Computable Analysis Over the Generalized Baire Space

MSc Thesis *(Afstudeerscriptie)*

written by

Lorenzo Galeotti
(born May 6th, 1987 in Viterbo, Italy)

under the supervision of **Prof. Dr. Benedikt Löwe** and **Drs. Hugo Nobrega**, and submitted to the Board of Examiners in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam.*

Date of the public defense:	**Members of the Thesis Committee:**
July 21st, 2015	Dr. Alexandru Baltag (Chair)
	Dr. Benno van den Berg
	Dr. Yurii Khomskii
	Prof. Dr. Benedikt Löwe
	Drs. Hugo Nobrega
	Dr. Arno Pauly
	Dr. Benjamin Rin

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Contents

Abstract

One of the main goals of computable analysis is that of formalizing the complexity of theorems from real analysis. In this setting Weihrauch reductions play the role that Turing reductions do in standard computability theory. Via coding, we can transfer computability and topological results from the Baire space ω^ω to any space of cardinality $2^{\aleph_0}$, so that e.g. functions over $\mathbb{R}$ can be coded as functions over the Baire space and then studied by means of Weihrauch reductions. Since many theorems from analysis can be thought to as functions between spaces of cardinality $2^{\aleph_0}$, computable analysis can then be used to study their complexity and to order them in a hierarchy.

Recently, the study of the descriptive set theory of the generalized Baire spaces κ^κ for cardinals $\kappa > \omega$ has been catching the interest of set theorists. It is then natural to ask if these generalizations can be used in the context of computable analysis.

In this thesis we start the study of *generalized computable analysis*, namely the generalization of computable analysis to generalized Baire spaces. We will introduce $\mathbb{R}_\kappa$, a Cauchy-complete real closed field of cardinality 2^κ with κ uncountable. We will prove that $\mathbb{R}_\kappa$ shares many features with $\mathbb{R}$ which have a key role in real analysis. In particular, we will prove that a restricted version of the intermediate value theorem and of the extreme value theorem hold in $\mathbb{R}_\kappa$.

We shall show that $\mathbb{R}_\kappa$ is a good candidate for extending computable analysis to the generalized Baire space κ^κ. In particular, we generalize many of the most important representations of $\mathbb{R}$ to $\mathbb{R}_\kappa$ and we show that these representations are well-behaved with respect to the interval topology over $\mathbb{R}_\kappa$.

In the last part of the thesis, we begin the study of the Weihrauch hierarchy in this generalized context. We generalize some of the most important choice principles which in the classical case characterize the Weihrauch hierarchy. Then we prove that some of the classical Weihrauch reductions can be extended to these generalizations. Finally we will start the study of the restricted version of the intermediate value theorem which holds for $\mathbb{R}_\kappa$ from a computable analysis prospective.

Chapter 1

Introduction

Computable Analysis

Computable analysis is the study of the computational properties of real analysis. We refer the reader to [28] and [6] for an introduction to classical computable analysis.

In classical computability theory one studies the properties of functions over natural numbers and then transfers these properties to arbitrary countable spaces via coding. The same approach is taken in computable analysis.

One of the main tools of computable analysis is the Baire space ω^ω, namely the space of sequence of natural numbers of length ω. Following the classical computability theory approach, computational and topological properties of ω^ω are studied and then transferred to spaces of cardinality $2^{\aleph_0}$ via coding.

$$\mathbb{R} \xleftarrow{\text{Coding}} \omega^\omega$$

Of particular interest in computable analysis is the study of the computational and topological content of theorems from classical analysis. The idea is that of formalizing the complexity of theorems by means similar to those used in computability theory to classify functions over the natural numbers. In this context, the Weihrauch theory of reducibility plays a predominant role. For an introduction to the theory of Weihrauch reductions see [5]. Weirauch reductions can be used to classify functions over the Baire space ω^ω. Intuitively, a function $f : \omega^\omega \to \omega^\omega$ is said to be Weihrauch reducible to $g : \omega^\omega \to \omega^\omega$ if there are two continuous functions which translate f into g as shown in the following commuting diagram:

$$\begin{array}{ccc} \omega^\omega & \xrightarrow{\text{Input Translation}} & \omega^\omega \\ f \downarrow & & \downarrow g \\ \omega^\omega & \xleftarrow{\text{Output Translation}} & \omega^\omega \end{array}$$

Many theorems from classical analysis can be stated as formulas of the type:

$$\forall x \in X \exists y \in Y.\ \varphi(x, y),$$

with φ a quantifier-free formula. These formulas can be formalized by using multi-valued functions. A multi-valued function $T : X \rightrightarrows Y$ is a function that given an element x of X returns a subset of Y. Let us consider two classical examples, namely the Intermediate Value Theorem and the Baire Category Theorem.

The statement of the Intermediate Value Theorem is the following:
For every continuous function $f : [a, b] \to \mathbb{R}$ such that $f(a) \cdot f(b) < 0$ there is a real number $c \in [a, b]$ such that $f(c) = 0$. Therefore it can be stated as follows:

$$\forall f \in \mathrm{C}_{[a,b]} \exists c \in [a, b].\ f(c) = 0,$$

where $C_{[a,b]}$ is the set of continuous functions $f : [a,b] \to \mathbb{R}$ such that $f(a) \cdot f(b) < 0$. We can formalize this formula by the following multi valued function:

$$\mathrm{IVT} : C_{[a,b]} \rightrightarrows [a,b],$$

where, given a function $f \in C_{[a,b]}$, the set $\mathrm{IVT}(f) \subset [a,b]$ is such that

$$c \in \mathrm{IVT}(f) \Rightarrow f(c) = 0.$$

The Baire Category Theorem can be stated as follows:
Given a countable sequence of closed nowhere dense subsets $(A_n)_{n\in\omega}$ of a complete separable metric space X, the set $X \setminus \bigcup_{n\in\omega} A_n$ is not empty.

Therefore it can be formalized by the following multi valued function:

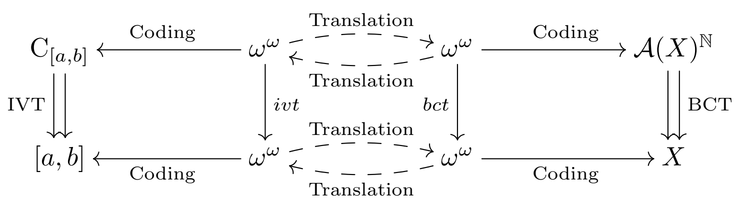
$$\mathrm{BCT} : \mathcal{A}(X)^{\mathbb{N}} \rightrightarrows X,$$

where $\mathcal{A}(X)^{\mathbb{N}}$ is the set of the countable sequences of closed nowhere dense subsets of X. Given a sequence $(A_n)_{n\in\omega}$, we have that:

$$\mathrm{BCT}((A_n)_{n\in\omega}) \in X \setminus \bigcup_{n\in\omega} A_n.$$

The previous two examples show that even though both the Intermediate Value Theorem and the Baire Category Theorem have a similar logical form, the multi-valued functions that represent them are quite different. It seems then really impractical to compare these two multi-valued functions directly. This apparent difficulty can be overcome by using the Baire space.

A multi-valued function $T : X \rightrightarrows Y$ is usually coded within the Baire space as the set of functions $t : \omega \to \omega$ such that for every $p \in \omega^\omega$, we have that $C(f(p)) \in T(C(p))$ where C is the function coding X in ω^ω. Given two multi-valued functions $T_1 : X_1 \rightrightarrows Y_1$ and $T_2 : X_2 \rightrightarrows Y_2$ one can therefore compare their complexity by studying the Weihrauch reducibility of their codings. In particular, one can study what is the relationship, with respect to Weihrauch reducibility, of the representations of T_1 and T_2. For this reason it is natural to use the Weihrauch theory of reducibility to compare theorems from analysis. The following diagram illustrates the situation for IVT and BCT:



By using this technique it is possible to arrange many theorems from classical real analysis in a complexity hierarchy called the Weihrauch hierarchy. A study of the Weihrauch degrees of the most important theorems from real analysis can be found in [5] and [2].

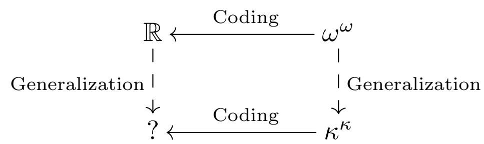
Generalized Baire Spaces

Recently, generalizations of the Baire space to uncountable cardinals have been of great interest for descriptive set theorists. We refer the reader to [13] for an introduction to generalized descriptive set theory. Even though the theory of generalized Baire spaces κ^κ with κ uncountable is not a new concept in set theory, many aspects of this theory are still unknown. In particular it is still unclear how these generalizations can be used in the context of computable analysis.

In this thesis we will begin for the first time the study of *generalized computable analysis*, namely the generalization of computable analysis to generalized Baire spaces. Given a space M of cardinality 2^κ, the idea is that of substituting the Baire space ω^ω with the generalized Baire space κ^κ and then of developing the machinery necessary in order to transfer topological properties form κ^κ to M. In particular we will be interested in the study of the Weihrauch hierarchy in the context of generalized Baire spaces.

Since in classical computable analysis and classical Weihrauch theory the field of real numbers has a central role, a question arises naturally:

What is the right generalization of $\mathbb{R}$ in the context of generalized computable analysis?



One of the main results of this thesis is the definition of $\mathbb{R}_\kappa$, a generalization of the real line which provides a well-behaved environment for generalizing real analysis and for developing generalized computable analysis.

Generalizations of the Real Line

The problem of generalizing the real line is not new in mathematics. Different approaches have been tried for very different proposes. A good introduction to these numbers systems can be found in [12]. Among the most influential contributions to this field particularly important are the works of Sikorski [26] and Klaua [18] on the *real ordinal numbers* and that of Conway [9] on the *surreal numbers*. Sikorski's idea was to repeat the classical Dedekind construction of the real numbers starting from an ordinal equipped with the Hessenberg operations (i.e., commutative operations over the ordinal numbers). Later Klaua extended Sikorski's work providing a complete study of this number system. Unfortunately the real ordinal numbers do not behave well in terms of analysis. In particular one can prove that these fields do not have the density properties that, as we will see, will have a central role in this context.

The surreal numbers were introduced by Conway in order to generalize both the Dedekind construction of real numbers and the Cantor construction of ordinal numbers. In his introduction to surreal numbers, Conway proved that they form a (class) real closed field (i.e., they have the same first order properties as the real numbers). Later, Dries and Ehrlich [18] proved that every real closed field is isomorphic to a subfield of the surreal numbers, showing therefore that they behave like a universal (class) model for real closed fields. It is then natural for us to use this framework in the development of $\mathbb{R}_\kappa$.

Our Results

As we will see, doing analysis over field extensions of $\mathbb{R}$ is not an easy task. In particular, this is due to the fact that no proper ordered field extension of $\mathbb{R}$ is connected. Intuitively this means that no such extension can be a linear continuum in the topological sense, namely it has many holes that can be detected by the interval topology. This is of course a problem if we want to do real analysis because many basic theorems of real analysis are in fact strongly related, sometimes even equivalent, to the fact that $\mathbb{R}$ is a connected space. To overcome this problem, instead of using standard topological tools, we will use a different mathematical framework which, under specific conditions over the density of $\mathbb{R}_\kappa$, will allow us to see our field extension of $\mathbb{R}$ as a linear continuum. By using these tools, we will prove some basic facts from classical analysis over $\mathbb{R}_\kappa$. In particular, since the Intermediate Value Theorem and the Extreme Value Theorem are two of the pillars of real analysis on which many others concepts rely, we will place particular attention on them.

The second part of this thesis will be devoted to the study of generalized computable analysis. In particular we will generalize the standard machinery from computable analysis by using generalized Baire spaces. Then we will start the study of $\mathbb{R}_\kappa$ from a computable analysis point of view, showing that, because of its properties, $\mathbb{R}_\kappa$ fits perfectly the role of extension of $\mathbb{R}$ to the generalized Baire space κ^κ. In particular we will show that many of the classical codings of $\mathbb{R}$ generalize naturally to $\mathbb{R}_\kappa$.

In the last part of this thesis, we will use all of the generalized tools we have developed to start the study of the Weihrauch hierarchy over $\mathbb{R}_\kappa$. We will show that some results from classical Weihrauch theory can be carried over to $\mathbb{R}_\kappa$ and κ^κ. In particular we will generalize some of the choice principles introduced by Brattka and Gherardi in [5] and we will show that, by generalizing the classical proofs, many classical results hold over these generalizations. Finally we will use these generalized choice principles to start the classification of the $\mathbb{R}_\kappa$ version the IVT.

Chapter 2

Basics

Before we start with the basic notions we will need to develop our theory of generalized computable analysis, we want to stipulate the following convention:

In this thesis, κ will refer to a fixed cardinal larger than ω. Moreover, since we are extending $\mathbb{R}$ to the generalized Baire space κ^κ, we will assume $\kappa^{<\kappa} = \kappa$. This is a standard requirement in generalized descriptive set theory. Moreover, since one of the essential features of ω that makes computable analysis work is that $\omega^{<\omega} = \omega$, it is natural for us to assume[1]:

ASSUMPTION: $\kappa^{<\kappa} = \kappa$.

2.1 Orders, Fields and Topology

Orders, ordered fields and topologies will be central concepts all over this thesis. In this section we will recall some of the basic definitions and properties of ordered sets, ordered fields and topological spaces. We start with the definition of partial order:

Definition 2.1.1 (Partial Order)**.** *Let P be a set and $\leq$ be a binary relation over P such that:*

- $\forall p \in P.\ p \leq p$ *(Reflexivity).*
- $\forall p, q \in P.\ p \leq q \wedge q \leq p \Rightarrow p = q$ *(Antisymmetry).*
- $\forall p, q, z \in P.\ p \leq q \wedge q \leq z \Rightarrow p \leq z$ *(Transitivity).*

then $(P, \leq)$ is called a partial order*. Moreover if*

$$\forall p, q \in P.\ p \leq q \vee q \leq p \vee p = q,$$

then $(P, \leq)$ is called a total *(or linear) order. A totally ordered subset of a partial order is called a* chain*.*

As usual if $p, q \in P$ are such that $p \leq q$ and $p \neq q$ then we will write $p < q$ (p is strictly smaller than q). Given two subsets A and B of a partial order $(P, \leq)$ we use the convention of writing $A < B$ if every element $a \in A$ is strictly smaller than every element of B.

Definition 2.1.2. *Let $(P, \leq)$ be a totally ordered set and A be a subset of P. Then we have:*

- *P is* dense *iff* $\forall p, q \in P.\ p < q \Rightarrow \exists r \in P.\ p < r < q$.
- *$A \subseteq P$ is* dense in P *iff* $\forall p, q \in P.\ p < q \Rightarrow \exists a \in A.\ p < a < q$.
- *$A \subseteq P$ is* cofinal *in P iff* $\forall p \in P. \exists a \in A.\ p \leq a$.
- *$A \subseteq P$ is* coinitial *in P iff* $\forall p \in P. \exists a \in A.\ a \leq p$.

[1] From now on, whenever we use the symbol κ we assume that it satisfies this assumption without further specification.

We will call cofinality *of P the smallest cardinal κ' such that there is a cofinal subset of P of cardinality κ'. We will denote the cofinality of P with* $\mathrm{Cof}(P)$. *Similarly, we will call* coinitiality *of P the smallest cardinal κ' such that there is a coinitial subset of P of cardinality κ'. We denote the coinitiality of P with* $\mathrm{Coi}(P)$. *Finally we will call* weight *of P,* $\mathrm{w}(P)$ *the smallest cardinal κ' such that there is a dense subset of P of cardinality κ'.*

Let us illustrate this notions by using a familiar example. Let $\mathbb{R}$ be the set of real numbers endowed with the usual order. Then $(\mathbb{R}, \leq)$ is a total order and $\mathbb{Q}$, the set of rational numbers, is dense in $\mathbb{R}$. Moreover $\mathbb{N}$, the set of natural numbers, is cofinal in $\mathbb{R}$ but is not coinitial, while $\mathbb{Z}$, the set of integer numbers, is both cofinal and coinitial in $\mathbb{R}$. As one can imagine cofinality, coinitiality and weight are three important properties of an ordered set, and as we will see they will be central in most of our constructions.

Definition 2.1.3. *Let $(P, \leq)$ be a totally ordered set. Then a* sequence *over P is an injective function $S = (x_i)_{i\in\alpha}$ whose domain is an ordinal α and codomain is P. α is the length of s and will be denoted as $|S|$. A sequence is* strictly increasing *if for all $\gamma, \beta < \alpha$, such that $\gamma < \beta$ then $x_\gamma < x_\beta$. Similarly, a sequence is* strictly decreasing *if for all $\gamma, \beta < \alpha$, such that $\gamma < \beta$ we have $x_\beta < x_\gamma$.*

Definition 2.1.4. *Let $(P, \leq)$ be a total order, α and β be two ordinals, $s_1 = (x_i)_{i\in\alpha}$ and $s_2 = (y_i)_{i\in\beta}$ be two sequences over P. Then we define:*

- *for $\gamma < \alpha$, $s_1 \restriction \gamma = (x_i)_{i\in\gamma}$, the* restriction *of s_1 to γ.*
- *For $p \in P$, $s_1^\frown p = (x_i)_{i\in\alpha+1}$ where $x_\alpha = p$, the* extension *of s_1 by p. More generally we define $s_1^\frown s_2$ as the* concatenation *of s_1 and s_2. We will sometimes omit the symbol $^\frown$, writing $s_1 s_2$ instead of $s_1^\frown s_2$.*
- *$s_1 \subseteq s_2$ iff there is $\gamma < \beta$ such that $s_1 = s_2 \restriction \gamma$, in this case we say that s_1 is a* prefix *of s_2.*
- *$s_1 \lhd s_2$ iff there are $\gamma < \beta$ such that for all $i < |s_1|$, $x_i = y_{\gamma+i}$, namely if s_1 is a* subsequence *of s_2.*

Let us illustrate the previous concepts with an example.

Example 2.1.5. *Let α be an ordinal and $\{0,1\}^{<\alpha}$ be the set of sequences with domain in κ. We have $0011010 \in \{0,1\}^{<\alpha}$ and the sequence 1_β of β ones is in $\{0,1\}^{<\alpha}$ if $\beta < \alpha$. Then $0011010^\frown 1_\beta \in \{0,1\}^{<\alpha}$ is the sequence 0011010 followed by β ones. We have that $00 \subset 0011010$, $1 \not\subset 0011010$, $101 \lhd 0011010$ and $111 \not\lhd 0011010$.*

Now we will recall two fundamental properties of orders introduced by Hausdorff, which will become extremely important later in this thesis.

Definition 2.1.6. *Let $(P, \leq)$ be a totally ordered set and κ' be a cardinal. Then we have:*

- *P is an $\alpha_{\kappa'}$-*set *iff every subset of P has a cofinal and coinitial subset of cardinality less than κ'.*
- *P is an $\eta_{\kappa'}$-*set *iff given $L, R \subseteq P$, such that $L < R$ and $|L| + |R| < \kappa'$ then there is $x \in P$ such that $L < \{x\} < R$.*

In particular $\eta_{\kappa'}$-sets for κ' uncountable are interesting. Intuitively a set X is an $\eta_{\kappa'}$-set if it is very dense, namely if in order to find an hole in the space unbounded sets of cardinality at least κ' are necessary.

Now that we have introduced all the basic definitions about orders we can start considering ordered groups and fields. We refer the reader to [8] for a complete introduction to field theory. We will recall some definitions that will be important in this thesis.

Definition 2.1.7 (Ordered Group)**.** *Let $(G, +, 0)$ be a group and $<$ be an order relation over G. Then $(G, +, 0, <)$ is an* ordered group *iff*

$$\forall a, b, c \in G.\ a \leq b \Rightarrow a + c \leq b + c.$$

We will denote the set of element of G which are strictly bigger than 0 with G^+. Moreover if G is an ordered group we will say that G has degree *κ' iff $\mathrm{Coi}(G^+) = \kappa'$. We will denote the degree of G by* $\mathrm{Deg}(G)$.

Let us illustrate these notions by two examples. The integers endowed with classical order and addition form an ordered group. Note, that $\mathbb{Z}^+$ has a minimum (i.e., 1), therefore $\mathrm{Deg}(\mathbb{Z}) = 1$. The rational numbers with their standard order and addition also form a group of degree ω. It is easy to see that the sequence $(\frac{1}{n})_{n\in\omega}$ is coinitial in $\mathbb{Q}^+$. Moreover, by the density of $\mathbb{Q}$ for every finite sequence of positive rational numbers $(q_n)_{n<m}$ there is $q \in \mathbb{Q}$ such that

$$0 < q < \{q_n \mid n < m\},$$

therefore $(q_n)_{n<m}$ can not be coinitial in $\mathbb{Q}^+$.

Given an ordered group we can define the *absolute value* of $a \in G$ as follows:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{otherwise.} \end{cases}$$

It is easy to see that

$$|a + b| \leq |a| + |b|$$

for every $a, b \in G$.

Definition 2.1.8 (Ordered Field)**.** *Let $(K, +, 0, 1, \cdot)$ be a field and $<$ be an order relation over K. Then $(K, +, \cdot, <)$ is an* ordered field *iff:*

- *$(K, +, <)$ is an ordered group.*
- *For every $a, b \in K$ bigger than 0, $0 \leq a \cdot b$.*

Using this definitions is not hard to see that many of the inequalities used in algebra hold for ordered fields. For example we have the following:

- $0 < 1$.
- For all $a, b, c \in K$, $a < b$ and $c > 0$ implies $a \cdot c < b \cdot c$.
- For all $a \in K$, $a < 0$ implies $-a > 0$.
- For all $a, b, c \in K$, $a < b$ implies $b - a > 0$.
- For all $a, b \in K$, $a < b$ and $a, b > 0$ implies $a^{-1} > b^{-1}$.

The most important examples of ordered fields are the set of rational numbers $\mathbb{Q}$ and the set of real numbers $\mathbb{R}$ endowed with the standard ordering and operations. As we said in the introduction one of the main aim of this thesis is that of finding a generalization of the field of real numbers which can be used in the context of computable analysis over the generalized Baire space κ^κ. It is natural then to focus on those fields which have the same (first order) properties of $\mathbb{R}$. Fields of this kind, form a special subclass of fields:

Definition 2.1.9 (Real Closed Field)**.** *A field K is* real closed *if every positive $a \in K$ is a square and if every polynomial of odd degree with coefficients in K has a root.*

It is a well known fact that the theory of real closed fields in the language $(+, \cdot, 0, 1, <)$ is model complete (i.e. every embedding of real closed fields is elementary). In particular it is easy to see that since the theory of real closed fields is model complete, every real closed field K is elementary equivalent to $\mathbb{R}$. In fact, let K be a real closed field. Since K has characteristic zero, $\mathbb{Q}$ is embedded in K. Therefore, the field of real algebraic numbers is an elementary submodel of K (note that the real algebraic numbers are the smallest real closed field containing $\mathbb{Q}$ see [19]). Now, since the field of real algebraic number is known to be elementary equivalent to $\mathbb{R}$ (see [20]), all the first order properties of $\mathbb{R}$ transfer to K. In particular this implies that the theory of real closed fields is complete. We refer a reader interested to the model theory of real closed fields to [20].

We conclude this section by recalling some basic notions from topology which will be particularly important for our constructions. We will use definitions and terminology from [22]. First recall that a topological space (X, τ) is T_0 if for every $x, y \in X$ there is an open set $U \in \tau$ such that $x \in U$ and $y \notin U$, is *second countable* if it has a countable base and is *separable* if it has a countable dense subset.

The order on $\mathbb{R}$ and the topology induced by this order have a central role in this field. Let $(X, \leq)$ be an ordered set. The *interval topology* over X is defined as the topology generated by the base B defined as follows:

- $(a, b) \in B$ for every $a, b \in X$ such that $a < b$.
- If b_0 is the maximum in M, then $(a, b_0] \in B$ for every $a \in X$.
- If a_0 is the minimum in M, then $[a_0, b) \in B$ for every $b \in X$.

The most important example of order topology is the topology on $\mathbb{R}$ generated by the open intervals of real numbers.

Another topology which will have a relevant role in our constructions is the subspace topology. Given a topology (X, τ) and a subset Y of X we define the *subspace topology* over Y as follows:

$$\tau_Y = \{U \cap Y \mid U \in \tau\}.$$

Naturally we have that the base of Y is related to that of X.

Lemma 2.1.10. *Let (X, τ) be a topology, B be a base of τ and $Y \subset X$. Then*

$$B_Y = \{B_1 \cap Y \mid B_1 \in B\},$$

is a base for the subspace topology.

Finally, let Y be a set, (X, τ) be a topological space and $f : X \to Y$ be a surjective function. Then the *final topology* induced by f is defined as follows:

$$O \in \tau \text{ iff } \delta^{-1}[O] \text{ is open in } \mathrm{dom}(f).$$

Note that since δ is surjective and continuous with respect to the final topology, then it is a quotient map. As we will see final topologies will have a central role both in classical and in generalized computable analysis.

2.2 Groups and Fields Completion

In this section we will recall some basic facts about group and field completions. A complete treatment of these subjects can be found in [8] and [10]. All the results in this section can be found in [10]. First we will present a general construction of *cut* completion over a group G.

Definition 2.2.1. *Let G be a totally ordered group and $L, R \subseteq G$ be subsets of G such that*

$$L < R.$$

We will call $\langle L, R\rangle$ a cut *over G.*

Definition 2.2.2. *Let G be a totally ordered group and C the set of all the cuts over G. Then we say that G is C-*complete *iff for every $\langle L, R\rangle \in C$ there is $x \in G$ such that $L < \{x\} < R$.*

Now we will define a general procedure which given a totally ordered dense group G and its set of cuts C, constructs a group G^C which contains G and is C-complete.

First we define an order relation over C as follows:

$$\langle L_1, R_1\rangle \leq \langle L_2, R_2\rangle \Leftrightarrow \forall \ell_1 \in L_1 \exists \ell_2 \in L_2.\ \ell_1 \leq \ell_2.$$

We define an equivalence relation $\sim$ over C as follows:

$$\langle L_1, R_1\rangle \sim \langle L_2, R_2\rangle \Leftrightarrow \langle L_1, R_1\rangle \leq \langle L_2, R_2\rangle \ \wedge\ \langle L_2, R_2\rangle \leq \langle L_1, R_1\rangle.$$

Now we define the underlying set of G^C as the quotient of C under $\sim$, namely

$$G^C = C/\sim.$$

First of all note that for all $x \in G$ we can define a cut $\langle L_x, R_x \rangle$ by taking

$$L_x = \{y \in G \mid y < x\}$$

and

$$R_x = \{y \in G \mid y > x\}$$

Then the mapping $x \longmapsto [\langle L_x, R_x \rangle]$ is an embedding of G in G^C.

It is easy to see that, if we define the order on G^C as follows:

$$[\langle L_1, R_1 \rangle] \leq [\langle L_2, R_2 \rangle] \Leftrightarrow \langle L_1, R_1 \rangle \leq \langle L_2, R_2 \rangle,$$

then the embedding preserves the order.

We define the addition over G^C in the following way:

$$[\langle L_1, R_1 \rangle] + [\langle L_2, R_2 \rangle] = [\langle L_1, R_1 \rangle + \langle L_2, R_2 \rangle],$$

where $\langle L_1, R_1 \rangle + \langle L_2, R_2 \rangle$ is defined as follows:

$$\langle L_1, R_1 \rangle + \langle L_2, R_2 \rangle = \langle \{\ell_1 + \ell_2 \mid \ell_1 \in L_1, \ell_2 \in L_2\}, \{r_1 + r_2 \mid r_1 \in R_1, r_2 \in R_2\} \rangle.$$

It is not hard to see that the embedding $x \longmapsto [\langle L_x, R_x \rangle]$ also preserves $+$. Indeed:

$$[\langle L_{x+y}, R_{x+y} \rangle] = [\langle L_x + L_y, R_x + R_y \rangle] = [\langle L_x, R_x \rangle] + [\langle L_y, R_y \rangle].$$

It is then clear that G^C is a totally ordered group. Finally we claim that G^C is complete. Let $\langle L, R \rangle$ be in C. Then defining

$$L' = \bigcup_{[\langle L_\alpha, R_\alpha \rangle] \in L} L_\alpha$$

and

$$R' = \bigcup_{[\langle L_\alpha, R_\alpha \rangle] \in R} R_\alpha,$$

we have that $L < [\langle L', R' \rangle] < R$ in G^C. Therefore G^C is complete.

Now, if G is an ordered field we can extend this completion in such a way that G^C is also an ordered field. We only need to define the multiplication over G^C. Let $x, y \in G^C$ with $x, y > 0$, $x = [\langle L_x, R_x \rangle]$ and $y = [\langle L_y, R_y \rangle]$. Then we define:

$$x \cdot y = [\langle L_x, R_x \rangle \cdot \langle L_y, R_y \rangle],$$

where $\langle L_x, R_x \rangle \cdot \langle L_y, R_y \rangle$ is defined as follows:

$$\langle L_x, R_x \rangle \cdot \langle L_y, R_y \rangle = \langle \{\ell_x \cdot \ell_y \mid \ell_x \in L_x, \ell_y \in L_y\}, \{r_x \cdot r_y \mid r_x \in R_x, r_y \in R_y\} \rangle.$$

Moreover we define:

$$x \cdot y = \begin{cases} (-x) \cdot (-y) & \text{iff } x, y < 0, \\ -((-x) \cdot y) & \text{iff } x < 0 \ y > 0, \\ -(x \cdot (-y)) & \text{iff } x > 0 \ y < 0, \\ 0 & \text{iff } x = 0 \ \vee \ y = 0. \end{cases}$$

Note that, if G is a real closed field, then G^C endowed with $\cdot$ fulfils all the properties of a real closed field, and that $x \longmapsto [\langle L_x, R_x \rangle]$ is a field morphism between G and G^C (see [10]).

Definition 2.2.3. *Let K be an ordered field and C the set of cuts over K. Then K' is a C-completion of K if it is C-complete and K is isomorphic to a dense subfield of K'.*

By what we have just seen we have:

Theorem 2.2.4. *Let K be an ordered field and C the set of cuts over K. Then K^C is a C-completion of K.*

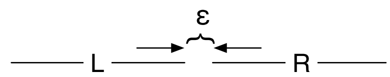


Figure 2.1: A Cauchy cut.

Now note that the previous construction is a generalization of the classical Dedekind construction of the real numbers. In particular by taking $G = \mathbb{Q}$ and restricting C to the set of Dedekind cuts over $\mathbb{Q}$ (i.e., imposing $L \neq \emptyset$ and $R \neq \emptyset$ for every $\langle L, R\rangle \in C$), we have that $G^C = \mathbb{R}$. Now we want to show that the classical Cauchy completion of a field is also just a particular case of the previous construction.

Definition 2.2.5 (Cauchy cuts)**.** *Let G be a totally ordered group and $\langle L, R\rangle$ be a cut over G. We will say that $\langle L, R\rangle$ is a* Cauchy cut *iff it is a cut such that, L has no maximum, R has no minimum and for each $\varepsilon \in G^+$ there are $\ell \in L$ and $r \in R$ such that $r < \ell + \varepsilon$. We will say that G is* Cauchy-complete *iff for each Cauchy cut $\langle L, R\rangle$, there is $x \in G$ such that $L < \{x\} < R$.*

Intuitively Cauchy cuts are cuts whose elements of the left and right sets get arbitrarily close to each other (Fig.2.1).

Definition 2.2.6. *Let K be an ordered field. We will say that K' is a* Cauchy cut completion *of K iff K is a dense subset of K' and K' is C-complete with C set of Cauchy cuts over K'.*

Theorem 2.2.7. *Let K be a field and C be the set of Cauchy cuts over K. Then K^C is a Cauchy cut completion of K.*

Proof. The construction of K^C we have just defined works perfectly also with C restricted to the set of Cauchy cuts over K. □

Classically a Cauchy completion of an ordered field is characterized in terms of sequences as follows:

Definition 2.2.8 (Cauchy sequences)**.** *Let G be a totally ordered group, and α an ordinal. Then a sequence $(x_i)_{i\in\alpha}$ of elements of G is* Cauchy *iff:*

$$\forall \varepsilon \in G^+ \exists \beta < \alpha \forall \gamma, \gamma' \geq \beta.\ |x_{\gamma'} - x_\gamma| < \varepsilon.$$

The sequence is convergent *if there is $x \in G$ such that:*

$$\forall \varepsilon \in G^+ \exists \beta < \alpha \forall \gamma \geq \beta.\ |x_\gamma - x| < \varepsilon.$$

We will call x the limit *of $(x_i)_{i\in\alpha}$. Given a group G it is said to be Cauchy complete iff every Cauchy sequence of length* $\mathrm{Deg}(G)$ *has a limit in G.*

It turns out that being Cauchy cut complete and being Cauchy complete are is equivalent notions.

Proposition 2.2.9 (Dales & Woodin)**.** *Let $\langle L, R\rangle$ a Cauchy cut in an ordered group G. Then $|L| = \mathrm{Deg}(G) = |R|$.*

Proof. See [10, Proposition 3.3]. □

Then we have the following:

Theorem 2.2.10 (Dales & Woodin)**.** *The group G is Cauchy cut complete iff G is Cauchy complete.*

Proof. Assume G Cauchy cut complete, and let $(x_i)_{i\in\alpha}$ be a Cauchy sequence in G. For each $\varepsilon \in G^+$ there is $\sigma_\varepsilon > 0$ such that for every $i, j \geq \sigma_\varepsilon$, we have $|x_i - x_j| < \varepsilon$. Define

$$L = \bigcup \{(-\infty, x_{\sigma_\varepsilon} - \varepsilon] \mid \varepsilon \in G^+\}$$

and

$$R = \bigcup \{ ([x_{\sigma_\varepsilon} + \varepsilon, +\infty) \mid \varepsilon \in G^+ \}.$$

Then for every $\varepsilon \in G^+$ take $0 < \varepsilon' < \varepsilon$, then take $\ell \in L$, $r \in R$ such that $\ell = x_{\sigma_{\varepsilon'}} - \varepsilon'$ and $r = x_{\varepsilon'} + \varepsilon'$, then $\ell + \varepsilon > r$. Hence $\langle L, R \rangle$ is Cauchy and there is $x \in G$ such that $L < \{x\} < R$ as desired. Now assume that every Cauchy sequence of length $\mathrm{Deg}(G)$ has a limit. Let $\langle L, R \rangle$ be a Cauchy cut in G, then by the previous proposition it $|L| = \mathrm{Deg}(G)$. Then there is a strictly increasing sequence cofinal in L of cardinality $\mathrm{Deg}(G)$ and it is trivially Cauchy, hence it converges to an element of $x \in G$. Then we have $L < \{x\} < R$ as desired. □

Given the previous theorem, we will use the two definitions of Cauchy completion interchangeably.

2.3 Surreal Numbers

The surreal numbers were introduced by Conway [9] in order to generalize both the Dedekind construction of real numbers and the Cantor construction of ordinal numbers. He realized that both Dedekind and Cantor were using a common pattern to define numbers.

As we will see even though their definition is simple, the surreal numbers form a very powerful tool for studying different number systems.

Conway's idea was that of generalize these two definition in order to generate both ordinals and real numbers on the same time.

2.3.1 Basic Definitions

The following definition of surreal numbers is due to Conway and it has been deeply studied by Gonshor in [14].

Definition 2.3.1 (Surreal Numbers)**.** *A* surreal number *is a function from an ordinal* $\alpha \in \mathrm{On}$ *to* $\{+, -\}$*, i.e., a sequence of pluses and minuses of ordinal length. We will denote the class of surreal numbers by* No*. The* length *of a surreal number* $x \in \mathrm{No}$ *is the smallest ordinal* $|x| \in \mathrm{On}$ *for which* x *is not defined.*

We can define a total order over No as follows:

Definition 2.3.2. *Let* $x, y \in \mathrm{No}$ *be two surreal numbers. Then we define the following* order*:*

$$x < y \text{ iff } x(\alpha) < y(\alpha) \qquad \text{where } \alpha \text{ is the smallest ordinal s.t. } x(\alpha) \neq y(\alpha),$$

here we are using the order $- < 0 < +$ *where* $x(\alpha) = 0$ *if* x *is not defined at* α*.*

Given the previous definition it easy to see that No has a natural binary tree structure (see Fig.2.2).

Note that each level of the tree corresponds to a set of surreal numbers with the same length. In particular we can define:

Definition 2.3.3. *Let* No *be the class of surreal numbers and* $\alpha \in \mathrm{On}$ *be an ordinal. We define the following sets:*

$$\mathrm{No}_\alpha = \{+, -\}^\alpha \text{ i.e. the set of sequences of length exactly } \alpha,$$

$$\mathrm{No}_{\leq\alpha} = \bigcup_{\beta \leq \alpha} \mathrm{No}_\beta \text{ i.e. the set of sequences of length less or equal to } \alpha,$$

$$\mathrm{No}_{<\alpha} = \bigcup_{\beta < \alpha} \mathrm{No}_{\leq\beta} \text{ i.e. the set of sequences of length less than } \alpha.$$

Note that from this definition it is not hard to see that $\mathrm{No}_{\leq\alpha} = \mathrm{No}_{<\alpha} \cup \mathrm{No}_\alpha$ and $\mathrm{No}_\alpha = \mathrm{No}_{\leq\alpha} \setminus \mathrm{No}_{<\alpha}$. Moreover these sets determine proper initial trees of the surreal number tree, as shown in Fig.2.3. Some of these subtrees will be of particular importance for our constructions. In particular as we will see, the following theorem will be central for the constructions of Chapter 3:

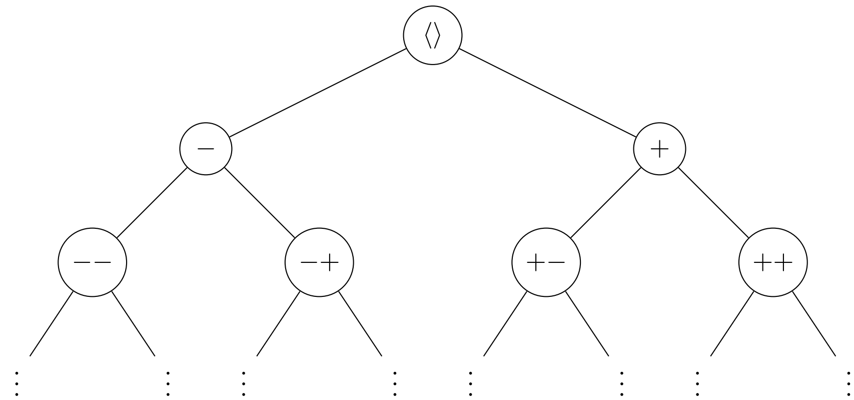


Figure 2.2: The surreal tree.

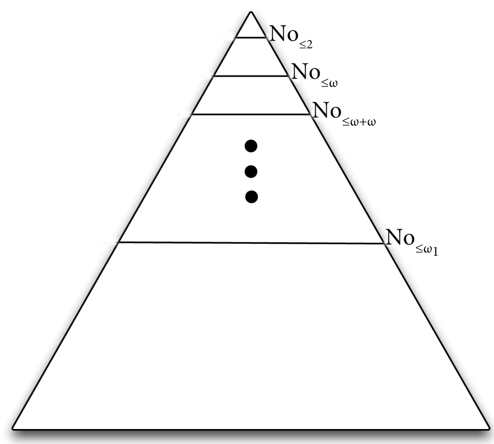


Figure 2.3: The subtrees of No.

Theorem 2.3.4 (Alling)**.** *Let κ' be a regular cardinal. Then $\mathrm{No}_{<\kappa'}$ is a real closed field.*

Proof. See [1, Theorem 6.22]. ☐

An extended study of these trees can be found in [11] and [1].

The following theorem will have a central role in the definition of operations over surreal numbers.

Theorem 2.3.5 (Gonshor, Simplicity Theorem)**.** *Let L and R be two sets of surreal numbers such that $L < R$. Then there is a unique surreal z, denoted by $[L|R]$, of minimal length such that $L < \{z\} < R$. We will call $[L|R]$ a representation of z.*

Proof. See [14, Theorem 2.1]. ☐

Given two finite family of sets of surreal numbers $S_0 \dots S_n$ and $S'_0 \dots S'_m$, we will use the following notation:

$$[S_0, \dots, S_n | S'_0, \dots, S'_m] = [\bigcup_{i \leq n} S_i | \bigcup_{i \leq m} S'_i].$$

Moreover, given two finite sequence of surreal numbers $x_0, \dots, x_n$ and $x'_0, \dots, x'_m$ we define:

$$[x_0, \dots, x_n | x'_0, \dots, x'_m] = [\{x_0, \dots, x_n\} | \{x'_0, \dots, x'_m\}].$$

Each surreal number has many different representations, the following theorem gives us a canonical representation.

Theorem 2.3.6 (Gonshor)**.** *Let $x \in \mathrm{No}$ be a surreal number, L and R be two subsets of* No *defined as follows:*

$$L = \{y \mid x < y \wedge y \subset x\},$$
$$R = \{y \mid x > y \wedge y \subset x\}.$$

Then $[L|R] = x$.

Proof. See [14, Theorem 2.8]. □

We will call the representation given by Theorem 2.3.6 the *canonical representation* of x. Note that the canonical representation just says that the elements of L are the proper initial segments y of x such that $x(|y|) = +$ and the elements of R are the proper initial segments y of x such that $x(|y|) = -$. For example the canonical representation of $+-+$ is $[(), (+-)|(+)]$, which means that $+-+$ is the shortest number between $+-$ and $+$. As we will see, representations have an important role in developing surreal numbers theory. For this reason we will introduce some theorems which allow to manipulate and characterize these representations.

Theorem 2.3.7 (Gonshor)**.** *Let L and R be two sets of surreal numbers such that $L < R$. Then $|[L|R]|$ is smaller or equal to the least ordinal α such that:*

$$\forall x \in L \cup R.\ |x| < \alpha.$$

Proof. Note that this follows trivially from the fact that $[L|R]$ is defined to be de shortest surreal number strictly between L and R, then if it is of length bigger than α. Hence $[L|R]\restriction\alpha$ would be shorter than $[L|R]$ and still in between L and R. □

Theorem 2.3.8 (Gonshor)**.** *Let $x, y \in \mathrm{No}$ be two surreal numbers and $[L_x|R_x]$, $[L_y|R_y]$ be respectively a representation of x and y. Then $x \leq y$ iff $\{x\} < R_y$ and $L_x < \{y\}$.*

Proof. We have $L_x < \{x\} < R_x$ and $L_y < \{y\} < R_y$. Assume $x \leq y$ then trivially $\{x\} \leq \{y\} < R_y$ and $L_x < \{x\} \leq \{y\}$. Assume $\{x\} < R_y$ and $L_x < \{y\}$ and $y < x$. We have $L_x < \{y\} < \{x\} < R_x$ hence x is an initial segment of y. Moreover $L_y < \{y\} < \{x\} < R_y$ then y is an initial segment of x. Hence $x = y$ which contradicts our assumption. □

Finally we present three theorems from [14] which are very useful to find out when two different representation represent the same surreal number.

Definition 2.3.9 (Cofinality)**.** $[L|R]$ *is* cofinal *in* $[L'|R']$ *iff:*

$$\forall x' \in R' \exists x \in R.\ x \leq x' \ \wedge\ \forall y' \in L' \exists y \in L.\ y \geq y'.$$

Moreover, given two representations $[L|R]$ *and* $[L'|R']$ *they are* mutually cofinal *iff* $[L|R]$ *is cofinal in* $[L'|R']$ *and* $[L'|R']$ *is cofinal in* $[L|R]$.

Note that this definition is totally consistent with the standard definition of cofinality (see Definition 2.1.2).

Theorem 2.3.10. *Suppose $x = [L|R]$, $L' < x < R'$ and $[L'|R']$ cofinal in $[L|R]$ then $x = [L'|R']$.*

Theorem 2.3.11. *Suppose $[L|R]$ and $[L'|R']$ are mutually cofinal then $[L|R] = [L'|R']$.*

2.3.2 Operations Over No

In this section we will define addition and multiplication over surreal numbers. First of all let us introduce some notation which will simplify the definition of the operations over surreal numbers. If S and S' are a sets of surreal numbers, x is a surreal number and Op a binary operation over the surreal numbers, then we define

$$S \operatorname{Op} x = \{s \operatorname{Op} x | s \in S\},\ x \operatorname{Op} S = \{x \operatorname{Op} s | s \in S\}$$

and

$$S \operatorname{Op} S' = \{x \operatorname{Op} y | x \in S \wedge y \in S'\}.$$

We begin the study of the surreal operations by defining the addition and its inverse.

Definition 2.3.12 (Surreal Sum and Inverse)**.** *Let x and y be two surreal numbers and $[L_x|R_x]$, $[L_y|R_y]$ be their canonical representations. Then we define the* sum *$x +_{\mathrm{s}} y$ as follows:*

$$x +_{\mathrm{s}} y = [L_x +_{\mathrm{s}} y, x +_{\mathrm{s}} L_y | R_x +_{\mathrm{s}} y, x +_{\mathrm{s}} R_y].$$

Moreover we define the inverse *of x as the surreal number obtained by reverting all the signs. It is easy to see that that $[-R_x| - L_x]$ where*

$$-R_x = \{-x_R \mid x_R \in R_x\}$$

and

$$-L_x = \{-x_L \mid x_L \in L_x\},$$

is a canonical representation of $-x$.

The previous definition was given by induction over the maximal length of the addends. Note that we defined $+_{\mathrm{s}}$ and $-_{\mathrm{s}}$ only for canonical representations. The following theorem tells us that the choice of the representations we used does not matters.

Theorem 2.3.13. *Let $[L_x|R_x]$ and $[L_y|R_y]$ be two representations respectively of x and y. Then*

$$x +_{\mathrm{s}} y = [L_x +_{\mathrm{s}} y, x +_{\mathrm{s}} L_y | R_x +_{\mathrm{s}} y, x +_{\mathrm{s}} R_y].$$

Proof. See [14, Theorem 3.2]. □

The intuition behind the definition of $+_{\mathrm{s}}$ is that $x +_{\mathrm{s}} y$ can be thought to be the smallest number such that the following inequalities hold:

$$L_x + y < x + y < R_x + y,$$
$$x + L_y < x + y < x + R_y.$$

Then the definition of $x +_{\mathrm{s}} y$ is exactly reflecting this intuition, in fact $x +_{\mathrm{s}} y$ is defined to be the shortest surreal number for which the previous inequalities hold. Let us consider some examples of sum.

Example 2.3.14. *Consider the sequence $+$ and its inverse $-$. Let $\langle\rangle$ be the empty sequence*[2]*. Then we have*

$$(+) = [\langle\rangle|\emptyset] \text{ and } (-) = [\emptyset|\langle\rangle],$$

where $\langle\rangle$ is the empty sequence. Then $(+) +_{\mathrm{s}} (-) = [\emptyset|\emptyset] = \langle\rangle$. Therefore it is natural to define $0 = \langle\rangle$. Now denote $(+)$ as 1. Finally we have

$$1 + 1 = (+) +_{\mathrm{s}} (+) = \{1 +_{\mathrm{s}} 0, 0 +_{\mathrm{s}} 1|\}.$$

We will denote this number by 2. It is not hard to convince yourself that we could interpret all the natural numbers in this way.

[2]Note that for the theory of surreal numbers $\langle\rangle = [\emptyset \mid \emptyset] \neq \emptyset$.

Definition 2.3.15 (Surreal Product). *Let x and y be two surreal numbers and $[L_x|R_x]$, $[L_y|R_y]$ be their canonical representations. Then we define the* product $x \cdot_{\mathrm{s}} y$ *as follows:*

$$x \cdot_{\mathrm{s}} y = [L_x \cdot_{\mathrm{s}} y +_{\mathrm{s}} x \cdot_{\mathrm{s}} L_y - L_x \cdot_{\mathrm{s}} L_y, R_x \cdot_{\mathrm{s}} y +_{\mathrm{s}} x \cdot_{\mathrm{s}} R_y - R_x \cdot_{\mathrm{s}} R_y \\ |L_x \cdot_{\mathrm{s}} y +_{\mathrm{s}} x \cdot_{\mathrm{s}} R_y - L_x \cdot_{\mathrm{s}} R_y, R_x \cdot_{\mathrm{s}} y +_{\mathrm{s}} x \cdot_{\mathrm{s}} L_y - R_x \cdot_{\mathrm{s}} L_y].$$

Also in this case the definition is by induction over the maximal length of the factors, and as before the definition is uniform (the interested reader is referred to [14] Theorem 3.5). Let us illustrate how this definition works with an example.

