Kant's Logic in the Critique of Practical Reason
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A Logical Formalization of
Kant's Practical Transcendental Argument

MSc Thesis *(Afstudeerscriptie)*

written by

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under the supervision of **Prof. dr. Michiel van Lambalgen**, and submitted to the Board of Examiners in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam.*

Date of the public defense:
August, 28, 2015

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Abstract

Logic plays a fundamental role in Kant's transcendental philosophy. The general presupposition is that Kant's logic can be subsumed under the Aristotelian tradition. However, with respect to this presupposition some problems arise in Kant's philosophy. This thesis takes a new approach to Kant's logic: Instead of imposing a logical framework on Kant's thought this thesis aims at deriving the logical apparatus underlying Kant's reasoning via the process of logical formalization. The formalization will be concerned with Kant's practical transcendental argument. In this thesis I will, firstly, provide a philosophical model for detecting, interpreting and evaluating Kantian transcendental arguments. Secondly, I will show that Kant's argument for 'the possibility of the moral law as a synthetic a priori proposition', as found in the Critique or Practical Reason, is a transcendental argument. The main aim of this thesis is to further our understanding of Kant's logical reasoning via the logical formalization of Kant's practical transcendental argument. Furthermore, this thesis aims to solve the philosophical problems of Kant's argument on the basis of this formalization. The argument will be formalized in an intuitionistic many-sorted type-free situation calculus. The encountered axioms underlying Kant's reasoning will turn out to be intuitionistic in nature. Furthermore, the logical formalization of Kant's argument will allow for the derivation of the necessary and sufficient definition of the concept of objective validity. As such, I hope to have shown that logical analysis can substantiate philosophical research.

Keywords: Kant, transcendental argument, Critique of Practical Reason, freedom, objective validity, logical analysis, formalization, intuitionistic logic, type-free logic.

Immanuel Kant (1724 - 1804)

Consistency is the greatest obligation of a philosopher and yet the most rarely found. [Kan96a, 5:24]

But whoever knows what a formula means to a mathematician, which determines quite precisely what is to be done to solve a problem [...], will not take a formula that does this with respect to all duty in general as something that is insignificant and can be dispensed with. [Kan96a, 5:8]

Contents

Chapter 1

Introduction

Readers acquainted with the work of Immanuel Kant will, most likely, not be surprised by the two quotes on the title page of this thesis. Kant's philosophy, especially his critical work, is a prime example of bringing mathematical rigour and discipline into philosophy. In the Critique of Pure Reason Kant introduces his critical philosophy as follows:

> Now and again one hears complaints about the superficiality of our age's way of thinking, and about the decay of well-grounded science. Yet I do not see that those sciences whose grounds are well laid, such as mathematics, physics, etc., in the least deserve this charge [...]. This same spirit would also prove itself effective in other species of cognition if only care had first been taken to correct their principles. [...] Our age is the genuine age of criticism to which everything must submit. [Kan00, Footnote to Axi]

At least Kant himself seems to agree that mathematical rigour could also benefit other sciences. What is more, Kant's critical philosophy is an attempt to bring this rigour into philosophy. It is therefore quite surprising that the attempts to look at Kantian philosophy from a logical or mathematical perspective are rather small in number.[1]

Kant and Logic

Logic plays a fundamental role in Kant's critical philosophy: A major part of the Critique of Pure Reason is devoted to the establishment of transcendental logic; that is, the science that "has to do merely with the laws of the understanding and reason, but solely insofar as they are related to objects a priori" [Kan00, A57/B81-82]. Furthermore, Kant's logic has a prominent role in the argumentative structure of the transcendental deductions of the pure concepts of the understanding in the first Critique and the transcendental deduction of

[1] The following articles and books treat Kant's philosophy from a logical point of view: [AVL], [Kro76], [AVL11], [Hin69], [Pos81].

the concept of freedom in the second Critique.[2] Nevertheless, Kant's logic has received relatively little attention in the secondary literature [Kan92a, p.xv].

Even more surprising is the fact that Kant only published one work on logic: "The False Subtlety of the Four Syllogistic Figures" (1762) [Kan92b, 2:47 – 2:61].[3] In this work Kant argues against the dominant logical tradition of his era.[4] In the 18th century the field logic was still dominated by the Aristotelian tradition. Kant's stance towards Aristotelian logic must at least have been critical. However, the general presupposition is that "Kant's approach to logic falls within what can broadly be called the Aristotelian tradition" [Kan92a, p.xv]. In the following quote from the Critique of Pure Reason Kant seems to confirm this view:

> [S]ince the time of Aristotle it [logic] has not had to go a single step backwards [...]. What is further remarkable about logic is that until now it has also been unable to take a single step forward, and therefore seems to all appearance to be finished and complete. [Kan00, Bviii]

On the basis of the above presupposition some serious problems have been detected in Kant's philosophy: For example, Kant claims that his table of pure logical forms consists of irreducible fundamental concepts only. However, Strawson criticizes Kant's list of pure logical concepts and claims that "this list includes the hypothetical and disjunctive forms, the analogues of which in modern logic are interdefinable with the help of negation" [Str02, p.80]. Strawson's findings contradict Kant's claim. Achourioti and Van Lambalgen seem to correctly identify the problematic character of the criticism of Kant's logic: "it is very much tied to classical logic" [AVL11, Footnote to p.4]. Instead of writing down Kant's logical concepts and reasoning as incorrect one might wonder whether the peculiarities that arise with respect to this part of Kant's philosophy do not suggest that Kant's view on logic deviates from the Aristotelian tradition. Whether a philosophical argument is inconsistent or obscure does not only depend on the logic endorsed by the philosopher, it also depends on the reader's logical framework (unconsciously) projected on the philosophy in question. In the light of the above, a proper investigation of Kant's underlying logical apparatus is necessary.

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[2] Throughout this thesis I will frequently refer to 'the Critique of pure reason', 'the Critique of practical reason' and 'the Groundwork of The metaphysics of morals' as, respectively, 'the first Critique', 'the second Critique' and 'the Groundwork'. Furthermore, I will omit the introduction of abbreviations.

[3] The 'Lectures on Logic' consists of lecture notes taken by Kant's students [Kan92a]. The Jäsche Logic however is an exception. This work is composed by Gottlob Benjamin Jäsche on the basis of Kant's own lecture notes and at Kant's request. However, none of these works is of Kant's own writing.

[4] In this essay Kant argues that although all four traditional Aristotelian syllogistic figures are valid the claim that they must be regarded as *simple* and *pure* inferences is false. That is, with respect to the last three figures Kant argues that "it is only ever the first figure which, concealed in a syllogism by means of covert inferences, has the power to generate the conclusion" [Kan92b, 2:58].

This thesis takes a new approach to Kant's logic. Instead of imposing a particular logic on Kant's thought this thesis aims at deriving the axioms underlying Kant's philosophical reasoning via the process of logical formalization. The derived axioms will subsequently provide the proper logical framework in which Kant's argumentation and, more importantly, the apparent problems of Kant's argumentation can be (re)evaluated. However, logical formalization is perhaps not the first thing that comes to mind when talking about philosophical problems; so why formalize at all?

The Benefits of Formalization

Formalizing philosophy is not only a fun thing that logicians 'just like to do'. Looking at philosophy from a logical perspective can benefit philosophical research in several ways. The logical formalization of a philosophical argument can have the following advantages:

1▶ A logical formalization of an argument's structure provides a tool for interpreting and evaluating the philosophical argument itself.

2▶ A proper formalization of a philosophical argument can function as a guiding thread for detecting, interpreting and evaluating similar arguments presented by the philosopher.

3▶ Logical analysis of a philosophical argument can reveal (or help to detect) underlying axioms and inference rules (implicitly) endorsed by the particular philosopher.

4▶ An adequate representation of the logical structure of a philosophical argument shows the relations between and implications of the philosophical concepts at stake.

5▶ An adequate logical formalization of a philosophical argument can support the determination of the consistency of both the argument and the underlying axiomatic system endorsed by that philosopher.

6▶ A logical formalization can help to detect missing links and ambiguities in the argument that obscure the validity of the original philosophical argument.

The above six points show the possible beneficiary roles that a logical formalization can play for the understanding of philosophical texts in general.[5] The

[5]The reader must keep in mind that one can always question whether it is the philosophical argument that gives rise to particular inconsistencies and problems or the proposed formalization of the argument. Every formalization of a philosophical argument however depends on the proposed *interpretation* of the argument. It is therefore of the utmost importance that the logician engaged in such an endeavour, always explicitly justifies every step of the interpretation. The acceptance of the consequences of the logical analysis will eventually depend on the plausibility of the provided interpretation.

following point supports a more practical interest in philosophy:

7▶ A logical model of a philosophical theory can help to calculate the logical consequences of that theory. Such a model can be applied to concrete problems provided by both the philosopher and everyday life.

Furthermore, on a larger scale the representation of a philosophical theory in a universal formal language facilitates the comparison and evaluation of different philosophical theories. Differences between philosophical theories can be more easily detected whenever the ambiguities of natural language(s) and writing style(s) have been overcome.[6]

8▶ The logical formalization of different philosophical theories in one universal language facilitates comparison.

I hope that the reader agrees with me that logical formalization can be of value to philosophical research.

Kant's Transcendental Arguments and the Second Critique

Kant is well-known for his 'transcendental shift': Kant moves away from the idea that "all our cognition must conform to the objects" because this idea has shown itself to be infertile [Kan00, Bxvi]. The shift is inspired by a similar move made by Copernicus (often called the 'Copernican Revolution'). According to Kant, Copernicus' move consisted of the idea that, because any attempt to explain the movements of celestial bodies by assuming that the entire celestial host would revolve around the observer's point of view (the earth) established no success, it would be more fertile to let "the observer revolve and let the stars at rest" [Kan00, Bxvi]. For Kant this move resulted in the idea to let objects conform to our cognitions instead of the other way around; positing the question,

> whether we do not get farther with the problems of metaphysics by assuming that the objects must conform to our cognition, which would agree better with the requested possibility of an a priori cognition of them, which is to establish something about objects before they are given to us. [Kan00, Bxvi]

One of the main aims of Kant's theoretical philosophy is to find and establish these cognitions that allow us to know something about objects prior to any experience, that is, *a priori*. Most of the 'Transcendental Analytic' of the Critique of Pure Reason is devoted to this undertaking. In order to succeed Kant makes use of arguments that are, what he calls, **transcendental** in nature. In

[6] For example, one can compare utilitarian and deontological theories in a single formal framework to determine whether and where they differ when applied to moral problems.

the secondary literature arguments of this form are conventionally called **transcendental arguments**.[7] There seems to be a general consensus that these transcendental arguments can be found in Kant's theoretical philosophy (e.g. in the Critique of Pure Reason). However, whether this form of argumentation also occurs in Kant's practical philosophy is obscure.

This thesis will be primarily concerned with the Critique of Practical Reason and the problems that arise in relation to Kant's transcendental reasoning in this Critique. Here I will name three of these problems: Firstly, the most central concept of a transcendental argument is the concept of *objective validity*. Unfortunately, an exact definition of this concept is lacking in both the primary and the secondary literature. Secondly, Kant claims that the possibility of the moral law as a synthetic a priori proposition is provided by the objective validity of freedom. However, there does not seem to be a general consensus about *how*, and even *that*, this possibility is attained. Lastly, any understanding of Kant's practical transcendental argument begins with the understanding of the logical apparatus underlying Kant's reasoning. However, which axioms and rules of inference underlie Kant's transcendental reasoning remains obscure. Moreover, only a proper determination of Kant's logical apparatus will allow for a proper determination of the first two points.

In this thesis I claim that a transcendental argument can be found in Kant's Critique of Practical Reason. A working model for detecting, interpreting and evaluating Kantian transcendental arguments in general will be provided and it will be shown that Kant's argument for *the possibility of the moral law as a synthetic a priori proposition*, as provided in the Critique of Practical Reason, conforms to this proposed model. This will justify the above claim. The first part of this thesis will be philosophical.

The Aim of the Thesis

The aim of this thesis is twofold: (1) The aim is to further our understanding of Kant's logical reasoning in the Critique of Practical Reason via a logical formalization of Kant's practical transcendental argument and (2) the aim is to address (and solve) the problems that arise during the philosophical analysis of the argument on the basis of this logical formalization.

With respect to the size of this thesis I deem it necessary to elaborate a little on the main results of the logical formalization. These results provide the proper context in which Kant's philosophical argument must be read and may serve as a guiding thread throughout the first part of this thesis. The result of this thesis is threefold: Firstly, the formalization of Kant's argument

[7] As far as I am aware of, Kant never refers to his arguments as 'transcendental arguments'. Although Kant often uses the term 'transcendental deduction' (mostly in the first Critique), I am cautious of using these terms interchangeably. However, it appears that the latter should at least be a part of the former. This will become clear in the next chapter.

allows for the determination of some crucial features of the syntax underlying Kant's original reasoning. Kant's reasoning in the second Critique turns out to be essentially intuitionistic in nature. This conclusion is substantiated by (i) Kant's restrictions on the use of negation in relation to concepts and modalities (e.g. 'not-impossible' does not imply 'possible'), (ii) the absence of the use of the law of excluded middle in Kant's argumentation and (iii) the requirement of 'positive construction' in relation to the possibility of a nature (to be more precise, the supersensible nature). What is more, it turns out that in relation to some of Kant's claims about the structure of his transcendental argument a 'classical' formal interpretation of the argument would generate some contradictions.

Secondly, the logical analysis of Kant's argument furthers our understanding of the concepts that remain obscure during the philosophical analysis of the argument: For example, the formalization will allow for the derivation of the necessary and sufficient definition of the concept of objective validity.

Thirdly, the logical formalization of Kant's argument shows how the possibility of the moral law as a synthetic a priori proposition emerges from the objective validity of its ground: The concept of freedom. The synthesis is established by connecting the formal implications of the concept of negative freedom with the formal implication of the concept of positive freedom. Furthermore, it will turn out that the possibility of this synthetic a priori proposition is formally provable from Kant's transcendental argument.[8]

The second part of this thesis will be formal. The logical framework that will be sufficient to formally represent Kant's vocabulary is an *intuitionistic many-sorted type-free situation calculus* called KL. Kant's practical transcendental argument will be formally represented in the system of *natural deduction*.

Of course, interpretations of Kant's practical philosophy vary widely and, although the aim of this thesis is not to establish any new interpretation or to disprove others, any formalization attempt necessitates a fixed interpretation at its base. The interpretation provided in this thesis will be my own. Whenever I strongly differ from common interpretations of Kant's second Critique I will justify this.

Lastly, although it will be to the reader's benefit to be already acquainted with Kant's terminology and the basic outlines of his practical philosophy, the way in which this thesis is presented should enable every reader, not familiar with Kantian philosophy as such but familiar with logic and philosophy in general, to read and understand the present undertaking.

[8]With respect to the 'list of benefits' provided in the previous section, the added value of the logical formalization of Kant's argument is expressed by point 1, 3, 4 and 6.

Reference and Quotation

The translations of Kant's works that I will be using in this thesis are from the *Cambridge edition of the Works of Immanuel Kant* as published by Cambridge University Press. Reference will be to the *volume* and *page number* of the standard German edition of Kant's works (deGruyter[9]) as found in the margins of the Cambridge translations. For example, the reference '4:345' refers to page 345 of the fourth volume of the standard German edition.[10] Because the title of Kant's original work cannot be inferred from the number of the volume in which it occurs, I will also refer to the English translation of this work. Reference to volume and page of the standard German edition is most common in the secondary literature on Kant and can be found printed on the margins of most English (and Dutch) translations of Kant's works. This form of reference will facilitate comparison and cross-reference for the reader. With respect to Kant's *Critique of Pure Reason* I will endorse the convention of referring to the different editions written by Kant. The first edition of this Critique is called the 'A' edition, the second the 'B' edition. For example, 'A211/B256' refers to page 211 and page 256 of, respectively, the A and B edition. Whenever I refer to notes on Kant's lectures I will refer to both the corresponding volume and page number of the standard German edition and the original author of the notes (e.g. '24:760 - Dohna-Wundlacken'). With respect to quotation I will reserve the use of the square brackets '[' and ']' to indicate the addition or omission of words in a quote.

[9]"[T]he standard German edition of Kant's works, Kant's Gesammelte Schriften, edited by the Royal Prussian (later German) Academy of Sciences (Berlin: Georg Reimer, later Walter deGruyter & Co., 1900-)" [Kan92a, p. xii].

[10]References in the digital version of this thesis are supplied with a hyperlink to the bibliography (back and forth).

Part I

The Philosophical Analysis

Chapter 2

Transcendental Arguments

The second half of the 20^{th} century witnessed an extensive debate on the general character and structure of transcendental arguments; both in the Kantian and the non-Kantian literature.[1] Elaborations on transcendental arguments as found in the non-Kantian literature might provide insight into the nature of a transcendental argument in general, but since the aim of this thesis is to establish a logical formalization of a Kantian transcendental argument, I will only be concerned with the validity of Kant's version of the argument.[2] Based on some well-known interpretations of Kant's transcendental arguments (both positive and negative) I will propose a model that consists of three major arguments. Before this threefold structure can be attended, the general aim of a transcendental argument must be determined.

2.1 The Aim of a Transcendental Argument

Kant's transcendental philosophy is primarily focussed on the possibility of **synthetic a priori cognition**; that is, focussed on the possibility of a priori knowledge. In the Critique of Pure Reason Kant states the problem very clearly: "How are synthetic judgments a priori possible?" [Kan00, B19]. In this section I will try to show that proving the possibility of such a synthetic a priori proposition is the main aim of any transcendental argument.

In the first section of *the Deduction of the Pure Concepts of the Understanding* (Critique of Pure Reason) Kant explains the general function of a transcendental argument by showing what kind of cognition belongs to tran-

[1] Consider for example the debate around Hintikka and Gram in the following series of articles: [Büb75], [Gra71], [Gra73], [Gra77], [Hin72], [Str68] and [Wil70]. For more Kantian literature on the validity and possibility of transcendental arguments the reader is referred to: [Ame78], [Ben78], [Ben77], [Bos77], [Bru96], [Hen69], [Kör67] and [Sac05].

[2] For articles on transcendental arguments outside Kantian literature, the reader is referred to: [Ben79], [Kör67], [PG57] and [Wat75]. Gram's articles could also be subsumed under the non-Kantian literature.

scendental philosophy. He starts with the exclusion of all cognition that is either (i) empirical, in which case the cognition receives its immediate justification from experience [Kan00, A84/B116], or (ii) incapable of any deduction whatsoever (e.g. fortune and fate) [Kan00, A84/B117]. For Kant transcendental philosophy is concerned with the possibility of cognition of objects that can be established a priori, that is, the possibility of synthetic a priori cognition. Kant provides the following definition of a transcendental cognition:

> I call all cognition **transcendental** that is occupied not so much with objects but rather with our mode of cognition of objects insofar as this is to be possible a priori. A system of such concepts [cognitions] would be called transcendental philosophy. [Kan00, B25][3]

Transcendental cognitions are thus a priori cognitions of objects and transcendental philosophy is the complete system of these cognitions. Such a system would contain both analytic and synthetic a priori cognitions of objects, but since the complete determination of such a system would be to extensive, Kant is mainly concerned with the possibility of synthetic a priori cognition:

> [W]e need to take the analysis only as far as is indispensably necessary in order to provide insight into the principles of a priori synthesis in their entire scope, which is our only concern. [Kan00, B25]

A transcendental argument would then be an argument that shows the principles that enable synthetic a priori cognition.[4] A transcendental argument aims at justifying the use of concepts that are on the one hand a priori, though on the other hand, related to objects. The justification of this usage, accordingly, should explain for the possibility of the synthetic a priori cognition(s) arising from these concepts. The following quote from the first Critique confirms this:

> Among the many concepts, however, that constitute the very mixed fabric of human cognition, there are some that are also destined for pure use a priori (completely independently of all experience), and these **always require a deduction of their entitlement**, since proofs from experience are not sufficient for the lawfulness of such a use, and yet one

[3]This quote is taken from the introduction to the B edition of the first Critique. In the introduction to the A edition Kant states the following: “I call all cognition **transcendental** that is occupied not so much with objects but rather with our a priori concepts of objects in general. A system of such concepts would be called transcendental philosophy” [Kan00, A11/A12]. For the B edition Kant rewrote the first sentence, but not the latter. The word ‘concept’, in the last sentence of both quotes, refers to ‘a priori concepts of objects’, which is to be equated with ‘a priori mode of cognition of objects’. I think that Kant rewrote this sentence to emphasize the role of ‘possibility’ with respect to his transcendental endeavour; namely, the possibility of synthetic a priori cognition.

[4]The authors of the following articles all seem to agree that the aim of a transcendental is to prove the possibility of synthetic a priori cognition: Ameriks [Ame78], Benton [Ben78], Henrich [Hen69], Hintikka [Hin72] and Sacks [Sac05].

> must know how these concepts can be related to objects that they do not derive from any experience. I therefore call the explanation of the way in which concepts can relate to objects *a priori* their **transcendental deduction**. [Kan00, A85/B117 - bold emphasis my own]

The entitlement of the use of these concepts in relation to objects a priori requires a deduction.[5] The entitlement of these a priori concepts is called their *objective validity*. A transcendental argument must therefore also prove the objective validity of the a priori concepts that allow for the possibility of the synthetic a priori cognitions at stake (I will come back to the notion of objective validity in section 2.2.3). Kant calls this part of the argument the transcendental deduction. In the first Critique Kant provides a transcendental deduction of the pure concepts of the understanding: The categories. The objective validity of these categories should accordingly explain for the possibility of synthetic a priori cognition with respect to the possibility of experience (that is, with respect to theoretical reason). Philosophers who disagree on the interpretation of the structure of a transcendental argument seem to agree, though, on the argument's general aim. Consider for example Gram interpretation of the argument: "a transcendental proof shows the truth of propositions claiming the application of primitive concepts to experience" [Gra73, p.254].[6] Henrich states something quite similar, according to him the argument needs to show how the objective validity of the categories provides the possibility of synthetic a priori cognition: "It is the task of a transcendental deduction to demonstrate that the categories of our understanding are qualified to provide knowledge of appearances" [Hen69, p.641].[7]

Thus, a transcendental argument needs to show the possibility of some synthetic a priori cognition on the basis of the objective validity of some a priori concept(s) from which the synthetic a priori cognition arises. I therefore propose the following postulate:

[5]Notice that the above does not imply the existence of a priori objects (a view which Hintikka seems to espouse). The argument is about the validity of the *a priori relation* between some a priori concept and possible objects, not about a priori objects.

[6]I only use this quote to highlight the common held opinion that Kant's transcendental arguments aim at establishing a relation between concepts and objects. Gram refers here to *truth* in relation to concepts and objects. I object to this interpretation. For Kant a priori synthesis has to do with the validity of *relating* concepts to objects. The determination of the validity of a relation and the determination of the truth of a cognition are not the same. Kant's theory of truth seems to be a correspondence theory of truth and therefore requires something more than the mere validity of synthetic a priori cognitions. I regard Gram's usage of truth here as incorrect. In this thesis I will not make use of the concept of truth in relation to Kant's transcendental arguments.

[7]Again, I use these quotes only to highlight the general consensus about the aim of Kant's transcendental arguments. Henrich talks about 'qualification' as possibility, though I will show that a transcendental argument does not only show possibility, it also shows necessity (this will become clear in the upcoming sections). In relation to Henrich's remark this means that Kant also needs to show that knowledge of appearances is *impossible* without the categories of the understanding.

Postulate 1. (The Aims of a Transcendental Argument) Every transcendental argument consists of the following two aims:

I. Prove the possibility of some synthetic a priori cognition.[8]
II. Derive the a priori concepts that make this synthetic a priori cognition possible and deduce the objective validity of these concepts.

(One remark must be made. The second aim serves only as an intermediate step for establishing the first aim, but since this second aim will eventually form the most central and fundamental part of the transcendental argument, I decided to explicitly state it as a distinct aim.)

2.2 The Structure of a Transcendental Argument

In this section I will propose an interpretation of the structure of a transcendental argument in general. I claim that through its generality this structure will be applicable to both Kant's theoretical and practical philosophy. Based on the two aims presented in the previous section, I propose an interpretation of the argument that consists of three major arguments.

Recall that the main aim of the transcendental argument (aim I) is to prove the possibility of an a priori synthetic cognition on the basis of the objective validity of some a priori concepts that make this cognition possible (aim II). In Kant's philosophy, an a priori concept that makes such a cognition possible is called a *ground*. In that sense, it is the objective validity of a cognition's ground that should make the cognition at stake possible. A transcendental argument needs to solve the following three problems with respect to the possibility of some synthetic a priori cognition ϕ:

1► Which a priori conditions, say $\psi_1, ..., \psi_n$, are necessary for the possibility of ϕ?

2► Which of these necessary conditions $\psi_1, ..., \psi_n$ form the a priori ground θ of ϕ?

3► How does the possibility of ϕ follow from the objective validity of its a priori ground θ?

(NB. The third problem requires a proof for the objective validity of the ground θ.)

The solutions to these problems are provided by three distinct arguments. I will call these corresponding arguments, respectively, 'the necessity argument',

[8]Kant uses transcendental arguments to prove the possibility of synthetic a priori cognitions, -propositions and -judgments. I have chosen to use the term cognition in the above formulation because it is the most general term. Though these three terms might be used interchangeably without any risk of altering the aim of the transcendental argument.

'the possibility argument' and the 'objective validity argument'. The introduction of these arguments will be based on the secondary literature as well as my own interpretation. Whenever I differ strongly from a common position in the literature, I will provide an argument to justify this deviation.

2.2.1 The Necessity Argument

There seems to be a consensus in the Kantian literature that transcendental arguments are (at least) about the necessary conditions of a cognition. Consider for example the following quotes:[9]

> What we must derive from such a premiss is a conclusion about a necessary condition of perceiving an object. [Gra71, p.25]

> This suggests that a transcendental deduction of a particular type of knowledge demonstrates its necessary and sufficient conditions. [Ame78, p.274]

> Transcendental arguments are supposed to demonstrate the impossibility or illegitimacy of this skeptical challenge by proving that certain concepts are necessary for thought or experience. [Str68, p.242]

> Understanding transcendental proofs in terms of the necessary conditions [...] explains why what makes them unique precisely fits them to be called transcendental proofs, in Kant's specific use of the term: as identifying that which is presuppositional to experience. [Sac05, p.452]

These quotes support the claim that any adequate representation of a Kantian transcendental argument should at least include a **necessity argument**, that is, an argument that determines the necessary conditions of some synthetic a priori cognition. A necessary condition is a condition that can be found in every instance of the cognition at stake. Consequently, if one of the cognition's necessary conditions would not be the case, the cognition itself would not be the case either. The above reading shows that the necessity argument is merely analytic in nature. I suggest the following formulation of this argument:

The Necessity Argument. A transcendental argument needs to prove what concepts $\psi_1, ..., \psi_n$ are necessary conditions for the possibility of some synthetic a priori cognition ϕ.

The necessity argument, though, is not sufficient to establish the aims of the transcendental argument.

[9]The following articles support this interpretation: [Ame78], [Ben77], [Gra71], [Hen69], [Rus96], [Sac05], [Ste82] and [Str68].

The Necessity Argument is either Incorrect or Incomplete

There is a substantial amount of secondary literature in which the necessity argument is interpreted as the core, and sometimes even sole, principle of a transcendental argument. Although the necessity argument does not seem to be wrong, I will argue that, when regarded as the *sole* argument of the transcendental argument, this interpretation of the necessity argument is incorrect. Consequently, I will argue that there must be (at least) a second argument at work in every transcendental argument.

Gram seems to correctly identify a problem that arises when one equates the transcendental argument with the necessity argument:

> But all such an argument could establish is which propositions are strictly implied by other propositions. And this alone will not suffice to distinguish those propositions which state necessary conditions of the meaning or sense of a proposition from any analytic propositions which follows from the same premisses just because it is implied by any proposition at all. [Gra71, p.20]

In Gram's line of thought, if the transcendental argument is to provide the necessary conditions of some cognition, this deduction will be merely an analytical inference, which would not distinguish the transcendental argument from any other analytic argument. In his article *Transcendental Arguments* (1971), Gram gives an extensive argument to prove the impossibility of such a deduction, which leads him to the conclusion that, if we do not want the transcendental argument to be merely analytic, we need to accept that "[w]hat we find in the conclusion of a transcendental argument, then, is not something that can be demonstrated by such an argument" [Gra71, p.26]. Because transcendental arguments, in Gram's interpretation, are analytic, they can never establish the possibility of something synthetic and therefore they must be impossible.

There are two conclusions possible: Either Gram's interpretation is correct and, hence, Kantian transcendental arguments are impossible, or, and this seems to be the more plausible conclusion, the above interpretation of the structure of transcendental arguments is incomplete. What leads Gram to his rather unfortunate conclusion is his interpretation of the structure of a transcendental argument as such: Gram's interpretation of the argument is incapable of proving the aim of a transcendental argument because it is incomplete.

If the aim of a transcendental argument would be to detect a necessary condition of some a priori cognition, Gram would be right. The necessity argument only points out the concepts that are necessary for the possibility of a cognition and, hence, such an argument can only show the *necessary relation* between a cognition and some concept. As a result, the argument would be a mere analytic inference. However, when we look at the two aims of a transcendental argument (section 2.1), we immediately see that a necessity argument does not show us anything about how the cognition is possible through its necessary conditions; that is, the necessity argument does not show the possibility of this derived concept as the cognition's ground; it only determines a relation between a concept and a cognition.

Moreover, it would be absurd to conclude the possibility of a *synthetic* a priori cognition as a consequence of only *one* analytic argument. Suppose this would be the case though, then the proposition would either be an immediate or mediate consequence of the argument. In the first case the cognition itself would be analytic as well and, hence, not synthetic (contradiction). The latter case, however, implies that there would be another irreducible argument at work in the argument and the cognition would not be the consequence of only one argument (contradiction). A transcendental argument can therefore never consist of only this interpretation of the necessity argument. Nevertheless, since the presented interpretation of the necessity argument is substantially endorsed in the Kantian literature, it seems the more plausible that, in order to prove the possibility of a synthetic a priori cognition, there must be another irreducible argument at work.[10] The question is: Can we find such an additional irreducible argument?

In this section I have tried to show, not only the acceptability of the necessity argument as the first part of the transcendental argument, but also that there must be another argument at work. In the next section I will provide a positive determination of this second argument.

2.2.2 The Possibility Argument

Every transcendental argument aims at proving the objective validity of the a priori ground of the synthetic a priori cognition at stake. The last section showed that, in order to prove the objective validity of this concept as ground, it is not sufficient to show its necessary relation to the cognition from which it is derived. The last section concluded, therefore, with the need for a second argument. I will call this second step of the transcendental argument the **possibility argument**. There have been several proposals for a two-step reading of Kant's transcendental arguments. My interpretation will be based on these readings.[11]

Henrich argues that the transcendental deduction from Kant's first Critique is essentially twofold in character. With respect to the second edition of this

[10] That these two arguments should not be reducible to one another is immediately clear from the following: If either one of these arguments could be reduced to the other, this would imply that one of the arguments must be somehow included in the other and therefore analytically derivable from the other. Consequently, we would again end up with a single analytic argument.

[11] The following articles, although quite different in effect, consist of a 'two-step' reading of the transcendental argument: [Ame78], [Ben77], [Sac05], [Hen69].

Critique he observes that, on first sight, "the conclusion of the deduction seems to be drawn twice in two completely different passages" [Hen69, p.641]. Henrich disagrees with this reading and argues that there are instead "two arguments, rather than two proofs, [that] are involved and that these together constitute the proof of the deduction" [Hen69, p.642].

Benton also suggests a two-step reading of this transcendental argument. He presents this dual character of the argument as a *must/can* structure. He states that, with respect to the must/can distinction, although the first step of the argument,

> shows that our intuitions *must* be subject to the categories, it does not show that they *can* be subject to the categories. So the second step has to [...] show, on that basis, that everything that can be given is capable of being brought to unity under the categories. [Ben77, p.15]

The 'must'-side of Benton's interpretation of the argument coincides with what I have called the necessity argument. The second step, that is the 'can'-side of the argument, addresses the possibility of the concept as a ground. Henrich proposes a similar two-step reading of Kant's first Critique. His first step corresponds to Benton's 'must'-side of the argument:

> The result of the proof in section 20 is therefore valid only for those intuitions which already contain unity. That is: wherever there is unity, there is a relation which can be thought according to the categories. [Hen69, p.645]

The proof shows thus that in every instance of unified intuitions, there must be a relation according to the categories present; in other words, the categories are the necessary conditions of unified intuitions. The second argument, according to Henrich, can be expressed as follows:

> [T]he second part of the deduction will show that the categories are valid for all objects of our senses (B161). The deduction is carried out with the help of the following reasoning: wherever we find unity, this unity is itself made possible by the categories and determined in relation to them. [Hen69, p.646]

This second argument clearly corresponds to the 'can'-side of Benton's interpretation of the argument, expressing the idea that the categories are not only necessary, they are also possible as grounds of these unified intuitions. Thus, following the interpretations of Benton and Henrich, any transcendental argument should also show which of the derived necessary concepts form in fact the synthetic a priori cognition's *ground*. This is done, firstly, by showing that and how these concepts are *possible* themselves and secondly, by showing that and how the cognition *can* be a consequence of these concepts. Based on the above analysis I therefore suggest the following reading of the second part of the transcendental argument (let ϕ be a synthetic a priori cognition):

The Possibility Argument. A transcendental argument needs to show which of ϕ's necessary conditions $\psi_1, ..., \psi_n$ form its ground ψ^*. That ψ^* is ϕ's ground is accomplished by (i) showing that ψ^* is (a priori) possible and (ii) by showing that ψ^* is sufficient to generate ϕ as its consequence.

(NB. The necessity argument singles out the cognition's possible ground, but only the possibility argument positively determines which of the derived conditions actually function as the cognition's ground.[12])

In order to avoid the possible reduction of both arguments to one analytical inference, the following needs to be shown:

Proposition 1. (Irreducibility) The necessity argument is irreducible to the possibility argument and vice versa.

Proof. 'Irreducible' means that the conclusion of the one argument must not follow analytically from the other and vice versa. The necessity argument only establishes a necessary *relation* between the cognition and a concept. From the determination of a relation one cannot infer any positive determination of the concept itself, let alone its possibility as a ground. (For example, if ϕ is necessary related to ψ it can still be that both ϕ and ψ are as concepts impossible, without influencing the necessity of their relation.) On the other hand, from the possibility argument one cannot infer that the concept, as a possible ground, must be necessary related to some cognition; that is, that the cognition can be a consequence of the concept does not necessarily imply that the cognition can *only* be possible through that concept. (For example, in other occurrences of the cognition there might be another ground possible that has not yet been determined.)

The conclusions of these two arguments together must yield the general conclusion of the transcendental argument. Since both arguments are regressive in character, neither of them separately can generate the desired conclusion. Hence, either the two arguments together will generate in a new argument the

[12] The distinction between the necessity argument and the possibility argument seems to coincide with Kant's distinction in syllogistic reasoning between *episyllogisms* and *prosyllogisms* (respectively): "In the series of composite inferences one can infer in two ways, either from the grounds down to the consequences, or from the consequences up to the grounds. The first occurs through *episyllogisms*, the other through *prosyllogisms*". Moreover, the relation between the above two forms of argumentation coincides with the relation between these two forms of syllogistic reasoning; Kant continues: "An episyllogism is that inference, namely, in the series of inferences, whose premise becomes the conclusion of a *prosyllogism*, hence of an inference that has the premises of the former as conclusion" [Kan92a, Jäsche Logic - 9:134 - emphasis Kant's own]. Unfortunately, explicit occurrence of this syllogistic terminology is rare in Kant's work and, to my knowledge, can only be found in the Critique of pure reason at A331/B388 and in some notes to Kant's lectures on logic. I will therefore omit further elaboration of the comparison.

desired conclusion of the transcendental argument (hence, progressively), or there must be another irreducible argument at play, such that the three together can generate the conclusion. In the next section I will provide a third argument which connects the conclusions from the necessity and the possibility argument progressively and yields the objective validity of the derived ground. I will call this third argument the **objective validity argument**.

2.2.3 The Objective Validity Argument

Together with a third argument the above two arguments must somehow provide the objective validity of the derived ground. It is clear that the first two arguments combined need to establish something completely different than what is contained under them: The objective validity of a ground can neither be concluded from the mere logical necessity of the relation to its consequence (the necessity argument), nor can it be derived from the mere possibility of an a priori concept as the cognition's ground (the possibility argument). Eventually, the transcendental argument must provide a proof for the possibility of *synthesis* in the a priori cognition at stake. Following Benton, "[i]t should be noted that although individual steps of the must/can argument may be analytic, the structure is essentially synthetic" [Ben77, p.17]. This third combining step must facilitate this synthesis. Consider the following remark by Sacks:

> [T]he conclusion of the transcendental argument is a substantial statement, one that says more than the premiss(es): the move from premisses to conclusion is, we might say, synthetic. But a deductive inference from premisses to conclusion could not in itself be responsible for the addition of substantive content along the way. [...] There must then be some point at which the process of simple deductive inference is disrupted, and a synthetic or ampliative move is made. [Sac05, p.440-441]

According to Sacks, the transcendental argument must contain a substantial move that disrupts the regressive argument(s) and facilitates the synthetic conclusion. A synthesis must consist of (at least) two premisses that somehow 'entail' a conclusion that does not follow from either one or both premisses immediately; in both cases the inference would be merely analytic (in the latter case the inference would be a syllogism). For this reason a third premiss that brings the two premisses together is needed. This third premiss, though, cannot be of the same kind as the former two. If this would be the case, the extra premiss would only turn the argument in a mere poly-syllogism instead of a syllogism; consequently, there would be no synthesis.[13] This third premiss must

[13]The two arguments alone cannot establish any synthesis. If there would be a third premiss at play in the argument of the same nature as the conclusions of the first two arguments, there would be no reason to alter the nature of the argument. Think of a poly-syllogism of the following form: P_1: SaP, P_2: MaR $\vdash$ C: SaR. In order to generate the result 'C' we need an extra (third) premiss, namely, P_3: PaM. However, this third premiss would

therefore be some sort of **meta-premiss**. Kant seems to be aware of the need of a meta-premiss that brings about a substantial move in the argument. Kant mysteriously defines this substantial move as the requirement of 'something more'. He states this as follows:

> But in order to ascribe objective validity to such a concept (**real possibility**, for the first sort of possibility was merely logical) something more is required. This "more", however, need not be sought in theoretical sources of cognition; it may also lie in practical ones. [Kan00, BXXVi - bold emphasis my own]

The key to this final step, that is the 'more' to which Kant refers, is **reality**; to be more precise, it is the reality of the synthetic a priori cognition at stake. The insertion of reality into the argument enables the derivation of the objective validity of the cognition's ground. Consider the following quote with respect to the objective validity of time:[14]

> Alterations are real (this is proved by the change of our own representations, even if one would deny all outer appearances together with their alterations). Now alterations are possible only in time, therefore time is something real. [...] I admit the entire argument. [Kan00, A37/B53-B54]

Moreover,

> Our assertions accordingly teach the empirical reality of time, i.e., objective validity in regard to all objects that may ever be given to our senses. [Kan00, A35/B52]

The above quote shows that time is objectively valid just because it is the ground of something which is regarded as real, more than that, because it is the ground of something whose reality seems *undeniable*. Alterations are real and for this reason one cannot deny the reality of their sole condition of possibility, that is, their ground (which is, in this case, time). With respect to transcendental arguments in general we have the following reasoning: A cognition is undeniably real. Some concept forms the necessary a priori ground of that cognition. Without this ground the cognition itself would be impossible

only finish the argument, but never change it from being merely analytic to synthetic. The third premiss must therefore be of a different nature.

[14]In relation to a priori (pure) concepts Kant uses the terms 'objective reality', 'objective validity' and plain 'reality'. Kant seems to use these terms interchangeably. Consider for example the following quote: "Our assertions accordingly teach the empirical reality of time, i.e., objective validity" [Kan00, A35/B52]. From Kant's usage of these terms in relation to the categories of the pure understanding it can be inferred that the terms 'objective validity' and 'objective reality' are interchangeable. Both terms express the validity of an a priori concept. Kant's use of plain 'reality', on the other hand, seems to be reserved for the assertion of undeniable (empirical) facts; e.g. the undeniable fact that 'we perceive alterations' and 'have experience'. Although Kant's own usage of these terms might seem obscure, for the sake of readability I will keep a strict distinction between, on the one hand, the use of 'reality' and, on the other hand, the use of 'objective validity' and 'objective reality'.

(let alone real). For this reason the cognition's ground must be granted objective reality as well (i.e. objective validity). Thus we can conclude that the **reality** of a cognition is the 'more' to which Kant refers that allows for the substantial move made in the objective validity argument.

The only thing uncovered thus far is the function and structure of the objective validity argument. In order to conclude *how* this argument is actually established a proper definition of both the concept of reality and objective validity is needed. It turns out that these two concepts are rather obscure in Kant's philosophy. In fact, Kant does not provide an explicit definition of these concepts in both the Critique of Pure Reason and the Critique of Practical Reason. Consider the following remark by Meerbote:

> Actual statements, in Kant's writings, of transcendental arguments also have their own complications, although it certainly should be possible to analyse the conclusion of such arguments in order to discover what Kant means when he says that such arguments establish the "objective validity" of some of our concepts. [Mee72, p.52]

In the line of Meerbote's thought I propose the postponement of the elaboration of the concept of reality and objective validity. In the upcoming chapters I will treat Kant's (transcendental) argumentation in the second Critique. Thorough analysis of Kant's reasoning might shed light on the definitions of these two concepts. For now, I will propose the following definition of the general function of the objective validity argument:

The Objective Validity Argument. A transcendental argument needs to show that the objective validity of the a priori ground ψ follows from the reality of the cognition ϕ from which ψ is derived.

This third argument uses the results of the previous arguments as follows: The necessity argument provides insight into the possible grounds of the cognition. The possibility argument, subsequently, determines which of these possible ground is the cognition's ground. The objective validity argument, accordingly, determines the objective validity of this derived ground via the reality of the cognition. Together, these three arguments provide the proof for the second aim of the transcendental argument. The objective validity of the ground (aim II) should, lastly, show how the synthetic a priori cognition itself is possible (aim I). The exposition of the possibility of the synthetic a priori cognition depends thus on two things: (i) the definition of objective validity and (ii) the definition of the cognition's ground. Thorough treatment of the transition from aim II to aim I must therefore be deferred to the analysis of Kant's argument in the second Critique. I will come back to this in chapter 5.

Relative Objective Validity

One last remark must be made about the objective validity argument. For Kant objective validity signifies a relative objective validity. That is, the objective validity of an a priori concept is the validity of this concept in relation to the cognition that it makes possible. For example, the concepts of space and time are objective validity just because they are valid in relation to every possible object given by the senses (both externally and internally):

> Our expositions accordingly teach the reality (i.e., objective validity) of space in regard to everything that can come before us externally as an object. [Kan00, A28/B44]
>
> Our assertions accordingly teach the empirical reality of time, i.e., objective validity in regard to all objects that may ever be given to our senses. [Kan00, A35/B52]

Space and time are therefore only objectively valid with respect to theoretical reason, since their objective validity is only provided by the reality of theoretical cognition (e.g. alteration). Thus, the objective validity of a concept is a relative objective validity. Furthermore, Kant emphasizes that the objective validity of a concept is also a limitation of the concept:

> [T]hese a priori sources of cognition determine their own boundaries by that very fact (that they are merely conditions of sensibility), namely that they apply to objects only so far as they are considered as appearances. [Kan00, A39/B56]

Hence, a transcendental argument not just proves the objective validity of a concept in general, the argument proves its objective validity relative (or limited) to the nature of the reality of the synthetic a priori cognition from which the ground is derived.[15][16]

[15]For example, in the Critique of Pure Reason Kant shows that the concept of causality is objectively valid with respect to theoretical reason only. On the other hand, in the Critique of Practical Reason Kant tries to show that the concept of freedom is only objectively valid with respect to practical reason. I will come back to this in chapter 4.

[16]A remark about Kant's usage of the term 'objective' is at place here. Kant's use of the term *objective* in his practical philosophy is different compared to his use of this term in his theoretical philosophy. In short, in his critical analysis of theoretical reason Kant uses the term objective as "having relation to an object" [AVL, Ch. 3]. In his practical philosophy, on the other hand, Kant has a different purpose for the term objective; here it means "objective, that is, as holding for the will of every rational being" [Kan96a, 5:19]. Abstracted from these forms of cognition, both definitions address a form of validity relative to the setting in which they are used. In that sense, with respect to theoretical reason, objective validity can be interpreted as a special form of validity of a concept as to how it applies to the objects of theoretical reason; namely, the possible objects of the senses. With respect to practical reason, objective validity can be interpreted as a special form of validity of a concept as to how it applies to the 'objects' of practical reason; namely, the 'wills' of practical beings. However, the exact definition of the concept of objective validity will remain unclear throughout the major part of this thesis. In fact, only the logical formalization of Kant's argumentation will shed (some) light on the concept.

