds for those geometric morphisms of toposes whose inverse image parts preserve constant objects. Johnstone in [Johnstone, 2013] showed that any geometric morphism of realizability toposes satisfies this condition (see Proposition 3.25). The question is, can this result be generalized to generalized relative realizability toposes, or to Krivine toposes in a suitable way? The proof given by Johnstone uses some essential properties of realizability toposes, e.g., the only non-degenerate Boolean subtopos of a realizability topos is Set, which is not true for generalized relative realizability toposes, neither for Krivine toposes. Hence, we do not expect this result can be extended to the full class of generalized relative realizability toposes or Krivine toposes. But I think it is possible to generalize it to some subclasses.

3. Streicher defined the notion of abstract Krivine structures in [Streicher, 2013] (see Definition 4.2), one can define morphisms of abstract Krivine structures to make a category. For any abstract Krivine structure $\mathcal{K}$, there is an associated filtered order-pca $\Sigma_{\mathcal{K}}$, it would be nice to define morphisms of abstract Krivine structures such that they correspond to applicative morphisms of the associated filtered order-pcas.

4. The last one concerns the example we give in section 4.4. In Theorem 4.19, I proved that when U is a non-empty finite set containing nonrecursive functions, then $[-, \Sigma_{\mathcal{K}^U_{A,A'}}]$ is not localic. Does this result extend to the case when U is countably infinite and contains only nonrecursive functions? What is the necessarily and sufficient condition for U, such that the resulting Krivine tripos $[-, \Sigma_{\mathcal{K}^U_{A,A'}}]$ is non-localic?[2]

[2]To solve this problem, the theory of *Medvedev degrees* [Medvedev, 1955] might be helpful.

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