Example 2.3.16. *We have already defined $0 = \langle\rangle$, $1 = +$ and $2 = ++$. Let us consider the multiplication $2 \cdot_{\mathrm{s}} 1$. First note that trivially:*

$$0 \cdot_{\mathrm{s}} 1 = 0 \cdot_{\mathrm{s}} 1 = [\emptyset|\emptyset] = 0$$

and

$$0 \cdot_{\mathrm{s}} 2 = 0 \cdot_{\mathrm{s}} 2 = [\emptyset|\emptyset] = 0.$$

Moreover we have

$$1 \cdot_{\mathrm{s}} 1 = [0 \cdot_{\mathrm{s}} 1 +_{\mathrm{s}} 1 \cdot_{\mathrm{s}} 0 - 0 \cdot_{\mathrm{s}} 0|\emptyset].$$

Therefore

$$1 \cdot_{\mathrm{s}} 2 = [0 \cdot_{\mathrm{s}} 2 +_{\mathrm{s}} 1 \cdot_{\mathrm{s}} 1 - 0 \cdot_{\mathrm{s}} 1|\emptyset].$$

In conclusion $1 \cdot_{\mathrm{s}} 2 = [1|\emptyset] = 2$.

The last operation we introduce is the inverse of the product. While the previous definitions are quite intuitive the definition for the inverse of $\cdot_{\mathrm{s}}$ is more complicated. First of all one should convince himself that the naive definition, namely

$$\frac{1}{y} = [0, \frac{1}{L_y}|\frac{1}{R_y}]$$

does not work. In particular this definition is such that $\frac{1}{y} \cdot_{\mathrm{s}} y \neq 1$ for some $y \in \mathrm{No}$. To see this it is enough to compute some left element of $\frac{1}{3} \cdot_{\mathrm{s}} 3$ and check that it is bigger than 1. In particular we would have $\frac{1}{3} = [0, 1, \frac{1}{2}|\emptyset]$ and $3 = [0, 1, 2|\emptyset]$. But then $3 +_{\mathrm{s}} \frac{1}{3} -_{\mathrm{s}} 1$ would be a left element of $\frac{1}{3} \cdot_{\mathrm{s}} 3$, and by using the definition of $+_{\mathrm{s}}$ and $-_{\mathrm{s}}$ we would have $3 +_{\mathrm{s}} \frac{1}{3} -_{\mathrm{s}} 1 > 1$.

The main idea behind the definition of inverse of $\cdot_{\mathrm{s}}$ is that of setting the values in $\frac{1}{y}$ in such a way that the left elements of $\frac{1}{y} \cdot_{\mathrm{s}} y$ are smaller than 1. We define the inverse of $\cdot_{\mathrm{s}}$ by induction over the length of y as follows:

Definition 2.3.17 (Product Inverse). *Let $y \in \mathrm{No}$ be a positive surreal number and let $\{L_y|R_y\}$ be a representation of y such that $L_y, R_y > \{0\}$*[3]*. By induction over the length of y assume that the inverse has already been defined for L_y and R_y. We define the following sequences:*

$$\langle\rangle = 0,$$
$$\langle y_0, \dots y_n\rangle = x \textit{ for every } y_0, \dots, y_n \in L_x \cup R_x \setminus \{0\},$$

where x is the solution of the equation $(y -_{\mathrm{s}} y_n) \cdot_{\mathrm{s}} \langle y_0, \dots y_{n-1}\rangle + y_n \cdot_{\mathrm{s}} x = 1$. Note that a solution for this equation exists by inductive hypothesis. Now we define:

$$\frac{1}{y} = [L_{\frac{1}{y}}|R_{\frac{1}{y}}],$$

where:

$$L_{\frac{1}{y}} = \{\langle y_0, \dots, y_n\rangle \mid n \in \mathbb{N} \textit{ the number of } 0 \leq i \leq n \textit{ such that } y_i \in L_y \textit{ is even}\}$$

and

$$R_{\frac{1}{y}} = \{\langle y_0, \dots, y_n\rangle \mid n \in \mathbb{N} \textit{ the number of } 0 \leq i \leq n \textit{ such that } y_i \in L_y \textit{ is odd}\}.$$

[3]Note that by cofinality argument such a representation always exist

As for the previous definitions also the definition of product inverse is uniform.

A class field is a proper class C whose members satisfy the axioms of the theory of real closed fields[4] (i.e., every axiom of the theory of real closed fields instantiated with members of C can be proved). Given this definition, we can now mention an important result:

Theorem 2.3.18. *The surreal numbers* No *endowed with* $+_{\mathrm{s}}$ *and* $\cdot_{\mathrm{s}}$ *form a class field.*

Proof. See [14, Theorem 3.7]. □

Since in this thesis we will mostly be dealing with surreal operations, when no confusion arise we will drop the s from $\cdot_{\mathrm{s}}$ and $+_{\mathrm{s}}$.

2.3.3 Real Numbers and Ordinals

In this section we will show how to interpret real numbers and ordinal numbers within the class field of surreal numbers.

Before we show how real numbers are represented we consider the easier case of integers. We have already given some basic example showing how to represent 0, 1 and 2. Our intuition lead us to think that natural numbers are just finite sequences of pluses. Formally we have the following theorem:

Theorem 2.3.19 (Gonshor)**.** *For all* $n \in \mathbb{N}$, $(+)^n$ *is the positive integer* n *and* $(-)^n$ *is its inverse.*

The *dyadic rational numbers* are those rational numbers of the form $\frac{n}{2^m}$ with $n \in \mathbb{Z}$ and $m \in \mathbb{N}$. The surreal numbers of finite length can be identified with the ring of dyadic rational numbers as shown by the following theorem:

Theorem 2.3.20 (Gonshor)**.** *Surreal numbers of finite length corresponds to dyadic numbers. Let* $d \in$ No *be a surreal number of finite length such that* n *is the smallest such that* $\forall i, j < n.\ d(i) = d(j) \wedge d(n) \neq d(0)$. *Define a sequence of dyadic numbers* s *as follows:*

$$\begin{aligned} s(i) &= +1 \text{ iff } i < n \wedge i = +, \\ s(i) &= -1 \text{ iff } i < n \wedge i = -, \\ s(i) &= +\frac{1}{2^{i-n+1}} \text{ iff } n \geq i \wedge i = +, \\ s(i) &= -\frac{1}{2^{i-n+1}} \text{ iff } n \geq i \wedge i = -. \end{aligned}$$

Then $d = \sum_{i=0}^{|d|-1} s(i)$.

Proof. See [14, Theorem 4.2]. □

Intuitively the previous theorem says that a surreal number d of finite length can be interpret as follows: take the longest prefix p of d in which there is no change in sign. Then $d = s(0)|p| \sum_{i=0}^{|d|-1} s(|p|+i)\frac{1}{2^i}$. For example consider the sequence $d = ++--+$, then we have that $d = 1 + 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{8} = \frac{11}{8}$.

Now we are ready to characterize the real numbers.

Definition 2.3.21 (Real Numbers)**.** *A surreal number* r *is a real number iff either* r *has a finite length or* $|r| = \omega$ *and for all* $i < \omega$ *exists* $j < \omega$ *such that* $i < j$ *and* $r(i) \neq r(j)$.

The previous definition says that a surreal number is a real number only if it is a dyadic or if it is of length ω and not eventually constant.

Theorem 2.3.22 (Conway)**.** *The real numbers form a Dedekind complete ordered subfield of* No.

Proof. See [14, Theorem 4.3]. □

[4]Note that the theorem as it is can be formalized in axiomatizations of set theory which allow the use of classes (e.g., Von Neumann-Beranys-Gödel set theory)

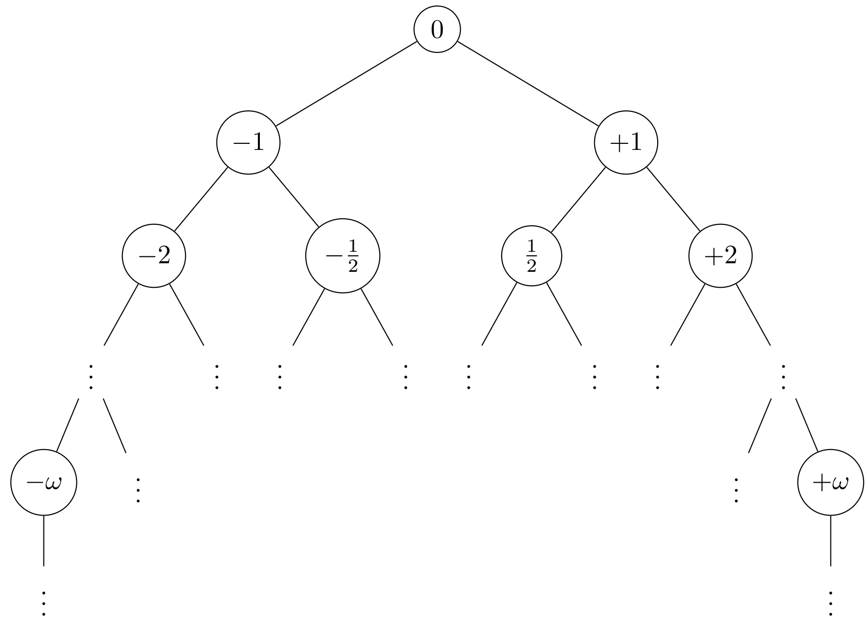


Figure 2.4: The surreal tree.

Finally will identify every ordinal α with the constant sequence of pluses of length α. We will denote such a sequence by $(+)^\alpha$.

Note that this fits completely with the definition of positive integers we have just given. Moreover the order is trivially preserved, namely $\alpha < \beta$ implies $(+)^\alpha < (+)^\beta$. Note that $\alpha + 1 = [\{\beta | \beta \leq \alpha\} | \emptyset]$ and if α is limit then $\alpha = [\{\beta | \beta < \alpha\} | \emptyset]$. Then from the order theoretic point of view we can identify ordinals and sequences of pluses.

Now if we look at the operations, the situation seems different. First of all we know that surreal operations are commutative while ordinal operations are not, for example $\omega +_{\mathrm{s}} 1 = 1 +_{\mathrm{s}} \omega$ while $\omega + 1 \neq \omega = 1 + \omega$. In his introduction to surreal numbers Gonshor proved that the surreal operations over ordinal numbers correspond to the *Hessenberg* (natural) operations.

2.3.4 Normal Form

In this section we will introduce a normal form for surreal numbers which is a generalization of Cantor's normal form for ordinal numbers.

Definition 2.3.23 (Archimedean Equivalence Relation)**.** *Given two positive surreal numbers x and y we define the following equivalence relation:*

$$x \sim_a y \text{ iff } \exists n \in \mathbb{Z} \,.\, n \cdot_{\mathrm{s}} y \geq x \ \wedge \ n \cdot_{\mathrm{s}} x \geq y.$$

The equivalence classes induced by this relation are called orders of magnitude.

One interesting fact of the orders of magnitudes is that they have canonical representatives.

Theorem 2.3.24 (Gonshor)**.** *Let x be a positive surreal number. Then there is a unique y of minimal length such that $x \sim_a y$.*

These canonical elements can be parametrized using surreal numbers and the ω-map. Intuitively, the ω-map is defined by letting ω^0 be the shortest canonical element, namely 1, ω^1 and ω^{-1} be respectively ω and $\frac{1}{\omega}$ and so on. Formally we have:

Definition 2.3.25 (ω-map)**.** *Let x be a surreal number. We define:*

$$\omega^x = [0, r \cdot_s \omega^{L_x} | s \cdot_s \omega^{R_x}],$$

where s and r are positive real numbers,

$$\omega^{L_x} = \{\omega^{x_L} \mid x_L \in L_x\}$$

and

$$\omega^{R_x} = \{\omega^{x_R} \mid x_R \in R_x\}.$$

The fact that the ω-map is represented as an exponentiation is because it behaves as one would expect from the exponentiation operator.

Theorem 2.3.26 (Gonshor)**.** *Let x,y be two surreal numbers. We have:*

- $\omega^x \cdot_s \omega^y = \omega^{x +_s y}$.
- *If x is an ordinal then ω^x is the same as the usual ordinal ω^x.*
- $\omega^x \cdot_s \omega^{-x} = 1$.

In order to define the normal form of surreal numbers we need transfinite sums. Let α be an ordinal, $(x_\gamma)_{\beta \in \alpha}$ be a strictly decreasing sequence of α surreal numbers and $(r_\beta)_{\beta \in \alpha}$ be a sequence of α non-zero real numbers. Then we define the sum $\sum_{\beta < \alpha} \omega^{x_\beta} \cdot_s r_\beta$ as follows:

$$\sum_{\beta < \gamma + 1} \omega^{x_\beta} \cdot_s r_\beta = \sum_{\beta \in \gamma} \omega^{x_\beta} \cdot_s r_\beta +_s \omega^{x_\gamma} \cdot_s r_\gamma \text{ if } \alpha = \gamma + 1,$$

$$\sum_{\beta < \alpha} \omega^{x_\beta} \cdot_s r_\beta = [L|R] \text{ if } \alpha \text{ limit},$$

where L and R are defined as follows:

$$L = \{\sum_{\gamma \leq \beta} \omega^{x_\gamma} \cdot_s s_\gamma : [\beta < \alpha] \wedge [\forall \gamma < \beta.\ s_\gamma = r_\gamma] \wedge$$
$$[s_\beta = r_b - t \text{ with } t \text{ any positive real number }]\},$$

$$R = \{\sum_{\gamma \leq \beta} \omega^{x_\gamma} \cdot_s s_\gamma : [\beta < \alpha] \wedge [\forall \gamma < \beta.\ s_\gamma = r_\gamma] \wedge$$
$$[s_\beta = r_b + t \text{ with } t \text{ any positive real number }]\}.$$

Theorem 2.3.27 (Conway, Normal Form Theorem)**.** *Every surreal number can be expressed uniquely in the form $\sum_{\beta \in \alpha} \omega^{x_\beta} \cdot_s r_\beta$.*

Proof. See [14, Theorem 5.6]. □

Note that the Cantor normal form is a special case of surreal numbers normal form.

2.4 Baire Space and Generalized Baire Space

In this section we will briefly recall some notion from basic descriptive set theory and generalized descriptive set theory. All the results in the first part of this section can be found in the first chapter of any introductory book of descriptive set theory such as [17].

Definition 2.4.1 (Baire Space)**.** *Let ω^ω be the set of sequences of natural numbers of length ω. For every finite sequence of natural numbers $w \in \omega^{<\omega}$ we define the following set:*

$$[w] = \{p \in \omega^\omega \mid w \subset p\},$$

namely $[w]$ is the set of infinite sequences that start with w. The set

$$B = \{[w] \mid w \in \omega^{<\omega}\}$$

is a base. We will call the set ω^ω equipped with the topology induced by B Baire space.

Note that ω^ω is by definition second countable.

Lemma 2.4.2 (Folklore)**.** *Baire space is Hausdorff.*

Proof. Let $p, p' \in \omega^\omega$ such that $p \neq p'$ and n be the smallest natural numbers such that $p(n) \neq p'(n)$. Now $[p{\upharpoonright}n]$ and $[p'{\upharpoonright}n]$ are open sets. By the fact that $p(n) \neq p'(n)$ we have that $[p{\upharpoonright}n] \cap [p'{\upharpoonright}n] = \emptyset$. Moreover, $p \in [p{\upharpoonright}n]$ and $p' \in [p'{\upharpoonright}n]$ as desired. □

Baire space is easily proved to be totally disconnected.

Lemma 2.4.3 (Folklore)**.** *Baire space is totally disconnected.*

Proof. We need to prove that for every $w \in \omega^{<\omega}$, the set $[w]$ is closed. Let W be defined as follows:

$$W = \{w' \in \omega^{<\omega} \mid \exists n \in \omega.\ w(n) \neq w'(n)\}.$$

Then $\omega^\omega \setminus [w] = \bigcup W$. Hence $\omega^\omega \setminus [w]$ is open and $[w]$ is closed as desired. □

Baire space is completely metrizable with the following metric[5]:

$$d(p, p') = \begin{cases} 0 & \text{if } p = p', \\ \frac{1}{n+1} & \text{if } n \text{ is the least such that } p(n) \neq p'(n). \end{cases}$$

One important property of Baire space is that it is homeomorphic to the product topology $\prod_{\alpha \in \omega} \omega$ where ω is endowed with the discrete topology.

Now we will recall some basic definitions and properties of generalized Baire spaces. In particular we will generalize the notions we have just seen to the cardinal κ we have fixed at the beginning of this chapter. All the notions that we will present in the rest of this section can be found in [13].

Definition 2.4.4 (Generalized Baire Space)**.** *Let κ^κ be the set of sequences of ordinals in κ of length κ. For every sequence $w \in \kappa^{<\kappa}$ of elements of κ of length less than κ, we define the following set:*

$$[w] = \{p \in \kappa^\kappa \mid w \subset p\}.$$

Then the set

$$B = \{[w] \mid w \in \kappa^{<\kappa}\}$$

is a base. We will call the set κ^κ equipped with the topology induced by B generalized Baire space*.*

Note that the assumption $\kappa^{<\kappa} = \kappa$, is necessary in order for generalized Baire space to have a base of cardinality κ and then a dense subset of cardinality κ. As we will see this will be crucial for generalize computable analysis. By using the same proofs of the classical case it is not hard to see that generalized Baire space κ^κ is Hausdorff and totally disconnected.

We want to end this section by mentioning to important differences between the Baire space ω^ω and its uncountable generalizations.

Theorem 2.4.5. *Generalized Baire space is not metrizable.*

Proof. First of all recall from topology that if a space X is metrizable, then for every $x \in X$, there is a countable set N_x of open sets such that for every open set U containing x there is $V \in N$ such that $V \subset U$. Assume that κ^κ is metrizable. Let p be an element of κ^κ. For every element of $U \in N_p$ take a basic open set $[w_U]$ which contains x and such that $[w_U] \subset O$. Since there are only countably many of these open sets and κ is a regular cardinal bigger than ω, there is $w \in \kappa^\kappa$ such that for every $U \in N_p$, we have $w_U \subset w$. But then $x \in [w]$ and for every $U \in N_p$ we have $U \not\subseteq [w]$. This contradicts our hypothesis, therefore κ^κ is not measurable. □

Hence there is no notion of a metric which induces κ^κ. In particular this means that all the notions that depend on the metrizability of κ^κ (e.g. the Borel hierarchy) have to be either reformulated in a non-metric way or cannot be generalized to κ^κ.

Finally, generalized Baire space is not homeomorphic to the product topology $\prod_{\alpha \in \kappa} \kappa$, where κ is endowed with the discrete topology (see [13]).

[5] Note that, even though this is not the standard definition of the metric over ω^ω, it is completely equivalent to the classical one from the topological point of view. As we will see in Chapter 3 this definition will generalize in a straightforward way to κ^κ by using $\mathbb{R}_\kappa$.

2.5 Computable Analysis

In this section we will present some basic notion from classical computable analysis and we will set up some conventions that we will use all over this thesis. A complete introduction to computable analysis can be found in [28] and a topological introduction to the more general theory of represented spaces can be found in [23]. Where it is possible, we will use the same notation as in [28].

2.5.1 Effective Topologies and Representations

The intuition behind computable analysis is that of generalizing computability theory to uncountable sets. In order to do this, the idea is that of study computational and topological properties of the Baire space ω^ω (or the Cantor space 2^ω) and then, transfer these results to any uncountable set by means of coding. These codings have a central role in computable analysis, therefore we recall their definition.

Definition 2.5.1 (Representation)**.** *Let M be a countable set. Then a notation over M is a surjective partial function from the set of finite sequences of natural numbers $\omega^{<\omega}$ to M. If M has cardinality $2^{\aleph_0}$ then a surjective partial function with domain the Baire space ω^ω and codomain M is called a representation of M. If δ is a representation over M, we will call (M, δ) a represented space.*

Definition 2.5.2 (Reductions)**.** *Let $\delta :\subseteq \omega^\omega \to M$ and $\delta' :\subseteq \omega^\omega \to M$ be two representations of M. Then we will say that δ* continuously reduces *to δ', in symbols $\delta \leq_{\mathrm{t}} \delta'$ iff there is a continuous function $h :\subseteq \omega^\omega \to \omega^\omega$ such that for every $x \in \mathrm{dom}(f)$, $\delta(x) = \delta'(h(x))$.*

If $\delta \leq_{\mathrm{t}} \delta'$ and $\delta' \leq_{\mathrm{t}} \delta$ we will say that δ and δ' are continuously equivalent and we will write $\delta \equiv_{\mathrm{t}} \delta'$.

Continuous reductions are a very useful tool, and as we will see they can be used to see how representations behave with respect to the topological properties of the space they represent.

Note that usually in computable analysis to any continuous notion correspond a computable notion, for example we could consider computable reductions instead of continuous reductions. The reason why we will only present the topological aspects of computable analysis is that it is still not clear how to define a notion of computability over κ^κ.

Effective topological spaces form a particularly well-behaved subclass of spaces. They induce naturally a standard representation which turns out to be a quotient map with respect to the topology of the space they represent.

Definition 2.5.3 (Effective Topological Space and Standard Representation)**.** *Let M be a set, σ be a countable family of subsets of M such that*

$$x = y \text{ iff } \{A \in \sigma \mid x \in A\} = \{A \in \sigma \mid y \in A\}$$

and $\nu :\subseteq \omega^{<\omega} \to \sigma$ be a naming on σ. Then $S = (M, \sigma, \nu)$ is an effective topological space. We will call τ_s the topology generated by taking σ as a subbase and $\delta_S :\subseteq \omega^\omega \to M$ the standard representation *of S defined as follows:*

$$\delta_S(p) = x \text{ iff } \{A \in \sigma \mid x \in A\} = \{\nu(w) \in \iota(w) \lhd p\} \qquad \forall p \in \omega^\omega.$$

Intuitively, given an effective topological space, we can think at σ as a list of properties that can distinguish elements of M and at ν as the way we can access them. From this point of view, $p \in \omega^\omega$ is a code for $x \in M$ according to the standard representation if and only if p codes the list of *all* the properties which characterize x.

As we have already said, effective topological spaces are a particularly well-behaved subclass of represented spaces. This is due to the fact that in this specific case the topological space τ_S and the final topology induced by δ_S are the same. This implies that some important properties of τ_S transfer to Baire space and vice versa. The following lemma shows how strong is the connection between an effective topological space and its induced topology.

Lemma 2.5.4. *Let $S = (M, \sigma, \nu)$ be an effective topological space, δ_S its standard representation and τ_S the induced topology. We have:*

- *τ_S is the final topology induced by δ_S.*

- δ_S *is continuous and open w.r.t* τ_S.

Proof. See [28, Lemma 3.2.5]. □

This lemma is important to establish a connection between Baire space and the topology τ_S. Let us illustrate how effective topological spaces work with an example.

Example 2.5.5. *Let us consider the set of real number* $\mathbb{R}$. *We already know that, in order to do analysis over* $\mathbb{R}$ *we will want to use the interval topology* $\tau_{\mathbb{R}}$ *over* $\mathbb{R}$. *Then it is natural look for a representation which induces this topology. We can use the well known fact that the set of open intervals with endpoints in* $\mathbb{Q}$ *is a base for* $\tau_{\mathbb{R}}$ *and the fact that* $\mathbb{Q}$ *is countable to define an effective topological space whose induced topology is* $\tau_{\mathbb{R}}$. *Let* $\nu_{\mathbb{Q}} :\subseteq \omega^{<\omega} \to \mathbb{Q}$ *be any notation over* $\mathbb{Q}$ *(it is not hard to explicitly define one). Moreover let* $\ulcorner \cdot, \cdot \urcorner : \omega^{<\omega} \times \omega^{<\omega} \to \omega^{<\omega}$ *be any pairing function. Then we can define a notation for the set of open intervals with rational endpoints* Cb *as follows:*

$$\mathrm{I}(\ulcorner i, j \urcorner) = B(\nu_{\mathbb{Q}}(i), \nu_{\mathbb{Q}}(j)),$$

where $B(q, q')$ *is the open ball with center* q *and radius* q'. *Define* $S = (\mathbb{R}, \mathrm{Cb}, \mathrm{I})$. *Now, since* Cb *is a base for the interval topology over* $\mathbb{R}$, *then* τ_S *is the interval topology over* $\mathbb{R}$. *Moreover, for what we have just shown,* δ_S *is continuous and open with respect to this topology.*

Lemma 2.5.6. *Let* M *be a set,* $\delta_0 :\subseteq \omega^\omega \to M$ *and* $\delta_1 :\subseteq \omega^\omega \to M$ *be two representations. Moreover, let* τ_0 *and* τ_1 *be respectively the final topology induced by* δ_0 *and* δ_1. *Then* $\delta_0 \leq_{\mathrm{t}} \delta_1$ *implies* $\tau_1 \subseteq \tau_0$. *Moreover, given* $\delta'_0 :\subseteq \omega^\omega \to M$ *and* $\delta'_1 :\subseteq \omega^\omega \to M$ *be other two representations of* M, *such that* $\delta'_0 \leq_{\mathrm{t}} \delta_0$ *and* $\delta'_1 \leq_{\mathrm{t}} \delta_1$. *Then every* (δ_0, δ_1)*-continuous function is* (δ_0, δ_1)*-continuous.*

Proof. Let $f :\subseteq \omega^\omega \to \omega^\omega$ be a continuous reduction of δ_0 to δ_1 and $O \in \tau_1$. By definition $\delta_1^{-1}(O)$ is open in $\mathrm{dom}(\delta_1)$. Moreover, since f is continuous, $f^{-1}\delta_1^{-1}(O)$ is open in $dom(f) \cap f^{-1}(dom(\delta_1))$. Then $f^{-1}\delta_1^{-1}(O)$ is open in $dom(f) \cap f^{-1}(dom(\delta_1)) \cap \mathrm{dom}(\delta_0)$. Now, since f is a reduction of δ_0 to δ_1, we have $dom(f) \cap f^{-1}(dom(\delta_1)) \cap \mathrm{dom}(\delta_0) = \mathrm{dom}(\delta_0)$ and $\delta_0^{-1}(O)$. Hence $O \in \tau_0$ as desired.

Now let f be a (δ'_0, δ'_1)-continuous function. Consider a continuous reduction h_0 of δ_0 to δ'_0 and a continuous reduction h_1 of δ_1 to δ'_1. Let F be a continuous realizer of f. Then $h_1 \circ F \circ h_0$ is a (δ_0, δ_1)-continuous realizer of f. □

Since continuous reductions preserve many topological properties we are interested in, it is natural to use them to characterize a well-behaved class of representations.

Definition 2.5.7 (Admissible Representation)**.** *Let* (M, τ) *be a topological space. Then a representation* $\delta :\subseteq \omega^\omega \to M$ *is* κ-admissible *w.r.t.* τ *iff* δ *is continuous and every continuous function* $\varphi :\subseteq \omega^\omega \to M$ *is continuously reducible to* δ.

Note that, as shown in [24], a representation δ of a topological space (M, τ) is admissible iff it is continuously equivalent to a standard representation of an effective topological space $S = (M, \sigma, \nu)$ with $\tau_S = \tau$.

In classical computability theory representations are used to transfer computability form the natural numbers to any countable space. The same approach is taken in computable analysis, where representations allow to transfer notions of continuity and computability from Baire space to any space of cardinality at most $2^{\aleph_0}$. Realizers have a central role in this construction.

Definition 2.5.8. *Let* $F :\subseteq M_1 \to M_0$ *be a function over two represented spaces* (M_1, δ_1) *and* (M_0, δ_0). *Then* $f :\subseteq \omega^\omega \to \omega^\omega$ *is a* realization *of* F *iff for every* $x \in \mathrm{dom}(\delta_1)$, $F(\delta_1(x)) = \delta_0(f(x))$. *If* f *is continuous we will say that* F *has a continuous realizer w.r.t.* δ_1 *and* δ_0 *or for short that* F *is* (δ_1, δ_0)*-continuous.*

Obviously it is important to define representations that induce notions of continuity and computability which are suitable for our purpose. For example, since we want to do computable analysis it makes little sense to have a representation that does not even make addition and product continuously representable. Therefore the following theorem is of main interest for computable analysis.

Theorem 2.5.9 (Main Theorem of Computable analysis)**.** *For $i = 0, \dots, n$ let $\delta_i :\subseteq \omega^\omega \to M_i$ be an admissible representation w.r.t. the topology τ_i. Then for any function $f :\subseteq M_1 \times \ldots \times M_n \to M_0$ we have:*

$$f \text{ is continuous } \Leftrightarrow f \text{ is } (\delta_1, \ldots, \delta_n, \delta_0)\text{-continuous.}$$

Proof. See [28, Lemma 3.2.11]. □

In particular the main theorem of computable analysis tells us that admissible representations respect continuous functions over the topological spaces they induce.

Example 2.5.10. *Let us continue our example on $\mathbb{R}$. Let $S = (\mathbb{R}, \mathrm{Cb}, \mathrm{I})$ be the effective topological space defined in the Example 2.5.5. We know that τ_S is the interval topology over $\mathbb{R}$. now, since is a well-known fact that $+$ and $\times$ are continuous over the interval topology, the main theorem of computable analysis tells us that $+$ and $\times$ are continuously represented over Baire space.*

The main theorem of computable analysis is important in computable analysis and in all those cases in which we already have a standard topology over the space we want to work with. In these cases, indeed, effective topological spaces and admissible representations give us a strict correspondence between continuous functions between represented spaces endowed with their intended topologies and the continuous functions over Baire space. This fact will turn out to be also important for representing the set of continuous functions between representable spaces.

Note that, in some cases a completely different approach is possible. In particular if we do not have a preferred candidate for the topology we want to use over the represented space we are working with, then we can just fix any representation and work with the final topology induced by this representation. In this case we would still have the main theorem of computable analysis w.r.t. the final topology and then a natural way to represent the space of continuous functions on our space.

2.5.2 Subspaces, Products and Continuous Functions

In this section we will briefly recall some constructions over representations.

First of all we consider subspaces of represented spaces. Note that, since restriction of surjective functions are still surjective, for every represented spaces (M, δ_M) and subset M_0 of M the restriction $\delta_M \restriction M_0$ of δ_M to M_0 is still a representation of M_0. Moreover, it turns out that the restriction of admissible representations is still admissible.

The second construction that we take into consideration is product. Before we can give the definition of product of representations we need to define some tupling functions. Fix a bijection $\ulcorner \cdot, \cdot \urcorner : \omega \times \omega \to \omega$. Then we define:

Definition 2.5.11 (Tupling Functions)**.** *Let $a_1, a_2 \ldots, a_i$ with $i < \omega$ be a sequence of element of ω. We define a* wrapping *function ι as follows:*

$$\iota(a_1, a_2, \ldots, a_i) = 110a_1 0a_2 0 \ldots 0a_i 011.$$

Moreover given $x_1, x_2, \ldots$ in $\omega^{<\omega}$ and $p_1, p_2 \ldots,$ in ω^ω, we define:

$$\begin{aligned}
&\ulcorner x_1, p_1 \urcorner = \ulcorner p_1, x_1 \urcorner = \iota(x_1) p_1 \in \omega^\omega, \\
&\ulcorner x_1, \ldots, x_i \urcorner = \iota(x_1) \ldots \iota(x_i) \text{ with } i < \omega, \\
&\ulcorner x_1, x_2 \ldots \urcorner = \iota(x_1)\iota(x_2) \ldots, \\
&\ulcorner p_1, \ldots, p_i \urcorner = p_1(0) \ldots p_i(0) p_1(1) \ldots p_i(1) \ldots \text{ with } i < \omega, \\
&\ulcorner p_1, p_2 \ldots \urcorner \ulcorner i, j \urcorner = p_i(j) \text{ for all } i, j \in \omega.
\end{aligned}$$

See [28] for further properties of these tupling functions.

Given these encodings, we can define products as follows:

Definition 2.5.12. *For every $i \in \omega$, let (M_i, δ_i) be a representation. Then we define the* product $\bigotimes_{i\in\omega} \delta_i$ *as follows:*

$$(\bigotimes_{i\in\omega} \delta_i)\ulcorner p_i \dots \urcorner_{i\in\omega} = (\delta_i(p_i))_{i\in\omega}.$$

For every $n \in \omega$, we define the product $\bigotimes_{i\in n} \delta_i$ as follows:

$$(\bigotimes_{i\in n} \delta_i)\ulcorner p_0, \dots, p_{n-1}\urcorner = (\delta_0(p_0), \dots, \delta_{n-1}(p_{n-1})).$$

Also in this case we have that the product of effective topological spaces is an effective topological space whose standard representation is the product representation and the induced topology is the product topology.

We will now consider the space of continuous functions between represented spaces. As we said the main theorem of computable analysis will turn out to be important in this context. Given two representable spaces (M_0, δ_0) and (M_1, δ_1) we want to represent the space of functions between M_1 and M_0 with a continuous realizer (note that this is the same as the space of continuous functions w.r.t the final topologies induced by δ_1 and δ_2). We will denote the set of continuous functions from M_1 to M_0 with $\mathrm{C}(M_1, M_0)$, sometimes the codomain is clear from the context in those cases we will write $\mathrm{C}(M_1)$.

Definition 2.5.13. *Let (M_0, δ_0) and (M_1, δ_1) be two represented spaces and $\mathrm{C}(\delta_1, \delta_0)$ be the space of (δ_1, δ_0)-continuous functions. Then we define a representation $[\delta_1 \to \delta_0]$ of $\mathrm{C}(\delta_1, \delta_0)$ as follows:*

$$[\delta_1 \to \delta_0](\langle n, p\rangle) = f \text{ iff } f \text{ is the function computed by the } n\text{-th Turing Machine with oracle } p,$$

for every $\langle n, p\rangle \in \omega^\omega$.

Definition 2.5.13 strongly depends on Turing machines and on the notion of computability over ω. As we will see, since we lack these notions for the generalized Baire space κ^κ, we will have to give a definition based on the topological properties rather than on computational notions.

2.5.3 The Weihrauch Hierarchy

As we said in the introduction,our main aim is that of study the complexity of theorems from classical analysis in the context of the generalized Baire space κ^κ. In the classical case, Weihrauch reductions are the main tools to compare and classify theorems. For an introduction to the theory of Weihrauch reductions see [5].

First, since we will be using multi-valued functions to represent theorems form analysis, we need to extend the definition of realizer:

Definition 2.5.14 (Multi-Valued Functions Realizers)**.** *Let $F :\subseteq M_1 \rightrightarrows M_0$ be a multi-valued function between the represented spaces (M_1, δ_{M_1}) and (M_0, δ_{M_0}). Then, $f :\subseteq \omega^\omega \to \omega^\omega$ is a realizer of F iff for every $x \in \mathrm{dom}(\mathrm{dom}(F \circ \delta_{M_1})$ we have*

$$\delta_{M_0}(f(x)) \in F(\delta_{M_1}(x)).$$

If F has a continuous realizer *we will say that it is $(\delta_{M_1}, \delta_{M_0})$-continuous.*

Weihrauch degrees can be used for classifying the complexity of functions over Baire space. This, together with the theory of representable spaces, makes them a natural tool for classifying functions between represented spaces.

Definition 2.5.15 (Weihrauch Reductions)**.** *Let $F :\subseteq M_1 \rightrightarrows M_0$ and $G :\subseteq N_1 \rightrightarrows N_0$ be two multi-valued functions between represented spaces. We will say that F is* Weihrauch reducible *to G, in symbols $F \leq_{\mathrm{w}} G$ iff there are two continuous functions $H :\subseteq \omega^\omega \to \omega^\omega$ and $K :\subseteq \omega^\omega \to \omega^\omega$ such that for every realizer $g :\subseteq \omega^\omega \to \omega^\omega$ of G there is a realizer $f :\subseteq \omega^\omega \to \omega^\omega$ of F such that*

$$f = H \circ \lceil \mathrm{ID}, g \circ K \rceil,$$

where ID : $\omega^\omega \to \omega^\omega$ *is the identity function. Moreover, if H and G are such that for every realizer $g :\subseteq \omega^\omega \to \omega^\omega$ of G there is a realizer $f :\subseteq \omega^\omega \to \omega^\omega$ of F such that*

$$f = H \circ g \circ K,$$

then we will say that F is strongly Weihrauch reducible *to G, in symbols $F \leq_{\mathrm{s,w}} G$.*

By using Weihrauch reductions one can study the complexity of functions between represented spaces. A particularly significant case of the use of Weihrauch reductions is that of computable analysis. Many theorems from analysis are in fact of the form:

$$\forall x \in X \exists y \in Y.P(x, y),$$

where P is quantifier free. In this case a theorem can be seen as a multi-valued function between X and Y which given an element of X returns an element of Y such that $P(x, y)$ holds. This fact makes Weihrauch reductions a natural tool for comparing theorem from real analysis. Let us illustrate this fact with an example:

Example 2.5.16. *Let us consider the Intermediate Value Theorem (IVT). It can be stated as follows:*

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(0) \cdot f(1) < 0$. Then there is $r \in [0, 1]$ such that $f(r) = 0$. Let $\mathrm{C}'[0, 1]$ be the set of continuous functions from $[0, 1]$ to $\mathbb{R}$. Since we have already seen that $\mathbb{R}$ is representable and $[0, 1] \subset \mathbb{R}$, the restriction of $\delta_{\mathbb{R}}$ is an admissible representation of $\mathrm{C}'[0, 1]$. By the Main Theorem of Computable Analysis we have that $\mathrm{C}'[0, 1]$ has a representation induced by the representation of continuous functions over Baire space. Then the set $\mathrm{C}[0, 1]$ of continuous functions such that $f(0) \cdot f(1) < 0$ is also represented. Then it is not hard to see that the IVT can formalized as follows:

$$IVT : \mathrm{C}[0, 1] \to [0, 1],\ IVT(f) = r \Leftrightarrow f(r) = 0.$$

This function has been extensively studied, then the interested reader is referred to [28], [5] and [7].

Chapter 3

Generalizing $\mathbb{R}$

Our aim in this chapter is that of defining an extension of $\mathbb{R}$ that we will call $\mathbb{R}_\kappa$. Instead of presenting directly the definition of $\mathbb{R}_\kappa$, we will have a quasi-axiomatic approach. In particular, we will first determine the properties we need on $\mathbb{R}_\kappa$ in order to prove some basic facts from classical analysis. Then we will show how it is possible to define $\mathbb{R}_\kappa$ as a subfield of the surreal numbers.

All over this chapter, we will put particular emphasis on the properties that have to hold over $\mathbb{R}_\kappa$ in order to prove the Intermediate Value Theorem (IVT) and the Extreme Value Theorem (EVT). We will end this chapter by showing how it is possible to use $\mathbb{R}_\kappa$ in order to extend classical results from descriptive set theory to κ^κ. In particular we will show that by using $\mathbb{R}_\kappa$ one can define a generalized version of the Borel hierarchy over κ^κ and we will show this hierarchy does not collapse.

3.1 Completeness and Connectedness of $\mathbb{R}_\kappa$

Let us consider some of the basic properties that we expect from $\mathbb{R}_\kappa$. First of all we want $\mathbb{R}_\kappa$ to be a generalization of $\mathbb{R}$ to the uncountable cardinal κ, therefore we require that $\mathbb{R}_\kappa$ is a proper ordered field extension of $\mathbb{R}$. As we said, we want to use $\mathbb{R}_\kappa$ to do analysis. For this reason, we expect $\mathbb{R}_\kappa$ to behave as much as possible like $\mathbb{R}$. Formally we will require that $\mathbb{R}_\kappa$ is a real closed field, in this way $\mathbb{R}_\kappa$ will have all the first order properties of $\mathbb{R}$.[1]

REQUIREMENT R1: $\mathbb{R}_\kappa$ has to be a real closed field extending $\mathbb{R}$.

Now, since we want to use $\mathbb{R}_\kappa$ to do computable analysis over sets of cardinality 2^κ, we require that $|\mathbb{R}_\kappa| = 2^\kappa$.

REQUIREMENT R2: $\mathbb{R}_\kappa$ has to have cardinality 2^κ.

Finally, since $\mathbb{Q}$ has a central role in the representation theory of $\mathbb{R}$ (the interested reader is referred to [28]), we want $\mathbb{R}_\kappa$ to have a dense subset which can play the same role as $\mathbb{Q}$. In particular we require that $\mathrm{w}(\mathbb{R}_\kappa) = \kappa$.

REQUIREMENT R3: $\mathbb{R}_\kappa$ has a dense subset of cardinality κ.

In general we define:

Definition 3.1.1 (κ-real extension of $\mathbb{R}$)**.** *Let K be a field satisfying* R1, R2, R3. *Then we will call K a κ-real extension of $\mathbb{R}$.*

From now on we will assume that $\mathbb{R}_\kappa$ is a κ-real extension of $\mathbb{R}$.

[1] In this chapter we will use grey boxes to highlight the requirements over $\mathbb{R}_\kappa$. These requirements will be assumed to be true over $\mathbb{R}_\kappa$. They will be discharged as assumptions while defining $\mathbb{R}_\kappa$.

Now we will focus on proving theorems from classical analysis over $\mathbb{R}_\kappa$. Many of these classical results depend on the order over $\mathbb{R}$ and on its interval topology. So we will start considering interval topologies over κ-real extensions of $\mathbb{R}$ and their properties.

Definition 3.1.2 (Interval Topology)**.** *Let K be a κ-real extension of $\mathbb{R}$. Then the* interval topology *over K is the topology generated by the base B of open intervals of $K \cup \{+\infty, -\infty\}$, where $+\infty$ and $-\infty$ are two new elements such that for all $r \in K$, $-\infty < r < +\infty$.*

First we recall few facts from field theory. The following property of the real numbers is crucial in analysis.

Definition 3.1.3 (Dedekind Completeness)**.** *An ordered set X is* Dedekind complete *if every bounded subset of X has a least upper bound in X.*

The following two theorems show that there is no Dedekind complete proper field extension of $\mathbb{R}$:

Theorem 3.1.4 (Folklore)**.** *Let K be an ordered field. If K is Dedekind complete then it is Archimedean.*

Proof. Let $r \in K$. We want to find $n \in \mathbb{N}$ such that $|r| < n$ (note that $n = \overbrace{1 + \ldots + 1}^{n \text{ times}}$). Assume by contradiction that for every $n \in \mathbb{N}$, $n \leq |r|$. Then r is an upper bound of $\mathbb{N}$ and by completeness $\sup \mathbb{N} \in K$. Now, we have that $n + 1 < \sup \mathbb{N}$ for all $n \in \mathbb{N}$(note that $\sup \mathbb{N}$ cannot be a natural number). Hence, $n < \sup \mathbb{N} - 1$ for all n and therefore $\sup \mathbb{N} - 1$ is an upper bound of $\mathbb{N}$ smaller than $\sup \mathbb{N}$. This is a contradiction therefore the field is Archimedean. □

Theorem 3.1.5 (Folklore)**.** *There are no Archimedean proper field extensions of $\mathbb{R}$.*

Proof. Let K be an Archimedean proper extension of $\mathbb{R}$. Assume $x \in K$. Since K is Archimedean, there is $n \in \mathbb{N}$ such that $|x| < n$. Consider the following set:

$$A = \{r \in \mathbb{R} \mid r < x\}.$$

Now $r = \sup A \in \mathbb{R}$ by Dedekind completeness of $\mathbb{R}$. Note that there is no rational number q such that $0 < q < |x - r|$. Indeed, assume such a q exists, we have

$$x > q + r$$

and

$$x < r - q.$$

We want to show that none of the previous inequalities has a solution. If there is a rational such that $x > q + r > r$, since $r = \sup A$ and $q + r \in A$ we would have a contradiction. Moreover, if there is a rational such that $x > q + b > b$, since $r = \sup A$ and $x > r - q$ therefore $r - q$ would be an upper bound of A and $r - q < r$ contradicting the fact that $r = \sup A$. Now, since there is no rational between 0 and $|x - r|$ we have that $n < \frac{1}{|x-r|}$ for every $n \in \mathbb{N}$ which contradicts the fact that K is Archimedean. □

By Theorem 3.1.4 and Theorem 3.1.5, since $\mathbb{R}_\kappa$ is a real closed extension of $\mathbb{R}$, it will not be Archimedean and therefore not Dedekind complete. More generally we have:

Corollary 3.1.6. *Let K be a κ-real extension of $\mathbb{R}$. Then K is not Dedekind complete.*

Another property which is central in mathematical analysis is connectedness. It turns out that connectedness and Dedekind completeness are equivalent properties. We have the following:

Theorem 3.1.7 (Folklore)**.** *Let K be an ordered field and τ be the interval topology over K. Then τ is connected iff the order over K is Dedekind complete.*

Proof. On the one hand, assume that K is not complete. Then there is a set U bounded in K such that $\sup U \notin K$. Let B be the set of upper bounds of U in K. Take $V = \bigcup_{r\in U}(-\infty, r)$ and $V' = \bigcup_{r\in B}(r, +\infty)$. Buy the definition, V and U partition K therefore τ is not connected.

On the other hand assume K Dedekind complete. Let V and V' two open sets in τ such that $V \cap V' = \emptyset$ and $V \cup V' = K$. Therefore $V = \bigcup_{i\in I}(a_i, b_i)$ and $V = \bigcup_{i\in I'}(a'_{i'}, b'_{i'})$ for some open intervals in K. Without loss of generality we can assume $b_i < a'_{i'}$ for all $i \in I$ and $i' \in I'$. Indeed, if this does not happen, take (a_i, b_i) and (a'_j, b'_j) such that $b_i < a'_j$ (note that they exist since $V \cap V' = \emptyset$) then define:

$$U = \bigcup\{(a, b) \subset V \mid b < a'_j\} \cup \{(a, b) \mid a < b \wedge b < b_i\}$$

and

$$U' = \bigcup\{(a, b) \subset V' \mid a > b_i\} \cup \{(a, b) \mid a < b \wedge a > a'_j\}.$$

Note that U and U' are still a partition of K with the desired properties.

Let $B = \{b_i \mid i \in I\}$. Then $\sup B \notin V$ since otherwise there would be a $i \in I$ such that $\sup B \in (a_i, b_i)$. Moreover $\sup B \notin V'$ since otherwise there would be an $i' \in I'$ such that $\sup B \in (a'_{i'}, b'_{i'})$, therefore $a'_{i'} < \sup B$ and by construction $a'_{i'}$ is an upper bound of B and this is a contradiction. □

By Corollary 3.1.6 and Theorem 3.1.4, it is easy to see that we will not be able to define $\mathbb{R}_\kappa$ in such a way that its interval topology is connected. Indeed we have:

Corollary 3.1.8. *Let K be a κ-real extension of $\mathbb{R}$. Then interval topology over K is not connected.*

As we said, our main purpose is that of proving basic facts from analysis over $\mathbb{R}_\kappa$. In particular we want to be able to prove the Intermediate Value Theorem (IVT) and the Extreme Value Theorem (EVT). It turns out that if the IVT holds on an ordered field K then K is connected.

Theorem 3.1.9 (Folklore)**.** *Let K be an ordered field. If IVT holds on the interval topology over K, then the interval topology over K is connected.*

Proof. Let U, V be two open sets such that $K = V \cup U$ and $V \cap U = \emptyset$. Moreover, let $f : K \longrightarrow K$ be a function such that $f[V] = -1$ and $f[U] = 1$. The function f is trivially continuous but there is no $r \in K$ such that $f(r) = 0$. Therefore f violates the IVT. □

In particular, Theorem 3.1.9 tells us that we cannot aim to prove the IVT over κ-real extensions of $\mathbb{R}$ in all its strength.

3.2 κ-Topologies

Given what we have proved in the previous section, it is quite hard to do analysis over κ-real extensions of $\mathbb{R}$ by using standard topological tools. To overcome this problem we will use a tool introduced by Alling called κ-topologies.

Definition 3.2.1 (κ-topology)**.** *A* κ-topology τ *over a set X is a collection of subsets of X such that:*

- $\emptyset, X \in \tau$.
- $\forall \alpha < \kappa$. *if $\{A_i\}_{i\in\alpha}$ is a collection of sets in τ, then $\bigcup_{i<\alpha} A_i \in \tau$.*
- $\forall A, B \in \tau.\ A \cap B \in \tau$.

The elements of τ are called κ-open *sets.*

Intuitively, the reason why we use κ-topologies is that, as we have seen in the previous section, interval topologies over κ-real extensions of $\mathbb{R}$ are too fine. As we will see κ-topologies will be coarser than topologies and will allow us to prove a weaker version of the Intermediate Value Theorem and of the Extreme Value Theorem over particularly well-behaved κ-real extensions of $\mathbb{R}$.