2.3 A Valid Transcendental Argument

Given the general aims of the transcendental argument and the three sub-arguments presented in the previous sections, the working model for detecting, interpreting and evaluating transcendental arguments in general can finally be presented. This model will serve as a guiding thread for the interpretation of Kant's argumentation in chapter I of 'The analytic of pure practical reason' in the Critique of Practical Reason. In this section I will present the criteria that constitute a valid Kantian transcendental argument.

Apart from the accomplishment of the two aims (Postulate 1) every transcendental argument must satisfy the following two criteria: Firstly, every transcendental argument needs to determine the objective validity of the *a priori* ground of some *a priori* synthetic cognition. Any empirical content would deprive the argument from its a priori status; that is, the use of experience (empirical premisses) would give the argument a contingent flavour. Thus, every transcendental argument must be established fully *a priori*:

> [O]ne cannot adduce experiences for the proof, for the objective validity of this a priori concept must be able to be demonstrated. [Kan00, A90/B122][17]

Secondly, the main aim of every transcendental argument is to prove the possibility of a *synthetic* a priori cognition. As shown in the previous sections, such a synthesis cannot be shown through a merely analytic argument. Hence, every transcendental argument itself must express a synthesis; that is, every transcendental argument must combine (at least) two arguments via a third argument such that the (substantive) synthetic conclusion can be obtained. Based on the above, I put forward the following postulate:

Postulate 2. (A Valid Transcendental Argument) An argument is a valid (Kantian) transcendental argument *if and only if* the argument satisfies the following four criteria:

1► The argument provides a proof for aim I of Postulate 1.
2► The argument provides a proof for aim II of Postulate 1.[18]
3► The argument is established a priori.
4► The argument brings about a synthesis.

[17]Kant provides the following definition of a 'demonstration': "A proof that is the ground of mathematical certainty is called a demonstration" [Kan92a, Jäsche Logic - 9:71]. Mathematical cognition, on the other hand, "carries with it thoroughly apodictic certainty (i.e., absolute necessity), [and] hence rests on no grounds of experience, and so is a pure product of reason" [Kan02, 4:280]. In other words, 'demonstration' implies 'a priori proof'.

[18]Recall that the second aim is established on the basis of the three proposed sub-arguments.

2.4 Kant's Practical Transcendental Argument

The above elaboration is mostly based on Kant's arguments for the possibility of a priori synthetic cognitions in relation to *theoretical* reason. While Kant tries to prove the possibility of these theoretical cognitions in his first Critique, in the 'Groundwork of The metaphysics of morals' he wonders whether this can be done for practical cognition as well:

> [I]n the case of this categorical imperative or law of morality the ground of the difficulty (of insight into its possibility) is also very great. It is an **a priori synthetic practical proposition**; and since it is so difficult to see the possibility of this kind of proposition in theoretical cognition, it can be readily gathered that the difficulty will be no less in practical cognition. [Kan96b, 4:420 – bold emphasis my own]

The above quote shows that Kant endorses the possibility of synthetic a priori propositions in relation to practical reason. The moral law (or categorical imperative) is such a synthetic a priori proposition.[19] According to Benton "it is precisely the synthetic character of the [categorical] imperative that makes a transcendental deduction necessary" [Ben78, p.226]. In the Critique of Practical Reason Kant emphasizes the possibility of such a practical synthetic a priori proposition as well. Moreover, Kant states that this practical proposition requires an exposition of its possibility similar to the method used in the first Critique:

> With [...] the discernment of the possibility of such a synthetic proposition a priori, one cannot hope to get on so well as was the case with the principles of the pure theoretical understanding. [Kan96a, 5:46]

That even the second Critique is concerned with transcendental philosophy is hinted by Kant in the introduction to this Critique: "With this faculty [the practical faculty] **transcendental** freedom is also established" [Kan96a, 5:3 - bold emphasis my own]. Hence, it is not quite unreasonable to look for transcendental arguments in Kant's practical philosophy.

In order to determine how such a practical synthetic a priori proposition is possible, Kant's use of 'possibility' must be determined for this context. Consider the following quote from the Groundwork:

> Now the question arises: how are all these imperatives possible? This question does not inquire how the performance of the action that the imperative commands can be thought, but only how the **necessitation of the will**, which the imperative expresses in the problem, can be thought. [Kan96b, 4:417 - bold emphasis my own]

[19] Throughout this thesis I will regard the moral law and the categorical imperative as different names for one and the same law. Both laws express the 'fundamental law of pure practical reason' with respect to a human being's will. Exact definitions of these terms will be given in the upcoming chapters.

Hence, with the discernment of the possibility of the moral law as a synthetic a priori proposition, Kant aims to show how such a law can necessitate the will. A practical law, such as the moral law, puts forward an action as necessary. The question is therefore: How can the moral law's necessary conduct apply immediately to a will? i.e., How can the moral law immediately determine a will? (Immediacy expresses the necessary and a priori nature of this relation.)

POSTULATE 3. (THE POSSIBILITY OF THE MORAL LAW) The aim of the transcendental argument of the Critique of Practical Reason is to prove how the moral law is possible as a synthetic a priori proposition; that is, the argument needs to prove how the moral law can immediately determine a human being's will.[20]

Postulate 3 represents Aim I (Postulate 1) of Kant's practical transcendental argument. The second aim is to establish the objective validity of this proposition's ground. This ground should eventually make the a priori synthesis possible. The ground in question is the concept of freedom. The following quote from the Critique of Practical Reason captures the two main aims of Kant's practical transcendental argument:

> [T]he human will is by virtue of its freedom immediately determinable by the moral law. [Kan96a, 5:38]

In the upcoming chapters I will treat Kant's proof of the above claim as presented in the Critique of Practical Reason. In the first part of 'The analytic of pure practical reason' (5:19 - 5:35) Kant derives the concept of freedom as the moral law's necessary condition. This necessity argument will be treated in chapter 3.[21] In the section 'On the deduction of the principles of pure practical reason' (5:42 - 5:50) Kant shows, on the one hand, that the concept of freedom is the moral law's ground and, on the other hand, that this concept is objectively valid with respect to practical reason. The possibility argument and the objective validity argument will be treated, respectively, in chapter 4 and 5. During the philosophical analysis of these arguments some problems and ambiguities will be encountered. These hazards will be addressed in the second part of this thesis: The logical formalization of Kant's argument.

[20]The moral law only applies to beings with a finite nature; that is, beings with sensible needs. This means that the moral law prescribes conduct to beings that do not necessarily put this conduct into action. Any philosophical result related to the moral law is therefore related to the will of a finite being; in this case, the will of a human being. Kant's proof for the relation between the moral law and human beings will be treated in section 3.8.

[21]The necessity argument will take up the largest part of the transcendental argument. It is nonetheless also the most important part of the argument and subsequent arguments will draw heavily on its results.

Chapter 3

The Necessity Argument

The previous chapter showed that Kant needs to establish the necessary conditions of the moral law in order to prove its possibility. In this chapter I will present Kant's argument for this necessity step. I will mainly follow the original structure of the analysis as presented by Kant in Chapter I of the 'Analytic of pure practical reason' of the second Critique (5:19 - 5:35). Recall what needs to be shown:

The Necessity Argument. A transcendental argument needs to prove what concepts $\psi_1, ..., \psi_n$ are necessary conditions for the possibility of some synthetic a priori cognition ϕ.

With respect to the necessity argument Kant needs to establish two things: Firstly, Kant needs to deduce a positive formulation of the moral law. This deduction will be treated in the sections 3.1 - 3.5 and 3.7 - 3.8. Sections 3.2 and 3.3 consist of Kant's negative determination, or limitation, of practical laws. In sections 3.5 and 3.7 these practical laws (including the moral law) will be determined positively. Secondly, Kant needs to derive from the positive determination of the moral law its necessary condition. This deduction will be treated in section 3.6 and 3.10. Kant's proof for the negative determination of the moral law's necessary condition will be provided in section 3.6. Its positive determination, and consequently the conclusion of the necessity argument, will be provided in section 3.10.[1] Let us turn to Kant's starting point: Practical principles.

[1] Every claim and premiss introduced in the upcoming chapters will be labelled. The digital version of this thesis is supplied with 'hyper-references', which will enable the reader to jump back from the referent to the original definition, theorem et cetera. At some of these definitions, theorems and propositions, the reader will encounter the following symbol: (KL). This symbol is a hyper-reference that enables the reader to jump back and forth between the informal definition and its formal representative during the logical formalization of the argument.

3.1 Practical Principles

Chapter I of the Analytic of the second Critique starts with the following definition:

Definition I. “Practical principles are propositions that contain a general determination of the will, having under it several practical rules. They are subjective, or maxims, when the condition is regarded by the subject as holding only for his will; but they are objective, or practical laws, when the condition is cognized as objective, that is, as holding for the will of every rational being.” [Kan96a, 5:19]

In this definition Kant presents one of the central concepts of the argument, namely, that of a practical principle. There are two distinct types: subjective and objective principles. But first, there are some more general terms in this definition that need clarification.

Practical Propositions. Kant makes a distinction between two types of propositions: On the one hand, propositions that are theoretical, that is, propositions that address knowledge, and on the other hand, propositions that address *acting* in any sense. The latter are called practical propositions [Kan92a, Dohna-Wundlacken Logic - 24:766].

Practical Rules. Practical principles can have several practical rules under them. Kant provides the following definition of a practical rule: “A practical rule is always a product of reason because it prescribes action as a means to an effect, which is its purpose” [Kan96a, 5:20]. These rules are a product of reason itself and express an action as the necessary action to some purpose.[2] In relation to Definition I it can be concluded that practical principles contain rules that allow a rational being to infer which particular actions are necessary in relation to some purpose that this being endorses.

Determination of the Will. A practical principle contains a general determination of the will. This means that on the basis of a principle, a will is determined to a particular action put forward by that principle.[3] The will is determined on the basis of a *ground of determination.* According to Kant, such a determining ground is a reason, motive, or cause “sufficient to determine the will” [Kan96a, 5:19].[4] A determination ground is, therefore, the reason on

[2]Kant calls such a practical rule a (hypothetical) *imperative.* An extensive treatment of imperatives, though, will be unnecessary for the purpose of the present undertaking.

[3]Notice that such a determination of the will is not about whether the particular being will actually perform the act. A practical principle only determines the will to an act as a sort of *disposition.*

[4]The following similar definition is provided in the ‘Notes on the lectures of Mr. Kant on the metaphysics of morals’ by Vigilantius: “The determining ground of choice [...] is causa impulsiva to the action, the motivating cause” [Kan97, 27:493].

which a will determines itself to a particular action. Hence, a practical principle is a practical proposition on which a will is determined to an action, which is derived from some practical rule(s), on the basis of that being's ground of determination. In the upcoming sections several types of determining grounds will be encountered.[5]

The following quote is an example of a practical principle provided by Kant in the second Critique:

> Tell someone, for example, that he must work and save in his youth in order not to want in his old age [1]; this is a correct and also important practical precept of the will. But it is readily seen that here the will is directed to something else which it is presupposed that it desires [2], and as to this desire, it must be left to the agent himself whether he foresees other resources than means acquired by himself, or does not hope to live to old age, or thinks that in case of future need he can make do with little [3]. [Kan96a, 5:20-5:21]

All elements of a practical principle can be distinguished in this example: The practical rule, marked with [1], is: 'in order not to want at old age, work and save in your youth'. The left side (the antecedent) expresses the purpose, the right side (the consequent) expresses the necessary action to realize this purpose. The possible determining ground [2] of this being's will is the desire to 'not want at old age'. The determination of the will [3] is therefore to either 'work and save in your youth' or not, depending on that being's endorsed determining ground. Based on the above example, the practical principle could be expressed as follows: '*because my will is not to want at old age, I will work and save in my youth*'. The above elaboration provides the following definition:

Definition 3.1. (Practical Principles) A practical proposition is a practical principle only if:

1► The proposition contains a general determination of the will.
2► The proposition contains a determining ground of the will.
3► The proposition can have (several) practical rules under it.
4► The proposition is either subjective or objective.

Every practical principle is either subjective or objective. I shall first elaborate on subjective principles.[6]

[5] I will explicitly refer to theses grounds as 'determining grounds' or 'grounds of determination' and reserve the more general term 'ground' for the possibility argument, where the possibility of a concept as a ground is proved.

[6] Kant already introduces this distinction between subjective and objective principles at the beginning of the analytic of the second Critique. The distinction, as we shall see, only arises though when the will of a finite being is considered; namely, the will of a being that is not mere reason. At first sight it might seem strange that Kant introduces the distinction in

3.1.1 Subjective Principles

In Definition I Kant states that the condition of a practical principle makes the principle subjective or objective. A principle is subjective only if its condition, that is determining ground, is only valid for the particular being that entertains the principle.[7] This definition implies two things: Firstly, the determining ground of such a principle cannot be universally valid and, secondly, the determining ground must arise from a faculty that generates grounds specific to a particular being. Because these grounds are dependent on a being's contingent faculty, any subjective principle based on such a ground will therefore only be contingently connected to that being's will. In the next section, we will see that the faculty that provides these contingent grounds is the faculty of desire.

A consequence of the above is that subjective principles do not make actions unavoidably necessary. That is, because the determining ground of such a principle is merely contingent, the action that follows from it must be contingent as well. As a consequence, "the principles that one makes for oneself are not yet laws to which one is unavoidably subject, because reason, in the practical, has to do with the subject [...] which by its special constitution can make various adjustments to the rule" [Kan96a, 5:20]. Hence, subjective principles can never be practical laws. The above elaboration provides the following (provisory) definition of a subjective principle:[8]

Definition 3.2. (Subjective Principles) A practical principle (Definition 3.1) is a subjective principle only if:

1▶ The principle's determining ground is only valid for the rational being that entertains it.
2▶ The principle's determining ground is contingent.
3▶ The principle allows for adjustments.

---

a chapter called 'The Analytic of *pure* practical reason'. However, I think that Kant has (at least) two reasons for the introduction. Firstly, in order to define a practical principle, Kant needs to consider its definition in general and, therefore, has to distinguish between every possible form of it. Secondly, in order to determine which practical principles belong to *pure* practical reason Kant needs to isolate them from practical principles that do not belong to it.

[7]According to Kant, maxims *are* the subjective principles of the will. Consider for example the following quotes: "A maxim is the subjective principle of acting" [Kan96b, 4:421] and "[a] maxim is the subjective principle of volition" [Kan96b, 4:401]. In the remainder of this thesis I shall use these terms interchangeably.

[8]The definitions of subjective and objective principles provided in this section are minimal definitions provided by Kant at the beginning of the Analytic. The upcoming theorems will determine the definitions of these principles more accurate.

3.1.2 Objective Principles

According to Kant, a practical principle is called objective only if its determining ground holds for the will of every rational being.[9] In order to hold for every rational being such a ground must be subtracted from everything that distinguishes one rational being from another. In other words, every objective principle's ground must be independent of the contingency of a particular being. As a consequence, such a practical principle does not allow for exceptions and therefore puts forward an action as absolutely necessary.

In Kant's philosophy terms like 'objective', 'universal' and 'necessary' are inevitably connected with 'being a priori'. It might therefore seem trivial that the ground of an objective principle must be found in *pure* reason. In definition I of the Analytic though, Kant only defines the minimal difference between subjective and objective principles on the basis of the properties of their determining grounds. Which faculties provide which determining grounds is something that Kant still needs to prove. Consequently, that every objective principle's determining ground must have its origin (a priori) in pure reason cannot simply be asserted: It needs to be proved. The above will be established by Kant's first three theorems.

Definition 3.3. (Objective Principles) A practical principle (Definition 3.1) is an objective principle only if:

1► The principle's determining ground holds for every rational being.
2► The principle's determining ground is independent of the subjective conditions of the rational being that entertains it.
3► The principle expresses an absolute necessity.

Some of the concepts used in the above analysis need some clarification. Firstly, consider the following statement made by Kant on *necessity* in relation to practical principles:

> [O]therwise they are not laws because they lack the necessity which, if it is to be practical, must be independent of conditions that are pathological and therefore only contingently connected with the will. [Kan96a, 5:20]

Hence, the necessity of a practical principle implies the principle's independence of any contingent condition. A rational being's pathological conditions depend on the being's sensibility and can therefore never generate a necessity. According to Kant, "[i]t is an outright contradiction to want to extract necessity from an empirical proposition" [Kan96a, 5:12].[10] This implies, accordingly,

[9] For Kant objective principles and practical laws are the same. I shall therefore use these terms interchangeably.

[10] In the proof of Theorem 1 Kant calls this form of necessity 'objective necessity': A principle containing an object of desire as the determining ground of the will can never serve as a law "because it is lacking in objective necessity" [Kan96a, 5:22]. Kant uses the term

that such a principle does not allow for any exceptions (as is the case for subjective principles). Hence, the practical necessity of a principle implies the unavoidable determination of a will to an action put forward by that principle. (However, the dual nature of a human being turns this practical necessity into a mere practical *necessitation*, that is, obligation. This point will be treated in section 3.8.1.) The above elaboration provides the following definition:[11]

Definition 3.4. (Necessity) A practical principle is necessary *if and only if* (i) it unavoidably determines a will to a particular action put forward by that principle and (ii) it is independent of all (empirical) conditions that are contingently connected with a rational being's will.

The second clause of the above definition guarantees that the principle is an a priori principle. A cognition is either a priori or a posteriori:[12]

Definition 3.5. (A Posteriori Cognitions) A cognition is called a posteriori, i.e. empirical, *if and only if* the cognition depends on experience, that is, sensibility (which is a rational being's subjective constitution).

Definition 3.6. (A Priori Cognitions) A cognition is called (purely) a priori *if and only if* the cognition is completely independent of anything empirical.

The concept of (strict) universality is defined by Kant as follows:

'objective' to distinguish between this form of necessity and a mere 'subjective' necessity. With respect to practical principles Kant makes a distinction between the absolute necessity of a practical law, that is objective necessity, and the necessity of a subjective principle that is only raised by some practical being(s) to the status of a law: "It would be better to maintain that there are no practical laws at all [...] than to **raise** merely subjective principles to the rank of practical laws, which absolutely must have objective and not merely subjective necessity" [Kan96a, 5:26]. Thus, the addition of the term 'objective' only serves to distinguish 'plain' necessity from a pseudo necessity, namely, subjective necessity. I will therefore use the terms objective necessity and necessity interchangeably.

[11] Again, for Kant the necessity of a principle implies that this principle must be a priori. Both concepts express an independence of contingent conditions; that is, experience. Consider for example the following remark by Kant: "[M]athematical propositions are always a priori and not empirical judgments, **because** they carry necessity with them, which cannot be taken from experience." [Kan02, 4:268 - bold emphasis my own]. Hence, if a practical principle is necessary, then it must be fully determinable a priori. In the corollary to the second theorem though, Kant proves that the origin of the determining ground of such a necessary practical principle must be pure reason. I will therefore leave the determination of this 'pure' character to Kant's first corollary. With respect to the present argument Definition 3.4 will suffice.

[12] Definition 3.5 and Definition 3.6 are mostly based on the following definition provided by Kant in the introduction to the B edition of the first Critique: "[W]e will understand by *a priori* cognitions not those that occur independently of this or that experience, but rather those that occur *absolutely* independently of all experience. Opposed to them are empirical cognitions, or those that are possible only *a posteriori*, i.e., through experience. Among *a priori* cognitions, however, those are called **pure** with which nothing empirical is intermixed" [Kan00, B2-B3].

> Thus if a judgment is thought in strict universality, i.e., in such a way that no exception at all is allowed to be possible, then it is not derived from experience, but is rather valid absolutely a priori. [Kan00, 5:4 - Introduction to the B edition]

Strict universality expresses a validity with respect to *every* possible case without exception. With respect to practical principles a universally valid principle would then be a principle that is valid for every being to which it can apply, that is, for every rational being.[13]

Definition 3.7. (Universality) A practical principle is universally valid *if and only if* the principle is valid for every rational being.

3.2 Theorem I

The upcoming sections will be concerned with the theorems that Kant proves on the basis of Definition I. Kant defines a theorem as follows:

> Theorems are theoretical propositions that are capable of and require proof [...]. Essential and universal moments of every theorem are the thesis and the demonstration. [Kan92a, Jäsche logic - 9:113]

I will construct every section in the light of the above quote; namely, every section will begin with Kant's original claim (thesis) and end with a representation of its proof (demonstration). The first theorem of the second Critique is formulated as follows:

Theorem 1. "All practical principles that presuppose an object (matter) of the faculty of desire as the determining ground of the will are, without exception, empirical and can furnish no practical laws." [Kan96a, 5:21]

Kant needs to prove two things: On the one hand, he must prove that a practical principle containing an object of desire as its determining ground must be empirical and, on the other hand, he must prove that such a principle cannot be a practical law. Before the proof of the first theorem can be attended, the faculty of desire needs to be properly defined.

3.2.1 The Faculty of Desire

Roughly, Kant distinguishes three levels of practically determination. Firstly, there is 'animal choice': "A faculty of choice, that is, is merely animal (arbit-

[13] Just as in the case of 'objective necessity' Kant uses the addition of the term 'strict' to universality to distinguish this form of universality from, what he calls, a mere assumed universality. The latter, which is based on induction and derived from experience, expresses the following thought: "[A]s far as we have yet perceived, there is no exception to this or that rule" [Kan00, B3/B4]. Kant's use of 'plain' universality coincides with strict universality and I will use these terms interchangeably.

rium brutum) which cannot be determined other than through sensible impulses, i.e., pathologically" [Kan00, A802/B830]. A being fully determined through its sensibility is called *pathological*.

On the second level, the level on which human beings abide, a being's choice is still pathologically affected, though it is not merely animal. Such a being has a pathologically affected *will*. This means that this being has a dual nature: On the one hand, it has a physical constitution, that is, a sensible nature:

> Now, however, we find our nature as sensible beings so constituted that the **matter** of the faculty of desire [...] first forces itself upon us, and we find our pathologically determinable self. [Kan96a, 5:74]
>
> By 'the matter of the faculty of desire' I understand an object whose reality is desired. [Kan96a, 5:21]

The objects of the faculty of desire arise from the sensible constitution of such a being. Consequently, the determination of these objects must be contingent. On the other hand though, we find that the will of such a being can be guided and determined by reason: The faculty of desire and reason interact whenever reason provides the will with practical rules to obtain the reality of a desired object. In this case, "reason supplies only the practical rule as to how to remedy the need of inclination" [Kan96b, 4: 413]. Such a being has therefore the ability to let reason interfere with its desires:

> [W]e have a capacity to overcome impressions on our sensory faculty of desire by representations of that which is useful or injurious even in a more remote way; [...] these considerations [...] depend on reason. [Kan00, A802/B830]

The third level of determination is the *pure* level. This level is concerned with the choice of a being that can only be determined by pure reason. Such a being can only consider determining grounds provided by pure reason and for this reason can only have grounds valid for every rational being. Consequently, the subjective principles of such a being necessarily coincide with objective principles.

This thesis will be (mostly) about the second level of determination, that is, about beings with both a faculty of reason and a faculty of desire. With respect to the present analysis I will use the following definition of the latter faculty provided by Kant in a footnote to the introduction of the Critique of Practical Reason:[14]

DEFINITION 3.8. (THE FACULTY OF DESIRE) "The faculty of desire is a being's faculty to be by means of its representations the cause of the reality of the objects of these representations"; [Kan96a, 5:9] which, because of its representations "has to do with the subject" [Kan96a, 5:20].

[14] It is obscure whether Kant's exposition of the faculty of desire is a definition or a postulate. However, Kant explicitly calls it a definition and I will therefore follow this approach as well.

The above definition is an abstract formulation of the following description: "Desire [...] is the self-determination of a subject's power through the representation of something in the future as an effect of this representation" [Kan08, 7:251]. In other words, the faculty of desire provides a rational being with representations of objects of desires that can serve as (subjective) determining grounds for that being's will. With this definition at hand, the proof of the first theorem can be provided.

3.2.2 Theorem I - the Argument

The argument runs as follows: Every practical principle contains a determining ground of the will. Hence, whenever an object of desire forms the determining ground of some practical principle, this principle depends on a ground provided by that being's faculty of desire. Consequently, the principle depends on the subjective constitution of the being that entertains it. As a consequence, this practical principle cannot be valid for every rational being. Hence, every practical principle that contains an object of desire as its determining ground cannot be a practical law. Moreover, because such a principle's ground depends on a representation of an object of desire, that is, on sensibility, the principle must be empirical as well.

The following is a structured representation of the above argument using the introduced definitions of the previous sections. The structured representations of the arguments corresponding to the upcoming sections will only be provided in the Appendix. The proof of Theorem 1 can be represented as follows:

Claim: If a practical principle contains an object of desire as the determining ground of the will, then the principle is not a practical law and the principle is empirical.

Take an arbitrary practical principle P.

Assume: An object of desire is the ground of determination of this principle P.

To prove: P cannot be a practical law and P is an empirical principle.

Proof: (of the left conjunct)

3.2.1 By Definition 3.8 we know that an object of desire is provided by the faculty of desire.

3.2.2 From 3.2.1 and the assumption we can infer that P depends on a ground provided by the faculty of desire.

3.2.3 From Definition 3.8 and the assumption we can infer that P's ground depends on a representation provided by the subjective constitution of a particular being.

3.2.4 From 3.2.2 and 3.2.3 we can infer that P depends on the subjective constitution of a particular being.

3.2.5 From 3.2.4 and Definition 3.4 we can infer that P lacks (objective) necessity.

3.2.6 From 3.2.5 and Definition 3.3 we can infer that P can never be a practical law. □

Proof: (of the right conjunct)

3.2.5 From 3.2.3 and Definition 3.5 we can infer that P is empirical. □

Since P is an arbitrary practical principle, we have proved the validity of Theorem 1. ■[15]

The contrapositive of this theorem tells us that every practical law cannot have an object of desire as its determining ground. Moreover, since every practical principle is either subjective or objective, we can conclude that (at least) every principle with an object of desire as its determining ground must be a subjective principle.

3.3 Theorem II

The first theorem provides the negative determination of a practical principle that contains objects of desire as its determining ground; namely, it is *not* a practical law. Kant's second theorem provides the positive determination:

THEOREM 2. "All material practical principles as such are, without exception, of one and the same kind and come under the general principle of self-love or one's own happiness." [Kan96a, 5:22]

The definition of a material practical principle is already (implicitly) provided by Kant in the formulation of the first theorem.

DEFINITION 3.9. (MATERIAL PRACTICAL PRINCIPLES) A practical principle is material *if and only if* it has an object of desire as its determining ground.

Before the proof of this second theorem can be attended the concepts of *Self-love* and *happiness* must be clarified.

3.3.1 Happiness and Self-Love

The faculty of desire provides a being with representations of objects whose reality can be desired (section 3.2.1). From the concept of this faculty the concept of *pleasure* arises: According to Kant, pleasure is the *agreement* between a representation of a desired object and the realization of that desired object. Pleasure and displeasure depend, therefore, respectively on the concepts of agreement and disagreement.

[15] I will use the symbol '■' to mark the end of a proof. The symbol '□' will be reserved to indicate the end of a sub-proof.

Definition 3.10. (Pleasure) "Pleasure is the representation of the agreement of an object [...] with the faculty of the causality of a representation with respect to the reality of its object." [Kan96a, Footnote to 5:9][16]

In the Groundwork Kant formulates the concept of happiness as follows: "[F]or the idea of happiness there is required an absolute whole, a maximum of well-being in my present condition and in every future condition" [Kan96b, 4:418]. Well-being can, subsequently, be defined in terms of pleasure: "Well-being or ill-being always signifies only a reference to our state of agreeableness or disagreeableness" [Kan96a, 5:60]. In other words, happiness is nothing but the agreement of the representations of a being's desired objects and their realization "uninterruptedly accompanying his whole existence" [Kan96a, 5:22]. The following definition will suffice:

Definition 3.11. (Happiness) Happiness is the entire satisfaction of a being's pleasure, that is, the being's total agreeableness with its own state (now and in the future).

Self-love is defined as a consequence of the above. According to Kant, "the principle of making this [happiness] the supreme determining ground of choice is the principle of self-love" [Kan96a, 5:22]. Consequently, every being that acts from self-love acts upon the principle of happiness.

Definition 3.12. (Self-Love) The principle of self-love is the principle of making happiness the supreme determining ground of the will.

The only concept that has not yet been properly defined is that of an 'object of desire'. In the first theorem Kant defines a material practical principle as a principle that has an object of desire as its determining ground. In a reformulation of the second theorem Kant seems to provide a definition of an 'object of desire regarded as the determining ground of a will':

> Thus all material principles, which place the determining ground of choice in the pleasure or displeasure to be felt in the reality of some object, are wholly *of the same kind* insofar as they belong without exception to the principle of self-love or one's own happiness. [Kan96a, 5:22 – emphasis Kant's own]

The above quote suggests that 'the object of desire as determining ground of the will' must be equated with 'the expected (dis)pleasure of the reality of some object as determining ground of the will' (the inclusion of 'expected' is justified by Kant's use of the future tense 'to be felt' in the above quote). An object of desire is therefore a representation of an object whose realization would generate the expected pleasure. With respect to the present argument

[16] In this very same footnote, Kant also defines pleasure in terms of 'the subjective conditions of life'. Happiness can be defined independent of the concept of life and I will therefore, for the sake of readability, omit elaboration of this concept.

Kant is only concerned with objects of desire in relation to possible determining grounds of the will and for this reason the following definition will suffice:

Definition 3.13. (Objects of Desire) Regarded as the determining ground of a being's will, an object of desire is defined as the representation of the expected pleasure of the reality of an object.

3.3.2 Theorem II - the Argument

All material practical principles have an object of the faculty of desire as their determining ground. As a determining ground, an object of desire is the representation of the expected pleasure of the reality of an object. Hence, all material practical principles have a representation of an expected pleasure as their determining ground. Pleasure, subsequently, is the agreement of the representation of an object of desire with the (actual) realization of that object. Pleasure is therefore "practical only insofar as the feeling of agreeableness that the subject expects from the reality of an object determines the faculty of desire" [Kan96a, 5:22]. Subsequently, happiness is nothing but a being's consciousness of the entire agreeableness with its own state now and in the future. Moreover, self-love is nothing but making happiness the supreme determining ground of the will. Hence, every material practical principle, whose determining ground depends solely on the representation of the expected pleasure of the reality of some object, is a material principle that serves the general principles of happiness and self-love. (The structured proof of Theorem 2 can be found in the Appendix at A1.)

With Theorem 1 Kant determined material practical principles negatively; namely, he proved that they can never be practical laws. Consequently, Kant also determined (some of) the boundaries of objective practical principles. That is, he excluded the faculty of desire as a possible origin of objective determining grounds. With Theorem 2 Kant determined material practical principles positively; namely, Kant showed that the faculty of desire can only provide principles that serve the principle of happiness or self-love. The following corollary is an immediate consequence of this result.

3.4 Corollary I - Pure Reason alone

At this point there are only two possibilities left: (i) either there are no objective principles, or (ii) if there are any, their determining ground must be found in pure reason. Kant proves this via the corollary to the second theorem. For Kant, "corollaries are immediate consequences from a preceding proposition" [Kan92a, Jäsche Logic - 9:113], where "[a]n immediate inference [...] is the derivation [...] of one judgment from the other without a mediating judgment" [Kan92a, Jäsche Logic - 9:114]. Although Kant's definition is straightforward, this definition does not seem to conform to the following proposition posited by Kant as a corollary:

COROLLARY 1. "All material practical rules put the determining ground of the will in the lower faculty of desire, and were there no merely formal laws of the will sufficient to determine it, then neither could any higher faculty of desire be admitted." [Kan96a, 5:22]

According to Kant this claim needs to follow immediately from Theorem 2. The question is: How is this immediate inference possible when the corollary consists of newly introduced terms such as 'higher' and 'lower' faculty of desire? Let us consider the argument.

The proof for Theorem 2 showed that every material practical principle depends on the expected feeling of agreeableness of the representation of some realized object; that is, it depends on some expected pleasure. According to Kant, a representation can have its origin either in the senses or in the understanding. However, since a material practical principle depends on the *agreeableness* of a representation, the origin of such a representation is not important (see [Kan96a, 5:23]). Thus, the feeling of pleasure that determines the object of the determination ground of the will, is always of one and the same kind regardless of the representation's origin:

> [T]he feeling of pleasure by which alone they properly constitute the determining ground of the will [...] is nevertheless of one and the same kind [...] insofar as it affects one and the same vital force that is manifested in the faculty of desire. [Kan96a, 5:23]

Hence, every material practical principle can contain no other determining ground than what is provided by the faculty of desire. So, where can the determining ground of an objective principle be found? For Kant a cognition can have two origins; a rational or an empirical origin:

> One can distinguish cognitions, [...] according to their objective origin, i.e., according to the sources from which alone a cognition is possible. In this respect all cognitions are either rational or empirical. [Kan92a, Jäsche Logic - 9:22-9:23][17]

Hence, if the determining ground of an objective principle cannot have an empirical origin, then *either* objective principles are just not possible and there

[17] According to Kant the origin of a cognition can be regarded from an objective or subjective point of view. The former is about the real origin of the cognition, that is, the origin from which the cognition in general arose. The latter is about the origin of the cognition with respect to how it arose in (the consciousness of) a particular individual being. For example, it is possible that a particular human being first hears about a pure cognition (which has an objective origin) via its senses (which is a subjective origin). With respect to the present undertaking I will only be concerned with the objective origin of a cognition.

is only one faculty (of desire) to be asserted; *or* the principle's determining ground must have a rational origin, that is, it must have its origin in pure reason. That an objective principle can never be empirically conditioned follows immediately from Kant's notion of experience; which is always conditioned on the subjective constitution of a being.

POSTULATE 4. (ORIGIN OF A COGNITION) According to the origin from which alone a cognition could arise a cognition's origin is either empirical or rational.[18]

The above argument shows that there is no distinction between a higher and a lower faculty of desire. Kant seems to introduce the distinction in this corollary to address a 'classical (philosophical) misunderstanding':

> It is surprising that men, otherwise acute, believe they can find a distinction between the lower and the higher faculty of desire. [Kan96a, 5:23]

This false distinction though arises when one talks about the origin of the *representations* of pleasure. Kant shows, though, that material determining grounds do not depend on these representations, but on the feeling of pleasure, which has only one origin. Consequently, the distinction between a higher and lower faculty of desire is superfluous.

The apparent 'lower' faculty of desire is therefore nothing but the faculty of desire itself. For this reason, Kant claims that "only insofar as reason of itself [...] determines the will, is reason a true higher faculty of desire" [Kan96a, 5:24]. Because Kant himself refutes the distinction, I will propose the following equivalent reformulation of the corollary, which preserves the structure of the original argument:[19]

COROLLARY 1'. All material practical principles have their determining ground in the faculty of desire and either there are no objective principles at all, or they must have their ground in pure reason alone.

(The structured proof of the corollary can be found in the Appendix at A2.)

3.5 Theorem III

Theorems 1 and 2 showed the negative property of the determining ground of an objective principle: It must *not* be matter. The corollary determined

[18] With respect to the philosophical analysis I will use the term 'postulate' instead of 'axiom'. The term 'axiom' will be reserved for the formal context. This division should make the distinction between the philosophical and logical part of the thesis more clear.

[19] In this corollary Kant introduces the term 'formal law' as well, though merely as a synonym for 'objective principle' and 'practical law'. Kant deduces the formal character of a practical law officially in the proof of his third theorem. I will therefore postpone the use of this term to the next section.

the only possible origin of such a ground: Pure reason. The aim of Kant's third theorem is to establish the properties of an objective determining ground positively. This theorem consists of Kant's derivation of 'the mere form of a practical principle' as the objective determining ground of the will. Before the proof of this theorem can be provided, Kant's general distinction between form and matter must be properly determined.

3.5.1 Form and Matter

The previous sections showed that, in Kant's practical philosophy, matter is inevitably connected with the objects of the faculty of desire. In the present section the main focus will be on Kant's notion of form. Consider the following remark by Kant on the distinction between form and matter with respect to concepts in general:

> With every concept we are to distinguish matter and form. The matter of the concept is the object, their form universality. [Kan92a, Jäsche Logic - 9:91]

The matter of a concept is that which is concerned with the object. The form of a concept is the concept's universality. The distinction coincides with Kant's distinction of the use of a cognition *in concreto*, with respect to its objects, and its use *in abstracto*, with respect to its universality: "Every concept can be used universally or particularly (in abstracto or in concreto)" [Kan92a, Jäsche Logic 9:99]. According to Kant,

> the expressions *abstract* and *concrete* relate not to concepts in themselves - for every concept is an abstract concept - but rather only to their use. And this use can in turn have various degrees. [Kan92a, Jäsche Logic - 9:99]

The distinction between matter and form must therefore be seen as a distinction in usage. That is, a concept can be regarded with respect to its sole matter or mere form, though the concept itself always contains both. With respect to the use of a concept matter and form are therefore mutually exclusive.

Postulate 5. Every practical principle can be regarded with respect to its *mere* form and with respect to its matter. If a practical principle is regarded with respect to its mere form, then it is not regarded as to its matter. If a practical principle is regarded with respect to its matter, then it is not regarded as to its mere form. (KL)

The question is: What in a concept constitutes its form? The following quote shows that the form of a concept is its universality and concerns that which is common to many; to be more precise, it concerns that what is universally represented in the extension of the concept:

> For since every concept, as a universally valid representation, contains that which is common to several representations of various things, all these things, which are to this extent contained under it, can be represented through it. [Kan92a, Jäsche Logic - 9:96]

The mere form of a concept represents an abstract rule that determines which objects belong to the extension of that concept and, subsequently, represents what is common to all objects in the concept's extension. Hence, the form of the concept of a practical principle is an abstract rule that represents what is common to all practical principles in the concept's extension; that is, its universal structure. Consequently, a practical law regarded as to its mere form must be regarded as to its universal lawgiving structure. The remaining question is: How can the mere form of such a principle be obtained? Kant provides a straightforward answer: "Now, all that remains of a law if one separates from it everything material [...] is the mere form of giving universal law" [Kan96a, 5:27]. The above elaboration provides the following axiom and definition:

POSTULATE 6. The mere form of a practical principle is obtained by subtracting the principle of all its matter.

DEFINITION 3.14. (MERE FORM) A practical principle's mere form is defined as the principle's use with respect to its universal structure; that is, with respect to what is commonly represented in all practical principles (of that sort).

3.5.2 Theorem III - the Argument

Kant formulates his third theorem as follows:

THEOREM 3. "If a rational being is to think of his maxims as practical universal laws, he can think of them only as principles that contain the determining ground of the will not by their matter but only by their form." [Kan96a, 5:27][20]

(KL)

In the above quote, 'them' and 'their' do not refer to maxims in general, but to 'maxims regarded as practical universal laws'. This theorem can therefore be seen as a thought-experiment to determine the necessary characteristics of a practical law: 'If we think of our maxims as practical laws, what should they

[20] The occurrence of the word 'only' twice in this theorem causes some confusion. Consider the German original: "Lehrsatz III. Wenn ein vernünftiges Wesen sich seine Maximen als praktische allgemeine Gesetze denken soll, so kann es sich dieselbe **nur** als solche Prinzipien denken, die, nicht der Materie, sondern **bloß** der Form nach, den Bestimmungsgrund des Willens enthalten" [Kan90, 5:27 - bold emphasis my own]. The first 'only' corresponds to the German 'nur', which implies that the term must be given a strict interpretation; that is, it signifies the determination of a necessary condition (only if). The second 'only' is a translation of the German 'bloß' which also translated as, for example, 'merely'. This last 'only' functions to highlight that the determining ground at stake must be *mere* form.

look like?'. That this theorem aims at determining the necessary characteristics of such a principle can be clearly seen through Kant's use of 'either ... or' in the reformulation of the claim at the end of the proof:

> Therefore, **either** a rational being cannot think of his subjectively practical principles, that is, his maxims, as being at the same time universal laws **or** he must assume that their mere form, by which they are fit for a giving of universal law, of itself and alone makes them practical laws. [Kan96a, 5:27 – bold emphasis my own]

I propose the following reading of the claim: 'If a rational being thinks of its maxim as a practical law then it must think of this principle as a principle containing a determining ground that consists merely of the form of that principle as a law'.[21] In the contrapositive this interpretation says, 'if a rational being does not think of its maxim as a principle containing a determining ground that consists of the mere form of a practical law, then it cannot think of its maxims as practical laws'. In other words, the mere form of a practical law as determining ground of the will is the necessary characteristic of a principle regarded as practical law.

The argument runs as follows: The matter of a principle is always the object of the will. There are two possibilities, this matter can be either (i) the determining ground of the principle or (ii) not (see [Kan96a, 5:27]). In the latter case matter *would* still be present in the principle, since the will needs something to be determined to (an action), though it would not be the ground on which the will is determined. If a rational being accordingly thinks of its maxim as a practical law, Theorem 1 tells us that, if (i) is the case this principle could never be a practical law. Hence, we must conclude that, if a rational being thinks of its maxim as a practical principle, then (ii) must be the case. That is, the determining ground of this principle must be non-matter. The question that rises is: What in this principle, instead of its matter, can constitute its determining ground? The ground must consist of what remains after subtracting everything material from this principle, which is nothing but the mere form of the principle as a law as such: The mere form of giving universal law. Thus, the determining ground of such a principle must be the mere form of this principle *as if* it would be a universal law. (The structured proof of Theorem 3 can be found in the Appendix at A3.)

3.6 The Constitution of a Lawgiving Will

The question that immediately rises with respect to the above result is: What kind of constitution of the will is needed, such that this will can be determined by a mere lawgiving form? This of course an enquiry to the *necessary conditions*

[21] The use of 'must' in this claim ensures the strictness of this claim and captures the idea of a necessary condition; that is, of an 'only if'-clause.

of a will, will it be determined by objective principles. For the first time in the second Critique we see explicit reference to the necessity argument. The problem addressed in this section, though, will only enable Kant to deduce this necessary condition *negatively.* In this section we will see that the results of the first three theorems enable Kant to derive this necessary condition positively.

At this point of the second Critique (5:28 - 5:30) Kant tries to find the constitution of a lawgiving will. He frames his endeavour in two problems. The first problem shows that 'if a will is to house such objective determining ground it must have the property of negative freedom'. The second problem shows that 'if a will is *only* negatively free (thus without a faculty of the desire), then it can only be determined through the mere form of universal lawgiving as a determining ground'.

3.6.1 Practical Lawgiving implies Freedom

PROBLEM 1. "Supposing that the mere lawgiving form of maxims is the only sufficient determining ground of a will: to find the constitution of a will that is determinable by it alone."[Kan96a, 5:28]

The argument runs as follows: 'mere form' implies subtraction from anything material, that is, from anything that belongs to sensibility. The mere form of a maxim can therefore never be an object of the senses; as a consequence, it can never be an appearance. That appearances belong only to the senses is something which Kant defines in the first Critique:

> The effect of an object on the capacity for representation, insofar as we are affected by it, is sensation. That intuition which is related to the object through sensation is called empirical. The undetermined object of an empirical intuition is called appearance. [Kan00, A20/B34]

In the Critique of Pure Reason Kant showed that the concept of causality is objectively valid with respect to theoretical cognition. On the one hand, Kant proved that all appearances fall under the law of causality (which is therefore called a natural causality) and, on the other hand, he proved that the concept of (natural) causality has no validity beyond the sensibility. In other words, all and only all appearances are subject to the concept of (natural) causality. Consider the following remark by Kant on causality:

> The concept of causality as natural necessity, as distinguished from the concept of causality as freedom, concerns **only** the existence of things insofar as it is determinable in time and hence as **appearances**. [Kan96a, 5:94 – bold emphasis my own]

From the above it can be concluded that, since the mere form of a practical principle can never be an appearance, it must be independent of natural causality. Consequently, a will that can *only* be determined through the mere lawgiving form of a practical principle, must be determined independent of natural causality. For Kant, this “independence, however, is freedom in the negative sense” [Kan96a, 5:33]. Thus, a will that takes the mere lawgiving form of its principles as its determining ground, must be negatively free.[22] The above solution to Problem 1 proves the following proposition:

Proposition 2. *If* the mere lawgiving form of a principle is the only sufficient determining ground of the will, *then*, if a will is determinable by this mere lawgiving form, this will must be negatively free.

(The structured proof of Proposition 2 can be found in the Appendix at A4.)

The above argument is based on the following result of the Critique of Pure Reason together with three corresponding definitions:[23]

Postulate 7. All appearances fall under the law of natural causality and this concept of causality does not apply beyond the sensibility.

Definition 3.15. (Appearances) An appearance is an (undetermined) object of the sensibility.

Definition 3.16. (Causality) Natural causality is the determination of an effect influenced by alien causes as appearances (with respect to a rational being's sensibility).[24]

Definition 3.17. (Negative Freedom) Negative freedom is independence of natural causality.[25] KL

(NB. Definition 3.16 and Definition 3.17 are mutually exclusive.)