Theorem 3.2.2 (Alling)**.** *Let X be a set B be a topological base. Then the set τ_κ defined as follows:*

- $\emptyset, X \in \tau_\kappa$,
- *union of less than κ elements of B is in τ_κ,*

is a κ-topology. We will call τ_κ the κ-topology generated by B. Moreover we will call B a base for the κ-topology.

Proof. We have to prove that the three properties of κ-topologies hold for τ_κ:

Note that, $\emptyset, X \in \tau_\kappa$ by definition.

Now we have to show that every $\bigcup_{\alpha\in\beta}\bigcup_{\alpha'\in\beta'_\alpha} B_\alpha$ with $\beta < \kappa$ and $\forall\alpha < \beta$, $\beta'_\alpha < \kappa$ and $B_\alpha \in B$ is a union of less than κ elements of B. We have

$$|\bigcup_{\alpha<\beta}\{\alpha' \in \kappa \mid \alpha' < \beta_\kappa\}| \leq \sum_{\alpha<\beta}|\{\alpha' \in \kappa \mid \alpha' < \beta_\kappa\}|$$

and by regularity of κ

$$\sum_{\alpha<\beta}|\{\alpha' \in \kappa \mid \alpha' < \beta_\kappa\}| \leq \max\{|\beta|, \sup_{\alpha<\beta}\{|\beta_\alpha|\}\} < \kappa$$

Then $\bigcup_{\alpha\in\beta}\bigcup_{\alpha'\in\beta'_\alpha} B_{\alpha'}$ is a union of less than κ intervals as desired.

Finally, let $A, B \in \tau_\kappa$. Then, there are $\alpha, \beta < \kappa$ and two sequences $(A_\gamma)_{\gamma\in\alpha}$ and $(B_\gamma)_{\gamma\in\beta}$ of elements of B such that $A = \bigcup_{\gamma<\alpha} A_\gamma$ and $B = \bigcup_{\gamma<\beta} B_\gamma$. Then we have $A \cap B = \bigcup_{\gamma<\alpha} A_\gamma \cap \bigcup_{\gamma<\beta} B_\gamma$ and therefore $A \cap B = \bigcup_{(\gamma,\gamma')\in\alpha\times\beta}(A_\gamma \cap B_{\gamma'})$. Since for all $(\gamma, \gamma') \in \alpha \times \beta$ the set $A_\gamma \cap B_{\gamma'}$ is either $\emptyset$ or in B and $\alpha, \beta < \kappa$ (cardinals are closed under ordinal multiplication), $A \cap B \in \tau_\kappa$ as desired.

□

Obviously many topological definitions can be relativized to κ-topologies. In particular we have the following:

Definition 3.2.3 (κ-continuity)**.** *Let X and Y be two sets and τ, τ' be two κ-topologies respectively on X and on Y. Then $f : X \to Y$ is a* κ-continuous *function iff $\forall U \in \tau'$. $f^{-1}[U] \in \tau$.*

Note that being a κ-continuous function is stronger than being continuous in the topology generated by the same base.

Lemma 3.2.4. *Let $f : X \to Y$ be κ-continuous over the κ-topologies induced by the bases B_X and B_Y. Then f is continuous over the topologies induced by B_X and B_Y.*

Proof. Note that every κ-open is trivially open. Moreover by definition basic κ-open and basic open sets are the same. Therefore every basic open is sent by f^{-1} to an open set as desired. □

Lemma 3.2.5. *Let X, Y be two ordered set and $f : X \to Y$ be a strictly monotonic surjective function. Then f is κ-continuous.*

Proof. Let (a, b) be an open interval in Y. Since

$$f^{-1}[(a, b)] = \{x \in X \mid f(x) \in (a, b)\},$$

for all $x \in f^{-1}[(a, b)]$, we have $a < f(x) < b$. Now, since f is strictly monotonic and surjective it is a bijection. Hence $f^{-1}(a) < f^{-1}(f(x)) < f^{-1}(b)$ for all $x \in f^{-1}[(a, b)]$ and $f^{-1}[(a, b)] = (f^{-1}(a), f^{-1}(b))$. But then, since f maps open intervals in open intervals, it is κ-continuous. □

Definition 3.2.6 (κ-connectedness)**.** *Let X be a set and τ be a κ-topology over X. Then X is* κ-connected *iff $\forall U, V \in \tau$. $X = U \cup V \wedge U \cap V = \emptyset \Rightarrow U = \emptyset \vee V = \emptyset$*

Definition 3.2.7 (κ-compactness)**.** *Let X be a set and τ be a κ-topology over X. Then X is* κ-compact *iff every κ-open cover of X by less than κ sets has a finite subcover.*

All these definitions behave quite well. Indeed, one can prove many relativized version of basic results from topology (see [1]). However, there are theorems from classical topology that do not transfer to κ-topologies. Typically in κ-topologies local properties do no transfer to global properties (e.g. in κ-topologies openness is not implied by local openness).

Now we will introduce a κ-topological analogous of the interval topology over an ordered set.

Definition 3.2.8 (Interval κ-Topology)**.** *Let X be an ordered set and B be the set of open intervals with end points in $X \cup \{+\infty, -\infty\}$. We will call* interval κ-topology *over X the κ-topology generated by B.*

From now on we will consider the interval κ-topology as the standard κ-topology over κ-real extensions of $\mathbb{R}$.

As we have seen, in order to be able to prove some basic theorems from analysis we need to work within a connected space. However, as we have already pointed out, we can not aim for connectedness of κ-real extensions of $\mathbb{R}$. The next result is due to Alling [1] and it makes precise the connection between the density of an ordered set and the connectedness of its interval κ-topology.

Theorem 3.2.9 (Alling)**.** *Let X be an η_κ-set endowed with the interval κ-topology and X' a subset of X. Then X' is κ-connected iff X' is an interval in X.*

In view of Theorem 3.2.9, it is natural to require:

REQUIREMENT R4: $\mathbb{R}_\kappa$ has to be an η_κ-set.

Definition 3.2.10. *Let K be a κ-real extension of $\mathbb{R}$. Then K is* super dense *iff K is an η_κ-set.*

For super dense κ-real extensions of $\mathbb{R}$ we have:

Corollary 3.2.11. *Let K be a super dense κ-real extension of $\mathbb{R}$. Then the interval κ-topology over K is κ-connected.*

Theorem 3.2.12 (Alling)**.** *Let X be an η_κ-set and X' an interval in X. Then, if one of the following holds:*

- $\mathrm{Cof}(X') = 1$ *or* $\mathrm{Cof}(X') \geq \kappa$.
- $\mathrm{Coi}(X') = 1$ *or* $\mathrm{Coi}(X') \geq \kappa$.

The interval κ-topology over X' is κ-compact.

Then we have:

Corollary 3.2.13. *Let K be a super dense κ-real extension of $\mathbb{R}$ and $[r, r']$ be a closed interval of K. Then $[r, r']$ is κ-compact.*

As in classical topology we have that κ-continuous functions preserve κ-connectedness and κ-compactness.

Theorem 3.2.14. *Let $f : X \to Y$ be a κ-continuous function. If X is κ-connected then $f(X)$ is κ-connected.*

Proof. Assume $f(X)$ not κ-connected. Therefore there are U, V κ-open subsets of Y which partition $f(X)$. By the κ-continuity of f we have that $f^{-1}(U)$ and $f^{-1}(V)$ are κ-open subsets of X. Moreover, since $f(X) = U \cup V$ we have that $f^{-1}(U)$ and $f^{-1}(V)$ separates X, but this contradicts our hypothesis therefore $f(X)$ is κ-connected. □

Theorem 3.2.15. *Let $f : X \to Y$ be a κ-continuous function. If X is κ-compact then $f(X)$ is κ-compact.*

Proof. Take $\{A_\beta\}_{\beta<\alpha}$ with $\alpha < \kappa$ be a κ-open cover of $f(X)$. By κ-continuity $\{f^{-1}(A_\beta)\}_{\beta<\alpha}$ is a κ-open cover of X. Now since X is κ-compact there is a finite subcover A' of A, but then A' is the finite subcover of $f(X)$ we where looking for. □

Given a κ-topological space (X, τ) and a subset Y of X, the κ-topology τ induces a κ-topology over Y then, by analogy with classical topology we will call subspace κ-topology.

Lemma 3.2.16. *Let X be a set, τ be a κ-topology over X and Y be a subset of X. Then the set τ_s defined as follows:*

$$\tau_s = \{U \cap Y \mid U \in \tau\},$$

is a κ-topology. We will call τ_s the subspace κ-topology *of Y.*

Proof. We will prove that the properties of κ-topologies holds on τ_s:

Trivially $\emptyset \in \tau_s$. Moreover, since $X \in \tau$, we have that $Y \in \tau_s$.

Let $\{A_\beta\}_{\beta\in\alpha}$ with $\alpha < \kappa$ be a family of elements of τ_s. Then for all $\beta \in \alpha$ there is an κ-open set U_β in τ such that $A_\beta = U_\beta \cap Y$. Now, since $\bigcup_{\beta\in\alpha} A_\beta = \bigcup_{\beta\in\alpha}(U_\beta \cap Y) = (\bigcup_{\beta\in\alpha} U_\beta) \cap Y$ and $(\bigcup_{\beta\in\alpha} U_\beta)$ is κ-open in τ, we have that $\bigcup_{\beta\in\alpha} A_\beta$ is κ-open in τ_s.

Let $\{A_i\}_{i<n}$ with $n \in \mathbb{N}$ be a finite family of κ-open sets in τ_s. For every $i < n$ we have $A_i = U_i \cap Y$ with $U_i \in \tau$. Therefore we have $\bigcap_{i<n} A_i = (\bigcap_{i<n} U_i \cap Y) = (\bigcap_{i<n} U_i) \cap Y$. Now, since $\bigcap_{i<n} U_i$ is κ-open in τ, we have that $(\bigcap_{i<n} A_i)$ is κ-open in τ_s as desired. ☐

It turns out that it is not true that the subspace κ-topology of an ordered set coincide with the κ-interval topology in general. But if we consider convex subsets, then interval κ-topologies and subspace κ-topologies coincide.

Lemma 3.2.17. *Let X be an ordered set and $Y \subset X$ such that for all $y_1, y_2 \in Y$ if there is $y_1 < x < y_2$ in X then $x \in Y$. Then the interval κ-topology over Y coincides with the subspace κ-topology over Y.*

Proof. Let (a, b) be an open interval with end points in $Y \cup \{+\infty, -\infty\}$. Since Y is convex, we have that (a, b) is an open interval in the order κ-topology over X. Therefore $(a, b) \cap Y = (a, b)$ is κ-open in the subspace topology over Y. On the other hand let $U = (a, b) \cap Y$ with (a, b) a κ-open interval in X. We claim that $U = \{y \in Y \mid a < y < b\}$ is either the empty set or an interval in Y. Indeed, since Y is convex, if $y \notin Y$ then y is either a lower or an upper bound of Y. If $\{y \in Y \mid a < y < b\} = \emptyset$ then it is κ-open in the κ-subspace topology over Y. If $\{y \in Y \mid a < y < b\} \neq \emptyset$ then we have the following cases:

- $a, b \in \{y \in Y \mid a < y < b\}$: then U is an open interval in Y.
- $a \in \{y \in Y \mid a < y < b\}$ and $b \notin \{y \in Y \mid a < y < b\}$: then $(a, +\infty) = Y \cap (a, b)$ and $(a, +\infty)$ is κ-open in Y.
- $a \notin \{y \in Y \mid a < y < b\}$ and $b \in \{y \in Y \mid a < y < b\}$: then $(-\infty, b) = Y \cap (a, b)$ and $(a, +\infty)$ is κ-open in Y.
- $a, b \notin \{y \in Y \mid a < y < b\}$: then $(-\infty, +\infty) = Y \cap (a, b)$ and $(a, +\infty)$ is κ-open in Y.

Then $(a, b) \cap Y$ is κ-open in the κ-subspace topology over Y. ☐

3.3 Analysis Over Super Dense κ-real Extensions of $\mathbb{R}$

By the results from the previous section we can modify the standard topological proof of the IVT to show that its restriction to κ-continuous functions holds over super dense κ-real extensions of $\mathbb{R}$.

Theorem 3.3.1 (IVT$_\kappa$)**.** *Let K be a super dense κ-real extension of $\mathbb{R}$, the set $[a, b] \subset K$ be a closed subinterval of K and $f : [a, b] \to K$ be a κ-continuous function. Then for every $r \in K$ such that r is in between $f(a)$ and $f(b)$, there is $c \in [a, b]$ such that $f(c) = r$.*

Proof. We can assume $f(a) \neq r \neq f(b)$. Assume that there is no $c \in [a, b]$ such that $f(c) = r$. We define two sets:

$$A = f([a, b]) \cap (-\infty, r) \text{ and } B = f([a, b]) \cap (r, +\infty).$$

They are non empty and disjoint ($f(a) \in A$ and $f(b) \in B$). By definition they are also κ-open sets in $f([a, b])$. Moreover, since there is no c such that $f(c) = r$, we have that $f([a, b]) = A \cup B$. Hence A and B separates $f([a, b])$. Now by Theorem 3.2.9, $[a, b]$ is κ-connected and by Theorem 3.2.14, $f([a, b])$ is κ-connected. Therefore we have a contradiction since we have shown that A and B separate $f([a, b])$ and $f([a, b])$ is κ-connected. ☐

The next theorem we want to prove is the extreme value theorem over super dense κ-real extensions of $\mathbb{R}$. One of the main features of $\mathbb{R}$ on which the classical topological proof of the EVT depends on is the fact that if the image of a closed interval $[a, b]$ of $\mathbb{R}$ under a continuous function f has no maximum then

$$\{(-\infty, a') \mid a' \in f([a, b])\},$$

is a covering of $f([a, b])$. This fact is not true in general for κ-continuous functions and κ-open covers of size less than κ. In particular, if the cofinality of $f([a, b])$ is bigger than or equal to κ, then

$$\{(-\infty, a') \mid a' \in f([a, b])\},$$

would not be a κ-cover of $f([a, b])$. For this reason we restrict ourself to a subclass of functions which preserve these κ-open covers.

Definition 3.3.2 (κ-super continuity)**.** *Let $f : X \to Y$ be a κ-continuous function. Then f is a right* κ-super continuous function *iff for every close interval $[a, b]$ in X, we have that $\mathrm{Cof}(f([a, b])) < \kappa$. Moreover f is a left κ-super continuous function iff for every close interval $[a, b]$ in X, we have that $\mathrm{Coi}(f([a, b])) < \kappa$. We will say that f is κ-super continuous if it is both left and right κ-super continuous.*

Now we have all the notions we need to prove the EVT over super dense κ-real extensions of $\mathbb{R}$.

Theorem 3.3.3 (EVT_κ)**.** *Let K be a super dense κ-real extension of $\mathbb{R}$, the set $[a, b]$ be a closed interval of K and $f : [a, b] \to K$ be a right (left) κ-super continuous function. Then there is a point $c \in [a, b]$ such that $f(c)$ is maximum (minimum) in $f([a, b])$.*

Proof. Let $A = f([a, b])$. Assume that A has no maximum. Consider the following set:

$$\{(-\infty, a) \mid a \in A\}.$$

By right κ-super continuity of f we have $\mathrm{Cof}(A) < \kappa$, therefore there are $\alpha < \kappa$ and a sequence $\{a_i\}_{i<\alpha}$ in A which is cofinal in A. Hence, the set

$$\{(-\infty, a_i) \mid i < \alpha\}$$

is a κ-open cover of A with less than κ intervals. Since $[a, b]$ is a closed interval it is κ-compact and since f is κ-continuous A is κ-compact. Therefore there is a finite κ-open subcover A' of A. Take $M = \max A'$, we have that $M \in f([a, b])$ and M would be the maximum in $f([a, b])$. But this contradicts our assumption. ☐

We end this section with some interesting properties of κ-continuous functions over super dense κ-real extensions of $\mathbb{R}$.

It is a well-known fact that in every real closed field the IVT holds for polynomials in one variable (see [20, Theorem 3.3.9]), therefore it is natural to ask if polynomials over super dense κ-real extensions of $\mathbb{R}$ are κ-continuous. We have the following result:

Theorem 3.3.4. *Let K be a super dense κ-real extension of $\mathbb{R}$ and p be a polynomial in one variable with coefficients in K. Then p is κ-continuous.*

Proof. Let p be a polynomial in K and (a, b) be an interval with endpoints in $K \cup \{+\infty, -\infty\}$. Note that since constant functions are κ-continuous we can assume that p is not the zero polynomial. Since K is a real closed field, the polynomials $p(x) - a$ and $p(x) - b$ have finitely many (possibly 0) roots in K. Let $(r_i)_{i\in n}$ be the strictly increasing listing of these roots. Define the set I as follows:

- if $n = 0$:
$$I = \begin{cases} \{(-\infty, +\infty)\} & \text{If there is } x \in K \text{ s.t. } p(x) \in (a, b), \\ \{\emptyset\} & \text{otherwise.} \end{cases}.$$

- If $n > 0$: define I as follows:
  - $(r_i, r_{i+1}) \in I$ iff $p(\frac{r_{i+1}-r_i}{2}) \in (a, b)$.

– $(-\infty, r_0) \in I$ iff $p(r_0 - 1) \in (a, b)$.

– $(r_{n-1}, +\infty) \in I$ iff $p(r_0 + 1) \in (a, b)$.

Now we claim that $p^{-1}[(a, b)] = \bigcup I$. We will first prove that $p^{-1}[(a, b)] \subseteq \bigcup I$.

Let $x \in p^{-1}[(a, b)]$. If $n = 0$ then trivially $x \in (-\infty, +\infty) = \bigcup I$. Assume $n > 0$. We have the following cases:

Assume that there is $i < n$ such that $r_i < x < r_{i+1}$, we want to prove $p(\frac{r_{i+1}-r_i}{2}) \in (a, b)$. Assume not. Since, K is a real closed field, by [20, Theorem 3.3.9] , the IVT holds for polynomials. In particular, since $p(\frac{r_{i+1}-r_i}{2}) \notin (a, b)$ and $p(x) \in (a, b)$ either $p(x) - a$ or $p(x) - b$ has a root in between r_i and r_{i+1}. But this is in contradiction whit the fact that $(r_i)_{i\in n}$ was strictly increasing. Therefore $p(\frac{r_{i+1}-r_i}{2}) \in (a, b)$ and $(r_i, r_{i+1}) \subseteq \bigcup I$.

Assume that for every $i < n$, we have $x < r_i$. We want to prove $p(r_0 - 1) \in (a, b)$. Assume not. As before the IVT holds for polynomials in K. In particular, since $p(r_0 - 1) \notin (a, b)$ and $p(x) \in (a, b)$ either $p(x) - a$ or $p(x) - b$ has a root in between $-\infty$ and r_0. But this is in contradiction whit the fact that $(r_i)_{i\in n}$ was the strictly increasing listing of all the roots of $p(x) - a$ and $p(x) - b$. Therefore $p(r_0 - 1) \in (a, b)$ and $(-\infty, r_0) \subseteq \bigcup I$.

Finally if for every $i < n$ we have $x > r_i$, then the proof is similar to the previous case.

Note that the case in which $x = r_i$ for some r_i is impossible since $p(x) \in (a, b)$.

Now we will prove that $p^{-1}[(a, b)] \supseteq \bigcup I$. Let $x \in \bigcup I$. If $n = 0$, then there is $y \in K$ such that $p(y) \in (a, b)$. Now, since the IVT hold for polynomials, if $p(x) \notin (a, b)$ we would have that either $p(x) - a$ or $p(x) - b$ has a root. This contradicts the assumptions. Assume $n > 0$. We will only consider the case in which $x \in (r_i, r_{i+1})$ for some $i < n$ and $p(\frac{r_{i+1}-r_i}{2}) \in (a, b)$. The other cases can be proved similarly. We want to prove that $p(x) \in (a, b)$. Assume not. Since the IVT hold for polynomials, we would have that either $p(x) - a$ or $p(x) - b$ has a root in between (r_i, r_{i+1}). But we assumed $(r_i)_{i\in n}$ strictly increasing. Therefore $p(x) \in (a, b)$ as desired. Therefore $p^{-1}[(a, b)] = \bigcup I$.

Now since I is a finite list of intervals with end points in $K \cup \{-\infty, +\infty\}$ we have that $\bigcup I$ is κ-open. Hence $p^{-1}[(a, b)]$ is κ-open and p is κ-continuous as desired. □

So far, one could get the impression that everything straightforwardly generalizes from the standard topological case to the case of κ-topologies. The following negative result can serve as a warning that one cannot just assume that everything works nicely.

Definition 3.3.5 (κ-continuity at a point)**.** *Let $f : X \to Y$ be a function and $x \in X$. Then we say that f is* κ-continuous at x *iff for every κ-open set V containing $f(x)$ there is a κ-open set U containing x such that $f(U) \subseteq V$.*

Theorem 3.3.6. *Let $f : X \to Y$ be a κ-continuous function. Then it is κ-continuous at every point $x \in X$.*

Proof. Let f be a κ-continuous function, x be an element of X and V be a κ-open set containing $f(x)$. By the κ-continuity of f, we have that $f^{-1}(V)$ is κ-open. Therefore there are $\alpha < \kappa$ and open intervals $\{U_\beta\}_{\beta\in\alpha}$ in X such that $f^{-1}(V) = \bigcup_{\beta\in\alpha} U_\beta$. Since $x \in f^{-1}(V)$ there is $\beta \in \alpha$ such that $x \in U_\beta$ and, since $f^{-1}(V) = \bigcup_{\beta\in\alpha} U_\beta$ therefore $f(U_\beta) \subset V$ as desired. □

Before we prove that there are functions which are κ-continuous at every point but not κ-continuous, we want to consider an example of a non κ-continuous function:

Let K be a super dense κ-real extension of $\mathbb{R}$. First of all we note that $+ : K \times K \to K$ is not κ-continuous (therefore not κ-super continuous). Let (a, b) be an open interval of K (without loss of generality we can assume $a \neq -\infty$ and $b \neq +\infty$). Hence we have

$$+^{-1}[(a, b)] = \{\langle x, y\rangle \mid x + y \in (a, b)\},$$

therefore

$$+^{-1}[(a, b)] = \bigcup_{x\in K} \bigcup_{y\in(a-x,b-x)} \{\langle x, y\rangle\}.$$

We claim that $+^{-1}[(a,b)]$ is not κ-open. By contradiction assume that $+^{-1}[(a,b)]$ is κ-open. Let $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ be the projections over $K \times K$. Since (a,b) is κ-open, $\mathbf{\Pi}_2(+^{-1}[(a,b)])$ is κ-open. Hence $\mathbf{\Pi}_2(+^{-1}[(a,b)]) = \bigcup_{\beta<\alpha}(c_\beta, d_\beta)$ for some $\alpha < \kappa$. Note that

$$\forall \beta < \alpha.\ d_\beta \neq +\infty$$

and

$$\forall \beta < \alpha.\ c_\beta \neq -\infty,$$

otherwise $a = -\infty$ or $b = +\infty$. Since $\{c_\beta\}_{\beta<\alpha}$ and $\{d_\beta\}_{\beta<\alpha}$ have cardinality less than κ, they are bounded in K (this follows by the fact that K is an η_κ-set). Hence there is d such that

$$\forall \beta \in \alpha.\ d > d_\beta.$$

Take $y = d$ and $x = \frac{a+b-2d}{2}$. We have that $\langle x, y\rangle \in +^{-1}[(a,b)]$ (i.e., $x + y = \frac{a+b}{2} = a + \frac{b-a}{2}$) and $y \notin \bigcup_{\beta<\alpha}(c_\beta, d_\beta)$. Hence $+ : K \times K \to K$ is not κ-continuous.

Note that if we restrict the addition to a subset of K of cardinality strictly less than κ, we obtain a κ-continuous function. Let X be a subset of K of cardinality less than κ. Define the restricted addition as follows:

$$+_X = +\restriction X.$$

We claim that $+_X$ is κ-continuous. Let (a,b) be an interval of K. Since

$$+_X^{-1}[(a,b)] = \bigcup_{(a',b')\subset X} (a',b') \times (a-a', b-b'),$$

therefore $+_X^{-1}[(a,b)]$ is κ-open as desired.

Now we are ready to prove that there are functions that are κ-continuous at every point but not κ-continuous.

Theorem 3.3.7. *Let K be a super dense κ-real extension of $\mathbb{R}$. Then there exists a function $f : K \to K$ which is κ-continuous at every point but not κ-continuous.*

Proof. We have just shown that $+$ is not κ-continuous. We will prove that $+$ is κ-continuous at every point. Let $x, y \in K$. We will prove the claim only for $x, y > 0$ the other cases follows similarly. Without loss of generality we can assume $x < y$. Let V be a κ-open set containing $x+y$. We have that $V = \bigcup_{\beta\in\alpha}(a_\beta, b_\beta)$ for $\alpha < \kappa$ many open intervals in K with end points in $K \cup \{\infty, -\infty\}$. Let (a_β, b_β) such that $x + y \in (a_\beta, b_\beta)$. Without loss of generality we can assume that $a_\beta \neq -\infty$ and $b_\beta \neq \infty$. Take $\varepsilon_a = a_\beta - (x+y)$ and $\varepsilon_b = b_\beta - (x+y)$. Now we have that

$$A = (x - \frac{\varepsilon_a}{2}, x + \frac{\varepsilon_b}{2}) \times (y - \frac{\varepsilon_a}{2}, y + \frac{\varepsilon_b}{2})$$

is κ-open and trivially $\langle x, y\rangle \in A$. Moreover for all $\langle x', y'\rangle \in A$ we have

$$x' + y' < x + \frac{\varepsilon_b}{2} + y + \frac{\varepsilon_b}{2} = b_\beta$$

and

$$x' + y' > x - \frac{\varepsilon_a}{2} + y - \frac{\varepsilon_a}{2} = a_\beta,$$

therefore $x' + y' \in V$ as desired. ☐

3.4 The Real Closed Field $\mathbb{R}_\kappa$

We are now ready to define $\mathbb{R}_\kappa$. Before we present our construction, let us list all the requirements we have introduced in the previous sections.

- R1: $\mathbb{R}_\kappa$ has to be a real closed field extension of $\mathbb{R}$. Since every real closed field is elementary equivalent to $\mathbb{R}$, this requirement will guarantee that $\mathbb{R}_\kappa$ behaves similarly to $\mathbb{R}$ with respect to the order and to the operations.
- R2: $|\mathbb{R}_\kappa| = 2^\kappa$. This is due to the fact that our generalized theory of computable analysis would have to be a starting point for the study of the theory of represented spaces of cardinality bigger than $2^{\aleph_0}$.
- R3: $\mathrm{w}(\mathbb{R}_\kappa) = \kappa$. This is needed since when we will generalize the classical representations of $\mathbb{R}$ to $\mathbb{R}_\kappa$ we will need a set that can play the role that the rationals played in the classical case.
- R4: $\mathbb{R}_\kappa$ is an η_κ-set. As we saw in the previous sections, this is needed in order for IVT_κ and EVT_κ theorems to hold in $\mathbb{R}_\kappa$.

A naive attempt to define such extension would be that of starting from κ endowed with the surreal operations (i.e., the Hessenberg operations) and try to repeat the standard construction of $\mathbb{Z}^\kappa$ and $\mathbb{Q}^\kappa$. Then, we could define $\mathbb{R}^\kappa$ as the Cauchy completion of $\mathbb{Q}^\kappa$ obtaining a Cauchy closed field $\mathbb{R}^\kappa$ as we showed in Section 2.2. Unfortunately this approach does not work. This is due to the following theorem:

Theorem 3.4.1 (Sikorski)**.** *The field $\mathbb{Q}^\kappa$ is Cauchy closed. In particular,*

$$\mathbb{R}^\kappa = \mathbb{Q}^\kappa .$$

Recall that $\mathbb{Q}^\kappa$ is a set of equivalence classes of pairs of elements in $\mathbb{Z}^\kappa$, hence it has cardinality at most κ. Therefore $\mathbb{R}^\kappa$ violates R2 and is not a good candidate for our purposes. This construction appeared for the first time in a paper from Sikorski in 1948 [26] (see also [18] for a complete study of this approach). During the work of this thesis, the Sikorski's construction has been re-discovered independently by Aspero and Tsapronis and by myself.

Recall from Section 2.3 that we have the following property of surreal numbers:

Theorem 3.4.2 (Alling)**.** *Let κ' be a regular cardinal. Then $\mathrm{No}_{<\kappa'}$ is a real closed field.*

In particular we have that $\mathrm{No}_{<\kappa}$ is a real closed field. Moreover since $\kappa > \omega$, we have that $\mathbb{R} \subset \mathrm{No}_{<\kappa}$. This means that R1 holds for $\mathrm{No}_{<\kappa}$. It is not hard to prove that $\mathrm{No}_{<\kappa}$ also satisfies R4. Indeed we have a more general result:

Proposition 3.4.3 (Folklore)**.** *Let κ' be a cardinal such that $\mathrm{Cof}(\kappa') = \alpha$. Then $\mathrm{No}_{<\kappa'}$ is a η_α-set.*

Proof. Assume $L, R \in \mathrm{No}_{<\kappa'}$ such that $|L| + |R| < \kappa'$. Then for every $x \in L$ and $y \in R$ we have $|x|, |y| < \kappa'$. But since $\mathrm{Cof}(\kappa') = \alpha$, we have that $\ell(L) = \sup\{|x| \mid x \in L\}$ and $\ell(R) = \sup\{|x| \mid x \in R\}$ are both smaller than κ'. But then Theorem 2.3.7 we have $|[L|R]| \leq \max\{l(L), l(R)\} < \kappa'$ which imply $[L|R] \in \mathrm{No}_{<\kappa'}$ as desired. □

Then we have:

Proposition 3.4.4. *The field $\mathrm{No}_{<\kappa}$ has the following properties:*

1) $|\mathrm{No}_{<\kappa}| = \kappa$.

2) $\mathrm{Cof}(\mathrm{No}_{<\kappa}) = \mathrm{Coi}(\mathrm{No}_{<\kappa}) = w(\mathrm{No}_{<\kappa}) = \kappa$.

3) $\mathrm{Coi}(\mathrm{No}^+_{<\kappa}) = \kappa$.

Proof. 1) Since we assumed $\kappa = \kappa^{<\kappa}$, the statement follows from the fact that $\mathrm{No}_{<\kappa}$ is the set of sequences of pluses and minuses of length less than κ.

2) Note that $\kappa \subset \mathrm{No}_{<\kappa}$ is a cofinal subset of $\mathrm{No}_{<\kappa}$ and $-\kappa$ is a coinitial subset of $\mathrm{No}_{<\kappa}$. By regularity of κ, $\mathrm{Cof}(\mathrm{No}_{<\kappa}) = \mathrm{Coi}(\mathrm{No}_{<\kappa}) = \kappa$. Moreover, note that every dense subset of $\mathrm{No}_{<\kappa}$ has to be cofinal in $\mathrm{No}_{<\kappa}$, therefore $\mathrm{w}(\mathrm{No}_{<\kappa}) \geq \kappa$ and since $|\mathrm{No}_{<\kappa}| = \kappa$, we have $\mathrm{w}(\mathrm{No}_{<\kappa}) = \kappa$.

3) We know that $\mathrm{No}_{<\kappa}$ is a real closed field. Consider the following sequence $S = \{\frac{1}{\alpha}\}_{\alpha\in\kappa}$. The sequence S is coinitial in $\mathrm{No}^+_{<\kappa}$. Indeed, take $x \in \mathrm{No}^+_{<\kappa}$. We can assume $x < 1$, therefore $x = \frac{1}{y}$ with $y \in \mathrm{No}^+_{<\kappa}$. Take $\alpha < \kappa$ such that $\alpha > y$ (note that α exists since κ is cofinal in $\mathrm{No}_{<\kappa}$), then $x > \frac{1}{\alpha} > 0$ and S is coinitial

in $\mathrm{No}^+_{<\kappa}$. Now, since $|S| = \kappa$, therefore $\mathrm{Coi}(\mathrm{No}^+_{<\kappa}) \leq \kappa$. Moreover, note that any subsequence S' of $\mathrm{No}^+_{<\kappa}$ of cardinality less than κ cannot be coinitial in $\mathrm{No}^+_{<\kappa}$. Indeed, let S' be such a sequence. Since $\mathrm{No}_{<\kappa}$ is an η_κ-set, if we take $L = \{0\}$ and $R = S'$, there is $x \in \mathrm{No}_{<\kappa}$ such that $L < \{x\} < R$. Trivially $x \in \mathrm{No}^+_{<\kappa}$ and $\{x\} < S'$. Hence S' is not coinitial in $\mathrm{No}^+_{<\kappa}$ as desired. In conclusion $\mathrm{Coi}(\mathrm{No}^+_{<\kappa}) = \kappa$. □

Proposition 3.4.4 tells us that $\mathrm{No}_{<\kappa}$ has almost all the properties that we want from $\mathbb{R}_\kappa$ but is still too small.

A possible approach to this problem would be that of repeating the construction of $\mathbb{R}$ over the surreal numbers starting from $\mathrm{No}_{<\kappa}$ instead that on $\mathrm{No}_{<\omega}$. In particular one can define $\mathbb{R}^{\mathrm{s}}_\kappa$ as follows:

$$\mathbb{R}^{\mathrm{s}}_\kappa = \mathrm{No}_{<\kappa} \cup D,$$

where D is the following set:

$$D = \{p \in \mathrm{No}_\kappa \mid \forall \alpha < \kappa \exists \kappa > \beta > \alpha.\ p(\beta) \neq p(\alpha)\}.$$

The problem with this construction is that, in the classical proof, in order to prove that the surreal operations behave correctly with respect to non eventually constant sequences of pluses and minuses, the Dedekind completeness of $\mathbb{R}$ is fundamental (see [14, Chapter 4 Section C]). Since we know that $\mathbb{R}_\kappa$ can not be Dedekind complete, a generalization of this proof can not work. Moreover, an alternative proof would require a characterization of the operations over the surreal numbers in terms of sequences of pluses and minuses[2]. Such a characterization is still missing in the theory of surreal numbers. This is partially due to the fact that the semantics of pluses and minuses in the the surreal numbers is not purely positional (see Section 2.3).

We can then define $\mathbb{R}_\kappa$ as the Cauchy completion of $\mathrm{No}_<$. In Section 2.2 we showed a standard way of completing a real closed field. In this section we will take a different prospective. Since we are working within No, we will not need to build any new number.

Recall from Section 2.2 that a cut over a group is a pair $\langle L, R\rangle$ of subsets of the group such that $L < R$. In view of the Simplicity theorem (Theorem 2.3.5), we know that every such cut identifies a unique surreal number. Moreover, we have seen in Section 2.2 that the Cauchy completion of a field can be defined in terms of closure under Cauchy cuts. For these reasons, we can define the Cauchy completion of $\mathrm{No}_{<\kappa}$ within the surreal numbers in a very natural way as follows:

Definition 3.4.5 ($\widetilde{\mathrm{No}}_{<\kappa}$)**.** *We define $\widetilde{\mathrm{No}}_{<\kappa}$ as follows:*

$$\widetilde{\mathrm{No}}_{<\kappa} = \mathrm{No}_{<\kappa} \cup \{x \mid x = [L|R] \text{ where } \langle L, R\rangle \text{ is a Cauchy cut in } \mathrm{No}_{<\kappa}\}.$$

Now we will show that $\widetilde{\mathrm{No}}_{<\kappa}$ is a super dense κ-real extension of $\mathbb{R}$. First of all we will prove that $\mathrm{No}_{<\kappa}$ is a dense subfield of $\widetilde{\mathrm{No}}_{<\kappa}$ and that $\widetilde{\mathrm{No}}_{<\kappa}$ is Cauchy complete (i.e., it is a Cauchy completion of $\mathrm{No}_{<\kappa}$).

Lemma 3.4.6. *The field $\mathrm{No}_{<\kappa}$ is dense in $\widetilde{\mathrm{No}}_{<\kappa}$.*

Proof. Let $x, y \in \widetilde{\mathrm{No}}_{<\kappa}$ be such that $x < y$. We can assume that at least one between x and y is not in $\mathrm{No}_{<\kappa}$, otherwise the statement follows trivially by the density of $\mathrm{No}_{<\kappa}$. Without loss of generality assume y is not in $\mathrm{No}_{<\kappa}$. Let $[L_x|R_x]$ be the standard representation of x and $[L_y|R_y]$ be a representation of y such that $\langle L_y, R_y\rangle$ is Cauchy. Since $x < y$, by [14, Theorem 2.5] we have $\{x\} < R_y$ and $\{y\} > L_x$. Moreover, since $x \neq y$, by [14, Theorem 2.6] we have that either there exists $x_R \in R_x$ such that $y \geq x_R$ or exists $y_L \in L_x$ such that $y_L \geq x$.

Assume that there is $x_R \in R_x$ such that $y \geq x_R$. Since $y_L \notin \mathrm{No}_{<\kappa}$ and $y \neq x_R$, we have $y > x_R > x$ as desired. On the other hand if there exists $y_L \in L_x$ such that $y_L \geq x$, then by the fact that L_y has no maximum we can take $y > y'_L > y_L \geq x$. Therefore y'_L is the desired element of $\mathrm{No}_{<\kappa}$. □

Lemma 3.4.7. *The set $\widetilde{\mathrm{No}}_{<\kappa}$ is Cauchy closed.*

[2]Note that all the operations over surreal numbers are defined in terms of their representations. It is not clear how and if these operations can be defined directly over surreal numbers.

Proof. Let $\langle L, R\rangle$ be Cauchy. We claim that there are two sequences $(\ell_\gamma)_{\gamma\in\alpha}$ and $(r_\gamma)_{\gamma\in\beta}$ of elements of $\mathrm{No}_{<\kappa}$ with $\alpha, \beta \leq \kappa$, which are respectively mutually cofinal in L and mutually coinitial in R. Moreover, since L has no minimum and R has no maximum, we can choose $(\ell_\gamma)_{\gamma\in\alpha}$ and $(r_\gamma)_{\gamma\in\beta}$ such that $\ell_\gamma < r_{\gamma'}$ for all $\gamma \in \alpha$ and $\gamma' \in \beta$. Let $\ell \in L$ and $r \in R$ be two elements respectively of L and R. Since L has no maximum and R has no minimum there exist $\ell' \in L$ and $r' \in R$ such that $\ell < \ell'$ and $r' < r$. By the density of $\mathrm{No}_{<\kappa}$ in $\widetilde{\mathrm{No}}_{<\kappa}$ there are $\ell_0, r_0 \in \mathrm{No}_{<\kappa}$ such that $\ell < \ell_0 < \ell' < r' < r_0 < r$. Now let $0 < \gamma < \kappa$ and assume we have already defined $\ell_{\gamma'}$, for every $\gamma' < \gamma$. We will define ℓ_γ, the same argument works for r_γ. We have two cases:

- if there exists $\ell \in L$ such that $\ell_{\gamma'} < \ell$ for all $\gamma' < \gamma$, then take $\ell' \in L$ such that $\ell < \ell'$ and $\ell_\gamma \in \mathrm{No}_{<\kappa}$ such that $\ell < \ell_\gamma < \ell'$.
- If for all $\ell \in L$ there is $\gamma' < \gamma$ such that $\ell_{\gamma'} < \ell$, then stop.

Now let α be the smallest ordinal on which the previous definition stops. Note that trivially $\alpha \leq |L| \leq \kappa$. It is an easy induction to prove that for every $\gamma < \alpha$ there are $\ell, \ell' \in L$ such that $\ell < \ell_\gamma < \ell'$ and that for every $\ell \in L$ there is $\gamma \in \alpha$ such that $\ell \leq \ell_\gamma$. Therefore $(\ell_{\gamma'})_{\gamma'<\alpha}$ is mutually cofinal with L as desired.

Now by [14, Theorem 2.3] we have $[\bigcup_{\gamma\in\alpha}\{\ell_\gamma\} \mid \bigcup_{\gamma\in\beta}\{r_\gamma\}] = [L|R]$. Moreover, since $\langle L, R\rangle$ is Cauchy, also $\langle\bigcup_{\gamma\in\alpha}\{\ell_\gamma\}, \bigcup_{\gamma\in\beta}\{r_\gamma\}\rangle$ is Cauchy in $\mathrm{No}_{<\kappa}$. Finally, since $[L|R] \in \widetilde{\mathrm{No}}_{<\kappa}$, we have that $[\bigcup_{\gamma\in\alpha}\{\ell_\gamma\} \mid \bigcup_{\gamma\in\beta}\{r_\gamma\}]$ is in $\widetilde{\mathrm{No}}_{<\kappa}$ as desired. ☐

Now we want to prove that our completion of $\mathrm{No}_{<\kappa}$ is an ordered field. Before we can do that we need the following lemma:

Lemma 3.4.8. *Let $y \in \widetilde{\mathrm{No}}^+_{<\kappa}$ and $[L|R]$ be a representation of y such that $\langle L, R\rangle$ is a Cauchy cut. We have:*

$$\exists\varepsilon_0 \in \widetilde{\mathrm{No}}^+_{<\kappa}\exists c \in L\setminus\{0\}\forall\varepsilon \in \widetilde{\mathrm{No}}^+_{<\kappa}.$$
$$(\varepsilon < \varepsilon_0) \to \exists y_L \in L\exists y_R \in R.\ (y_L + \varepsilon > y_R \wedge y_L > c).$$

Proof. By contradiction assume:

$$\forall\varepsilon_0 \in \widetilde{\mathrm{No}}^+_{<\kappa}\forall c \in L\setminus\{0\}\exists\varepsilon \in \widetilde{\mathrm{No}}^+_{<\kappa}.$$
$$(\varepsilon < \varepsilon_0 \wedge \forall y_L \in L\forall y_R \in R.\ y_L + \varepsilon > y_R) \to y_L \leq c.$$

Choose $\varepsilon_0 > 0$ such that the set $X = \{x \in L \mid x < y - \varepsilon_0\} \neq \emptyset$. Note that ε_0 exists since L has no maximum. Let $c \in X$. Since No is a real closed field we have $c + \varepsilon_0 < y$. Now by density of $\mathrm{No}_{<\kappa}$ we can choose $0 < \varepsilon < \varepsilon_0$. Since $\langle L, R\rangle$ is a Cauchy cut, there are $y_L \in L$ and $y_R \in R$ such that $y_L + \varepsilon > y_R$. Hence $y_L \leq c$ and $\varepsilon < \varepsilon_0$ imply $y_L + \varepsilon < c + \varepsilon < y < y_R$ which is a contradiction. ☐

Theorem 3.4.9. *The set $\widetilde{\mathrm{No}}_{<\kappa}$ is a field.*

Proof. Note that by density of $\mathrm{No}_{<\kappa}$ in $\widetilde{\mathrm{No}}_{<\kappa}$ every time we need to prove a statement of the form

$$\forall\varepsilon \in \widetilde{\mathrm{No}}^+_{<\kappa}\exists y \in Y.\ y < \varepsilon,$$

we can prove

$$\forall\varepsilon \in \mathrm{No}^+_{<\kappa}\,\exists y \in Y.\ y < \varepsilon$$

instead.

We only need to show the closure properties since we already know from the theory of surreal numbers that the operations respect the ordered field axioms.

+: let $x, y \in \widetilde{\mathrm{No}}_{<\kappa}$. Note that since $\mathrm{No}_{<\kappa}$ is a real closed field we can assume that at least one of the two elements is not in $\mathrm{No}_{<\kappa}$. Without loss of generality we assume $y \notin \mathrm{No}_{<\kappa}$ (the proof for $x \notin \mathrm{No}_{<\kappa}$ is analogous). We want to prove that $x + y$ is represented by a Cauchy cut. Let us recall the definition of + over surreal numbers:

$$x + y = [L_x + y, x + L_y|R_x + y, x + R_y].$$

Recall that by uniformity of plus, we can use any representation for y, in particular we can assume that we used a representation of y induced by a Cauchy cut. Let $\varepsilon \in \mathrm{No}^{+}_{<\kappa}$. Since y is represented by a Cauchy cut $[L_y|R_y]$, there exist $y_L \in L_y$ and $y_R \in R_y$ such that $y_L + \varepsilon > y_R$. Hence, since plus over surreal respects the axioms of real closed field theory, we have $x + y_L + \varepsilon > x + y_R$. Now $x + y_L$ is on the left side of $x + y$ and $x + y_R$ is on the right side, therefore $x + y$ is represented by a Cauchy cut as desired.

$\cdot$: As before we will assume $x, y \in \widetilde{\mathrm{No}}_{<\kappa}$ and $y \notin \mathrm{No}_{<\kappa}$. First we recall the definition of $x \cdot y$:

$$x \cdot y = [L_x \cdot y + x \cdot L_y - L_x \cdot L_y, R_x \cdot y + x \cdot R_y - R_x \cdot R_y \\ |L_x \cdot y + x \cdot R_y - L_x \cdot R_y, R_x \cdot y + x \cdot L_y - R_x \cdot L_y].$$

As before by uniformity of $\cdot$, we can assume that we used a representation of y induced by a Cauchy cut. Given $\varepsilon \in \mathrm{No}^{+}_{<\kappa}$, we will prove that there exist x_L, y_L and y_R such that

$$x_L \cdot y + x \cdot y_L - x_L \cdot y_L + \varepsilon > x_L \cdot y + x \cdot y_R - x_L \cdot y_R.$$

Note that it is enough to prove $x \cdot y_L - x_L \cdot y_L + \varepsilon > x \cdot y_R - x_L \cdot y_R$. Let x_L be any element on the left in the representation of x. Since $x > x'$ we have $x - x_L > 0$, we can take $\varepsilon' = \frac{\varepsilon}{x - x_L}$. Moreover $\varepsilon > 0$ implies $\varepsilon' > 0$. Now y was represented by a Cauchy gap therefore there exist y_L and y_R, respectively left and right elements of a representation, such that $y_L + \varepsilon' > y_R$. We have

$$(x - x_L) \cdot (y_L + \varepsilon') > (x - x_L) \cdot y_R$$

and

$$x \cdot y_L - x_L \cdot y_L + \varepsilon > x \cdot y_R - x_L \cdot y_R$$

as desired.

$-$: let $y \in \widetilde{\mathrm{No}}_{<\kappa}$ and $y \notin \mathrm{No}_{<\kappa}$. Recall that if $[L|R]$ is a representation of y, therefore $[-R| - L]$ is a representation for $-y$. As before we can take a Cauchy representation of y. Let $\varepsilon \in \mathrm{No}^{+}_{<\kappa}$. Hence, there exist y_L and y_R such that $y_L + \varepsilon > y_R$. Therefore $y_L > y_R - \varepsilon$ and $-y_R + \varepsilon > -y_L$ as desired.