3.6.2 Freedom implies Practical Lawgiving

The first problem showed that negative freedom is a necessary condition in order to be determinable by objective principles. The solution to the second

[22] Since we are dealing with the a priori concepts of natural causality and negative freedom, the reader must keep in mind that, whenever Kant speaks about a negatively free will, he speaks about a will *subject* to the concept of negative freedom.

[23] To show Kant's proof for Postulate 7 would, of course, be beyond the scope of this thesis. For this reason I pose it as a postulate. Notice that this postulate expresses the implications of Kant's proof for the objective validity of the concept of causality in relation to theoretical reason as found in his first Critique. I will come back to this result in 4.2.

[24] Definition 3.16 is based on the following remarks by Kant: “Since the mere form of a law can be represented only by reason [...] as the determining ground of the will [it] is distinct from all determining grounds of events in nature in accordance with the law of causality, because in their case the determining grounds must themselves be appearances” [Kan96a, 5:29] and “natural necessity is the property of the causality of all nonrational beings to be determined to activity by the influence of alien causes” [Kan96b, 4:446].

[25] Definition 3.17 is based on the following quotes: “[S]uch a will [a free will] must be thought as altogether independent of the natural law of appearances in their relations to one another, namely the law of causality” [Kan96a, 5:29], subsequently, “[t]hat independence, however, is freedom in the negative sense” [Kan96a, 5:33].

problem will show that a mere negatively free will can *only* be determined by objective principles. The elaboration of this problem belongs to the *possibility argument* because it contains a proof for the possibility of negative freedom as the (partial) ground of the possibility of being determined by practical law. Although the result of this problem will only be addressed in the next chapter, I will follow Kant and treat these two problems successively. This will, subsequently, facilitate comparison of the two problems and their results. Kant formulates the second problem as follows:

PROBLEM 2. "Supposing that a will is [negatively] free: to find the law that alone is competent to determine it necessarily."[Kan96a, 5:29]

The argument runs as follows: The matter of a practical law, that is the object of a principle, is always empirical. A mere (negatively) free will is a will that is completely independent of empirical conditions. Hence, matter can never determine such a will. Consequently, this will must find a determining ground for its principles that is independent of sensibility. What law is competent to determine this will? It must be the law subtracted from anything material; in other words, it must be the 'mere lawgiving form'. For this reason Kant concludes that, "[t]he lawgiving form, insofar as this is contained in the maxim, is therefore **the only thing** that can constitute a determining ground of the [negatively free] will" [Kan96a, 5:29 - bold emphasis my own]. The proposition proved by the solution to this problem can be formulated accordingly:

PROPOSITION 3. If a will is negatively free, then it can be determined by a practical law *if and only if* the determining ground of this will is the mere lawgiving form of this principle. KL

(The structured proof of Proposition 3 can be found in the Appendix at A5.)

3.6.3 Ratio Cognoscendi versus Ratio Essendi

One of the conclusions of the above two problems seems to be that, "freedom and unconditional practical law reciprocally imply each other" [Kan96a, 5:29]. However, Kant stresses that there is an important distinction to be made. Kant tries to find out where the idea of something as the 'unconditional practical' first arises:

> I ask instead from what our cognition of the unconditionally practical starts, whether from freedom or from the practical law. [Kan96a, 5:29]

Kant's aim here is not to determine the 'actual' origin of the unconditional practical, but to determine whether freedom or the practical law makes us first conscious of this practicality.

Consciousness of the unconditional practical cannot start from the concept of freedom derived thus far. This concept of freedom is only negatively defined, namely as an independence and for this reason Kant concludes that "[t]he preceding definition of freedom is negative and therefore unfruitful for insight into its essence" [Kan96b, 4:446]. A negative concept can only show itself in relation to something else. Experience, though, can only generate a consciousness of the law of natural causality because all experience is subject to this law. This law of natural causality is merely the opposite of freedom and for this reason it cannot provide any sign of freedom whatsoever. Thus, cognition of the unconditionally practical cannot start from the concept of negative freedom.

Kant concludes that cognition of the unconditional practical must arise from the immediate consciousness of the practical law, which presents itself to human beings as the moral law.[26] It is this form of the unconditionally practical that eventually leads to the idea of freedom. So, how is consciousness of this moral law possible? Kant provides the following positive justification: We, human beings, become aware of the moral law through (i) attending to the necessity with which reason prescribes moral principles to us, (ii) the awareness of reason that directs us to set aside our needs and inclinations and, lastly, (iii) through our experience in moral situations in which we become aware of the idea of having 'possibilities'. These three points give rise to the idea that we *ought to do* something. In this *ought* we recognize the idea of a moral law; namely, the idea of a practical law that tells us unconditionally what to do. Accordingly, this consciousness of the moral law leads us to the concept of freedom. Based on the above, Kant proposes the following distinction:

> [W]hereas freedom is indeed the **ratio essendi** of the moral law, the moral law is the **ratio cognoscendi** of freedom. For, had not the moral law already been distinctly thought in our reason, we should never consider ourselves justified in assuming such a thing as freedom (even though it is not self-contradictory). But were there no freedom, the moral law would not be encountered at all in ourselves. [Kan96a, 5:4]

Hence, although negative freedom is the (partial) ground of the possibility of being determined by practical laws (Problem 2), the idea of a practical law is the ground from which we become conscious of the idea of freedom at all (Problem 1). Kant calls these grounds, respectively, *ratio essendi* and *ratio cognoscendi*.

3.7 The Fundamental Practical Law

At this point in the second Critique Kant officially presents the formulation of the *fundamental law of pure reason*. Kant does not provide an explicit proof for this formulation of the law as he did with respect to the previous claims. Two questions arise: (1) Is the formulation of this law a (logical) consequence

[26]The remark to problem I and II contains Kant's first, though informal, use of the term *moral law* in the argument (see 5:29).

of the previous analysis? (2) Why does Kant present his formulation at this stage of the Analytic? To answer these questions, let us first consider the law itself:

FLoP. "So act that the maxim of your will could always hold at the same time as a principle in a giving of universal law." [Kan96a, 5:30][27] KL

Kant's formulation of FLoP shows that this law is completely independent of any subjective conditions of a particular being's will. The law's only condition is that the principle at stake *could hold* as a universal law. This possibility of 'holding as a universal law', though, can be determined a priori by pure reason since it consists of only form. A practical being's determination of the will, therefore, solely depends on the principle's agreement with the formal structure of a universal law a priori. For this reason Kant concludes that FLoP is an **unconditional** law expressing the thought that "one ought absolutely to proceed in a certain way" [Kan96a, 5:31].

There are three reasons why FLoP must be the fundamental law of practical reason. Every practical principle that consists of FLoP as its determining ground can be called an objective principle, because (i) it does not have any object of desire as its determining ground (Theorem 1), (ii) the principle does not depend on a desired effect of happiness (Theorem 2) and (iii) because FLoP determines the will in pure reason with respect to the form of the principle as a law (Corollary 1' and Theorem 3). Moreover, Theorem 3 showed that the mere form of a principle regarded as a universal law is the *only* possible objective determining ground of a will. In other words, a practical being can only be determined by practical law, if it regards FLoP as the determining ground of its will. For these reasons Kant concludes that as a "determining ground", FLoP "is regarded as the supreme condition of all maxims" [Kan96a, 5:31]. In relation to question (1), it can be concluded that Kant regards his formulation of the fundamental law of pure reason as a consequence of the previously established theorems. That is, for Kant FLoP is the only possible formulation of an objective determining ground given the results of Theorems 1, 2, and 3.

The following is a consequence of Theorem 3 and FLoP:

Proposition 4. A practical principle is an objective principle *if and only if* the principle consists of the fundamental law of pure practical reason (FLoP) as its determining ground.

[27] I will use the abbreviation 'FLoP' to refer to this Fundamental Law of Practical Reason. As the reader already might have noticed, this formulation of FLoP is identical to Kant's first formulation of the categorical imperative as found in the Groundwork of the Metaphysics of Morals (4:421). Kant calls an unconditional practical law, such as FLoP, a categorical practical proposition. In the next section I will address Kant's proof for the relation between FLoP and the moral law (i.e. the categorical imperative).

Proof: Trivial. By Definition 3.3, Theorem 3 and FLoP. ■

So far, the following two results in relation to Kant's practical transcendental argument have been established: Firstly, negative freedom is the moral law's necessary condition and, secondly, FLoP is the positive formulation of the fundamental law of practical reason. Negative freedom is only a negative concept and in order to complete the necessity argument Kant still needs to determine this necessary condition positively. In the proof of the fourth, and last, theorem of the second Critique Kant derives, from this positive formulation of FLoP, the concept of *freedom* as the moral law's positive necessary condition. Consequently, this answers question (2). Before this last theorem can be addressed, the relation between FLoP and the moral law needs to be established.

3.8 Corollary II - the Moral Law

Up till now, Kant's analysis has been devoted mostly to the differences between objective and subjective principles. The distinction between these two types of principles depends on the fact that the determining grounds associated with these principles have different origins (respectively, pure reason and the faculty of desire). A human being is a rational being that has these two possible origins at its disposal; that is, a human being consists of both a faculty of desire and reason. Because of this dual nature, a human being might find itself easily in conflict. The question is therefore:

> How can a being's will be determined by reason alone when it is at the same time affected by needs and desires?

Although the answer to this question is precisely what the practical transcendental argument needs to establish, it is the specific constitution of a human being that shows how the moral law is related to FLoP. The corollary to FLoP establishes this relation.

3.8.1 Human Beings

Kant regards the human being as a subspecies of the rational beings. He divides the set of rational beings into, on the one hand, the set of finite beings that consist of both reason and a faculty of desire and, on the other hand, the infinite being which consists of pure reason only.[28] In the first case such a being is called imperfect, in the latter perfect.[29] Human beings are subsumed under the first class of rational beings.

[28]This distinction corresponds to Kant's distinction between the second and the third level of practical determination (see 3.2.1).

[29]The idea of a perfect will (a supreme intelligence) is the idea of a rational being that is merely reason (i.e. reason without any sensible restrictions). Such a perfect will can, thus,

In this section I will only be concerned with the necessary characteristics of a human being. A human being is a finite being which by its special constitution has a dual nature. From a practical perspective, Kant describes the human being as follow:

> The human being is a being with needs, insofar as he belongs to the sensible world, and to this extent his reason certainly has a commission from the side of his sensibility which it cannot refuse [...]. But he is nevertheless not so completely an animal as to be indifferent to all that reason says on its own and to use reason merely as a tool for the satisfaction of his needs as a sensible being. [Kan96a, 5:61]

From this quote the following two points become clear: Firstly, as a consequence of its physical constitution a human being has needs that, to some extent, must be satisfied. These needs, that is desires, are the objects provided by the faculty of desire (see section 3.2.1). Practical principles arising from these needs are called subjective principles (Theorem 1). With respect to a human being's physical constitution, reason is used only as a tool for inferring the necessary means to realize the being's needs.

Secondly, a human being also seems to have the ability to use its sole reason in a practical sense. Here Kant refers to the apparent ability of a human being to act on moral grounds, namely, the ability to set aside desires and act upon what it thinks that is morally correct. (How this is possible is, of course, exactly what Kant aims to prove in his second Critique.)

Thus a human being has a twofold nature. On the one hand, it has a **sensible** nature. From the viewpoint of this nature a human being considers its "existence under empirically conditioned laws" [Kan96a, 5:43]. On the other hand, it has an intelligible nature. Kant calls this nature a human being's **supersensible** nature. With respect to this nature a human being regards itself as pure reason not affected by any sensibility.[30] For the present argument this will suffice.

Postulate 8. (The Human Being) From a practical perspective, a human being is a finite rational being with the following dual constitution:

(i) It is bound by needs provided by its faculty of desire.
(ii) It can formulate practical principles of its own with the use of its (pure) reason.

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solely be determined by grounds arising from (pure) reason. As a consequence, a rational being with a perfect will could only act in accordance with practical laws. For this reason Kant states that "[i]n the supremely self-sufficient intelligence, choice is rightly represented as incapable of any maxim that could not at the same time be objectively a law" [Kan96a, 5:32]. In other words, for a supreme intelligence there is no difference between its subjective principles and what is prescribed by practical law.

[30] Kant still needs to prove that this supersensible nature is actually possible. The proof will be treated in section 4.2.

3.8.2 Corollary II - the Argument

Corollary 2. "Pure reason is practical of itself alone and gives (to the human being) a universal law which we call the moral law." [Kan96a, 5:31] (KL)

The corollary consists of two claims: (i) pure reason *is* practical and (ii) for every human being FLoP is the moral law. For the first time, as a strict part of the analysis, Kant uses the term 'moral law'. Although he claims that this proposition is an immediate inference of FLoP, some justification for the official introduction of morality is needed.

According to Kant, there are four major questions that philosophy needs to answer: 'What can I know?' 'What ought I to do?' 'What may I hope?' and lastly, 'What is man?' [Kan92a, Jäsche Logic - 9:25]. The answer to the second question is the main concern of moral philosophy. Morality is thus concerned with a special form of conduct, namely, with normative conduct. Normative conduct prescribes what ought to be done. Moral philosophy is therefore concerned with the "laws in accordance with which everything ought to happen" [Kan96b, 4:388]. For this reason Kant states that the law(s) of morality prescribe obligation: "Everyone must grant that a law, if it is to hold morally, that is, as a ground of an obligation, must carry with it absolute necessity" [Kan96b, 4:389]. Hence, for Kant the question of 'What ought to be done?' inevitably leads to an inquiry to moral laws. Based on the above I propose the following definition:

Definition 3.18. (Moral Philosophy) Moral philosophy is defined as the philosophy of 'what ought to be'; i.e. moral philosophy is the philosophy that is concerned with the laws in accordance with which everything ought to happen from a practical point of view.

Kant's argument for the second corollary runs as follows: First, that pure practical reason is practical of itself alone, Kant showed by formulating the fundamental law. FLoP is a principle formulated by pure reason alone and conforms to the definition of an objective determining ground. With the formulation of FLoP Kant showed that pure reason can supply the will with practical principles; in other words, Kant showed that pure reason is practical. Secondly, a human being has the ability to create practical principles of its own. Moreover, it has sensible needs to satisfy. A human being's will can therefore be determined by subjective principles that would conflict practical law. Although a practical law puts forward a particular action as necessary, the actual determination of a will to that action is something completely different. The actions prescribed by objective principles are therefore not necessarily put into conduct by a human being. Consequently, with respect to human beings a practical law only expresses what 'has to be done'; i.e. it expresses what **ought to be**. Morality is about what ought to be done and, thus, for human beings pure reason presents the fundamental law as a moral law. (The structured proof of Corollary 2 can be found in the Appendix at A6.)

3.9 Objection: Pure Sensibility(?)

Based on the argument for Corollary 2 one might pose the following objection: If Kant conditions his arguments on the definition of a human being, doesn't he, as a consequence, deprive his transcendental argument from its purity by smuggling empirical cognition into the argument? After all, the answer to the question 'what is man?' belongs to the field of anthropology [Kan92a, Jäsche Logic - 9:25]. Moreover, in the Critique of Pure Reason Kant explicitly states that the above elaboration of the fundamental principle of practical reason does not belong to transcendental philosophy at all, since it builds on the empirical concepts of, for example, pleasure and desire:

> Hence, although the supreme principles of morality and the fundamental concepts of it are a priori cognitions, they still do not belong in transcendental philosophy since the concepts of pleasure and displeasure, of desires and inclinations, of choice, etc., which are all of empirical origin, must there be presupposed. [Kan00, A14/B28 - A15/B29]

At first sight the above objection seems to be a serious problem for Kant's transcendental endeavour. Empirical concepts, such as the concepts mentioned in the above quote, have been used throughout the whole analysis. Kant seems to have anticipated this problem and clarified it in a footnote in the introduction of the second Critique (5:9). He says the following:

> However, the definition there could admittedly be so framed that the feeling of pleasure would ground the determination of the faculty of desire [...], and thus the supreme principle of practical philosophy would necessarily turn out to be empirical. [Kan96a, footnote to 5:9]

One can question the purity of Kant's undertaking by positing the question whether pleasure, arising from empirical cognition, grounds the faculty of desire. (An affirmative answer means that Kant's usage of the concept of the faculty of desire would make the results of Theorems 1 and 2, and consequently FLoP itself, empirical.) Kant solves this problem by stating that the question in total must be refuted. He provides the following definitions of the concept's associated with the faculty of desire:

> **Life** is the faculty of a being to act in accordance with laws of the faculty of desire. The **faculty of desire** is a being's faculty to be by means of its representations the cause of the reality of the objects of these representations. **Pleasure** is the representation of the agreement of an object or of an action [...] with the faculty of the causality of a representation [...] with respect to the determination of the powers of the subject to action in order to produce the object. [Kan96a, footnote to 5:9]

The above definitions, the ones that Kant uses in the Analytic of the second Critique, are not based on whether pleasure grounds the determination of the faculty of desire or not; they are solely based on their functions and the relations between these concepts. The concepts and relations used in these definitions are not of an empirical nature, but merely constructed from concepts of the pure understanding (e.g. consider 'in accordance with laws', 'being the cause of the reality of an object' and 'by means of representation'). Consequently, Kant's concept of a human being does not have an empirical origin: The concept is merely based on the human being's faculties which are defined in terms of pure concepts of the understanding. Kant therefore concludes the following:

> For the purposes of this Critique [the second Critique] I have no further need of concepts borrowed from psychology; [...] for it [the definition of the faculty of desire] is **composed only of marks belonging to the pure understanding**, i.e., categories, which contain nothing empirical. [Kan96a, 5:10 - bold emphasis my own]

Benton argues that only through this explicit pure definition of the faculty of desire, Kant is allowed to carry out a Critique of practical reason without the risk of transforming his argument to an argument of empirical psychology [Ben77, p.24-25]. Hence, only with respect to the definitions provided in the footnote of the introduction to the second Critique Kant is justified in using the concepts of the faculty of desire in the analytical part of this Critique. On the basis of the above elaboration it can be concluded that Kant's argument thus far is free of any empirical cognition.

Proposition 5. The concept of the faculty of desire and its associated concepts, as used in Kant's practical transcendental argument, are defined on the basis of a priori concepts and are independent of any empirical cognition.

Proof. By definition Definitions 3.8, 3.10, 3.11, and 3.12. ■

With respect to the first quote of this section, it seems that Kant has changed his mind regarding the possibility of transcendental arguments in a critique of practical reason. The footnote in the introduction to the second Critique supports this conjecture. In the first Critique Kant still expresses the conviction that a transcendental undertaking in practical philosophy would be impossible because of the empirical nature of the faculty of desire. In the second Critique Kant explicitly states that one might object to any transcendental endeavour in this Critique for the same reasons that he endorsed in his first Critique. Only now Kant advocates the refutation of this objection: That is, Kant explicitly shows that the faculty of desire is *not* based on any empirical cognition.

3.10 Theorem IV

At this point of the argument Kant established the sufficient results to determine the moral law's necessary condition positively. So far this condition has only been determined negatively (Problem 1). The fourth, and last, theorem of the second Critique provides the positive determination. Unfortunately, in contrast to the previous theorems, Kant does not formulate his fourth theorem through a single proposition. What Kant aims to prove here can be understood through the following quote:

> [A]utonomy of pure practical reason, that is, freedom, [...] is itself the formal condition of all maxims, under which alone they can accord with the supreme practical law. [Kan96a, 5:33]

First of all, in the above quote Kant states that the concept of autonomy is the necessary condition through which a being can be determined in accordance with the fundamental law. Furthermore, the above quote shows that the concept of 'autonomy' must be equated with the concept of 'freedom'.[31] The concept of autonomy is defined by Kant as follows:

> Autonomy of the will is the sole principle of all moral laws [...] [and] the sole principle of morality consists in independence of all matter of the law [...] and at the same time in the determination of choice through the mere form of giving universal law that a maxim must be capable of. [Kan96a, 5:33]

Recall though that this "independence, however, is freedom in the negative sense", moreover "whereas this lawgiving of its own on the part of pure and, as such, practical reason is freedom in the positive sense" [Kan96a, 5:33]. Hence, it can be concluded that the concept of autonomy *is* the concept of freedom defined as a conjunction of the concepts of positive and negative freedom.

Definition 3.19. (Positive Freedom) Positive freedom is the idea of a being capable of lawgiving of its own. (KL)

Definition 3.20. (Freedom) The concept of freedom is defined as the conjunction of the concept of negative freedom and positive freedom. (KL)

3.10.1 Theorem IV - the Argument

Kant's fourth theorem can be reformulated as follows:

Theorem 4. The concept of freedom is the (a priori) necessary condition through which a human will can be determined by the moral law, that is, through which a being's maxims can accord with the moral law.

[31] To enhance readability I will omit explicit usage of the term 'autonomy' and consistently use the term 'freedom' instead.

The proof of this claim is provided by the conjunction of the following results. Firstly, Problem 1 showed that, for a will to be determined by a practical law, it needs to be determined independently from natural causality. This independence is nothing but the concept of freedom in its *negative sense*. Hence, it can be concluded that a will can be determined by a practical law, only if it is negatively free. Secondly, Theorem 3 and FLoP showed that, for a rational being's will to be determined by a practical law, that being must regard itself as capable of giving universal law through its maxims; that is, it must regard his maxims as fit for universal lawgiving. This idea of 'universal lawgiving of its own' however is nothing but the concept of freedom in its *positive sense*. Hence, a will can be determined by practical law, only if it is both negatively and positively free. The conjunction of negative and positive freedom however is the concept of freedom and it can therefore be concluded that a will can be determined by practical law, only if that will has the property of freedom. The above result is established for rational beings in general. In combination with Corollary 2 the above result implies the following: A will of a human being can be determined by the moral law only if that will is free. (The structured proof of Theorem 4 can be found in the Appendix at A7.)

The proof of Kant's fourth theorem finishes the practical necessity argument:

PROPOSITION 6. (THE PRACTICAL NECESSITY ARGUMENT) The concept of freedom is the necessary condition of the possibility of the moral law as a synthetic a priori proposition; that is, the concept of freedom is the necessary condition of a human will in order to be (immediately) determinable by the moral law.

Proof: Follows immediately from Theorem 4. ■

This finishes the elaboration of the necessity argument as found in the Critique of Practical Reason. This chapter treated Kant's argument for arriving at a positive formulation of the moral law (section 3.5 and 3.7) and the argument for the concept of freedom as the necessary condition of the moral law (section 3.6 and 3.10). Moreover, up till now Kant's argument has been established completely a priori (section 3.9). There are still two arguments left: The possibility argument and the objective validity argument. The former will be treated in the next chapter.

Chapter 4

The Possibility Argument

In the previous chapter I have shown Kant's derivation of the moral law and the proof of the concept of freedom as this law's necessary condition (Theorem 4). If my interpretation is correct, Kant did not only provide a positive determination of the fundamental law of pure reason (FLoP), he also showed that pure reason is practical because it provides an a priori determining ground (Corollary 2). Although we know that freedom is a necessary condition for the moral law, we do not know whether it can be the law's sufficient condition as well, that is, whether the concept of freedom is sufficient to generate the moral law as its consequence. To prove this is to prove that freedom is the moral law's *ground*. In this chapter I will provide an interpretation of, what I have called, the possibility argument.[1] Let us first recall what needs to be established:

The Possibility Argument. A transcendental argument needs to show which of ϕ's necessary conditions $\psi_1, ..., \psi_n$ form its ground ψ^*. That ψ^* is ϕ's ground is accomplished by (i) showing that ψ^* is (a priori) possible and (ii) by showing that ψ^* is sufficient to generate ϕ as its consequence.

In relation to the Critique of Practical Reason the possibility argument can be reformulated as follows:

Postulate 9. The concept of freedom is the ground of the moral law *if and only if* (1) it is possible as concept as such and (2) it is sufficient to generate the moral law as its consequence. KL

[1]Both the possibility and objective validity argument, as encountered in the second Critique, are substantially shorter than Kant's necessity argument. Moreover, Kant provides both arguments (almost simultaneously) in section I 'On the deduction of the principles of pure practical reason' of chapter I of the Analytic (5:42 - 5:50). The arguments for the possibility of freedom as a ground and the concept's objective validity, though, are essentially different. In order to enhance readability, I have decided to treat these two steps in two separate chapters.

Some of the premisses used in the possibility argument have already been proved in the previous chapter. I will omit explicit treatment of these arguments here, but only refer to them when necessary. Section 4.2 will be spend on Kant's argument for part (1) of Postulate 9. Section 4.3 will consist of the proof for part (2) of Postulate 9. Let us first consider a possible objection to the possibility argument.

4.1 Objection: Infinite Regression(?)

Before the practical possibility argument can be presented some attention must be paid to a problem that seems to arise in relation to this form of argumentation. One could object to the possibility argument by claiming the following: 'By trying to prove the possibility of freedom we have arrived at a regressive argument. Asking for the possibility of freedom is nothing more than asking for a new transcendental argument'. The objector would continue by stating that 'the possibility of freedom would call for another necessity argument which determines the necessary conditions of the concept of freedom, whose ground, subsequently, would be in need for another necessity argument et cetera'. If this objection is correct the possibility argument would initiate an infinite regression.

To show that there is no risk of ending up in an infinite regression, the function of the possibility argument (and how the argument differs from the necessity argument) must be emphasized. Recall that the aim of the possibility argument is to prove the possibility of some concept as a *ground*. This a priori concept is derived from an (undeniable) a priori synthetic cognition (e.g. '5 + 7 = 12' or 'the moral law'). Via the possibility argument it must be shown how the derived concept is possible, not with respect to its own cause (ground), but in relation to the synthetic a priori cognition as a cause. Moreover, since the possibility of the concept must be shown relative to its consequence and since its consequence is something which must be regarded as given, the concept that functions as a ground is *only* given relative to the consequence. This relation between ground and consequence is expressed in Kant's distinction between *ratio essendi* and *ratio cognoscendi* (see section 3.6.3). The ground is therefore unknowable independent from the synthetic a priori cognition in question and does not require a necessity argument of its own possibility: The necessity argument requires a deduction of the possibility of a cognition relative to its necessary cause, the possibility argument requires a deduction of the possibility of an a priori ground relative to its consequence. A search for the possibility of an a priori ground can therefore never end up in an infinite search for necessary conditions.

4.2 The Possibility of the Concept of Freedom

To show *how* the concept of freedom is possible is to show *when* the concept of freedom is possible; i.e. the how-question should determine in what setting the concept of freedom is possible. This means that the proof for freedom's possibility (as a concept) must determine which *limitations* to reason allow for this particular possibility.[2] Let us look at Kant's argument for the possibility of freedom.[3]

4.2.1 The Negative Possibility

For Kant, valid usage of a concept is restricted to viewpoints of reason. For example, when regarded from a logical point of view, reason is only regarded as governed by the laws and rules of logic, without considering for example reason in relation to how it processes experience. Showing the possibility of a concept requires therefore, on the one hand, a determination of the viewpoint from which the concept is not impossible (a negative determination) and, on the other hand, a proof for the possibility of the viewpoint itself (a positive determination).[4] The former part of the argument consists of determining the boundaries of the concept and is again twofold in nature: Firstly, the argument must show from which viewpoint(s) the concept is impossible, such that secondly, the remaining viewpoint(s) from which the concept would not be impossible can be determined.

Sensible Nature and Supersensible Nature

With respect to transcendental arguments, there are two particular viewpoints (or perspectives) that are to Kant's concern. These viewpoints concern reason with respect to its theoretical and practical use. That these viewpoints are mere 'applications' of one and the same reason is expressed by Kant in the following quote:[5]

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[2]The limited possibility of a concept implies that the objective validity of that concept must be relative to these limits as well. Namely, a concept's objective validity can only be proved for a context in which the concept is at least possible. This point will be addressed in the next chapter where Kant's proof for the objective validity of the concept of freedom will be provided.

[3]Recall that I will sometimes omit explicit reference to freedom as a concept, though the reader must keep in mind that Kant's argument is fully directed to prove the possibility of freedom *as a concept*.

[4]The last clause excludes the possibility that the only viewpoint from which the concept is possible is itself impossible. Without this clause the argument would remain merely hypothetical, namely based on the assumption of the possibility of a particular viewpoint, and the possibility of the concept itself could never be established positively.

[5]There seems to be an agreement in the Kantian literature that theoretical and practical reason are different applications of one and the same reason. Disagreement, though, arises mostly in relation to (i) the status of this distinction between theoretical and practical reason,

> But if pure reason of itself can be and really is practical, as the consciousness of the moral law proves it to be, it is still only one and the same reason which, whether from a theoretical or a practical perspective, judges according to a priori principles. [Kan96a, 5:121]

The a priori principles of the theoretical viewpoint have been firmly established by Kant in the first Critique and concerns rational beings when regarded with respect to their *sensible* nature. Opposite to this sensible nature is the *supersensible* nature. Before the boundaries of the concept of freedom can be determined, these two natures must be properly defined.

Kant defines a nature as follows: "Now, nature in the most general sense is the existence of things under laws" [Kan96a, 5:43]. Hence, regarded from a particular nature, a being is considered as existing under a particular set of laws belonging to that nature. For example, if reason could be regarded with respect to its logical nature, it would be considered as existing under the laws of logic (only). Thus, when we look at a being's theoretical nature, we look at this being's existence as regarded under the laws of sensibility. All these different natures from which a rational being can be regarded are therefore viewpoints from which a being can be regarded as existing under a set of laws particular to that nature. In this sense, Kant's Critique of Pure Reason can be seen as an attempt to establish the fundamental laws that make the sensible nature possible; that is, an attempt to prove the possibility of theoretical (synthetic a priori) cognition. Kant defines the sensible nature as follows:

> The sensible nature of rational beings in general is their existence under empirically conditioned laws. [Kan96a, 5:43]

A being's supersensible nature is defined as the sensible nature's complement:

> The supersensible nature **of the same beings**, on the other hand, is their existence in accordance with laws that are independent of any empirical condition. [Kan96a, 5:43 - bold emphasis my own]

Hence when we consider a rational being's supersensible nature we regard its existence as under a set of laws independent of sensibility. The emphasis in the last quote shows that a being's nature is only a way in which we can regard a rational being: It is a viewpoint. Based on Kant's elucidation of these two natures, I propose the following definitions:

Definition 4.1. (Sensible Nature) A being's sensible nature is the being's existence under empirical laws. It is a viewpoint established when reason (KL)

(ii) Kant's method of establishing this distinction and (iii) the status of the distinction with respect to reason's *unity* (see for example [Kle98]). The above three points, though, do not concern the present analysis and I will only use the following two less controversial points: Firstly, theoretical reason and practical reason are different 'perspectives' or 'applications' of one and the same reason; which I will call *viewpoints* of reason. Secondly, these two perspectives are distinct perspectives. That is, theoretical reason and practical reason are mutually exclusive. With respect to the present undertaking these two results will suffice.

is considered from a theoretical point of view.

DEFINITION 4.2. (SUPERSENSIBLE NATURE) A being's supersensible nature is the being's existence independently of any empirical laws. It is a viewpoint established when reason is considered from a non-theoretical point of view. (KL)

The following is a consequence of Definitions 4.1 and 4.2:

PROPOSITION 7. The sensible and supersensible nature are mutually exclusive: (i) If we look at a rational being's sensible nature, then we do not look at that being's supersensible nature. (ii) If we look at a rational being's supersensible nature, then we do not look at that being's sensible nature. (KL)

Proof. Trivial, by Definitions 4.1 and 4.2. ■

That there is such a sensible nature with its own set of laws, Kant has tried to shown in the Critique of Pure Reason. One of the main results of the first Critique is that the concept of causality is granted objective validity, though only *restricted* to the theoretical point of view. In the second Critique (5:42 - 5:50), Kant emphasizes the role of this result in relation to the possibility, and consequently the objective validity, of the concept of freedom. In the first Critique Kant proved that the concept of causality is objectively valid with respect to a rational being's sensible nature. One of the consequences of the objective validity of causality is that every appearance must be subject to the law of (natural) causality. Recall:

> [An] object should be taken in a twofold meaning, namely as appearance or as things in itself; [and] if its [the first Critique's] deduction of the pure concepts of the understanding is correct [...] the principle of causality applies **only** to things taken in the first sense, namely insofar as they are objects of experience. [Kan00, Bxxvii - bold emphasis my own]

The concept of causality is not only granted objective validity with respect to this particular viewpoint, its objective validity is also a *restricted* or *limited* validity of the concept:

> Even the concept of causality, which has application and so too significance strictly speaking only in reference to appearances, in order to connect them into experiences (as the Critique of Pure Reason proves) is not enlarged in such a way as to extend its use beyond the boundaries mentioned. [Kan96a, 5:49]

Why the concept of causality cannot be granted objective validity with respect to a being's supersensible nature, is explained by Kant as follows:

> For, if reason sought to do this [extending the application of the concept of causality beyond the theoretical viewpoint] it would have to try to show how the logical relation of ground and consequence could be used synthetically with a kind of intuition different from the sensible [...], this it cannot do. [Kan96a, 5:49]

According to Kant, a causal synthesis based on a non-sensible intuition would be impossible because it would require a causality of objects (that is, an intuition) for a nature in which such objects could not be intuited, namely, the supersensible nature. Again I will assume the validity of Kant's argument for the objective validity of the concept of causality as presented in the first Critique.[6] I propose the following extension of Postulate 7:

Postulate 10. The concept of causality is objectively valid with respect to the theoretical use of reason (the theoretical viewpoint) and its validity cannot exceed this use.[7] KL

With the above results at hand, the argument for the negative determination of freedom's boundaries can be represented: The concept of natural causality is contrary to the concept of freedom. The former expresses a dependence on sensibility, the latter an independence of sensibility. In the first Critique Kant showed that, with respect to a rational being's sensible nature, the concept of causality is objectively valid. Consequently, Kant showed the impossibility of the concept of freedom from a theoretical point of view: "The determination of the causality of beings in the sensible world can as such never be unconditioned" [Kan96a, 5:48]. At this point of the argument Kant seems to use the following assumption: Only natural causality makes freedom impossible. According to Kant it is the restriction of natural causality that creates room for the concept of freedom. That is, through *excluding* freedom from a rational being's sensible nature and *limiting* causality to this very same nature, Kant is able to secure 'the possibility of the possibility' of freedom from another point of view:

> I grant the mechanism of natural necessity [causality] the justice of going back from the conditioned to the condition ad infinitum, while on the other side **I keep open** for speculative reason the place which for it is vacant, namely the intelligible, in order to transfer the unconditioned into it. [Kan96a, 5:49 - bold emphasis my own]

This allows Kant to,

> preserve against all objections the assumption of freedom, regarded negatively, as quite compatible with those principles and limitations of pure theoretical reason. [Kan96a, 5:49]

Hence, because the concept of natural causality does not apply beyond a rational being's sensible nature, the concept of freedom could at least not be impossible in a being's non-sensible nature; which is a non-theoretical point of view. This argument is based on the following implicit assumption:

Postulate 11. The concept of natural causality is the *only* obstruction to the possibility of the concept of freedom.[8] KL

[6] The proof of the objective validity of the concept of causality would require its own transcendental argument. Such an elaboration is (of course) beyond the scope of this thesis.

[7] Notice that Postulate 7 is implied by Postulate 10.

Thus far the following proposition has been proved:

Proposition 8. The concept of freedom is impossible from a theoretical point of view and only from a non-theoretical point of view it is not impossible. (KL)

(The structured proof of Proposition 8 can be found in the Appendix at A8.)

However, Kant still needs to show that this supersensible nature itself is possible. Only through this Kant can show that the concept of freedom is not only not-impossible, but even possible as such. Kant's proof for the possibility of this supersensible nature will be addressed in the next section.[9]

4.2.2 The Positive Possibility

The previous argument showed that freedom is at least not impossible from a non-theoretical point of view. In order to prove that the concept of freedom is possible, Kant first of all needs to show that this non-theoretical viewpoint is possible at all. Is there an argument in the second Critique that proves the possibility of a viewpoint different from the theoretical viewpoint? The answer is affirmative. In this section, I will present Kant's argument for the positive determination of this viewpoint: *the practical viewpoint.* The possibility of this viewpoint is nothing but the possibility of a supersensible nature. The deductive principle that allows Kant to infer the possibility of this supersensible nature is the moral law:

> The moral law is, in fact, a law of causality through freedom and hence a law of the possibility of a supersensible nature. [Kan96a, 5:48]

In the previous chapter we saw that Kant is able to show that pure reason is practical, by showing that pure reason can formulate an objective determining ground of its own (FLoP). Moreover, we also saw that, for human beings, this fundamental law of pure practical reason is nothing but the moral law (Corollary 2). Hence, the moral law seems to point out a supersensible nature, namely, a nature of pure practical reason:

> [T]he moral law [...] provides a fact absolutely inexplicable from any data of the sensible world and from the whole compass of our theoretical use

[8]During the logical formalization I will come back to this assumption and treat it more thoroughly.

[9]Notice that, as an idea, the concept of causality may be freely used in both viewpoints. The concept of freedom is even based on the concept of causality: 'The idea of a causality with an unconditioned cause'. To our concern here is the limitation of the concept of causality with respect to its valid application to a being's will; that is, as a regulative principles. Kant does not aim to prove that the concept of causality cannot be used as an idea outside the sensible viewpoint, "since this concept is always found a priori in the understanding, even independently of any intuition" [Kan96a, 5:49]. He only excludes causality as a concept from being applicable to objects beyond the sensible nature.

> of reason, a fact that points to a pure world of the understanding and, indeed, even determines it positively and lets us cognize something of it, namely a law. [Kan96a, 5:43]

The moral law therefore enables Kant to deduce and justify the possibility of a practical viewpoint. The argument runs as follows: We cognize the moral law. The moral law is nothing but a practical law that is fully established in pure reason. This law points therefore to a pure world; namely, one in which we regard our maxims as at the same time being universal laws. Hence, the moral law points out a viewpoint of pure practical reason. Moreover, since this viewpoint is pure, it is (by definition) independent of any sensibility. Thus, since the moral law is the fundamental law of this pure world and since this world expresses an independence of sensibility, it can be concluded that this viewpoint must be the supersensible nature:

> Yet we are conscious through reason of a law to which all our maxims are subject, as if a natural order must at the same time arise from our will. This law must therefore be the idea of a nature not given empirically and yet possible through freedom, hence a supersensible nature to which we give objective reality at least in a practical respect. [Kan96a, 5:44]

From the possibility of the moral law, Kant deduces the possibility of a supersensible nature. Consequently, by showing the possibility of a supersensible nature Kant has provided a positive determination of a nature in which the concept of freedom is at least not impossible.

PROPOSITION 9. A supersensible nature is possible; namely, as a viewpoint of (pure) practical reason.[10] (KL)

(The structured proof of Proposition 9 can be found in the Appendix at A9.)

The above philosophical argument might seem obscure. The logical formalization of the argument in the second part of this thesis will shed light on

[10] For Kant the purely practical viewpoint *is* the practical viewpoint. Kant's distinction between the theoretical and practical viewpoint will play an important role throughout his transcendental argument and for this reason the reader must keep the following in mind: With respect to Kant's practical philosophy both the theoretical and practical viewpoint are concerned with the *practical* rational being, that is, with the rational being's *will*. The theoretical point of view is concerned with a rational being's will regarded as existing under the law's of that being's sensible nature. The practical point of view is concerned with a rational being's will regarded as existing independently of the law's of that being's sensible nature. From the former point of view a rational being is bound by the law of natural causality and its will is regarded as determined by the grounds arising from its faculty of desire. From the latter point of view, on the other hand, a rational being's will is regarded as determined by the grounds arising independently of that being's sensibility. Thus, in relation to Kant's practical philosophy, both viewpoints signify a different perspective of a being's nature with respect to that being's *will*. (Kant calls the pure viewpoint the practical viewpoint because he has shown that pure reason can be practical; that is, because he has shown that pure reason can generate a determining ground of its own.)

the Kant's reasoning with respect to the possibility of a supersensible nature. In short, it seems that Kant regards the proof of (some) *positive* content of an undetermined viewpoint as the positive construction of that viewpoint's possibility.

4.2.3 The Practical Possibility of Freedom

With the possibility of a supersensible nature Kant has shown that,

> it is not self-contradictory to regard all its [a human being's] actions as physically conditioned insofar as they are appearances and yet also to regard their causality as physically unconditioned insofar as the acting being is a being of the understanding. [Kan96a, 5:48]

Kant established this result by positively determining two different viewpoints that can coexist without infringing on one another: The theoretical and practical viewpoint. Kant derived the possibility of the practical viewpoint from the possibility of the moral law. This practical viewpoint signifies an independence of natural causality. This independence, however, is nothing but freedom in its negative sense. Hence, Kant derived the possibility of negative freedom. Moreover, Kant showed that FLoP is the fundamental law of this viewpoint. This fundamental law expresses the idea of freedom in its positive sense. Hence, by showing the possibility of a viewpoint of which FLoP is the fundamental law, Kant has shown the possibility of positive freedom as well. In other words, Kant proved the possibility of the concept of freedom.

PROPOSITION 10. The concept of freedom is possible from a practical point of view. (KL)

(The structured proof of Proposition 10 can be found in the Appendix at A10.)

The only thing left to show is that, from a practical point of view, the concept of freedom can generate the a priori synthesis expressed by the moral law.

4.3 The Possibility of Freedom as Cause

In order to show whether or not the concept of freedom can be the moral law's ground, Kant needs to show that the concept is sufficient to generate the a priori synthesis expressed by the moral law. In other words, Kant needs to show that the possibility of being immediately determinable by the moral law is a consequence of the freedom of the will.

That freedom can be the ground of the moral law has already been proved in the previous chapter: The solution to Problem 2 showed that a will that is negatively free, can be determined by an objective determining ground, and hence by practical law. Theorem 4, on the other hand, showed that the concept of the fundamental practical law (FLoP), that is the concept of universal legislation, is the positive concept of freedom. Consequently, it can be concluded that, on the one hand, the concept of freedom provides the fundamental law of practical reason (positive freedom) and, on the other hand, the concept generates the possibility of being determinable by this law (negative freedom). In other words, from the concept of freedom alone both the law and the possibility to be determined by this law can be derived. Together with Corollary 2 it can be concluded that, with respect to human beings, freedom can *sufficiently* generate both the moral law and the possibility to be determined by it; i.e. on the basis of a human being's own freedom the moral law can immediately determine that being's will.

PROPOSITION 11. With respect to human beings, the concept of freedom is sufficient to generate both the moral law and the possibility of being determined by this law. (KL)

(The structured proof of Proposition 11 can be found in the Appendix at A11.)

The result from this section (Proposition 11) and the previous section (Proposition 10) together provide the proof of the following proposition:

PROPOSITION 12. The concept of freedom is the moral law's ground. (KL)

Proof. Follows immediately from the results of Proposition 10 and Proposition 11 together with Postulate 9. ■

In this chapter I have represented Kant's practical possibility argument as can be found in the Critique of Practical Reason. The obscurities that arose in the above analysis will be addressed during the logical formalization of Kant's argument.

Chapter 5

The Objective Validity Argument

5.1 What it is not About

The moral law transfers a finite rational being *in idea* into a supersensible world, though, only as "if it were accompanied with suitable power" to determine its will and be its own unconditioned cause in the sensible world [Kan96a, 5:43]. The if-clause is very important in the above statement. Kant does not claim that he is about to prove that human beings are actually free, that is, that they can be an original (or unconditioned) cause in the sensible world. As we saw in the last chapter, this possibility of being a free cause in the sensible world has been excluded by the first Critique (Postulate 10):

> The determination of the causality of beings in the sensible world can as such never be unconditioned. [Kan96a, 5:48]

The above quote highlights Kant's emphasis on the fact that human beings *are* sensibly determined. What Kant wants to show, though, is whether these beings can *regard* themselves as being necessarily practically free. I think that Kant is aware of the risk of wrongly interpreting a proof for the 'objective validity of freedom' as a proof for 'the possibility of being a free cause in the sensible world' and for this reason repeatedly emphasizes the *relative* objective validity that needs to be shown through the second Critique's transcendental deduction. Consider for example the following quotes:

> Whether the causality of the will is adequate for the reality of the objects or not is left to the theoretical principles of reason to estimate. [Kan96a, 5:45]
>
> For, the concept [of freedom] receives significance apart from this [apart from theoretical reason] - though **only** for practical use. [Kan96a, 5:50 - bold emphasis my own]

Kant proved that the concept of freedom is restricted to the supersensible nature (section 4.2) and it is from this point of view that the concept must shown to be objectively valid. Hence, the transcendental question at stake is not whether conduct can be realized on the basis of pure reason, but "whether and how it [reason] can determine the will immediately" [Kan96a, 5:46].

5.2 The Objective Validity of Freedom

A consequence of restricting the validity of the concept of freedom to the practical viewpoint is that the deduction of the objective validity of this concept cannot be based on the same deductive principles as used in the transcendental arguments for theoretical reason in the first Critique. For this reason Kant states that "one cannot hope to get on so well as was the case with the principles of the pure theoretical understanding" [Kan96a, 5:46]. In the first Critique Kant was able to derive the objective validity of the categories on the basis of the reality of experience. However, the concept of freedom cannot get its significance from experience, that is, from the experience of an actually free produced deed. All experienced deeds fall under the law of natural causality (Postulate 10) and therefore belong to the heteronomy of the will (as opposed to autonomy) [Kan96a, 5:43]. That this practical transcendental deduction cannot be based on a priori intuition follows from the fact that the deduction is restricted to the supersensible nature, where an "a priori intuition (of an intelligible world), [...] as supersensible, would also have to be impossible for us" [Kan96a, 5:45]. Hence the question rises: What is the deductive principle from which Kant starts his practical transcendental deduction?

> For, the concept [of freedom] receives significance apart from this [theoretical reason] - though only for practical use, namely, through the moral law. [Kan96a, 5:50]

It is the moral law itself that provides objective validity to the concept of freedom. According to Kant, the reality of the moral law must be taken as the deductive principle of the transcendental deduction of practical reason. The following quotes confirm this:

> In this undertaking the Critique [of Practical Reason] can therefore not be censured for beginning with pure practical laws and their **reality**, and it must begin there. Instead of intuition, however, it **takes as its basis** those laws. [Kan96a, 5:46]

> Moreover the moral law is given, as it were, as a **fact of pure reason** of which we are a priori conscious and which is apodictically certain [...]. Hence the objective reality of the moral law cannot be proved by any deduction. [Kan96a, 5:47]

> One cannot provide objective reality for any theoretical idea, or prove it, except for the idea of freedom, because this is the condition of the moral law, whose **reality is an axiom**. [Kan92a, Jäsche Logic - 9:93][1]

For Kant the moral law is real (or at least must be assumed to be real). The moral law's reality allows Kant to ascribe the same 'deductive powers' to the moral law as the reality of experience had for the transcendental deductions in the first Critique. According to Kant, the objective reality of the moral law cannot be proved; its reality must be assumed as an axiom or fact of reason. For this reason the reality of the moral law can be the basic principle from which the objective validity of its ground may be deduced. In other words, the transcendental deduction of the objective validity of freedom starts with the reality of the moral law.

Postulate 12. The moral law is real. (KL)

Recall what needs to be proved:

The Objective Validity Argument. A transcendental argument needs to show that the objective validity of the a priori ground ψ follows from the reality of the cognition ϕ from which ψ is derived.

The cognition in question is the moral law. The necessity and possibility argument provided the positive determination of the moral law's ground: the concept of freedom. In relation to the second Critique the argument's aim can thus be translated accordingly: The objective validity argument needs to show that the (practical) objective validity of the concept of freedom follows from the reality of the moral law. How is this established?

Kant's argument is twofold. Firstly, the moral law points out a supersensible nature. Consequently, the reality of the moral law shows us that, for every rational being who regards the moral law as real, this supersensible nature must be objectively valid. This supersensible nature, though, is a nature independent of sensibility and expresses therefore nothing but the concept of negative freedom. In other words, the reality of the moral law implies the objective validity of the concept of *negative* freedom:

> This law must therefore be the idea of a nature not given empirically and yet possible through freedom, hence a supersensible nature to which we give objective reality at least in a practical respect. [Kan96a, 5:44][2]

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[1] In all three quotes bold emphasis is my own.

[2] With respect to the concept of freedom Kant uses the terms 'objective reality' and 'objective validity' interchangeably. I will therefore regard this quote as being about the objective validity of the supersensible nature. For the sake of readability I will only use the term objective validity. Notice though that plain 'reality' and 'objective validity' are not synonyms (section 2.2.3).

Secondly, the moral law implies the concept of a being capable of lawgiving of its own. Consequently, the reality of the moral law shows us that, for every rational being who regards the moral law as real, the concept of being your own legislator must be objectively valid. This concept, though, is the concept of positive freedom. In other words, the reality of the moral law implies the objective validity of the concept of positive freedom:

> [T]he moral law thus determines that which speculative philosophy had to leave undetermined, namely the law for a causality the concept of which was only negative in the latter, and thus for the first time provides objective reality to this concept [of freedom]. [Kan96a, 5:47][3]

Thus far the following result has been established: Objective validity is implemented in the concept of freedom via two channels, namely, (i) via a deduction of the objective validity of negative freedom and (ii) via a deduction of the objective validity of positive freedom.

Unfortunately, the part of the Critique in which Kant claims to have shown the objective validity of the concept of freedom (5:42 – 5:50) shows us nothing of *how* this objective validity is actually generated (and justified) by the reality of the moral law. The question remains: What in the concept of reality establishes the objective validity of a pure concept? Kant, however, does not provide an explicit definition of the concept of reality or an elaboration of its function in relation to the present argument. How Kant actually proves the objective validity of freedom can only be expressed by conjectures. I therefore propose to postpone treatment of this argument. In chapter 7 of part two of this thesis, I will provide a logical formalization of both the possibility and objective validity argument. It will turn out that the logical formalization of Kant's argument(s) allows us to determine the relations between, and eventually even the definitions of, the concepts of reality and objective validity.

The following proposition needs to be proved:

Proposition 13. (The Objective Validity of Freedom) The concept of freedom is objectively valid from a practical point of view. (KL)

Based on the above analysis I will propose the following minimal criteria for the validity of Kant's practical objective validity argument:

Postulate 13. (The Practical Objective Validity Argument) Kant's argument for the objective validity of the concept of freedom (i.e. the proof of Proposition 13) must at least satisfy the following criteria:

[3]In this quote Kant refers to a law of causality determined by the moral law. This law of causality is the formulation of FLoP. FLoP, on the other hand, expresses the positive concept of freedom. For this reason Kant claims that the moral law provides a positive determination of a law that provides the objective validity (reality) of freedom: "For, the moral law proves its reality, so as even to satisfy the Critique of speculative reason, by adding a **positive determination** to a causality thought only negatively [...]; it adds, namely, the concept of a reason determining the will immediately (by the condition of a universal lawful form of its maxims)" [Kan96a, 5:48 - bold emphasis my own].

1► The reality of the moral law (i.e. Postulate 12) is the (primary) deductive principle of the argument.
2► The objective validity of the concept of negative freedom is derived from the reality of the moral law.
3► The objective validity of the concept of positive freedom is derived from the reality of the moral law.
4► The objective validity of freedom is concluded from the objective validity of the concept of negative freedom and the objective validity of the concept of positive freedom.

5.3 The Possibility of the Moral Law

Kant's argument treated thus far covers the second aim of a transcendental argument in general (Postulate 1). Recall the argument's primary aim:

Postulate 3 (The Possibility of the Moral Law) The aim of the transcendental argument of the Critique of Practical Reason is to prove how the moral law is possible as a synthetic a priori proposition; that is, the argument needs to prove how the moral law can immediately determine a human being's will.

The demonstration of the above aim should result from the objective validity of the concept of freedom. That is, Kant needs to show that "the human will is **by virtue of its freedom** immediately determinable by the moral law" [Kan96a, 5:38 – bold emphasis my own]. Hence, the main aim of Kant's practical transcendental argument is to prove the following proposition:

Proposition 14. From the objective validity of the concept of freedom it follows that the will of every human being is immediately determinable by the moral law.

Unfortunately, the last section ended with the postponement of the thorough treatment of Kant's argument for the objective validity of the concept of freedom. Proposition 14, though, is based on the objective validity of this concept. As a consequence, at this stage of the analysis Kant's proof for Proposition 14 cannot be provided. In the next part of this thesis Kant's argument(s) will be regarded from a logical point of view. The consequences of the logical analysis will accordingly be used to provide an interpretation of Kant's argument for the possibility of the moral law as a synthetic a priori proposition. The aim of this section is to establish the minimal criteria to which any interpretation of the argument must conform. These criteria may also serve as a guiding thread in interpreting the formal results of the logical analysis.

There are two criteria: Firstly, the argument for Proposition 14 should establish an a priori synthesis. That is, the exposition of the proof for this proposition should generate a necessary connection between, on the one hand, the moral law and, on the other hand, the will of every human being. This connection, accordingly, expresses the possibility of the immediate determinability of every human being by the moral law. Secondly, this connection must be established on the basis of the objective validity of the concept of freedom *only*; that is, it must be shown that the possibility of the a priori synthesis is a consequence of the objective validity of the ground in question.

POSTULATE 14. (THE MAIN AIM'S ARGUMENT) Kant's argument for the possibility of the moral law as a synthetic a priori proposition (i.e. the proof of Proposition 14) must at least satisfy the following criteria:

1► The argument must establish an a priori necessary connection between the moral law and every human being's will.

2► The a priori synthesis must be derived from the objective validity of the concept of freedom only.

To sum up the first part of this thesis: The argument provided by Kant in book I of the Critique of Practical Reason has all the appearance of a genuine (Kantian) transcendental argument. That is, the analysis of Kant's argument(s), as provided in chapters 3, 4 and 5, conforms to the criteria of transcendental arguments in general as provided in chapter 2. During the philosophical analysis, though, some problems arose with respect to Kant's argumentation. Moreover, insufficient information about Kant's definitions of reality and objective validity forced us to postpone any thorough treatment of the objective validity argument and Kant's proof for the transcendental argument's main aim. In the second part of this thesis, which will consist of a logical formalization of the proposed interpretation of Kant's argument, I will address the philosophical problems encountered in part one. The logical formalization will consist of a formalization of the possibility and objective validity argument. Eventually, with the (philosophical) results of this logical analysis, I will reconsider Kant's exposition of the possibility of the moral law as a synthetic a priori proposition.

Before the logical part of this thesis can be attended, a brief remark must be made. Any formalization presupposes an interpretation of the phenomenon it aims to formalize. The reader might disagree with the present interpretation of Kant's argument, but in order to understand, and eventually judge, the validity of the logical formalization relative to the proposed interpretation, I ask the reader to follow me in my train of thought and assume (for now) that this interpretation is at least coherent.

Part II

The Logical Formalization

Chapter 6

The Logical Framework

In this chapter I will present the logic that will be used for the logical formalization of Kant's practical transcendental argument. The formalization itself will be provided in chapter 7. Although the logical framework will be properly defined, I assume that the reader has some basic knowledge of the terminology and principles of intuitionistic predicate logic and the system of natural deduction.

Determining the logical framework

Kant's vocabulary is far richer than classical logic would allow us to express. Firstly, in the Kantian language a variety of objects can be encountered (e.g. 'concepts', 'beings', 'actions' and 'viewpoints'). This variety of objects suggests the need of a logic with several distinct sets of ground-level objects. The many-sorted calculus is such a logic. Secondly, Kant's use of concepts is twofold: On the one hand, Kant uses concepts as predicates that can apply to objects (e.g. 'a being x is free'). On the other hand, he uses concepts as objects of reasoning to which other concepts might apply (e.g. 'the concept of freedom is a ground'). We therefore need a logic that allows us to treat concepts as predicates as well as objects. The type-free calculus is such a logic. In the type-free calculus formulas (and concepts) can be treated as objects via the formal method of 'reification'. Reification, or objectification, enables predication over formulas and concepts as objects. In this thesis I will use the type-free calculus as developed by Feferman in the article '*Toward useful Type-Free Theories. I*' [Fef84]. The type-free calculus and many-sorted calculus will be introduced, respectively, in section 6.1 and 6.2.

Kant's modalities are related to viewpoints (e.g. practical necessity). Necessity and possibility must therefore be evaluated in a 'viewpoint-context'. For this reason I will propose a 'soft' version of the situation calculus (based on [Rei01]). The calculus will enable us to treat sets of situations as representatives of the different viewpoints from which we can regard rational beings.

This version of the calculus will be introduced in section 6.3.

During the formalization we will encounter several instances of Kant's reasoning that are similar to intuitionistic reasoning. The logic that will be used in this formalization will therefore have an intuitionistic base. I have chosen to represent Kant's argumentation via the system of *natural deduction*. There are several advantages to the use of this system. For example, natural deduction allows the reader to follow every inference step in the reasoning process. Moreover, it forces us to uncover every necessary step in order to obtain the original conclusion of Kant's argument. The system will be introduced in section 6.4.

To sum up, the logic that will be used for the formalization of Kant's transcendental argument is an *intuitionistic many-sorted type-free situation calculus*. I will refer to this logic as KL. What makes the logic so especially Kantian is the inclusion of Kant's modalities and the set of inference rules and axioms justified by Kant's original reasoning. Lastly, the formalization of Kant's argument will only concern the argument's syntax. For the sake of the size of the present undertaking I will therefore omit explicit elaboration of KL models and the corresponding semantics.

6.1 The Type-Free Calculus

In a type theory every term is assigned a type of a particular level. This causes a hierarchy in the language. Different types can subsequently be restricted to particular operations such that paradoxes of self-reference can be averted.[1] A type-free theory provides an alternative to such an approach:

> [O]ne usually refers to a formal framework (a theory or structure) as being type-free if it admits significant instances of self-application. [Fef84, p.76-77]

A type-free theory uses *reification* instead of a complex hierarchy of types. Via the formal device of reification a formula can be regarded as a term in a predicate or a function. This enables direct and indirect self-reference. In such a framework one can avoid paradoxes of self-reference by adopting a many-valued truth system.[2] One of the advantages of using a type-free theory instead of a higher-order type theory is that the former keeps the complexity of a formula low. By reducing complex formulas to objects the former method

[1] For example, in type theory sets can be restricted to contain only elements of type-level lower than itself. This is one of the proposed solutions to Russell's paradox: Let Φ be the set of all sets such that $\Phi = \{x | \ x \notin x\}$. Without the restriction this would imply that $\Phi \in \Phi \Leftrightarrow \Phi \notin \Phi$, which is of course a genuine contradiction.

[2] For example, in a three-valued logic (e.g. a bivalent logic extended with the value 'undefined') one can account for Russell's paradox by restricting self-referential statements to the truth-value 'undefined'.

keeps the logic on a 'first-order level', while the latter creates a hierarchy of types that increases with the complexity of every formula.

The calculus that will be used in this thesis consists of the type-free calculus as proposed by Feferman [Fef84]. The reification device of the calculus contains the formal machinery of *Gödel Numbering.* A Gödel Numbering is a function that maps every element of the language under consideration to a unique natural number. By taking the natural numbers as terms of the language, Gödel-numbers can be regarded as the representatives of the formulas of that language. As the representative of a formula, a Gödel-number can accordingly serve as a term in functions and predicates. For the present purpose of this thesis, exact understanding of the device of Gödel Numbering is unnecessary. In the next section I will informally represent Feferman's denotation device.[3] The main purpose of that section is to provide the reader with a general idea of how reification is obtained in Feferman's type-free calculus.

The language of the type-free calculus is a first-order language consisting of the following logical symbols: The binary connectives $\wedge, \vee, \rightarrow$ (respectively: 'and', 'or' and 'implication') and the unary connectives $\neg, \forall, \exists$ (respectively: 'not', 'universal quantifier' and 'existential quantifier'). Negation is defined as follows: $\neg\phi := \phi \rightarrow \bot$. The binary connective $\leftrightarrow$ is defined as $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$. Set-theoretical operations such as $\in, \setminus$ and $\subseteq$ (respectively 'inclusion', 'complement' and 'subset relation') will be used in the meta-language.[4] The parentheses '[' and ']' indicate the positioning of formulas as conventionally defined. '(' and ')' will be used for the determination of parameters in functions and predicates . '{' and '}' are reserved to indicate the use of sets. Let $\bot$ and $\top$ stand for contradiction and tautology respectively. $\bot$ will be regarded as a zero-place logical operator, that is, a logical constant (following [TS00, Ch.1]). The following abbreviation will be used for tautology: $\top := \bot \rightarrow \bot$. The symbol $\dashv$ will be used to mark the end of a formal definition, proposition, theorem et cetera.[5]

Let Φ^n be the (countably infinite) set of n-ary predicate symbols: P^n, Q^n, R^n... . I will be informal in the notation of these predicate symbols by omitting the arity and writing P, Q and R instead whenever the context is clear. Let X be the (countably infinite) set of variables: x, x_1, y, z... . Lastly, let $\mathcal{F}^n$ be the (countably infinite) set of n-ary function symbols: f^n, f_1^n, f_2^n... . Individual constants (ground-level objects) are defined as 0-ary function symbols.

[3]For a formal treatment of the device of Gödel Numbering the reader is referred to [Raa15].

[4]In the type-free calculus a reified formula can be treated as an argument of a predicate. Set-theoretical operations, such as the member relation $\in$, must therefore be used with caution. In order to evade unnecessary confusion I will therefore only use these operations in the meta-language.

[5]The identity symbol $=$ will be used in the meta-language, but not in the logic itself.

The Denotation Device

Feferman's reification device is strictly formal and a full elaboration of the device will be unnecessary for the present undertaking. The sole aim of this section is to provide a glimpse of how this formal machinery is obtained. The upcoming informal exposition is based on [Fef84, §§7-10] and [VLH08, Ch.6].[6]

A logic L fit for the device must include a first-order language $\mathcal{L}$. The language $\mathcal{L}$ must contain (at least) the ground-level object 0. Furthermore, L must consist of the following three operations: P, P_1 and P_2. Let S be some set of objects of L such that S does not contain 0. P functions as a binary paring from S^2 into S. (Explicit reference to P will be omitted which means that (τ_1, τ_2) will be written instead of $P(\tau_1, \tau_2)$.) P_1 and P_2 are unary projection operations. The behaviour of these operations is expressed by the following axioms:

- $(x, y) \neq 0$
- $[P_1(x, y) = x] \wedge [P_2(x, y) = y]$

(P_1 and P_2 can be seen as operations that 'single out' a term from a pair.) Subsequently, L must be extended with the following successor operation '$*$' such that $x^* = (x, 0)$. (In relation to the $*$-operation P_1 functions as a predecessor operation.) A logic that satisfies all of the above can sufficiently represent the natural numbers structure. Consequently, the set of natural numbers can be identified with a subset S' of S.

Tuples will be defined recursively: $(\tau_1) = \tau$ and $(\tau_1, ..., \tau_{k+1}) = ((\tau_1, ..., \tau_k), \tau_{k+1})$. The corresponding projection operation $P_i^k(1 \leq i \leq k)$ 'singles out' the i-th term of a tuple: $P_i^k(\tau_1, ..., \tau_k) = \tau_i$. This finishes the basic framework that enables the introduction of Feferman's denotation device.

Definition 6.1. (Denotation via Gödel-Numbering) Let ϕ be a formula of the language $\mathcal{L}$ at stake. $\ulcorner \phi \urcorner$ represents the Gödel-number of ϕ or its corresponding numeral in $\mathcal{L}$. ($\ulcorner \phi \urcorner$ will be used both for the term in $\mathcal{L}$ and for the object denoted by that term in a model for $\mathcal{L}$.)

If ϕ has free variable occurrences among its variables $x_1, ..., x_k, y_1, ., y_n$, then the term $(\ulcorner \phi \urcorner, y_1, ..., y_n)$ serves as an operation of the logic that *abstracts* the bound variables $x_1, ..., x_k$ and treats the free variables $y_1, ..., y_n$ as parameters (a bound variable x is indicated by $\hat{x}$). This operation provides the following definition:

$$\Delta_k \qquad \phi[\hat{x}_1, ..., \hat{x}_k, y_1, ..., y_n] = (\ulcorner \phi \urcorner, y_1, .., y_n)$$

(The denotation device allows for the introduction of set-like objects of the following form (k=1, n=0): $\phi[\hat{x}]$, that is $\{x \mid \phi(x)\}$. The set-like object represents x by abstraction and hence can be seen as an object that represents a set containing all x that belong to ϕ.)

[6] For an elaboration of the complete formal system the reader is referred to Feferman's original article [Fef84].

The following truth axiom provides meaning to the 'Gödel-numbers' of Feferman's denotation device Δ_k (see [Fef84, p.91]):

$$(T_k A) \quad T_k(x_1, ..., x_k, \phi[\hat{u}_1, ..., \hat{u}_k, y_1, ..., y_n]) \leftrightarrow \phi(x_1, ..., x_k, y_1, ..., y_n)$$

T_k is a $(k+1)$-place predicate symbol. $T_k(x_1, ..., x_k, z)$ must be read as follows: The tuple $(x_1, ..., x_k)$ satisfies z. (z represents a particular formula. That is, z is a Gödel-number.) Hence, the formula $T_k(x_1, ..., x_k, z)$ expresses that the tuple $(x_1, ..., x_k)$ satisfies the formula coded by the Gödel-number z. How should the axiom be interpreted?

During the formalization of Kant's argument we will only consider formulas without free variables; that is, we will only consider *sentences*. In relation to Feferman's device this means that we will only be using the following version of the axiom of truth $(T_k A)$ (n=0):

$$(T_k^* A) \quad T_k(x_1, ..., x_k, \phi[\hat{u}_1, ..., \hat{u}_k]) \leftrightarrow \phi(x_1, ..., x_k)$$

The axiom $(T_k^* A)$ can be interpreted as follows: A tuple $(x_1, ..., x_k)$ satisfies the Gödel-number for ϕ if and only if $(x_1, ..., x_k)$ belongs to the extension of ϕ, that is, $(x_1, ..., x_k) \in \{(y_1, ..., y_k) \mid \phi(y_1, ..., y_k)\}$. In other words, the truth axiom $(T_k A)$ provides meaning to a Gödel-number $\ulcorner \phi \urcorner$ by associating every tuple that satisfies the original formula ϕ with $\ulcorner \phi \urcorner$. Hence, via the axiom of truth $(T_k A)$ the term $\ulcorner \phi \urcorner$ can be correctly identified with the formula ϕ. The above elaboration shows how a Gödel-number can represent a formula.

A logic that consists of a truth predicate and a method for reification opens its doors to the *liar paradox*: 'This sentence is false'. Consequently, such a logic cannot be consistently expressed via the use of a classical two-valued semantics. After all, the sentence 'this sentence is false' can be neither true nor false. The present calculus is not an exception. One way to account for this paradox is to use a formal framework that takes as its base partial predicates (and corresponding partial structures) in a three-valued environment. Feferman provides such an interpretation.[7] The calculus used in this thesis will consist of Feferman's version of a type-free system in a three-valued logic. (The logical connectives are provided with a Kleene interpretation of three-valued connectives; see [Fef84, p.87].[8]) In short, its semantics consists of three truth-values: 'true', 'false' and 'undefined' (respectively t, f and u). These truth-values are *partially ordered*: $u \leq t$, $u \leq f$, $t \leq t$, $f \leq f$ and $u \leq u$. Informally

[7] To be more precise, Feferman's work takes as its starting point the Liar paradox and Russell's paradox. The present logical framework represents one of the possible formal frameworks, provided by Feferman, that deals in a type-free way with these paradoxes.

[8] A three-valued Kleene semantics does not alter the formal apparatus represented in this section. Only the logical equivalence '$\leftrightarrow$' in the truth axiom $(T_k A)$ requires a Kleene interpretation of the connective. This means that whenever the left and right side of the equivalence are both 'undefined' the formula itself is true (see [Fef84, §8]).

this means that at some undecided stage the truth-value of some object can still evolve from undecided to either true or false: "u is not to be thought of in this respect as a definite truth-value on a par with t, f [...], but rather as a lack of such" [Fef84, p.87]. The partial order implies that after an object has been determined true (or false), the object remains true (or false); i.e. the logic is monotonic (see [Fef84, §§7-8]).[9] The aim of this thesis is to present Kant's reasoning via de system of natural deduction and for this reason the present exposition of Feferman's 'three-valued' interpretation of the denotation device will suffice.[10]

The Type-Free First Order Language

In the present framework neither Gödel Numbering, nor terms, nor formulas can be regarded as fundamental because their recursive definitions are interdependent. These three notions will therefore be defined simultaneously.[11]

Definition 6.2. (The Type-Free Language) Let $\mathcal{L}$ be a type-free first order language. $\mathcal{L}$ is formally defined by Definitions 6.1, 6.3, and 6.4. ⊣

Definition 6.3. (Terms of $\mathcal{L}$) Let $\mathcal{L}$ be the language in question. Terms (denoted by τ_i) of $\mathcal{L}$ are defined as follows:

(i)▶ If $x \in X$ is an individual variable, then x is a term of $\mathcal{L}$. The free variable occurrence of x is x.

(ii)▶ If $a \in \mathcal{F}^0$ is an individual constant, then a is a term of $\mathcal{L}$. a has no free variable occurrences.

(iii)▶ If $\tau, ...\tau_n$ are terms and $f \in \mathcal{F}^n$ is an n-ary function symbol (s.t. $0 < n$), then $f(\tau, ...\tau_n)$ is a term (τ_{n+1}) of $\mathcal{L}$. The free variable occurrences of $f(\tau, ...\tau_n)$ are those of $\tau, ...\tau_n$.

(iv)▶ If ϕ is a formula of $\mathcal{L}$ as defined in Definition 6.4, then the Gödel-number $\ulcorner\phi\urcorner$ of ϕ, as defined in Definition 6.1, is a term of $\mathcal{L}$.

(NB. A Gödel-number $\ulcorner\phi\urcorner$ does not contain free variables. Any free variables $y_1, .., y_n$ of the corresponding formula ϕ are represented in Δ_k via the tuple $(\ulcorner\phi\urcorner, y_1, ...y_n)$ which is a function that consists of $\ulcorner\phi\urcorner$ and $(y_1, ..., y_n)$. The free variable occurrences of the function $(\ulcorner\phi\urcorner, y_1, ...y_n)$ are already defined at point (iii) of the present definition.) ⊣

[9] Feferman's partial structures correspond to the interpretation of intuitionistic logic in Kripke models. (Originally proposed by Kripke in '*Semantical Analysis of Intuitionistic Logic*' [Kri65].) For an introduction to these models the reader is referred to [Kri65] and [Min02, Ch.7].

[10] For the complete formal framework the reader is referred to [Fef84, §§7 – 10]. For an adaptation of Feferman's device the reader is referred to [VLH08, Ch.6].

[11] The 'trick' of simultaneous-defining has been used before in, for example, high-order type theory (see [Fit02, p.5]).

Definition 6.4. (Formulas of $\mathcal{L}$) Let $\mathcal{L}$ be the language at stake. Formulas of $\mathcal{L}$ are recursively defined as follows:

(i)▶ If $\tau, ...\tau_n$ are terms and $P \in \Phi^n$ is an n-ary predicate, then $P(\tau, ...\tau_n)$ is an atomic formula of $\mathcal{L}$. The free variable occurrences of $P(\tau, ..\tau_n)$ are those of $\tau, ...\tau_n$.

(ii)▶ If ϕ is a formula of $\mathcal{L}$, then $\neg\phi$ is a formula of $\mathcal{L}$. The free variable occurrences of $\neg\phi$ are those of ϕ.

(iii)▶ If ϕ and ψ are formulas of $\mathcal{L}$, then $\phi \wedge \psi$ is a formula of $\mathcal{L}$. The free variable occurrences of $\phi \wedge \psi$ are those of ϕ and ψ.

(iv)▶ If ϕ and ψ are formulas of $\mathcal{L}$, then $\phi \vee \psi$ is a formula of $\mathcal{L}$. The free variable occurrences of $\phi \vee \psi$ are those of ϕ and ψ.

(v)▶ If ϕ and ψ are formulas of $\mathcal{L}$, then $\phi \to \psi$ is a formula of $\mathcal{L}$. The free variable occurrences of $\phi \to \psi$ are those of ϕ and ψ.

(vi)▶ If ϕ is a formula of $\mathcal{L}$, then $\forall x\phi$ is a formula of $\mathcal{L}$. The free variable occurrences of $\forall x\phi$ are those of ϕ minus the variables bound by $\forall$.

(vii)▶ If ϕ is a formula of $\mathcal{L}$, then $\exists x\phi$ is a formula of $\mathcal{L}$. The free variable occurrences of $\exists x\phi$ are those of ϕ minus the variables bound by $\exists$. $\dashv$

All connectives in Definition 6.4 are defined independently. In classical logic it would suffice to define rules for the connectives $\neg, \wedge$ and $\forall$ because every other connective can be defined in terms of these. One of the consequences of having an intuitionistic base of the logic is that none of the connectives is interdefinable. I will come back to this in section 6.4.

The above definitions contain the notion of 'free variable occurrence'. A variable x is bound in a formula ϕ if and only if it falls within the scope of a quantifier for x (either $\forall x$ or $\exists x$). A variable x occurs free in ϕ if and only if otherwise. During the formalization of Kant's argument we will only be dealing with formulas in which all variables are bound. These formulas are called *sentences*.

Definition 6.5. (Sentences) For every $\phi \in \mathcal{L}$ we get:

'ϕ is a sentence' $\Leftrightarrow$ 'there are no free variable occurrences in ϕ'

$\dashv$

6.2 The Many-Sorted First Order Calculus

In a many-sorted logic the idea of a heterogeneous universe of discourse can be formally expressed. In 'real life' we treat many objects as being essentially different in nature; for example, consider 'rational beings' and 'natural numbers' (I retain myself from any ontological commitment). It seems therefore reasonable to maintain these distinctions between sorts of objects in logic as well. In a many-sorted calculus the universe of discourse is partitioned. The groups of the partitioning can be considered as separate sets of objects, each set with its own set of quantifiers. In such a logic predicates and functions will be syntactically defined such that their arguments are restricted to specific sorts.

The many-sorted first order language that I will be using is based on Reiter's [Rei01, p.8-10]. The type-free language from the previous section will be extended to a many-sorted type-free language: Let $\mathcal{D}$ be the set whose elements are sorts d_i and let $\mathcal{D} \neq \emptyset$. The logical symbols are extended such that (i) for every sort $d_i \in \mathcal{D}$ we have a (countably infinite) set X^{d_i} consisting of individual variables $x_1^{d_i}, x_2^{d_i}...$ belonging to that sort and (ii) for every sort $d_i \in \mathcal{D}$ we have a corresponding universal and existential quantification symbol, respectively $\forall_{d_i}$ and $\exists_{d_i}$, ranging over that sort.

Definition 6.6. (The Many-Sorted Type-Free Language) Let $\mathcal{L}_M$ be a many-sorted type-free first order language. $\mathcal{L}_M$ is formally defined by Definitions 6.1, 6.7, 6.8, and 6.9.

$\dashv$

Definition 6.7. (Minimal sorts of $\mathcal{L}_M$) Let $\mathcal{L}_M$ be the language in question. Let $\mathcal{D}$ be the set of sorts d_i. The minimal set of sorts $\mathcal{D}$ consists of the natural numbers $\mathbb{N}$ and the sorts generated by the following recursive definition:

- ▶ For every $n \geq 0$, if $(d_1, ..., d_n)$ is a n-tuple of sorts $d_1, ..., d_n$, then there is a sort $(d_1, ..., d_n)$ such that $(d_1, ...d_n) \in \mathcal{D}$.
- ▶ For every $n \geq 0$ and every n-tuple $(d_1, ..., d_n)$ of sorts, there is a set $\Phi^n_{(d_1,...,d_n)}$ of n-ary predicate symbols, such that every $P^n \in \Phi^n_{(d_1,...,d_n)}$ is of the sort $(d_1, ..., d_n)$.
- ▶ For every $n \geq 0$ and every $(n+1)$-tuple $(d_1, ..., d_n, d_{n+1})$ of sorts, there is a set $\mathcal{F}^n_{(d_1,...,d_n,d_{n+1})}$ of n-ary function symbols, such that every $f^n \in \mathcal{F}^n_{(d_1,...,d_n,d_{n+1})}$ is of the sort $(d_1, ..., d_n, d_{n+1})$.

(The first clause ensures that there is a corresponding sort d_i in $\mathcal{D}$ for every object generated by either a predicate or a function.)

$\dashv$

By treating the set of natural numbers as a sort of the logic, Feferman's denotation device (Definition 6.1) can remain unaltered. Consequently, this enables the type-free variant of the many-sorted calculus. Terms and formulas are defined accordingly:

Definition 6.8. (Terms of $\mathcal{L}_M$) Let $\mathcal{L}_M$ be the language in question. Terms of $\mathcal{L}_M$ are recursively defined as follows:

(i)▶ If $x \in X$ is an individual variable of sort d_i, then x is a term τ of sort d_i of $\mathcal{L}_M$. The free variable occurrence of x is x.

(ii)▶ If $a \in \mathcal{F}^0$ is an individual constant of sort d_i, then a is a term τ of sort d_i of $\mathcal{L}_M$. a has no free variable occurrences.

(iii)▶ If $\tau_1, ...\tau_n$ are terms of sort $d_1, ..., d_n$ respectively and $f \in \mathcal{F}^n$ is an n-ary function symbol of sort $(d_1, ..., d_n, d_{n+1})$ (s.t. $0 < n$), then $f(\tau_1, ...\tau_n)$ is a term τ_{n+1} of sort d_{n+1} of $\mathcal{L}_M$. The free variable occurrences of $f(\tau_1, ...\tau_n)$ are those of $\tau_1, ...\tau_n$.

(iv)▶ If ϕ is a formula of $\mathcal{L}_M$, then the Gödel-number $\ulcorner\phi\urcorner$ of ϕ as defined in Definition 6.1 is a term of $\mathcal{L}_M$ of sort $\mathbb{N}$. ⊣

Definition 6.9. (Formulas of $\mathcal{L}_M$) Let $\mathcal{L}_M$ be the language in question. Formulas of $\mathcal{L}_M$ are defined as in Definition 6.4 with the following adjustments to (i), (vi) and (vii) respectively:

(i')▶ If $\tau_1, ...\tau_n$ are terms of sort $d_1, ..., d_n$ respectively and $P \in \Phi^n$ is an n-ary predicate of sort $(d_1, ..., d_n)$, then $P(\tau_1, ...\tau_n)$ is an atomic formula of $\mathcal{L}_M$. The free variable occurrences of $P(\tau_1, ..\tau_n)$ are those of $\tau_1, ...\tau_n$.

(vi')▶ If ϕ is a formula and d_i is a sort of $\mathcal{L}_M$, then $\forall_{d_i} x^{d_i} \phi$ is a formula of $\mathcal{L}_M$. The free variable occurrences of $\forall_{d_i} x^{d_i} \phi$ are those of ϕ minus the variables bound by $\forall_{d_i}$.

(vii')▶ If ϕ is a formula and d_i is a sort of $\mathcal{L}_M$, then $\exists_{d_i} x^{d_i} \phi$ is a formula of $\mathcal{L}_M$. The free variable occurrences of $\exists_{d_i} x^{d_i} \phi$ are those of ϕ minus the variables bound by $\exists_{d_i}$. ⊣

With respect to the quantifiers, the use of variables is restricted to the sort of the quantifier; that is, a quantifier cannot range over objects of a different sort. I will omit explicit reference to the indices of quantifiers and predicates because their sorts will be clear from the context.

Notational Convention 1. For all variables x^{d_i} of the sort d_i in X^{d_i} we get:

$$\forall_{d_i} x^{d_i} \phi(x^{d_i}) \;\Leftrightarrow\; \forall x^{d_i} \phi(x^{d_i})$$

Moreover, if d_i and d_j are sorts such that the predicate symbol P is of sort (d_i, d_j), we get:

$$\forall x^{d_i} \forall x^{d_j} P^{(d_i,d_j)}(x^{d_i}, x^{d_j}) \;\Leftrightarrow\; \forall x^{d_i} \forall x^{d_j} P(x^{d_i}, x^{d_j})$$ ⊣

The many-sorted language has no more expressive power than a classical 'one-sorted' first order language. However, the partitioning of sorts does have the advantage of enhancing readability. Many-sorted logic can be rewritten in terms of first-order logic: Every sentence of $\mathcal{L}_M$ can be transformed into a first-order sentence.[12] I will introduce the following formal rewrite rules for $\mathcal{L}_M$:

[12] For a full elaboration of the translation and the corresponding theorem the reader is referred to [Rei01, p.10].

Axiom 1. (First-Order Translations of $\mathcal{L}_M$) Let the language $\mathcal{L}_M$ be extended with a new unary predicate symbol D_{d_i} for every sort $d_i \in \mathcal{D}$. $D_{d_i}(\tau)$ must be read as 'τ is a term of sort d_i'. Every sentence ϕ of $\mathcal{L}_M$ can then be rewritten accordingly. For every sub-formula ψ of ϕ we get:

$$\forall x^{d_i}\psi \;\Leftrightarrow\; \forall y[D_{d_i}(y) \rightarrow \psi[x^{d_i}/y]]$$

$$\exists x^{d_i}\psi \;\Leftrightarrow\; \exists y[D_{d_i}(y) \wedge \psi[x^{d_i}/y]]$$

Let $\forall y$ and $\exists y$ range over $\bigcup_{d_i \in \mathcal{D}} d_i$ (the union of all sorts in $\mathcal{D}$).

$\dashv$

The rewrite rule expressed in Axiom 1 will play a fundamental role in the formalization of Kant's argument. The rule will allow us to transfer the formal results of a particular sort to any subset of that sort.

6.3 The Type-Free Many-Sorted Situation Calculus

For the purpose of this thesis a soft version of the situation calculus will suffice. The main feature of the situation calculus is that it enables reasoning about situations. The core elements of the calculus are *situations* and *fluents*.[13] In the situation calculus a fluent is either a predicate or a function that contains a situation variable as its last argument. The former is called a *relational* fluent, the latter a *functional* fluent [Rei01, p.19].[14] The last argument of a fluent makes the fluent *situation-dependent*; its value can therefore vary across situations. In this thesis I will treat fluents as predicates that *describe* a situation. For example, the fluent $free(Joan, s)$ would describe that 'Joan is free in situation s'. Kant's distinction between viewpoints of reason can be expressed in the situation calculus. This framework will enable the introduction of (context-dependent) *modalities*. The formal Kantian modalities will be

[13] The *Situation Calculus* was first introduced by McCarthy in [MH68]. Its best known contemporary version is probably that of Reiter [Rei01]. The main difference between these two versions is that in the former situations are considered *static* and in the latter *dynamic*. I will follow the first approach. In the former a situation is "the complete state of the universe at an instant of time" [MH68, p.18], though, in the latter it is "[a] possible world history, which is simply a sequence of actions" [Rei01, p.19]. In Reiter's situation calculus fluents are actions that can change a situation rather than predicates or functions describing a situation.

[14] The difference between a relational fluent and a functional fluent can best be expressed by an example: The former fluent describes a situation. For example, the relational fluent '$OnTable(x, s)$' expresses that some object x is 'on the table' in situation s (this is a standard first-order interpretation). A functional fluent, on the other hand, denotes an object. In this case '$OnTable(x, s)$' denotes the object x as 'being on the table' in situation s. The difference between these two fluent-forms can thus be explained as a difference between describing and denoting. With respect to the present undertaking I will only be concerned with relational fluents. Note well that a functional fluent is a term (object) that can serve as an argument in other functions or predicates. Feferman's denotation device allows us to treat relational fluents as objects as well. The version of the language that I will be using here can therefore be seen as a situation-dependent first-order language.

introduced in section 6.5. The language of the many-sorted type-free situation calculus is an extension of the language presented in the last section.[15]

Definition 6.10. (The Many-Sorted Type-Free Situation Language) Let $\mathcal{L}_S$ be a many-sorted type-free first order situation language. $\mathcal{L}_S$ is formally defined by Definitions 6.1, 6.8, and 6.9 together with the following extension of Definition 6.7:

- The set $\mathcal{D}$, consisting of all sorts of $\mathcal{L}_S$ also contains the sort S of situations.
- The language $\mathcal{L}_S$ is (at least) extended with the variables $x^s, x_1^s, y^s...$ belonging to S and the quantifiers $\forall_S$ and $\exists_S$ ranging over the elements of S.

$\dashv$

6.4 The Intuitionistic Many-Sorted Type-Free Situation Calculus

In order to obtain the desired calculus the many-sorted type-free situation language will be provided with an intuitionistic natural deduction system. The introduction of intuitionistic logic will be short and informal.[16] Intuitionistic logic was first developed by A. Heyting [Hey30] and has its origin in L.E.J. Brouwer's approach to intuitionistic mathematics. In short, an intuitionistic logic is a classical logic without the *law of excluded middle*:

$$\text{(LEM)} \quad \phi \vee \neg\phi$$

Excluding LEM as a valid principle of the logic is motivated by a lack of *proof*, that is, by the lack of a general proof that determines for every ϕ whether ϕ or $\neg\phi$ is the case. Proof requirement can be seen as the gist of intuitionistic reasoning. The following classically valid principles are *not* valid in intuitionistic (first-order) logic:

- $\phi \vee \neg\phi$
- $\neg\neg\phi \to \phi$
- $\neg\exists x \neg P(x) \to \forall x P(x)$
- $\neg\forall x \neg P(x) \to \exists x P(x)$

[15] Note well that the language $\mathcal{L}_S$ has no more expressive power than the original first-order language provided for Feferman's denotation device. This can be seen as follows: Firstly, the 'many-sorted' part of the language can be reduced to a classical first-order language (see [Rei01, p.19]). Secondly, the present version of the situation language contains only relational fluents. These relational fluents are in fact 'ordinary predicates' that contain an extra parameter that belongs to one of the many sorts of $\mathcal{L}_S$; namely, a situation parameter.

[16] For an extensive introduction to intuitionistic logic the reader is referred to [Hey71, Ch.7], [Mos14] and [Min02].

The above enumeration shows that the use of negation is restricted in intuitionistic logic: Most of the classically valid inferences from negative to positive formulas are blocked as a consequence of the rejection of LEM (this explains why the logical connectives are not interdefinable).

Intuitionistic Logic in Relation to Kant

The main justification for setting up the logic with an intuitionistic base is provided by Kant's use of impossibility and negation in the practical possibility argument (chapter 4). In the Critique of Pure Reason Kant claims to have shown that for a human being it is at least,

> **not self-contradictory** to regard all its actions as physically conditioned insofar as they are appearances and yet also to regard their causality as physically unconditioned insofar as the acting being is a being of the understanding. [Kan96a, 5:48 - bold emphasis my own]

Kant showed the above by restricting natural causality to the theoretical point of view. In the first Critique though, Kant only proved that freedom is *not impossible* from a non-theoretical point of view. That the concept of freedom is in fact *possible*, is determined by Kant in his second Critique through the possibility of FLoP. In other words, with respect to Kant's reasoning the inference from not impossible (read: 'not not possible') to possible is not trivial. Hence, the formula

$$\neg\neg\phi \rightarrow \phi$$

seems to be rejected by Kant as a valid principle. In both classical logic and intuitionistic logic the principle '$(\neg\neg\phi \rightarrow \phi) \leftrightarrow (\neg\phi \vee \phi)$' is provable [Mos14]. Consequently, the above implies that LEM must be rejected as a valid principle of a Kantian logic. For this reason I will provide the present logic with an intuitionistic base. (We shall see that the formal representation of Kant's argument does not require any inference that is intuitionistically invalid but classically valid.)

The Intuitionistic Natural Deduction System

Kant's transcendental reasoning will be represented in the formal system of *natural deduction* as elaborated by Troelstra and Schwichtenberg [TS00].[17] Deductions will be represented as trees. Every node of a tree is assigned a formula ϕ such that its immediate successor contains the premiss of which ϕ is derived as its (sub-)conclusion. The final conclusion of a deduction will be represented at the root of the tree. A formula is derived by *rule application.* Rules are represented as schemas and the application of a rule is called an inference. Formulas with superscript u, v, w or u_i indicate the introduction of

[17] For a thorough introduction to the natural deduction system the reader is referred to [TS00, Ch.2.1].

an assumption. The indexed symbol $\mathcal{D}_i$ is reserved to indicate a deduction. An example of a proof tree is represented in Figure 6.1 (I will use the notational convention as found on the right).

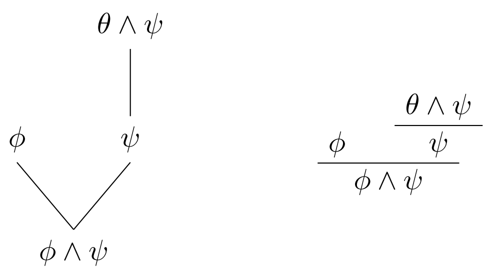

Figure 6.1: Example of a Proof Tree

The natural deduction system for intuitionistic logic is defined as follows:

Definition 6.11. (The Intuitionistic Natural Deduction System NI) Let $\mathcal{L}_S$ be the language in question. The intuitionistic natural deduction system NI for the many-sorted type-free situation calculus is defined as follows.

- Assumptions are formulas of the language indexed by u, v, w, u_i... and appear at the top of a branch in the deduction. Distinct assumptions have distinct indices. In a set of assumptions (i.e. an assumption class $[\phi]^u$) every assumption in that set is indexed by the same letter.
- Assumptions may be closed or left open, though in the case of assumption classes at every inference either *all* assumptions of that class are closed or left open. Closure of an assumption is indicated by repeating the index of the assumption at the label of the inference.