$\frac{1}{y}$: let $y \in \widetilde{\mathrm{No}}_{<\kappa}$ and $y \notin \mathrm{No}_{<\kappa}$. By the definition of $\frac{1}{y}$ it is not hard to see that $\frac{1}{y_R}$ and $\frac{1}{y_L}$ are respectively left and right members of the representation of $\frac{1}{y}$. Now by Lemma 3.4.8 there exist $\varepsilon_0 \in \mathrm{No}^{+}_{<\kappa}$ and $c > 0$ such that for every $\varepsilon \in \mathrm{No}^{+}_{<\kappa}$ such that $\varepsilon < \varepsilon_0$ there are y_L and y_R such that $y_L + \varepsilon > y_R$ and $y_L > c$. Therefore we have the following:

$$\frac{1}{y_L} - \frac{1}{y_R} < \varepsilon \Leftrightarrow \frac{y_R - y_L}{y_L \cdot y_R} < \varepsilon.$$

Now, by the fact that y has a Cauchy representation, we can choose $y_R - y_L$ arbitrary small. Moreover, since $\frac{y_R - y_L}{y_L \cdot y_R} < \frac{y_R - y_L}{c^2}$, it is enough to take $\varepsilon' = \varepsilon \cdot c^2$ and $y_L + \varepsilon' > y_R$ to get

$$\frac{1}{y_R} - \frac{1}{y_L} < \varepsilon$$

as desired. □

Now we want to prove that $\widetilde{\mathrm{No}}_{<\kappa}$ is a real closed field. Before we can do that we need to prove that polynomial over $\widetilde{\mathrm{No}}_{<\kappa}$ are continuous with respect to the interval topology over $\widetilde{\mathrm{No}}_{<\kappa}$. Note that the following lemma is true in general for ordered fields:

Lemma 3.4.10. *Let $f : \widetilde{\mathrm{No}}_{<\kappa} \to \widetilde{\mathrm{No}}_{<\kappa}$ and $g : \widetilde{\mathrm{No}}_{<\kappa} \to \widetilde{\mathrm{No}}_{<\kappa}$ be two continuous functions. Then $f + g$ and $f \cdot g$ are continuous.*

Proof. Note that since $f \times g$ is continuous, $f + g = + \circ (f \times g)$ and $f \cdot g = \cdot \circ (f \times g)$, we only need to prove that sum and product over $\widetilde{\mathrm{No}}_{<\kappa}$ are continuous. Let (a, b) be an open interval in $\widetilde{\mathrm{No}}_{<\kappa}$. We have that

$$+^{-1}[(a, b)] = \{\langle x, y\rangle \in \widetilde{\mathrm{No}}_{<\kappa} \times \widetilde{\mathrm{No}}_{<\kappa} | x + y \in (a, b)\},$$

therefore $a < x + y < b$. Moreover, since $\widetilde{\mathrm{No}}_{<\kappa}$ is an ordered field, $a - y < x < b - y$. Now we have

$$+^{-1}[(a, b)] = \bigcup_{(a', b') \subset \widetilde{\mathrm{No}}_{<\kappa}} (a', b') \times (a - a', b - b')$$

therefore $+$ is continuous in $\widetilde{\mathrm{No}}_{<\kappa}$. A similar reasoning works for $\cdot$. Let (a,b) be an open interval in $\widetilde{\mathrm{No}}_{<\kappa}$, we have that

$$\cdot^{-1}[(a,b)] = \{\langle x,y\rangle \in \widetilde{\mathrm{No}}_{<\kappa} \times \widetilde{\mathrm{No}}_{<\kappa} \mid x \cdot y \in (a,b)\},$$

therefore $a < x \cdot y < b$. We will only prove that case $x, y > 0$, the other cases follows similarly. Since $\widetilde{\mathrm{No}}_{<\kappa}$ is an ordered field, $\frac{a}{y} < x < \frac{b}{y}$ and

$$\cdot^{-1}[(a,b)] = \bigcup_{\substack{(a',b')\subseteq\widetilde{\mathrm{No}}_{<\kappa} \\ b'\neq 0}} (a',b') \times (\frac{a}{a'}, \frac{b}{b'}),$$

therefore $\cdot$ is continuous as desired. □

Now it is easy to see that every polynomial over $\widetilde{\mathrm{No}}_{<\kappa}$ is continuous.

Theorem 3.4.11. *Let $p(x)$ be a polynomial with coefficients in $\widetilde{\mathrm{No}}_{<\kappa}$. Then p is continuous.*

Proof. By induction over the degree of $p(x)$. Note that every constant function is trivially continuous. Moreover if $p(x)$ is of degree $n+1$ then

$$p(x) = \sum_{i=0}^{i=n} a_i x^i = x(\sum_{i=1}^{i=n} a_i x^{i-1}) + a_0.$$

Let $p'(x) = \sum_{i=1}^{i=n} a_i x^{i-1})$, it is continuous by inductive hypothesis. Moreover $p = (id \cdot p') + c_{a_0}$ where c_{a_0} is the constant function with value a_0. Hence by the previous lemma we have that p is continuous. □

Finally we are ready to prove that $\widetilde{\mathrm{No}}_{<\kappa}$ is a real closed field.

Theorem 3.4.12. *The field $\widetilde{\mathrm{No}}_{<\kappa}$ is real closed.*

Proof. It is enough to show that $\mathrm{No}_{<\kappa}$ is an elementary substructure of $\widetilde{\mathrm{No}}_{<\kappa}$ with respect to the language of real closed field theory, $(+, \cdot, 0, 1, <)$. We will prove that the Tarski-Vaught test holds. Let $\phi(x)$ be a formula in one variable with parameters in $\mathrm{No}_{<\kappa}$. Note that, since $\mathrm{No}_{<\kappa}$ is a real closed field, for any formula of this type there are $(p_i)_{i\in n}$ and $(q_i)_{i\in m}$ with $m, n \in \mathbb{N}$ and p_i,q_i polynomials in $\mathrm{No}_{<\kappa}$ such that

$$\phi(x) \Leftrightarrow \bigwedge_{i\in n} p_i(x) = 0 \wedge \bigwedge_{i\in m} q_i(x) > 0.$$

Now let $b \in \widetilde{\mathrm{No}}_{<\kappa}$ be such that $\phi(b)$ holds. If there is $i < n$ such that p_i is not the trivial zero polynomial, then $p_i(b) = 0$ and b is algebraic over $\mathrm{No}_{<\kappa}$. But since $\mathrm{No}_{<\kappa}$ is a real closed field, it has no proper ordered algebraic extensions, therefore $b \in \mathrm{No}_{<\kappa}$. If for all i the polynomial p_i is the zero polynomial, then b is a solution of the system $\bigwedge_{i\in m} q_i(x) > 0$. Now every q_i is continuous in $\widetilde{\mathrm{No}}_{<\kappa}$ hence there are $a_i < b < c_i$ such that $q_i(x) > 0$ for all $x \in (a_i, c_i)$. By the density of $\mathrm{No}_{<\kappa}$ in $\widetilde{\mathrm{No}}_{<\kappa}$, there are $a_i < a'_i < b < c'_i < c_i$ such that $q_i(x) > 0$ for all $x \in (a'_i, c'_i)$. Take $c = \min\{c'_i \mid i \in m\}$ and $a = \max\{a'_i \mid i \in m\}$. Hence, for every $x \in (a,c)$ we have $\bigwedge_{i\in m} q_i(x) > 0$ as desired. Then by the Tarski-Vaught test we have that $\mathrm{No}_{<\kappa}$ is an elementary substructure of $\widetilde{\mathrm{No}}_{<\kappa}$. In conclusion, since $\mathrm{No}_{<\kappa}$ is real closed, we have that $\widetilde{\mathrm{No}}_{<\kappa}$ is a real closed. □

In particular the fact that $\widetilde{\mathrm{No}}_{<\kappa}$ is a real closed field makes the field satisfy all the first order properties valid over $\mathbb{R}$.

Now that we have shown that $\widetilde{\mathrm{No}}_{<\kappa}$ is a real closed field extending $\mathbb{R}$ we want to check that also all the other properties of super dense extensions of $\mathbb{R}$ hold for $\widetilde{\mathrm{No}}_{<\kappa}$.

Lemma 3.4.13. *Let $\mathrm{No}_{\leq\kappa}$ be the set of surreal numbers of length at most κ. Then $\widetilde{\mathrm{No}}_{<\kappa} \subseteq \mathrm{No}_{\leq\kappa}$.*

Proof. We will prove that $\mathrm{No}_{\leq\kappa}$ contains the Dedekind closure of $\mathrm{No}_{<\kappa}$. This implies by definition that $\mathrm{No}_{\leq\kappa}$ also contains the Cauchy closure of $\mathrm{No}_{<\kappa}$, namely $\widetilde{\mathrm{No}}_{<\kappa}$. Let $\langle L, R\rangle$ be a cut in $\mathrm{No}_{<\kappa}$, we claim that $[L|R] \in \mathrm{No}_{\leq\kappa}$. Note that for every $x \in L \cup R$, since $L \cup R \subset \mathrm{No}_{<\kappa}$, $|x| < \kappa$. Hence, by Theorem 2.3.7 we have $|[L|R]| \leq \kappa$. Then $[L|R] \in \mathrm{No}_{\leq\kappa}$ as desired. Since $\widetilde{\mathrm{No}}_{<\kappa} = \mathrm{No}_{<\kappa} \cup C$ where C is the set of Cauchy cuts therefore $\widetilde{\mathrm{No}}_{<\kappa} \subseteq \mathrm{No}_{\leq\kappa}$ as desired. □

Theorem 3.4.14. *The real closed field $\widetilde{\mathrm{No}}_{<\kappa}$ has the following properties:*

1) $\mathrm{Deg}(\widetilde{\mathrm{No}}_{<\kappa}) = \kappa$

2) $\widetilde{\mathrm{No}}_{<\kappa}$ *is an η_κ-set.*

3) $|\widetilde{\mathrm{No}}_{<\kappa}| = 2^\kappa$, $\mathrm{Cof}(\widetilde{\mathrm{No}}_{<\kappa}) = \mathrm{Coi}(\widetilde{\mathrm{No}}_{<\kappa}) = w(\widetilde{\mathrm{No}}_{<\kappa}) = \kappa$.

Proof. 1) We will prove that $\mathrm{Deg}(\mathrm{No}_{<\kappa}) = \mathrm{Deg}(\widetilde{\mathrm{No}}_{<\kappa})$. Since $\mathrm{No}_{<\kappa}$ is dense in $\widetilde{\mathrm{No}}_{<\kappa}$, then we have that $\mathrm{Deg}(\widetilde{\mathrm{No}}_{<\kappa}) \geq \mathrm{Deg}(\mathrm{No}_{<\kappa})$. Now, assume that every sequence of length κ in $\widetilde{\mathrm{No}}_{<\kappa}$ is such that there is $x \in \widetilde{\mathrm{No}}^{+}_{<\kappa}$ smaller than every element of the sequence. Then, by the density of $\mathrm{No}_{<\kappa}$, there is $x' \in \mathrm{No}_{<\kappa}$ such that $0 < x' < x$, but this is absurd because $\mathrm{Deg}(\mathrm{No}_{<\kappa}) = \kappa$.

2) Take $L, R \subset \widetilde{\mathrm{No}}_{<\kappa}$ such that $|L| + |R| < \kappa$ and $L < R$. We have the following possibilities:

- L has no maximum and R has no minimum: by the density of $\mathrm{No}_{<\kappa}$ in $\widetilde{\mathrm{No}}_{<\kappa}$, as in the proof of Lemma 3.4.7, there are two sequences $(\ell_\gamma)_{\gamma\in\alpha}$ and $(r_\gamma)_{\gamma\in\beta}$ with $\alpha, \beta < \kappa$ of elements of $\mathrm{No}_{<\kappa}$ which are respectively cofinal in L and and coinitial in R and such that
$$\forall\alpha' \in \alpha \forall\beta' \in \beta.\ \ell_{\alpha'} < r_{\beta'}.$$
Hence, since $\mathrm{No}_{<\kappa}$ is an η_κ-set, we have that there is $x \in \widetilde{\mathrm{No}}_{<\kappa}$ such that $L < \{x\} < R$ as desired.
- L has maximum M and R has minimum m: it is enough to take $x = \frac{m-M}{2}$.
- L has maximum M and R has no minimum: consider the sequence $(r - M)_{r\in R}$. Note that
$$\forall r \in R.\ r - M > 0,$$
therefore, since $\mathrm{Deg}(\widetilde{\mathrm{No}}_{<\kappa}) = \kappa$ and $|R| < \kappa$ there is $x \in \widetilde{\mathrm{No}}_{<\kappa}$ such that $\forall r \in R.\ 0 < x < r - M$ but then $M < x + M$ and $\forall r \in R. x + M < r$ as desired.
- L has no maximum M and R has minimum m: a proof similar to the previous case applies.

3) Note that by the construction of $\widetilde{\mathrm{No}}_{<\kappa}$, since $\mathrm{No}_{<\kappa}$ is a dense subfield of $\widetilde{\mathrm{No}}_{<\kappa}$, we have
$$\mathrm{Cof}(\widetilde{\mathrm{No}}_{<\kappa}) = \mathrm{Coi}(\widetilde{\mathrm{No}}_{<\kappa}) = w(\widetilde{\mathrm{No}}_{<\kappa}) = \kappa.$$

Now, we want to prove $2^\kappa \leq |\widetilde{\mathrm{No}}_{<\kappa}| \leq 2^\kappa$. On the one hand, by Lemma 3.4.13 we have that $\widetilde{\mathrm{No}}_{<\kappa} \subset \mathrm{No}_{\leq\kappa}$. Indeed, $\mathrm{No}_{\leq\kappa}$ contains the Dedekind completion of $\mathrm{No}_{<\kappa}$, hence also its Cauchy completion $\widetilde{\mathrm{No}}_{<\kappa}$. Then, since $|\mathrm{No}_{\leq\kappa}| = 2^\kappa$, we have that $|\widetilde{\mathrm{No}}_{<\kappa}| \leq 2^\kappa$.

On the other hand let $\{0,1\}^{<\kappa}$ be the full binary tree of height κ, we define a tree T which is in bijection with $\{0,1\}^{<\kappa}$ and whose nodes are in $\mathrm{No}_{<\kappa} \times \mathrm{No}_{<\kappa}$. We define the tree by recursion as follows:

let $x_0, y_0 \in \mathrm{No}_{<\kappa}$ such that $x_0 < y_0 < 1$ set (x_0, y_0) as the root of the tree.

Assume that the tree is already defined for all $\beta < \alpha < \kappa$, for every $w \in \{0,1\}^\alpha$ define x_w and y_w as follows:

(A) $y_w - x_w < \dfrac{1}{\alpha}$.

(B) $x_{w'} < x_w < y_w < y_{w'}$ for all $w' \subset w$.

(C) $(x_w, y_w) \cap (x_{w'}, y_{w'}) = \emptyset$ for all $w' \in \{0,1\}^\alpha,\ w' \neq w$.

Note that the tree is well defined since $\mathrm{No}_{<\kappa}$ is an η_κ-set and every path from an element to the root has strictly less than κ elements. Now by property (B) and by the fact that $\mathrm{No}_{<\kappa}$ is a real closed field, we have the following:

$$\forall p \in \{0,1\}^\kappa \forall \alpha < \kappa \forall \beta > \alpha.\ x_{p\restriction\beta} \in (x_{p\restriction\alpha}, y_{p\restriction\alpha}).$$

Moreover by construction for every $p \in \{0,1\}^\kappa$, the sequence $(x_{p\restriction\alpha})_{\alpha<\kappa}$ is Cauchy. Indeed, let $\varepsilon \in \mathbb{Q}^+_\kappa$. Take α such that $\frac{1}{\alpha} \leq \varepsilon$. By (A) we have

$$y_{p\restriction\alpha} - x_{p\restriction\alpha} < \frac{1}{\alpha} \leq \varepsilon$$

and

$$\forall \beta > \alpha.\ x_{p\restriction\beta} \in (x_{p\restriction\alpha}, y_{p\restriction\alpha}).$$

Then, since $\mathrm{No}_{<\kappa}$ is a real closed field,

$$\forall \beta, \beta' > \alpha.\ |x_{p\restriction\beta} - x_{p\restriction\beta'}| < x_{p\restriction\alpha} - y_{p\restriction\alpha} < \varepsilon$$

as desired.

Finally we have that if $p, p' \in \{0,1\}^\kappa$ and $p \neq p'$ then $(x_{p\restriction\alpha})_{\alpha\in\kappa}$ and $(x_{p'\restriction\alpha})_{\alpha\in\kappa}$ converge to different points. Let α be the least such that $p(\alpha) \neq p(\alpha)$. Then by construction $x_{p\restriction\alpha+1}$ and $x_{p'\restriction\alpha+1}$ are such that

$$(x_{p\restriction\alpha+1}, y_{p\restriction\alpha+1}) \cap (x_{p'\restriction\alpha+1}, y_{p'\restriction\alpha+1}) = \emptyset.$$

Now, since

$$\forall \beta > \alpha.\ x_{p\restriction\beta} \in (x_{p\restriction\alpha+1}, y_{p\restriction\alpha+1})$$

and

$$\forall \beta > \alpha.\ x_{p'\restriction\beta} \in (x_{p'\restriction\alpha+1}, y_{p'\restriction\alpha+1})$$

we have

$$\lim_{\beta\in\kappa} x_{p\restriction\beta} \in (x_{p\restriction\alpha+1}, y_{p\restriction\alpha+1})$$

and

$$\lim_{\beta\in\kappa} x_{p'\restriction\beta} \in (x_{p'\restriction\alpha+1}, y_{p'\restriction\alpha+1}).$$

Hence $\lim_{\beta\in\kappa} x_{p\restriction\beta} \neq \lim_{\beta\in\kappa} x_{p'\restriction\beta}$ as desired. Then we have $2^\kappa \leq |\widetilde{\mathrm{No}}_{<\kappa}|$ as desired. □

Therefore all the properties listed at the beginning of this section are fulfilled by $\widetilde{\mathrm{No}}_{<\kappa}$. We are now ready to define our generalization of $\mathbb{R}$.

Definition 3.4.15 ($\mathbb{R}_\kappa$ and $\mathbb{Q}_\kappa$)**.** *We define $\mathbb{R}_\kappa$ as follows:*

$$\mathbb{R}_\kappa = \widetilde{\mathrm{No}}_{<\kappa}.$$

We will use $+\infty$ and $-\infty$ to denote respectively the least ordinal not in $\mathbb{R}_\kappa$ and its inverse namely $+\infty = \kappa$ and $-\infty = -\kappa$. Moreover we define:

$$\mathbb{Q}_\kappa = \mathrm{No}_{<\kappa}\,.$$

From now on, we will call *κ-reals* the elements of $\mathbb{R}_\kappa$ and *κ-rationals* the elements of $\mathbb{Q}_\kappa$.

Note that the construction of $\mathbb{R}_\kappa$ we have just presented seems to be really different form the construction of $\mathbb{R}$ within the theory of surreal numbers. The following theorems show some connection between the definition of $\mathbb{R}$ and $\mathbb{R}_\kappa$.

Definition 3.4.16. *Let $x \in \mathrm{No}$ be an eventually constant surreal number. We will call* stabilization point *of x the smallest ordinal α such that $\forall \beta \geq \alpha . x(\alpha) = x(\beta)$.*

Theorem 3.4.17 (Alling)**.** *A surreal number $x \in \mathrm{No}$ is in the Dedekind closure $\mathrm{No}^{\mathrm{D}}_{<\kappa}$ of $\mathrm{No}_{<\kappa}$ iff one of the following holds:*

- $x \in \mathrm{No}_{<\kappa}$.
- $x \in \mathrm{No}_\kappa$ *and is not eventually constant.*
- $x \in \mathrm{No}_\kappa$ *and its stabilization point is a limit ordinal.*

Since $\mathbb{R}_\kappa \subset \mathrm{No}^{\mathrm{D}}_{<\kappa}$, we have the following corollary:

Corollary 3.4.18. *Let $x \in \mathbb{R}_\kappa$. Then one of the followings holds:*

1) x has length smaller than κ.

2) x has length κ but is not eventually constant.

3) x has length κ and its stabilization point is a limit ordinal.

Note that the converse is not true (i.e., there are elements that satisfies 2) or 3) which are not in $\mathbb{R}_\kappa$). It is still unclear if there is a characterization of $\mathbb{R}_\kappa$ similar to that of $\mathbb{R}$ (i.e., in terms of surreal numbers). In particular, we don't know if the fact of being a limit of a Cauchy sequence can be characterized in terms of sequences of pluses and minuses. We leave the following open question:

Open Question 3.4.19. *Is there any characterization of $\mathbb{R}_\kappa$ in terms of sequences of pluses and minuses?*

We conclude this chapter with some result on the interval topology and the interval κ-topology over $\mathbb{R}_\kappa$. The first two results are a generalized version of a classical result over $\mathbb{R}$.

Lemma 3.4.20. *Every open interval of $\mathbb{R}_\kappa$ has cardinality 2^κ.*

Proof. Let $(r, r') \subset \mathbb{R}_\kappa$. On the one hand, since $|\mathbb{R}_\kappa| = 2^\kappa$, we have that $|(r, r')| \leq 2^\kappa$. On the other hand, by the same construction of the third point of Theorem 3.4.14, we can build a tree with root (r, r') of Cauchy sequences converging to different limits. Note that, since all the points of these Cauchy sequences are in (r, r'), their limits are also in (r, r'). Then we have $2^\kappa \leq |(r, r')|$ as desired. ☐

Theorem 3.4.21. *Let τ be the interval topology over $\mathbb{R}_\kappa$. Then the set of open intervals of $\mathbb{R}_\kappa$ with end points in $\mathbb{Q}_\kappa \cup \{+\infty, -\infty\}$ is a base for τ.*

Proof. It is enough to show that every open interval with end points in $\mathbb{R}_\kappa \cup \{+\infty, -\infty\}$ is a union open intervals with κ-rational end points. Let (a, b) an interval with end points in $\mathbb{R}_\kappa \cup \{+\infty, -\infty\}$. If $a = -\infty$ and $b = +\infty$ then trivially (a, b) has end points in $\mathbb{Q}_\kappa \cup \{+\infty, -\infty\}$. If $a \in \mathbb{R}_\kappa$, and $b = +\infty$ then by density of $\mathbb{Q}_\kappa$ in $\mathbb{R}_\kappa$, we have that $(a, b) = \bigcup_{q \in \{\mathbb{Q}_\kappa | q > a\}} (q, b)$ (a similar argument proves the case $a = -\infty$, $b \in \mathbb{R}_\kappa$). Finally if $a, b \in \mathbb{R}_\kappa$ then by density of $\mathbb{Q}_\kappa$ there are $q, q' \in \mathbb{Q}_\kappa$ such that $a < q < q' < b$. Then $(a, b) = \bigcup_{\{q'' \in \mathbb{Q}_\kappa | q'' > a\}} (q'', q') \cup \bigcup_{\{q'' \in \mathbb{Q}_\kappa | q'' < b\}} (q, q'')$ as desired. ☐

Note that the previous proof can not be used to prove that intervals with κ-rational end points are a base for the interval κ-topology over $\mathbb{R}_\kappa$. In fact, since $\mathbb{R}_\kappa$ is an η_κ-set, in general we can not write intervals (a, b) with end points in $\mathbb{R}_\kappa \cup \{+\infty, -\infty\}$ as a union of less than κ open intervals with κ-rational end points. This is a consequence of the more general fact that κ-topologies depends on the cardinality of their bases. In general for κ-topology we have:

Lemma 3.4.22. *Let B and B' two bases for the topology τ over a set X. If B and B' generate the same κ-topology then $(|B|)^{<\kappa} = (|B'|)^{<\kappa}$.*

Proof. Let B, B' two bases for τ. Assume that they generate the same κ-topology τ_κ. By definition, we have $(|B|)^{<\kappa} = |\tau_\kappa| = (|B'|)^{<\kappa}$ as desired. ☐

Note that the other direction does not hold. Indeed, consider the topology generated by the set B of open intervals with end points in $\mathbb{R}_\kappa$, and the topology generated by the base $B' = B \cup \{(1, +\infty)\}$. It is not hard to see that the two bases both generates the interval topology over $\mathbb{R}_\kappa$ and that they have the same cardinality. But while $(1, +\infty)$ is κ-open in the κ-topology generated by B' it is not κ-open in the κ-topology generated by B (recall that $\mathrm{Cof}(\mathbb{R}_\kappa) = \kappa$). The fact that $\mathbb{R}_\kappa$ is an η_κ-set gives us more information about the interval κ-topology over $\mathbb{R}_\kappa$.

Theorem 3.4.23. *Let B be a base for the interval κ-topology τ over $\mathbb{R}_\kappa$. Then $|B| = 2^\kappa$.*

Proof. First note that every base B of τ is of the form $B = B' \cup B''$ where

$$B' \subseteq \{(a,b) \subset \mathbb{R}_\kappa \mid a, b \in \mathbb{R}_\kappa \cup \{+\infty, -\infty\} \wedge a < b\}$$

and

$$B'' \subset \{O \subset \mathbb{R}_\kappa \mid O \ \kappa - \text{open in the interval } \kappa\text{-topology}\}.$$

Assume $|B| < 2^\kappa$. Now, define

$$A = \{a \mid (a,b) \in B'\}$$

and

$$A' = \{a_i \mid O \in B'' \wedge O = \bigcup_{\beta<\alpha} (a_\beta, b_\beta) \wedge \alpha < \kappa\}.$$

Since $|B| < 2^\kappa$, there exists $r \notin A \cup A'$. Consider the interval $(r, r+1)$. Since $(r, r+1)$ is an open interval in $\mathbb{R}_\kappa$ and B is a base for the interval κ-topology, we have

$$(r, r+1) = \bigcup_{\beta<\alpha} O_\beta \cup \bigcup_{\beta<\alpha'} (a_\beta, b_\beta)$$

for some $\alpha, \alpha' < \kappa$ and $O_\beta \in B''$ such that for all $\beta \in \alpha$ and $(a_\beta, b_\beta) \in B'$ for all $\beta \in \alpha'$. Now, take the sets $\{r\}$ and $(A \cap (r, r+1)) \cup (A' \cap (r, r+1))$. We have

$$|(A \cap (r, r+1)) \cup (A' \cap (r, r+1))| < \max\{|\alpha|, |\alpha'|\} < \kappa$$

and

$$\{r\} < (A \cap (r, r+1)) \cup (A' \cap (r, r+1)).$$

But since $\mathbb{R}_\kappa$ is an η_κ-set, there exists $x \in \mathbb{R}_\kappa$ such that

$$\{r\} < \{x\} < (A \cap (r, r+1)) \cup (A' \cap (r, r+1)),$$

therefore we have $x \in (r, r+1)$ and $x \notin \bigcup_{\beta<\alpha} O_\beta \cup \bigcup_{\beta<\alpha'} (a_\beta, b_\beta)$ which is a contradiction. □

The last theorem we prove in this section can be seen as an amalgamation theorem for κ-continuous functions. It will be a very useful tool to build κ-continuous functions.

Theorem 3.4.24. *Let $(x_i)_{i\in n}$ be a strictly increasing sequence in $\mathbb{Q}_\kappa$ such that, defining $(X_i)_{i\in n}$ by:*

$$X_i = \begin{cases} (-\infty, x_0] & i = 0, \\ [x_i, x_{i+1}] & 0 < i < n-1, \\ [x_{n-1}, +\infty) & i = n-1, \end{cases}$$

we have $\mathbb{R}_\kappa = \bigcup_{i\in n} X_i$. Then every function $f : \mathbb{R}_\kappa \to \mathbb{R}_\kappa$ such that $f \restriction X_i$ is κ-continuous for every $0 < i < n$ is κ-continuous.

Proof. Let (a,b) be an open interval with end points in $\mathbb{R}_\kappa \cup \{+\infty, -\infty\}$. Define $U = f^{-1}(a,b)$. We want to prove that U is κ-open. For every $0 < i < n$ set $U_i = U \cap X_i$. By κ-continuity of $f \restriction X_i$, every U_i is κ-open in X_i. Then, for every $i < n$, there is $O_i = \bigcup_{\alpha\in\beta_i} (a^i_\alpha, b^i_\alpha)$ such that $U_i = O_i \cap X_i$. Moreover note that $U = \bigcup_{i\in n} U_i$. We want to show that U is κ-open. To do that we will extract from $(U_i)_{i\in n}$ a sequence $(U'_i)_{i\in n}$ such that U'_i is a union of less than κ κ-open intervals with end points in $\mathbb{R}_\kappa \cup \{+\infty, -\infty\}$ such that $U = \bigcup_{i\in n} U'_i$. For every $i < n$ we define U'_i as follows:

- if $U_i = \emptyset$ then $U_i = U'_i$.
- If for all $\alpha < \beta_i$ $(a^i_\alpha, b^i_\alpha) \subset X_i$ then $U_i = U'_i$.

If none of the previous applies we define:

$$I = \{(a^i_\alpha, b^i_\alpha) \mid \alpha < \beta_i \ \wedge \ (a^i_\alpha, b^i_\alpha) \subset X_i\}.$$

Moreover we define a set I' as follows:

- for every (a^i_α, b^i_α) such that $(a^i_\alpha, b^i_\alpha) \cap X_i = (a^i_\alpha, x_{i+1}]$. Then $a < f(x_{i+1}) < b$. We have two cases:
  - If for all $i < j \leq n$ and for every $x \in X_j$ we have $f(x) \in (a, b)$. We have

    $$U \supseteq (a^i_\alpha, +\infty) \supseteq (a^i_\alpha, x_{i+1}].$$

    In this case $(a^i_\alpha, +\infty) \in I'$.
  - If exists $i < j \leq n$ and there exists $x \in X_i$ such that $f(x) \notin (a, b)$. Then take m least of these j. Hence, there is $b_m \in X_m$ such that $f(b_m) \in (a, b)$. Therefore

    $$U \supseteq (a^i_\alpha, b_m) \supseteq (a^i_\alpha, x_{i+1}].$$

    Note that we can pick $b_m \neq x_{i+1}$ because (a, b) is an open interval, therefore $f(x_{i+1}) \neq b$ and $f{\restriction}X_m$ is κ-continuous. In this case $(a^i_\alpha, b_m) \in I'$.
- for every (a^i_α, b^i_α) such that $(a^i_\alpha, b^i_\alpha) \cap X_i = [x_i, b^i_\alpha)$. Then $a < f(x_i)) < b$. We have can use the same construction of the previous case.
- for every (a^i_α, b^i_α) such that $(a^i_\alpha, b^i_\alpha) \cap X_i = [x_i, x_{i+1}]$. By using the previous two cases with $[x_i, x_{i+1})$ and $(x_i, x_{i+1}]$, we can find (a', b') such that

  $$U \supseteq (a', b') \supset [x_i, x_{i+1}]$$

  and take $(a', b') \in I'$.

Then we define $U'_i = I \cup I'$. Note that every U'_i is a κ-open set in $\mathbb{R}_\kappa$.

Now, note that, by the definition of $\{U'_i\}_{i<n}$ we have $U_i \subseteq U'_i$ and $U_i \subset U$ for every $0 < i < n$. Hence $U = \bigcup_{i \in n} U'_i$ and therefore U is κ-open as desired. □

3.5 Generalized Descriptive Set Theory

Metrizability is one of the main ingredients of descriptive set theory. As we noticed in Section 2.4, since generalized Baire space is not metrizable, all those notions from descriptive set theory that depend on metrizability cannot be easily generalized to uncountable cardinals. In this section we will use $\mathbb{R}_\kappa$ to fill this gap between classical and generalized descriptive set theory. In particular we will define a generalization of metrizability to κ which can be used to study generalized Baire space from a metric point of view. Then we will show that, as in the classical case, κ-Borel sets form a hierarchy.

Note that in this section we will focus our attention on topologies rather than κ-topologies. This is due to the following facts:

We want to use $\mathbb{R}_\kappa$ to generalize results from classical descriptive set theory to generalized Baire space. In descriptive set theory the topology over κ^κ plays a central role, the use of κ-topologies would then make our work harder and sometimes impossible.

The results at the end of this section will be very important in Chapter 4 where we will be forced to consider the standard topology over κ^κ.

Definition 3.5.1 (κ-metric Space)**.** *A* κ-metric space *is a pair (X, d) where X is a set and $d : X \to \mathbb{R}_\kappa$ is a function such that:*

- *$d(x, y) \geq 0$.*
- *$d(x, y) = 0 \Leftrightarrow x = y$.*

- $d(x, y) = d(y, x)$.
- $d(x, y) + d(y, z) \geq d(x, z)$.

We will call d a κ-metric. Moreover a κ-metric d is called a κ-ultrametric if the following hold:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

Given a κ-metric space (X, d), $x \in X$ and $r \in \mathbb{R}_\kappa$, we define

$$\mathrm{B}(x, r) = \{y \mid d(x, y) < r\}$$

and

$$\mathrm{B}[x, r] = \{y \mid d(x, y) \leq r\}.$$

We will call $\mathrm{B}(x, r)$ the open ball of centre x and radius r, and $B[x, r]$ the closed ball of centre x and radius r.

Note that the absolute value is definable in the language of real closed fields as follows:

$$|x| = y \Leftrightarrow (x \geq 0 \to y = x) \wedge (x < 0 \to y = -x).$$

Now, since $\mathbb{R}_\kappa$ is a real closed field and the absolute value over $\mathbb{R}$ is a metric, the absolute value over $\mathbb{R}_\kappa$ is a κ-metric (note that all the properties required by metrics are sentences in $(+, \cdot, 0, 1, <)$).

Definition 3.5.2 (κ-metrizability)**.** *Let τ be a topology over X. Then τ is said to be* κ-metrizable *if there is a κ-metric $d : X \to \mathbb{R}_\kappa$ such that*

$$B = \{\mathrm{B}(x, r) \mid x \in X \text{ and } r \in \mathbb{R}_\kappa\}$$

is a base for τ. Moreover, we say that τ is completely κ-metrizable *if every Cauchy sequence in X induced by d has a limit in X.*

As we have just seen, the classical metric over $\mathbb{R}$ extends naturally to a κ-metric for $\mathbb{R}_\kappa$. It is not hard to see that the absolute value induces the interval topology over $\mathbb{R}_\kappa$.

Theorem 3.5.3. *The interval topology over $\mathbb{R}_\kappa$ is κ-metrizable.*

Proof. We want to show that $|\cdot| : \mathbb{R}_\kappa \to \mathbb{R}_\kappa$ induces the interval topology over $\mathbb{R}_\kappa$. We will show that the set of open balls B is equal to the set of open intervals with end points in $\mathbb{R}_\kappa$. Since $\mathbb{R}_\kappa$ is a real closed field, the proof is the same as in the classical case.

On the one hand let (r_1, r_2) be an open interval in $\mathbb{R}_\kappa$. Then

$$\mathrm{B}(\frac{r_2 + r_1}{2}, \frac{r_2 - r_1}{2}) = (r_1, r_2).$$

Indeed, we have:

$$\begin{aligned} x \in \mathrm{B}(\frac{r_2 + r_1}{2}, \frac{r_2 - r_1}{2}) &\Leftrightarrow \frac{r_2 + r_1}{2} - \frac{r_2 - r_1}{2} < x < \frac{r_2 + r_1}{2} + \frac{r_2 - r_1}{2} \\ &\Leftrightarrow r_1 < x < r_2 \\ &\Leftrightarrow x \in (r_1, r_2). \end{aligned}$$

On the other hand, let $\mathrm{B}(x, r)$ be an open ball, then trivially $\mathrm{B}(x, r) = (x - r, x + r)$.

In conclusion B is the set of open sets with end points in $\mathbb{R}_\kappa$, hence B generates the interval topology over $\mathbb{R}_\kappa$ □

Proposition 3.5.4. *Let (X, τ) be κ-metrizable and C be a closed subset of X. Then C is the intersection of κ many open sets.*

Proof. Let d be a κ-metric which induces τ and C a closed set. Define:

$$C_\alpha = \{x \mid \exists c \in C.\ d(x,c) < \frac{1}{\alpha}\}.$$

Note that for every $\alpha \in \kappa$ we have that $C_\alpha = \bigcup_{c\in C} \mathrm{B}(c, \frac{1}{\alpha})$ is open. We claim $C = \bigcap_{\alpha\in\kappa} C_\alpha$.

On the one hand, assume $x \in C$. Then $d(x,x) = 0$ and $x \in C_\alpha$ for every $\alpha \in \kappa$. Therefore $x \in \bigcap_{\alpha\in\kappa} C_\alpha$. On the other hand if $x \in \bigcap_{\alpha\in\kappa} C_\alpha$ then for all $\alpha \in \kappa$ there exists $c \in C_\alpha$ such that $d(x,c) < \frac{1}{\alpha}$. For each α choose $c_\alpha \in C_\alpha \cap C$. Since $\lim_{\alpha\in\kappa} \frac{1}{\alpha} = 0$, we have $\lim_{\alpha\in\kappa} d(x, c_\alpha) = 0$ and since C is closed, $x \in C$ as desired. □

Analogously to the classical case, where ω^ω was completely metrizable, generalized Baire space can be proved to be completely κ-metrizable.

Theorem 3.5.5. *The generalized Baire space κ^κ is completely κ-metrizable.*

Proof. First we define the following κ-metric:

$$d(x,y) = \begin{cases} 0 & \text{iff } x = y, \\ \frac{1}{\alpha+1} & \alpha \text{ minimal s.t. } x(\alpha) \neq y(\alpha), \end{cases}$$

for $x, y \in \kappa^\kappa$.

We claim that d is a κ-metric:

Trivially $d(x,y) \geq 0$, moreover $d(x,y) = 0$ if and only if $x = y$ and $d(x,y) = d(y,x)$. We want to prove the following:

$$\forall x \forall y \forall z. d(x,y) + d(y,z) \geq d(x,z).$$

If $d(x,y) = 0$ or $d(y,z) = 0$ the statement follows trivially. Assume $d(x,y) = \frac{1}{\alpha+1}$, $d(y,z) = \frac{1}{\beta+1}$ and $d(x,z) \neq 0$, in particular $d(x,z) = \frac{1}{\gamma+1}$. First assume $\alpha \leq \beta$. Since $y{\restriction}\beta = z \restriction \beta$ and $\alpha \leq \beta$, we have that $d(x,z) = d(x,y) \leq d(x,y) + d(y,z)$ as desired. Now assume $\beta < \alpha$. Since $x{\restriction}\alpha = y \restriction \alpha$ and $\beta < \alpha$, we have that $d(x,z) = d(y,z) \leq d(x,y) + d(y,z)$ as desired. Note that we actually proved that d is a κ-ultrametric.

Now we want to show that the set of open balls induced by d is a base for generalized Baire space.

On the one hand let $w \in \kappa^{<\kappa}$. We claim that

$$[w] = \mathrm{B}(p, \frac{1}{|w|+1}) \text{ where } w \subset p.$$

By definition we have

$$p' \in [w] \Leftrightarrow w \subseteq p' \Leftrightarrow d(p,p') < \frac{1}{|w|+1} \Leftrightarrow p' \in \mathrm{B}(p, \frac{1}{|w|+1})$$

as desired. On the other hand let $\mathrm{B}(p,r)$ with $r \in \mathbb{R}_\kappa$ be an open ball. Note that by definition of d, we have $\mathrm{B}(p,r) = \mathrm{B}(p, \frac{1}{\alpha_r+1})$ with α_r greatest ordinal such that $\frac{1}{\alpha_r+1} \leq r$. Now let $w = p{\restriction}\alpha_r$ as before. We have

$$p' \in [w] \Leftrightarrow w \subset p' \Leftrightarrow d(p,p') < \frac{1}{|w|+1} = \frac{1}{\alpha_r+1} \Leftrightarrow p' \in \mathrm{B}(p, \frac{1}{\alpha_r+1}) = \mathrm{B}(p,r)$$

as desired.

Finally we want to prove that κ^κ is complete w.r.t. d. Let $(p_\alpha)_{\alpha\in\kappa}$ a Cauchy sequence in κ^κ. Note that

$$\forall \varepsilon \in \mathbb{R}^+_\kappa\ \exists \gamma < \kappa \forall \alpha, \alpha' \geq \gamma.\ d(p_\alpha, p_{\alpha'}) < \varepsilon.$$

Now, define the following sequence

$$\begin{aligned} &p_\ell{\restriction}0 = \langle\rangle, \\ &p_\ell{\restriction}\alpha = p_\gamma{\restriction}\alpha \qquad \gamma \text{ smallest s.t. } \forall \alpha, \alpha' \geq \gamma.\ d(p_\alpha, p_{\alpha'}) < \frac{1}{\alpha}. \end{aligned}$$

Since the starting sequence was Cauchy, p_γ always exists. We claim that $(p_\ell\restriction\alpha)_{\alpha\in\kappa}$ is monotone. Let $\alpha \leq \beta$. If $p_\ell\restriction\alpha = p_\ell\restriction\beta$ then trivially $p_\ell\restriction\alpha \subseteq p_\ell\restriction\beta$. Hence assume $p_\ell\restriction\alpha \neq \langle\rangle \neq p_\ell\restriction\beta$. Let γ and γ' be such that

$$p_\ell\restriction\alpha = p_\gamma\restriction\alpha \qquad p_\ell\restriction\beta = p_{\gamma'}\restriction\beta.$$

Without loss of generality assume $\gamma' \geq \gamma$ a similar proof works for $\gamma' < \gamma$. By the definition we have

$$d(p_\gamma, p_{\gamma'}) < \frac{1}{\alpha+1},$$

and therefore

$$p_\ell\restriction\alpha = p_\gamma\restriction\alpha \subseteq p_{\gamma'}\restriction\beta = p_\ell\restriction\beta$$

as desired. Then the sequence $p_\ell = \bigcup_{\alpha\in\kappa} p_\ell\restriction\alpha$ is well defined.

Now we want to prove that $\lim_{\alpha\in\kappa} p_\alpha = p_\ell$, that is

$$\forall\varepsilon \in \mathbb{R}^+_\kappa\, \exists\gamma < \kappa \forall\beta \geq \gamma.\ d(p_\beta, p_\ell) < \varepsilon.$$

Fix $\varepsilon \in \mathbb{R}^+_\kappa$. Let α such that $\frac{1}{\alpha+1} \leq \varepsilon$. We have that

$$p_\ell\restriction\alpha = p_\gamma\restriction\alpha,$$

with γ smallest ordinal such that

$$\forall\beta, \beta' \geq \gamma.\ d(p_\beta, p_{\beta'}) < \frac{1}{\alpha+1}$$

In particular, we have that

$$\forall\beta \geq \gamma.\ d(p_\beta, p_\gamma) < \frac{1}{\alpha+1}.$$

Then, since $p_\ell\restriction\alpha = p_\gamma\restriction\alpha$, we have that

$$\forall\beta \geq \gamma.\ d(p_\beta, p_\ell) < \frac{1}{\alpha+1} \leq \varepsilon,$$

therefore $\lim_{\alpha\in\kappa} p_\alpha = p_\ell$ as desired. □

Definition 3.5.6 (κ-separability)**.** *Let τ be a topological space over X. Then τ is* κ-separable *iff it has a dense subset of cardinality κ.*

Since in the previous section we have proved that the set of intervals with end points in $\mathbb{Q}_\kappa$ is a base for the interval topology over $\mathbb{R}_\kappa$, we have the following:

Proposition 3.5.7. *The interval topology over $\mathbb{R}_\kappa$ is κ-separable.*

Recall that we assumed $\kappa^{<\kappa} = \kappa$, therefore the standard base of κ^κ is of cardinality κ. In particular we have:

Proposition 3.5.8. *The generalized Baire space κ^κ is κ-separable.*

Polish spaces have a central role in descriptive set theory, it is natural then to generalize them to uncountable cardinals.

Definition 3.5.9 (κ-Polish Space)**.** *Let τ be a topological space over X. Then, τ is* κ-Polish *iff it is κ-separable and completely κ-metrizable.*

By what we have proved before, we have:

Corollary 3.5.10. *The generalized Baire space κ^κ is κ-Polish.*

Now we are ready to define the generalization to κ of the Borel hierarchy over κ-metrizable sets. First of all we extend the notion of a Borel set to our generalized setting. The following definition can be found in [13], [21] and [15]:

Definition 3.5.11 (κ-Borel Sets)**.** *The collection* $\mathbf{B}(X)$ *of* κ-Borel *sets over a topological space* (X, τ) *is the smallest containing the topology and closed under unions of size* κ *and complementation.*

Since we have generalized metrizability to $\mathbb{R}_\kappa$, we can now show that the κ-Borels form a hierarchy as in the classical case.

Definition 3.5.12 (κ-Borel hierarchy)**.** *Let* (X, τ) *be a* κ*-metrizable space. Then for* $1 < \alpha < \kappa^+$ *we define:*

$$\begin{aligned}
\mathbf{\Sigma}_1^{(\kappa,0)}(X) &= \textit{ open sets},\\
\mathbf{\Pi}_1^{(\kappa,0)}(X) &= \textit{ closed sets},\\
\mathbf{\Sigma}_\alpha^{(\kappa,0)}(X) &= \{\bigcup_{\beta\in\kappa} A_\beta \mid \forall\beta<\kappa\exists\gamma<\alpha.\ A_\beta \in \mathbf{\Pi}_\gamma^{(\kappa,0)}(X)\},\\
\mathbf{\Pi}_\alpha^{(\kappa,0)}(X) &= \{\bigcap_{\beta\in\kappa} A_\beta \mid \forall\beta<\kappa\exists\gamma<\alpha.\ A_\beta \in \mathbf{\Sigma}_\gamma^{(\kappa,0)}(X)\}.
\end{aligned}$$

As usual $\mathbf{\Delta}_\alpha^{(\kappa,0)}(X) = \mathbf{\Sigma}_\alpha^{(\kappa,0)}(X) \cap \mathbf{\Pi}_\alpha^{(\kappa,0)}(X)$*.*

Proposition 3.5.13. *If* (X, τ) *is* κ*-metrizable, then for every* $\alpha \in \mathrm{On}$ *we have* $\mathbf{\Sigma}_\alpha^{(\kappa,0)}(X) \subseteq \mathbf{\Sigma}_{\alpha+1}^{(\kappa,0)}(X)$ *and* $\mathbf{\Pi}_\alpha^{(\kappa,0)}(X) \subseteq \mathbf{\Pi}_{\alpha+1}^{(\kappa,0)}(X)$*.*

Proof. We will proceed by induction over α:

Assume $\alpha = 1$. Let $A \in \mathbf{\Pi}_1^{(0,\kappa)}(X)$. Then A is closed in X. By Proposition 3.5.4 we have that $A = \bigcap_{\beta\in\kappa} A_\beta$ with A_β open for every $\beta \in \kappa$. But then $A \in \mathbf{\Pi}_2^{(\kappa,0)}(X)$. A similar proof works for $\mathbf{\Sigma}_1^{(\kappa,0)}(X)$.

Assume $\alpha > 1$. Let $A \in \mathbf{\Pi}_\alpha^{(\kappa,0)}(X)$. By definition $A = \bigcap_{\beta\in\kappa} A_\beta$ with $A_\beta \in \bigcup_{\gamma\in\alpha} \mathbf{\Sigma}_\gamma^{(\kappa,0)}(X)$ for all $\beta < \kappa$. By inductive hypothesis, for all $\beta < \kappa$, $A_\beta \in \mathbf{\Sigma}_\alpha^{(\kappa,0)}(X)$ and by definition $A = \bigcap_{\beta\in\kappa} A_\beta \in \mathbf{\Pi}_{\alpha+1}^{(\kappa,0)}(X)$. A similar proof works for $\mathbf{\Sigma}_\alpha^{(\kappa,0)}(X)$. □

Then we trivially have:

Corollary 3.5.14. *If* (X, τ) *is* κ*-metrizable, then for every* $\alpha \in \mathrm{On}$ *we have*

$$\mathbf{\Sigma}_\alpha^{(\kappa,0)}(X) \cup \mathbf{\Pi}_\alpha^{(\kappa,0)}(X) \subseteq \Delta_{\alpha+1}^{(\kappa,0)}(X).$$

Proof. It is enough to prove $\mathbf{\Sigma}_\alpha^{(\kappa,0)}(X) \subseteq \mathbf{\Pi}_{\alpha+1}^{(\kappa,0)}(X)$ and $\mathbf{\Pi}_\alpha^{(\kappa,0)}(X) \subseteq \mathbf{\Sigma}_{\alpha+1}^{(\kappa,0)}(X)$. If $A \in \mathbf{\Sigma}_\alpha^{(\kappa,0)}(X)$, then A is trivially the intersection of κ copies of itself therefore $A \in \mathbf{\Pi}_{\alpha+1}^{(\kappa,0)}(X)$. A similar proof shows $\mathbf{\Pi}_\alpha^{(\kappa,0)}(X) \subseteq \mathbf{\Sigma}_{\alpha+1}^{(\kappa,0)}(X)$. □

As in the classical case if X is κ-metrizable, one can show that the hierarchy over X contains all and only κ-Borel sets.

Theorem 3.5.15. *Let* (X, τ) *be a* κ*-metrizable space. Then we have*

$$\mathbf{B}(X) = \bigcup_{\alpha<\kappa^+} \mathbf{\Pi}_\alpha^{(\kappa,0)}(X) = \bigcup_{\alpha<\kappa^+} \mathbf{\Sigma}_\alpha^{(\kappa,0)}(X) = \bigcup_{\alpha<\kappa^+} \mathbf{\Delta}_\alpha^{(\kappa,0)}(X).$$

Proof. Note that

$$\bigcup_{\alpha<\kappa^+} \mathbf{\Pi}_\alpha^{(\kappa,0)}(X) = \bigcup_{\alpha<\kappa^+} \mathbf{\Sigma}_\alpha^{(\kappa,0)}(X) = \bigcup_{\alpha<\kappa^+} \mathbf{\Delta}_\alpha^{(\kappa,0)}(X)$$

follows trivially from what we have just proved. Now we will prove

$$\mathbf{B}(X) = \bigcup_{\alpha<\kappa^+} \mathbf{\Sigma}_\alpha^{(\kappa,0)}(X).$$

First note that $\bigcup_{\alpha<\kappa^+} \mathbf{\Sigma}_\alpha^{(\kappa,0)}(X)$ is closed under unions of size κ and complementation. Let $\{A_\alpha\}_{\alpha\in\kappa}$ be a family of elements in $\bigcup_{\alpha<\kappa^+} \mathbf{\Sigma}_\alpha^{(\kappa,0)}(X)$. By regularity of κ^+ there is $\beta < \kappa^+$ such that for all $\alpha < \kappa$ we have $A_\alpha \in \mathbf{\Sigma}_\beta^{(\kappa,0)}(X)$. But then $\bigcup_{\alpha\in\kappa} A_\alpha \in \mathbf{\Pi}_{\alpha+1}^{(\kappa,0)}(X) \subseteq \mathbf{\Sigma}_{\alpha+2}^{(\kappa,0)}(X)$. Then $\bigcup_{\alpha\in\kappa} A_\alpha \in \bigcup_{\alpha<\kappa^+} \mathbf{\Sigma}_\alpha^{(\kappa,0)}(X)$. If $A \in \bigcup_{\alpha<\kappa^+} \mathbf{\Sigma}_\alpha^{(\kappa,0)}(X)$, then there is β such that $A \in \mathbf{\Sigma}_\beta^{(\kappa,0)}(X)$. Therefore, the complement of A is in $\mathbf{\Pi}_\beta^{(\kappa,0)}(X) \subseteq \mathbf{\Sigma}_{\beta+1}^{(\kappa,0)}(X) \subseteq \bigcup_{\alpha<\kappa^+} \mathbf{\Sigma}_\alpha^{(\kappa,0)}(X)$. Then $\mathbf{B}(X) \subseteq \bigcup_{\alpha<\kappa^+} \mathbf{\Sigma}_\alpha^{(\kappa,0)}(X)$.