Deductions '$\mathcal{D}_i$' are defined by the following recursion (see [TS00, p.36-37]):

Base case: A single-node tree with the formula ϕ^u at its only node is a natural deduction $\mathcal{D}_i$ from the open assumption ϕ^u such that every node (only one) is left open at the conclusion.
Recursive step: Let $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$ be deductions. Let $[\phi]^u$ and $[\psi]^v$ be sets of assumptions such that $[\phi]^u$ and $[\psi]^v$ are closed at the root. Let x and y be the variables at stake and let τ be some term. A natural deduction $\mathcal{D}_i$ can be constructed according to the following *introduction* and *elimination* rules (*I*- and *E*-rules):

(NB. NI requires quantifier I- and E-rules for every sort $d_i \in \mathcal{D}$.)

$$\frac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ \phi & \psi\end{array}}{\phi \wedge \psi}\wedge\text{I} \qquad \frac{\begin{array}{c}\mathcal{D}_1\\ \phi \wedge \psi\end{array}}{\phi}\wedge\text{E} \qquad \frac{\begin{array}{c}\mathcal{D}_1\\ \phi \wedge \psi\end{array}}{\psi}\wedge\text{E} \qquad \frac{\begin{array}{c}[\phi]^u\\ \mathcal{D}_1\\ \psi\end{array}}{\phi \to \psi}\to\text{I},u$$

$$\frac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ \phi \to \psi & \phi\end{array}}{\psi}\to\text{E} \qquad \frac{\begin{array}{c}\mathcal{D}_1\\ \phi\end{array}}{\phi \vee \psi}\vee\text{I} \qquad \frac{\begin{array}{c}\mathcal{D}_1\\ \psi\end{array}}{\phi \vee \psi}\vee\text{I}$$

$$\frac{\begin{array}{ccc} & [\phi]^u & [\psi]^v\\ \mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3\\ \phi \vee \psi & \theta & \theta\end{array}}{\theta}\vee\text{E},u,v \qquad \frac{\begin{array}{c}\mathcal{D}_1\\ \phi[x^{d_i}/y^{d_i}]\end{array}}{\forall_{d_i}x^{d_i}\phi}\forall_{d_i}\text{I} \qquad \frac{\begin{array}{c}\mathcal{D}_1\\ \forall_{d_i}x^{d_i}\phi\end{array}}{\phi[x^{d_i}/\tau]}\forall_{d_i}\text{E}$$

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \phi[x^{d_i}/\tau]\end{array}}{\exists_{d_i}x^{d_i}\phi}\exists_{d_i}\text{I} \qquad \frac{\begin{array}{cc} & [\phi[x^{d_i}/y^{d_i}]]^u\\ \mathcal{D}_1 & \mathcal{D}_2\\ \exists_{d_i}x^{d_i}\phi & \psi\end{array}}{\psi}\exists_{d_i}\text{E},u \qquad \frac{\begin{array}{c}\mathcal{D}_1\\ \perp\end{array}}{\phi}\perp_i$$

There are some restrictions to the use of the quantifier rules:

- (In general) For every application of $\forall_{d_i}$I, $\forall_{d_i}$E, $\exists_{d_i}$I and $\exists_{d_i}$E the following holds: A variable x of sort d_i can only be substituted for a variable y or a term τ of that same sort d_i. A term τ of sort d_i can only be substituted for a variable x of that same sort d_i.

We have the following restrictions to $\forall_{d_i}$I and $\exists_{d_i}$E in particular:

- $\forall_{d_i}$I Either $y^{d_i} = x^{d_i}$ or y^{d_i} is not a free variable of ϕ. And y^{d_i} does not occur free in any assumption open in $\mathcal{D}_1$.
- $\exists_{d_i}$E Either $y^{d_i} = x^{d_i}$ or y^{d_i} is not a free variable of ϕ. And y^{d_i} neither occurs free in the conclusion ψ nor in any assumption open in $\mathcal{D}_2$ other than $\phi[x^{d_i}/y^{d_i}]^u$.

The rules for assumption closure can be stipulated as follows:

- The application of '$\to$I,u' closes the set $[\phi]^u$ of open assumptions ϕ in $\mathcal{D}_1$. All other assumptions remain open.
- The application of '$\vee$E,u, v' closes the set $[\phi]^u$ of open assumptions ϕ in $\mathcal{D}_2$ and closes the set $[\psi]^v$ of open assumptions ψ in $\mathcal{D}_3$. All other assumptions remain open.
- The application of $\exists_{d_i}$E,u closes the set $[\phi[x^{d_i}/y^{d_i}]]^u$ of open assumptions ϕ in $\mathcal{D}_2$. All other assumptions remain open.

(NB. The principle of *reductio ad absurdum* is not valid in NI. This means that LEM cannot be derived in the system. The principle of *ex falso quodlibet* is valid in NI and is represented by the $\perp_i$-rule.) $\dashv$

I will provide deductions of three intuitionistic valid principles that will be frequently used in the formalization of Kant's argument. Deduction (b) shows for example that *modus tollens* is valid in NI. Having proved them once, I will omit explicit elaboration of these principles in the upcoming chapter. Deduction (c) allows the reader to get acquainted with the use of negation in the present natural deduction system. In the remainder of this thesis I will omit explicit reference to the definition of negation. (Recall that $\neg\phi := \phi \to \bot$.)

$$\dfrac{\dfrac{\dfrac{\phi^u \qquad \neg\phi^v}{\bot}\,{\to}\mathrm{E}}{\neg\neg\phi}\,{\to}\mathrm{I},v}{\phi \to \neg\neg\phi}\,{\to}\mathrm{I},u$$

(a) IL.1

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\psi^v \qquad \dfrac{\phi \to \psi^u \qquad \phi^w}{\psi}\,{\to}\mathrm{E}}{\bot}\,{\to}\mathrm{E}}{\neg\phi}\,{\to}\mathrm{I},w}{\neg\psi \to \neg\phi}\,{\to}\mathrm{I},v}{[\phi \to \psi] \to [\neg\psi \to \neg\phi]}\,{\to}\mathrm{I},u$$

(b) IL.2

$$\dfrac{\dfrac{\dfrac{\dfrac{\exists x P(x)^v \qquad \dfrac{\dfrac{\dfrac{\forall x \neg P(x)^u}{\neg P(x)}\,\forall\mathrm{E}}{P(x) \to \bot}\,\mathrm{def.}\neg \qquad P(x)^w}{\bot}\,{\to}\mathrm{E}}{\bot}\,\exists\mathrm{E},w}{\exists x P(x) \to \bot}\,{\to}\mathrm{I},v}{\neg\exists x P(x)}\,\mathrm{def.}\neg}{\forall x \neg P(x) \to \neg\exists x P(x)}\,{\to}\mathrm{I},u$$

(c) IL.3

Figure 6.3

6.5 The Kantian Logic - KL

At this point the logic **KL** can be properly defined. KL is an *intuitionistic many-sorted type-free situation calculus* extended with a set of sorts $\mathcal{D}$ that coincides with the sorts encountered in Kant's philosophical argument. (The specific relational fluents and predicates of KL will be introduced in the next chapter.) Rational beings and human beings are the most central (ground-level) objects in Kant's transcendental reasoning. However, Kant regards human beings as a subspecies of rational beings. For this reason I will only treat rational beings as a proper sort of the logic (the set of human beings will be introduced in section 7.2.4). In order to regard rational beings (and the concepts that might apply to them) from different points of view a set of situations is needed.[18] The list of (primitive) sorts is short, but we shall see that most of the objects used in Kant's reasoning are of 'higher complexity'; that is, they are reified formulas.

[18] Although Kant's original argument is concerned with 'being practical', I have decided to omit the inclusion of a set of actions. One might argue that, since practical principles contain relations between actions and determining grounds, the inclusion of a set of actions will be appropriate. However, Kant's argument will be about the properties of practical principles in general, not about the content of some practical principle in particular. I have therefore decided to treat practical principles as unspecified (reified) formulas.

Definition 6.12. (The Language of KL) The language $\mathcal{L}_{KL}$ is a *many-sorted type-free situation language* as defined in Definition 6.10 extended with the sorts of Definition 6.13. $\dashv$

Definition 6.13. (Many Sorts of KL) Let $\mathcal{L}_{KL}$ be the language of KL. $\mathcal{D}$ of $\mathcal{L}_{KL}$ consists of the following sorts:

- RB - The set of rational beings. (variables x^{rb}, x_1^{rb}, y^{rb})
- S - The set of situations. (variables x^s, x_1^s, y^s)
- $\mathbb{N}$ - The set of natural numbers n.
- Every sort $(d_1, ..., d_n)$ generated by Definition 6.7 on $\mathcal{L}_{KL}$ (together with the corresponding variables $x^{(d, ..., d_n)}$).

(I will refer to the Gödel-number of some formula ϕ as $\ulcorner\phi\urcorner$ instead of the unique natural number n assigned to ϕ by Definition 6.1. Quantification over Gödel-numbers will be of the form $\forall\ulcorner\phi\urcorner$. This will be clear from the context.) $\dashv$

Kantian Modalities

Kant's reasoning contains two forms of necessity and possibility. On the one hand there is *context-independent* necessity. This form of necessity pertains to analytic truths, (a priori) definitions and axioms. Formulas with this form of necessity hold in *every* situation. In the logic KL this modality will be interpreted as follows:

Definition 6.14. (Context-Independent Necessity in KL) Let ϕ be some formula of $\mathcal{L}_{KL}$. Let $(x_1, .., x_k, x^s)$ be the list of the $(k+1)$-many free variable occurrences in ϕ. We have the following informal interpretation of necessity in KL:

- 'ϕ is necessary' $\Leftrightarrow$ $\forall x^s \phi(x_1, .., x_k, x^s)$ $\dashv$

On the other hand there is *context-dependent* necessity (and possibility).[19] In the second Critique Kant is mostly concerned with the theoretical and practical 'context'. For example, the concept of natural causality is necessary from a theoretical point of view, but not from a practical point of view. With respect to the present framework this means that there must be a division of contextualized situations in S. That is, S must (at least) consist of a set S^t consisting of theoretical situations and a set S^p consisting of practical situations.

[19] It seems that Kant's conceptual apparatus does not contain a *context-independent possibility*. Namely, if something is possible, then it must be clear in what context it is possible.

Axiom 2. (The Practical and Theoretical Viewpoint) Let S be the set of situations as defined in Definition 6.13. Let ϕ be some formula of $\mathcal{L}_{KL}$. Let $S^t \subseteq S$ and $S^p \subseteq S$. For every S^i such that $S^i \subseteq S$ we have the following two rewrite rules (let x^{s_i}, y^{s_i}... be the corresponding variables):

- $\forall x^{s_i}\phi \Leftrightarrow \forall x[S^i(x) \rightarrow \phi[x^{s_i}/x]]$
- $\exists x^{s_i}\phi \Leftrightarrow \exists x[S^i(x) \wedge \phi[x^{s_i}/x]]$

The relation between S and its subsets S^i is defined as follows:

- $\forall x[S^i(x) \rightarrow S(x)]$

$\dashv$

The last clause of Axiom 2 expresses the relation between the context-dependent modalities and the context-independent modality. Specific properties of Kant's viewpoints will be introduced during the formalization. The context-dependent modalities are provided with the following interpretation:

Definition 6.15. (Context-Dependent Modalities in KL) Let ϕ be some formula of $\mathcal{L}_{KL}$. Let $(x_1, .., x_k, x^{s_i})$ be the list of the $(k+1)$-many free variable occurrences in ϕ. We have the following informal interpretation of the viewpoint modalities in KL:

- 'ϕ is necessary from a theoretical point of view' $\Leftrightarrow$ $\forall x^{s_t}\phi(x_1, .., x_k, x^{s_t})$
- 'ϕ is necessary from a practical point of view' $\Leftrightarrow$ $\forall x^{s_p}\phi(x_1, .., x_k, x^{s_p})$
- 'ϕ is possible from a theoretical point of view' $\Leftrightarrow$ $\exists x^{s_t}\phi(x_1, .., x_k, x^{s_t})$
- 'ϕ is possible from a practical point of view' $\Leftrightarrow$ $\exists x^{s_p}\phi(x_1, .., x_k, x^{s_p})$

$\dashv$

Kant's Definition of Definition

A large part of Kant's argument is based on definitions and for this reason the logical status of a definition must be determined. According to Kant every definition is either analytic or synthetic. Every analytic or synthetic definition is, subsequently, either a priori or a posteriori (see for example [Kan92a, Jäsche Logic - 9:140-9:142]). The transcendental nature of Kant's argument implies that every definition of the argument must be a priori. Consequently, these definitions express necessity. Translated to the present framework this means that Kant's definitions must be treated as context-independent formulas; that is, as formulas that hold in every situation. Every definition ϕ will therefore be provided with a situation-parameter x^s and a universal quantifier $\forall$ that bounds x^s. Axiom 2 ensures accordingly that Kant's definitions can be freely applied in every possible viewpoint $S^i \subseteq S$ in KL. This finishes the elaboration of the required formal framework.

Definition 6.16. (The Logic KL) The Logic KL is a *intuitionistic many-sorted type-free situation calculus* constructed from the language $\mathcal{L}_{KL}$ (Definition 6.12) together with the intuitionistic natural deduction system NI (Definition 6.11). Let KL satisfy Axioms 1 and 2. $\dashv$

Chapter 7

A Formalization of Kant's Practical Transcendental Argument

We have come to the point at which Kant's practical transcendental argument can be formalized. The formalization will be in the logic KL. In the upcoming section I will first provide a formalization of the main concepts of a transcendental argument in general. The formal concepts will correspond to the three main argumentative steps of Kant's argument. The subsequent sections consist of the formalization of Kant's practical transcendental argument. The formalization will be restricted to the *possibility* and *objective validity* argument as elaborated in chapters 4 and 5 respectively. I have chosen to formalize these two arguments for three reasons: Firstly, in both arguments Kant uses all the proposed modalities (in contrast to the necessity argument). Secondly, in these two arguments several axioms underlying Kant's reasoning can be detected. Lastly, the formalization of the two arguments will provide a glimpse of the 'mechanism' behind the concept of objective validity. This part of the formalization will result in a provisory definition of the concept of objective validity. At the end of this chapter I will show that the provisory definition is in fact sufficient to prove the main claim of Kant's transcendental argument. In other words, the formalization of Kant's practical transcendental argument will provide a sufficient formal definition of the concept of objective validity.

The formal language consists (at least) of the relational fluents and predicates represented in table 7.1. The formal definitions of these fluents and predicates will be introduced in the course of this chapter.[1]

[1] I will use the informal reading 'x^{rb} is subject to the concept of negative freedom in x^{s}' and 'x^{rb} is negatively free in situation x^{s}' interchangeably. The same holds for the other concepts in table 7.1.

List of Fluents	Informal reading
$NF(x^{rb}, x^s)$	'x^{rb} is subject to the concept of negative freedom in x^s'
$PF(x^{rb}, x^s)$	'x^{rb} is subject to the concept of positive freedom in x^s'
$FR(x^{rb}, x^s)$	'x^{rb} is subject to the concept of freedom in x^s'
$caus(x^{rb}, x^s)$	'x^{rb} is subject to the concept of natural causality in x^s'
$DG(\ulcorner\phi\urcorner, x^{rb}, x^s)$	'ϕ is a (possible) determining ground for x^{rb} in x^s'
$DB(x^{rb}, \ulcorner\phi\urcorner, x^s)$	'x^{rb} is determinable by ϕ in x^s'
$sens(x^s)$	'x^s is a situation in the sensible nature'
$supersens(x^s)$	'x^s is a situation in the supersensible nature'
$form(\ulcorner\phi\urcorner, x^s)$	'ϕ is regarded as to its mere form in x^s'
$matter(\ulcorner\phi\urcorner, x^s)$	'ϕ is regarded as to its matter in x^s'
List of Predicates	
$PP(\ulcorner\phi\urcorner)$	'ϕ is a practical principle'
$NC_\phi(\ulcorner\psi\urcorner)$	'ψ is a necessary condition of ϕ'
$G_\phi(\ulcorner\psi\urcorner)$	'ψ is the ground of ϕ'
$R_i(\ulcorner\phi\urcorner)$	'ϕ is real from the i-viewpoint'
$O_i(\ulcorner\phi\urcorner)$	'ϕ is objectively valid from the i-viewpoint'

Table 7.1: Fluents and Predicates of $\mathcal{L}_{KL}$

7.1 The Transcendental Argument's Main Concepts

Recall that every transcendental argument has two aims (Postulate 1):

I. Prove the possibility of some synthetic a priori cognition.
II. Derive the a priori concepts that make this synthetic a priori cognition possible and deduce the objective validity of these concepts.

'Aim I' should be obtainable from the results of 'Aim II'. That is, the exposition of the possibility of the a priori synthesis of the cognition at stake must result from the objective validity of that cognition's ground. The formal treatment of this aim will be postponed to the end of this chapter, where I will treat the relation between the first and the second aim with respect to Kant's second Critique. In this section I will provide a formalization of the general concepts involved in the establishment of the second aim. Aim II will be the result of the transcendental argument's three major arguments.

THE NECESSITY ARGUMENT. A transcendental argument needs to prove what concepts $\psi_1, ..., \psi_n$ are necessary conditions for the possibility of some synthetic a priori cognition ϕ.

DEFINITION 7.1. (FORMALIZING NECESSARY CONDITIONS) Let $NC_\phi(\ulcorner\psi\urcorner)$ be interpreted as in table 7.1. Let ϕ and ψ be formulas of $\mathcal{L}_{KL}$. Let $(x_1, .., x_k, x^s)$ be the list of the (k+1)-many free variable occurrences in ϕ and let $(y_1, .., y_l, x^s)$ be the list of (l+1)-many free variable occurrences in ψ (the same situation variable x^s must occur free in both ϕ and ψ).[2]

$$NC_\phi := \{\ulcorner\psi\urcorner \mid \forall x^s[\phi(x_1, ..., x_k, x^s) \rightarrow \psi(y_1, ..., y_l, x^s)]\ \}$$

⊣

The above definition expresses the following thought: In every situation x^s in which ϕ is the case, ψ is the case as well. In other words, if ϕ is the case, then ψ *must* be the case. Consequently, if ψ is not the case, ϕ cannot be the case either. Hence, ψ is a necessary condition of ϕ.

THE POSSIBILITY ARGUMENT. A transcendental argument needs to show which of ϕ's necessary conditions $\psi_1, ..., \psi_n$ form its ground ψ^*. That ψ^* is ϕ's ground is accomplished by (i) showing that ψ^* is (a priori) possible and (ii) by showing that ψ^* is sufficient to generate ϕ as its consequence.

In short, a ground is a necessary condition that can also be the cognition's sufficient condition. A necessary condition does not belong to the cognition's ground if it can be omitted without the loss of generating this cognition. Moreover, it seems (philosophically) reasonable to assume that self-reference is not allowed in the ground-cognition relation; after all, $\phi \rightarrow \phi$ would be trivial.

DEFINITION 7.2. (FORMALIZING A GROUND) Let $G_\phi(\ulcorner\psi\urcorner)$ and $NC_\phi(\ulcorner\psi\urcorner)$ be interpreted as in table 7.1. Let ϕ, ψ and θ_i be formulas of $\mathcal{L}_{KL}$. Let $(x_1, .., x_k, x^s)$ be the list of the (k+1)-many free variable occurrences in ϕ and let $(y_1, .., y_l, x^s)$ be the list of (l+1)-many free variable occurrences in ψ (the same situation variable x^s must occur free in both ϕ and ψ).

$$\begin{aligned} G_\phi := \{\ulcorner\psi\urcorner \mid \psi = \theta_1\wedge ... \wedge\theta_n \text{ s.t. } & \text{(i) } \forall\theta_i(1\leq i\leq n) \text{ we get: } NC_\phi(\ulcorner\theta_i\urcorner) \\ & \text{(ii) } \forall\theta_i(1\leq i\leq n) \text{ we get: } \theta_i \neq \phi \\ & \text{(iii) } \exists x^s\ \psi(y_1, ..., y_l, x^s) \\ & \text{(iv) } \forall x^s[\psi(y_1, ..., y_l, x^s) \rightarrow \phi(x_1, ..., x_k, x^s)] \\ & \text{(v) } \forall\psi^* \text{ s.t. } \psi^* = \theta_i\wedge ... \wedge\theta_j\ (1\leq i, j\leq n) \text{ s.t.} \\ & \quad \psi \vdash \psi^* \text{ and } \psi^* \nvdash \psi \text{ we get:} \\ & \quad \neg\forall x^s[\psi^*(y_1, ..., y_l, x^s) \rightarrow \phi(x_1, ..., x_k, x^s)]\ \} \end{aligned}$$

⊣

[2]This definition of N_ϕ does not rule out the inclusion of ϕ itself and the tautologies of KL (it even seems that N_ϕ is an infinite set of necessary conditions). This will not pose any problem on the present formalization; the definition of a ground will filter out these 'extra formulas'.

The first clause of Definition 7.2 ensures that the ground consists only of necessary conditions. The second clause ensures that ϕ can never be its own ground (this clause excludes the possibility that ϕ itself is the minimal sufficient condition that can generate ϕ). The third clause ensures that the concept, which forms the cognition's ground, is at least possible. The fourth clause ensures that the cognition ϕ can be generated by the derived ground. The last clause ensures that this sufficient condition is a minimal sufficient condition (this clause prevents the inclusion of superfluous formulas, such as tautologies, in the cognition's ground).

THE OBJECTIVE VALIDITY ARGUMENT. A transcendental argument needs to show that the objective validity of the a priori ground ψ follows from the reality of the cognition ϕ from which ψ is derived.

A formal definition of neither the concept of objective validity nor the concept of reality can be given at this stage of the analysis. The uncovering of these (formal) definitions is left to the formalization of Kant's practical transcendental argument. What can be provided at this stage of the analysis, though, are the rough outlines of the third argument's structure to which the final formalization must conform. With respect to its structure two things are clear: Firstly, the departure point of the formal deduction must be the reality of the cognition at stake and secondly, the objective validity of the cognition's ground must be derivable from the reality of the cognition. Hence, any formal derivation that represents the objective validity argument must (at least) conform to the following:

$$\vdash_{KL} \; R_i(\ulcorner\phi\urcorner) \;\Rightarrow\; [G_\phi(\ulcorner\psi\urcorner) \Rightarrow O_i(\ulcorner\psi\urcorner)]$$

7.2 The Formal Practical Possibility Argument

The practical possibility argument can be divided in *five* phases. I call every substantial argumentative step from chapter 4 a distinct phase. All five phases have their corresponding central claims and premisses. A premiss is either a definition, axiom or previously derived proposition. The main claims of the five phases of the possibility argument are represented in Table 7.2. Proposition 10 must result from Propositions 8 and 9. Proposition 10 together with Proposition 11 implies the validity of Proposition 12. The last inference depends on the following axiom:

> Postulate 9 (Chapter 4) The concept of freedom is the ground of the moral law (5) $\iff$ it is possible as a concept as such (3) and it is sufficient to generate the moral law as its consequence (4).

The relations between the propositions of Table 7.2 are represented in Figure 7.1.

Phase	**Main Claim** (section)	**Content**
(1)	Proposition 8 (4.2.1)	The concept of freedom is impossible from a theoretical point of view and only from a non-theoretical point of view it is not impossible.
(2)	Proposition 9 (4.2.2)	A supersensible nature is possible; namely, as a viewpoint of practical reason.
(3)	Proposition 10 (4.2.3)	The concept of freedom is possible from a practical point of view.
(4)	Proposition 11 (4.3)	The concept of freedom is sufficient to generate the moral law as its consequence.
(5)	Proposition 12 (4.3)	The concept of freedom is possible as ground of the moral law from a practical viewpoint.

Table 7.2: The Main Claims of the Practical Possibility Argument

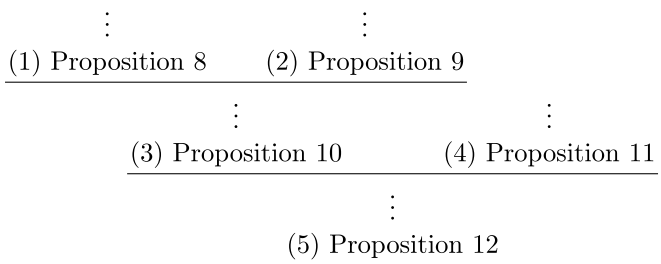

Figure 7.1: General Structure of the Practical Possibility Argument

Since the formalization of Kant's transcendental argument will start at the possibility argument, I will take the results of the necessity argument as given.

Proposition 15. (Freedom as Necessary Condition of the Moral Law) Let $NC_\phi(\ulcorner\psi\urcorner)$ and $FR(x^{rb}, x^s)$ be interpreted as in table 7.1. Let ML stand for the formula expressing the moral law.

$$NC_{ML}(\ulcorner FR\urcorner)$$

(Proposition 15 will be assumed throughout the formalization.) ⊣

With respect to Kant's practical endeavour, this proposition (philosophically) implies that 'in every situation in which a human being is determinable by the moral law, that being must be subject to the concept of freedom'. The necessity argument expresses therefore the following proposition:

$$\forall x^{rb}\forall x^s[DB(x^{rb}, \ulcorner ML\urcorner, x^s) \rightarrow FR(x^{rb}, x^s)]$$

Let us first look at a formal representation of the most central concept of the present argument: The concept of *freedom*. Freedom is defined in terms of negative and positive freedom.

DEFINITION 7.3. (FORMALIZING INFORMAL Definition 3.17) Let $NF(x^{rb}, x^s)$ and $caus(x^{rb}, x^s)$ be interpreted as in table 7.1. NF is defined as follows:

$$NF := \{(x^{rb}, x^s) \mid \neg caus(x^{rb}, x^s)\ \}$$

$\dashv$

The above definition expresses the following thought: A rational being is negatively free in a situation if and only if the being is not subject to (i.e. independent of) natural causality in that situation. Positive freedom is formalized as follows:

DEFINITION 7.4. (FORMALIZING INFORMAL Definition 3.19) Let $PF(x^{rb}, x^s)$, $PP(\ulcorner\phi\urcorner)$, $O_i(\ulcorner\phi\urcorner)$ and $DG(\ulcorner\phi\urcorner, x^{rb}, x^s)$ be interpreted as in table 7.1. PF is defined as:

$$PF := \{(x^{rb}, x^s) \mid \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner, x^{rb}, x^s)]\ \}$$

$\dashv$

Definition 7.4 captures Kant's idea that the concept of positive freedom consists of the notion of 'being your own legislator'. This notion of 'being your own legislator', subsequently, is nothing but the idea of 'having an objective practical principle as the determining ground of your own will'. Hence, a rational being is positively free in a situation if and only if a practical law is possible as the determining ground of that being's will in that situation.

DEFINITION 7.5. (FORMALIZING INFORMAL Definition 3.20) Let $FR(x^{rb}, x^s)$, $NF(x^{rb}, x^s)$ and $PF(x^{rb}, x^s)$ be interpreted as in table 7.1. FR is defined as follows:

$$FR := \{(x^{rb}, x^s) \mid NF(x^{rb}, x^s) \wedge PF(x^{rb}, x^s)\ \}$$

$\dashv$

May the informal interpretation of Definition 7.5 be clear. One brief remark must be made: The extension of the concept of positive freedom is identical to the extension of the concept of negative freedom because both concepts are derived from the very same principle (namely, the moral law). Their meaning, though, clearly differs. One of the benefits of using Feferman's logic is that setlike-objects do not necessarily satisfy the following axiom of extensionality [VLH08, p.73]:

$$\text{(EX)} \quad \forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b$$

In other words, two predicates that are equal with respect to their extension can still differ with respect to their intension. Consequently, although the concepts of negative and positive freedom have identical extensions, as to their meaning they are still conceptually different.

7.2.1 Phase 1

In the first phase of the possibility argument Kant determines the *boundaries* of the concept of freedom; first negatively and thereafter positively. With the boundaries of freedom Kant determines the restricted framework in which the possibility of freedom can be shown. The following proposition must be proved:

PROPOSITION 16. (FORMALIZING INFORMAL Proposition 8) Let FR be interpreted as in table 7.1.

$$\forall x^{rb} \neg \exists x^{s_t} FR(x^{rb}, x^{s_t}) \wedge \forall x^{rb} \forall x^{S \setminus S^t} \neg\neg FR(x^{rb}, x^{S \setminus S^t})$$

$\dashv$

The proof of the left side of the conjunction in Proposition 16 determines the impossibility of freedom from the theoretical point of view (the negative determination of freedom's boundaries). The proof of the right side of the conjunction determines the sphere in which the concept of freedom is at least not impossible (the positive determination of freedom's boundaries). I will treat these proofs respectively.

Part 1 of Proposition 16

The first part of the proof leans heavily on the following result from the Critique of Pure Reason: With respect to the theoretical point of view every rational being is necessarily subject to (the concept of) natural causality. Moreover, outside this viewpoint natural causality is impossible (Postulate 10).[3] Postulate 10 is formalized as follows:

PROPOSITION 17. (FORMALIZING INFORMAL Postulate 10) Let $caus(x^{rb}, x^s)$ be interpreted as in table 7.1:

$$\forall x^{rb} \forall x^{s_t} caus(x^{rb}, x^{s_t}) \wedge \forall x^{rb} \forall x^{S \setminus S^t} \neg caus(x^{rb}, x^{S \setminus S^t})$$

(Proposition 17 will be assumed throughout the formalization.) $\dashv$

One last remark must be made with respect to the present logical framework: In the system of natural deductions every definition, axiom or proposition must be regarded as either an assumption or a formula proved in a previous deduction. Consider the following example:

[3]Proposition 17 is implied by the theoretical objective validity of the concept of causality (i.e. $O_t(\ulcorner caus \urcorner)$). However, at this point of the formalization a formal definition of the concept of objective validity is lacking. Consequently, the use of $O_t(\ulcorner caus \urcorner)$ would be meaningless. Proposition 17 will therefore only express some logical consequences of the objective validity of causality. For the purpose of the present argument this proposition will suffice.

$$\cfrac{\overset{\text{Proposition A}}{\phi} \qquad\qquad \overset{\text{Definition B}}{\phi \to \psi}}{\psi}\ {\to}\text{E}$$

Strictly speaking, the above deduction only ends after discharging the open assumptions ‘Proposition A’ and ‘Definition B’:

$$\cfrac{\cfrac{\cfrac{\overset{\text{Proposition A}}{\phi^u} \qquad\qquad \overset{\text{Definition B}}{\phi \to \psi^v}}{\psi}\ {\to}\text{E}}{[\phi \to \psi] \to \psi}\ {\to}\text{I},v}{\phi \to [[\phi \to \psi] \to \psi]}\ {\to}\text{I},u$$

In other words, every conclusion of the upcoming deductions is conditional on the validity of the definitions and propositions introduced in that particular deduction. With respect to the above example, the validity of the conclusion would be expressed accordingly:

$$\vdash_{KL} \{\text{Proposition A},\ \text{Definition B}\} \Rightarrow \psi,$$

or

$$\text{Proposition A},\ \text{Definition B} \vdash_{KL} \psi.$$

For the sake of readability I have chosen to omit explicit discharge of these formulas at the end of every deduction. The reader must keep in mind though that the validity of the final conclusion of Kant's transcendental argument will be conditional on the validity of these formulas. Let us turn to the first formal deduction.

Proposition 18. (The Left Side of Proposition 16) Let FR be interpreted as in table 7.1.

$$\forall x^{rb} \neg \exists x^{st} FR(x^{rb}, x^{st})$$

$\dashv$

Proof. The natural deduction provided in Figure 7.2 establishes Proposition 18. (The sub-deductions can be found in the Appendix at B1.) ■

Definition 7.3

$$\dfrac{\dfrac{\dfrac{\forall x^{rb}\forall x^{s}[NF(x^{rb},x^{s}) \leftrightarrow \neg caus(x^{rb},x^{s})]}{\forall x^{rb}\forall x^{s}[NF(x^{rb},x^{s}) \rightarrow \neg caus(x^{rb},x^{s})]}\wedge\text{E}}{NF(x^{rb},x^{s}) \rightarrow \neg caus(x^{rb},x^{s})}\forall\text{E}}{\neg\neg caus(x^{rb},x^{s}) \rightarrow \neg NF(x^{rb},x^{s})}\text{IL.2}$$

IL.1

$$\dfrac{caus(x^{rb},x^{s}) \rightarrow \neg\neg caus(x^{rb},x^{s}) \qquad caus(x^{rb},x^{s})^{u}}{\neg\neg caus(x^{rb},x^{s})}\rightarrow\text{E}$$

$$\dfrac{\dfrac{\neg\neg caus(x^{rb},x^{s}) \rightarrow \neg NF(x^{rb},x^{s}) \qquad \neg\neg caus(x^{rb},x^{s})}{\neg NF(x^{rb},x^{s})}\rightarrow\text{E}}{caus(x^{rb},x^{s}) \rightarrow \neg NF(x^{rb},x^{s})}\rightarrow\text{I},u$$

$$\mathcal{D}_1 \quad \vdots$$

$$caus(x^{rb},x^{s_t}) \rightarrow \neg NF(x^{rb},x^{s_t})$$

Proposition 17

$$\dfrac{\dfrac{\forall x^{rb}\forall x^{s_t} caus(x^{rb},x^{s_t}) \wedge \forall x^{rb}\forall x^{S\setminus S^t}\neg caus(x^{rb},x^{S\setminus S^t})}{\forall x^{rb}\forall x^{s_t} caus(x^{rb},x^{s_t})}\wedge\text{E}}{caus(x^{rb},x^{s_t})}\forall\text{E}$$

$$\dfrac{caus(x^{rb},x^{s_t}) \rightarrow \neg NF(x^{rb},x^{s_t}) \qquad caus(x^{rb},x^{s_t})}{\neg NF(x^{rb},x^{s_t})}\rightarrow\text{E}$$

Definition 7.5

$$\dfrac{\dfrac{\forall x^{rb}\forall x^{s}[FR(x^{rb},x^{s}) \leftrightarrow [NF(x^{rb},x^{s}) \wedge PF(x^{rb},x^{s})]]}{FR(x^{rb},x^{s}) \leftrightarrow [NF(x^{rb},x^{s}) \wedge PF(x^{rb},x^{s})]}\forall\text{E}}{FR(x^{rb},x^{s}) \rightarrow [NF(x^{rb},x^{s}) \wedge PF(x^{rb},x^{s})]}\wedge\text{E}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{FR(x^{rb},x^{s})^{v} \qquad FR(x^{rb},x^{s}) \rightarrow [NF(x^{rb},x^{s}) \wedge PF(x^{rb},x^{s})]}{NF(x^{rb},x^{s}) \wedge PF(x^{rb},x^{s})}\rightarrow\text{E}}{NF(x^{rb},x^{s})}\wedge\text{E}}{FR(x^{rb},x^{s}) \rightarrow NF(x^{rb},x^{s})}\rightarrow\text{I},v}{\neg NF(x^{rb},x^{s}) \rightarrow \neg FR(x^{rb},x^{s})}\text{IL.2}$$

$$\mathcal{D}_2 \quad \vdots$$

$$\dfrac{\dfrac{\dfrac{\neg NF(x^{rb},x^{s_t}) \qquad \neg NF(x^{rb},x^{s_t}) \rightarrow \neg FR(x^{rb},x^{s_t})}{\neg FR(x^{rb},x^{s_t})}\rightarrow\text{E}}{\forall x^{rb}\forall x^{s_t}\neg FR(x^{rb},x^{s_t})}\forall\text{I}}{\forall x^{rb}\neg\exists x^{s_t} FR(x^{rb},x^{s_t})}\text{IL.3}$$

Figure 7.2: The Deduction of Proposition 18.

Part 2 of Proposition 16

Kant's reasoning in the second part of this first phase seems to contain an implicit additional inference step. The argument captures the following train of thought: 'Because natural causality is objectively valid *only* from the theoretical viewpoint, freedom is not impossible from a *non*-theoretical viewpoint'. What makes this inference valid? Kant seems to use the following abstract reasoning step (1):

> If from some point of view all conditions that could make a concept impossible do not hold, then from that particular viewpoint the concept at stake is not impossible.

The structure of Kant's reasoning here is very similar to the structure of a Kantian *disjunctive judgment*:[4]

> If the concept ϕ is impossible in situation x^s, then either θ_1 or ... or θ_n must be the case at x^s.

('$\theta_1, ..., \theta_n$' is the exhaustive and mutually exclusive list of conditions that make ϕ impossible.) The contrapositive of this 'disjunctive judgment' expresses Kant's reasoning in (1). Namely, if none of the concepts '$\theta_1, ..., \theta_n$' is the case at x^s, then ϕ is not impossible at x^s'.

Thus, provided that the only obstruction to freedom is natural causality (see section 4.2.1 Postulate 11), the concept of freedom is not impossible from any viewpoint in which natural causality does not hold.[5] Since natural causality only holds with respect to the theoretical viewpoint (Postulate 10), it must

[4]Kantian disjunctive judgments express "[t]he relations of members of the divided concept, of parts of the sphere to the whole sphere" [Kan92a, Dohna-Wundlacken Logic 24:765]. The relation between the whole and its parts is that of a partitioning. The concept in question makes up the whole sphere such that its disjuncts consist of the exhaustive and mutually exclusive division of that particular sphere. A disjunctive judgment can be represented as an implication of the form: $\forall x[x \in \phi \rightarrow [$ either $x \in \psi_1$ or $x \in \psi_i$ or ... or $x \in \psi_n]]$ (ϕ is a disjoint union of its subsets $\psi_1, \psi_i ... \psi_n$). The disjunctive judgment might give rise to problems with respect to the intuitionistic nature of Kant's reasoning. A partitioning requires a general proof that determines for every possible object of the sphere to which of the sphere's disjoint subsets it belongs. If the disjunctive judgment is regarded as a 'partition function', then this judgment form is (at least) intuitionistically dubious. However, the inclusion of an antecedent in Kant's judgment seems to condition this 'partitioning' on the objects affirmed, or proved, as belonging to the total sphere of the concept in question. The disjunctive judgment would then express the following: If an object can be affirmed to belong to the total sphere, then it can be affirmed to belong to one of its disjoint subsets. In a three valued logic this would keep the possibility of an 'unconfirmed' (i.e. undefined) object open. However, since Kant's notion of the disjunctive judgment will not be essential to the present formalization I will omit any further elaboration of its apparent problems. For a more extensive treatment of Kant's disjunctive judgment the reader is referred to [Kan92a, Jäsche Logic - 9:106-9:109], [AVL, Ch.1 and Ch.5] and [Lon05, Ch.7].

[5]That natural causality is the only obstruction to freedom implies that the sphere of not-freedom (i.e. $\neg FR = \{(x^{rb}, x^s) \mid \neg FR(x^{rb}, x^s)\}$) is fully exhausted by the concept of natural causality. There might be other 'obstructions' to freedom though, but these obstructions only hold in situations in which natural causality holds as well.

be concluded that freedom is at least not impossible from a non-theoretical viewpoint.[6]

PROPOSITION 19. (FORMALIZING INFORMAL Postulate 11) Let $caus(x^{rb}, x^s)$ and $FR(x^{rb}, x^s)$ be interpreted as in table 7.1.

$$\forall x^{rb}\forall x^s[\neg FR(x^{rb}, x^s) \rightarrow caus(x^{rb}, x^s)]$$

(Proposition 19 will be assumed throughout the formalization.) ⊣

(NB. Proposition 19 can only 'make room' for the concept of freedom. The concept of natural causality can neither positively nor negatively imply the concept of freedom itself.)[7]

PROPOSITION 20. (THE RIGHT SIDE OF Proposition 16) Let FR be interpreted as in table 7.1.

$$\forall x^{rb}\forall x^{S\backslash S^t}\neg\neg FR(x^{rb}, x^{S\backslash S^t})$$ ⊣

Proof. The deduction provided in Figure 7.3 establishes Proposition 20. ■

This finishes the proof of Proposition 16:

Proof. Trivial, Propositions 18 and 20 establish Proposition 16. ■

---

[6]The above interpretation of Kant's reasoning explains Kant's emphasis on the limits of the concept of natural causality in the proof of the possibility of the concept of freedom in the second Critique (see section 4.2.1).

[7]That natural causality is the only obstruction to freedom must be either presupposed or proved. In both cases this leads to the same problem: The exhaustiveness of the list of possible obstructions to the concept of freedom (in the first case the list must be assumed, in the latter determined). The problem arises with respect to Kant's *principle of determinability* (not to be confused with Kant's principle of thoroughgoing determination, which can be seen as a stronger version of this principle). In the first Critique Kant states that: "[O]f **every two** contradictorily opposed predicates *only one can* apply to it [a concept], which rests on the principle of contradiction and hence is a merely logical principle" [Kan00, A571/B599 - italics my own]. Kant's use of 'only one can' in the above quote emphasizes the possibility of the application of *at most one* predicate of every two contradictorily opposed predicates. It therefore implies that the quote must not be read as an affirmation of the validity of the law of excluded middle, but as an affirmation of the law of non-contradiction: $[\phi \wedge \neg\phi] \rightarrow \bot$ i.e., $\neg[\phi \wedge \neg\phi]$. However, according to Kant there does not exist a 'smallest species', "because such a one cannot possibly be determined" [Kan92a, Jäsche Logic - 9:97]. This means that there is no smallest predicate that applies to a concept. Consequently, there are infinitely many predicates that can apply to a concept. This makes the exhaustiveness of the possible obstructions to the concept of freedom, as expressed in Postulate 11, problematic. Namely, any exhaustive list of obstructions to a concept presupposes a proof that determines for every possible predicate (concept) (i) whether it applies to the concept or not, or (ii) whether the predicate is already covered by the sphere of an already determined obstruction to the concept in question. It seems that Kant presupposes the exhaustiveness of this list with respect to the concept of freedom. Although the principle of determinability causes some doubts with respect to Kant's argumentation here, for the sake of the size of this thesis I will take the exhaustiveness of this disjunctive judgment as given.

Proposition 19

$$\frac{\forall x^{rb}\forall x^{s}[\neg FR(x^{rb},x^{s}) \to caus(x^{rb},x^{s})]}{\neg FR(x^{rb},x^{s}) \to caus(x^{rb},x^{s})}\ \forall\text{E}$$

$$\frac{\neg FR(x^{rb},x^{s}) \to caus(x^{rb},x^{s})}{\neg caus(x^{rb},x^{s}) \to \neg\neg FR(x^{rb},x^{s})}\ \text{IL.2}$$

$$\frac{\neg caus(x^{rb},x^{s}) \to \neg\neg FR(x^{rb},x^{s})}{\forall x^{rb}\forall x^{s}[\neg caus(x^{rb},x^{s}) \to \neg\neg FR(x^{rb},x^{s})]}\ \forall\text{I}$$

$$\frac{\forall x^{rb}\forall x^{s}[\neg caus(x^{rb},x^{s}) \to \neg\neg FR(x^{rb},x^{s})]}{\forall x^{s}[\neg caus(x^{rb},x^{s}) \to \neg\neg FR(x^{rb},x^{s})]}\ \forall\text{E}$$

$$\frac{\forall x^{s}[\neg caus(x^{rb},x^{s}) \to \neg\neg FR(x^{rb},x^{s})]}{\forall x[S(x) \to [\neg caus(x^{rb},x) \to \neg\neg FR(x^{rb},x)]]}\ \text{Axiom 1}$$

$$\frac{\forall x[S(x) \to [\neg caus(x^{rb},x) \to \neg\neg FR(x^{rb},x)]]}{S(x) \to [\neg caus(x^{rb},x) \to \neg\neg FR(x^{rb},x)]}\ \forall\text{E}$$

Axiom 2

$$\frac{\forall x[S\backslash S^{t}(x) \to S(x)]}{S\backslash S^{t}(x) \to S(x)}\ \forall\text{E}$$

$$\frac{S\backslash S^{t}(x) \to S(x) \qquad S\backslash S^{t}(x)^{u}}{S(x)}\ \to\text{E}$$

$$\frac{S(x) \to [\neg caus(x^{rb},x) \to \neg\neg FR(x^{rb},x)] \qquad S(x)}{\neg caus(x^{rb},x) \to \neg\neg FR(x^{rb},x)}\ \to\text{E}$$

$$\frac{\neg caus(x^{rb},x) \to \neg\neg FR(x^{rb},x)}{S\backslash S^{t}(x) \to [\neg caus(x^{rb},x) \to \neg\neg FR(x^{rb},x)]}\ \to\text{I},u$$

$$\frac{S\backslash S^{t}(x) \to [\neg caus(x^{rb},x) \to \neg\neg FR(x^{rb},x)]}{\forall x[S\backslash S^{t}(x) \to [\neg caus(x^{rb},x) \to \neg\neg FR(x^{rb},x)]]}\ \forall\text{I}$$

$$\frac{\forall x[S\backslash S^{t}(x) \to [\neg caus(x^{rb},x) \to \neg\neg FR(x^{rb},x)]]}{\forall x^{S\backslash S^{t}}[\neg caus(x^{rb},x^{S\backslash S^{t}}) \to \neg\neg FR(x^{rb},x^{S\backslash S^{t}})]}\ \text{Axiom 2}$$

$$\frac{\forall x^{S\backslash S^{t}}[\neg caus(x^{rb},x^{S\backslash S^{t}}) \to \neg\neg FR(x^{rb},x^{S\backslash S^{t}})]}{\neg caus(x^{rb},x^{S\backslash S^{t}}) \to \neg\neg FR(x^{rb},x^{S\backslash S^{t}})}\ \forall\text{E}$$

Proposition 17

$$\frac{\forall x^{rb}\forall x^{s_t} caus(x^{rb},x^{s_t}) \wedge \forall x^{rb}\forall x^{S\backslash S^{t}} \neg caus(x^{rb},x^{S\backslash S^{t}})}{\forall x^{rb}\forall x^{S\backslash S^{t}} \neg caus(x^{rb},x^{S\backslash S^{t}})}\ \wedge\text{E}$$

$$\frac{\forall x^{rb}\forall x^{S\backslash S^{t}} \neg caus(x^{rb},x^{S\backslash S^{t}})}{\neg caus(x^{rb},x^{S\backslash S^{t}})}\ \forall\text{E}$$

$$\frac{\neg caus(x^{rb},x^{S\backslash S^{t}}) \qquad \neg caus(x^{rb},x^{S\backslash S^{t}}) \to \neg\neg FR(x^{rb},x^{S\backslash S^{t}})}{\neg\neg FR(x^{rb},x^{S\backslash S^{t}})}\ \to\text{E}$$

$$\frac{\neg\neg FR(x^{rb},x^{S\backslash S^{t}})}{\forall x^{rb}\forall x^{S\backslash S^{t}} \neg\neg FR(x^{rb},x^{S\backslash S^{t}})}\ \forall\text{I}$$

Figure 7.3: The Deduction of Proposition 20.