Now we want to show that $\bigcup_{\alpha<\kappa^+} \mathbf{\Sigma}_\alpha^{(\kappa,0)}(X) \subseteq \mathbf{B}(X)$. We will prove that $\mathbf{\Sigma}_\alpha^{(\kappa,0)}(X) \subset \mathbf{B}(X)$ and $\mathbf{\Pi}_\alpha^{(\kappa,0)}(X) \subset \mathbf{B}(X)$ for every $\alpha \in \kappa^+$. We will proceed by induction over α.

Assume $\alpha = 1$. Then by definition $\mathbf{\Sigma}_\alpha^{(\kappa,0)}(X) \subset \mathbf{B}(X)$ and $\mathbf{\Pi}_\alpha^{(\kappa,0)}(X) \subset \mathbf{B}(X)$.

Assume $\alpha > 1$. Let $A \in \mathbf{\Sigma}_\alpha^{(\kappa,0)}(X)$. Then $A = \bigcup_{\beta\in\kappa} A_\beta$, where for all $\beta \in \kappa$ we have $A_\beta \in \bigcup_{\gamma\in\alpha} \mathbf{\Pi}_\gamma^{(\kappa,0)}(X)$. Then by inductive hypothesis $A_\beta \in \mathbf{B}(X)$ for every $\beta \in \kappa$. Now since $\mathbf{B}(X)$ is closed under union of size κ, $A \in \mathbf{B}(X)$ as desired. □

We will now follow Kechris [17] to prove that the hierarchy over κ^κ does not collapse.

Definition 3.5.16 (Universal Sets)**.** *Let (X,τ) be a κ-metrizable space. Moreover let*

$$\Gamma(X) \in \{\mathbf{\Sigma}_\alpha^{(\kappa,0)}(X), \mathbf{\Pi}_\alpha^{(\kappa,0)}(X) \mid \alpha \in \kappa^+\}$$

*and Y be a set. We will call U a Y-*universal set for $\Gamma(X)$ *if $U \in \Gamma(Y \times X)$ and*

$$\Gamma(X) = \{U_y \mid y \in Y\}$$

where

$$U_y = \{x \mid \langle y, x\rangle \in U\}.$$

Theorem 3.5.17. *Let (X,τ) be a κ-metrizable and κ-separable space. Then for every $\alpha \in \mathrm{On}$, the sets $\mathbf{\Sigma}_\alpha^{(\kappa,0)}(X)$ and $\mathbf{\Pi}_\alpha^{(\kappa,0)}(X)$ have κ^κ-universal sets.*

Proof. We will only prove the theorem for $\mathbf{\Sigma}_\alpha^{(\kappa,0)}(X)$. We will proceed by induction over α.

Assume $\alpha = 1$. Since (X,τ) is κ-separable, it has a base B of cardinality at most κ. Let $(B_\beta)_{\beta\in\kappa}$ be a listing of B. Define

$$\langle y, x\rangle \in U^1 \Leftrightarrow y \in \kappa^\kappa \wedge x \in X \wedge x \in \bigcup_{y(\alpha)=1} B_\alpha.$$

Therefore U^1 is trivially open in $\kappa^\kappa \times X$. Moreover, let $A \in \mathbf{\Sigma}_1^{(\kappa,0)}(X)$. Since A is open, $A = \bigcup_{\beta\in I} B_\beta$ for some $I \subseteq \kappa$. Now define

$$y(\beta) = 1 \Leftrightarrow \beta \in I.$$

We have

$$x \in U_y^1 \Leftrightarrow \langle y, x\rangle \in U^1 \Leftrightarrow x \in \bigcup_{\beta\in I} B_\beta \Leftrightarrow x \in A.$$

Therefore U^1 is κ^κ-universal for $\mathbf{\Sigma}_1^{(\kappa,0)}(X)$. A similar proof works for $\mathbf{\Pi}_1^{(\kappa,0)}(X)$.

Assume $\alpha > 1$. Let $\eta : \kappa \to \alpha$ be a monotone function such that

$$\sup\{\eta(\beta) + 1 \mid \beta \in \kappa\} = \alpha$$

(note that every ordinal $\kappa < \alpha < \kappa^+$ has cardinality κ). For every $y \in \kappa^\kappa$ and $\beta \in \kappa$ define

$$(y)_\beta(\gamma) = y(\lceil \beta, \gamma\rceil)$$

Note that the function $y \mapsto (y)_\beta$ is continuous. Now, let $w \in \kappa^{<\kappa}$. We have

$$(-)_\beta^{-1}[w] = \{p \in \kappa^\kappa \mid \forall\gamma < |w|.\ p(\lceil \beta, w(\gamma)\rceil) = w(\gamma)\}.$$

Take $m = \sup\{\lceil \beta, w(\gamma) \rceil \mid \gamma < |w|\}$. We claim

$$(-)_\beta^{-1}[w] = \bigcup_{p \in P} [p \restriction m]$$

where $P = \{p \in \kappa^\kappa \mid \forall \gamma < |w|.\ p(\lceil \beta, w(\gamma) \rceil) = w(\beta)\}$.

On the one hand assume $p \in (-)_\beta^{-1}[w]$ then $p(\lceil \beta, w(\gamma) \rceil) = w(\gamma)$ for every $\beta < |w|$. Hence $p \in P$ and $p \in \bigcup_{p' \in P} [p' \restriction m]$.

On the other hand, if $p \in \bigcup_{p' \in P} [p' \restriction m]$, then there is $p' \in P$ such that $p \restriction m = p' \restriction m$, hence

$$\forall \gamma < |w|.\ p(\lceil \beta, w(\gamma) \rceil) = w(\gamma),$$

therefore $p' \in (-)_\beta^{-1}[w]$ as desired.

By inductive hypothesis for every $\beta < \alpha$ we have that $\mathbf{\Pi}_\beta^{(\kappa,0)}(X)$ has a κ^κ-universal set U^β. Now define

$$\langle y, x \rangle \in U^\alpha \Leftrightarrow \exists \beta \langle (y)_\beta, x \rangle \in U^{\eta(\beta)}.$$

Since every $(-)_\beta$ is continuous, and every $U^{\eta(\beta)}$ is κ^κ-universal for $\mathbf{\Sigma}_{\eta(\beta)}^{(\kappa,0)}(X)$, then U^α is in $\mathbf{\Sigma}_\alpha^{(\kappa,0)}(X)$. Moreover, let $A \in \mathbf{\Sigma}_\alpha^{(\kappa,0)}(X)$. Then A is a union of at most κ elements in $\bigcup_{\beta<\alpha} \mathbf{\Pi}_\beta^{(\kappa,0)}(X)$, call them $\{A_\beta\}_{\beta \in \kappa}$. Now for every A_β let γ_β be the smallest such that $A_\beta \in \mathbf{\Pi}_{\eta(\gamma_\beta)}^{(\kappa,0)}(X)$ and define

$$y_A(\lceil \beta, \gamma \rceil) = y_\beta(\gamma) \text{ where } U_{y_\beta}^{\gamma_\beta} = A_\beta.$$

Then $U_{y_A}^\alpha = A$. Now we have

$$\begin{aligned} x \in \bigcup_{\beta \in \kappa} A_\beta &\Leftrightarrow \exists \beta \in \kappa.\ x \in A_\beta \\ &\Leftrightarrow x \in U_{y_\beta}^{\gamma_\beta} \\ &\Leftrightarrow \forall \gamma.\ y_A(\lceil \beta, \gamma \rceil) = y_\beta(\gamma) \\ &\Leftrightarrow \langle (y_A)_\beta, x \rangle \in U^{\gamma_\beta} \\ &\Rightarrow \langle y, x \rangle \in U^\alpha. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \langle y, x \rangle \in U^\alpha &\Leftrightarrow \exists \beta.\ \langle (y_A)_\beta, x \rangle \in U^{\gamma_\beta} \\ &\Leftrightarrow x \in U_{(y_A)_\beta}^{\gamma_\beta} \\ &\Rightarrow x \in A_\beta \\ &\Leftrightarrow x \in \bigcup_{\beta \in \kappa} A_\beta. \end{aligned}$$

Hence U^α is κ^κ-universal for $\mathbf{\Sigma}_\alpha^{(\kappa,0)}(X)$ as desired. □

We can finally prove that, as in the classical case the hierarchy does not collapse.

Theorem 3.5.18. *For every $\alpha < \kappa^+$, we have:*

$$\mathbf{\Sigma}_\alpha^{(\kappa,0)}(\kappa^\kappa) \not\subseteq \mathbf{\Pi}_\alpha^{(\kappa,0)}(\kappa^\kappa) \qquad \mathbf{\Pi}_\alpha^{(\kappa,0)}(\kappa^\kappa) \not\subseteq \mathbf{\Sigma}_\alpha^{(\kappa,0)}(\kappa^\kappa).$$

Proof. We will only prove that

$$\mathbf{\Sigma}_\alpha^{(\kappa,0)}(\kappa^\kappa) \not\subseteq \mathbf{\Pi}_\alpha^{(\kappa,0)}(\kappa^\kappa),$$

the fact that

$$\mathbf{\Pi}_\alpha^{(\kappa,0)}(\kappa^\kappa) \not\subseteq \mathbf{\Sigma}_\alpha^{(\kappa,0)}(\kappa^\kappa),$$

can be proved analogously.

Assume $\mathbf{\Sigma}^{(\kappa,0)}_\alpha(\kappa^\kappa) \subseteq \mathbf{\Pi}^{(\kappa,0)}_\alpha(\kappa^\kappa)$. Let U be κ^κ-universal for $\mathbf{\Sigma}^{(\kappa,0)}_\alpha(\kappa^\kappa)$. Define

$$y \in A \Leftrightarrow \langle y, y \rangle \notin U,$$

therefore $A \in \mathbf{\Pi}^{(\kappa,0)}_\alpha(\kappa^\kappa) = \mathbf{\Sigma}^{(\kappa,0)}_\alpha(\kappa^\kappa)$, and there exists $y \in \kappa^\kappa$ such that $U_y = A$. But this is a contradiction since we have $\langle y, y \rangle \in U$ and $y \in A$. □

Note that, now that we have a generalized version of Polish spaces we can ask if, as in the classical case, the Theorem 3.5.18 also holds for κ-Polish spaces of cardinality at least 2^κ. In the classical case this is proved by using the Cantor space.

Definition 3.5.19 (Generalized Cantor space)**.** *We will call* generalized Cantor space *the set* 2^κ *of binary sequences of length* κ *endowed with the topology generated by the base*

$$B = \{[w] \mid w \in 2^{<\kappa}\},$$

where $2^{<\kappa}$ *denotes as usual the set of binary sequences of length less than* κ*.*

First note that the proof of Theorem 3.5.17 also works for generalized Cantor space. Then, by using the same argument of Theorem 3.5.18 we have:

Theorem 3.5.20. *For every* $\alpha < \kappa^+$*, we have:*

$$\mathbf{\Sigma}^{(\kappa,0)}_\alpha(2^\kappa) \not\subseteq \mathbf{\Pi}^{(\kappa,0)}_\alpha(2^\kappa) \qquad \mathbf{\Pi}^{(\kappa,0)}_\alpha(2^\kappa) \not\subseteq \mathbf{\Sigma}^{(\kappa,0)}_\alpha(2^\kappa).$$

From Theorem 3.5.20 we have the following:

Theorem 3.5.21. *Let* (X, τ) *be a* κ*-Polish spaces. If* 2^κ *is a subspace of* X *then, for every* $\alpha < \kappa^+$ *we have*

$$\mathbf{\Sigma}^{(\kappa,0)}_\alpha(X) \not\subseteq \mathbf{\Pi}^{(\kappa,0)}_\alpha(X) \text{ and } \mathbf{\Pi}^{(\kappa,0)}_\alpha(X) \not\subseteq \mathbf{\Sigma}^{(\kappa,0)}_\alpha(X).$$

Proof. We will only prove that

$$\mathbf{\Sigma}^{(\kappa,0)}_\alpha(X) \not\subseteq \mathbf{\Pi}^{(\kappa,0)}_\alpha(X),$$

the fact that

$$\mathbf{\Pi}^{(\kappa,0)}_\alpha(X) \not\subseteq \mathbf{\Sigma}^{(\kappa,0)}_\alpha(X),$$

can be proved analogously.

Assume $\mathbf{\Sigma}^{(\kappa,0)}_\alpha(X) \subseteq \mathbf{\Pi}^{(\kappa,0)}_\alpha(X)$. Then we have:

$$\mathbf{\Sigma}^{(\kappa,0)}_\alpha(2^\kappa) = \{Y \cap 2^\kappa \mid Y \in \mathbf{\Sigma}^{(\kappa,0)}_\alpha(X)\} \subseteq \{Y \cap 2^\kappa \mid Y \in \mathbf{\Pi}^{(\kappa,0)}_\alpha(X)\} = \mathbf{\Pi}^{(\kappa,0)}_\alpha(2^\kappa).$$

But this contradicts Theorem 3.5.20. □

Note that it is still unclear if 2^κ can be embedded in every κ-Polish space of cardinality at least 2^κ. The classical analogous of this fact is usually proved by using a Cantor scheme (see [17, Corollary 6.5]). In particular, in the classical proof given a Polish space X one constructs a binary tree T of height ω with the following properties:

(1) Every node of T is a nonempty open subset of X.

(2) Every node U of T and its closure $\overline{U}$ are a subset of the predecessors of U in T.

(3) Every branch $(U_i)_{i\in\omega}$ of T starting at the root of the tree is such that $\bigcap_{i\in\omega} \overline{U_i}$ is a singleton.

In the generalized case we would have to extend this tree to 2^κ (i.e., the generalized tree should have height κ). While the definitions for base and successor stages of T can essentially be the same as in the classical proof, it is unclear how to define the tree on limit ordinals. In particular, on limit stages nothing guarantees that the intersection of less than κ nested open sets is nonempty. This property is needed to satisfy (2). Therefore it is natural to ask the following question:

Open Question 3.5.22. *Does Theorem 3.5.18 generalize to κ-Polish spaces of cardinality at least 2^κ?*

Our generalizations of metrizable and Polish spaces give rise to many open questions. It would be interesting to study if the standard relationship between Polish spaces and the Baire space holds in the generalized case. In particular:

Open Question 3.5.23. *Is every κ-Polish space the continuous image of the generalized Baire space κ^κ?*

We conclude this section proving that, by using κ-metrics, one can prove a generalized version of the Baire Category Theorem.

Theorem 3.5.24 (κ-Baire Category Theorem)**.** *Let (X, τ) be a complete κ-metric space such that for every family of nested non empty open balls $\{B_\alpha\}_{\alpha\in\beta}$ with $\beta < \kappa$ we have:*

$$\bigcap_{\alpha\in\beta} B_\alpha \neq \emptyset.$$

Then for every family $\{A_\alpha\}_{\alpha\in\kappa}$ of closed subsets of X with empty interior, we have that $\bigcup_{\alpha\in\kappa} A_\alpha$ has empty interior.

Proof. Given a nonempty open subset U_0 of X, we want to find an element $x \in U_0$ such that for every $\alpha \in \kappa$ we have $x \notin A_\alpha$. We will build a sequence of nested open balls $(B_\alpha)_{\alpha\in\kappa}$ such their intersection contains the element we are looking for. We proceed by recursion. For $\alpha = 0$, by hypothesis $U_0 \setminus A_0 \neq \emptyset$. Choose $x \in U_0 \setminus A_0$. Since U_0 is open and the complement A_0^c of A_0 is open, $U_0 \cap A_0^c$ is open. Hence $U_0 \cap A_0^c = \bigcup_{O\in B} O$ with B a set of open balls in X. Let $\mathrm{B}(y, r) \in B$ be such that $x \in \mathrm{B}(y, r)$. Note that, we can choose an open ball $\mathrm{B}(x, r')$ with $r \leq \frac{1}{2}$ such that:

- the closure $\overline{\mathrm{B}(x, r')}$ of $\mathrm{B}(x, r')$ has empty intersection with A_0.
- $\overline{\mathrm{B}(x, r')} \subset \mathrm{B}(x, r')$.

therefore we set $B_0 = \mathrm{B}(x, r')$. In general for $\alpha = \beta + 1$ take the open ball B_β and define B_α such that:

- $\overline{B_\alpha} \cap A_\alpha = \emptyset$.
- $\overline{B_\alpha} \subseteq B_\beta$.
- The radius r of B_α is smaller than $\frac{1}{\alpha+1}$.

Now for the limit case $\alpha = \lambda$, let $U = \bigcap_{\beta\in\alpha} B_\alpha$. We claim U open. By the assumptions over X we know that U is non empty. For every $x \in U$ define:

$$R_x = \{r \mid \mathrm{B}(x, r) \subseteq U\}.$$

Note that since $\mathbb{R}_\kappa$ is an η_κ-set and $\alpha < \kappa$, therefore R_x is not empty (i.e., every lower bound of $(r_\beta)_{\beta\in\alpha}$ where r_β is the radius of B_β is in R_x). Then we have:

$$U = \bigcup_{x\in U} \bigcup_{r\in R_x} \mathrm{B}(x, r).$$

On the one hand if $y \in U$, since R_y is not empty, $y \in \bigcup_{x\in U} \bigcup_{r\in R_x} \mathrm{B}(x, r)$. On the other hand, if $y \in \bigcup_{x\in U} \bigcup_{r\in R_x} \mathrm{B}(x, r)$ then there are $x \in U$ and $r \in R_x$ such that $y \in \mathrm{B}(x, r)$. But by definition $\mathrm{B}(x, r) \subset U$, then $y \in U$ as desired.

Then, by following the same construction we used for the case $\alpha = 0$, we can start from U and define an open ball B_α such that:

- $\overline{B_\alpha} \cap A_\alpha = \emptyset$.
- $\overline{B_\alpha} \subseteq U$.
- The radius r of B_α is smaller than $\frac{1}{\lambda}$.

Now note that $(\overline{B_\alpha})_{\alpha\in\kappa}$ is a sequence of closed balls. Moreover since $\lim_{\alpha\in\kappa}\frac{1}{\alpha}=0$ and for every B_α its radius $r_\alpha<\frac{1}{\alpha}$, therefore $\lim_{\alpha\in\kappa} r_\alpha=0$. Hence the sequence $(c_\alpha)_{\alpha\in\kappa}$, such that c_α is the centre of B_α is a Cauchy sequence. By completeness of X, $\ell=\lim_{\alpha\in\kappa} c_\alpha\in X$. Moreover note that by definition, $\ell\in U_0$, and for every $\alpha\in\kappa$ we have that $\ell\notin A_\alpha$ as desired. □

Finally it is not hard to see that the previous theorem holds for the generalized Baire space κ^κ and for $\mathbb{R}_\kappa$.

Corollary 3.5.25. *For every family $\{A_\alpha\}_{\alpha\in\kappa}$ of closed subsets of κ^κ with empty interior, we have that $\bigcup_{\alpha\in\kappa} A_\alpha$ has empty interior.*

Proof. Since we have already proved that κ^κ is κ-Polish, it is enough to show that for every for every family of nested non empty open balls $\{\mathrm{B}(x_\alpha,r_\alpha)\}_{\alpha\in\beta}$ with $\beta<\kappa$ we have:

$$U=\bigcap_{\alpha\in\beta}\mathrm{B}(x_\alpha,r_\alpha)\neq\emptyset.$$

Without loss of generality we can assume that every r_α is of the type $\frac{1}{\gamma_\alpha}$. Define the following sequence:

$$y_\alpha=x_\alpha\restriction\gamma_\alpha.$$

Note that the sequence $(y_\alpha)_{\alpha\in\beta}$ is monotone. Then $y=\bigcup_{\alpha<\beta} y_\alpha\in\kappa^{<\kappa}$. We claim that $[y]\subseteq U$. If $x\in[y]$, then for every $\alpha<\beta$ we have $x(\alpha)=y(\alpha)$. Hence for every $\alpha<\beta$, $d(x,x_\alpha)<\frac{1}{\gamma_\alpha}$ therefore $x\in\mathrm{B}(x_\alpha,r_\alpha)$ as desired. □

Corollary 3.5.26. *For every family $\{A_\alpha\}_{\alpha\in\kappa}$ of closed subsets of $\mathbb{R}_\kappa$ with empty interior, we have that $\bigcup_{\alpha\in\kappa} A_\alpha$ has empty interior.*

Proof. Since we have already proved that $\mathbb{R}_\kappa$ is κ-Polish, it is enough to show that for every for every family of nested non empty open balls $\{\mathrm{B}(x_\alpha,r_\alpha)\}_{\alpha\in\beta}$ with $\beta<\kappa$ we have

$$U=\bigcap_{\alpha\in\beta}\mathrm{B}(x_\alpha,r_\alpha)\neq\emptyset.$$

Define the following sets:

$$L=\{x_\alpha-r_\alpha\mid\alpha<\beta\}$$

and

$$R=\{x_\alpha+r_\alpha\mid\alpha<\beta\}.$$

Then $|R|+|L|<\kappa$. Moreover by the fact that $\{\mathrm{B}(x_\alpha,r_\alpha)\}_{\alpha\in\beta}$ is a family of nested open balls we have $L<R$. But then, by the fact that $\mathbb{R}_\kappa$ is an η_κ-set, we have that there is $x\in\mathbb{R}_\kappa$ such that $L<\{x\}<R$. Now, since $L<\{x\}<R$ it is easy to see that for every $\alpha\in\beta$ we have that $x\in\mathrm{B}(x_\alpha,r_\alpha)$, therefore $x\in\bigcap_{\alpha\in\beta}\mathrm{B}(x_\alpha,r_\alpha)$ as desired. □

Chapter 4

Generalized Computable Analysis

This chapter is devoted to the study of notions from classical computable analysis in the context of the generalized Baire space κ^κ. Our goal is to study the Weihrauch hierarchy over $\mathbb{R}_\kappa$ and κ^κ. In order to do that, we will first generalize the standard tools of computable analysis. In particular, in Section 4.1 we will prove that, as in the classical case, Wadge strategies characterize continuous functions over κ^κ. As we will see the fact that we can prove a generalized version of the Main Theorem of Computable Analysis will have a central role in defining the representation of continuous functions over $\mathbb{R}_\kappa$. For this reason, in Section 4.2 we will characterize a subclass of topological spaces for which the Main Theorem of Computable Analysis generalizes. In Section 4.3, we will generalize the constructions we presented in Section 2.5.2 and we show how to use Wadge strategies to represent continuous functions over κ^κ. After having introduced these general results, in Section 4.4 we will focus on $\mathbb{R}_\kappa$ and its representation theory, generalizing classical representations of $\mathbb{R}$ and showing that many result from the classical theory hold over these generalizations. Finally we will dedicate the last three sections to the study of the Weihrauch hierarchy. In particular, in Section 4.5 and 4.6 we will study how some of the classical choice principles generalize to $\mathbb{R}_\kappa$ and to κ^κ. Finally in Section 4.7 we will start the study of the Weihrauch degree of IVT_κ.

4.1 Wadge Strategies

In classical computable analysis, Turing machines are used as an intuitive tool for proving computability of functions over Baire space. Moreover, they have a central role in the coding of continuous functions over representable spaces (See Section 2.5.2).

As we have said before, we do not have a notion of computability over κ, therefore focus on the topological properties of κ^κ and $\mathbb{R}_\kappa$. In this context, Wadge strategies will provide a good substitute Turing machines. They will, indeed, both serve as a tool to prove the continuity of functions over κ^κ and as the main ingredient we will use to code the spaces of continuous functions between represented spaces.

Wadge strategies were introduced by Wadge in his PhD thesis [27] and have become one of the fundamental tools of classical computable analysis. On the one hand they are used to classify sets in terms of Wadge degrees, on the other hand they offer an intuitive way of characterizing continuous functions over the Baire space. We will now extend Wadge strategies to generalized Baire space, then we will show that they can be used to characterize continuous functions over κ^κ.

Definition 4.1.1 (Generalized Wadge strategy)**.** *A* generalized Wadge strategy *is a monotone function* $\theta :\subseteq \kappa^{<\kappa} \to \kappa^{<\kappa}$.

As we said we want to use Wadge strategies as an intuitive tool in order to check if functions over generalized Baire space are continuous.

Theorem 4.1.2. *The function* $f :\subseteq \kappa^\kappa \to \kappa^\kappa$ *is continuous over generalized Baire space iff there is* $\theta :\subseteq \kappa^{<\kappa} \to \kappa^{<\kappa}$ *monotone such that for all* $p \in \kappa^\kappa$*:*

$$f(p) = \bigcup_{\alpha\in\kappa} \theta(p\restriction\alpha).$$

Proof. First we prove that every continuous function over κ^κ has a Wadge strategy. Let $f :\subseteq \kappa^\kappa \to \kappa^\kappa$ be a continuous function. Given $w \in \kappa^{<\kappa}$, we define the following set:

$$W_w = \{w' \in \kappa^{<\kappa} \mid [w] \subseteq f^{-1}([w'])\}.$$

Note that for every $w \in \kappa^{<\kappa}$, the set W_w is totally ordered. Now we define $\theta :\subseteq \kappa^{<\kappa} \to \kappa^{<\kappa}$ as follows:

$$\theta(w) = w' \text{ where } w' \text{ is the longest element of } W_w \text{ s.t. } |w'| \leq |w|.$$

First we will prove that θ is monotone. Let $w, w' \in \kappa^{<\kappa}$ such that $w \subseteq w'$. Then $[w] \subseteq f^{-1}[\theta(w)]$. Moreover, since $w \subset w'$, we have $[w'] \subseteq [w]$ and $[w'] \subseteq f^{-1}[\theta(w)]$. Therefore $\theta(w) \in W_w$ and $\theta(w) \subseteq \theta(w')$ as desired.

Now we will prove the following:

$$f(p) = \bigcup_{\alpha<\kappa} \theta(p{\restriction}\alpha).$$

First we prove $f(p) \supseteq \bigcup_{\alpha<\kappa} \theta(p{\restriction}\alpha)$. By the definition of θ we have

$$[p{\restriction}\alpha] \subseteq f^{-1}([\theta(p{\restriction}\alpha]).$$

Hence, since $p \in [p{\restriction}\alpha]$, we have that $f(p) \in [\theta(p{\restriction}\alpha]$. Therefore $\theta(p{\restriction}\alpha) \subseteq f(p)$ for every α and $\bigcup_{\alpha<\kappa} \theta(p{\restriction}\alpha) \subseteq f(p)$ as desired.

Now we want to prove that $f(p) \subseteq \bigcup_{\alpha<\kappa} \theta(p{\restriction}\alpha)$. Let $\alpha < \kappa$. By the continuity of f we have

$$[p{\restriction}\beta] \subseteq f^{-1}([f(p){\restriction}\alpha]),$$

for some $\beta < \kappa$. We can assume $\beta \geq \alpha$, otherwise we could take $\beta' > \beta$ and we would have

$$[p{\restriction}\beta'] \subseteq [p{\restriction}\beta] \subset f^{-1}[f(p){\restriction}\alpha]$$

and therefore $f(p){\restriction}\alpha \in W_{p{\restriction}\beta}$. Now, since $\theta(p{\restriction}\beta)$ is the element of maximum length at most β in $W_{p{\restriction}\beta}$ and $\beta > \alpha$, we have that

$$f(p){\restriction}\alpha \subseteq \theta(p{\restriction}\beta)$$

as desired.

Now we will prove that every Wadge strategy give rise to a continuous function. Let $\theta :\subseteq \kappa^{<\kappa} \to \kappa^{<\kappa}$ be a monotone function and $f :\subseteq \kappa^\kappa \to \kappa^\kappa$ be defined as follows:

$$f(p) = \bigcup_{\alpha\in\kappa} \theta(p{\restriction}\alpha) \text{ for every } p \in \kappa^\kappa.$$

We want to show that f is continuous. Let $w \in \kappa^\kappa$. We will prove:

$$f^{-1}[w] = \bigcup_{w'\subseteq\theta^{-1}(w)} [w'].$$

First we will prove $f^{-1}[w] \subseteq \bigcup_{w'\subseteq\theta^{-1}(w)}[w']$. Let $p \in f^{-1}[w]$. Then we have

$$f(p) = \bigcup_{\alpha\in\kappa} \theta(p{\restriction}\alpha),$$

hence $f(p){\restriction}|w| = (\bigcup_{\alpha\in\kappa} \theta(p{\restriction}\alpha)){\restriction}|w| = w$. Therefore there exists $\alpha \in \kappa$ such that $p{\restriction}\alpha \in \theta^{-1}(w)$. Hence we have $p \in \bigcup_{w'\subset\theta^{-1}(w)}[w']$ as desired.

It remains to prove that $f^{-1}[w] \supseteq \bigcup_{w'\subseteq\theta^{-1}(w)}[w']$. Let $p \in \bigcup_{w'\subset\theta^{-1}(w)}[w']$. Then there exists w' such that $p \in [w']$ and $w' \in \theta^{-1}(w)$. Moreover, since $f(p) = \bigcup_{\alpha\in\kappa} \theta(p{\restriction}\alpha)$, by setting $\alpha = |w'|$ we have

$$\theta(p{\restriction}\alpha) = \theta(w') = w.$$

Hence, by the monotonicity of θ, we have $w \subset f(p)$ and $p \in f^{-1}[w]$ as desired. □

The intuition behind the previous theorem is that continuous functions on generalized Baire space are those whose behaviour over a small (shorter than κ) initial segment of the output is determined only by a small initial segment of the input. Given this, from now on we will feel free to use this result without mentioning it, in particular sentences such as "f is continuous because we are using only a small portion of the input to determine a small portion of the output" will be used to refer to this theorem.

4.2 Computable Analysis Over κ^κ

The main goal of this section is that of proving a generalized version of the Main Theorem of Computable Analysis. First we will generalize many basic notions from classical computable analysis. After having introduced these tools, we will show how to characterize a class of topological spaces for which a generalized version of the the Main Theorem of Computable Analysis holds.

At this point the reader could be surprised to see that instead of using κ-topologies in generalizing computable analysis we will use standard topological tools. To see why recall that any base of the κ-interval topology over $\mathbb{R}_\kappa$ has cardinality 2^κ (see Theorem 3.4.23) and that the κ-topology on κ^κ generated by the classical base has cardinality κ (this is true for any base of cardinality κ). This means that, there will be no representation of $\mathbb{R}_\kappa$ whose induced topology is the κ-interval topology over $\mathbb{R}_\kappa$ (see Section 2.5). As we have seen in Section 2.5, the fact that the topology induced by the representation we are using is the one we want to work with is essential for computable analysis. As we will see, using topologies we will be able to define representations whose induced topology is the interval topology over $\mathbb{R}_\kappa$. Note that, since the κ-interval topology over $\mathbb{R}_\kappa$ is contained in the interval topology, every set of objects we introduced in Section 3.3 (e.g., κ-continuous functions) will be representable.

Before we start our generalization let us fix some useful coding functions and some notations. Let $(w_\alpha)_{\alpha\in\kappa}$ be a sequence of elements of $\kappa^{<\kappa}$. We will use the following notation:

$$[\![w_\alpha]\!]_{\alpha\in\kappa} = w_0^\frown w_1^\frown \dots .$$

Now we fix a bijection $\lceil\cdot,\cdot\rceil : \kappa\times\kappa \to \kappa$ and we generalize the tupling functions we have defined in Section 2.5.

Definition 4.2.1 (Tupling). *Let $a : \alpha \to \kappa$ be a sequence in κ^α with $\alpha < \kappa$. We will write a_β for $a(\beta)$ to reduce the number of parentheses. We define a* wrapping *function ι as follows:*

$$\iota(a) = 11[\![0a_\beta 0]\!]_{\beta\in\alpha}11.$$

Moreover given $w_1, w_2, \dots$ in $\kappa^{<\kappa}$ and $p_1, p_2 \dots,$ in κ^κ, we define:

$$\lceil w_1, p_1\rceil = \lceil p_1, w_1\rceil = \iota(w_1)p_1 \in \kappa^\kappa$$
$$\lceil w_1, \dots, w_\alpha\rceil = \iota(w_1)\dots\iota(w_\alpha) \text{ with } \alpha\in\kappa$$
$$\lceil p_1, \dots, p_\alpha\rceil = p_1(0)\dots p_\alpha(0)p_1(1)\dots p_i(1)\dots \text{ with } \alpha\in\kappa$$

Moreover, let $(p_\alpha)_{\alpha\in\kappa}$ be a sequence of elements in κ^κ and $(w_\alpha)_{\alpha\in\kappa}$ be a sequence of elements in $\kappa^{<\kappa}$. We define

$$\lceil (w_\alpha)_{\alpha\in\kappa}\rceil = [\![\iota(w_\alpha)]\!]_{\alpha\in\kappa}$$

and

$$\lceil (p_\alpha)_{\alpha\in\kappa}\rceil\lceil\alpha,\beta\rceil = p_\alpha(\beta) \text{ for all } \alpha,\beta\in\kappa.$$

Note that the wrapping function ι can now deal with limit lengths. In particular if $a : \alpha \to \kappa$ is a sequence in κ^α with α limit ordinal, then $\iota(a)$ will have length $\alpha + 2$ and will be the sequence $110a_0 0a_1 \dots 0a_\beta 0 \dots 11$.

In classical computable analysis, a notation of a set M is a surjective function from $\omega^{<\omega}$ to M, and a representation of M is a surjective function from ω^ω to M. We can easily generalize these notions to κ^κ.

Definition 4.2.2 (Notations). *Let M be a set and $\nu :\subseteq \kappa^{<\kappa} \to M$ be a surjective function. We will call ν a* notation *of M.*

Definition 4.2.3 (Representation). *Let M be a set and $\delta_M :\subseteq \kappa^\kappa \to M$ be a surjective function. We will call δ a* representation *of M.*

Since, as we said, we want to generalize the the Main Theorem of Computable Analysis it is natural to start generalizing the concept of effective topological space and their standard representation.

Definition 4.2.4 (Effective Topological Space)**.** *An* effective topological space *is a triple* $S = (M, \sigma, \nu)$ *where* M *is a set,* $\sigma \subseteq \mathcal{P}(M)$ *is a family of subsets of* M *of cardinality at most* κ *such that*

$$x = y \Leftrightarrow \{A \in \sigma \mid x \in A\} = \{A \in \sigma \mid y \in A\}$$

and $\nu :\subseteq \kappa^{<\kappa} \to \sigma$ *is a notation on* σ*. We will call* τ_S *the topology generated by taking* σ *as a subbase.*

Definition 4.2.5 (Standard Representation)**.** *Let* $S = (M, \sigma, \nu)$ *be an effective topological space. We define the* standard representation $\delta_S :\subseteq \kappa^\kappa \to M$ *of* M *as follows:*

$$\delta_S(p) = x \Leftrightarrow \{A \in \sigma \mid x \in A\} = \{\nu(w) \mid \iota(w) \lhd p\},$$

where $p \in \mathrm{dom}(\delta_S)$ *and* $\iota(w) \lhd p$ *implies that* $w \in \mathrm{dom}(\nu)$*. We will denote the final topology induced by* δ_S *with* τ_{F_S}*.*

As we have seen in Section 2.5, for effective topological spaces we have that $\tau_S = \tau_{\mathrm{F}_s}$. This fact turns out to be crucial in proving the Main Theorem of Computable Analysis. Unfortunately as we will see, this property does not hold in general for effective topological spaces over κ^κ.

First let us prove that δ_S is continuous and open with respect to the *final topology* over S.

Lemma 4.2.6. *Let* $S = (M, \sigma, \nu)$ *be an effective topological space. Then we have:*

(1) δ_S *is continuous in* τ_{F_S}*.*

(2) δ_S *is open in* τ_{F_S}*.*

Proof. Continuity follows directly by the definition. Now let $\gamma \in \kappa^{<\kappa}$. Note that we can assume γ ends with 11, otherwise it is enough to define $\gamma' = \gamma^\frown 2211$ and we would have $\delta_S[\gamma] = \delta_S[\gamma']$. Moreover if $\delta_S[\gamma] = \emptyset$ then it is trivially open, hence we can assume $\delta_S[\gamma]$ not empty. We claim that

$$\delta_S([\gamma]) = \bigcap\{\nu(w) \mid \iota(w) \lhd \gamma\}.$$

First we will prove $\delta_S([\gamma]) \subseteq \bigcap\{\nu(w) \mid \iota(w) \lhd \gamma\}$. Take $x \in \delta_S([\gamma])$. Then there is $p \in [\gamma]$ such that

$$\{A \in \sigma \mid x \in A\} = \{\nu(w) \mid \iota(w) \lhd p\}.$$

Since $\gamma \subset p$ we have $x \in \bigcap\{\nu(w) \mid \iota(w) \lhd \gamma\}$.

Now we want to prove that $\delta_S([\gamma]) \supseteq \bigcap\{\nu(w) \mid \iota(w) \lhd \gamma\}$. Let $x \in \bigcap\{\nu(w) \mid \iota(w) \lhd \gamma\}$. Define $p = \gamma^\frown\gamma'$ with $\gamma' = [\![\iota(w_i)]\!]_{i \in \kappa}$ such that

$$\forall i < \kappa.\ x \in \nu(w_i) \text{ and } \forall A \in \sigma.\ x \in A \Rightarrow \exists i < \kappa.\ \nu(w_i) = A.$$

Therefore, we have $p \in [\gamma]$ and $\delta_S(p) = x$ as desired.

Now we need to show that $\bigcap\{\nu(w) \mid \iota(w) \lhd \gamma\}$ is open in τ_{F_S}. Let us define the following set:

$$G = \{\gamma' \in \kappa^{<\kappa} \mid \forall \iota(w) \lhd \gamma \exists \iota(w') \lhd \gamma'.\ \nu(w) = \nu(w')\}.$$

We claim that

$$\delta_S^{-1}(\bigcap\{\nu(w) \mid \iota(w) \lhd \gamma\}) = \bigcup_{\gamma' \in G} [\gamma'] \cap \mathrm{dom}(\delta_S).$$

First we will prove the following:

$$\delta_S^{-1}(\bigcap\{\nu(w) \mid \iota(w) \lhd \gamma\}) \subseteq \bigcup_{\gamma' \in G} [\gamma'] \cap \mathrm{dom}(\delta_S).$$

Let $p \in \delta_S^{-1}(\bigcap\{\nu(w) \mid \iota(w) \lhd \gamma\})$. Then, by definition there exists x in $\bigcap\{\nu(w) \mid \iota(w) \lhd \gamma\}$ such that $\delta_S(p) = x$ and

$$\{A \in \sigma \mid x \in A\} = \{\nu(w) \mid \iota(w) \lhd p\}.$$

Since $\delta_S(p) \in \bigcap\{\nu(w) \mid \iota(w) \lhd \gamma\}$, we have

$$\{\nu(w) \mid \iota(w) \lhd \gamma\} \subseteq \{A \in \sigma \mid x \in A\} = \{\nu(w) \mid \iota(w) \lhd p\},$$

which implies

$$\forall \iota(w) \lhd \gamma \exists \iota(w') \lhd p.\ \nu(w) = \nu(w').$$

But then $p \in \bigcup_{\gamma' \in G}[\gamma'] \cap \mathrm{dom}(\delta_S)$.

Finally it remains to prove that

$$\delta_S^{-1}(\bigcap\{\nu(w) \mid \iota(w) \lhd \gamma\}) \supseteq \bigcup_{\gamma' \in G}[\gamma'] \cap \mathrm{dom}(\delta_S).$$

Let $p \in \bigcup_{\gamma' \in G}[\gamma'] \cap \mathrm{dom}(\delta_S)$. Then $p \in \mathrm{dom}(\delta_S)$ and there is $x \in M$ such that $\delta_S(p) = x$, namely

$$\{A \in \sigma \mid x \in A\} = \{\nu(w) \mid \iota(w) \lhd p\}.$$

Now, since $p \in \bigcup_{\gamma' \in G}[\gamma']$ we have

$$\forall \iota(w) \lhd \gamma \exists \iota(w') \lhd p.\ \nu(w) = \nu(w').$$

Hence we have that

$$\{\nu(w) \mid \iota(w) \lhd \gamma\} \subseteq \{\nu(w) \mid \iota(w) \lhd p\} = \{A \in \sigma \mid x \in A\}.$$

Therefore $x \in \bigcap\{\nu(w) \mid \iota(w) \lhd \gamma\}$ as desired.

Now note that, since $\bigcup_{\gamma' \in G}[\gamma']$ is open in κ^κ, we have that $\delta_S^{-1}(\bigcap\{\nu(w) \mid \iota(w) \lhd \gamma\})$ is open in $\mathrm{dom}(\delta_S)$ as desired. □

As we said, in generalized computable analysis the fact that $\tau_S = \tau_{\mathrm{F}_S}$ is not true in general for effective topological spaces. To see this let us give a better characterization of τ_{F_S}.

Lemma 4.2.7. *Let $S = (M, \sigma, \nu)$ be an effective topological space. Then τ_{F_S} contains τ_S and is closed under intersections of less than κ elements of σ.*

Proof. First we want to show that $\tau_{\mathrm{F}_S} \subseteq \tau_S$. Note that it is enough to show that δ_S is continuous w.r.t. τ_S. For every $X \in \tau_S$ we have

$$\delta_S^{-1}(X) = \{p \in \mathrm{dom}(\delta_S) \mid \iota(w) \lhd p \text{ for some } w \text{ with } \nu(w) \subseteq X\}.$$

Therefore, $\delta_S^{-1}(X)$ is trivially open in $\mathrm{dom}(\delta_S)$ and δ_S is continuous w.r.t. τ_S.

Now we want to prove that τ_{F_S} is closed under intersection of less than κ elements of σ. Let $A \subset \sigma$ such that $|A| < \kappa$. Let $\gamma \in \kappa^{<\kappa}$ be defined as follows

$$\gamma = [\![\iota(w_a)]\!]_{a \in A}.$$

where for all $a \in A$, $\nu(w_a) = a$. Then, as we proved in the previous lemma, we have:

$$\delta_S([\gamma]) = \bigcap\{\nu(w) \mid \iota(w) \lhd \gamma\},$$

and by the fact that δ_S is open with respect to τ_{F_S}, we have that

$$\bigcap\{\nu(w) \mid \iota(w) \lhd \gamma\}$$

is open in τ_{F_S} as desired. □

It is not hard to see that there are many effective topological spaces for which τ_S does not have the necessary closure property to be the final topology induced by their standard representations. Let us illustrate this fact by simple example:

Consider the standard order topology over the ordinal κ. Since $\kappa > \omega$, the interval topology τ_S over κ is not the discrete topology. By contradiction, assume τ_S to be the discrete topology. Then for every ordinal

$\alpha < \kappa$, the set $\{\alpha\}$ is open. Hence, there exists a subset B of intervals with end points in κ such that $\{\alpha\} = \bigcup_{(\beta,\beta')\in B}(\beta, \beta')$ and there exists an interval $(\beta, \beta') \in B$ such that $(\beta, \beta') = \{\alpha\}$.

Consider the case in which α is a limit ordinal. By the fact that $\alpha \in (\beta, \beta')$ we have $\beta < \alpha < \beta'$. Since α is a limit ordinal, $\beta < \beta + 1 < \alpha < \beta$ and $\beta + 1 \in (\beta, \beta')$.

Now let σ be the set of open intervals in κ, note that σ has cardinality κ. Then there is a notation $\nu :\subseteq \kappa^{<\kappa} \to \sigma$ over σ. Consider the effective topological space $S = (\kappa, \sigma, \nu)$. We want to show that τ_{F_S} is the discrete topology. By the previous theorem we have that the intersections of less than κ open intervals in κ is open in the final topology. Moreover for every $\alpha < \kappa$ we have that

$$\{\alpha\} = \bigcap_{\beta\in\alpha}(\beta, \alpha + 1).$$

Hence for every element α of κ we have $\{\alpha\} \in \tau_{\mathrm{F}_S}$ as desired. In conclusion since we proved that τ_S is not the discrete topology, we have $\tau_S \neq \tau_{\mathrm{F}_S}$.

We will now characterize a subclass of topological spaces for which this $\tau_S = \tau_{\mathrm{F}_S}$.

Definition 4.2.8 (κ-effective Space)**.** *Let (M, τ) be a topological space. Then it is* κ-effective *w.r.t. σ iff σ is a subbase of τ of cardinality at most κ and τ is closed under intersections of strictly less than κ elements of σ.*

We will say that (M, τ) is κ-effective *if there is σ such that (M, τ) is κ-effective w.r.t. σ.*

If (M, τ) is κ-effective w.r.t. σ and ν is a notation of σ, we will call $S = (M, \sigma, \nu)$ a κ-effective space*. Note that in this case $\tau = \tau_S$.*

Now, recall form Section 2.5, that reductions can be used to characterize those representation who are particularly well-behaved. We will now follow this intuition to characterize a class of represented spaces on which we can prove a generalized version of the Main Theorem of Computable Analysis.

We will start generalizing the definition of continuous reduction to κ^κ:

Definition 4.2.9 (Reductions)**.** *Let $\delta :\subseteq \kappa^\kappa \to M$ and $\delta' :\subseteq \kappa^\kappa \to M$ be two representations of M. Then we will say that δ* continuously reduces *to δ', in symbols $\delta \leq_{\mathrm{t}} \delta'$ iff there is a continuous function $h :\subseteq \kappa^\kappa \to \kappa^\kappa$ such that for every $x \in \mathrm{dom}(f)$, $\delta(x) = \delta'(h(x))$.*

If $\delta \leq_{\mathrm{t}} \delta'$ and $\delta' \leq_{\mathrm{t}} \delta$ we will say that δ and δ' are continuously equivalent *and we will write $\delta \equiv_{\mathrm{t}} \delta'$.*

Now, following the classical proof we have:

Lemma 4.2.10. *Let M be a set, $\delta_0 :\subseteq \kappa^\kappa \to M$ and $\delta_1 :\subseteq \kappa^\kappa \to M$ be two representations;oreover, let τ_0 and τ_1 be respectively the final topology induced by δ_0 and δ_1. Then $\delta_0 \leq_{\mathrm{t}} \delta_1$ implies $\tau_1 \subseteq \tau_0$. Moreover, given $\delta'_0 :\subseteq \kappa^\kappa \to M$ and $\delta'_1 :\subseteq \kappa^\kappa \to M$ be other two representations of M, such that $\delta'_0 \leq_{\mathrm{t}} \delta_0$ and $\delta'_1 \leq_{\mathrm{t}} \delta_1$. Then every (δ_0, δ_1)-continuous function is (δ_0, δ_1)-continuous.*

Proof. See the proof of Lemma 2.5.6. ☐

For κ-effective topologies the standard theory applies. In particular we have that $\tau_S = \tau_{\mathrm{F}_S}$.

Lemma 4.2.11. *Let $S = (M, \sigma, \nu)$ be an effective topological space. Then we have:*

(1) if (M, τ_S) is κ-effective w.r.t. σ, then $\tau_S = \tau_{\mathrm{F}_S}$.

(2) $\xi \leq_{\mathrm{t}} \delta_S$ for all τ_{F_S}-continuous functions $\xi :\subseteq \kappa^\kappa \to M$.

(3) for every topological space (M', τ') and function $H :\subseteq M \to M'$ such that $H \circ \delta_S$ is τ'-continuous we have that H is $(\tau_{\mathrm{F}_S}, \tau')$-continuous.

Proof. (1) By Lemma 4.2.6 and Lemma 4.2.7, we have $\tau_S \subseteq \tau_{\mathrm{F}_S}$ and δ_S continuous w.r.t. τ_{F_S}. Hence, it is enough to show that δ_S is open in τ_S. Let $\gamma \in \kappa^{<\kappa}$, as before we can assume γ ends with 11. Therefore we have that

$$\delta_S([\gamma]) = \bigcap\{\nu(w) \mid \iota(w) \lhd \gamma\},$$

and since τ_S is κ-effective w.r.t. σ, it is closed under intersections of less than κ elements of σ. Therefore $\bigcap\{\nu(w) \mid \iota(w) \lhd \gamma\}$ is open in τ_S and δ_S is open in τ_S.

(2) Let $\xi :\subseteq \kappa^\kappa \to M$ be continuous in τ_{F_S}. Then for every $p \in \kappa^{<\kappa}$ and $X \in \sigma$ there is $\alpha < \kappa$ such that $\xi([p\restriction\alpha]) \subseteq X$ iff $\xi(p) \in X$. Now let $(w_i)_{i\in\kappa}$ be an enumeration of σ. For $w \in \kappa^{<\kappa}$, we define the following Wadge strategy:

$$\theta(w)(\alpha) = \begin{cases} \iota(w_\alpha) & \text{if } \xi([w]) \subseteq \nu(w_\alpha), \\ 2 & \text{otherwise.} \end{cases}$$

Define $f(p)(\alpha)(\beta) = (\bigcup_{\alpha\in\kappa} h(p\restriction\alpha))(\gamma)(\beta)$ where γ is the smallest such that:

$$|(\sup_{\alpha\in\kappa} h(p\restriction\alpha))(\gamma)| > \beta.$$

Since θ is monotone then f is well defined and continuous (note that the function $g : \beta \mapsto \gamma$ is trivially continuous). The fact that f translates ξ to δ_S follows by the definition, indeed, $f(p)$ is the list of $w_i \in \sigma$ such that $\xi(p) \in \nu(w_i)$. Finally, since every small portion of the output of the function only depends on a small portion of the input, f is continuous as desired.

(3) Let $T \in \tau'$. Then $(H \circ \delta_S)^{-1}(T)$ is open in $\mathrm{dom}(H \circ \delta_S)$. Therefore $\delta_S^{-1}(H^{-1}(T)) = V \cap \mathrm{dom}(H \circ \delta_S)$ for some open set $V \subseteq \kappa^\kappa$. Hence $H^{-1}(T) = \delta_S(V \cap \mathrm{dom}(H \circ \delta_S)) = \delta_S[V \cap \delta_S^{-1}(\mathrm{dom}(H))] = \delta_S(V) \cap \mathrm{dom}(H)$. Now since δ_S is an open map therefore $\delta_S(V)$ is open in τ_{F_S} and $H^{-1}(T)$ is open in $\mathrm{dom}(H)$. Hence H is continuous. □

Hence every κ-effective topological space S has the property that $\tau_S = \tau_{\mathrm{F}_S}$.

Since continuously equivalent representations share the same final topology, every representation which is continuously equivalent to a standard representation of a κ-effective topological space $S = (M, \sigma, \nu)$ has τ_S as final topology. Given this, it is natural to consider the following class of representations:

Definition 4.2.12 (κ-admissible Representation)**.** *Let (M, τ) be a topological space. Then a representation $\delta :\subseteq \kappa^\kappa \to M$ is* κ-admissible *w.r.t. τ iff δ is continuous and every continuous function $\varphi :\subseteq \kappa^\kappa \to M$ is continuously reducible to δ.*

Note that as in the classical case if a representation δ of a topological space (M, τ) is continuously equivalent to a standard representation of a κ-effective topological space $S = (M, \sigma, \nu)$ with $\tau_S = \tau$, then δ is κ-admissible.

Finally we are ready to prove a generalized version of the Main Theorem of Computable Analysis.

Theorem 4.2.13 (Generalized Main Theorem of Computable Analysis)**.** *For $i \in \{0, 1\}$, let (M_i, τ_i) be an effective topological space and $\delta_i :\subseteq \kappa^\kappa \to M_i$ be a set of κ-admissible representation of M_i w.r.t. τ_i. Then for any function $f :\subseteq M_1 \to M_0$ we have:*

$$f \text{ is continuous } \Leftrightarrow f \text{ has a continuous realizer.}$$

Proof. Since $\delta_i \equiv \delta_{S_i}$ for some κ-effective topological space with final topology τ_i, we can prove the theorem on δ_{S_i} instead of δ_i. Let f be continuous. Then by Lemma 4.2.6 (1) $f \circ \delta_{S_1}$ is continuous and by Lemma 4.2.11 (2) $f \circ \delta_1 \leq_{\mathrm{t}} \delta_{S_0}$. Namely, there is a continuous function f_0 such that for every $p \in \mathrm{dom}(f \circ \delta_1)$, we have $f(\delta_1(p)) = \delta_{S_0}(f_0(p))$. Then g is a continuous realizer of f.

On the other hand, let f be a function with a continuous realizer f_0. By definition, for every $p \in \mathrm{dom}(\delta_{S_1})$, $f(\delta_{S_1}(p)) = \delta_{S_0}(f_0(p))$ therefore $f \circ \delta_1$ is continuous. Now by Lemma 4.2.11 (3) we have that f is continuous as desired. □

4.3 Restrictions, Products and Continuous Functions Representations

As we said in the beginning of this chapter our main goal is that of generalize the Weihrauch hierarchy to $\mathbb{R}_\kappa$ and κ^κ. To do that, it is very important to be able to represent different type of spaces. For this reason, in this section will generalize the constructions we introduced in Section 2.5.2.

We start with restrictions. Consider a representation $\delta_M :\subseteq \kappa^\kappa \to M$ of some space M, and $N \subseteq M$. Then we define the restriction of δ_M to N as follows:

$$(\delta_M \upharpoonright N)(p) = \delta_M(p) \text{ iff } p \in \mathrm{dom}(\delta_M) \ \wedge \ \delta_M(p) \in N.$$

Note that $\delta_M \upharpoonright N$ is just the restriction of δ_M to those elements whose image is in N. Now we want to prove that restrictions preserve κ-admissibility.

Lemma 4.3.1. *Let $\delta_M :\subseteq \kappa^\kappa \to M$ be a κ-admissible representation of M, and $N \subseteq M$. Then $\delta_M \upharpoonright N :\subseteq \kappa^\kappa \to N$ is κ-admissible.*

Proof. Since δ_M is κ-admissible w.r.t. τ, there is a κ-effective topological space $S = (M, \sigma, \nu)$ such that $\delta_M \equiv_{\mathrm{t}} \delta_S$ and $\tau = \tau_S$. Now, σ is a subbase for τ_S, therefore $\tau_S^N = \{A \cap N \mid A \in \sigma\}$ is a subbase for the subspace topology over N. Define ν_N as follows:

$$\nu_N(w) = \nu(w) \cap N.$$

Therefore, we have that $S_N = (N, \sigma_N, \nu_N)$ is an effective topological space and τ_{S_N} is the subspace topology over N. It remains to be proved that $\delta_N \equiv_{\mathrm{t}} \delta_M \upharpoonright N$ and S_N is κ-effective. The fact that $\delta_N \equiv_{\mathrm{t}} \delta_M \upharpoonright N$ follows directly by unravelling the definitions. In fact, for all $p \in \mathrm{dom}(\delta_S \upharpoonright N)$ we have

$$(\delta_S \upharpoonright N)(p) = x \Leftrightarrow \{A \in \sigma \mid x \in A\} = \{\nu(w) \mid \iota(w) \triangleleft p\}$$

and since $x \in N$

$$\begin{aligned}\{A \in \sigma \mid x \in A\} = \{\nu(w) \mid \iota(w) \triangleleft p\} &\Leftrightarrow \{A \cap N \mid x \in A \ \wedge \ A \in \sigma\} = \{\nu(w) \mid \iota(w) \cap N \triangleleft p\} \\ &\Leftrightarrow \{A \in \sigma_N \mid x \in A\} = \{\nu_N(w) \mid \iota(w) \cap N \triangleleft p\} \\ &\Leftrightarrow \delta_{S_N}(p) = x.\end{aligned}$$

The same proof works for $p \in \mathrm{dom}(\delta_{S_N})$, therefore $\delta_S \upharpoonright N = \delta_{S_N}$. Finally, since $\delta_M \equiv_{\mathrm{t}} \delta_S$, we have $\delta_M \upharpoonright N \equiv_{\mathrm{t}} \delta_S \upharpoonright N = \delta_{S_N}$.

Now we want to prove that S_N is κ-effective. We have to prove that given a family $\{A_\alpha\}_{\alpha \in \beta}$ of elements of σ_N with $\beta < \kappa$, we have that $\bigcap_{\alpha \in \beta} A_\alpha$ is open in τ_N. We have

$$\{A_\alpha\}_{\alpha \in \beta} = \bigcap_{\alpha \in \beta} (A'_\alpha \cap N),$$

for some family $\{A'_\alpha\}_{\alpha \in \beta}$ of elements in σ. But then:

$$\bigcap_{\alpha \in \beta} (A'_\alpha \cap N) = (\bigcap_{\alpha \in \beta} A'_\alpha) \cap N$$

and since $\tau = \tau_S$ was κ-effective, $\bigcap_{\alpha \in \beta} A'_\alpha$ is open in τ_S. Hence we have that

$$(\bigcap_{\alpha \in \beta} A'_\alpha) \cap N$$

is open in τ_{S_N} as desired. □

This proof shows that the topology induced by the restriction of a representation to a subset N of M is the subspace topology over N. Therefore, we can represent subspaces of κ-effective topological spaces just by restricting the standard representation.

Products of spaces will appear very often in the study of generalized computable analysis over $\mathbb{R}_\kappa$. It is natural then to ask if the product of κ-admissible representations is still κ-admissible. First of all let us define what a product of representations is.

Definition 4.3.2 (Product Representation). *Let $(\delta_i)_{i\in\alpha}$, with $\alpha \le \kappa$ be a sequence of representations for the spaces $(M_i)_{i\in\alpha}$. Then we define the* product *of $(\delta_i)_{i\in\alpha}$ as follows:*

$$(\bigotimes_{i\in\alpha}(\delta_i))\lceil p_0, p_1, \dots\rceil = (\delta_i(p_i))_{i\in\alpha}$$

for every $(p_i)_{i\in\alpha}$.

Note that, since every δ_i is surjective, $\bigotimes_{i\in\alpha}(\delta_i)$ is trivially a representation of $\prod_{i\in\alpha} M_i$.

While admissibility in standard computable analysis was preserved by arbitrary products, κ-admissibility may not be.

Lemma 4.3.3. *Let $(\delta_i)_{i\in n}$ be finitely many κ-admissible representations. Then $\bigotimes_{i\in n}(\delta_i)$ is κ-admissible.*

Proof. Since every $(\delta_i)_{i\in n}$ is κ-admissible w.r.t some τ_i, there are

$$S_i = (M_i, \sigma_i, \nu_i)$$

κ-effective topological spaces such that $\tau_{S_i} = \tau_i$ and $\delta_{S_i} \equiv_{\mathrm{t}} \delta_i$.

Now, define $S = (\prod_{i\in\alpha} M_i, \sigma, \nu)$ as follows:

$$\sigma = \{\prod_{i\in n} A_i \mid (A_i)_{i\in n} \in \prod_{i\in n}\sigma_i \text{ s.t. there is only one } i\in n \text{ s.t. } A_i \neq M_i\ \},$$

$$\nu(0^i 1w) = \prod_{j\in n} \nu_j(w_j) \text{ where } w_j = \begin{cases} w' & \text{With } \nu_j(w') = M_j \text{ if } j \neq i, \\ w & \text{if } j = i. \end{cases}$$

Trivially σ is a subbase of the product topology $\prod_{i\in n}\tau_i$. On the one hand, every set of the type $\prod_{i\in n} A_i$ in the standard base of the product topology is such that every A_i is a basic open in τ_i. Then $\prod_{i\in n} A_i$ is trivially the finite intersection of elements of σ. On the other hand if $A = \bigcup_{i\in m} A_i$ with $A_i \in \sigma$ then $A = \bigcup_{i\in m}\prod_{j\in n} A'_{i,j}$ where every $A'_{i,j} \in \sigma_j$. Now, we have that

$$A = \bigcup_{i\in m}\prod_{j\in n} A'_{i,j} = \prod_{j\in n}\bigcup_{i\in m} A'_{i,j}.$$

Moreover, for every $j < n$, we have that $\bigcup_{i\in m} A'_{i,j}$ is open in τ_j. Then A is open in the product topology. In conclusion τ_S is the product topology $\prod_{i\in n}\tau_i$.

Now we want to prove $\bigotimes_{i\in n}(\delta_i) \equiv_{\mathrm{t}} \delta_S$. Note that the two representations are listing the same properties.

First will see that $\delta_S \le_{\mathrm{t}} \bigotimes_{i\in n}(\delta_i)$. Let $\gamma \in \kappa^{<\kappa}$. Without loss of generality we can assume that γ is already coding a tuple $\langle \gamma_0, \dots, \gamma_m\rangle$ with $m \le n$ and $\gamma_i \in \kappa^{<\kappa}$ for all $i \le m$. We define W_i^γ as follows:

$$W_i^\gamma = \{w \mid \iota(w) \triangleleft \gamma_i\}.$$

Now, define the following Wadge strategy:

$$\theta(\gamma) = [\![\iota(0^{i_j}1w_j)]\!]_{j\in|\bigcup_{i\in m} W_i^\gamma|},$$

where $(w_j)_{i\in|\bigcup_{i\in m} W_i^\gamma|}$ is a listing of $\bigcup_{i\in m} W_i^\gamma$ in the same order they appear in γ[1], and every i_j is such that $w_j \in W_{i_j}^\gamma$. Note that since θ is monotonic, the following function is continuous:

$$h(p) = \bigcup_{\alpha\in\kappa}\theta(p\restriction\alpha).$$

Moreover for every $p \in \mathrm{dom}(\bigotimes_{i\in\alpha}(\delta_i))$ we have

$$\bigotimes_{i\in\alpha}(\delta_i)(p) = \delta_S(h(p)).$$

[1] Namely w appears before w' in γ iff the position $\iota(w)(|\iota(w)|-1)$ is smaller than the position of $\iota(w')(|\iota(w')|-1)$ (note that, since $\iota(w)$ and $\iota(w')$ are wrapped, they have successor length).

Indeed, let $p = \ulcorner p_0, \dots, p_n \urcorner \in \mathrm{dom}(\delta_S)$. Then we have

$$\begin{aligned}\delta_S(h(p)) = x &\Leftrightarrow \{A \in \sigma \mid x \in A\} = \{\nu(w) \mid \iota(w) \triangleleft h(p)\} \\ &\Leftrightarrow \{A \in \sigma \mid x \in A\} = \{\nu(0^i 1 w) \mid \iota(w) \triangleleft p_i\} \\ &\Leftrightarrow \forall i \in n.\ \{A \in \sigma_i \mid \pi_i(x) \in A\} = \{\nu_i(w) \mid \iota(w) \triangleleft p_i\} \\ &\Leftrightarrow \forall i \in n.\ \delta_{S_i}(p_i) = \pi_i(x) \Leftrightarrow \bigotimes_{j \in n} \delta_{S_j}(p) = x.\end{aligned}$$

Hence h reduces $\bigotimes_{i \in n}(\delta_i)$ to δ_S.

On the other hand $\bigotimes_{i \in n}(\delta_i) \leq_{\mathrm{t}} \delta_S$. Indeed, let $\gamma \in \kappa^{<\kappa}$. We define

$$W_i^\gamma = \{w \mid \iota(0^i 1 w) \triangleleft \gamma\}$$

and $w_i' = [\![\iota(w_i)]\!]_{i<m}$ to be the concatenation of the elements of W_i^γ in the same order they appear in γ. We define the following Wadge strategy:

$$\theta(\gamma) = [\![w_i']\!]_{i \in m'},$$

where $m' = |\bigcup_{i \in |\gamma|} W_i^\gamma|$. Then the following function is continuous[2]:

$$h(p) = \ulcorner \bigcup_{\alpha \in \kappa} \theta(p \restriction \alpha) \urcorner \text{ for every } p \in \kappa^\kappa.$$

Moreover for every $p \in \mathrm{dom}(\delta_S)$ we claim that:

$$\delta_S(p) = \bigotimes_{i \in n}(\delta_i)(h(p))$$

Indeed, the function h transforms the tabular coding given by using prefixes of the type $0^i 1$ to a tabular coding given by $\ulcorner \cdot, \dots, \cdot \urcorner$. Formally we have:

$$\begin{aligned}\delta_S(p) = x &\Leftrightarrow \{A \in \sigma \mid x \in A\} = \{\nu(w) \mid \iota(w) \triangleleft p\} \\ &\Leftrightarrow \{A \in \sigma \mid x \in A\} = \{\nu(0^i 1 w) \mid \iota(0^i 1 w) \triangleleft p\} \\ &\Leftrightarrow \forall i \in n.\ \{\pi_i(A) \in \sigma \mid x \in A\} = \{\nu_i(w) \mid \iota(w) \triangleleft \pi_i(h(p))\} \\ &\Leftrightarrow \forall i \in n.\ \delta_{S_i}(\pi_i(h(p))) = \pi_i(x) \Leftrightarrow \bigotimes_{j \in n} \delta_{S_j}(h(p)) = x.\end{aligned}$$

Finally we want to prove that $(\prod_{i \in n} M_i, \tau_S)$ is κ-effective w.r.t. σ. Let $(A_\beta)_{\beta \in \alpha}$ with $\alpha < \kappa$ be a sequence in σ. Every A_β is a product $\prod_{j \in n} A'_{\beta,j}$ with $(A'_{\beta,j})_{j \in n} \in \prod_{\beta \in \kappa} \sigma_\beta$. Now, we have that

$$\bigcap_{\beta \in \alpha} A_\beta = \bigcap_{\beta \in \alpha} \prod_{j \in n} A'_{\beta,j} = \prod_{j < n} \bigcap_{\beta \in \alpha} A'_{\beta,j}.$$

But then, by κ-effectiveness of the S_i we have that $\bigcap_{\beta \in \alpha} A'_{\beta,j}$ is open in σ_β. Therefore $\bigcap_{\beta \in \alpha} A_\beta$ is open in the product topology. □

Note that for the last part of the previous proof the finiteness of was essential. Indeed, in the case of infinite products we would not necessarily have that $\prod_{\beta \in \kappa} A_\beta$ with $A_\beta \in \tau_\beta$ is open in $\prod_{\beta \in \kappa} \tau_\beta$.

The set of continuous functions is central in classical analysis. For this reason, it important for us to be able to represent it. As we have seen for κ-effective topological spaces we have that continuous functions are realized by continuous functions over generalized Baire space. For this reason, the problem of representing continuous functions over κ-effective topological spaces reduces to that of representing continuous functions over κ^κ itself.

[2]Note that the tupling functions are trivially continuous.

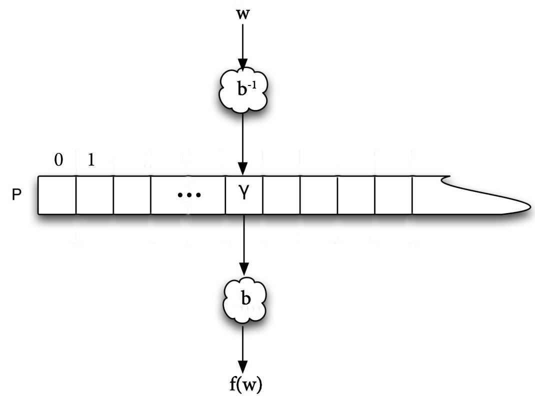


Figure 4.1: The encoding of a Wadge strategy.

First of all note that the set of partial continuous function from κ^κ to κ^κ is of cardinality $2^{(\kappa^\kappa)}$. In fact it is enough to consider all the constant functions and their restrictions. Therefore we can not aim to represent all the continuous functions over generalized Baire space.

For this reason we will restrict to the set of total continuous functions over κ^κ. Note that since generalized Baire space is Hausdorff and has a dense subset of cardinality κ, there are at most 2^κ total continuous functions over κ^κ. Hence a representation for these functions exists.

We will use the fact that every total continuous function over κ^κ can be characterized by using a Wadge strategy. In order to define a representation of the set Ws of Wadge strategies, we fix a bijection $b : \kappa^{<\kappa} \to \kappa$. Note that b exists due to our assumptions over κ. Then we define $\delta_{\mathrm{Ws}} :\subseteq \kappa^\kappa \to \mathrm{Ws}$ as follows:

$$\delta_{\mathrm{Ws}}(p) = \theta \Leftrightarrow \forall w \in \mathrm{dom}(\theta).\ \theta(w) = b(p(b^{-1}(w))).$$

The function δ_{Ws} is trivially surjective. Intuitively a code for a function over $\kappa^{<\kappa}$ is obtained by coding the function f by using b (i.e., the output for f applied to w is in given by decoding via b^{-1} the sequence in position $b(w)$).

We can extend this representation to a encode the set $\mathrm{C}(\kappa^\kappa, \kappa^\kappa)$ of (total) continuous functions over κ^κ. We define the following representation $\delta_{\mathrm{C}(\kappa^\kappa,\kappa^\kappa)} :\subseteq \kappa^\kappa \to \mathrm{C}(\kappa^\kappa, \kappa^\kappa)$:

$$\delta_{\mathrm{C}(\kappa^\kappa,\kappa^\kappa)}(p) = f \Leftrightarrow \forall p' \in \kappa^\kappa.\ f(p') = \bigcup_{\alpha \in \kappa} \delta_{\mathrm{Ws}}(p)(p' \restriction \alpha).$$

Intuitively, $\delta_{\mathrm{C}(\kappa^\kappa,\kappa^\kappa)}(p) = f$ if p is a code of a Wadge strategy for f. Now, following the definition from classical computable analysis, we can use these representations to define a standard representation for the set of continuous functions between two represented spaces.

Definition 4.3.4 (Functions Representations)**.** *Let M_1 and M_2 be two sets and $\delta_{M_1} :\subseteq \kappa^\kappa \to M_1$, $\delta_{M_2} :\subseteq \kappa^\kappa \to M_2$ be two representations respectively for M_1 and M_2. Then the set of $(\delta_{M_1}, \delta_{M_2})$-continuous functions from M_1 to M_2 with domain $A \subseteq M_1$, $\mathrm{C}(A, M_2)$ can be represented as follows:*

$$[\delta_{M_1} \to \delta_{M_2}]_A(p) = f \Leftrightarrow f(\delta_{M_1}(p')) = \delta_{M_2}(\delta_{\mathrm{C}(\kappa^\kappa,\kappa^\kappa)}(p)(p')) \forall p' \in A.$$

Note that while in the classical case the coding for the Wadge strategies was very important to connect computability and continuity, in our case this relationship is completely absent. Moreover, since the existence of a representation of Ws depends on the cardinality of $\kappa^{<\kappa}$, we can not aim for a constructive definition of δ_{Ws}. In particular we can not hope to have a complete constructive definition in ZFC of a representation of Ws. Indeed, since $|\mathrm{Ws}| = \kappa^{<\kappa}$, if we could define a surjective function from κ to Ws then we would have proved $\kappa^{<\kappa} = \kappa$ which is independent from ZFC. In particular this means that any representation of the functions over κ^κ based on the fact that $\kappa = \kappa^{<\kappa}$ will always have a non constructive part. Note that, as we have just seen, the fact that $\kappa^{<\kappa}$ is fundamental in order for the set of continuous functions over κ^κ to be representable.

4.4 Representations for $\mathbb{R}_\kappa$

In this section we will present the generalization to $\mathbb{R}_\kappa$ of some of the most common representations of $\mathbb{R}$. We will focus in particular to those representations which are needed to formalize the IVT_κ and those choice principles which we will consider in the rest of this thesis.

We will start by showing that, by the fact that $\mathbb{Q}_\kappa$ is a dense subset of $\mathbb{R}_\kappa$ follows that $\mathbb{R}_\kappa$ can still be represented by using κ-rational open balls (i.e., open balls in $\mathbb{R}_\kappa$ with radius and center in $\mathbb{Q}_\kappa$).

We have already showed that $\mathbb{Q}_\kappa$ has cardinality κ, hence $|\mathbb{Q}_\kappa \times \mathbb{Q}_\kappa| = \kappa$. This means that there is a bijection from $\kappa^{<\kappa}$ to $\mathbb{Q}_\kappa \times \mathbb{Q}_\kappa$. In this section we will fix one of such bijections $\delta_{\mathbb{Q}_\kappa \times \mathbb{Q}_\kappa} : \kappa^{<\kappa} \to \mathbb{Q}_\kappa \times \mathbb{Q}_\kappa$.

Let B be the set of κ-rational open balls of $\mathbb{R}_\kappa$. We define the following notation $\mathrm{I} : \kappa^{<\kappa} \to \mathrm{B}$:

$$\mathrm{I}(w) = \mathrm{B}(q_1, q_2) \qquad \text{iff } \delta^{-1}_{\mathbb{Q}_\kappa \times \mathbb{Q}_\kappa}(q_1, q_2) = w.$$

Then $S_{\mathbb{R}_\kappa} = (\mathbb{R}_\kappa, \mathrm{B}, \mathrm{I})$ is an effective topological space. By following the definition of standard representation we have:

$$\delta_{\mathbb{R}_\kappa}(p) = x \text{ iff } \{A \in \mathrm{B} \mid x \in A\} = \{\mathrm{I}(w) \mid \iota(w) \triangleleft p\}.$$

Intuitively every element of $\mathbb{R}_\kappa$ is represented by *all* the κ-rational open balls in which it is contained. Note that in view of the fact that we have proved that intervals and open balls are the same in $\mathbb{R}_\kappa$, we have that p can also be thought to be the list of *all* the open intervals with κ-rational end points containing x.

Lemma 4.4.1. *Let $\tau_{\mathbb{R}_\kappa}$ be the standard topology induced by $S_{\mathbb{R}_\kappa}$. Then $\tau_{\mathbb{R}_\kappa}$ is the interval topology over $\mathbb{R}_\kappa$.*

Proof. Follows trivially by Theorem 3.5.3. □

Now we will prove that the interval topology over $\mathbb{R}_\kappa$ is κ-effective w.r.t. the base of κ-rational open balls. Before we can prove this we need some preliminary work. First of all, since $\mathbb{R}_\kappa$ is η_κ-set, we have that intersections of less than κ many κ-rational open balls are either empty or of the same cardinality as $\mathbb{R}_\kappa$.

Lemma 4.4.2. *Let $\alpha < \kappa$ be an ordinal and $\{\mathrm{B}(q_\beta, q'_\beta)\}_{\beta\in\alpha}$ be a family of open balls of $\mathbb{R}_\kappa$ with radius and center in $\mathbb{Q}_\kappa$. Then $\bigcap_{\beta\in\alpha} \mathrm{B}(q_\beta, q'_\beta)$ is either empty or of cardinality 2^κ.*

Proof. Assume $\bigcap_{\beta\in\alpha} \mathrm{B}(q_\beta, q'_\beta)$ is not the empty set. Then there is $x \in \mathbb{R}_\kappa$ such that

$$\forall \beta \in \alpha.\ q_\beta - q'_\beta < x < q_\beta + q'_\beta$$

Let us define the following sets:

$$Q = \{q_\beta - q'_\beta \mid \beta \in \alpha\}$$

and

$$Q' = \{q_\beta + q'_\beta \mid \beta \in \alpha\}.$$

Therefore we can define the following surreal numbers:

$$r = [Q \cup \{x\} \mid Q']$$

and

$$r' = [Q \mid Q' \cup \{x\}].$$

Note that since $\mathbb{R}_\kappa$ is an η_κ-set, r' and r are in $\mathbb{R}_\kappa$. Moreover $r < r'$ and $(r, r') \subset \bigcap_{\beta\in\alpha} \mathrm{B}(q_\beta, q'_\beta)$. Now, by Lemma 3.4.20, open intervals in $\mathbb{R}_\kappa$ have cardinality 2^κ therefore $2^\kappa \le |\bigcap_{\beta\in\alpha} \mathrm{B}(q_\beta, q'_\beta)| \le 2^\kappa$ as desired. □

In particular the previous lemma shows that intersections of less than κ open balls cannot be a singleton.

Lemma 4.4.3. *Let X be a subset of $\mathbb{R}_\kappa$ with cardinality less than κ be such that X has no maximum (minimum) then $\sup X \notin \mathbb{R}_\kappa$ ($\inf X \notin \mathbb{R}_\kappa$).*

Proof. Assume $\sup X = r \in \mathbb{R}_\kappa$. By the fact that X has no maximum, we have $r \notin X$. Then, since $\mathbb{R}_\kappa$ is an η_κ-set we have $r' = [X \mid r] \in \mathbb{R}_\kappa$. Moreover by the definition r' is an upper bound of X and $r' < r$. □

We are now ready to prove that the interval topology over $\mathbb{R}_\kappa$ is the same as the final topology induced by $\delta_{\mathbb{R}_\kappa}$.

Theorem 4.4.4. *The real closed field $\mathbb{R}_\kappa$ endowed with the interval topology is κ-effective w.r.t. the base of κ-rational open balls.*

Proof. It is enough to show that the interval topology is closed under intersection of less than κ open balls with κ-rational radius and center. Let $\alpha < \kappa$ be an ordinal and $\{\mathrm{B}(q_\beta, q'_\beta)\}_{\beta\in\alpha}$ be a family of κ-rational open balls of $\mathbb{R}_\kappa$. If $\bigcap_{\beta\in\alpha} \mathrm{B}(q_\beta, q'_\beta)$ is empty then it is open. Assume $\bigcap_{\beta\in\alpha} \mathrm{B}(q_\beta, q'_\beta)$ not empty. Define

$$Q = \{q_\beta - q'_\beta \mid \beta \in \alpha\}$$

and

$$Q' = \{q_\beta + q'_\beta \mid \beta \in \alpha\}.$$

Note that since $\bigcap_{\beta\in\alpha} \mathrm{B}(q_\beta, q'_\beta) \neq \emptyset$, we have $Q < Q'$. By Lemma 4.4.2, $\bigcap_{\beta\in\alpha} \mathrm{B}(q_\beta, q'_\beta)$ is not a singleton. Without loss of generality we can assume Q has no minimum and Q' has no maximum, a similar argument works for the other cases. Moreover, $\sup Q, \inf Q' \notin \mathbb{R}_\kappa$ by Lemma 4.4.3. Let us define the following set:

$$R = \{(r, r') \subset \mathbb{R}_\kappa \mid \{r\} > Q \wedge \{r'\} < Q' \wedge r < r\}.$$

We claim:

$$\bigcap_{\beta\in\alpha} \mathrm{B}(q_\beta, q'_\beta) = \bigcup R.$$

The right to left inclusion is trivial.

Let $x \in \bigcap_{\beta\in\alpha} \mathrm{B}(q_\beta, q'_\beta)$. Then $Q < \{x\} < Q'$. Define

$$r = [Q \cup \{x\} \mid Q']$$

and

$$r' = [Q \mid Q' \cup \{x\}].$$

Since $\mathbb{R}_\kappa$ is an η_κ-set, r' and r are in $\mathbb{R}_\kappa$. Moreover $r < r'$ hence $(r, r') \in R$ and $x \in (r, r')$ which implies $x \in \bigcup R$ as desired. $\square$

Even if the interval representation is really intuitive, many times we need a representation which is more practical. As is done in the classical case we will prove that the representation of $\mathbb{R}_\kappa$ given by Cauchy sequences is continuously equivalent to the open balls representation that we have just defined.

Definition 4.4.5 (Fast Convergent Cauchy Sequences)**.** *Let $(r_\alpha)_{\alpha\in\kappa}$ be a Cauchy sequence such that:*

$$\forall\alpha \in \kappa \forall\beta \in \kappa.\ \alpha \in \beta \Leftrightarrow |r_\alpha - r_\beta| \leq \frac{1}{\alpha + 1}.$$

Then we will call $(r_\alpha)_{\alpha\in\kappa}$ a fast convergent Cauchy sequence*.*

Let us define the following representation:

$$\delta_{\mathrm{C}}(p) = x \Leftrightarrow \begin{cases} p = [\![\iota(w_\alpha)]\!]_{\alpha\in\kappa} \text{ where } w_1, w_2, \ldots \in \mathrm{dom}(\nu_{\mathbb{Q}_\kappa}), \\ (\nu_{\mathbb{Q}_\kappa}(w_\alpha))_{\alpha\in\kappa} \text{ fast convergent Cauchy sequence with} \\ \lim_{\alpha\to\kappa} \nu_{\mathbb{Q}_\kappa}(w_\alpha) = x. \end{cases}$$

First of all we want to show that δ_{C} is surjective. By definition every $x \in \mathbb{R}_\kappa$ is a limit of a Cauchy sequence of length κ. We only need to show that given a Cauchy sequence we can extract a sequence which converges with the desired rate. Let $(r_\alpha)_{\alpha\in\kappa}$ be a Cauchy sequence with limit x. Without loss of generality we can assume $(r_\alpha)_{\alpha\in\kappa}$ strictly monotonic (otherwise we can take either $L = \{r_\alpha \mid r_\alpha < \lim_{\beta\in\kappa} r_\beta\}$ or $R = \{r_\alpha \mid r_\alpha > \lim_{\beta\in\kappa} r_\beta\}$). Let us define the following sequence:

$$r'_\alpha = r_\beta \text{ with } \beta \text{ is the least such that } |r_\beta - x| < \frac{1}{\alpha + 1}.$$

Note that by the fact that $\lim_{\alpha\to\kappa} r_\alpha = x$ the sequence is well defined. Since the starting sequence was strictly monotonic we have that $(r'_\alpha)_{\alpha\in\kappa}$ has the desired property.

As in in classical computable analysis we have that $\delta_{\mathbb{R}_\kappa} \equiv_{\mathrm{t}} \delta_{\mathrm{C}}$.

Theorem 4.4.6. *The open balls representation $\delta_{\mathbb{R}_\kappa} :\subseteq \kappa^\kappa \to \mathbb{R}_\kappa$ and the Cauchy representation $\delta_{\mathrm{C}} :\subseteq \kappa^\kappa \to \mathbb{R}_\kappa$ are continuously equivalent.*

Proof. We will first show that $\delta_{\mathrm{C}} \leq_{\mathrm{t}} \delta_{\mathbb{R}_\kappa}$. We want to find a continuous map $f :\subseteq \kappa^\kappa \to \kappa^\kappa$ such that:

$$\forall p \in \mathrm{dom}(\delta_{\mathrm{C}}).\ \delta_{\mathbb{R}_\kappa}(f(p)) = \delta_{\mathrm{C}}(p).$$

Let $p \in \kappa^\kappa$ be such that $p = [\![\iota(w_\alpha)]\!]_{\alpha\in\kappa}$ and $(\nu_{\mathbb{Q}_\kappa}(w_\alpha))_{\alpha\in\kappa}$ is a fast convergent Cauchy sequence converging at x. Take $(w'_\alpha)_{\alpha\in\kappa}$ be a listing of $\kappa^{<\kappa}$. We define a function $h : \kappa \to k^{<\kappa}$ as follows:

$$h^p(\lceil \alpha, \beta \rceil) = \begin{cases} \iota(w'_\alpha) & \text{If } w'_\alpha \in \mathrm{dom}(I) \wedge [\nu_{\mathbb{Q}_\kappa}(w_\beta) - \frac{1}{\beta}, \nu_{\mathbb{Q}_\kappa}(w_\beta) + \frac{1}{\beta}] \subset \mathrm{I}(w'_\alpha), \\ \langle\rangle & \text{otherwise.} \end{cases}$$

Then we define the function $f :\subseteq \kappa^\kappa \to \kappa^\kappa$ reducing δ_{C} to $\delta_{\mathbb{R}_\kappa}$ as follows:

$$f(p) = [\![h^p(w'_\alpha)]\!]_{\alpha\in\kappa}.$$

Note that since every portion of length less than κ of the output of f uses only a portion of length less than κ of the input, f is continuous.

Now since $\lim_{\alpha\to\kappa} \nu_{\mathbb{Q}_\kappa}(w_\alpha) = x$, for every $w \in \mathrm{dom}(\mathrm{I})$ we have

$$x \in \mathrm{I}(w) \Leftrightarrow \exists \alpha < \kappa.\ [\nu_{\mathbb{Q}_\kappa}(w_\alpha) - \frac{1}{\alpha}, \nu_{\mathbb{Q}_\kappa}(w_\alpha) + \frac{1}{\alpha}] \subseteq \mathrm{I}(w).$$

Assume $x \in \mathrm{I}(w)$ and

$$\forall \alpha > \kappa.\ [\nu_{\mathbb{Q}_\kappa}(w_\alpha) - \frac{1}{\alpha}, \nu_{\mathbb{Q}_\kappa}(w_\alpha) + \frac{1}{\alpha}] \not\subseteq \mathrm{I}(w).$$

Then we have

$$\forall \alpha > \kappa.\ |\nu_{\mathbb{Q}_\kappa}(w_\alpha) - x| > \frac{1}{\alpha}.$$

But then $(\nu_{\mathbb{Q}_\kappa}(w_\alpha))_{\alpha\in\kappa}$ does not converges at x. This is a contradiction.

On the other hand assume

$$\exists \alpha < \kappa.\ [\nu_{\mathbb{Q}_\kappa}(w_\alpha) - \frac{1}{\alpha}, \nu_{\mathbb{Q}_\kappa}(w_\alpha) + \frac{1}{\alpha}] \subseteq \mathrm{I}(w).$$

Let α_{m} be such an element. Since $(\nu_{\mathbb{Q}_\kappa}(w_\alpha))_{\alpha\in\kappa}$ is fast convergent we have

$$\forall \alpha > \alpha_{\mathrm{m}}.\ \nu_{\mathbb{Q}_\kappa}(w_\alpha) \in [\nu_{\mathbb{Q}_\kappa}(w_{\alpha_{\mathrm{m}}}) - \frac{1}{\alpha_{\mathrm{m}}}, \nu_{\mathbb{Q}_\kappa}(w_{\alpha_{\mathrm{m}}}) + \frac{1}{\alpha_{\mathrm{m}}}].$$

Moreover, since x is the limit of $(\nu_{\mathbb{Q}_\kappa}(w_\alpha))_{\alpha\in\kappa}$, we have:

$$x \in [\nu_{\mathbb{Q}_\kappa}(w_{\alpha_{\mathrm{m}}}) - \frac{1}{\alpha_{\mathrm{m}}}, \nu_{\mathbb{Q}_\kappa}(w_{\alpha_{\mathrm{m}}}) + \frac{1}{\alpha_{\mathrm{m}}}] \subseteq \mathrm{I}(w),$$

as desired.

Now, since for every $w \in \mathrm{dom}(I)$ we have

$$x \in \mathrm{I}(w) \Leftrightarrow \exists \alpha < \kappa.\ [\nu_{\mathbb{Q}_\kappa}(w_\alpha) - \frac{1}{\alpha}, \nu_{\mathbb{Q}_\kappa}(w_\alpha) + \frac{1}{\alpha}] \subset \mathrm{I}(w).$$

Then, by the definition of f, we have $\delta_{\mathbb{R}_\kappa}(f(p)) = \delta_{\mathrm{C}}(p)$ for every $p \in \mathrm{dom}(\delta_{\mathrm{C}})$ as desired.

Now we will prove $\delta_{\mathbb{R}_\kappa} \leq_{\mathrm{t}} \delta_{\mathrm{C}}$. Let $p \in \mathrm{dom}(\delta_{\mathbb{R}_\kappa})$. We define a function $h^p : \kappa \to \kappa^{<\kappa}$ by recursion. For the base case we define

$$h^p(0) = w_1,$$

where w_1 is such that $\iota(w_0)$ is the first appearing substring of p with the following properties:

$$\mathrm{I}(w_0) = \mathrm{B}(\nu_{\mathbb{Q}_\kappa}(w_1), \nu_{\mathbb{Q}_\kappa}(w_2)),$$
$$|\nu_{\mathbb{Q}_\kappa}(w_1) - \nu_{\mathbb{Q}_\kappa}(w_2)| < \frac{1}{2}.$$

Now, for $\alpha > 0$ we define h^p as follows:

$$h^p(\alpha) = w_1$$

where w_1 is such that $\iota(w_0)$ is the first appearing substring of p with the following properties:

$$\begin{aligned} &\mathrm{I}(w_0) = \mathrm{B}(\nu_{\mathbb{Q}_\kappa}(w_1), \nu_{\mathbb{Q}_\kappa}(w_2)),\\ &|\nu_{\mathbb{Q}_\kappa}(w_1) - \nu_{\mathbb{Q}_\kappa}(w_2)| < \frac{1}{\alpha+1},\\ &\nu_{\mathbb{Q}_\kappa}(w_1) > \{\nu_{\mathbb{Q}_\kappa}(h^p(\beta)) \mid \beta < \alpha\}. \end{aligned}$$

Note that h^p is well defined since p is the listing of *all* the κ-rational open balls containing $\delta_{\mathbb{R}_\kappa}(p)$. In particular h^p contains open balls arbitrary small around $\delta_{\mathbb{R}_\kappa}(p)$. Now we define the function $h : \kappa^\kappa \to \kappa^\kappa$ as follows:

$$h(p) = [\![h^p(\alpha)]\!]_{\alpha\in\kappa}.$$

Note that, since every time we are using only a small portion of the input to define a small portion of the output, h is continuous. Moreover it is not hard to see that for every $p \in \mathrm{dom}(\delta_{\mathbb{R}_\kappa})$, we have that $h(p)$ is the coding of a strictly monotone Cauchy sequence converging to $\delta_{\mathbb{R}_\kappa}(p)$. Since every $h(p)$ is monotone, for all $\alpha < \kappa$ and $\beta > \alpha$, we have $|(\nu_{\mathbb{Q}_\kappa} h^p(\alpha)) - \nu_{\mathbb{Q}_\kappa}(h^p(\beta))| < \frac{1}{\alpha+1}$. Then h reduces $\delta_{\mathbb{R}_\kappa}$ to δ_{C}. In conclusion we have $\delta_{\mathbb{R}_\kappa} \equiv_{\mathrm{t}} \delta_{\mathrm{C}}$ as desired. □

We end this section by presenting two other κ-admissible representations of $\mathbb{R}_\kappa$. These representations, will have a crucial role in studying the Weihrauch hierarchy over $\mathbb{R}_\kappa$. As in classical computable analysis, we can define the following effective topological spaces:

$$S_< = (\mathbb{R}_\kappa, \sigma_<, \nu_<) \text{ and } S_> = (\mathbb{R}_\kappa, \sigma_>, \nu_>)$$

where

$$\begin{aligned} \sigma_< &= \{(q, +\infty) \mid q \in \mathbb{Q}_\kappa\},\ \nu_<(w) = (\nu_{\mathbb{Q}_\kappa}(w), +\infty),\\ \sigma_> &= \{(-\infty, q) \mid q \in \mathbb{Q}_\kappa\},\ \nu_>(w) = (-\infty, \nu_{\mathbb{Q}_\kappa}(w)). \end{aligned}$$

First note that:

Theorem 4.4.7. *Let $\mathbb{Q}_\kappa^{\mathrm{D}}$ be the Dedekind completion of $\mathbb{Q}_\kappa$. Then we have*

$$\tau_{S_<} = \{(m, +\infty) \mid m \in \mathbb{Q}_\kappa^{\mathrm{D}}\}$$

and

$$\tau_{S_>} = \{(-\infty, m) \mid m \in \mathbb{Q}_\kappa^{\mathrm{D}}\}.$$

Proof. We will prove only the statement for $S_<$, the other proof is completely analogous.

First we prove $\tau_{S_<} \supseteq \{(m, +\infty) \mid m \in \mathbb{Q}_\kappa^{\mathrm{D}}\}$. Let $m \in \mathbb{Q}_\kappa^{\mathrm{D}}$. Define $Q = \{q \in \mathbb{Q}_\kappa \mid q \geq m\}$. By definition m is a lower bound of Q. Let $m' \in \mathbb{Q}_\kappa^{\mathrm{D}}$ be a lower bound of Q such that $m < m'$. Since $\mathbb{Q}_\kappa$ is dense in $\mathbb{Q}_\kappa^{\mathrm{D}}$, there exists $q \in \mathbb{Q}_\kappa$ such that $m < q < m'$ but then $q \in Q$. This is absurd since m' was a lower bound of Q. Then we have

$$(m, +\infty) = (\inf Q, +\infty) = \bigcup_{q\in Q} (q, +\infty).$$

Now we want to prove $\tau_{S_<} \subseteq \{(m, +\infty) \mid m \in \mathbb{Q}_\kappa^{\mathrm{D}}\}$. By the definition every element A of $\tau_{S_<}$ is of the form $\bigcup_{q\in Q}(q, +\infty)$ with $Q \subset \mathbb{Q}_\kappa$. But then we have

$$A = \bigcup_{q\in Q} (q, +\infty) = (\inf Q, +\infty),$$

and since $\mathbb{Q}_\kappa^{\mathrm{D}}$ is the Dedekind completion of $\mathbb{Q}_\kappa$, we have that $\sup Q \in \mathbb{Q}_\kappa^{\mathrm{D}}$ as desired. □

Note that these topologies are very different from those obtained in the classical case (see [28, Lemma 4.1.4]). As we have done for $S_{\mathbb{R}_\kappa}$,we will prove that $S_<$ and $S_>$ are κ-effective.

Lemma 4.4.8. *The representations $S_<$ and $S_>$ are κ-effective.*

Proof. We will prove the theorem only for $S_<$, the other proof is analogous. We want to prove that $\tau_{S_<}$ is closed under conjunction of less than κ elements of $\sigma_<$. Let Q be a subset of $\mathbb{Q}_\kappa$ of cardinality less than κ. We have that

$$\bigcap_{q\in Q}(q,+\infty) = (\sup Q, +\infty),$$

therefore, by Theorem 4.4.7, $\bigcap_{q\in Q}(q,+\infty) \in \tau_{S_<}$ as desired. □

We will denote the standard representations of $S_<$ and $S_>$ respectively with $\delta_{S_<}$ and with $\delta_{S_<}$. Note that by the definition:

$$\delta_{S^<}(p) = x$$

if and only if p is the code for *all* the κ-rationals smaller than x (similarly for $\delta_{S_>}$).

As for the representation $\delta_{\mathbb{R}_\kappa}$, we introduce now two Cauchy representations which are continuously equivalent respectively to $\delta_{S_<}$ and to $\delta_{S_>}$. Let us focus on $\delta_{S_<}$, analogous considerations can be applied to $\delta_{S_>}$. We define the following representation:

$$\delta_{\mathbb{R}_\kappa^<}(p) = x \Leftrightarrow \begin{cases} p = [\![\iota(w_\alpha)]\!]_{\alpha\in\kappa}, \\ (\nu_{\mathbb{Q}_\kappa}(w_\alpha))_{\alpha\in\kappa} \text{ strictly increasing}, \\ \lim_{\alpha\in\kappa}\nu_{\mathbb{Q}_\kappa}(w_\alpha) = x. \end{cases}$$

for every $p \in \kappa^\kappa$ and $x \in \mathbb{R}_\kappa$. As we said $\delta_{\mathbb{R}_\kappa^<}$ is continuously equivalent to $\delta_{S_<}$.

Theorem 4.4.9. $\delta_{\mathbb{R}_\kappa^<} \equiv_{\mathrm{t}} \delta_{S_<}$.