7.2.2 Phase 2

The proofs from the previous phase showed the possibility of freedom only negatively. That is, they only revealed the viewpoint from which the concept of freedom would *not* be *impossible*. It can still be the case, though, that the viewpoint itself is impossible. This would imply that the concept of freedom is neither a priori impossible nor a priori possible (formally this means that the concept would be undefined). The aim of the second phase is to prove that this viewpoint is possible. For Kant the possibility of such a viewpoint means that a *supersensible nature* is possible. He defines this nature as a (pure) *practical viewpoint* (see section 4.2.2). The possibility of this nature is derived from the results of Theorem 3 and the possibility of FLoP. In the present argument five axioms can be encountered.

AXIOM 3. (FORMALIZING INFORMAL Definition 4.1, Definition 4.2 AND Proposition 7) Let $sens(x^s)$ and $supersens(x^s)$ be interpreted as in table 7.1. Let S, S^t and S^p de defined as in Axiom 2. The sensible and supersensible nature are mutually exclusive:

$$\forall x^s[sens(x^s) \rightarrow \neg supersens(x^s)]$$

$$\forall x^s[supersens(x^s) \rightarrow \neg sens(x^s)]$$

$sens(x)$ and $supersens(x)$ correspond, respectively, to $S^t(x)$ and $S^p(x)$:

$$\forall x[S(x) \rightarrow [[S^t(x) \leftrightarrow sens(x)] \wedge [S^p(x) \leftrightarrow supersens(x)]]$$

Let the complement of S^t be defined as the negative sphere of S^t:

$$\forall x[S \backslash S^t(x) \leftrightarrow \neg S^t(x)]$$

⊣

Although the sensible and supersensible nature are mutually exclusive, they do not (necessarily) form one another's complement. As a consequence, the direct inference from a negatively determined nature to its positive determination is not valid. This means that the mere possibility of a non-sensible nature is insufficient for the proof of the possibility of a supersensible nature. That is,

$$\exists x^s \neg sens(x^s) \not\rightarrow \exists x^s supersens(x^s).$$

Hence to obtain the positive determination of this supersensible nature, something more is needed. Kant seems to be aware of this requirement:

> [T]he moral law [...] provides a fact absolutely inexplicable from any data of the sensible world and from the whole compass of our theoretical use of reason, a fact that points to a pure world of the understanding [the supersensible nature] and, indeed, even *determines* it *positively* and lets us cognize something of it, namely a law. [Kan96a, 5:43]

In the above quote Kant emphasizes the positive determination of the supersensible nature. The moral law is the law of this supersensible nature and for this reason provides the *positive content* of this nature. In other words, the determination of the possibility of a nature does not only require a negative determination, it also requires a positive determination of (some of) the nature's content.[8] The following definition enables the identification of positive formulas in the logic KL.[9]

Definition 7.6. (Positive Formulas in KL) Let $\mathcal{P}$ be a one-place predicate in the language $\mathcal{L}_{KL}$ of KL. $\mathcal{P}(\ulcorner\phi\urcorner)$ must be interpreted as 'ϕ is a positive formula'. For every formula ϕ in $\mathcal{L}_{KL}$ we have the following:

$\mathcal{P}\ (\ulcorner\phi\urcorner)\ \Leftrightarrow$ (i) for every unary operator '$\otimes$' present in ϕ: $\otimes = \forall$ or $\exists$, *and*
(ii) for every binary operator '$\circledast$' present in ϕ: $\circledast = \wedge, \vee$ or $\rightarrow$ ⊣

Positive formulas are thus formulas that are '$\neg, \bot$'-free. On the basis of the above definition the second axiom of this phase can be introduced.

Axiom 4. (Positive Determination of the Supersensible Nature) Let $\neg sens(x^s)$ and $supersens(x^s)$ be interpreted as in table 7.1. Let $\mathcal{P}(\ulcorner\phi\urcorner)$ be as defined in Definition 7.6. Let ϕ be some formula of $\mathcal{L}_{KL}$ such that (i) ϕ is bound by $\exists x^s$ (i.e. there is a free-variable occurrence of x^s in ϕ) and (ii) $\forall x^s \phi$ does not hold.

$$\exists x^s supersens(x^s) \leftrightarrow \exists x^s \exists \ulcorner\phi\urcorner[\neg sens(x^s) \wedge \mathcal{P}(\ulcorner\phi\urcorner) \wedge \phi\ ]$$

(Clause (ii) ensures that the tautologies and axioms of KL, which hold in *every* possible situation, do not satisfy this axiom. If clause (ii) would be omitted Axiom 4 would immediately validate $\exists x^s \neg sens(x^s) \rightarrow \exists x^s supersens(x^s)$.) ⊣

The proof of the possibility of a non-sensible nature is based on the result of Theorem 3: 'A practical law can be thought of as a determining ground of the will only if it is regarded as to its mere form'.[10]

Proposition 21. (Formalizing informal Theorem 3) Let $PP(\ulcorner\phi\urcorner)$, $O_i(\ulcorner\phi\urcorner)$, $DG(\ulcorner\phi\urcorner, x^{rb}, x^s)$ and $form(\ulcorner\phi\urcorner)$ be interpreted as in table 7.1.

$$\forall\ulcorner\phi\urcorner[[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner)] \rightarrow \forall x^{rb} \forall x^s[DG(\ulcorner\phi\urcorner, x^{rb}, x^s) \rightarrow form(\ulcorner\phi\urcorner, x^s)]]$$

(Proposition 21 will be assumed throughout the formalization.) ⊣

---

[8] Kant's approach is similar to the idea of positive construction, or proof requirement, in intuitionistic reasoning. Consider for example the following remark by Heyting: "$(\exists x)p(x)$ will be true if and only if an element a of Q [the domain of the logic] for which $p(a)$ is true has actually been constructed" [Hey71, p.107].

[9] The definition of a positive formula is based on [DJZ14].

[10] The clause 'only if' is interpreted as a strict implication $\forall x^s[\phi \rightarrow \psi]$. The strict implication captures the following thought: In every situation in which ϕ is the case ψ is the case as well; i.e., if ϕ is the case, then ψ *must* be the case. For an introduction to the strict implication the reader is referred to [CH96, Ch.11].

Subsequently, Proposition 21 is based on Kant's distinction between mere form and matter. According to Kant, every practical principle can be regarded as to its mere form or matter.[11] Hence, Proposition 21 justifies the introduction of the third axiom:

Axiom 5. (Formalizing Informal Postulate 5) Let $PP(\ulcorner\phi\urcorner)$, $form(\ulcorner\phi\urcorner, x^s)$ and $matter(\ulcorner\phi\urcorner, x^s)$ be interpreted as in table 7.1.

$$\forall\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \rightarrow \forall x^s[form(\ulcorner\phi\urcorner, x^s) \rightarrow \neg matter(\ulcorner\phi\urcorner, x^s)]]$$

$$\forall\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \rightarrow \forall x^s[matter(\ulcorner\phi\urcorner, x^s) \rightarrow \neg form(\ulcorner\phi\urcorner, x^s)]]$$

⊣

One of the main results of Theorems 1 and 2 is that the sensible nature can only facilitate practical principles that contain matter. In other words, with respect to a rational being's sensible nature all practical principles are material practical principles (see sections 3.2 and 3.3). The fourth axiom can be formulated accordingly:

Axiom 6. (Formal Material Practical Principles) Let $PP(\ulcorner\phi\urcorner)$, $sens(x^s)$ and $matter(\ulcorner\phi\urcorner, x^s)$ be interpreted as in table 7.1.

$$\forall\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \rightarrow \forall x^s[sens(x^s) \leftrightarrow matter(\ulcorner\phi\urcorner, x^s)]]$$

⊣

The positive determination of the supersensible nature depends on the following result of FLoP: An objective practical principle as possible determining ground of a rational being's will is possible; namely, FLoP.[12]

Proposition 22. (Formalizing informal FLoP 3.7) Let $PP(\ulcorner\phi\urcorner)$, $O_i(\ulcorner\phi\urcorner)$ and $DG(\ulcorner\phi\urcorner, x^{rb}, x^s)$ be interpreted as in table 7.1. FLoP 3.7 formally implies the following:

$$\exists x^s \exists\ulcorner\phi\urcorner\forall x^{rb}[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner, x^{rb}, x^s)]$$

(Proposition 22 will be assumed throughout the formalization.) ⊣

[11]Notice that every practical principle can also be regarded as to its form and matter simultaneously. The concepts at stake here are '*mere* form' and 'matter' and these are mutually exclusive. Furthermore, mere form and matter are not necessarily exhaustively exclusive.

[12]Two remarks must be made. Firstly, the argument presented in this phase can also be made with respect to the set of human beings only, but for the sake of readability I will use the more general result expressed by Proposition 22. The transition from rational beings to human beings will be provided in section 7.2.4. Secondly, I will omit explicit formalization of FLoP itself. Any serious attempt of formalizing FLoP (i.e., the first formulation of the categorical imperative) would be to controversial and perhaps to ambitious for the present undertaking. Proposition 22 will suffice.

Following Kant, the possibility of the moral determines the supersensible nature positively. That the moral law represents the positive content of this nature is expressed by the fifth, and last, axiom:

AXIOM 7. (FLOP IS POSITIVE) Let $PP(\ulcorner\phi\urcorner)$, $O_i(\ulcorner\phi\urcorner, x^s)$ and $DG(\ulcorner\phi\urcorner, x^{rb}, x^s)$ be interpreted as in table 7.1. Let $\mathcal{P}$ be as defined in Definition 7.6. Let $\theta = \forall x^{rb}[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner, x^{rb}, x^s)]$.

$$\mathcal{P}(\ulcorner\theta\urcorner)$$

⊣

The following formal proposition must be proved:

PROPOSITION 23. (FORMALIZING INFORMAL Proposition 9) Let $S^p(x)$ be interpreted as in table 7.1.

$$\exists x S^p(x)$$

⊣

Proof. The natural deduction provided in the Appendix at B2 establishes Proposition 23. ■

7.2.3 Phase 3

Propositions 16 and 23, together with the results established in the necessity argument, should be sufficient to prove the possibility of the concept of freedom. Recall the informal argument: The supersensible nature is possible. In this supersensible nature a rational being is independent of natural causality. In other words, negative freedom is possible in this nature. Moreover, FLoP is the fundamental law of this nature. FloP is possible and therefore, the idea of 'having an objective principle as determining ground of the will' is possible. In other words, positive freedom is possible in this nature and hence, the concept of freedom as well. The following formal proposition must be proved:

PROPOSITION 24. (FORMALIZING INFORMAL Proposition 10) Let $FR(x^{rb}, x^s)$ be interpreted as in table 7.1.

$$\exists x^s \forall x^{rb} FR(x^{rb}, x^s)$$

⊣

Proof. The natural deduction provided in the Appendix at B3 establishes Proposition 24. ■

(NB. The natural deduction of Proposition 24 formally captures the twofold nature of Kant's original argument very accurately.)

7.2.4 Phase 4

The aim of this phase is to prove the following informal proposition: With respect to human beings, the concept of freedom is sufficient to generate both the moral law and the possibility of being determined by this law (Proposition 11). At this point in the transcendental argument Kant moves from rational beings in general to human beings in particular. Eventually this move will be necessary because the only law who's reality can be asserted is the moral law (i.e. FLoP for human beings). Kant treats human beings as a subspecies of rational beings.

Axiom 8. (Formalizing Human Beings) Let the language $\mathcal{L}_{KL}$ be extended with the one-place predicate $HB(x)$. Let $HB(x)$ be interpreted as 'x is a human being'. Let ϕ be some formula of $\mathcal{L}_{KL}$. KL is extended with the following rewrite rule:

$$\forall x^{hb}\phi \;\Leftrightarrow\; \forall x[HB(x) \rightarrow \phi[x^{hb}/x]]$$

The relation between HB and RB is expressed by the following axiom:

$$\forall x[HB(x) \rightarrow RB(x)]$$

(The results thus far relate to rational beings in general. Axiom 8 allows us to rewrite these results, such that they relate to human beings in particular.) $\dashv$

The proof of the claim of this phase is primarily based on Proposition 3 and Corollary 2. The first proposition expresses that a negatively free will can be determined by practical law if and only if the determining ground of this will is the mere lawgiving form of this principle (see section 3.6.2).

Proposition 25. (Formalizing informal Proposition 3) Let $NF(x^{rb}, x^s)$, $PP(\ulcorner\phi\urcorner)$, $O_i(\ulcorner\phi\urcorner)$, $DB(x^{rb}, \ulcorner\phi\urcorner, x^s)$, $DG(\ulcorner\phi\urcorner, x^{rb}, x^s)$ and $form(\ulcorner\phi\urcorner, x^s)$ be interpreted as in table 7.1. Let $\theta = DB(x^{rb}, \ulcorner\phi\urcorner, x^s) \leftrightarrow [DG(\ulcorner\phi\urcorner, x^{rb}, x^s) \wedge form(\ulcorner\phi\urcorner, x^s)]$.

$$\forall x^{rb}\forall x^s[NF(x^{rb}, x^s) \rightarrow \forall\ulcorner\phi\urcorner[[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner)] \rightarrow \theta\ ]] \qquad \dashv$$

Only the following result of Corollary 2 will be used: With respect to human beings an objective practical principle, that is a practical law, as determining ground of the will is nothing but the moral law. The following proposition expresses a 'substitution rule' that allows us to treat the moral law as the fundamental law of practical reason (FLoP) whenever the argument is only concerned with human beings.

Proposition 26. (Formalizing informal Corollary 2) Let $PP(\ulcorner\phi\urcorner)$, $O_i(\ulcorner\phi\urcorner)$ and $DG(\ulcorner\phi\urcorner, x^{hb}, x^s)$ be interpreted as in table 7.1. Let $\ulcorner ML\urcorner$ stand for the Gödel-number corresponding to the formula ML expressing the moral law. Let $\theta = PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner, x^{hb}, x^s)$. Let χ be some formula of $\mathcal{L}_{KL}$ such that θ is a *sub-formula* of χ:

$$\forall x^{hb} \forall x^s \forall \ulcorner\phi\urcorner[\chi \Rightarrow \chi[\ulcorner\phi\urcorner / \ulcorner ML \urcorner]]$$

(Provided that $\ulcorner\phi\urcorner$ occurs free in χ.[13]) ⊣

The following proposition must be proved:

Proposition 27. (Formalizing informal Proposition 11) Let $FR(x^{rb}, x^s)$ and $DB(x^{rb}, \ulcorner\phi\urcorner, x^s)$ be interpreted as in table 7.1. Let $\ulcorner ML \urcorner$ stand for the Gödel-number corresponding to the formula ML expressing the moral law.

$$\forall x^{hb} \forall x^s [FR(x^{hb}, x^s) \rightarrow DB(x^{hb}, \ulcorner ML \urcorner, x^s)]$$

⊣

Proof. The deduction provided in Figure 7.4 establishes Proposition 27. $[u]$ is the open assumption $FR(x^{hb}, x^s)$ and $[v]$ is the open assumption $PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner, x^{hb}, x^s)$. (The sub-deductions can be found in the Appendix at B4.) ■

$$\dfrac{\dfrac{\dfrac{\begin{matrix}[u]\\ \mathcal{D}_9\\ \vdots\\ \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner, x^{hb}, x^s)]\end{matrix} \qquad \begin{matrix}[u]\\ [v]\\ \mathcal{D}_{10}\\ \vdots\\ DB(x^{hb}, \ulcorner ML \urcorner, x^s)\end{matrix}}{DB(x^{hb}, \ulcorner ML \urcorner, x^s)}\ \exists\text{E},v}{FR(x^{hb}, x^s) \rightarrow DB(x^{hb}, \ulcorner ML \urcorner, x^s)}\ \rightarrow\text{I},u}{\forall x^{hb} \forall x^s [FR(x^{hb}, x^s) \rightarrow DB(x^{hb}, \ulcorner ML \urcorner, x^s)]}\ \forall\text{I}$$

Figure 7.4: The Deduction of Proposition 27.

The deduction of Figure 7.4 shows two things. Firstly, from the sole concept of freedom we can derive the formulation of the moral law (as a possible determining ground). This derivation is represented by $\mathcal{D}_9$. Secondly, on the basis of the concept of freedom, together with the derived law, we can derive that the free human will is determinable by this law. This derivation is represented by $\mathcal{D}_{10}$. Together these two deductions prove Proposition 27.
(NB. Again, the provided natural deduction represents the twofold nature of Kant's original argument very accurately.)[14]

[13] A sub-formula can be defined as follows: A string of symbols occurring in a proper part of some formula ϕ is called a sub-formula if and only if that string of symbols satisfies the recursive definition of a formula in Definition 6.9.

[14] The necessity argument (Proposition 15) and this last phase (Proposition 27) proved respectively (1) $\forall x^{hb} \forall x^s [DB(x^{hb}, \ulcorner ML \urcorner, x^s) \rightarrow FR(x^{hb}, x^s)]$ and (2) $\forall x^{hb} \forall x^s [FR(x^{hb}, x^s) \rightarrow$

7.2.5 Phase 5

The results of phase 3 and 4 are sufficient to prove that the concept of freedom is the ground of the moral law (Proposition 12). Before the proof can be addressed the general definition of a ground, as provided in section 7.1, must be adjusted to Kant's practical vocabulary. In clause (v) of Definition 7.2 the informal proposition 'ψ is sufficient to generate ϕ as its consequence' is formally defined as: $\forall x^s[\psi(y_1, ..., y_l, x^s) \rightarrow \phi(x_1, ..., x_k, x^s)]$. In relation to the present framework it seems reasonable to adjust this general clause. How it must be adjusted depends on what needs to be 'generated' by the ground in question. Kant's practical philosophy concerns a priori grounds in relation to the *determinability of a will* and I will therefore use the more accurate formula

$$\forall x^{hb}\forall x^s[FR(x^{rb}, x^s) \rightarrow DB(x^{hb}, \ulcorner ML \urcorner, x^s)]$$

as the representative of clause (v) of Definition 7.2. The following axiom is a reformulation of Definition 7.2 in relation to Kant's second Critique:

AXIOM 9. (FORMALIZING INFORMAL Postulate 9) Let $NC_\phi(\ulcorner \psi \urcorner)$, $FR(x^{hb}, x^s)$, $DB(x^{hb}, \ulcorner \psi \urcorner, x^s)$ and $G_\phi(\ulcorner \psi \urcorner)$ be interpreted as in table 7.1. Let $\theta = NC_{ML}(\ulcorner FR \urcorner) \wedge \exists x^s \forall x^{hb} FR(x^{hb}, x^s) \wedge \forall x^{hb}\forall x^s[FR(x^{hb}, x^s) \rightarrow DB(x^{hb}, \ulcorner ML \urcorner, x^s)]$.

$$G_{ML}(\ulcorner FR \urcorner) \leftrightarrow \theta$$

(Axiom 9 explicitly satisfies clause (i)-(iv) of Definition 7.2. Clause (v) is ensured by the fact that all components of freedom, i.e. negative and positive freedom, are necessary to derive $DB(x^{hb}, \ulcorner ML \urcorner, x^s)$: see Figure 7.4.) ⊣

The following formal proposition must be proved:

PROPOSITION 28. (FORMALIZING INFORMAL Proposition 12) Let $G_\phi(\psi)$ be interpreted as in table 7.1.

$$G_{ML}(\ulcorner FR \urcorner)$$

⊣

Proof. Trivial, by Axiom 9 and Propositions 15, 24, and 27. (The formal deduction of Proposition 28 can be found in the Appendix at B5.) ■

This finishes the formalization of Kant's practical possibility argument.

$DB(x^{hb}, \ulcorner ML \urcorner, x^s)]$. Logically we are justified in concluding (3) $\forall x^{hb}\forall x^s[FR(x^{hb}, x^s) \leftrightarrow DB(x^{hb}, \ulcorner ML \urcorner, x^s)]$. The informal reading of (3) coincides with Kant's remark that "freedom and unconditional practical law reciprocally imply each other" [Kan96a, 5:29]. However there is a distinction between (1) and (2) that must be made clear. The distinction can be found in the function of these formulae. The former captures Kant's notion of a *ratio cognoscendi*, the latter of a *ratio essendi* (see section 3.6.3). In other words, (1) expresses the way in which we *first* come to *know* the ground and (2) expresses the way in which the cognition *comes to being* through the ground. The distinction can therefore be seen as a 'temporal' distinction; namely, (2) can only be established after the consequent of (1) has been determined. To preserve Kant's distinction I have restricted the definition of G_{ML} to the results of NC_{ML} (see Definition 7.2).

7.3 The Formal Practical Objective Validity Argument

In this section I will try to reconstruct Kant's argument for the objective validity of freedom. This will be done on the basis of the criteria of Postulate 13 in relation to the formal results derived thus far. The following proposition must be proved:

PROPOSITION 29. (FORMALIZING INFORMAL Proposition 13) Let $O_i(\ulcorner\phi\urcorner)$ be interpreted as in table 7.1.

$$O_p(\ulcorner FR \urcorner)$$

⊣

With respect to Kant's practical objective validity argument the following is clear: The objective validity of the concept of freedom must be derived from the reality of the moral law; that is, the reality of the moral law is the argument's primary deductive principle. This reality though is restricted to the practical viewpoint. Kant treats the reality of the moral law as an axiom (section 5.2).

AXIOM 10. (FORMALIZING INFORMAL Postulate 12) Let $R_i(\ulcorner\phi\urcorner)$ be interpreted as in table 7.1. Let $\ulcorner ML \urcorner$ stand for the Gödel-number corresponding to the formula ML expressing the moral law.

$$R_p(\ulcorner ML \urcorner)$$

⊣

As such Axiom 10 is meaningless. The formal implications of the concept of reality still need to be determined. In the Critique of Practical Reason Kant states the following:

> [T]he moral law is given, as it were, as a fact of pure reason of which we [human beings] are **a priori** conscious and which is **apodictically certain** [...]. Hence the objective **reality** of the moral law cannot be proved by any deduction. [Kan96a, 5:47 - bold emphasis my own]

From the above quote the following two points can be inferred: (1) The reality of the moral implies that every human being must be a priori conscious of this law. That every human being is a priori conscious of this law implies that the moral law must be universally valid (see Definition 3.7). (2) The reality of the moral law implies that the law is apodictically certain. Apodictic certainty though is defined by Kant as absolute necessity.[15] In other words, the reality of the moral law implies the universal validity and absolute necessity of this

[15] See for example [Kan00, A75-B100] and [Kan02, 4:280].

law. The above elaboration justifies the introduction of the quantifiers $\forall x^{hb}$ and $\forall x^{s_p}$ in the following axiom:[16]

Axiom 11. (The Concept of Reality) Let $R_i(\ulcorner\phi\urcorner)$, $PP(\ulcorner\phi\urcorner)$, $O_i(\ulcorner\phi\urcorner)$ and $DG(\ulcorner\phi\urcorner, x^{rb}, x^{s})$ be interpreted as in table 7.1. Let $\ulcorner ML\urcorner$ be the Gödel-number for the formula ML expressing the moral law.

$$R_p(\ulcorner ML\urcorner) \;\Rightarrow\; \forall x^{hb}\forall x^{s_p}[PP(\ulcorner ML\urcorner) \wedge O_i(\ulcorner ML\urcorner) \wedge DG(\ulcorner ML\urcorner, x^{hb}, x^{s_p})] \dashv$$

The results established by Kant's practical necessity and possibility argument, together with the implications of the reality of the moral law, should be sufficient to prove the objective validity of freedom (see section 5.2). If we bring together proposition $R_p(\ulcorner ML\urcorner)$ and the formal results of section 7.2 in a single deduction, then the natural deduction represented in Figure 7.5 results.

($[u]$ is the open assumption $\forall x^{hb}[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{hb}, x^{s_p})$. The sub-deductions of Figure 7.5 can be found in the Appendix at B6.)

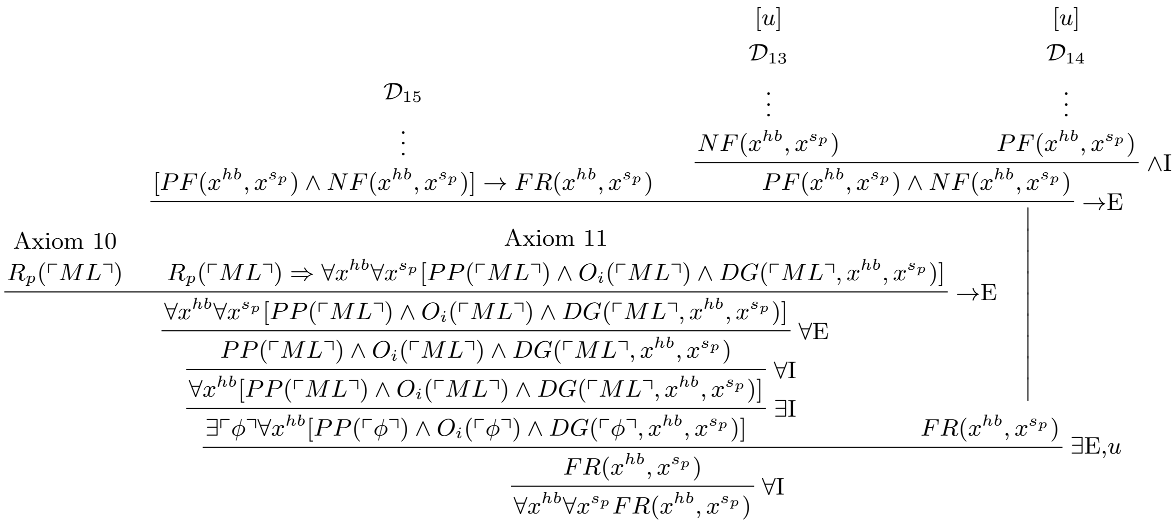

Figure 7.5: The Deduction of Proposition 24.

The deduction of Figure 7.5 shows that (at least) the following proposition can be derived from the introduction of $R_p(\ulcorner ML\urcorner)$ to the argument:

[16] The possibility of FLoP signifies the possibility of an objective practical principle as a determining ground of a rational being's will (see section 3.7). This proposition is formally expressed by Proposition 22 (see section 7.2.2). I have therefore chosen to formally represent the reality of the moral (Axiom 11) as an universal validity and necessity of the possibility of an objective practical principle, that is the moral law, as a determining ground of a human will.

Proposition 30. (A Formal Result of the Objective Validity Argument) Let $FR(x^{hb}, x^{s_p})$ be interpreted as in table 7.1.

$$\forall x^{hb} \forall x^{s_p} FR(x^{hb}, x^{s_p})$$

⊣

Proof. The natural deduction provided in Figure 7.5 establishes Proposition 30. (The sub-deductions can be found in the Appendix at B6.) ■

Thus, the reality of the moral law implies that the concept of freedom is *universally* and *necessarily* valid.[17] The objective validity of freedom must be deduced from the very same premisses from which Proposition 30 is deduced. It seems therefore very likely that the result of this section is at least part of the desired result. In other words, it seems likely that the universal validity and necessity of the concept of freedom is part of the objective validity of that concept. The main aim of Kant's practical transcendental argument must be established on the basis of the objective validity of freedom: The objective validity of freedom must establish the (a priori) synthesis between the human will and the moral law. In the next section I will determine whether or not Proposition 30 is sufficient to prove the main claim of Kant's argument.

7.4 The Possibility of the Moral Law as an A Priori Synthetic Proposition

Kant claims that the objective validity of the concept of freedom is derivable from the reality of the moral law. The previous section showed that from the reality of this law at least the formal proposition $\forall x^{hb} \forall x^{s_p} FR(x^{hb}, x^{s_p})$ can be derived. With respect to Kant's claim it seems reasonable to assume that the above proposition is (at least) a part of the definition of the objective validity of freedom.

Assumption 1. (The Objective Validity of Freedom)(A provisory partial definition) Let $FR(x^{hb}, x^{s_p})$ and $O_p(\ulcorner \phi \urcorner)$ be interpreted as in table 7.1.

$$O_p(\ulcorner FR \urcorner) \rightarrow \forall x^{hb} \forall x^{s_p} FR(x^{hb}, x^{s})$$

⊣

Whether Proposition 30 is in fact the sufficient definition of the objective validity of freedom will be determined in this section. (That is, it will be determined whether the implication of Assumption 1 should be a logical

[17]The deduction from Figure 7.5 is quite similar to the deduction of the possibility of freedom as provided in phase 3 of the possibility argument. In the latter deduction the modality 'possibility' is transferred from FLoP to the concept of freedom. In the former the modalities 'necessity' and 'universality' are transferred from the moral law to its ground, namely the concept of freedom. Both deductions take the same path.

equivalence.) I will call the definition of the objective validity of freedom sufficient *if and only if* the main claim of Kant's practical transcendental argument can be proved on the basis of that definition. Recall what Kant needs to prove:

Postulate 3 (The Possibility of the Moral Law) The aim of the transcendental argument of the Critique of Practical Reason is to prove how the moral law is possible as a synthetic a priori proposition: The argument needs to prove how the moral law can immediately determine a human being's will.

The (a priori) synthesis consists of the (a priori) connection between the human will and the moral law. This connection, subsequently, is defined in terms of *determinability*. In other words, the argument must show that (and how) every human being is necessarily and a priori determinable by the moral law.[18]

Proposition 31. (The Main Claim) Let $DB(x^{hb}, \ulcorner\phi\urcorner, x^{s_p})$ be interpreted as in table 7.1. Let $\ulcorner ML\urcorner$ stand for the Gödel-number corresponding to the formula ML expressing the moral law.

$$\forall x^{hb}\forall x^{s_p} DB(x^{hb}, \ulcorner ML\urcorner, x^{s_p}) \qquad \dashv$$

Subsequently, the a priori synthesis is established on the basis of the objective validity of the moral law's ground: "[T]he human will is by virtue of its freedom immediately determinable by the moral law" [Kan96a, 5:38]. Kant's use of 'immediately' implies that the synthesis at stake can be established *solely* on the basis of the objective validity of freedom; that is, without the use of any mediating premiss (see [Kan92a, Jäsche Logic – 9:114]). This immediacy of the determinability of the will implies therefore that both the moral law and the possibility of being determined by this law, must be derived directly from the objective validity of freedom. Formally this can be expressed as follows:

$$O_p(\ulcorner FR\urcorner) \vdash_{KL} \forall x^{hb}\forall x^{s_p} DB(x^{hb}, \ulcorner ML\urcorner, x^{s_p})$$

It turns out that Proposition 31 is derivable from the provisory definition of objective validity. The deduction of Proposition 31 is represented in Figure 7.6. This deduction is solely based on the following assumptions: $[v]$ $O_p(\ulcorner FR\urcorner)$, $[u]$ $O_p(\ulcorner FR\urcorner) \rightarrow \forall x^{hb}\forall x^{s_p} FR(x^{hb}, x^{s_p})$ and Definition 7.5. (The sub-deductions of Figure 7.6 can be found in the Appendix at B7.)

[18] The a priority of the synthesis is guaranteed by the a priori nature of the transcendental argument itself.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}[u]\\ [v]\\ \mathcal{D}_{16}\\ \vdots\\ \forall x^{hb}\forall x^{s_p}\exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner)\wedge O_i(\ulcorner\phi\urcorner)\wedge DG(\ulcorner\phi\urcorner,x^{hb},x^{s_p})]\end{array}}{\exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner)\wedge O_i(\ulcorner\phi\urcorner)\wedge DG(\ulcorner\phi\urcorner,x^{hb},x^{s_p})]}\forall\text{E} \qquad \begin{array}{c}[w]\\ [u]\\ [v]\\ \mathcal{D}_{17}\\ \vdots\\ DB(x^{hb},\ulcorner ML\urcorner,x^{s_p})\end{array}}{DB(x^{hb},\ulcorner ML\urcorner,x^{s_p})}\exists\text{E},w}{\forall x^{hb}\forall x^{s_p}DB(x^{hb},\ulcorner ML\urcorner,x^{s_p})}\forall\text{I}}{O_p(\ulcorner FR\urcorner)\to\forall x^{hb}\forall x^{s_p}DB(x^{hb},\ulcorner ML\urcorner,x^{s})}\to\text{I},v}{[O_p(\ulcorner FR\urcorner)\to\forall x^{hb}\forall x^{s_p}FR(x^{hb},x^{s_p})]\to[O_p(\ulcorner FR\urcorner)\to\forall x^{hb}\forall x^{s_p}DB(x^{hb},\ulcorner ML\urcorner,x^{s})]}\to\text{I},u$$

Figure 7.6: The Formal Implications of Assumption 1.

(NB. $\mathcal{D}_{16}$ represents the deduction of the universal necessity of the moral itself and $\mathcal{D}_{17}$ represents (the main part of) the deduction of the universal necessity of being determinable by this law. With respect to $\mathcal{D}_{16}$ notice that: $\forall x^{hb}\forall x^{s_p}\exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner)\wedge O_i(\ulcorner\phi\urcorner)\wedge DG(\ulcorner\phi\urcorner,x^{hb},x^{s_p})] \vdash_{KL} \forall x^{hb}\forall x^{s_p}[PP(\ulcorner ML\urcorner)\wedge O_i(\ulcorner ML\urcorner)\wedge DG(\ulcorner ML\urcorner,x^{hb},x^{s_p})]$.)

The conclusion of the above deduction expresses the following: If the objective validity of freedom implies the universal validity and the necessity of the concept of freedom, then the objective validity of freedom implies the universal necessity of being determinable by the moral law. Thus far the only meaning provided to $O_p(\ulcorner FR\urcorner)$ has been $\forall x^{hb}\forall x^{s_p}FR(x^{hb},x^{s_p})$ and for this reason it can be concluded that $\forall x^{hb}\forall x^{s_p}DB(x^{hb},\ulcorner ML\urcorner,x^{s_p})$ is already provable from the provisory definition of objective validity. In other words, the universal validity and necessity of the concept of freedom is sufficient to prove the main claim of Kant's practical transcendental argument (Proposition 31).

Proof. The deduction that establishes Proposition 31 is identical to the natural deduction provided in Figure 7.6 except for the following adjustment: Proposition 30 must be substituted for Proposition 29 and Assumption 1. ■

In the previous section we saw that Proposition 30 is derivable from the reality of the moral law. In this section we saw that Proposition 30 can prove the main claim of Kant's transcendental argument. Furthermore, Proposition 30 exhibits the exact behaviour that is expected from the objective validity of freedom. Hence, this proposition is not merely a part of the definition of the objective validity of the concept freedom, it is in fact the sufficient definition.

Definition 7.7. (The Objective Validity of Freedom) Let $FR(x^{hb},x^{s_p})$ and $O_p(\ulcorner\phi\urcorner)$ be interpreted as in table 7.1.

$$O_p(\ulcorner FR\urcorner) \Leftrightarrow \forall x^{hb}\forall x^{s_p}FR(x^{hb},x^{s}) \qquad \dashv$$

Hence, we are justified in concluding that the concept of freedom is objectively valid from a practical point of view (Proposition 29):

Proof. The natural deduction provided in Figure 7.5 together with Definition 7.7 establishes Proposition 29. ■

This finishes the formalization of Kant's practical transcendental argument as found in the Critique of Practical Reason.

A Philosophical Recapitulation

The above formal elaboration provides the following philosophical definition: The concept of freedom is objectively valid (from a practical point of view) if and only if the concept of freedom is *universally and necessarily valid* (from a practical point of view). Accordingly, that freedom is universally and necessarily valid means that the concept of freedom necessarily applies to every human being's reason (again restricted to the practical viewpoint).

Definition 7.8. (Objective Validity) With respect to Kant's practical philosophy, the objective validity of a concept is defined as the universal validity and necessity of that concept.

With this definition at hand the following can be concluded with respect to Kant's objective validity argument. From the reality of the moral law the objective reality of the concepts of negative and positive freedom can be derived separately.[19] Together these two derivations enable the deduction of the objective validity of freedom. Hence, the formal structure of the objective validity argument as presented in the previous section confirms to the criteria of Postulate 13.

Furthermore, the formal analysis provided in this chapter suggests the following philosophical reading of Kant's proof for the possibility of the moral law as a synthetic a priori proposition: The objective validity of freedom implies that, from a practical point of view, the concept of freedom necessarily applies to every human being's will. The concept of freedom implies that a free will can be determined independently of natural causality and solely by practical law. Moreover, the concept of freedom provides a practical law as the product of a being's own (pure) reason. Thus, the objective validity of the concept of freedom makes it a priori possible that a human being is determinable by a law that originated from its own reason; namely, the moral law. That both the possibility of being determined by the moral law and the law itself result from a human being's own freedom implies that the synthesis between will and law is determined immediately. This proves Proposition 14. The above argument satisfies the criteria of Postulate 14.

[19]The two derivations are represented by the formal deductions $\mathcal{D}_{13}$ and $\mathcal{D}_{14}$ of Figure 7.5.

Intuitionistic versus Classical Reasoning

Why is Kant's reasoning intuitionistic in nature?[20] There are two main reasons for this claim. Firstly, if Kant's argument would have been represented in a classical logic, then the objective validity of freedom could already have been derived in the first phase of the possibility argument:

$$\vdash_{class.} \forall x^{S \setminus S^t} \neg\neg FR(x^{rb}, x^{S \setminus S_t}) \Rightarrow \forall x^{s_p} FR(x^{rb}, x^{S_p})$$

(NB. $\forall x[S^p(x) \rightarrow S \setminus S^t(x)]$)

From a philosophical point of view this means that the objective validity of freedom could already have been derived from the restriction of natural causality to the theoretical viewpoint. If this would have been the case, then Kant could already have shown the objective validity of freedom in his first Critique. Consequently, a classical framework would make both Kant's possibility and objective validity argument superfluous. The following two points must be made. On the one hand, the above 'classical' result contradicts Kant's statement that the first Critique could only show the possibility of freedom logically (see section 4.2.3). On the other hand, the above result contradicts Kant's statement that the reality of the moral law is the primary deductive principle of the objective validity of freedom. Hence, a classical interpretation contradicts the essential structure of Kant's transcendental argument.

The first reason is only negative and one might easily object to it by stating that Kant's reasoning is in fact classical and his arguments are just superfluous. However, there are also positive reasons for the refutation of a classical reading and the endorsement of an intuitionistic reading. The second reason concerns Kant's explicit restriction of the use of negation. Kant states that our consciousness of the unconditionally practical cannot arise from the concept of negative freedom *because* the concept is merely negative [Kan96a, 5:29]. A negative concept cannot provide positive insight into a concept:

> Through negation I have not extended the concept and cannot thereby have more distinct insight into the concept. An affirmative concept must be added, and deeper distinctness must be provided. [Kan92a, 24:836 - The Vienna Logic]

Furthermore, Kant states that in order to prove the possibility of the supersensible nature a negative determination of this nature is insufficient. Only the positive formulation of the moral law, as the fundamental law of this nature, can determine this nature positively [Kan96a, 5:42 and 5:48]. With respect to the second Critique Kant's restriction of the (logical) implications of negative

[20] Intuitionistic reasoning originated in the beginning of the 20^{th} century in the work of L.E.J. Brouwer [Iem15] and it would therefore be an anachronism to call Kant's logic intuitionistic. However, Kant's reasoning coincides with intuitionistic reasoning and for this reason I will refer to Kant's reasoning as being intuitionistic in *nature*.

concepts necessitates (1) the uncovering of the formulation of FLoP and (2) the positive and complete determination of the supersensible nature.[21]

The imposed restriction on negation substantiates a non-classical interpretation of Kant's reasoning. This can be explained in relation to Kant's distinction between the *affirmative, negative* and *infinite* judgment [Kan00, A70/B95]. The following three examples represent the judgment forms in question:

(i)▶ The soul is mortal.
(ii)▶ The soul is not mortal.
(iii)▶ The soul is non-mortal.

The intuitionistic nature of Kant's distinction becomes clear when we regard (i)-(iii) from a set theoretical point of view: Let M be the set of 'mortal things'. Let s be 'the soul'. The affirmative judgment (i) assigns the soul to the set of mortal things: $s \in M$. The negative judgment (ii) excludes the soul from this set of mortal things: $s \notin M$. The infinite judgment (iii) positively assigns the soul to the complement of the set of mortal things: $s \in \neg M$. Consider the following remark by Kant: "The infinite judgment indicates not merely that a subject is not contained under the sphere of a predicate", as is the case with the negative judgment, "but that it lies somewhere in the infinite sphere outside its sphere; consequently this judgment represents the sphere of the predicate as restricted" [Kan92a, 9:104 - Jäsche Logic]. In other words, Kant distinguishes an infinite judgment that *places* an object in the complement of a certain sphere from a negative judgment that *denies* an object a certain sphere.

Two remarks must be made: Firstly, Kant's distinction between negative and infinite judgments cannot be maintained in a classical framework, but can be maintained in an intuitionistic framework. Namely, the law of excluded middle implies the merging of these judgment forms: $\forall x[M(x) \vee \neg M(x)]$ implies $\forall x[x \notin M \rightarrow x \in \neg M]$. Secondly, Kant's approach to negation coincides with the intuitionistic definition of negation (i.e. $\neg\phi = \phi \rightarrow \bot$). Namely, Kant's negative judgment only functions to prevent contradiction: "[I]n regard to the content of our cognition in general [...] negative judgments have the task solely of preventing error" [Kan00, A709/B737]. Hence, the restriction of negation shows that Kant's reasoning must be regarded as intuitionistic in nature.[22]

The above elaboration shows the importance of a proper determination of Kant's underlying syntax. A classical interpretation generates a completely different (and even contradictory) interpretation of Kant's transcendental reasoning and only via an intuitionistic interpretation the original course of Kant's argument can be maintained.

[21] In chapter III of the Groundwork Kant states that the concept of freedom (i.e. autonomy) cannot be derived from the negative concept of freedom alone because the concept is only "*negative* and therefore unfruitful for insight into its essence" [Kan96b, 4:446].

[22] The interested reader is referred to [Kan00, A77/B98], [Kan92a, 9:103/9:104 - Jäsche Logic] and [Han14].

A Possible Contradiction

Let the set TA be the collection of definitions, axioms and assumed propositions used in the formal deductions of the possibility and objective validity argument:

TA = { Definition 7.3, Definition 7.4, Definition 7.5, Definition 7.6, Definition 7.7, Axiom 1, Axiom 2, Axiom 3, Axiom 4, Axiom 5, Axiom 6, Axiom 7, Axiom 8, Axiom 9, Axiom 10, Axiom 11, Proposition 15, Proposition 17, Proposition 19, Proposition 21, Proposition 22, Proposition 25, Proposition 26 }

This chapter showed that the proposition $O_p(\ulcorner FR \urcorner)$ is provable from TA in the logic KL (Aim II of Postulate 1). Furthermore, at the end of this section we saw that $\forall x^{hb} \forall x^{s_p} DB(x^{hb}, \ulcorner ML \urcorner, x^{s_p})$ is provable from $O_p(\ulcorner FR \urcorner)$ (Aim I of Postulate 1). The following can be concluded:

$$\text{TA} \vdash_{KL} \forall x^{hb} \forall x^{s_p} DB(x^{hb}, \ulcorner ML \urcorner, x^{s_p})$$

Hence, given the formal interpretation provided in this chapter, it can be concluded that in the logic KL the possibility of the moral law as a synthetic a priori proposition is provable from Kant's practical transcendental argument. If we can also prove that the set of derived Kantian axioms is consistent, then we have shown that Kant's argument is consistent.[23] However, to prove the consistency of the axioms would be beyond the scope of the present undertaking. (In fact, it could even be impossible to prove the consistency of these axioms.) As a last remark I would like to address a philosophical problem put forward by Nelkin in the article *Two Standpoints and the Belief in Freedom.* The problem concerns an inconsistency that seems to arise in relation to Kant's distinction between the sensible and supersensible nature. I will show why at least the inconsistency of this known problem does not arise in the present formal framework. With respect to consistency and the purpose of this thesis this will suffice.