Proof. First note that the identity proves that $\delta_{\mathbb{R}_\kappa^<} \leq_{\mathrm{t}} \delta_{S_<}$. By definition, $p \in \mathrm{dom}(\delta_{\mathbb{R}_\kappa^<})$ if and only if p is the listing of codes of a sequence of strictly increasing κ-rational numbers whose limit is $\delta_{\mathbb{R}_\kappa^<}(p)$. Moreover claim that $\delta_{S_<}(p) = x$ iff $x = \sup\{\nu_{\mathbb{Q}_\kappa}(w) \mid \iota(w) \triangleleft p\}$. Indeed, assume $\delta_{S_<}(p) = x$ and $x \neq \sup\{\nu_{\mathbb{Q}_\kappa}(w) \mid \iota(w) \triangleleft p\}$. By definition we have

$$\delta_{S_<}(p) = x \Leftrightarrow \{(q,+\infty) \mid q \in \mathbb{Q}_\kappa \wedge x \leq q\} = \{\nu_{\mathbb{Q}_\kappa}(w) \mid \iota(w) \triangleleft p\},$$

therefore x is an upper bound of $\{\nu_{\mathbb{Q}_\kappa}(w) \mid \iota(w) \triangleleft p\}$. Assume $y \in \mathbb{R}_\kappa$ upper bound of $\{\nu_{\mathbb{Q}_\kappa}(w) \mid \iota(w) \triangleleft p\}$ smaller than x. Then there is $q \in \mathbb{Q}_\kappa$ such that $y < q < x$ but then there is $\iota(w) \triangleleft p$ such that $\nu_{\mathbb{Q}_\kappa}(w) = q$. This is a contradiction, therefore

$$\delta_{S_<}(p) = x \Rightarrow x = \sup\{\nu_{\mathbb{Q}_\kappa}(w) \mid \iota(w) \triangleleft p\}.$$

For the other direction assume $x = \sup\{\nu_{\mathbb{Q}_\kappa}(w) \mid \iota(w) \triangleleft p\}$. Then for every $\iota(w) \triangleleft p$, we have that $x \in (\nu_{\mathbb{Q}_\kappa}(w), +\infty)$. Moreover, since x is the limit of the sequence represented by p, for every $q \in \mathbb{R}_\kappa$ with $x \geq q$ there is $\iota(w) \triangleleft p$ such that $\nu_{\mathbb{Q}_\kappa}(w) \geq q$. Then

$$\{(q,+\infty) \mid q \in \mathbb{Q}_\kappa \wedge x \leq q\} = \{\nu_{\mathbb{Q}_\kappa}(w) \mid \iota(w) \triangleleft p\},$$

which means that $\delta_{S_<}(p) = x$ as desired.

Hence ID reduces $\delta_{\mathbb{R}_\kappa^<}$ to $\delta_{S_<}$.

Now to prove that $\delta_{S_<} \leq_{\mathrm{t}} \delta_{\mathbb{R}_\kappa^<}$. Let $w \in \kappa^{<\kappa}$ and

$$Q = \{\nu_{\mathbb{Q}_\kappa}(w') \mid \iota(w') \triangleleft w\}.$$

Moreover let $(q_\alpha)_{\alpha\in\beta}$ be the listing of the elements of Q in the same order they appear in w. We define the following two sequences:

$$\begin{aligned} w_0^Q &= \nu_{\mathbb{Q}_\kappa}^{-1}(q_0), & Q'_0 &= \{q_0\}, \\ w_{\alpha+1}^Q &= \begin{cases} w_\alpha^Q \text{ if } q_{\alpha+1} \leq Q'_\alpha, \\ w_\alpha^Q{}^\frown \nu_{\mathbb{Q}_\kappa}^{-1}(q_{\alpha+1}) \text{ otherwise}, \end{cases} & Q'_{\alpha+1} &= \begin{cases} Q'_\alpha \text{ if } q_{\alpha+1} \leq Q'_\alpha, \\ Q'_\alpha \cup \{q_{\alpha+1}\} \text{ otherwise}, \end{cases} \\ w_\lambda^Q &= \textstyle\bigcup_{\alpha\in\lambda} w_\alpha^Q, & Q'_\lambda &= \textstyle\bigcup_{\alpha\in\lambda} Q'_\alpha \text{ if } \lambda \text{ is a limit ordinal.} \end{aligned}$$

Now define a Wadge strategy $\theta :\subseteq \kappa^{<\kappa} \to \kappa^{<\kappa}$ as follows

$$\theta(w) = w^Q_{|Q|}.$$

Note that θ is monotone, hence the function

$$h'(p) = \bigcup_{\alpha\in\kappa} \theta(p\restriction\alpha)$$

is continuous with domain $\mathrm{dom}(\delta_{S_<})$. Now let $p \in \mathrm{dom}(\delta_{S_<})$. By definition p contains the codes for a sequence of rationals $(q_\alpha)_{\alpha\in\kappa}$ whose least upper bound is $\delta_{S_<}(p)$. Moreover $h'(p)$ is a listing of codes for a streactly increasing cofinal subsequence of $(q_\alpha)_{\alpha\in\kappa}$. Hence h' reduces $\delta_{S_<}$ to $\delta_{\mathbb{R}^<_\kappa}$ as desired. □

We end this section by showing how, thanks to the Generalized Main Theorem of Computable Analysis, we can represent the set of continuous functions over $\mathbb{R}_\kappa$. We start showing a negative result. If we consider the κ-topology over κ^κ generated by the standard base, then there is no κ-continuous representation.

Lemma 4.4.10. *Let κ^κ be equipped with the κ-topology induced by the standard base of generalized Baire space. Then is no κ-continuous representation of $\mathbb{R}_\kappa$ w.r.t. the base κ-rational open balls.*

Proof. Let δ be a κ-continuous representation of $\mathbb{R}_\kappa$. Then as we have seen the interval κ-topology over $\mathbb{R}_\kappa$ has 2^κ open sets. But since $\kappa^{<\kappa} = \kappa$, the κ-topology over κ^κ has cardinality κ. Then there cannot be κ-continuous map between κ^κ and $\mathbb{R}_\kappa$. □

In particular this implies that we cannot use the generalized main theorem of computable analysis in order to represent κ-continuous functions over $\mathbb{R}_\kappa$ with κ-continuous functions over κ^κ.

Even if our representation may not preserve κ-continuity, note that by Lemma 3.2.4, κ-continuous functions are continuous, then they are still realized by continuous functions. In particular since we know that we can represent continuous functions over $\mathbb{R}_\kappa$, we can also represent κ-continuous functions.

Following the classical, case for every $A \subseteq \mathbb{R}_\kappa$ we can now define a standard representation for the set $\mathrm{C}(A, \mathbb{R}_\kappa)$ of continuous functions from A to $\mathbb{R}_\kappa$. By Lemma 4.3.1 and by the generalized main theorem of computable analysis , we have that continuous functions are realized by continuous functions over κ^κ. In the previous chapter we have seen that, in this case, we can represent continuous functions as follows:

$$[\delta_\mathrm{C} \to \delta_\mathrm{C}]_A(p) = f \Leftrightarrow f(\delta_\mathrm{C}(p')) = \delta_\mathrm{C}(\delta_{\mathrm{C}(\kappa^\kappa,\kappa^\kappa)}(p)(p'))\forall p' \in A.$$

We will call $[\delta_\mathrm{C} \to \delta_\mathrm{C}]_A$ the standard representation of $\mathrm{C}(A)$.

4.5 Generalized Choice Principles

Finally we are ready to begin the study of the generalization of the Weihrauch hierarchy to κ^κ and $\mathbb{R}_\kappa$. In this section we will focus on choice principles.

Choice principles have a central role in classical Weihrauch reducibility theory. In particular, as shown in [5], one can use them to characterize different Weihrauch degrees which are of main interest from the computable analysis point of view. In this section we will generalize some of these choice principles and we will start their classification within the Weihrauch hierarchy. For a complete introduction to classical choice principles we refer the reader to [5] and [4]. Before we begin the study of generalized choice principles, let us generalize the definition of Weihrauch reducibility and fix some conventions.

Definition 4.5.1 (Realizers)**.** *Let $F :\subseteq M_1 \rightrightarrows M_0$ be a multi-valued function over the represented spaces (M_1, δ_{M_1}) and (M_0, δ_{M_0}). Then $f :\subseteq \kappa^\kappa \to \kappa^\kappa$ is a* realizer *of F iff for every $x \in \mathrm{dom}(F \circ \delta_{M_1})$ we have*

$$\delta_{M_0}(f(x)) \in F(\delta_{M_1}(x)).$$

*If F has a continuous realizer we will say that it is $(\delta_{M_1}, \delta_{M_0})$-*continuous*.*

Definition 4.5.2 (Generalized Weihrauch Reducibility)**.** *Let $F :\subseteq M_1 \rightrightarrows M_0$ and $G :\subseteq N_1 \rightrightarrows N_0$ be two multi-valued functions between represented spaces. Then we will say that F is* Weihrauch reducible *to G, in symbols $F \leq_{\mathrm{w}} G$, iff there are two (partial) continuous functions $H :\subseteq \kappa^\kappa \to \kappa^\kappa$ and $K :\subseteq \kappa^\kappa \to \kappa^\kappa$ such that for every realizer $g :\subseteq \kappa^\kappa \to \kappa^\kappa$ of G there is a realizer $f :\subseteq \kappa^\kappa \to \kappa^\kappa$ of F such that*

$$f = H \circ \lceil \mathrm{ID}, g \circ K \rceil,$$

where $\mathrm{ID} : \kappa^\kappa \to \kappa^\kappa$ is the identity function. Moreover, if F and G are such that for every realizer $g :\subseteq \kappa^\kappa \to \kappa^\kappa$ of G there is a realizer $f :\subseteq \kappa^\kappa \to \kappa^\kappa$ of F such that

$$f = H \circ g \circ K,$$

then we will say that F is strongly Weihrauch reducible *to G, in symbols $F \leq_{\mathrm{s,w}} G$.*

As usual if F is (strongly) Weihrauch reducible to G and G is (strongly) Weihrauch reducible to F then we will say that F is (strongly) Weihrauch equivalent *to G and we will write $F \equiv_{\mathrm{w}} G$ ($F \equiv_{\mathrm{s,w}} G$).*

From now on we will consider $\mathbb{R}_\kappa$, $\mathbb{R}_\kappa^<$ and $\mathbb{R}_\kappa^>$ as represented by $\delta_{\mathbb{R}_\kappa}$, $\delta_{\mathbb{R}_\kappa^<}$ and $\delta_{\mathbb{R}_\kappa^>}$, respectively. Moreover, κ will be represented by

$$\delta_\kappa(p) = p(0).$$

Let $\mathcal{P}^-(\kappa)$ be the power set of κ minus κ itself. We will represent the hyperspace $\mathcal{P}^-(\kappa)$ by

$$\delta_{\mathcal{P}^-(\kappa)}(p) = \{\alpha \mid \forall \beta.\ p(\beta) \neq \alpha\}.$$

Finally, the set $\mathrm{Cl}_{[0,1]}$ of closed subsets of $[0,1]$ will be represented by the following function:

$$\delta_{[0,1]}(p) = A \Leftrightarrow A = \mathbb{R}_\kappa \setminus \bigcup \{\mathrm{I}'(w) \mid \iota(w) \lhd p\},$$

where $\mathrm{I}' : \kappa^{<\kappa} \to \kappa$ is defined as follows:

$$\mathrm{I}'(w) = (q_1, q_2) \text{ iff } \nu^{-1}_{\mathbb{Q}_\kappa \times \mathbb{Q}_\kappa}(w) = \lceil q_1, q_2 \rceil.$$

Namely p is a code for A iff it is a list of open intervals in $\mathbb{R}_\kappa$ with κ-rational end points whose union is the complement of A.

Definition 4.5.3 (Choice Principles)**.** *We will consider the Weihrauch degrees of the following choice principles:*

- $\mathrm{C}_{\mathrm{I}}^\kappa$ *interval choice: given a non empty closed interval in $[0,1]$, the function $\mathrm{C}_{\mathrm{I}}^\kappa$ chooses an element in the interval. Formally $\mathrm{C}_{\mathrm{I}}^\kappa$ is defined as the following multi-valued function:*

$$\mathrm{C}_{\mathrm{I}}^\kappa : \mathrm{Cl}_{[0,1]} \rightrightarrows \mathbb{R}_\kappa \qquad \mathrm{dom}(\mathrm{C}_{\mathrm{I}}^\kappa) = \{[a,b] \mid 0 \leq a \leq b \leq 1\}.$$

- C_κ *discrete choice: given a non empty subset of κ, the function C_κ chooses an element of the set. Formally C_κ is defined as the following multi-valued function:*

$$\mathrm{C}_\kappa : \mathcal{P}^-(\kappa) \to \kappa \qquad \mathrm{dom}(\mathrm{C}_\kappa) = \{A \subset \kappa \mid A \neq \emptyset\}.$$

- *For every $U \subseteq \kappa^\kappa$ we consider the following multi-valued function $\mathrm{C}_U : \kappa^\kappa \rightrightarrows \kappa^\kappa$ given by:*

$$\mathrm{C}_U(p)(\alpha) = \begin{cases} 1 & (p)_\alpha \in U, \\ 0 & \textit{otherwise}, \end{cases}$$

where $(p)_\alpha(\beta) = p(\lceil \alpha, \beta \rceil)$.

In this section we will begin the study of the diagram in Figure 4.2[3].

First of all we will show that C_κ and $\mathrm{C}_{\mathrm{I}}^\kappa$ are not continuous:

[3]In the figure an arrow $A \to B$ represents the fact that A can be Weihrauch reduced to B, while an arrow $A \nrightarrow B$ represents the fact that $A \not\leq_w B$.

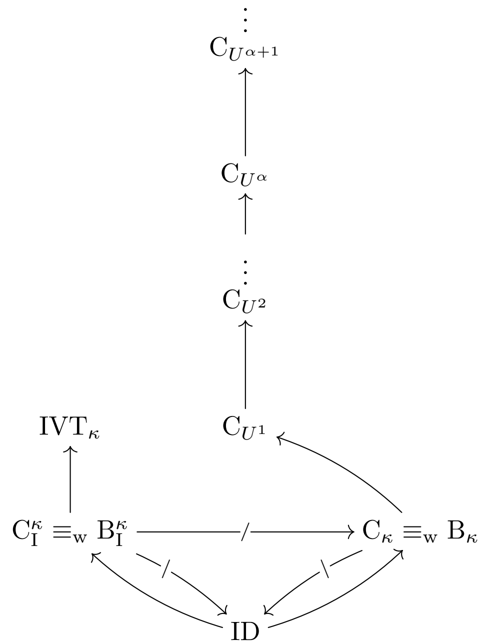


Figure 4.2: A part of the Weihrauch hierarchy.

Proposition 4.5.4. *The following hold:*

$$\mathrm{ID} <_\mathrm{w} \mathrm{C}_\kappa \text{ and } \mathrm{ID} <_\mathrm{w} \mathrm{C}_\mathrm{I}^\kappa.$$

Proof. Note that the identity ID can be trivially reduced to any function, so it is enough to show that C_κ and $\mathrm{C}_\mathrm{I}^\kappa$ do not reduce to ID.

We will prove $\mathrm{C}_\kappa \not\leq_\mathrm{w} \mathrm{ID}$. By contradiction assume $\mathrm{C}_\kappa \leq_\mathrm{w} \mathrm{ID}$. Then there are two continuous functions H and K such that $H \circ \lceil \mathrm{ID}, \mathrm{ID} \circ K \rceil$ is a realizer of C_κ. Note that, since ID, H and K are continuous we have that

$$H \circ \lceil \mathrm{ID}, \mathrm{ID} \circ K \rceil$$

is continuous. Therefore, there is a Wadge strategy $\theta :\subseteq \kappa^{<\kappa} \to \kappa^{<\kappa}$ such that for every $p \in \kappa^\kappa$

$$H \circ \lceil \mathrm{ID}, \mathrm{ID} \circ K \rceil(p) = \bigcup_{\alpha \in \kappa} \theta(p \restriction \alpha).$$

Now, let $\alpha \in \kappa$ be such that

$$\delta_\kappa([\theta(p \restriction \alpha)]) \subseteq X$$

where $X \subset \kappa$. Define $p'' = (p \restriction \alpha)^\frown p'$ where p' is any map in κ^κ such that $X \subseteq \{\beta \in \kappa \mid \exists \alpha < \kappa.\ p(\alpha) = \beta\}$. By the monotonicity of θ, we have

$$\delta_\kappa(\bigcup_{\alpha \in \kappa} \theta(p'' \restriction \alpha)) \in X.$$

But by the definition of p'' we have $X \subseteq \delta_{\mathcal{P}^-(\kappa)}(p'')$, therefore $H \circ \lceil \mathrm{ID}, \mathrm{ID} \circ K \rceil$ would not be a realizer of C_κ.

Now we want to prove that $\mathrm{C}_\mathrm{I}^\kappa \not\leq_{\mathrm{s,w}} \mathrm{ID}$. Assume $\mathrm{C}_\mathrm{I}^\kappa \leq_{\mathrm{s,w}} \mathrm{ID}$. Then there are H and K continuous functions such that $H \circ \lceil \mathrm{ID}, \mathrm{ID} \circ K \rceil$ is a realizer of $\mathrm{C}_\mathrm{I}^\kappa$. As before $H \circ \lceil \mathrm{ID}, \mathrm{ID} \circ K \rceil$ is continuous, therefore there is a Wadge strategy $\theta :\subseteq \kappa^{<\kappa} \to \kappa^{<\kappa}$ such that

$$H \circ \lceil \mathrm{ID}, \mathrm{ID} \circ K \rceil(p) = \bigcup_{\alpha \in \kappa} \theta(p \restriction \alpha).$$

Now define the following sequence

$$p = [\![\iota(w_\alpha)^\frown\iota(w'_\alpha)]\!]_{\alpha\in\kappa},$$

where w_α and w'_α are such that

$$\mathrm{I}'(w_\alpha) = (\frac{1}{\alpha}, \frac{1}{2} - \frac{1}{\alpha}) \text{ and } \mathrm{I}'(w'_\alpha) = (\frac{1}{3} + \frac{1}{\alpha}, 1 - \frac{1}{\alpha}).$$

Note that $\delta_{[0,1]}(p) = [\frac{1}{2}, \frac{2}{3}] \cup \{0, 1\}$. Then we have:

$$\delta_{\mathbb{R}_\kappa}(H \circ \lceil \mathrm{ID}, \mathrm{ID} \circ K\rceil(p)) \in [\frac{1}{2}, \frac{2}{3}] \cup \{0, 1\}.$$

Without loss of generality we can assume

$$\delta_{\mathbb{R}_\kappa}(H \circ \lceil \mathrm{ID}, \mathrm{ID} \circ K\rceil(p)) \in [\frac{1}{2}, \frac{2}{3}]$$

the other cases can be proved similarly. Let α be such that

$$\delta_{\mathbb{R}_\kappa}([\theta(p{\restriction}\alpha)]) \subseteq [\frac{1}{2}, \frac{2}{3}].$$

Now, define

$$p' = [\![\iota(w_\alpha)]\!]_{\alpha\in\kappa},$$

where w_α are such that $\mathrm{I}(w_\alpha) = (\frac{1}{\alpha}, 1]$ (note that they are open intervals in $[0,1]$). Let $p'' = (p{\restriction}\alpha)^\frown p'$. Then trivially $\delta_{[0,1]}(p'') = \{0\}$. But by monotonicity of θ and by the fact that $p{\restriction}\alpha = p''{\restriction}\alpha$, it follows that $\delta_{\mathbb{R}_\kappa}([\theta(p''{\restriction}\alpha)]) \in [\frac{1}{2}, \frac{2}{3}]$ and then

$$H \circ \lceil \mathrm{ID}, \mathrm{ID} \circ K\rceil(p'') \in [\frac{1}{2}, \frac{2}{3}].$$

But this means that $H \circ \lceil \mathrm{ID}, \mathrm{ID} \circ K\rceil$ is not a realizer of $\mathrm{C}^\kappa_\mathrm{I}$. □

One important tool for studying the Weihrauch degrees of choice principles are the boundedness principles. In particular, reductions with boundedness principles usually result to be easier to construct and will then simplify our proofs.

Definition 4.5.5 (Boundedness Principles)**.** *We define the following boundedness principles:*

- $\mathrm{B}^\kappa_\mathrm{I}$*: given two Cauchy sequences,* $(q_\alpha)_{\alpha\in\kappa}$ *and* $(q'_\alpha)_{\alpha\in\kappa}$ *in* $\mathbb{Q}_\kappa$ *such that*

  $$\sup_{\alpha\in\kappa} q_\alpha \le \inf_{\alpha\in\kappa} q'_\alpha,$$

  there is $x \in \mathbb{R}_\kappa$ *such that*

  $$\sup q_\alpha \le x \le \inf q'_\alpha.$$

  Formally we define B^κ_I *as follows:*

  $$\mathrm{B}^\kappa_\mathrm{I} :\subseteq \mathbb{R}^<_\kappa \times \mathbb{R}^>_\kappa \rightrightarrows \mathbb{R}_\kappa\ ,\ (x, y) \mapsto [x, y] \qquad \mathrm{dom}(\mathrm{B}_\mathrm{I}) = \{(x, y) \mid x \le y\}.$$

- B_κ*: given a Cauchy sequence* $(q_\alpha)_{\alpha\in\kappa}$ *of* κ*-rationals such that*

  $$\sup_{\alpha\in\kappa} q_\alpha \le r \text{ for some } r \in \mathbb{R}_\kappa.$$

  Intuitively, B_κ *chooses a* κ*-real which is greater than or equal to* $\sup_{\alpha\in\kappa} q_\alpha$*. Formally:*

  $$\mathrm{B}_\kappa : \mathbb{R}^<_\kappa \rightrightarrows \mathbb{R}_\kappa, \qquad x \mapsto [x, +\infty).$$

Now we will focus on proving that as in the classical case we have

$$\mathrm{B}_\kappa \equiv_{\mathrm{w}} \mathrm{C}_\kappa \text{ and } \mathrm{C}_{\mathrm{I}}^{\kappa} \equiv_{\mathrm{s,w}} B_{\mathrm{I}}^{\kappa}.$$

The formal proofs of these facts can be slightly convoluted. For this reason, before we prove these results we will give an informal intuition of the reason why they work.

First we will prove that $\mathrm{B}_\kappa \equiv_{\mathrm{w}} \mathrm{C}_\kappa$. While it is not complicated to prove that $\mathrm{B}_\kappa \leq_{\mathrm{w}} \mathrm{C}_\kappa$, the proof for the other direction is less intuitive.

We are given an enumeration of a set $A \subset \kappa$ and we want to find an element in $\kappa \setminus A$ by using a realizer of B_κ. To find this element, we read the enumeration of A and we build two sequences of rational numbers h_p and f_p. The sequence h_p will keep track of the smallest ordinal number not in the portion of A we have read so far. The sequence f_p, instead, will be a strictly Cauchy sequence whose limit is the position in which h_p becomes constant. Intuitively h_p is defined as follows:

Start by setting $h_p(0)$. Then look at a_0, namely the first element of the enumeration of A. If $a_0 > 0$ then $h_p(1) = a_0$ otherwise $h_p(1) = h_p(0)$. In general for a_α we have $h_p(\alpha) = a_\alpha$ if $a_\alpha > h_p(\beta)$ for every $\beta < \alpha$, and $h_p(\alpha) = \bigcup_{\beta\in\alpha} h_p(\beta)$ otherwise.

By using this sequence we can define f_p. Intuitively f_p will guess the smallest position on which h_p stabilizes.

We start defining $f_p(0) = -\frac{1}{2}$, guessing that the h_p is the constant 0. Now we check $h_p(1)$, if it is 0, then our guess is sill valid and we set $f_p(1) = -\frac{1}{3}$. Otherwise we guess that h_p stabilizes at position 1 and set $f_p(1) = \frac{1}{3}$. In general, we check $h_p(\alpha)$, if this does not contradict our guess g we keep defining a sequence converging at g by setting $f_p(\alpha) = g - \frac{1}{\alpha+1}$, otherwise we change our guess to α and we set $f_p(\alpha) = \alpha - \frac{1}{\alpha+1}$.

Note that since $\kappa \setminus A$ is nonempty, then f_p is Cauchy. Now we can feed a realizer of B_κ with h_p and it will find an upper bound r of f_p. The κ-real r will be given to us as a fast converging Cauchy sequence. Therefore we can easily find an ordinal α_{m} which is bigger than r. Now we can go through the enumeration of A again and find the least upper bound of the first α_{m} elements of A. This number will be by definition in $\kappa \setminus A$ as desired.

Proposition 4.5.6. $\mathrm{B}_\kappa \equiv_{\mathrm{w}} \mathrm{C}_\kappa$.

Proof. First we will prove $\mathrm{C}_\kappa \leq_{\mathrm{w}} \mathrm{B}_\kappa$. Let p be an enumeration of the set A. We want to find an element not in A. We define the following sequence:

$$\begin{aligned} &h_p(0) = 0, \\ &h_p(\alpha) = \min \kappa \setminus \{p(\beta) \mid \beta \leq \alpha\}. \end{aligned}$$

Intuitively $h_p(\alpha)$ is the minimum ordinal which is not in A according to the information in $p{\restriction}(\alpha+1)$. Now by using h_p we can define the following sequence:

$$\begin{aligned} &s_p(0) = 0, \\ &s_p(\alpha+1) = \begin{cases} s_p(\alpha) & \text{If } h_p(\alpha) = h_p(\alpha+1), \\ \alpha+1 & \text{otherwise}, \end{cases} \\ &s_p(\lambda) = \bigcup_{\alpha\in\lambda} s_p(\alpha). \end{aligned}$$

The intuition is that s_p keeps track of the changes in h_p. Since A is non empty, s_p is increasing and eventually constant, hence Cauchy. In particular s_p is a Cauchy sequence whose limit ℓ is an index of p such that $h_p(\ell)$ is the smallest ordinal not in A. We want to extract form s_p a strictly increasing subsequence. We define:

$$f_p(\alpha) = s_p(\alpha) - \frac{1}{\alpha+1}.$$

The Cauchy sequence f_p is trivially increasing and has limit ℓ. Therefore we can define K as follows:

$$K(p) = [\![\iota(\nu_{\mathbb{Q}_\kappa}^{-1}(f_p(\alpha)))]\!]_{\alpha\in\kappa}.$$

Note that, for every α, we use only a small portion of p in order to define $f_p(\alpha)$, therefore K is continuous.

Note that $\mathrm{B}_\kappa(K(p))$ is a κ-real number greater than every element in A. It is not hard to see that, since $\mathrm{B}_\kappa(K(p))$ is fast convergent, by considering an initial portion of $\mathrm{B}_\kappa(K(p))$ of length less than κ we can find an ordinal α_{m} which is not in A (note that is enough to pick α_{m} such that $\alpha_{\mathrm{m}} > \mathrm{B}_\kappa(K(p))$). Then we can define the function H as follows:

$$H(\lceil p, \mathrm{B}_\kappa(K(p))\rceil) = \min \kappa \setminus \{p(\alpha) \mid \alpha < \alpha_{\mathrm{m}}\}^\frown \mathbf{0},$$

where $\mathbf{0}$ is the constant 0 sequence in κ^κ. Now, note that H is continuous and $\delta_\kappa(H(\lceil p, \mathrm{B}_\kappa(K(p))\rceil)) \in \kappa \setminus A$ as desired.

Now we want to show that $\mathrm{B}_\kappa \leq_{\mathrm{s,w}} \mathrm{C}_\kappa$. Let p be the code of a strictly increasing Cauchy sequence $(q_\alpha)_{\alpha\in\kappa}$ of κ-rationals. We want to find a κ-real which is greater than or equal to the least upper bound of the sequence. Define K as follows:

$$K(p)(\lceil \alpha, \beta \rceil) = \begin{cases} \beta & \text{If } p\restriction\alpha \text{ codes the sequence } (q_\gamma)_{\gamma\in\gamma'}, \\ & \text{and there exists } \gamma \leq \gamma' \text{ such that } \beta \leq q_\gamma, \\ \gamma_0 & \text{otherwise,} \end{cases}$$

where γ_0 is the smallest ordinal such that $\gamma_0 \leq q_0$. Since for defining an initial segment of $K(p)$ we only need a small portion of p, we have that K is continuous. Now define $H(\alpha) = \alpha^\frown\mathbf{0}$. We claim that $K(p)$ is a code for the set of *all* the ordinal numbers smaller than or equal to $\ell = \lim_{\alpha\in\kappa} q_\alpha$. Indeed, if $\beta \leq \ell$ there is q_α such that $\beta \leq q_\alpha$. Take γ be such that $p\restriction\gamma$ codes the sequence $(q'_\gamma)_{\gamma'\in\gamma''}$ with $\alpha \leq \gamma''$. Therefore $K(p)(\lceil \gamma, \beta\rceil) = \beta$. On the other hand, if $K(p)(\lceil \alpha, \beta\rceil) = \gamma$, either $\gamma = \gamma_0 \leq q_0$ or there is an element $q_{\gamma'}$ of the sequence coded by p such that $\gamma \leq q_{\gamma'}$.

Now by using C_κ, we obtain an ordinal which is greater than or equal to ℓ. Then we can define a continuous function $H(p) = p'$, where p' is any code for $p(0)$ in $\mathbb{R}_\kappa$. Therefore we have $\delta_{\mathbb{R}_\kappa}(H \circ \mathrm{C}_\kappa \circ K(p)) \in [\ell, +\infty)$ as desired. □

We will now prove that $\mathrm{C}_{\mathrm{I}}^\kappa \equiv_{\mathrm{s,w}} B_{\mathrm{I}}^\kappa$. Let us illustrate the idea behind the proof of $\mathrm{C}_{\mathrm{I}}^\kappa \leq_{\mathrm{s,w}} B_{\mathrm{I}}^\kappa$. We are given a closed subinterval $[r, r']$ of $[0,1]$ as the listing of all the open intervals with κ-rational end points which have empty intersection with it. We have to define two sequences $(q_\alpha)_{\alpha\in\kappa}$ and $(q'_\alpha)_{\alpha\in\kappa}$ of κ-rationals, respectively strictly increasing and strictly decreasing such that every element x in between $\sup_{\alpha\in\kappa} q_\alpha$ and $\inf_{\alpha\in\kappa} q'_\alpha$ is in $[r, r']$. Let us consider the construction of $(q_\alpha)_{\alpha\in\kappa}$, a similar reasoning works for $(q'_\alpha)_{\alpha\in\kappa}$. The idea is that of building a Cauchy sequence for r. We can start by setting $q_0 = 0$. Now we start reading the description of $[r, r']$ until we find enough open intervals to cover $[0, q]$ with q any κ-rational. Then we are sure that $q < r$ and we can set $q_1 = q$. In general to define q_α we read an initial portion of the code of $[r, r']$ long enough to cover $[0, q]$ with q strictly bigger than all the q_β for $\beta < \alpha$ and we set $q_\alpha = q$. It is not hard to see that this is a strictly increasing Cauchy sequence converging to r as we wanted.

Proposition 4.5.7. $\mathrm{C}_{\mathrm{I}}^\kappa \equiv_{\mathrm{s,w}} B_{\mathrm{I}}^\kappa$.

Proof. First we prove $\mathrm{C}_{\mathrm{I}}^\kappa \leq_{\mathrm{s,w}} B_{\mathrm{I}}^\kappa$. Let p code the closed interval $[r, r'] \subset [0,1]$. In particular assume that p is the listing of open intervals $(J_\alpha)_{\alpha\in\kappa}$ such that $\bigcup_{\alpha\in\kappa} J_\alpha = \mathbb{R}_\kappa \setminus [r, r']$. We want to define two Cauchy sequences $(q_\alpha)_{\alpha\in\kappa}$ and $(q'_\alpha)_{\alpha\in\kappa}$ in $\mathbb{R}_\kappa$ respectively strictly increasing and strictly decreasing, such that $\sup_{\alpha\in\kappa} q_\alpha \leq \inf_{\alpha\in\kappa} q'_\alpha$. Let $(q''_\alpha)_{\alpha\in\kappa}$ be a listing of $\mathbb{Q}_\kappa$. We define

$$\begin{aligned} & q_0 = 0,\ q'_0 = 1, \\ & q_\alpha = q''_m \text{where } m = \min\{\beta \mid [0, q''_\beta] \subseteq \bigcup_{\gamma\in\alpha_{\mathrm{m}}} J_\gamma \wedge q''_\beta > \{q_\gamma \mid \gamma < \alpha\}\}, \\ & q'_\alpha = q''_m \text{where } m = \min\{\beta \mid [q''_\beta, 1] \subseteq \bigcup_{\gamma\in\alpha'_{\mathrm{m}}} J_\gamma \wedge q''_\beta < \{q'_\gamma \mid \gamma < \alpha\}\}, \end{aligned}$$

where α_{m} and α'_{m} are the smallest ordinals such that

$$\{\beta \mid [0, q''_\beta] \subseteq \bigcup_{\gamma\in\alpha_{\mathrm{m}}} J_\gamma \wedge q''_\beta > \{q_\gamma \mid \gamma < \alpha\}\} \neq \emptyset$$

and

$$\{\beta \mid [q''_\beta, 1] \subseteq \bigcup_{\gamma \in \alpha'_{\mathrm{m}}} J_\gamma \wedge q''_\beta < \{q'_\gamma \mid \gamma < \alpha\}\} \neq \emptyset.$$

Note that α_{m} and α'_{m} exist by definition of $(J_\alpha)_{\alpha\in\kappa}$ and by the fact that the κ-rationals are dense in $\mathbb{R}_\kappa$. Hence the sequences are well defined. Moreover, by definition, $(q_\alpha)_{\alpha\in\kappa}$ and $(q'_\alpha)_{\alpha\in\kappa}$ are Cauchy and respectively strictly increasing and decreasing. We claim $\sup_{\alpha\in\kappa} q_\alpha = r$ and $\inf_{\alpha\in\kappa} q'_\alpha = r'$. Since every q_α is in some J_γ, it follows that r is an upper bound of $(q_\alpha)_{\alpha<\kappa}$. Moreover let $r'' < r$ be such that r'' is an upper bound of $(q_\alpha)_{\alpha<\kappa}$. Then there is a rational $r'' < q''_{\gamma'} < r$, and $q''_{\gamma'} \in J_\gamma$ for some $\gamma < \kappa$. Then for $\alpha > \gamma$ there is $q_\alpha = q''_{\gamma''}$ with $\gamma'' > \gamma'$. But since $q''_{\gamma'} > r'' \leq (q_\alpha)_{\alpha\in\kappa}$ and $[0, q''_{\gamma'}] \subset \bigcup_{\beta<\alpha_{\mathrm{m}}} J_\beta$, it follows that $q_\alpha = q''_{\gamma''}$ implies $\gamma'' \leq \gamma'$ which contradicts our assumption. A similar proof shows $\inf_{\alpha\in\kappa} q'_\alpha = r'$. Then the function $K : p \mapsto \lceil p', p'' \rceil$ where p' and p'' are respectively $[\![\iota(\nu_{\mathbb{Q}_\kappa}^{-1}(q_\alpha))]\!]_{\alpha\in\kappa}$ and $[\![\iota(\nu_{\mathbb{Q}_\kappa}^{-1}(q'_\alpha))]\!]_{\alpha\in\kappa}$ is continuous. Moreover, taking b^κ_{I} any realizer of $\mathrm{B}^\kappa_{\mathrm{I}}$ we have $\delta_{\mathbb{R}_\kappa}(b^\kappa_{\mathrm{I}} K(p)) \in [r, r']$ as desired.

Now we want to prove $B^\kappa_{\mathrm{I}} \leq_{\mathrm{s,w}} \mathrm{C}^\kappa_{\mathrm{I}}$. First of all note that the function $f : (0,1) \to \mathbb{R}_\kappa$ defined as

$$f(x) = \frac{2x-1}{x-x^2}$$

is a homeomorphism between $(0,1)$ and $\mathbb{R}_\kappa$. Indeed, since f is definable in the language of real closed fields and it is a strictly increasing bijection over $\mathbb{R}$ it is a strictly increasing bijection over $\mathbb{R}_\kappa$. Moreover, since $(r, r') \in \mathbb{R}_\kappa$, we have $f((r,r')) = (\frac{2r-1}{r-r^2}, \frac{2r'-1}{r'-r'^2})$, it follows that f is open in $\mathbb{R}_\kappa$.

Now let p be the code of the two Cauchy sequences $(q_\alpha)_{\alpha\in\kappa}$ and $(q'_\alpha)_{\alpha\in\kappa}$, respectively strictly increasing and strictly decreasing. Define the following sequence:

$$p' = \lceil [\![I'^{-1}([0, f(q_\alpha))]\!]_{\alpha\in\kappa}, [\![I'^{-1}((f(q'_\alpha), 1])]\!]_{\alpha\in\kappa} \rceil.$$

Since to define a small prefix of p' only a small prefix of p is needed, the function $K : p \mapsto p'$ is continuous. Now by the fact that f^{-1} is continuous in $\mathbb{R}_\kappa$ it has a continuous realizer F^{-1}. Define $H = F^{-1}$. Then, given c^κ_{I} a realizer of $\mathrm{C}^\kappa_{\mathrm{I}}$, we have that $c^\kappa_{\mathrm{I}} K(p)$ is the code for a real number in $[f(\sup_{\alpha\in\kappa} q_\alpha), f(\inf_{\alpha\in\kappa} q'_\alpha)]$ and then $\delta_{\mathbb{R}_\kappa}(H c^\kappa_{\mathrm{I}} K(p)) \in [\sup_{\alpha\in\kappa} q_\alpha, \inf_{\alpha\in\kappa} q'_\alpha]$ as desired. □

Now we define the following set:

$$U^1 = \bigcup_{\alpha<\kappa} \{p \in \kappa^\kappa \mid p(\alpha) \neq 0\}.$$

We end this section by showing that C_{U^1} is strictly more complex than C_κ.

Proposition 4.5.8. $\mathrm{C}_\kappa \leq_{\mathrm{s,w}} \mathrm{C}_{U^1}$.

Proof. Define K as follows:

$$K(p)(\lceil \alpha, \beta \rceil) = \begin{cases} 1 & \text{if } \exists \gamma < \alpha.\ p(\gamma) = \beta, \\ 0 & \text{otherwise,} \end{cases}$$

therefore K is continuous. Moreover let H be defined as follows:

$$H(p) = \beta^\frown \mathbf{0} \text{ iff } \beta < \kappa \text{ is the least such that } p(\beta) = 0,$$

where $\mathbf{0}$ is the constant 0 function in κ^κ. Note that H is trivially continuous in its domain. Then for every realizer c_{U^1} of C_{U^1} and $p \in \mathrm{dom}(\mathrm{C}_{U^1})$ such that p codes the complement of $A \subset \kappa^\kappa$, we have:

$$\begin{aligned}(H \circ c_{U^1} \circ K)(p) = \beta &\Leftrightarrow c_{U^1} K(p)(\beta) = 0 \\ &\Leftrightarrow \forall \alpha < \kappa.\ K(p)(\lceil \alpha, \beta \rceil) = 0 \\ &\Leftrightarrow \forall \gamma < \kappa.\ p(\gamma) \neq \beta \\ &\Leftrightarrow \beta \notin A,\end{aligned}$$

therefore $\mathrm{C}_\kappa \leq_{\mathrm{s,w}} \mathrm{C}_{U^1}$ as desired. □

Now we want to prove that $\mathrm{C}_{\mathrm{I}}^{\kappa}$ and C_{κ} characterize two different Weihrauch degrees. By following Brattka and Gherardi we can prove that $\mathrm{C}_{\mathrm{I}}^{\kappa}$ does not reduce to C_{κ} (the interested reader is referred to [5, Proposition 4.9]). The only difference between our proof and the classical proof is in the use of the κ-Baire Category Theorem. While in the standard proof the BCT could be used without any problem, in this case we need to show that all the requirements of the BCT_{κ} we presented in Section 3.5 are fulfilled by the space we are working with.

Proposition 4.5.9. $\mathrm{C}_{\mathrm{I}}^{\kappa} \not\leq_{\mathrm{w}} \mathrm{C}_{\kappa}$.

Proof. By Propositions 4.5.7 and 4.5.6, it is enough to prove $B_{\mathrm{I}}^{\kappa} \not\leq_{\mathrm{w}} \mathrm{B}_{\kappa}$. We define $\delta = \delta_{\mathbb{R}_{\kappa}^{<}} \otimes \delta_{\mathbb{R}_{\kappa}^{>}}$. By contradiction assume $B_{\mathrm{I}}^{\kappa} \leq_{\mathrm{w}} \mathrm{B}_{\kappa}$. Then there are two continuous functions H and K such that for every realizer b_{κ} of B_{κ} we have that

$$H \circ \lceil \mathrm{ID}, b_{\kappa} \circ K \rceil \text{ is a realizer of } B_{\mathrm{I}}^{\kappa}.$$

Consider the following closed set

$$C = \{\lceil p, q \rceil \in \kappa^{\kappa} \mid \delta_{\mathbb{R}_{\kappa}^{<}}(p) \leq \delta_{\mathbb{R}_{\kappa}^{>}(q)}\}.$$

Without loss of generality we can assume $\mathrm{dom}(K) = C$. Then define the following sequence:

$$P_{\alpha} = \{p \in \kappa^{\kappa} \mid \delta_{\mathbb{R}_{\kappa}^{<}}(K(p)) \leq \alpha\},$$

for $\alpha \in \kappa$. Note that, since K is continuous, we have that for every $\alpha \in \kappa$ the set P_{α} is closed in $\mathrm{dom}(K)$. Moreover, since C is closed and κ^{κ} is κ-Polish, C is completely κ-metrizable. We claim that C satisfies the requirement of the κ-Baire Category theorem (see Theorem 3.5.25). As we have just seen, $C = \bigcup_{\alpha \in \kappa} P_{\alpha}$. Moreover by the definition for every $\alpha < \beta < \kappa$, we have $P_{\alpha} \subseteq P_{\beta}$. Now let $\{I_{\alpha}\}_{\alpha \in \gamma}$ with $\gamma < \kappa$ be a family of nested open balls such that for every $\alpha < \gamma$, we have $C \cap I_{\alpha} \neq \emptyset$. We want to prove that $C \cap \bigcap_{\alpha \in \gamma} I_{\alpha} \neq \emptyset$. Then we have

$$\forall \alpha < \gamma \exists \eta_{\alpha} < \kappa.\ P_{\eta_{\alpha}} \cap I_{\alpha} \neq \emptyset$$

and by regularity of κ, we have

$$\exists \eta < \kappa \forall \alpha.\ \eta_{\alpha} < \eta.$$

Moreover, by monotonicity of $(P_{\alpha})_{\alpha \in \kappa}$

$$\forall \alpha < \gamma.\ P_{\eta_{\alpha}} \subseteq P_{\eta},$$

therefore

$$\forall \alpha < \gamma.\ I_{\alpha} \cap P_{\eta} \neq \emptyset,$$

which by Corollary 3.5.25 implies

$$\bigcap_{\alpha \in \gamma} I_{\alpha} \cap P_{\eta} \neq \emptyset.$$

In conclusion $\bigcap_{\alpha \in \gamma} I_{\alpha} \cap C = C \cap \bigcap_{\alpha \in \gamma} I_{\alpha} \neq \emptyset$ as desired.

By the κ-Baire Category Theorem (i.e., Theorem 3.5.24), we have that there are $\alpha \in \kappa$ and $w \in \kappa^{<\kappa}$ such that $\emptyset \neq [w] \cap C \subset P_{\alpha}$. Now we fix a realizer b_{κ} of B_{κ} defined as follows

$$\delta_{\kappa}(b_{\kappa}(p)) = \max\{\alpha, \min\{\beta \mid \beta \geq \delta_{\mathbb{R}_{\kappa}^{<}}(p)\}\}.$$

Then we have

$$P_{\alpha} = \{p \in \kappa^{\kappa} \mid \delta_{\kappa}(b_{\kappa}(K(p))) = \alpha\}.$$

Without loss of generality we can assume that w is long enough to code the interval $[a, b]$ (i.e., $\delta([w]) \subset [a, b]$). Now take $p \in \kappa^{\kappa}$ such that $w \subset p$ and $I = \mathrm{B}_{\mathrm{I}}^{\kappa}\delta(p) = [a', b']$ with $a < a' < b' < b$. Then by definition $x = \delta_{\mathbb{R}_{\kappa}}(H(\lceil p, \mathrm{B}_{\kappa}K(p)\rceil)) \in I$. This means that either $x \in [a', b')$ or $x \in (a', b']$. Assume that $x \in (a', b']$, a similar proof works for the other case. Take an open interval J such that $a' < J$. By continuity of $\delta_{\mathbb{R}_{\kappa}} \circ H$, there is $v \in \kappa^{<\kappa}$ such that $w \subset v \subset p$ and $\delta_{\mathbb{R}_{\kappa}}(H\lceil [v], \mathrm{B}_{\kappa}H(p)\rceil) \subset J$. As before assume v long enough to represent an interval $[a'', b'']$ such that

$$a < a'' < a' < b' < b'' < b.$$

Then there is $p' \in \kappa^\kappa$ such that $\mathrm{B}_{\mathrm{I}}^{\kappa}\delta(p') \subseteq (a'', a']$. Hence

$$\delta_{\mathbb{R}_\kappa}(H(\lceil p', b_\kappa K(p')\rceil)) \notin J.$$

But since $p' \in [w]$ we have $\delta_{\mathbb{R}_\kappa}(H(\lceil p', b_\kappa K(p)\rceil)) \in J$ and $b_\kappa K(p) = b_\kappa K(p')$. This contradicts the hypothesis. Therefore H and K do not reduce B_{I}^{κ} to B_κ. □

In Figure 4.2 many arrows are still missing. In particular we have:

Open Question 4.5.10. *Are the following true:*

Q1*:* $\mathrm{C}_\kappa \not\leq_{\mathrm{w}} \mathrm{C}_{\mathrm{I}}^{\kappa}$.

Q2*:* $\mathrm{C}_{\mathrm{I}}^{\kappa} \leq_{\mathrm{w}} \mathrm{C}_{U^1}$.

Note that that a positive answer to Q1 would imply $\mathrm{C}_{U^1} \not\leq_{\mathrm{w}} \mathrm{C}_\kappa$ and a positive answer to Q2 would imply $\mathrm{C}_{U^1} \not\leq_{\mathrm{w}} \mathrm{C}_{\mathrm{I}}^{\kappa}$. Moreover, note that in the literature there are no direct proofs for the classical version for Q1 and Q2. Indeed, Q1 is usually proved by using omniscience principles and parallelization, while Q2 is proved by showing that $\mathrm{C}_{\mathrm{I}} <_{\mathrm{w}} \mathrm{C}_{\mathrm{K}} <_{\mathrm{w}} \mathrm{C}_{\mathrm{A}}$, where C_{I}, C_{K} and C_{A} are closed intervals, closed sets and dense sets choice, respectively. A generalization of these notions is needed to start the study of Q1 and Q2 (for an overview on the classical approaches see [5]).

4.6 Baire Choice Functions

In this section we will start the study of the choice functions C_U for a particular class of sets U which characterize the Borel measurable functions over the generalized Borel hierarchy. These choice functions are the generalization of those introduced by Brattka in [3].

Let us fix for every $\lambda < \kappa^+$ limit a surjective function $f_\lambda : \kappa \to \lambda$. Define the following sets:

$$\begin{aligned} U^1 &= \bigcup_{\alpha\in\kappa} \{p \in \kappa^\kappa \mid p(\alpha) \neq 0\}, \\ U^{\alpha+1} &= \{p \in \kappa^\kappa \mid \exists\beta < \kappa.\ (p)_\beta \notin U^\beta\}, \\ U^\lambda &= \{p \in \kappa^\kappa \mid \exists\alpha < \kappa.\ (p)_\alpha \notin U^{f_\lambda(\alpha)}\}. \end{aligned}$$

First of all let us generalize the definition of Wadge reducibility from classical descriptive set theory.

Definition 4.6.1 (Wadge Reducibility)**.** *Let U and V be two subsets of κ^κ. Then U is said to be* Wadge reducible *to V, in symbols $U \leq_{\mathrm{wa}} V$ iff there is a continuous function $f :\subseteq \kappa^\kappa \to \kappa^\kappa$ such that*

$$x \in U \Leftrightarrow f(x) \in V.$$

As usual, if $U \leq_{\mathrm{wa}} V$ and $V \leq_{\mathrm{wa}} U$ we will say that U is Wadge equivalent *to V, in symbols $V \equiv_{\mathrm{wa}} U$. Moreover, let $\mathcal{A} \subset \mathcal{P}^-(\kappa^\kappa)$. If $U \in \mathcal{A}$ and for every $V \in \mathcal{A}$ we have that $V \leq_{\mathrm{wa}} U$, then U is said to be* Wadge complete *with respect to $\mathcal{A}$.*

Now we want to prove that for every $\alpha < \kappa^+$, the set U^α is Wadge complete w.r.t. the κ-Borel class $\Sigma_\alpha^{(\kappa,0)}(\kappa^\kappa)$. First note that the κ-Borel classes are closed under Wadge reducibility:

Proposition 4.6.2. *For every $\alpha < \kappa^+$, the sets $\mathbf{\Sigma}_\alpha^{(\kappa,0)}(\kappa^\kappa)$ and $\mathbf{\Pi}_\alpha^{(\kappa,0)}(\kappa^\kappa)$ are closed under Wadge reduction.*

Proof. We will proceed by induction over α.