There are two apparent contradictory propositions in Kant's philosophy: (i) A rational being's will is subject to the law of natural causality (and can as such never be free) and (ii) a rational being's will is free and can be determined independently of natural causality. Kant's distinction between the supersensible and sensible nature dissolves the contradiction: From a theoretical point of view a rational being is not free and from a practical point of view a rational being is free. Proposition (i) can only be asserted from a viewpoint at which proposition (ii) cannot be asserted and vice versa. In the literature Kant's solution for the compatibility of natural causality and freedom has been called

[23]NB. If the set of axioms is inconsistent, then everything would follow from this set by *ex falso quodlibet*; i.e. the principle of explosion '$\perp \rightarrow \phi$'.

a ‘two-worlds account’, ‘two-aspects account’ and a ‘two-standpoints account’ [Nel00].[24] In the present formal framework Kant’s solution is generated by the mutual exclusiveness of the sets S^t and S^p: Proposition (i) can only be asserted in $s \in S^t$ and proposition (ii) can only be asserted in $s \in S^p$. Consequently, there does not exist a situation $s \in S$ in which both (i) and (ii) can be asserted. However, indirectly it still seems to be possible to construct a situation $s \in S$ in which both $sens(s) \wedge supersens(s)$ hold; this would be a contradiction.

According to Nelkin, Kant’s solution for the incompatibility of freedom and natural causality (i.e. the introduction of the viewpoint distinction) causes the following problem:

> [I]f I am deliberating about whether to sound a fire alarm, one of the things I rely on is my belief about what effects that action is likely to have. Does this mean, then, that my belief about the causal role of alarm sounding is a belief from the standpoint of the deliberator [the practical viewpoint]? It certainly seems so, for the belief seems quite “relevant” to my deliberative task. And if so, then it would appear that either I have two beliefs with similar contents that are distinguished by the points of view from which they are held, or I have a single belief that floats freely back and forth between standpoints. [Nel00, p.]

With respect to the present framework the problem can be reformulated as follows: Whenever a rational being regards the moral law as the determining ground of its will this is done with respect to that being’s supersensible nature. That the moral law is the determining ground of the will implies that the will of this being is determined to the action put forward by its maxim if and only if this maxim ‘could hold at the same time as a principle in a giving of universal law’. However, the maxim in question might consist of propositions that depend on natural causality (e.g. practical rules concerning the effects of ‘sounding a fire alarm’). That is, the maxim might depend on theoretical propositions. In that case the maxim would depend on the rational being’s sensible nature. Hence, with respect to such a maxim the proposition ‘the moral law is the determining ground of the will’ can only be evaluated in a

[24]The above accounts differ in the status of the distinction [Nel00, p.564-565]. For example, on the two-worlds account Kant’s distinction is regarded as an ontological distinction. The accounts provided by Nelkin and Beck reject the ontological reading of Kant’s distinction. Nelkin’s two-standpoints reading expresses a *doxastic* distinction: “[W]e are not irrational in believing that we are free and undetermined, on the one hand, and believing that we are determined and so unfree, on the other, because we hold the apparently contradictory beliefs from different standpoints” [Nel00, p.567]. Beck, on the other hand, advocates that Kant’s distinction is purely *methodological*: “Instead of thinking of two worlds, one noumenal and one phenomenal, Kant is here thinking of one world under two aspects. [...] The noumenal and the phenomenal are not ontologically distinct [...] but are aspects determined by methodological procedures chosen with regard to the divergent purposes of two kinds of inquiry” [Bec87, p.44]. (In the same paragraph Beck refers to these aspects as ‘the scientific point of view’ and the ‘practical point of view’.) The interpretation of Kant’s distinction between the sensible and supersensible nature provided in this thesis seems to coincide with that of Beck.

nature that is both sensible and supersensible. In the Kantian framework this would be a contradiction.

In a formal context the above problem can be sketched as follows:

1► Let ϕ^* represent a practical principle (maxim) that consists of some theoretical proposition(s): $PP(\ulcorner \phi^* \urcorner)$ and $matter(\ulcorner \phi^* \urcorner, x^s)$.

2► That $PP(\ulcorner \phi^* \urcorner)$ consists of theoretical propositions, that is matter, implies that $PP(\ulcorner \phi^* \urcorner)$ depends on the sensible nature: $sens(x^s)$.

3► Let ML^* be the application of the moral law with respect to the practical principle ϕ^* and let the moral law be the determining ground the will: $DG(\ulcorner ML^* \urcorner, x^{rb}, x^s)$.[25]

4► To have the moral law as the determining ground of the will, that is $DG(\ulcorner ML^* \urcorner, x^{rb}, x^s)$, implies that this determining ground is regarded with respect to the being's supersensible nature: $supersens(x^s)$.

5► To have the moral law as the determining ground of the will also implies that it must be determined whether $PP(\ulcorner \phi^* \urcorner)$ can hold as a principle in a universal lawgiving. Hence, $DG(\ulcorner ML^* \urcorner, x^{rb}, x^s)$ depends on $PP(\ulcorner \phi^* \urcorner)$ and therefore $DG(\ulcorner ML^* \urcorner, x^{rb}, x^s)$ depends on $sens(x^s)$.

6► Point 4 and 5 together imply that, with respect to $DG(\ulcorner ML^* \urcorner, x^{rb}, x^s)$, x^s must validate both $sens(x^s)$ and $supersens(x^s)$: $\bot$.

The argument depends on the following assumption: If the moral law is the determining ground of a being's will with respect to that being's maxim, then the maxim in question is asserted in the same nature in which the moral law is asserted as a determining ground of that being's will. This causes a single maxim to, what Nelkin calls, 'float freely back and forth between standpoints'.

Whether the above contradiction is generated in the present logic depends on the status of a practical principle in relation to the moral law. There are two inferences at work: Firstly, if a practical principle $PP(\ulcorner \phi^* \urcorner)$ is regarded in an application of the moral law this can only be done in a situation that belongs to the supersensible nature. Secondly, if $PP(\ulcorner \phi^* \urcorner)$ contains theoretical propositions, the principle enforces that the situation in question belongs to the sensible nature. The contradiction only arises when $PP(\ulcorner \phi^* \urcorner)$ is actually *asserted* in the situation in question. Only then the formal consequences of this practical principle can be derived. However, the present formal framework makes use of Gödel-numbers as representatives of formulas. When $PP(\ulcorner \phi^* \urcorner)$ is only referred to via the use of its Gödel-number $\ulcorner PP(\ulcorner \phi^* \urcorner) \urcorner$ the second inference will not be initiated.

[25]The application of the moral law is of course much more complicated, but with respect to the present problem the above will suffice.

The use of Gödel-numbers allows us to 'talk about' or 'refer to' a proposition from another viewpoint without asserting the proposition in the viewpoint in question (e.g. from the theoretical point of view we can talk about the unknowability of the proposition 'the will of a rational being is free'). Thus, we can talk about the possible universality of some maxim in relation to the moral law without asserting the maxim itself in the supersensible nature. Hence, whether the contradiction is generated in the present logic depends on whether the material practical principle in question is only *referred to* or actually *asserted* when regarded in relation to the moral law as the determining ground of the will. However, there seems to be no explicit justification why the practical principle in question should actually be asserted.

If the distinction between 'asserting' and 'talking about' is not made, then Nelkin's two options arise: (i) Either there are two identical maxims that are isomorphic, but which belong to two different natures or (ii) a maxim can float freely between the supersensible and the sensible nature. The first possibility seems to be highly artificial and raises only new questions, while the latter possibility generates the contradiction. Finally, that the contradiction, implied by Nelkin's problem, does not occur within the present logical framework, does not mean that the philosophical questions concerning this problem are answered. On the contrary, the distinction between, as well as the status of, Kant's theoretical and practical viewpoint remains obscure. Nevertheless, with respect to the present thesis the above will suffice.[26]

[26]I will not elaborate on the philosophical debate on Kant's theoretical and practical reason. The exposition of Kant's theoretical and practical viewpoint provided in section 4.2.1 will suffice for the purpose of this undertaking. The interested reader is referred to [Bec87], [Guy89], [Kle98] and [Nel00].

Chapter 8

Conclusion

This thesis started with one claim and one aim. In the philosophical part of the thesis I provided a proof for the claim: There is a transcendental argument to be found in Kant's Critique of Practical Reason. I proposed a model for detecting, interpreting and evaluating transcendental arguments in general. The model consists of two aims: The primary aim of a transcendental argument is to show the possibility of some synthetic a priori cognition. This possibility is shown on the basis of the objective validity of that cognition's ground. The secondary aim of the argument is to deduce the objective validity of that ground. This is established on the basis of three arguments: a necessity argument, a possibility argument and an objective validity argument.

On the basis of this model I argued that Kant's argument for 'the possibility of the moral law as a synthetic a priori proposition' is in fact a transcendental argument. I showed that Kant's reasoning in book I of the Critique of Practical Reason (to be more precise 5:19 – 5:35 and 5:42 – 5:50) conforms to the three major arguments of the proposed model. In short, Kant first shows that the concept of freedom is the necessary condition of the moral law. Secondly, he shows that this concept of freedom is the ground of the moral law. Thirdly, he proves the objective validity of this concept from the reality of the law. On the basis of these arguments Kant concludes that "the human will is by virtue of its freedom immediately determinable by the moral law" [Kan96a, 5:38].

Proposition 32. The exposition of the possibility of the moral law as a synthetic a priori proposition as presented by Kant in book I of the Critique of Practical Reason is a transcendental argument.

Proof. With respect to the criteria presented in Postulate 2 the following can be concluded: The proof of Proposition 14 satisfies criterion (1). The arguments presented in chapter 3, 4 and 5, which provided the proof for Proposition 13, satisfy criterion (2). Criterion (3) is satisfied by Proposition 5. Lastly, criterion (4) is satisfied by the argument for Proposition 14, which showed the necessary and a priori connection (i.e. synthesis) between the will and the moral law. Hence, Kant's argument for Proposition 14 satisfies the criteria provided in Postulate 2 and can therefore be called a transcendental argument. ■

The main aim of this thesis was established in the second part of this thesis. In this part the logical formalization of Kant's practical transcendental argument was provided. The logic used for the formalization is an *intuitionistic many-sorted type-free situation calculus* named KL.

One of the main results of the logical analysis of Kant's argument is that Kant's reasoning in the second Critique is essentially intuitionistic. This can be concluded on the basis of three reasons. The first two reasons are direct evidence for intuitionistic reasoning, the last one is only indirect. Firstly, with respect to the possibility of the concept of freedom, Kant makes a clear distinction between 'not-impossible' and 'possible'. The inference from the former to the latter is not trivial and Kant provides an additional (positive) proof that validates the inference. Consequently, with respect to Kant's reasoning the formula '$\neg\neg\phi \rightarrow \phi$' is not an axiom. Secondly, Kant's proof for the possibility of the supersensible nature requires a positive determination of this nature's content. This is another instance of Kant's reasoning in which a negative result can only be turned into a positive result via a positive construction. Both reasons depend on Kant's restriction of negation. Lastly, in the Critique of Practical Reason explicit usage of the law of excluded middle does not occur. Moreover, Kant's practical transcendental argument can be formally represented without the use of LEM. Furthermore, if LEM would have been valid, several of Kant's arguments in the second Critique would be superfluous.

Finally, as a major result of this endeavour, the logical formalization of Kant's argument allowed for the formal deduction and determination of the definition of the concept of (practical) objective validity. Although this concept remained obscure during the philosophical analysis of the argument, and treatment of the objective validity argument had to be postponed, the logical formalization of the argument provided proper insight into its essence. The logical formalization showed that the practical objective validity of the concept of freedom must be defined in terms of the *universal validity* and *necessity* of the concept. Its demonstration was twofold: (i) The logical formalization showed that the universal validity and necessity of the concept of freedom can be formally deduced from the reality of the moral law and (ii) it showed that the universal validity and necessity of the concept is sufficient to prove the main claim of Kant's practical transcendental argument.

With this thesis I hope to have shown that logical analysis of philosophical texts can support and further understanding of philosophical arguments. Proper understanding of philosophical theories begins with a proper understanding of the underlying syntax and conceptual-apparatus. For example, if the law of excluded middle would have been valid in Kant's reasoning, many sub-arguments of the above represented transcendental argument would have been superfluous. Logical analysis can at least help to clarify these points. (Not every philosophical theory is as suitable for formal analysis as Kant's philosophy. Nevertheless, many theories in metaphysics and analytical philosophy will lend themselves for such an endeavour, since they lean heavily on structured argumentation.)

The present undertaking has some weaknesses. In every research there is a compromise to be made in relation to time, size and readability. I have tried to minimize the loss of detail throughout this thesis, though I am aware that crucial (and perhaps even controversial) details can always be found. Nonetheless, I think that with respect to the choices made in this thesis, some very interesting conclusion arose.

Problems and Further Research

The above undertaking is not problem-free. I will briefly discuss two problems that arose during the formalization process. These problems might be relevant for further research on Kant's (logical) reasoning. The first problem relates to Kant's notion of a *disjunctive judgment*. Recall that a disjunctive judgment expresses the relation between the elements of a whole sphere and its parts. The disjunctive judgment seems to expresses a partitioning: $\forall x[x \in \phi \rightarrow [x \in \psi_1 \vee x \in \psi_i \vee ... \vee x \in \psi_n]]$ (provided ϕ is a disjoint union of its subsets $\psi_1, \psi_i ... \psi_n$). From an intuitionistic point of view such a judgment would be invalid because it presupposes a general proof that can decide for every (possible) element of the whole sphere to which of its parts it belongs. Kant's distinction between matter and form and between the empirical and the rational origin of a cognition can be interpreted in the light of a disjunctive judgment (see Postulate 5 and Postulate 4). The present analysis only employs that part of Kant's distinction that is compatible with intuitionistic reasoning. However, if a Kantian disjunctive judgment turns out to be a genuine partitioning, the relation between this judgment form and Kant's reasoning in the second Critique would have to be re-evaluated.

The second problem relates to the first one. In the first Critique Kant introduces the principle of *determinability*. The principle states that only one of every two contradictory predicates *can* apply to a concept. Apart from its relation to Kant's remark that there is no lowest species, this principle does not seem to be intuitionistically problematic (see footnote to page 98). However, in a footnote to the passage in which the principle is introduced Kant states the following: "The determinability of every single concept is the universality [...] of the principle of excluded middle between two opposed predicates" [Kan00, Footnote to A572/B600]. Kant's use of 'determinability' in the above quote refers to the *complete* determinability of a concept in relation to every possible pair of these contradictorily opposed predicates. Kant calls this form determinability the universality of the law of excluded middle. Does

Kant endorse this universal determinability? In relation to this passage the answer remains undetermined. In the Jäsche logic, though, Kant states that the law of excluded middle is one of the three principles of the logical criteria of truth [Kan92a, Jäsche Logic - 9:53]. In Kant's first Critique LEM is mentioned only once in a footnote (see the above quote). In Kant's second Critique there is no mention of this principle at all. So, what is the status of this statement from the Jäsche Logic? And most of all, what is its status with respect to Kant's Critiques? Future research might shed light on this confusing observation.[1]

Future research might also be directed to the determination of the consistency of the set of Kantian axioms derived and used in the formalization of Kant's argument.

Furthermore, future research might be directed to the development of a logical system for the evaluation of (Kantian) transcendental arguments in general. On a more philosophical level it would be interesting to determine whether the transcendental arguments as found in Kant's Critique of Pure Reason conform to the working model as presented at the beginning of this thesis. Moreover, logical analysis of these theoretical transcendental arguments might further understanding of the (logical) structure of transcendental arguments in general and the differences between Kant's use of the argument with respect to his practical and theoretical philosophy.

Lastly, I would like to stress the following interesting phenomenon that occurs in Kant's first Critique. The arguments provided by Kant in the 'Analogies of Experience' (see [Kan00, A176-A218/B218-B265]) are transcendental in nature. In short, in the Analogies Kant proves that the categories of relation form the a priori grounds of the possibility of experience. In the B edition of this Critique though, Kant added a completely modified proof at the beginning of every analogy (four in total). The proofs from the A edition remained unaltered. Why did Kant include new proofs? Apparently there is nothing wrong with the proofs from the A edition because they are not excluded from the B edition. Moreover, the A edition proofs do not seem to be incomplete; both series of proofs use the same premisses and have the same conclusions.[2] The answer must be found in the logical structure of these series of proofs. In Kant's proofs of the A edition several instances of the use of *reductio ad absurdum* can be encountered. Whereas this form of reasoning still plays a role for the validity of the arguments of the first edition, use of this proof method has been completely omitted in the reformulations of the proofs of the second edition: The latter proofs are (more) direct, positive, proofs. In intuitionistic logic the principle of *reductio ad absurdum* is invalid. Did Kant rewrite the

[1] An explanation might be found in relation to the idea that Kant uses different forms of logic with respect to different parts of his philosophy. In the Kantian literature there has already been a proposal for a dual logical interpretation of Kant's philosophy; namely, with respect to Kant's mathematical and theoretical reasoning (see [Pos92]).

[2] For the B edition Kant also reformulated the four claims made in the analogies. It can be shown though that the A and B claims eventually express the same proposition.

proofs of the analogies of experience because he changed his mind about the validity of *reductio ad absurdum* (with respect to transcendental arguments)? The above phenomenon could be explained as a change in the syntax underlying Kant's reasoning; a change that must have occurred somewhere between the publication of the first edition of the first Critique (1781) and the second edition (1787). Could this change be explained as an awareness of the necessity of positive construction in proofs? It would be most interesting for further research to examine whether there is in fact a switch in Kant's reasoning from a classical form of reasoning to a reasoning form that is, what we call, intuitionistic in nature.

How (un)Fortunate

After I finished the first final draft of this thesis I stumbled on the following two quotes from Kant's *Prolegomena to any future metaphysics*:

> [O]bjective validity of a judgment of experience signifies nothing other than its necessary universal validity. [Kan02, 4:298]
>
> Objective validity and necessary universal validity (for everyone) are therefore interchangeable concepts, and although we do not know the object in itself, nonetheless, if we regard a judgment as universally valid and hence necessary, objective validity is understood to be included. [Kan02, 4:298]

On first sight this discovery struck me as rather unfortunate: One of the main conclusions of the present formalization is the formal deduction of the definition of the concept of objective validity; and Kant just provides an exact definition himself (!). It is rather strange though that such a fundamental concept of Kant's critical works is so rarely explicitly defined. I realize that the discovery of Kant's definition can actually be seen as a confirmation of the accurateness of the presented logical formalization: The provided logical formalization of Kant's practical transcendental argument allowed for the deduction of Kant's exact definition. From this point of view the late discovery can be seen as a fortunate one. Moreover, the above definition does not only show something about the plausibility of the present formalization, it also shows that from a logical point of view the definition of the concept of objective validity is not just a posited definition of Kant's philosophy, but it is a strict logical consequence of Kant's transcendental reasoning. In other words, Kant's definition of objective validity is *necessary* with respect to his use of transcendental arguments.

Appendix A

Part I - The Structured Arguments

A1. (Theorem 2)

Claim: If a practical principle is material, then it belongs to the general principle of happiness or self-love.

(The principle of self-love consists of making happiness the supreme determining ground of the will [Definition 3.12]. A practical principle therefore belongs to the principle of self-love *only if* it serves the principle of happiness as well. It will therefore suffice to show that every material practical principle belongs to the principle of happiness.)

Take an arbitrary practical principle P.

Assume: P is a material principle.

To prove: P belongs to the general principle of happiness.

Proof:

3.3.1 From Definition 3.9 and the assumption we can infer that P has an object of desire as its determining ground.

3.3.2 From 3.3.1 and Definition 3.13 we can infer that P's ground is determined by the representation of the expected pleasure of the reality of an object.

3.3.3 From 3.3.2 and Definition 3.10 we can infer that P's practicality serves the agreeableness of one's state.

3.3.4 By Definition 3.11 we know that happiness is the sum of the agreeableness with one's state.

3.3.5 From 3.3.4 and 3.3.5 we can infer that P belongs to the general principle of happiness.

Since P is an arbitrary practical principle, we can conclude that every material practical principle belongs to the general principle of happiness, which proves the validity of Theorem 2. ■

A2. (Corollary 1')

Claim: All material practical principles have their determining ground in the faculty of desire and there are objective principles *if and only if* pure reason has a determining ground.

Proof: (of the left conjunct)

3.4.1 Follows immediately from Theorem 2 and Definition 3.9. □

Proof: (of the right conjunct)(from left to right)

Assume: There are objective principles.

To prove: Pure reason has a determining ground.

3.4.2 From Definition 3.3 and the assumption we can infer that there must be an objective determining ground.

3.4.3 From 3.4.2 and Definition 3.3 we can infer that the objective determining ground must be independent of any subjective conditions.

3.4.4 From 3.4.3 and Definition 3.5 we can infer that the objective determining ground cannot be empirical.

3.4.5 From 3.4.4 and Postulate 4 we can infer that the objective determining ground must have its origin in pure reason.

Proof: (of the right conjunct)(from right to left)

Assume: Pure reason has a determining ground.

To prove: There are objective principles.

3.4.6 From Definition 3.6 and the assumption we can infer that this determining ground must be independent of any subjective conditions.

3.4.7 From Definition 3.6 and the assumption we can infer that this determining ground must be necessary and universally valid.

3.4.8 From 3.4.6, 3.4.7 and Definition 3.3 we can infer that this determining ground must be an objective determining ground.

3.4.9 From 3.4.8 and Definition 3.3 we can infer that there is an objective principle. □

Hence Corollary 1' is valid. ■

A3. (Theorem 3)

Claim: If a rational being thinks of its maxim as a practical law, then it must think of this principle as a principle containing a determining ground that consists merely of the form of that principle as a law.

Let P *be a maxim and* x *a rational being.*

Assume: x thinks of P as a practical law.

To prove: x thinks of P as a principle containing a determining ground that consists merely of the form of that principle as a law.

Proof:

3.5.1 From Theorem 1 and the assumption we can infer (modes tollens) that x does not think of P as having an object of desire as its determining ground.

3.5.2 From 3.5.1 and Definition 3.9 we can infer that P's matter cannot be its determining ground.

3.5.3 From Definition 3.1 and the assumption we can infer that P must have a determining ground.

3.5.4 From 3.5.2 and 3.5.3 we can infer that P's ground must be non-material.

3.5.5 From 3.5.4 and Postulate 6 we can infer that only P's mere form can be its determining ground.

3.5.6 From 3.5.5, Definition 3.14 and the assumption we can infer that P's ground must be the mere form of P as a universal law.

Since x and P are arbitrary we have shown the validity of Theorem 3. ■

A4. (Problem 1)

Assumption: The mere lawgiving form of a maxim is the only sufficient determining ground of the will.[3]

To Prove: There is a constitution of the will that is determinable by the mere lawgiving form of a maxim.

Proof:

3.6.1 By Corollary 1' and Theorem 3 we know that 'mere form' can only be represented by pure reason.

3.6.2 From 3.6.1 and Definition 3.14 we can infer that 'mere form' cannot be an object of the senses.

3.6.3 From 3.6.2 and Definition 3.15 we can infer that this 'mere form' cannot be an appearance.

3.6.4 From 3.6.3 and Postulate 7 we can infer that this 'mere form' does not stand under the law of natural causality.

3.6.5 From 3.6.4 and the assumption we can infer that a will that can only be determined by this mere form, must be determined independent of natural causality.

3.6.6 From 3.6.5 and Definition 3.17 we can infer that this will must have the property of negative freedom.

Hence there is a constitution of a will that is determinable by the ground of the above assumption; namely, the negatively free will. ■

[3]The function of 'only sufficient' is to ensure that the result of the argument expresses the necessity of the derived property of the will with respect to this determining ground only.

A5. (Problem 2)

Assumption: Consider a will that has the property of negative freedom.

To Prove: There is a formulation of a practical law that alone can determine this will.

Proof:

3.6.7 By Definition 3.9 we know that the matter of a practical law is always empirical.

3.6.8 From Definition 3.17 and the assumption we can infer that this negatively free will is independent of empirical conditions.

3.6.9 From 3.6.7 and 3.6.8 we can infer that the matter of this law cannot determine this will.

3.6.10 From 3.6.9 and Postulate 6 we can infer that the determining ground of this will must consist of mere form.

3.6.11 From 3.6.10 and Theorem 3 we can infer that this will must be determined by 'the mere form of a law' contained in the principle.

Hence the mere lawgiving form, insofar it is contained in a principle of this will, is the only thing that can constitute the determining ground of a mere negatively free will. ■

A6. (Corollary 2)

Claim: Pure reason is practical of itself alone *and* for human beings FLoP is the moral law.

Proof: (left side of the conjunction)

3.8.1 From Definition 3.3 we know that objective principles must have an objectively valid ground.

3.8.2 From Corollary 1' we know that the ground of an objective principle must have its origin in pure reason.

3.8.3 From 3.8.1, 3.8.2 and FLoP we can infer that pure reason provides an objective ground.

3.8.4 From 3.8.3 we can infer that pure reason can be practical of its own. □

Proof: (right side of the conjunction)

3.8.5 From Definition 3.3 we know that an objective principle puts forward an action as necessary.

3.8.6 From Postulate 8 we know that a human being has the ability to create its own principles.

3.8.7 From Postulate 8 we know that a human being has sensible needs which it must satisfy.

3.8.8 From 3.8.6 and 3.8.7 we can infer that a human being can create principles that have objects of desire as their determining ground.

3.8.9 From 3.8.8 and Theorem 1 we can infer that a human being can create maxims that conflict with objective principles.

3.8.10 From 3.8.3 and 3.8.9 we can infer that a human being is not necessarily determined by an objective principle.

3.8.11 From 3.8.5 and 3.8.10 we can infer that for a human being the necessary action provided by the fundamental law of pure practical reason is only what ought to be.

3.8.12 from 3.8.11 and Definition 3.18 we can infer that for human beings FLoP is the moral law. □

Hence we can conclude that Corollary 2 is valid. ■

A7. (Theorem 4)

Claim: The will of a human being can be determined according to the moral law, only if that will is free.

Proof:

3.10.1 By Problem 1 we know that a will can be determined by a practical law only if the will is negatively free.

3.10.2 By FLoP and Theorem 3 we know that a will can be determined by a practical law only if it regards its maxims as capable of being a universal law.

3.10.3 From 3.10.2 and Definition 3.19 we can infer that a will can be determined by a practical law only if the will is positively free.

3.10.4 From 3.10.1, 3.10.3 we can infer that a will can be determined by a practical law only if the will is negatively and positively free.

3.10.5 From 3.10.4 and Definition 3.20 we can infer that a will can be determined by a practical law only if the will is free.

3.10.6 From 3.10.5 and Corollary 2 we can infer that a human being can be determined by the moral law only if the will is free.

Hence we have shown the validity of Theorem 4. ■

A8. (Proposition 8)

Claim: The concept of freedom is impossible from a theoretical point of view *and* only from a non-theoretical point of view it is not impossible.

Proof: (Left side of the conjunction.)

4.2.1 By Postulate 10 we know that the concept of (natural) causality has objective validity from a theoretical point of view.

4.2.2 From Definitions 3.16 and 3.20 we can infer that the concepts of (natural) causality and freedom are mutually exclusive.

4.2.3 From 4.2.1 and 4.2.2 we can infer that the concept of freedom is impossible from a theoretical point of view. □

Proof: (Right side of the conjunction.)

4.2.4 By Postulate 10 we know that the concept of (natural) causality does not apply beyond the theoretical point of view.

4.2.5 By Postulate 11 we know that only the concept of (natural) causality makes the concept of freedom impossible.

4.2.6 From 4.2.4 and 4.2.5 we can infer that the concept of freedom is not impossible beyond the theoretical point of view; that is, from a non-theoretical point of view. □

Hence, we can conclude to the validity of Proposition 8. ■

A9. (Proposition 9)

Claim: A supersensible nature is possible.

Proof:

4.2.7 By Corollary 2 we know that pure reason is practical and FLoP is its fundamental law.

4.2.8 By Corollary 2 we know that the moral law is FLoP for human beings.

4.2.9 From 4.2.7 and 4.2.8 we can infer that for human beings the moral law is the fundamental law of pure practical reason.

4.2.10 By FLop and Corollary 2 we know that the moral law is possible.

4.2.11 From 4.2.9 and 4.2.10 we can infer that a viewpoint of pure practical reason is possible.

Hence, we can infer the validity of Proposition 9. ■

A10. (Proposition 10)

Claim: The concept of freedom is possible from a practical point of view.

Proof:

4.2.12 By Proposition 9 we know that a practical viewpoint is possible.

4.2.13 From Postulate 10 and 4.2.12 we can infer that a viewpoint independent from natural causality is possible.

4.2.14 By Corollary 2 we know that FLoP is the fudamental law of this viewpoint.

4.2.15 By FLop and Definition 3.19 we know that FLoP is the idea of a being capable of lawgiving of its own.

4.2.16 From 4.2.13, 4.2.14 and 4.2.15 we can infer that this idea of a being capable of lawgiving of its own and an independence from natural causality are possible from this point of view.

4.2.17 From Definition 3.17, Definition 3.19 and 4.2.16 we can infer that positive and negative freedom are possible from this point of view.

4.2.18 From 4.2.17 and Definition 3.20 we can infer that the concept of freedom is possible from this point of view.

Hence, we can conclude that Proposition 10 is valid. ■

A11. (Proposition 11)

Claim: With respect to human beings, the concept of freedom is sufficient to generate both the moral law and the possibility of being determined by this law.

Let x be a human being

Assume: x has a free will.

To Prove: x generates the moral law and x can be determined by this law.

Proof: (left side of the conjunction)

4.3.1 By Definition 3.20 we know that freedom consists of positive and negative freedom.

4.3.2 From 4.3.1 and the assumption we can infer that x is positively free.

4.3.3 By Definition 3.19 and FLoP we know that the fundamental law of pure reason can be derived from the concept of positive freedom.

4.3.4 From 4.3.2, 4.2.3 and Corollary 2 we can infer that the moral law can be derived from x's positive freedom (as the law of its pure reason). □

Proof: (right side of the conjunction)

4.3.5 From 4.3.1 and the assumption we can infer that x is negatively free.

4.3.6 By Proposition 3 we know that a will that is negatively free can be determined practical law.

4.3.7 From 4.3.5 and 4.3.5 we can infer that x can be determined by practical law.

4.3.8 From 4.3.4 and By 4.3.7 we can infer that x can be determined the moral law. □

Since x is arbitrary we have shown the validity of Proposition 11. ■

Appendix B

Part II - The Natural Deductions

B1.

(Section 7.2.1) The formal deduction $\mathcal{D}_1$ belonging to the proof of Proposition 18. (Deduction $\mathcal{D}_1$ is similar to $\mathcal{D}_2$ and for this reason I will omit explicit elaboration of the latter.)

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\dfrac{\vdots}{caus(x^{rb}, x^{s}) \rightarrow \neg NF(x^{rb}, x^{s})}
}{\forall x^{s}[caus(x^{rb}, x^{s}) \rightarrow \neg NF(x^{rb}, x^{s})]}\;\forall\text{I}
}{\forall x[S(x) \rightarrow [caus(x^{rb}, x) \rightarrow \neg NF(x^{rb}, x)]]}\;\text{Axiom 1}
}{S(x) \rightarrow [caus(x^{rb}, x) \rightarrow \neg NF(x^{rb}, x)]}\;\forall\text{E}
\qquad
\dfrac{
\dfrac{
\dfrac{\text{Axiom 2}}{\forall x[S^{t}(x) \rightarrow S(x)]}
}{S^{t}(x) \rightarrow S(x)}\;\forall\text{E}
\qquad S^{t}(x)^{u}
}{S(x)}\;\rightarrow\text{E}
}{caus(x^{rb}, x) \rightarrow \neg NF(x^{rb}, x)}\;\rightarrow\text{E}
}{S^{t}(x) \rightarrow [caus(x^{rb}, x) \rightarrow \neg NF(x^{rb}, x)]}\;\rightarrow\text{I},u
}{\forall x[S^{t}(x) \rightarrow [caus(x^{rb}, x) \rightarrow \neg NF(x^{rb}, x)]]}\;\forall\text{I}
}{\forall x^{s_t}[caus(x^{rb}, x^{s_t}) \rightarrow \neg NF(x^{rb}, x^{s_t})]}\;\text{Axiom 2}
}{\mathcal{D}_1 :\; caus(x^{rb}, x^{s_t}) \rightarrow \neg NF(x^{rb}, x^{s_t})}\;\forall\text{E}
$$

B2.

(Section 7.2.2) The formal deduction that establishes Proposition 23.

$\theta = \forall x^{rb}[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner, x^{rb}, x^s)]$

$$\dfrac{\theta^v \text{ (rep.)}}{PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner, x^{rb}, x^s)}\ \forall\text{E}$$

$$\dfrac{PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner, x^{rb}, x^s)}{PP(\ulcorner\phi\urcorner)}\ \wedge\text{E}$$

$$\dfrac{\begin{matrix}\mathcal{D}_3\\ \vdots\\ PP(\ulcorner\phi\urcorner) \to \neg sens(x^s)\end{matrix} \qquad PP(\ulcorner\phi\urcorner)}{\neg sens(x^s)}\ \to\text{E}$$

$$\dfrac{\theta^v \text{ (rep.)} \qquad \neg sens(x^s)}{\neg sens(x^s) \wedge \theta}\ \wedge\text{I}$$

$$\dfrac{\neg sens(x^s) \wedge \theta \qquad \begin{matrix}\text{Axiom 7}\\ \mathcal{P}(\ulcorner\theta\urcorner)\end{matrix}}{\neg sens(x^s) \wedge \theta \wedge \mathcal{P}(\ulcorner\theta\urcorner)}\ \wedge\text{I}$$

$$\dfrac{\neg sens(x^s) \wedge \theta \wedge \mathcal{P}(\ulcorner\theta\urcorner)}{\exists\ulcorner\phi\urcorner[\neg sens(x^s) \wedge \theta \wedge \mathcal{P}(\ulcorner\theta\urcorner)]}\ \exists\text{I}$$

$$\dfrac{\exists\ulcorner\phi\urcorner[\neg sens(x^s) \wedge \theta \wedge \mathcal{P}(\ulcorner\theta\urcorner)]}{\exists x^s\exists\ulcorner\phi\urcorner[\neg sens(x^s) \wedge \theta \wedge \mathcal{P}(\ulcorner\theta\urcorner)]}\ \exists\text{I}$$

$$\dfrac{\exists\ulcorner\phi\urcorner\forall x^{rb}[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner, x^{rb}, x^s)]^u \qquad \exists x^s\exists\ulcorner\phi\urcorner[\neg sens(x^s) \wedge \theta \wedge \mathcal{P}(\ulcorner\theta\urcorner)]}{\exists x^s\exists\ulcorner\phi\urcorner[\neg sens(x^s) \wedge \theta \wedge \mathcal{P}(\ulcorner\theta\urcorner)]}\ \exists\text{E},v$$

$$\dfrac{\begin{matrix}\text{Proposition 22}\\ \exists x^s\exists\ulcorner\phi\urcorner\forall x^{rb}[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner, x^{rb}, x^s)]\end{matrix} \qquad \exists x^s\exists\ulcorner\phi\urcorner[\neg sens(x^s) \wedge \theta \wedge \mathcal{P}(\ulcorner\theta\urcorner)]}{\exists x^s\exists\ulcorner\phi\urcorner[\neg sens(x^s) \wedge \theta \wedge \mathcal{P}(\ulcorner\theta\urcorner)]}\ \exists\text{E},u$$

$$\dfrac{\begin{matrix}\text{Axiom 4}\\ \exists x^s\exists\ulcorner\phi\urcorner[\neg sens(x^s) \wedge \theta \wedge \mathcal{P}(\ulcorner\theta\urcorner)] \leftrightarrow \exists x^s supersens(x^s)\end{matrix}}{\exists x^s\exists\ulcorner\phi\urcorner[\neg sens(x^s) \wedge \theta \wedge \mathcal{P}(\ulcorner\theta\urcorner)] \to \exists x^s supersens(x^s)}\ \wedge\text{E}$$

$$\dfrac{\exists x^s\exists\ulcorner\phi\urcorner[\neg sens(x^s) \wedge \theta \wedge \mathcal{P}(\ulcorner\theta\urcorner)] \qquad \exists x^s\exists\ulcorner\phi\urcorner[\neg sens(x^s) \wedge \theta \wedge \mathcal{P}(\ulcorner\theta\urcorner)] \to \exists x^s supersens(x^s)}{\exists x^s supersens(x^s)}\ \to\text{E}$$

$$\dfrac{\exists x^s supersens(x^s)}{\exists x[S(x) \wedge supersens(x)]}\ \text{Axiom 2}$$

$$\dfrac{\begin{matrix}[w]\\ \mathcal{D}_4\ \vdots\\ S^p(x)\end{matrix}}{\exists x S^p(x)}\ \exists\text{I}$$

$$\dfrac{\exists x[S(x) \wedge supersens(x)] \qquad \exists x S^p(x)}{\exists x S^p(x)}\ \exists\text{E},w$$

Proposition 21

$$\dfrac{\forall\ulcorner\phi\urcorner[[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner)] \rightarrow \forall x^{rb}\forall x^{s}[DG(\ulcorner\phi\urcorner, x^{rb}, x^{s}) \rightarrow form(\ulcorner\phi\urcorner, x^{s})]]}{[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner)] \rightarrow \forall x^{rb}\forall x^{s}[DG(\ulcorner\phi\urcorner, x^{rb}, x^{s}) \rightarrow form(\ulcorner\phi\urcorner, x^{s})]}\ \forall\text{E}$$

$$\dfrac{\dfrac{\theta^{v}}{PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner, x^{rb}, x^{s})}\ \forall\text{E}}{PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner)}\ \wedge\text{E}$$

$$\dfrac{[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner)] \rightarrow \forall x^{rb}\forall x^{s}[DG(\ulcorner\phi\urcorner, x^{rb}, x^{s}) \rightarrow form(\ulcorner\phi\urcorner, x^{s})] \qquad PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner)}{\dfrac{\forall x^{rb}\forall x^{s}[DG(\ulcorner\phi\urcorner, x^{rb}, x^{s}) \rightarrow form(\ulcorner\phi\urcorner, x^{s})]}{DG(\ulcorner\phi\urcorner, x^{rb}, x^{s}) \rightarrow form(\ulcorner\phi\urcorner, x^{s})}\ \forall\text{E}}\ \rightarrow\text{E}$$

$$\dfrac{\dfrac{\theta^{v}\ (\text{rep.})}{PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner, x^{rb}, x^{s})}\ \forall\text{E}}{DG(\ulcorner\phi\urcorner, x^{rb}, x^{s})}\ \wedge\text{E}$$

$$\dfrac{DG(\ulcorner\phi\urcorner, x^{rb}, x^{s}) \qquad DG(\ulcorner\phi\urcorner, x^{rb}, x^{s}) \rightarrow form(\ulcorner\phi\urcorner, x^{s})}{form(\ulcorner\phi\urcorner, x^{s})}\ \rightarrow\text{E}$$

Axiom 5

$$\dfrac{\forall\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \rightarrow \forall x^{s}[form(\ulcorner\phi\urcorner, x^{s}) \rightarrow \neg matter(\ulcorner\phi\urcorner, x^{s})]]}{PP(\ulcorner\phi\urcorner) \rightarrow \forall x^{s}[form(\ulcorner\phi\urcorner, x^{s}) \rightarrow \neg matter(\ulcorner\phi\urcorner, x^{s})]}\ \forall\text{E}$$

$$\dfrac{PP(\ulcorner\phi\urcorner)^{z} \qquad PP(\ulcorner\phi\urcorner) \rightarrow \forall x^{s}[form(\ulcorner\phi\urcorner, x^{s}) \rightarrow \neg matter(\ulcorner\phi\urcorner, x^{s})]}{\dfrac{\forall x^{s}[form(\ulcorner\phi\urcorner, x^{s}) \rightarrow \neg matter(\ulcorner\phi\urcorner, x^{s})]}{form(\ulcorner\phi\urcorner, x^{s}) \rightarrow \neg matter(\ulcorner\phi\urcorner, x^{s})}\ \forall\text{E}}\ \rightarrow\text{E}$$

$$\dfrac{form(\ulcorner\phi\urcorner, x^{s}) \rightarrow \neg matter(\ulcorner\phi\urcorner, x^{s}) \qquad form(\ulcorner\phi\urcorner, x^{s})}{\neg matter(\ulcorner\phi\urcorner, x^{s})}\ \rightarrow\text{E}$$

Axiom 6

$$\dfrac{\forall\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \rightarrow \forall x^{s}[sens(x^{s}) \leftrightarrow matter(\ulcorner\phi\urcorner, x^{s})]]}{PP(\ulcorner\phi\urcorner) \rightarrow \forall x^{s}[sens(x^{s}) \leftrightarrow matter(\ulcorner\phi\urcorner, x^{s})]}\ \forall\text{E}$$

$$\dfrac{PP(\ulcorner\phi\urcorner) \rightarrow \forall x^{s}[sens(x^{s}) \leftrightarrow matter(\ulcorner\phi\urcorner, x^{s})] \qquad PP(\ulcorner\phi\urcorner)^{z}\ (\text{rep.})}{\forall x^{s}[sens(x^{s}) \leftrightarrow matter(\ulcorner\phi\urcorner, x^{s})]}\ \rightarrow\text{E}$$

$$\dfrac{\dfrac{\dfrac{\forall x^{s}[sens(x^{s}) \leftrightarrow matter(\ulcorner\phi\urcorner, x^{s})]}{sens(x^{s}) \leftrightarrow matter(\ulcorner\phi\urcorner, x^{s})}\ \forall\text{E}}{sens(x^{s}) \rightarrow matter(\ulcorner\phi\urcorner, x^{s})}\ \wedge\text{E}}{\neg matter(\ulcorner\phi\urcorner, x^{s}) \rightarrow \neg sens(x^{s})}\ \text{IL.2}$$

$$\dfrac{\dfrac{\neg matter(\ulcorner\phi\urcorner, x^{s}) \qquad \neg matter(\ulcorner\phi\urcorner, x^{s}) \rightarrow \neg sens(x^{s})}{\neg sens(x^{s})}\ \rightarrow\text{E}}{\mathcal{D}_3 : \quad PP(\ulcorner\phi\urcorner) \rightarrow \neg sens(x^{s})}\ \rightarrow\text{I},z$$

Axiom 3

$$\dfrac{S(x) \wedge supersens(x)^{w}}{S(x)}\ \wedge\text{E} \qquad \dfrac{\forall x[S(x) \rightarrow [[S^{t}(x) \leftrightarrow sens(x)] \wedge [S^{p}(x) \leftrightarrow supersens(x)]]]}{S(x) \rightarrow [[S^{t}(x) \leftrightarrow sens(x)] \wedge [S^{p}(x) \leftrightarrow supersens(x)]]}\ \forall\text{E}$$

$$\dfrac{S(x) \qquad S(x) \rightarrow [[S^{t}(x) \leftrightarrow sens(x)] \wedge [S^{p}(x) \leftrightarrow supersens(x)]]}{\dfrac{\dfrac{[S^{t}(x) \leftrightarrow sens(x)] \wedge [S^{p}(x) \leftrightarrow supersens(x)]}{S^{p}(x) \leftrightarrow supersens(x)}\ \wedge\text{E}}{supersens(x) \rightarrow S^{p}(x)}\ \wedge\text{E}}\ \rightarrow\text{E}$$

$$\dfrac{\dfrac{S(x) \wedge supersens(x)^{w}\ (\text{rep.})}{supersens(x)}\ \wedge\text{E} \qquad supersens(x) \rightarrow S^{p}(x)}{\mathcal{D}_4 : \quad S^{p}(x)}\ \rightarrow\text{E}$$

B3.

(Section 7.2.3) The formal deduction that establishes Proposition 24.