Let $\alpha = 1$. Assume $U \in \mathbf{\Sigma}_1^{(\kappa,0)}(\kappa^\kappa)$ and let $V \subset \kappa^\kappa$ be such that $V \leq_{\mathrm{wa}} U$. Then there is a continuous f such that $f^{-1}[U] = V$. Now since by definition U is open in κ^κ and f is continuous, V is open in κ^κ. Therefore $V \in \mathbf{\Sigma}_1^{(\kappa,0)}(\kappa^\kappa)$ as desired. A similar proof works for $\mathbf{\Pi}_\alpha^{(\kappa,0)}(\kappa^\kappa)$.

Now let $\alpha > 1$. Assume $U \in \mathbf{\Sigma}_\alpha^{(\kappa,0)}(\kappa^\kappa)$, then there are $(U_\beta)_{\beta\in\kappa}$ and $(\eta_\beta)_{\beta\in\kappa}$ such that $U = \bigcup_{\beta<\kappa} U_\beta$ and for every $\beta < \kappa$ we have $\eta_\beta < \alpha$ and $U_\beta \in \mathbf{\Pi}_{\eta_\beta}^{(\kappa,0)}(\kappa^\kappa)$. Moreover assume $V \subset \kappa^\kappa$ and f continuous such that $V \leq_{\mathrm{wa}} U$. By inductive hypothesis we have $f^{-1}[U_\beta] \in \mathbf{\Pi}_{\eta_\beta}^{(\kappa,0)}(\kappa^\kappa)$ for every $\beta \in \kappa$. Then $V = \bigcup_{\beta\in\kappa} g^{-1}[U_\beta]$ and $V \in \mathbf{\Sigma}_\alpha^{(\kappa,0)}(\kappa^\kappa)$ as desired. □

Note that by the previous proposition and by the fact that the κ-Borel hierarchy over κ^κ does not collapse, we have the following:

Corollary 4.6.3. *If U is Wadge complete for $\mathbf{\Sigma}_\alpha^{(\kappa,0)}(\kappa^\kappa)$, V is Wadge complete for $\mathbf{\Sigma}_\beta^{(\kappa,0)}(\kappa^\kappa)$ and $\alpha < \beta$, then $V \not\leq_{\mathrm{wa}} U$.*

Proof. Assume $V \leq_{\mathrm{wa}} U$. Then $\mathbf{\Sigma}_\alpha^{(\kappa,0)}(\kappa^\kappa) = \mathbf{\Sigma}_\beta^{(\kappa,0)}(\kappa^\kappa)$. By Theorem 3.5.18 this is a contradiction. □

Now we will prove that for every α, the set U^α is Wadge complete for $\mathbf{\Sigma}_\alpha^{(\kappa,0)}(\kappa^\kappa)$.

Theorem 4.6.4. *Let $\alpha < \kappa^+$. Then U^α is Wadge complete for $\mathbf{\Sigma}_\alpha^{(\kappa,0)}(\kappa^\kappa)$.*

Proof. We proceed by induction over α.

For $\alpha = 1$, we have:

$$U^1 = \bigcup_{\gamma\in\kappa} \{p \in \kappa^\kappa \mid p(\gamma) \neq 0\}.$$

Note that U_1 is trivially open. Therefore $U_1 \in \mathbf{\Sigma}_1^{(\kappa,0)}(\kappa^\kappa)$. Moreover, let $V \in \mathbf{\Sigma}_1^{(\kappa,0)}(\kappa^\kappa)$. Since V is open there is a sequence $(w_\gamma)_{\gamma\in\kappa}$ in $\kappa^{<\kappa}$ such that $V = \bigcup_{\gamma\in\kappa}[w_\gamma]$. Define the following function:

$$g(p)(\alpha) = \begin{cases} 1 & \text{If } \exists\gamma < \kappa.\ w_\gamma = p\restriction\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then g is continuous. Moreover we have

$$\begin{aligned} g^{-1}[U^1] &= \bigcup_{\beta\in\kappa} \{p \in \kappa^\kappa \mid g(p)(\beta) \neq 0\} \\ &= \bigcup_{\beta\in\kappa} \{p \in \kappa^\kappa \mid \exists\gamma < \kappa.\ p\restriction\beta = w_\gamma\} \\ &= \bigcup_{\beta\in\kappa} \bigcup_{\gamma\in\kappa} \{p \in \kappa^\kappa \mid p\restriction\beta = w_\gamma\} \\ &= \bigcup_{\gamma\in\kappa} [w_\gamma] = V \end{aligned}$$

as desired.

For $\alpha = \beta + 1$, we have

$$U^{\beta+1} = \{p \in \kappa^\kappa \mid \exists\gamma < \kappa.\ (p)_\gamma \notin U^\beta\}.$$

By inductive hypothesis $U^\beta \in \mathbf{\Sigma}_\beta^{(\kappa,0)}(\kappa^\kappa)$. Moreover by definition

$$U^\alpha = \bigcup_{\gamma\in\kappa} \{p \in \kappa^\kappa \mid (p)_\gamma \notin U^\beta\},$$

therefore $U^\alpha \in \mathbf{\Sigma}_\alpha^{(\kappa,0)}(\kappa^\kappa)$. Now let $V \in \mathbf{\Sigma}_\alpha^{(\kappa,0)}(\kappa^\kappa)$. Then there are $(V_\gamma)_{\gamma\in\kappa}$ such that $V_\gamma \in \mathbf{\Pi}_\beta^{(\kappa,0)}(\kappa^\kappa)$ for every $\gamma \in \kappa$ and $V = \bigcup_{\gamma\in\kappa} V_\gamma$. Moreover, since U^β is Wadge complete for $\mathbf{\Sigma}_\beta^{(\kappa,0)}(\kappa^\kappa)$, there are continuous functions $(f_\gamma)_{\gamma\in\kappa}$ such that $f_\gamma^{-1}[(\kappa^\kappa \setminus U^\beta)] = V_\gamma$ for every $\gamma \in \kappa$. Now define a continuous function f as follows:

$$(f(p))_\gamma = f_\gamma(p) \qquad \forall p \in \kappa^\kappa \forall\gamma \in \kappa.$$

We have that

$$\begin{aligned} p \in V &\Leftrightarrow \exists\gamma < \kappa. p \in V_\gamma \\ &\Leftrightarrow \exists\gamma < \kappa.(f(p))_\gamma = f_\gamma(p) \notin U^\beta \\ &\Leftrightarrow f(p) \in U^\alpha \end{aligned}$$

as desired.

Finally if $\alpha = \lambda$ limit. By definition we have

$$U^\lambda = \{p \in \kappa^\kappa \mid \exists \gamma < \kappa.\ (p)_\gamma \notin U^{f_\lambda(\gamma)}\}.$$

First note that

$$U^\lambda = \bigcup_{\gamma \in \kappa} \{p \in \kappa^\kappa \mid (p)_\gamma \notin U^{f_\lambda(\gamma)}\}.$$

Moreover, by inductive hypothesis we have

$$\{p \in \kappa^\kappa \mid (p)_\gamma \notin U^{f_\lambda(\gamma)}\} \in \mathbf{\Pi}^{(\kappa,0)}_{f_\lambda(\gamma)}(\kappa^\kappa)$$

and therefore $U^\lambda \in \mathbf{\Sigma}^{(\kappa,0)}_\lambda(\kappa^\kappa)$. Now let $V \in \mathbf{\Sigma}^{(\kappa,0)}_\lambda(\kappa^\kappa)$. By definition there are $(V_\gamma)_{\gamma\in\kappa}$ with $V_\gamma \in \mathbf{\Pi}^{(\kappa,0)}_{\eta_\gamma}(\kappa^\kappa)$ and $\eta_\gamma < \lambda$ for every $\gamma < \lambda$, such that $V = \bigcup_{\gamma\in\kappa} V_\gamma$. Define the sequence $(V'_\alpha)_{\alpha\in\kappa}$ as follows:

$$V'_\beta = \begin{cases} V_\beta & \text{iff } \eta_\beta \leq f_\lambda(\beta), \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $V = \bigcup_{\gamma\in\kappa} V'_\gamma$ and for all $\gamma < \kappa$ we have $V'_\gamma \in \mathbf{\Pi}^{(\kappa,0)}_{f_\lambda(\gamma)}(\kappa^\kappa)$. Hence by inductive hypothesis there are $(f_\gamma)_{\gamma\in\kappa}$ continuous such that $f_\gamma^{-1}[(\kappa^\kappa \setminus U^{f_\lambda(\gamma)}] = V'_\gamma$ for every $\gamma \in \kappa$. Define a continuous function f as follows:

$$(f(p))_\gamma = f_\gamma(p).$$

Hence we have

$$\begin{aligned} p \in V &\Leftrightarrow \exists \gamma < \kappa. p \in V'_\gamma \\ &\Leftrightarrow \exists \gamma < \kappa. (f(p))_\gamma = f_\gamma(p) \notin U^{f_\lambda(\gamma)} \\ &\Leftrightarrow f(p) \in U^\lambda \end{aligned}$$

as desired. □

Now we want to use the fact that for all α, the set U^α is Wadge complete for $\mathbf{\Sigma}^{(\kappa,0)}_\alpha(\kappa^\kappa)$ in order to prove that every C_{U^α} characterize set of $\mathbf{\Sigma}^{(\kappa,0)}_{\alpha+1}(\kappa^\kappa)$-measurable functions w.r.t. Weihrauch reductions. First we extend the definitions we need from descriptive set theory and from computable analysis to generalized Baire space.

Definition 4.6.5 (Weihrauch Completeness)**.** *A function* $f :\subseteq \kappa^\kappa \to \kappa^\kappa$ *is* Weihrauch complete *with respect to the set* $\mathcal{A} \subset (\kappa^\kappa)^{\kappa^\kappa}$ *iff* $f \in \mathcal{A}$ *and for every* $g \in \mathcal{A}$, $g \leq_{\mathrm{w}} f$.

Definition 4.6.6. *A function* $f :\subseteq \kappa^\kappa \to \kappa^\kappa$ *is* $\mathbf{\Sigma}^{(\kappa,0)}_\alpha(\kappa^\kappa)$-measurable *iff* $f^{-1}[O] \in \mathbf{\Sigma}^{(\kappa,0)}_\alpha(\kappa^\kappa)$ *for every open set* O *in* κ^κ*. We will denote the set of* $\mathbf{\Sigma}^{(\kappa,0)}_\alpha(\kappa^\kappa)$*-measurable functions with* M_α*.*

Note that the sequence of $(M_\alpha)_{\alpha<\kappa^+}$ is increasing.

Theorem 4.6.7. *The function* C_{U^α} *is Weihrauch complete for* $M_{\alpha+1}$*.*

Proof. First of all to prove that C_{U^α} is in $M_{\alpha+1}$, let $w \in \kappa^{<\kappa}$. We have:

$$\begin{aligned} \mathrm{C}_{U^\alpha}^{-1}[w] &= \{p \in \kappa^\kappa \mid \mathrm{C}_{U^\alpha}(p) \in [w]\} \\ &= \{p \in \kappa^\kappa \mid \forall \beta < |w|.\ (p)_\beta \in U^\alpha \Leftrightarrow w(\beta) = 0\} \\ &= \bigcap_{\substack{\beta<|w| \\ w(\beta)=0}} \{p \in \kappa^\kappa \mid (p)_\alpha \in U^\alpha\} \cap \bigcap_{\substack{\beta<|w| \\ w(\beta)=1}} \{p \in \kappa^\kappa \mid (p)_\alpha \notin U^\alpha\}. \end{aligned}$$

Since in κ^κ intersections of less than κ open set are open, we have $\mathrm{C}_{U^\alpha}^{-1}[w] \in \mathbf{\Sigma}_{\alpha+1}^{(\kappa,0)}(\kappa^\kappa)$ as desired.

Now we want to show that C_{U^α} is Weihrauch complete for $M_{\alpha+1}$. Let $f :\subseteq \kappa^\kappa \to \kappa^\kappa$ be a function in $M_{\alpha+1}$. Fix a bijection $b : \kappa \to \kappa^{<\kappa}$. Since f is in $M_{\alpha+1}$, there is a sequence $(V_\gamma)_{\gamma\in\kappa}$ of elements of $\mathbf{\Sigma}_{\alpha}^{(\kappa,0)}(\kappa^\kappa)$, such that

$$f^{-1}[b(\gamma)] = \bigcup_{\beta\in I_\gamma} (\kappa^\kappa \setminus V_\beta),$$

for some sequence $(I_\gamma)_{\gamma\in\kappa}$ with $I_\gamma \subset \kappa$ for every $\gamma < \kappa$. Since we know that U^α is Wadge complete for $\mathbf{\Sigma}_{\alpha}^{(\kappa,0)}(\kappa^\kappa)$, for each $\gamma < \kappa$, there is $g_\gamma :\subseteq \kappa^\kappa \to \kappa^\kappa$ continuous such that

$$f^{-1}[b(\gamma)] = \bigcup_{\beta\in I_\gamma} (\kappa^\kappa \setminus g_\beta^{-1}[U^\alpha]).$$

Now define the continuous function $K :\subseteq \kappa^\kappa \to \kappa^\kappa$ as follows:

$$(K(p))_\gamma = g_\gamma(p),$$

for every $\gamma \in \kappa$ and $p \in \kappa^\kappa$. Then we have

$$\begin{aligned} f(p) \in [b(\gamma)] &\Leftrightarrow p \in f^{-1}[b(\gamma)] \\ &\Leftrightarrow \exists \beta \in I_\gamma.\ p \in (\kappa^\kappa \setminus g_\beta^{-1}[U^\alpha]) \\ &\Leftrightarrow \exists \beta \in I_\gamma.\ g_\beta(p) \notin U^\alpha \\ &\Leftrightarrow \exists \beta \in I_\gamma.\ (K(p))_\beta \notin U^\alpha \\ &\Leftrightarrow \exists \beta \in I_\gamma.\ \mathrm{C}_{U^\alpha}(K(p))(\beta) = 1. \end{aligned}$$

Now define $H :\subseteq \kappa^\kappa \to \kappa^\kappa$ as follow:

$$H(p) = \bigcup \{b(\gamma) \mid \exists \beta \in I_\gamma.\ p(\beta) = 1\}.$$

Therefore H and K are continuous in their domains. Moreover we have

$$\begin{aligned} H(\mathrm{C}_{U^\alpha}(K(p))) &= \bigcup \{b(\gamma) \mid \exists \beta \in I_\gamma.\ \mathrm{C}_{U^\alpha}(K(p)) = 1\} \\ &= \bigcup \{b(\gamma) \mid f(p) \in [b(\gamma)]\} \\ &= f(x). \end{aligned}$$

Hence H and K reduce f to C_{U^α} as desired. □

From the previous theorem we have:

Corollary 4.6.8. *For every $1 \le \alpha, \beta < \kappa^+$, if $\alpha \le \beta$, then $\mathrm{C}_{U^\alpha} \le_{\mathrm{w}} \mathrm{C}_{U^\beta}$.*

Note that, since we are not guaranteed that the sequence $(M_\alpha)_{\alpha\in\kappa^+}$ does not collapse, we do not know if the sequence $(\mathrm{C}_{U^\alpha})_{\alpha\in\kappa^+}$ collapses. In particular we leave the following open question:

Open Question 4.6.9. *Is it true that for every $1 \le \alpha, \beta < \kappa^+$ such that $\alpha < \kappa$, we have $\mathrm{C}_{U^\beta} \not\le_{\mathrm{w}} \mathrm{C}_{U^\alpha}$?*

4.7 Representation of the IVT

In this last section we will start the study of the IVT_κ by means of Weihrauch degrees.

First of all note that the IVT_κ can be stated as follows:

$$\forall f \in \mathrm{C}_{[a,b]} \exists c \in [a,b].\ f(c) = 0,$$

where $\mathrm{C}_{[a,b]}$ is the set of continuous functions with domain $[a,b]$ over $\mathbb{R}_\kappa$ such that $f(a)\cdot f(b)$ is smaller than 0. Then it can be represented by the following multi-valued function:

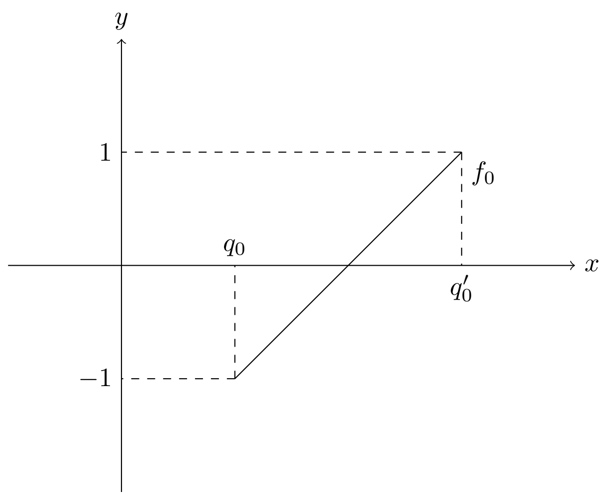
$$\mathrm{IVT}_\kappa : \mathrm{C}_{[a,b]} \rightrightarrows [a,b],\ \mathrm{IVT}_\kappa(f) = \{c \in [a,b] \mid f(c) = 0\},$$

where $\mathrm{C}_{[a,b]}$ is endowed with the standard representation $[\delta_\mathrm{C} \to \delta_\mathrm{C}]$ restricted to $\mathrm{C}_{[a,b]}$ and $[a,b]$ is represented by $\delta_\mathrm{C}{\restriction}[a,b]$.

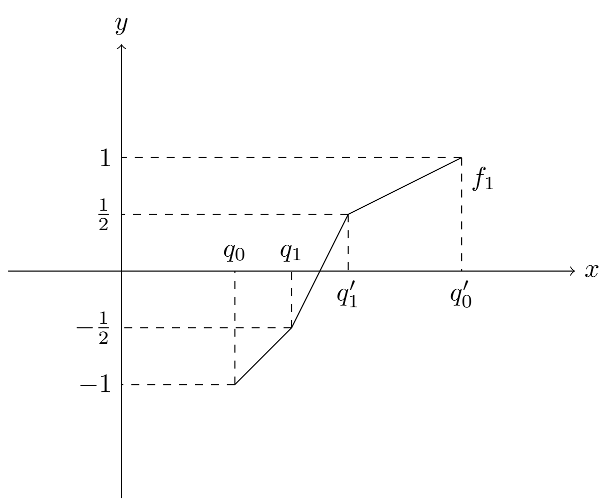
We will prove that $\mathrm{B}_\mathrm{I}^\kappa \leq_{\mathrm{s,w}} \mathrm{IVT}_\kappa$. We will follow the same strategy used in classical computable analysis (see [5]). Since the proof of the following theorem is particularly complicated due to the representation of κ-continuous functions over $\mathbb{R}_\kappa$, we will first give an intuition of why it works. First of all, we are given two Cauchy sequences $Q = (q_\alpha)_{\alpha\in\kappa}$ and $Q' = (q'_\alpha)_{\alpha\in\kappa}$ such that Q is strictly increasing, Q' is strictly decreasing and $\sup Q \leq \inf Q'$. We want to produce the code of a κ-continuous function f which has the following property:

$$f(r) = 0 \Rightarrow \sup Q \leq r \leq \inf Q' \text{ and } \forall r \in \mathbb{R}_\kappa\,.$$

To do so we will define f as the point-wise limit of a sequence of polygons in $\mathbb{R}_\kappa$. In particular we will start taking $q_0 \in Q$ and $q'_0 \in Q'$ and we define the following polygon f_0:



Now we take $q_1 \in Q$ and $q'_1 \in Q'$ (note that $q_0 < q_1 < q'_1 < q'_0$) and define the following polygon f_1:



We can iterate this process building a sequence of κ polygons (note that there is no difficulty in defining these polygons also at limit stages). Our function f would be the limit of the polygons $(f_\alpha)_{\alpha\in\kappa}$. Note that at every stage $\alpha \in \kappa$, the value of the function f for real numbers $r \leq q_\alpha$ and $r' \geq q'_\alpha$ is already determined by its value on f_α. We can then use this fact to build a continuous function that given a code for Q and Q' returns a code for f. In particular if we start reading Q and Q' at some point we read the first $\alpha < \kappa$ elements of both the sequences. Then we will have enough information to determine the values of f for all those κ-reals which are either smaller than q_α or greater than q'_α. Following the intuition given by Wadge strategies, the process we have just described defines a continuous reduction of $\mathrm{B}^\kappa_{\mathrm{I}}$ to IVT_κ.

Theorem 4.7.1. $\mathrm{B}^\kappa_{\mathrm{I}} \leq_{\mathrm{s,w}} \mathrm{IVT}_\kappa$.

Proof. Let $(q_\alpha)_{\alpha\in\kappa}$ and $(q'_\alpha)_{\alpha\in\kappa}$ be two sequences of rational numbers such that $\sup_{\alpha\in\kappa} q_\alpha \leq \inf_{\alpha\in\kappa} q'_\alpha$. Note that by the definition of $\mathrm{B}^\kappa_{\mathrm{I}}$, the sequence $(q_\alpha)_{\alpha\in\kappa}$ is strictly increasing and the sequence $(q'_\alpha)_{i\in\kappa}$ is strictly decreasing. We want to define a κ-continuous function f such that for every $x \in \mathbb{R}_\kappa$, if $f(x) = 0$, then $x \in [\sup_{\alpha\in\kappa} q_\alpha, \inf_{\alpha\in\kappa} q'_\alpha]$. We define a sequence of functions $(f_\alpha)_{\alpha\in\kappa}$ such that $\lim_{\alpha\in\kappa} f_\alpha = f$ as follows:

$$f_0(x) = \frac{2}{q'_0 - q_0}(x - q_0) - 1,$$

$$f_{\alpha+1}(x) = \begin{cases} f_\alpha(x) & \text{iff } x \leq q_\alpha \vee x \geq q'_\alpha, \\ \frac{1}{(\alpha^2+\alpha)(q_{\alpha+1}-q_\alpha)}(x - q_\alpha) - \frac{1}{\alpha} & \text{iff } q_\alpha < x < q_{\alpha+1}, \\ \frac{1}{(\alpha^2+\alpha)(q'_\alpha-q'_{\alpha+1})}(x - q'_{\alpha+1}) + \frac{1}{\alpha+1} & \text{iff } q'_{\alpha+1} < x < q'_\alpha, \\ \frac{2}{(\alpha+1)(q'_{\alpha+1}-q_{\alpha+1})}(x - q_{\alpha+1}) - \frac{1}{\alpha+1} & \text{otherwise} \end{cases}$$

$$f_\lambda(x) = \begin{cases} f_\beta(x) & \text{iff } \exists\beta < \lambda.\ q_\beta \leq x \leq q_{\beta+1}, \\ f_\beta(x) & \text{iff } \exists\beta < \lambda.\ q'_{\beta+1} \leq x \leq q'_\beta, \\ \frac{2}{\lambda(q'_\lambda-q_\lambda)}(x - q_\lambda) - \frac{1}{\lambda} & \text{otherwise.} \end{cases}$$

Note that for every x if $x \in (q_\alpha, q_{\alpha+1})$ or $x \in (q'_{\alpha+1}, q'_\alpha)$ for some $\alpha < k$ then for all $\beta \geq \alpha$ we have $f_\beta(x) = f_\alpha(x)$. Hence $(f_\alpha)_{\alpha\in\kappa}$ is well defined. By an easy induction one can prove the following properties:

If $x \in [q_\beta, q_{\beta+1}]$ for some $\beta < \alpha$ then $-\frac{1}{\beta} < f_\alpha(x) < -\frac{1}{\beta+1}$.

If $x \in [q'_{\beta+1}, q'_\beta]$ for some $\beta < \alpha$ then $\frac{1}{\beta+1} < f_\alpha(x) < \frac{1}{\beta}$.

If $x \in (q_\alpha, q'_\alpha)$ then $-\frac{1}{\alpha} < f_\alpha(x) < \frac{1}{\alpha}$.

Now let $\alpha \in \kappa$. We claim that for every $x \in \mathbb{R}_\kappa$ and $\beta, \gamma > 2\alpha$ the following holds:

$$|f_\beta(x) - f_\gamma(x)| < \frac{1}{\alpha}.$$

Without loss of generality assume $\beta < \gamma$. We have the following cases:
If $x \in [q_{\beta'}, q_{\beta'+1}]$ or $x \in [q'_{\beta'+1}, q'_{\beta'}]$ for some $\beta' < \beta$, then

$$|f_\beta(x) - f_\gamma(x)| = 0 < \frac{1}{\alpha}.$$

If $x \in (q_\gamma, q'_\gamma)$, then we have

$$-\frac{1}{\gamma} < f_\gamma(x) < \frac{1}{\gamma} \text{ and } -\frac{1}{\beta} < f_\beta(x) < \frac{1}{\beta}.$$

Now, since $2\alpha < \beta < \gamma$ we have

$$|f_\beta(x) - f_\gamma(x)| \leq \frac{2}{\gamma} < \frac{1}{\alpha}$$

as desired.

If $x \in [q_{\gamma'}, q_{\gamma'+1}]$ for $\beta < \gamma' < \gamma$, then we have that

$$-\frac{1}{\beta} < f_\beta(x) < \frac{1}{\beta}$$

and

$$-\frac{1}{\gamma'} < f_\gamma(x) < -\frac{1}{\gamma'+1},$$

therefore

$$|f_\beta(x) - f_\gamma(x)| \leq \frac{2}{\gamma} < \frac{1}{\alpha}$$

as desired.

Hence for every x, $(f_\alpha(x))_{\alpha\in\kappa}$ is a Cauchy sequence and $f(x) = \lim_{\alpha\in\kappa} f_\alpha(x)$ is well defined over $\mathbb{R}_\kappa$.

Moreover we claim that $f\restriction[q_0, \sup_{\alpha\in\kappa} q_\alpha]$ and $f\restriction[\inf_{\alpha\in\kappa} q'_\alpha, q'_0]$ are strictly increasing. First we prove that for every $\alpha < \kappa$, the polygon f_α is strictly increasing. By induction over α:

Assume $\alpha = 0$. Note that f_0 is trivially strictly increasing.

Assume $\alpha = \alpha' + 1$. Take $x, y \in [q_0, q'_0]$ with $x < y$. We only prove some cases (the others are proved similarly):

- if $x, y \leq q_{\alpha'}$ or $x, y \geq q'_{\alpha'}$ or $x \leq q_{\alpha'}$ and $y \geq q_{\alpha'}$, then the statement follows by inductive hypothesis.
- if $x, y \in (q_{\alpha'}, q_\alpha)$, we have

$$\frac{1}{(\alpha'\cdot\alpha'+\alpha)(q_{\alpha'+1}-q_{\alpha'})}(x - q_{\alpha'}) - \frac{1}{\alpha'} < \frac{1}{(\alpha'\cdot\alpha'+\alpha)(q_{\alpha'+1}-q_{\alpha'})}(y - q_{\alpha'}) - \frac{1}{\alpha'}$$

  iff $x < y$.

- if $x \in (q_{\alpha'}, q_\alpha)$ and $y \in (q'_\alpha, q'_{\alpha'})$ the statement follows form the fact that

$$-\frac{1}{\alpha'} < f_\alpha(x) < -\frac{1}{\alpha}$$

  and

$$\frac{1}{\alpha'} < f_\alpha(y) < \frac{1}{\alpha}.$$

For $\alpha = \lambda$ limit, the proof is analogous to that for the successor case.

Now we have that $f\restriction[q_0, \sup_{\alpha\in\kappa} q_\alpha]$ and $f\restriction[\inf_{\alpha\in\kappa} q'_\alpha, q'_0]$ are strictly increasing. Indeed, let $x, y \in [q_0, \sup_{\alpha\in\kappa} q_\alpha]$ (a similar argument works for x and y in $[\inf_{\alpha\in\kappa} q'_\alpha, q'_0]$). Then we have the following cases:

If $y = \sup_{\alpha\in\kappa} q_\alpha$ then, since $\lim_{\alpha\in\kappa} \frac{1}{\alpha} = 0$ and for every $\beta \leq \alpha$

$$f_\beta(q_\alpha) = -\frac{1}{\alpha},$$

hence we have that

$$f(\sup_{\alpha\in\kappa} q_\alpha) = 0.$$

Moreover, since $x < \sup_{\alpha\in\kappa} q_\alpha$, there is $\alpha < \kappa$ such that $q_\alpha \leq x \leq q_{\alpha+1}$ therefore for every $\beta \geq \alpha$ we have

$$f_\beta(q_\alpha) \leq f_\beta(x) \leq f_\beta(q_{\alpha+1}) < 0$$

and for every $\beta \geq \alpha$ we have

$$f_\beta(q_\alpha) = f_\alpha(q_\alpha), f_\beta(x) = f_\alpha(x),\ f_\beta(q_{\alpha+1}) = f_\alpha(q_{\alpha+1}).$$

Hence

$$f(q_\alpha) \leq f_\beta(x) \leq f(q_{\alpha+1}) < 0 = f(y).$$

Let $\alpha, \beta \in \kappa$ such that $q_\alpha \leq x \leq q_{\alpha+1}$ and $q_\beta \leq y \leq q_{\beta+1}$. For the same reasons used in the previous case we have:

$$f_m(x) \leq f_m(q_{\alpha+1}) \leq f_m(q_\beta) \leq f_m(y),$$

where $m = \max\{\alpha, \beta\}$. Now, since f_m is strictly monotonic, one of the inequalities is strict. Moreover, since $\alpha' \geq m$ we have

$$f_{\alpha'}(x) = f_m(x),\ f_{\alpha'}(y) = f_m(y),$$
$$f_{\alpha'}(q_{\alpha+1}) = f_m(q_{\alpha+1}),\ f_{\alpha'}(q_\beta) = f_m(q_\beta).$$

Therefore, $f(x) < f(y)$ as desired.

Now note that $f\restriction[q_0, \sup_{\alpha\in\kappa} q_\alpha]$ and $f\restriction[\inf_{\alpha\in\kappa} q'_\alpha, q'_0]$ are also surjective respectively over $[-1, 0]$ and $[0, 1]$. Hence by Lemma 3.2.5 they are κ-continuous. Moreover $f\restriction[\sup_{\alpha\in\kappa} q_\alpha, \inf_{\alpha\in\kappa} q'_\alpha]$ is the constant function 0 hence it is κ-continuous. But then, by Theorem 3.4.24 f is κ-continuous as desired.

Finally we will prove that the function reducing $\mathrm{B}^\kappa_\mathrm{I}$ to IVT_κ is continuous.

Let us fix a listing of codes for Cauchy sequences $(p_r)_{r\in\mathbb{R}_\kappa}$ such that $\delta_\mathrm{C}(p_r) = r$. Let $w \in \kappa^{<\kappa}$ be such that it is an initial segment for a code of the sequences $(q_\alpha)_{\alpha\in\kappa}$ and $(q'_\alpha)_{\alpha\in\kappa}$ in $\mathbb{Q}_\kappa$ such that $\sup_{\alpha\in\kappa} q_\alpha \leq \inf_{\alpha\in\kappa} q'_\alpha$. Without loss of generality we can assume that w is long enough to contain the code for two initial subsequences $(q_\alpha)_{\alpha\in\gamma}$ and $(q'_\alpha)_{\alpha\in\gamma'}$ respectively of $(q_\alpha)_{\alpha\in\kappa}$ and $(q'_\alpha)_{\alpha\in\kappa}$. Note that $(q_\alpha)_{\alpha\in\gamma}$ is strictly increasing and $(q'_\alpha)_{\alpha\in\gamma'}$ is strictly decreasing. We define a Wadge strategy $\theta :\subseteq \kappa^{<\kappa} \to \kappa^{<\kappa}$ as follows:

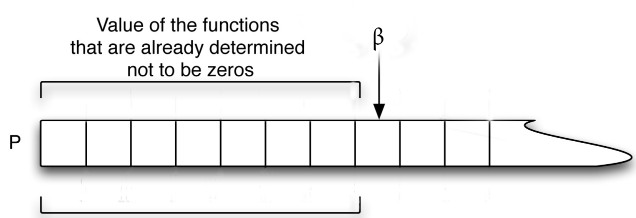
$$\theta(w) = w',$$

where w' is defined as follows:

Let $b : \kappa^{<\kappa} \to \kappa$ be the bijection used to define δ_Ws (see Section 4.3). First we want to determine the length of w'. Let B be the following set:

$$B = \{\beta' \mid \forall\alpha < \beta'(\exists\alpha' < \gamma.\ \delta_\mathrm{C}[b^{-1}(\alpha)] \subset [q_{\alpha'}, q_{\alpha'+1}])\vee$$
$$(\exists\alpha'' < \gamma'.\delta_\mathrm{C}[b^{-1}(\alpha)] \subset [q'_{\alpha''+1}, q'_{\alpha''}])\}.$$

Intuitively B is the set of ordinals β' such that every $\alpha < \beta'$ is the code of a prefix of a real number which has already been determined not to be in between $\sup_{\gamma\in\kappa} q_\gamma$ and $\inf_{\gamma\in\kappa} q'_\gamma$. The situation is illustrated by the following figure:



Now, by using B, we can determine the length of w'. Let the length of w' be the following ordinal:

$$\ell = \begin{cases} \max B & \text{If } B \text{ is bounded,} \\ |w| & \text{otherwise.} \end{cases}$$

Then for every $\alpha < \ell$, the sequence $w'(\alpha)$ is defined as follows:

If $\delta_\mathrm{C}([b^{-1}(\alpha)]) = \emptyset$: then $w'(\alpha)$ is the prefix of length $|w|$ of p_0 (the Cauchy sequence for 0). Note that these entries are meaningless for $[\delta_\mathrm{C} \to \delta_\mathrm{C}]$.

If $\delta_{\mathrm{C}}([b^{-1}(\alpha)]) \neq \emptyset$: in this case we know that $f(\delta_{\mathrm{C}}([b^{-1}(\alpha)]))$ is already partially determined. In fact by definition of ℓ, we know that $\delta_{\mathrm{C}}([b^{-1}(\alpha)])$ is going to be in between two of the κ-rationals of the sequences we have processed so far. Hence, we have that

$$\delta_{\mathrm{C}}([b^{-1}(\alpha)]) \subseteq [q_{\alpha'}, q_{\alpha'+1}] \text{ or } \delta_{\mathrm{C}}([b^{-1}(\alpha)]) \subseteq [q'_{\alpha'+1}, q'_{\alpha'}],$$

for some $\alpha' < \gamma$. Let α_{m} be the minimal such α'. Then $w'(\alpha)$ is defined as the longest common prefix of the sequences in the following set:

$$\{p_r \mid r \in f_{\alpha_{\mathrm{m}}}(\delta_{\mathrm{C}}([b^{-1}(\alpha)]))\}.$$

Note that, by the definition of ℓ and since the κ-rational sequences are monotone, then θ is monotone. Let $h' : \kappa^\kappa \to \kappa^\kappa$ be the continuous function:

$$h'(p) = \bigcup_{\alpha\in\kappa} (\theta(p{\restriction}\alpha)),$$

where $\mathrm{dom}(h)$ is the set of codes of sequences of κ-rationals $(q_\alpha)_{\alpha\in\kappa}$ and $(q'_\alpha)_{\alpha\in\kappa}$. Then for every $p \in \mathrm{dom}(h')$ and $p' \in \kappa^\kappa$, we have

$$f(\delta_{\mathrm{C}}(p')) = \delta_{\mathrm{C}}(\bigcup_{\alpha\in\kappa} h'(p)(p'{\restriction}\alpha))$$

and therefore $[\delta_{\mathrm{C}} \to \delta_{\mathrm{C}}](h'(p)) = f$ as desired.

□

In the classical case the converse also holds, namely we have $\mathrm{IVT} \leq_{\mathrm{w}} \mathrm{B_I}$. In the classical proof, given a continuous function f, the sequences $(q_\alpha)_{\alpha\in\kappa}$ and $(q'_\alpha)_{\alpha\in\kappa}$ such that $\sup_{\alpha\in\kappa} q_\alpha \leq \inf_{\alpha\in\kappa} q'_\alpha$ are defined by using the division algorithm. One can start from two rationals q_0 and q'_0 such that $q_0 < q'_0$ and $f(q_0)\cdot f(q'_0) < 0$. Then q_1 and q'_1 can be choose by looking at the sign of f at $\frac{q'_1+q_1}{2}$. In general, to define the rationals q_{n+1} and q'_{n+1} given the fact that q_n and q'_n have already been defined, it is enough to look at the sign of $f(\frac{q'_n+q_n}{2})$ and set

$$q_{n+1} = q_n \text{ and } q'_{n+1} = \frac{q'_n + q_n}{2}$$

if $f(\frac{q'_n+q_n}{2}) \cdot f(q_n) < 0$, and

$$q_{n+1} = \frac{q'_n + q_n}{2} \text{ and } q'_{n+1} = q'_n$$

otherwise. Note that, since $\mathbb{R}_\kappa$ is not Dedekind complete we cannot use the same approach. In particular, for $\mathbb{R}_\kappa$, given two sequences of κ-rationals $(q_\alpha)_{\alpha\in\lambda}$ and $(q'_\alpha)_{\alpha\in\lambda}$ with λ limit smaller than κ, such that

$$\forall\alpha \in \lambda.\ q_\alpha < q'_\alpha \ \wedge\ f(q_\alpha)\cdot f(q'_\alpha) < 0 \text{ and}$$
$$\forall\alpha,\beta \in \lambda.\ \alpha < \beta \Rightarrow q_\alpha < q_\beta \ \wedge\ q'_\alpha > q'_\beta,$$

we are not guaranteed that there is a zero $z \in \mathbb{R}_\kappa$ of f such that

$$\forall\alpha < \lambda.\ q_\alpha < z < q'_\alpha.$$

Indeed it could happen that $\sup_{\alpha\in\lambda} q_\alpha, \inf_{\alpha\in\lambda} q'_\alpha \notin \mathbb{R}_\kappa$ and for every $x, y \in (\sup_{\alpha\in\lambda} q_\alpha, \inf_{\alpha\in\lambda} q'_\alpha)$, we have $f(x)\cdot f(y) \leq 0$.

For this reason, we think that a deeper study of κ-continuous functions is needed in order to determine if $\mathrm{IVT}_\kappa \leq_{\mathrm{w}} \mathrm{B}^\kappa_{\mathrm{I}}$. Therefore we leave the following open question:

Open Question 4.7.2. *Is the* IVT_κ *Weihrauch equivalent to* $\mathrm{C}^\kappa_{\mathrm{I}}$*?*

Chapter 5

Conclusions and Open Questions

5.1 Summary

In this thesis we have started the study of generalized computable analysis, namely the generalization of computable analysis to generalized Baire spaces. In particular we have defined $\mathbb{R}_\kappa$, a field extension of the real numbers which has the following properties:

R1: $\mathbb{R}_\kappa$ is a real closed field.

R2: $\mathbb{R}_\kappa$ has cardinality 2^κ.

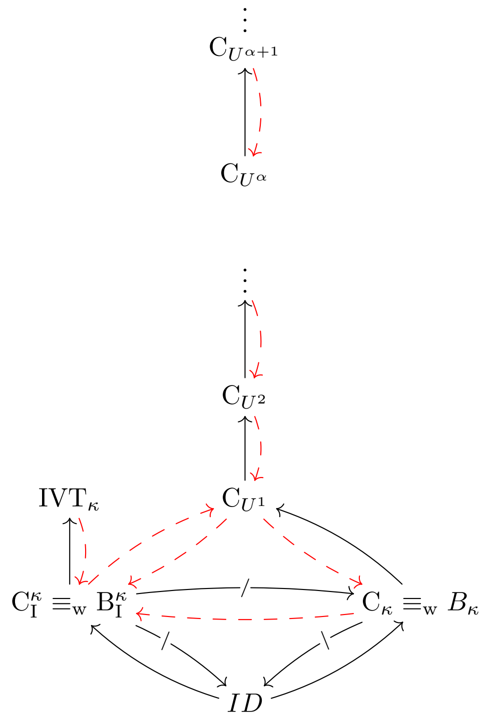
R3: $\mathbb{R}_\kappa$ has a dense subset of cardinality κ.

R4: $\mathbb{R}_\kappa$ is an η_κ-set.

Moreover, we have proved that $\delta(\mathbb{R}_\kappa) = \kappa$ and that $\mathbb{R}_\kappa$ is Cauchy complete. Then we have showed that it is possible to prove basic results from analysis over $\mathbb{R}_\kappa$. In particular, we proved that the IVT holds for κ-continuous functions over $\mathbb{R}_\kappa$ and the EVT holds for κ-super continuous functions over $\mathbb{R}_\kappa$.

We extended the basic machinery of computable analysis by using the generalized Baire space κ^κ and we started the study of $\mathbb{R}_\kappa$ from a computable analysis prospective. We showed that the most important classical representations of $\mathbb{R}$ can be naturally generalized to $\mathbb{R}_\kappa$ and that a generalized version of the Main Theorem of Computable Analysis holds for these representations. Finally we have begun the study of the Weihrauch hierarchy over $\mathbb{R}_\kappa$. In particular we proved the reductions showed in the following diagram[1]:

[1]In the diagram an arrow $A \to B$ means that $A \leq_{\mathrm{w}} B$, an arrow $A \nrightarrow B$ means that $A \nleq_{\mathrm{w}} B$ and a dashed arrow from A to B means that it is still unknown if $A \leq_{\mathrm{w}} B$. Note that many other arrows could be added to de diagram.



5.2 Future Work

Many questions are left open. In particular we highlight the following possible directions for future research in this area:

- Computability: in this thesis we have mainly focused on the topological aspects of generalized computable analysis. It is then natural to ask for their computable counterparts. A starting point in this direction will be that of providing a κ generalization of Type Two Turing Machines, which are the extension of Turing Machines used in the context of classical computable analysis.

  The problem of generalizing computability to uncountable cardinality is not a new one, of particular interest in this context are the Infinite Time Turing Machines introduced by Hamkins and Lewis in [16]. A more general introduction to the problem of computability over uncountable cardinals can be found in [25].

- Real Analysis over $\mathbb{R}_\kappa$: even though we have started the study of real analysis over $\mathbb{R}_\kappa$, many fundamental notions of real analysis and computable analysis have still to be generalized. In particular a study of the κ versions of the theorems listed in [5] would be of major interest both for the theory of real analysis over $\mathbb{R}_\kappa$ and for generalized computable analysis.

- Generalized Computable Analysis: many open questions are left in this area. In particular, the Weihrauch hierarchy is still missing many important arrows (see the previous diagram). Moreover, even though we have proved a κ version of the Baire Category Theorem and of the EVT, we haven't started studying them from a computational point of view.

- Generalized Descriptive Set Theory: in this thesis we have introduced the concept of κ-Polish spaces, and we have seen how $\mathbb{R}_\kappa$ can be used to generalize concept from standard descriptive set theory. A complete theory of these generalizations is still missing. In particular it would be interesting to see what is the relationship between $\mathbb{R}_\kappa$ and κ^κ. In the classical case, ω^ω is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$. Does this generalize to κ? Namely, is it true that κ^κ is homeomorphic to $\mathbb{R}_\kappa \setminus \mathbb{Q}_\kappa$. Note that if we set $\mathbb{R}_\omega = \mathbb{R}$ and $\mathbb{Q}_\omega$ to be the dyadic numbers the homeomorphism between $\mathbb{R}_\omega \setminus \mathbb{Q}_\omega$ and ω^ω can still be proved.

5.3 Open Questions

During this thesis many questions have been left open. In particular we have:

- Theory of $\mathbb{R}_\kappa$:

  Open Question 3.4.19: Is there any characterization of $\mathbb{R}_\kappa$ in terms of sequences of pluses and minuses?

  Open Question 3.5.22: Does Theorem 3.5.18 generalize to κ-Polish spaces of cardinality at least 2^κ?

  Open Question 3.5.23: Is every κ-Polish space the continuous image of the generalized Baire space κ^κ?

- Generalized Computable Analysis:

  Open Question 4.5.10: Are the following true:

  Q1: $\mathrm{C}_\kappa \not\leq_{\mathrm{w}} \mathrm{C}_{\mathrm{I}}^\kappa$.

  Q2: $\mathrm{C}_{\mathrm{I}}^\kappa \leq_{\mathrm{w}} \mathrm{C}_{U^1}$.

  Open Question 4.6.9: Is it true that for every $1 \leq \alpha, \beta < \kappa^+$ such that $\alpha < \kappa$, we have $\mathrm{C}_{U^\beta} \not\leq_w \mathrm{C}_{U^\alpha}$?

  Open Question 4.7.2: Is the IVT_κ Weihrauch equivalent to C_I^κ?

Bibliography

[1] N.L. Alling. *Foundations of Analysis over Surreal Number Fields.* North-Holland Mathematics Studies. Elsevier Science, 1987.

[2] V. Brattka. Computable versions of Baire's category theorem. In Jiří Sgall, Aleš Pultr, and Petr Kolman, editors, *Mathematical Foundations of Computer Science 2001*, volume 2136 of *Lecture Notes in Computer Science*, pages 224–235. Springer, 2001.

[3] V. Brattka. Effective Borel measurability and reducibility of functions. *Mathematical Logic Quarterly*, 51(1):19–44, 2005.

[4] V. Brattka, M. de Brecht, and A. Pauly. Closed choice and a uniform low basis theorem. *arXiv preprint arXiv:1002.2800*, 2010.

[5] V. Brattka and G. Gherardi. Effective choice and boundedness principles in computable analysis. *Bulletin of Symbolic Logic*, 17(1):73–117, 2011.

[6] V. Brattka, P. Hertling, and K. Weihrauch. A tutorial on computable analysis. In S. B. Cooper, B. Löwe, and A. Sobri, editors, *New computational paradigms*, pages 425–491. Springer, 2008.

[7] V. Brattka, S. Le Roux, and A. Pauly. Connected choice and the brouwer fixed point theorem. *arXiv preprint arXiv:1206.4809*, 2012.

[8] P.M. Cohn. *Basic Algebra: Groups, Rings, and Fields.* Springer, 2003.

[9] J.H. Conway. *On Numbers and Games.* Ak Peters Series. Taylor & Francis, 2000.

[10] H.G. Dales and W.H. Woodin. *Super-real Fields: Totally Ordered Fields with Additional Structure.* London Mathematical Society Monographs. Clarendon Press, 1996.

[11] L. Dries and P. Ehrlich. Fields of surreal numbers and exponentiation. *Fundamenta Mathematicae*, 167:173–188, 2011.

[12] P. Ehrlich. *Real Numbers, Generalizations of the Reals, and Theories of Continua.* Synthese Library. Springer, 1994.

[13] S.D. Friedman, T. Hyttinen, and V. Kulikov. *Generalized Descriptive Set Theory and Classification Theory*, volume 230 of *Memoirs of the American Mathematical Society.* American Mathematical Society, 2014.

[14] H. Gonshor. *An Introduction to the Theory of Surreal Numbers.* London Mathematical Society Lecture Note Series. Cambridge University Press, 1986.

[15] A. Halko. *Negligible Subsets of the Generalized Baire Space* $\omega_1^{\omega_1}$. Annales Academiae Scientiarum Fennicae: Mathematica. Suomalainen Tiedeakatemia, 1996.

[16] J. D. Hamkins and A. Lewis. Infinite time Turing machines. *The Journal of Symbolic Logic*, 65:567–604, 6 2000.

[17] A. Kechris. *Classical Descriptive Set Theory*, volume 156 of *Graduate Texts in Mathematics*. Springer, 2012.

[18] D. Klaua. Rational and real ordinal numbers. In P. Ehrlich, editor, *Real Numbers, Generalizations of the Reals, and Theories of Continua*, volume 242 of *Synthese Library*, pages 259–276. Springer, 1994.

[19] S. Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer, 2002.

[20] D. Marker. *Model Theory : An Introduction*. Graduate Texts in Mathematics. Springer, 2006.

[21] A. Mekler and J. Väänänen. Trees and-subsets of $\omega_1^{\omega_1}$. *The Journal of Symbolic Logic*, 58(03):1052–1070, 1993.

[22] J.R. Munkres. *Topology*. Featured Titles for Topology Series. Prentice Hall, 2000.

[23] A. Pauly. On the topological aspects of the theory of represented spaces. *arXiv preprint arXiv:1204.3763*, 2012.

[24] M. Schröder. Extended admissibility. *Theoretical Computer Science*, 284(2):519 – 538, 2002.

[25] B. Seyfferth. *Three Models of Ordinal Computability*. PhD thesis, University of Bonn, 2013.

[26] R. Sikorski. On an ordered algebraic field. *La Societe des Sciences et Lettres Varsovie Comptes Rendus de la Classe III Sciences Mathematiques et Physiques*, 41:69–96, 1948.

[27] W.W. Wadge. *Reducibility and determinateness on the Baire space*. PhD thesis, University of California, Berkeley, 1983.

[28] K. Weihrauch. *Computable Analysis: An Introduction*. Texts in Theoretical Computer Science. An EATCS Series. Springer, 2012.