$[v]$ is the open assumption $\forall x^{rb}[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{rb}, x^s)$.

$$\dfrac{\exists\ulcorner\phi\urcorner\forall x^{rb}[PP(\ulcorner\phi\urcorner)\wedge O_i(\ulcorner\phi\urcorner)\wedge DG(\ulcorner\phi\urcorner,x^{rb},x^s)]^u \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}[v]\\ \mathcal{D}_5\\ \vdots\\ PF(x^{rb},x^s)\end{matrix}\qquad\begin{matrix}[v]\\ \mathcal{D}_6\\ \vdots\\ NF(x^{rb},x^s)\end{matrix}}{PF(x^{rb},x^s)\wedge NF(x^{rb},x^s)}\wedge\mathrm{I}}{\forall x^{rb}[PF(x^{rb},x^s)\wedge NF(x^{rb},x^s)]}\forall\mathrm{I}}{\exists x^s\forall x^{rb}[PF(x^{rb},x^s)\wedge NF(x^{rb},x^s)]}\exists\mathrm{I}}{\exists x^s\forall x^{rb}[PF(x^{rb},x^s)\wedge NF(x^{rb},x^s)]}\exists\mathrm{E},v$$

$$\dfrac{\overset{\text{Proposition 22}}{\exists x^s\exists\ulcorner\phi\urcorner\forall x^{rb}[PP(\ulcorner\phi\urcorner)\wedge O_i(\ulcorner\phi\urcorner)\wedge DG(\ulcorner\phi\urcorner,x^{rb},x^s)]} \qquad \exists x^s\forall x^{rb}[PF(x^{rb},x^s)\wedge NF(x^{rb},x^s)]}{\exists x^s\forall x^{rb}[PF(x^{rb},x^s)\wedge NF(x^{rb},x^s)]}\exists\mathrm{E},u$$

$$\dfrac{\exists x^s\forall x^{rb}[PF(x^{rb},x^s)\wedge NF(x^{rb},x^s)] \qquad \dfrac{\dfrac{\dfrac{\dfrac{\forall x^{rb}[PF(x^{rb},x^s)\wedge NF(x^{rb},x^s)]^w}{PF(x^{rb},x^s)\wedge NF(x^{rb},x^s)}\forall\mathrm{E} \qquad \dfrac{\dfrac{\overset{\text{Definition 7.5}}{\forall x^{rb}\forall x^s[FR(x^{rb},x^s)\leftrightarrow[PF(x^{rb},x^s)\wedge NF(x^{rb},x^s)]]}}{FR(x^{rb},x^s)\leftrightarrow[PF(x^{rb},x^s)\wedge NF(x^{rb},x^s)]}\forall\mathrm{E}}{[PF(x^{rb},x^s)\wedge NF(x^{rb},x^s)]\rightarrow FR(x^{rb},x^s)}\wedge\mathrm{E}}{FR(x^{rb},x^s)}\rightarrow\mathrm{E}}{\forall x^{rb}FR(x^{rb},x^s)}\forall\mathrm{I}}{\exists x^s\forall x^{rb}FR(x^{rb},x^s)}\exists\mathrm{I}}{\exists x^s\forall x^{rb}FR(x^{rb},x^s)}\exists\mathrm{E},w$$

$$\dfrac{\dfrac{\forall x^{rb}[PP(\ulcorner\psi\urcorner)\wedge O_i(\ulcorner\psi\urcorner)\wedge DG(\ulcorner\psi\urcorner,x^{rb},x^s)]^v}{\exists\ulcorner\psi\urcorner\forall x^{rb}[PP(\ulcorner\psi\urcorner)\wedge O_i(\ulcorner\psi\urcorner)\wedge DG(\ulcorner\psi\urcorner,x^{rb},x^s)]}\;\exists\text{I} \qquad \dfrac{\dfrac{\forall x^{rb}[PP(\ulcorner\psi\urcorner)\wedge O_i(\ulcorner\psi\urcorner)\wedge DG(\ulcorner\psi\urcorner,x^{rb},x^s)]^w}{PP(\ulcorner\psi\urcorner)\wedge O_i(\ulcorner\psi\urcorner)\wedge DG(\ulcorner\psi\urcorner,x^{rb},x^s)}\;\forall\text{E}}{\exists\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner)\wedge O_i(\ulcorner\psi\urcorner)\wedge DG(\ulcorner\psi\urcorner,x^{rb},x^s)]}\;\exists\text{I}}{\exists\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner)\wedge O_i(\ulcorner\psi\urcorner)\wedge DG(\ulcorner\psi\urcorner,x^{rb},x^s)]}\;\exists\text{E},w$$

Definition 7.4

$$\dfrac{\exists\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner)\wedge O_i(\ulcorner\psi\urcorner)\wedge DG(\ulcorner\psi\urcorner,x^{rb},x^s)] \qquad \dfrac{\dfrac{\forall x^s\forall x^{rb}[PF(x^{rb},x^s)\leftrightarrow\exists\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner)\wedge O_i(\ulcorner\psi\urcorner)\wedge DG(\ulcorner\psi\urcorner,x^{rb},x^s)]]}{PF(x^{rb},x^s)\leftrightarrow\exists\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner)\wedge O_i(\ulcorner\psi\urcorner)\wedge DG(\ulcorner\psi\urcorner,x^{rb},x^s)]}\;\forall\text{E}}{\exists\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner)\wedge O_i(\ulcorner\psi\urcorner)\wedge DG(\ulcorner\psi\urcorner,x^{rb},x^s)]\rightarrow PF(x^{rb},x^s)}\;\wedge\text{E}}{\mathcal{D}_5:\quad PF(x^{rb},x^s)}\;\rightarrow\text{E}$$

Proposition 21

$$\dfrac{\dfrac{\forall\ulcorner\psi\urcorner[[PP(\ulcorner\psi\urcorner)\wedge O_i(\ulcorner\psi\urcorner)]\rightarrow\forall x^{rb}\forall x^s[DG(\ulcorner\psi\urcorner,x^{rb},x^s)\rightarrow form(\ulcorner\psi\urcorner,x^s)]]}{[PP(\ulcorner\psi\urcorner)\wedge O_i(\ulcorner\psi\urcorner)]\rightarrow\forall x^{rb}\forall x^s[DG(\ulcorner\psi\urcorner,x^{rb},x^s)\rightarrow form(\ulcorner\psi\urcorner,x^s)]}\;\forall\text{E} \qquad \dfrac{\dfrac{\forall x^{rb}[PP(\ulcorner\psi\urcorner)\wedge O_i(\ulcorner\psi\urcorner)\wedge DG(\ulcorner\psi\urcorner,x^{rb},x^s)]^v\ \text{(rep.)}}{PP(\ulcorner\psi\urcorner)\wedge O_i(\ulcorner\psi\urcorner)\wedge DG(\ulcorner\psi\urcorner,x^{rb},x^s)}\;\forall\text{E}}{PP(\ulcorner\psi\urcorner)\wedge O_i(\ulcorner\psi\urcorner)}\;\wedge\text{E}}{\forall x^{rb}\forall x^s[DG(\ulcorner\psi\urcorner,x^{rb},x^s)\rightarrow form(\ulcorner\psi\urcorner,x^s)]}\;\rightarrow\text{E}$$

$$\dfrac{\dfrac{\forall x^{rb}\forall x^s[DG(\ulcorner\psi\urcorner,x^{rb},x^s)\rightarrow form(\ulcorner\psi\urcorner,x^s)]}{DG(\ulcorner\psi\urcorner,x^{rb},x^s)\rightarrow form(\ulcorner\psi\urcorner,x^s)}\;\forall\text{E} \qquad \dfrac{\dfrac{\forall x^{rb}[PP(\ulcorner\psi\urcorner)\wedge O_i(\ulcorner\psi\urcorner)\wedge DG(\ulcorner\psi\urcorner,x^{rb},x^s)]^v\ \text{(rep.)}}{PP(\ulcorner\psi\urcorner)\wedge O_i(\ulcorner\psi\urcorner)\wedge DG(\ulcorner\psi\urcorner,x^{rb},x^s)}\;\forall\text{E}}{DG(\ulcorner\psi\urcorner,x^{rb},x^s)}\;\wedge\text{E}}{form(\ulcorner\psi\urcorner,x^s)}\;\rightarrow\text{E}$$

Proposition 17

Identical to $\mathcal{D}_3$

$$\dfrac{PP(\ulcorner\psi\urcorner)\rightarrow\neg sens(x^s) \qquad \dfrac{\dfrac{\forall x^{rb}[PP(\ulcorner\psi\urcorner)\wedge O_i(\ulcorner\psi\urcorner)\wedge DG(\ulcorner\psi\urcorner,x^{rb},x^s)]^v\ \text{(rep.)}}{PP(\ulcorner\psi\urcorner)\wedge O_i(\ulcorner\psi\urcorner)\wedge DG(\ulcorner\psi\urcorner,x^{rb},x^s)}\;\forall\text{E}}{PP(\ulcorner\psi\urcorner)}\;\wedge\text{E}}{\neg sens(x^s)}\;\rightarrow$$

$$\dfrac{\dfrac{\forall x^{rb}\forall x^{s_t}caus(x^{rb},x^{s_t})\wedge\forall x^{rb}\forall x^{S\setminus S^t}\neg caus(x^{rb},x^{S\setminus S^t})}{\forall x^{rb}\forall x^{S\setminus S^t}\neg caus(x^{rb},x^{S\setminus S^t})}\;\wedge\text{E}}{\vdots\ \mathcal{D}_7}$$

$$\dfrac{\neg sens(x^s)\rightarrow\neg caus(x^{rb},x^s) \qquad \neg sens(x^s)}{\neg caus(x^{rb},x^s)}\;\rightarrow\text{E}$$

Definition 7.3

$$\dfrac{\neg caus(x^{rb},x^s) \qquad \dfrac{\dfrac{\forall x^s\forall x^{rb}[\neg caus(x^{rb},x^s)\leftrightarrow NF(x^{rb},x^s)]}{\neg caus(x^{rb},x^s)\leftrightarrow NF(x^{rb},x^s)}\;\forall\text{E}}{\neg caus(x^{rb},x^s)\rightarrow NF(x^{rb},x^s)}\;\wedge\text{E}}{\mathcal{D}_6:\quad NF(x^{rb},x^s)}\;\rightarrow\text{E}$$

Axiom 3

$$\forall x[S(x) \rightarrow [[S^t(x) \leftrightarrow sens(x)] \wedge [S^p(x) \leftrightarrow supersens(x)]]]$$

$$\forall\text{E}: \quad S(x) \rightarrow [[S^t(x) \leftrightarrow sens(x)] \wedge [S^p(x) \leftrightarrow supersens(x)]] \qquad S(x)^z$$

$$\rightarrow\text{E}: \quad [S^t(x) \leftrightarrow sens(x)] \wedge [S^p(x) \leftrightarrow supersens(x)]$$

$$\wedge\text{E}: \quad S^t(x) \leftrightarrow sens(x)$$

$$\wedge\text{E}: \quad S^t(x) \rightarrow sens(x)$$

$$\text{IL.2}: \quad \neg sens(x) \rightarrow \neg S^t(x) \qquad \neg sens(x)^w$$

$$\rightarrow\text{E}: \quad \neg S^t(x)$$

Axiom 3

$$\forall x[S\backslash S^t(x) \leftrightarrow \neg S^t(x)]$$

$$\forall\text{E}: \quad S\backslash S^t(x) \leftrightarrow \neg S^t(x)$$

$$\wedge\text{E}: \quad \neg S^t(x) \rightarrow S\backslash S^t(x)$$

$$\rightarrow\text{E}: \quad S\backslash S^t(x)$$

$$\vdots$$

$$\forall x^{rb} \forall x^{S\backslash S^t} \neg caus(x^{rb}, x^{S\backslash S^t})$$

$$\forall\text{E}: \quad \forall x^{S\backslash S^t} \neg caus(x^{rb}, x^{S\backslash S^t})$$

$$\text{Axiom 2}: \quad \forall x[S\backslash S^t(x) \rightarrow \neg caus(x^{rb}, x)]$$

$$\forall\text{E}: \quad S\backslash S^t(x) \rightarrow \neg caus(x^{rb}, x) \qquad S\backslash S^t(x)$$

$$\rightarrow\text{E}: \quad \neg caus(x^{rb}, x)$$

$$\rightarrow\text{I},w: \quad \neg sens(x) \rightarrow \neg caus(x^{rb}, x)$$

$$\rightarrow\text{I},z: \quad S(x) \rightarrow [\neg sens(x) \rightarrow \neg caus(x^{rb}, x)]$$

$$\forall\text{I}: \quad \forall x[S(x) \rightarrow [\neg sens(x) \rightarrow \neg caus(x^{rb}, x)]]$$

$$\text{Axiom 2}: \quad \forall x^s[\neg sens(x^s) \rightarrow \neg caus(x^{rb}, x^s)]$$

$$\forall\text{E}: \quad \mathcal{D}_7 : \quad \neg sens(x^s) \rightarrow \neg caus(x^{rb}, x^s)$$

B4.

(Section 7.2.4) The formal deductions $\mathcal{D}_8$, $\mathcal{D}_9$ and $\mathcal{D}_{10}$ as belonging to the proof of Proposition 27.

$\chi = \forall x^{rb} \forall x^s [NF(x^{rb}, x^s) \rightarrow \forall \ulcorner \phi \urcorner [[PP(\ulcorner \phi \urcorner) \wedge O_i(\ulcorner \phi \urcorner)] \rightarrow [DB(x^{rb}, \ulcorner \phi \urcorner, x^s) \leftrightarrow [DG(\ulcorner \phi \urcorner, x^{rb}, x^s) \wedge form(\ulcorner \phi \urcorner, x^s)]]]]$

$\theta = PP(\ulcorner \phi \urcorner) \wedge O_i(\ulcorner \phi \urcorner) \wedge DG(\ulcorner \phi \urcorner, x^{hb}, x^s)$

Definition 7.5

$$\dfrac{\forall x^{rb}\forall x^{s}[FR(x^{rb},x^{s}) \leftrightarrow [NF(x^{rb},x^{s}) \wedge PF(x^{rb},x^{s})]]}{FR(x^{rb},x^{s}) \leftrightarrow [NF(x^{rb},x^{s}) \wedge PF(x^{rb},x^{s})]}\ \forall\text{E}$$

$$\dfrac{FR(x^{rb},x^{s}) \leftrightarrow [NF(x^{rb},x^{s}) \wedge PF(x^{rb},x^{s})]}{\forall x^{rb}[FR(x^{rb},x^{s}) \leftrightarrow [NF(x^{rb},x^{s}) \wedge PF(x^{rb},x^{s})]]}\ \forall\text{I}$$

$$\dfrac{\forall x^{rb}[FR(x^{rb},x^{s}) \leftrightarrow [NF(x^{rb},x^{s}) \wedge PF(x^{rb},x^{s})]]}{\forall x[RB(x) \rightarrow [FR(x,x^{s}) \leftrightarrow [NF(x,x^{s}) \wedge PF(x,x^{s})]]]}\ \text{Axiom 1}$$

$$\dfrac{\forall x[RB(x) \rightarrow [FR(x,x^{s}) \leftrightarrow [NF(x,x^{s}) \wedge PF(x,x^{s})]]]}{RB(x) \rightarrow [FR(x,x^{s}) \leftrightarrow [NF(x,x^{s}) \wedge PF(x,x^{s})]]}\ \forall\text{E}$$

Axiom 8

$$\dfrac{\forall x[HB(x) \rightarrow RB(x)]}{HB(x) \rightarrow RB(x)}\ \forall\text{E}$$

$$\dfrac{HB(x) \rightarrow RB(x) \qquad HB(x)^{z}}{RB(x)}\ \rightarrow\text{E}$$

$$\dfrac{RB(x) \rightarrow [FR(x,x^{s}) \leftrightarrow [NF(x,x^{s}) \wedge PF(x,x^{s})]] \qquad RB(x)}{FR(x,x^{s}) \leftrightarrow [NF(x,x^{s}) \wedge PF(x,x^{s})]}\ \rightarrow\text{E}$$

$$\dfrac{FR(x,x^{s}) \leftrightarrow [NF(x,x^{s}) \wedge PF(x,x^{s})]}{HB(x) \rightarrow [FR(x,x^{s}) \leftrightarrow [NF(x,x^{s}) \wedge PF(x,x^{s})]]}\ \rightarrow\text{I},z$$

$$\dfrac{HB(x) \rightarrow [FR(x,x^{s}) \leftrightarrow [NF(x,x^{s}) \wedge PF(x,x^{s})]]}{\forall x[HB(x) \rightarrow [FR(x,x^{s}) \leftrightarrow [NF(x,x^{s}) \wedge PF(x,x^{s})]]]}\ \forall\text{I}$$

$$\dfrac{\forall x[HB(x) \rightarrow [FR(x,x^{s}) \leftrightarrow [NF(x,x^{s}) \wedge PF(x,x^{s})]]]}{\forall x^{hb}[FR(x^{hb},x^{s}) \leftrightarrow [NF(x^{hb},x^{s}) \wedge PF(x^{hb},x^{s})]]}\ \text{Axiom 8}$$

$$\dfrac{\forall x^{hb}[FR(x^{hb},x^{s}) \leftrightarrow [NF(x^{hb},x^{s}) \wedge PF(x^{hb},x^{s})]]}{FR(x^{hb},x^{s}) \leftrightarrow [NF(x^{hb},x^{s}) \wedge PF(x^{hb},x^{s})]}\ \forall\text{E}$$

$$\dfrac{FR(x^{hb},x^{s}) \leftrightarrow [NF(x^{hb},x^{s}) \wedge PF(x^{hb},x^{s})]}{FR(x^{hb},x^{s}) \rightarrow [NF(x^{hb},x^{s}) \wedge PF(x^{hb},x^{s})]}\ \wedge\text{E}$$

$$\dfrac{FR(x^{hb},x^{s}) \rightarrow [NF(x^{hb},x^{s}) \wedge PF(x^{hb},x^{s})] \qquad FR(x^{hb},x^{s})^{u}}{NF(x^{hb},x^{s}) \wedge PF(x^{hb},x^{s})}\ \rightarrow\text{E}$$

$$\dfrac{NF(x^{hb},x^{s}) \wedge PF(x^{hb},x^{s})}{PF(x^{hb},x^{s})}\ \wedge\text{E}$$

Definition 7.4

$$\dfrac{\forall x^{rb}\forall x^{s}[PF(x^{rb},x^{s}) \leftrightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{rb},x^{s})]]}{PF(x^{rb},x^{s}) \leftrightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{rb},x^{s})]}\ \forall\text{E}$$

$$\dfrac{PF(x^{rb},x^{s}) \leftrightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{rb},x^{s})]}{\forall x^{rb}[PF(x^{rb},x^{s}) \leftrightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{rb},x^{s})]]}\ \forall\text{I}$$

$$\dfrac{\forall x^{rb}[PF(x^{rb},x^{s}) \leftrightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{rb},x^{s})]]}{\forall x[RB(x) \rightarrow [PF(x,x^{s}) \leftrightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x,x^{s})]]]}\ \text{Axiom 1}$$

$$\dfrac{\forall x[RB(x) \rightarrow [PF(x,x^{s}) \leftrightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x,x^{s})]]]}{RB(x) \rightarrow [PF(x,x^{s}) \leftrightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x,x^{s})]]}\ \forall\text{E}$$

Axiom 8

$$\dfrac{\forall x[HB(x) \rightarrow RB(x)]}{HB(x) \rightarrow RB(x)}\ \forall\text{E}$$

$$\dfrac{HB(x) \rightarrow RB(x) \qquad HB(x)^{w}}{RB(x)}\ \rightarrow\text{E}$$

$$\dfrac{RB(x) \rightarrow [PF(x,x^{s}) \leftrightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x,x^{s})]] \qquad RB(x)}{PF(x,x^{s}) \leftrightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x,x^{s})]}\ \rightarrow\text{E}$$

$$\dfrac{PF(x,x^{s}) \leftrightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x,x^{s})]}{HB(x) \rightarrow [PF(x,x^{s}) \leftrightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x,x^{s})]]}\ \rightarrow\text{I},w$$

$$\dfrac{HB(x) \rightarrow [PF(x,x^{s}) \leftrightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x,x^{s})]]}{\forall x[HB(x) \rightarrow [PF(x,x^{s}) \leftrightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x,x^{s})]]]}\ \forall\text{I}$$

$$\dfrac{\forall x[HB(x) \rightarrow [PF(x,x^{s}) \leftrightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x,x^{s})]]]}{\forall x^{hb}[PF(x^{hb},x^{s}) \leftrightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{hb},x^{s})]]}\ \text{Axiom 8}$$

$$\dfrac{\forall x^{hb}[PF(x^{hb},x^{s}) \leftrightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{hb},x^{s})]]}{PF(x^{hb},x^{s}) \leftrightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{hb},x^{s})]}\ \forall\text{E}$$

$$\dfrac{PF(x^{hb},x^{s}) \leftrightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{hb},x^{s})]}{PF(x^{hb},x^{s}) \rightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{hb},x^{s})]}\ \wedge\text{E}$$

$$\dfrac{PF(x^{hb},x^{s}) \rightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{hb},x^{s})] \qquad PF(x^{hb},x^{s})}{\mathcal{D}_9 : \quad \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{hb},x^{s})]}\ \rightarrow\text{E}$$

Definition 7.5

$$\forall x^{rb}\forall x^{s}[FR(x^{rb},x^{s}) \leftrightarrow [NF(x^{rb},x^{s}) \wedge PF(x^{rb},x^{s})]]$$

$\vdots$ Similar to the right branch of $\mathcal{D}_9$

$$\forall x^{hb}[FR(x^{hb},x^{s}) \leftrightarrow [NF(x^{hb},x^{s}) \wedge PF(x^{hb},x^{s})]]$$

$$\frac{}{FR(x^{hb},x^{s}) \leftrightarrow [NF(x^{hb},x^{s}) \wedge PF(x^{hb},x^{s})]}\ \forall E$$

$$\frac{}{FR(x^{hb},x^{s}) \rightarrow [NF(x^{hb},x^{s}) \wedge PF(x^{hb},x^{s})]}\ \wedge E$$

$$\frac{FR(x^{hb},x^{s}) \rightarrow [NF(x^{hb},x^{s}) \wedge PF(x^{hb},x^{s})] \qquad FR(x^{hb},x^{s})^{u}\ \text{(rep.)}}{NF(x^{hb},x^{s}) \wedge PF(x^{hb},x^{s})}\ \rightarrow E$$

$$\frac{NF(x^{hb},x^{s}) \wedge PF(x^{hb},x^{s})}{NF(x^{hb},x^{s})}\ \wedge E$$

Proposition 25

$$\frac{\chi}{}$$

$\vdots\ \mathcal{D}_8$

$$\frac{\forall x^{hb}[NF(x^{hb},x^{s}) \rightarrow \forall\ulcorner\phi\urcorner[\,\theta \rightarrow DB(x^{hb},\ulcorner\phi\urcorner,x^{s})]]}{NF(x^{hb},x^{s}) \rightarrow \forall\ulcorner\phi\urcorner[\,\theta \rightarrow DB(x^{hb},\ulcorner\phi\urcorner,x^{s})]}\ \forall E$$

$$\frac{NF(x^{hb},x^{s}) \rightarrow \forall\ulcorner\phi\urcorner[\,\theta \rightarrow DB(x^{hb},\ulcorner\phi\urcorner,x^{s})] \qquad NF(x^{hb},x^{s})}{\forall\ulcorner\phi\urcorner[\,\theta \rightarrow DB(x^{hb},\ulcorner\phi\urcorner,x^{s})]}\ \rightarrow E$$

$$\frac{\forall\ulcorner\phi\urcorner[\,\theta \rightarrow DB(x^{hb},\ulcorner\phi\urcorner,x^{s})]}{\theta \rightarrow DB(x^{hb},\ulcorner\phi\urcorner,x^{s})}\ \forall E$$

Proposition 26

$$\frac{\forall x^{hb}\forall x^{s}\forall\ulcorner\phi\urcorner[[\,\theta \rightarrow DB(x^{hb},\ulcorner\phi\urcorner,x^{s})] \rightarrow [\,\theta \rightarrow DB(x^{hb},\ulcorner\phi\urcorner,x^{s})][\ulcorner\phi\urcorner/\ulcorner ML\urcorner]]}{[\,\theta \rightarrow DB(x^{hb},\ulcorner\phi\urcorner,x^{s})] \rightarrow [\,\theta \rightarrow DB(x^{hb},\ulcorner\phi\urcorner,x^{s})][\ulcorner\phi\urcorner/\ulcorner ML\urcorner]}\ \forall E$$

$$\frac{}{[\,\theta \rightarrow DB(x^{hb},\ulcorner\phi\urcorner,x^{s})] \rightarrow [[PP(\ulcorner ML\urcorner) \wedge O_i(\ulcorner ML\urcorner) \wedge DG(\ulcorner ML\urcorner,x^{hb},x^{s})] \rightarrow DB(x^{hb},\ulcorner ML\urcorner,x^{s})]}\ [\ulcorner\phi\urcorner/\ulcorner ML\urcorner]$$

$$\frac{[\,\theta \rightarrow DB(x^{hb},\ulcorner\phi\urcorner,x^{s})] \rightarrow [[PP(\ulcorner ML\urcorner) \wedge O_i(\ulcorner ML\urcorner) \wedge DG(\ulcorner ML\urcorner,x^{hb},x^{s})] \rightarrow DB(x^{hb},\ulcorner ML\urcorner,x^{s})] \qquad \theta \rightarrow DB(x^{hb},\ulcorner\phi\urcorner,x^{s})}{[PP(\ulcorner ML\urcorner) \wedge O_i(\ulcorner ML\urcorner) \wedge DG(\ulcorner ML\urcorner,x^{hb},x^{s})] \rightarrow DB(x^{hb},\ulcorner ML\urcorner,x^{s})}$$

Proposition 26

$$\frac{\forall x^{hb}\forall x^{s}\forall\ulcorner\phi\urcorner[\,\theta \rightarrow \theta[\ulcorner\phi\urcorner/\ulcorner ML\urcorner]]}{\forall x^{hb}\forall x^{s}\forall\ulcorner\phi\urcorner[\,\theta \rightarrow [PP(\ulcorner ML\urcorner) \wedge O_i(\ulcorner ML\urcorner) \wedge DG(\ulcorner ML\urcorner,x^{hb},x^{s})]]}\ [\ulcorner\phi\urcorner/\ulcorner ML\urcorner]$$

$$\frac{}{\theta \rightarrow [PP(\ulcorner ML\urcorner) \wedge O_i(\ulcorner ML\urcorner) \wedge DG(\ulcorner ML\urcorner,x^{hb},x^{s})]}\ \forall E$$

$$\frac{\theta^{v} \qquad \theta \rightarrow [PP(\ulcorner ML\urcorner) \wedge O_i(\ulcorner ML\urcorner) \wedge DG(\ulcorner ML\urcorner,x^{hb},x^{s})]}{PP(\ulcorner ML\urcorner) \wedge O_i(\ulcorner ML\urcorner) \wedge DG(\ulcorner ML\urcorner,x^{hb},x^{s})}\ \rightarrow E$$

$$\frac{[PP(\ulcorner ML\urcorner) \wedge O_i(\ulcorner ML\urcorner) \wedge DG(\ulcorner ML\urcorner,x^{hb},x^{s})] \rightarrow DB(x^{hb},\ulcorner ML\urcorner,x^{s}) \qquad PP(\ulcorner ML\urcorner) \wedge O_i(\ulcorner ML\urcorner) \wedge DG(\ulcorner ML\urcorner,x^{hb},x^{s})}{\mathcal{D}_{10}:\quad DB(x^{hb},\ulcorner ML\urcorner,x^{s})}\ \rightarrow E$$

$$\vdots$$

$$\dfrac{NF(x^{rb},x^s)^w \qquad \dfrac{\forall x^{rb}\forall x^s[NF(x^{rb},x^s) \to \forall\ulcorner\phi\urcorner[[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner)] \to [DB(x^{rb},\ulcorner\phi\urcorner,x^s) \leftrightarrow [DG(\ulcorner\phi\urcorner,x^{rb},x^s) \wedge form(\ulcorner\phi\urcorner,x^s)]]]]}{NF(x^{rb},x^s) \to \forall\ulcorner\phi\urcorner[[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner)] \to [DB(x^{rb},\ulcorner\phi\urcorner,x^s) \leftrightarrow [DG(\ulcorner\phi\urcorner,x^{rb},x^s) \wedge form(\ulcorner\phi\urcorner,x^s)]]]}\,\forall E}{\forall\ulcorner\phi\urcorner[[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner)] \to [DB(x^{rb},\ulcorner\phi\urcorner,x^s) \leftrightarrow [DG(\ulcorner\phi\urcorner,x^{rb},x^s) \wedge form(\ulcorner\phi\urcorner,x^s)]]]}\,\to E$$

$$\dfrac{\dfrac{PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{rb},x^s)^z}{PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner)}\,\wedge E \qquad \dfrac{\forall\ulcorner\phi\urcorner[[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner)] \to [DB(x^{rb},\ulcorner\phi\urcorner,x^s) \leftrightarrow [DG(\ulcorner\phi\urcorner,x^{rb},x^s) \wedge form(\ulcorner\phi\urcorner,x^s)]]]}{[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner)] \to [DB(x^{rb},\ulcorner\phi\urcorner,x^s) \leftrightarrow [DG(\ulcorner\phi\urcorner,x^{rb},x^s) \wedge form(\ulcorner\phi\urcorner,x^s)]]}\,\forall E}{DB(x^{rb},\ulcorner\phi\urcorner,x^s) \leftrightarrow [DG(\ulcorner\phi\urcorner,x^{rb},x^s) \wedge form(\ulcorner\phi\urcorner,x^s)]}\,\to E$$

Proposition 21

$$\dfrac{\dfrac{\forall\ulcorner\phi\urcorner[[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner)] \to \forall x^{rb}\forall x^s[DG(\ulcorner\phi\urcorner,x^{rb},x^s) \to form(\ulcorner\phi\urcorner,x^s)]]}{[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner)] \to \forall x^{rb}\forall x^s[DG(\ulcorner\phi\urcorner,x^{rb},x^s) \to form(\ulcorner\phi\urcorner,x^s)]}\,\forall E \qquad \dfrac{PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{rb},x^s)^z \text{ (rep.)}}{PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner)}\,\wedge E}{\forall x^{rb}\forall x^s[DG(\ulcorner\phi\urcorner,x^{rb},x^s) \to form(\ulcorner\phi\urcorner,x^s)]}\,\to E$$

$$\dfrac{\dfrac{PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{rb},x^s)^z \text{ (rep.)}}{DG(\ulcorner\phi\urcorner,x^{rb},x^s)}\,\wedge E \qquad \dfrac{\forall x^{rb}\forall x^s[DG(\ulcorner\phi\urcorner,x^{rb},x^s) \to form(\ulcorner\phi\urcorner,x^s)]}{DG(\ulcorner\phi\urcorner,x^{rb},x^s) \to form(\ulcorner\phi\urcorner,x^s)}\,\forall E}{form(\ulcorner\phi\urcorner,x^s)}\,\to E$$

$$\dfrac{\dfrac{DB(x^{rb},\ulcorner\phi\urcorner,x^s) \leftrightarrow [DG(\ulcorner\phi\urcorner,x^{rb},x^s) \wedge form(\ulcorner\phi\urcorner,x^s)]}{[DG(\ulcorner\phi\urcorner,x^{rb},x^s) \wedge form(\ulcorner\phi\urcorner,x^s)] \to DB(x^{rb},\ulcorner\phi\urcorner,x^s)}\,\wedge E \qquad \dfrac{form(\ulcorner\phi\urcorner,x^s) \qquad \dfrac{PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{rb},x^s)^z \text{ (rep.)}}{DG(\ulcorner\phi\urcorner,x^{rb},x^s)}\,\wedge E}{DG(\ulcorner\phi\urcorner,x^{rb},x^s) \wedge form(\ulcorner\phi\urcorner,x^s)}\,\wedge I}{DB(x^{rb},\ulcorner\phi\urcorner,x^s)}\,\to E$$

$$\dfrac{DB(x^{rb},\ulcorner\phi\urcorner,x^s)}{[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{rb},x^s)] \to DB(x^{rb},\ulcorner\phi\urcorner,x^s)}\,\to I,z$$

$$\dfrac{[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{rb},x^s)] \to DB(x^{rb},\ulcorner\phi\urcorner,x^s)}{\forall\ulcorner\phi\urcorner[[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{rb},x^s)] \to DB(x^{rb},\ulcorner\phi\urcorner,x^s)]}\,\forall I$$

$$\dfrac{\forall\ulcorner\phi\urcorner[[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{rb},x^s)] \to DB(x^{rb},\ulcorner\phi\urcorner,x^s)]}{NF(x^{rb},x^s) \to \forall\ulcorner\phi\urcorner[[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{rb},x^s)] \to DB(x^{rb},\ulcorner\phi\urcorner,x^s)]}\,\to I,w$$

$$\dfrac{NF(x^{rb},x^s) \to \forall\ulcorner\phi\urcorner[[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{rb},x^s)] \to DB(x^{rb},\ulcorner\phi\urcorner,x^s)]}{\forall x^{rb}[NF(x^{rb},x^s) \to \forall\ulcorner\phi\urcorner[[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner,x^{rb},x^s)] \to DB(x^{rb},\ulcorner\phi\urcorner,x^s)]]}\,\forall I$$

Similar to the left branch of $\mathcal{D}_9$ $\vdots$

$$\mathcal{D}_8 : \quad \forall x^{hb}[NF(x^{hb},x^s) \to \forall\ulcorner\phi\urcorner[\,\theta \to DB(x^{hb},\ulcorner\phi\urcorner,x^s)]]$$

B5.

(Section 7.2.5) The formal deduction of Proposition 28.

$$\theta = NC_{ML}(\ulcorner FR \urcorner) \wedge \exists x^s \forall x^{hb} FR(x^{hb}, x^s) \wedge \forall x^{hb} \forall x^s [FR(x^{hb}, x^s) \rightarrow DB(x^{hb}, \ulcorner ML \urcorner, x^s)]$$

$$\dfrac{\dfrac{\forall x^{rb} FR(x^{rb}, x^s)^u}{\forall x[RB(x) \rightarrow FR(x, x^s)]}\ \text{Axiom 1}}{RB(x) \rightarrow FR(x, x^s)}\ \forall\text{E}$$

$$\dfrac{\dfrac{\overset{\text{Axiom 8}}{\forall x[HB(x) \rightarrow RB(x)]}}{HB(x) \rightarrow RB(x)}\ \forall\text{E} \qquad HB(x)^v}{RB(x)}\ \rightarrow\text{E}$$

$$\dfrac{RB(x) \rightarrow FR(x, x^s) \qquad RB(x)}{FR(x, x^s)}\ \rightarrow\text{E}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{FR(x, x^s)}{HB(x) \rightarrow FR(x, x^s)}\ \rightarrow\text{I},v}{\forall x[HB(x) \rightarrow FR(x, x^s)]}\ \forall\text{I}}{\forall x^{hb} FR(x^{hb}, x^s)}\ \text{Axiom 8}}{\exists x^s \forall x^{hb} FR(x^{hb}, x^s)}\ \exists\text{I}$$

$$\dfrac{\overset{\text{Proposition 24}}{\exists x^s \forall x^{rb} FR(x^{rb}, x^s)} \qquad \exists x^s \forall x^{hb} FR(x^{hb}, x^s)}{\exists x^s \forall x^{hb} FR(x^{hb}, x^s)}\ \exists\text{E},u$$

$$\dfrac{\exists x^s \forall x^{hb} FR(x^{hb}, x^s) \qquad \overset{\text{Proposition 27}}{\forall x^{hb} \forall x^s [FR(x^{hb}, x^s) \rightarrow DB(x^{hb}, \ulcorner ML \urcorner, x^s)]}}{\exists x^s \forall x^{hb} FR(x^{hb}, x^s) \wedge \forall x^{hb} \forall x^s [FR(x^{hb}, x^s \rightarrow DB(x^{hb}, \ulcorner ML \urcorner, x^s)]}\ \wedge\text{I}$$

$$\dfrac{\overset{\text{Proposition 15}}{NC_{ML}(\ulcorner FR \urcorner)} \qquad \exists x^s \forall x^{hb} FR(x^{hb}, x^s) \wedge \forall x^{hb} \forall x^s [FR(x^{hb}, x^s \rightarrow DB(x^{hb}, \ulcorner ML \urcorner, x^s)]}{\theta}\ \wedge\text{I}$$

$$\dfrac{\theta \qquad \dfrac{\overset{\text{Axiom 9}}{G_{ML}(\ulcorner FR \urcorner) \leftrightarrow \theta}}{\theta \rightarrow G_{ML}(\ulcorner FR \urcorner)}\ \wedge\text{E}}{G_{ML}(\ulcorner FR \urcorner)}\ \rightarrow\text{E}$$

B6.

A large part of the upcoming deductions consists of rewriting some of the already established results in order to either formally relate them to the practical point of view or to condition them on human beings in specific. To preserve readability I will abbreviate the following deductions to $\mathcal{D}_{S^p}$ and $\mathcal{D}_{HB}$.

(Notice that the deductions $\mathcal{D}_{S^p}$ and $\mathcal{D}_{HB}$ always introduce, respectively, the assumptions $S^p(x)^v$ and $HB(y)^w$ to the deduction in question.)

$$\dfrac{\dfrac{\overset{\text{Axiom 2}}{\forall x[S^p(x)\to S(x)]}}{S^p(x)\to S(x)}\,\forall\text{E} \qquad S^p(x)^v}{\mathcal{D}_{S^p}:\quad S(x)}\,\to\text{E}$$

(a) Deduction $\mathcal{D}_{S^p}$

$$\dfrac{\dfrac{\overset{\text{Axiom 8}}{\forall y[HB(y)\to RB(y)]}}{HB(y)\to RB(y)}\,\forall\text{E} \qquad HB(y)^w}{\mathcal{D}_{HB}:\quad RB(y)}\,\to\text{E}$$

(b) Deduction $\mathcal{D}_{HB}$

(Section 7.3) The formal deductions $\mathcal{D}_{15}$, $\mathcal{D}_{14}$, $\mathcal{D}_{13}$, $\mathcal{D}_{12}$ and $\mathcal{D}_{11}$ as belonging to the proof of Proposition 30.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{Definition 7.5}}{\forall x^{rb}\forall x^s[FR(x^{rb},x^s)\leftrightarrow[PF(x^{rb},x^s)\wedge NF(x^{rb},x^s)]]}}{\forall x[RB(x)\to\forall x^s[FR(x,x^s)\leftrightarrow[PF(x,x^s)\wedge NF(x,x^s)]]]}\,\text{Axiom 1}}{RB(x)\to\forall x^s[FR(x,x^s)\leftrightarrow[PF(x,x^s)\wedge NF(x,x^s)]]}\,\forall\text{E} \qquad \begin{matrix}[w]\\ \mathcal{D}_{HB}\\ RB(x)\end{matrix}}{\forall x^s[FR(x,x^s)\leftrightarrow[PF(x,x^s)\wedge NF(x,x^s)]]}\,\to\text{E}}{\forall y[S(y)\to[FR(x,y)\leftrightarrow[PF(x,y)\wedge NF(x,y)]]]}\,\text{Axiom 1}}{S(y)\to[FR(x,y)\leftrightarrow[PF(x,y)\wedge NF(x,y)]]}\,\forall\text{E} \qquad \begin{matrix}[v]\\ \mathcal{D}_{S^p}\\ S(y)\end{matrix}}{FR(x,y)\leftrightarrow[PF(x,y)\wedge NF(x,y)]}\,\to\text{E}}{S^p(y)\to[FR(x,y)\leftrightarrow[PF(x,y)\wedge NF(x,y)]]}\,\to\text{I},v}{\forall y[S^p(y)\to[FR(x,y)\leftrightarrow[PF(x,y)\wedge NF(x,y)]]]}\,\forall\text{I}}{\forall x^{s_p}[FR(x,x^{s_p})\leftrightarrow[PF(x,x^{s_p})\wedge NF(x,x^{s_p})]]}\,\text{Axiom 2}}{HB(x)\to\forall x^{s_p}[FR(x,x^{s_p})\leftrightarrow[PF(x,x^{s_p})\wedge NF(x,x^{s_p})]]}\,\to\text{I},w}{\forall x[HB(x)\to\forall x^{s_p}[FR(x,x^{s_p})\leftrightarrow[PF(x,x^{s_p})\wedge NF(x,x^{s_p})]]]}\,\forall\text{I}}{\forall x^{hb}\forall x^{s_p}[FR(x^{hb},x^{s_p})\leftrightarrow[PF(x^{hb},x^{s_p})\wedge NF(x^{hb},x^{s_p})]]}\,\text{Axiom 8}}{FR(x^{hb},x^{s_p})\leftrightarrow[PF(x^{hb},x^{s_p})\wedge NF(x^{hb},x^{s_p})]}\,\forall\text{E}}{\mathcal{D}_{15}:\quad [PF(x^{hb},x^{s_p})\wedge NF(x^{hb},x^{s_p})]\to FR(x^{hb},x^{s_p})}\,\wedge\text{E}$$

$$\dfrac{\dfrac{\dfrac{\text{Definition 7.4}}{\forall x^{s}\forall x^{rb}[PF(x^{rb},x^{s}) \leftrightarrow \exists\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{rb}, x^{s})]]}}{\forall x[S(x) \rightarrow \forall x^{rb}[PF(x^{rb},x) \leftrightarrow \exists\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{rb}, x)]]]}\text{ Axiom 1}}{S(x) \rightarrow \forall x^{rb}[PF(x^{rb},x) \leftrightarrow \exists\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{rb}, x)]]}\ \forall\text{E} \qquad \begin{matrix}[v]\\ \mathcal{D}_{S^p}\\ S(x)\end{matrix}$$

$$\rightarrow\text{E}$$

$$\forall x^{rb}[PF(x^{rb},x) \leftrightarrow \exists\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{rb}, x)]]$$

$$\text{Axiom 1}$$

$$\forall y[RB(y) \rightarrow [PF(y,x) \leftrightarrow \exists\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, y, x)]]]$$

$$\forall\text{E}$$

$$RB(y) \rightarrow [PF(y,x) \leftrightarrow \exists\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, y, x)]] \qquad \begin{matrix}[w]\\ \mathcal{D}_{HB}\\ RB(y)\end{matrix}$$

$$\rightarrow\text{E}$$

$$PF(y,x) \leftrightarrow \exists\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, y, x)]$$

$$\rightarrow\text{I},w$$

$$HB(y) \rightarrow [PF(y,x) \leftrightarrow \exists\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, y, x)]]$$

$$\forall\text{I}$$

$$\forall y[HB(y) \rightarrow [PF(y,x) \leftrightarrow \exists\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, y, x)]]]$$

$$\text{Axiom 8}$$

$$\forall x^{hb}[PF(x^{hb},x) \leftrightarrow \exists\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{hb}, x)]]$$

$$\rightarrow\text{I},v$$

$$S^p(x) \rightarrow \forall x^{hb}[PF(x^{hb},x) \leftrightarrow \exists\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{hb}, x)]]$$

$$\forall\text{I}$$

$$\forall x[S^p(x) \rightarrow \forall x^{hb}[PF(x^{hb},x) \leftrightarrow \exists\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{hb}, x)]]]$$

$$\text{Axiom 2}$$

$$\forall x^{s_p}\forall x^{hb}[PF(x^{hb},x^{s_p}) \leftrightarrow \exists\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{hb}, x^{s_p})]]$$

$$\forall\text{E}$$

$$PF(x^{hb},x^{s_p}) \leftrightarrow \exists\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{hb}, x^{s_p})]$$

$$\wedge\text{E}$$

$$\dfrac{\dfrac{\dfrac{\forall x^{hb}[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{hb}, x^{s_p})]^{u}}{PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{hb}, x^{s_p})}\ \forall\text{E}}{\exists\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{hb}, x^{s_p})]}\ \exists\text{I} \qquad \exists\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{hb}, x^{s_p})] \rightarrow PF(x^{hb},x^{s_p})}{\mathcal{D}_{14}: \quad PF(x^{hb},x^{s_p})}\ \rightarrow\text{E}$$

$$\mathcal{D}_{12}$$
$$\vdots$$
$$PP(\ulcorner\psi\urcorner) \rightarrow \neg sens(x^{s_p})$$

$$\dfrac{\dfrac{\forall x^{hb}[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{hb}, x^{s_p})]^u \text{ (rep.)}}{PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{hb}, x^{s_p})}\ \forall E}{PP(\ulcorner\psi\urcorner)}\ \wedge E$$

$$\dfrac{PP(\ulcorner\psi\urcorner) \rightarrow \neg sens(x^{s_p}) \qquad PP(\ulcorner\psi\urcorner)}{\neg sens(x^{s_p})}\ \rightarrow$$

Proposition 17

$$\dfrac{\forall x^{rb}\forall x^{s_t} caus(x^{rb}, x^{s_t}) \wedge \forall x^{rb}\forall x^{S\setminus S^t} \neg caus(x^{rb}, x^{S\setminus S^t})}{\forall x^{rb}\forall x^{S\setminus S^t} \neg caus(x^{rb}, x^{S\setminus S^t})}\ \wedge E$$

$$\mathcal{D}_7 \text{ (rep.)} \quad \vdots$$

$$\dfrac{\neg sens(x^s) \rightarrow \neg caus(x^{rb}, x^s)}{\forall x^s \forall x^{rb}[\neg sens(x^s) \rightarrow \neg caus(x^{rb}, x^s)]}\ \forall I$$

$$\dfrac{\forall x^s \forall x^{rb}[\neg sens(x^s) \rightarrow \neg caus(x^{rb}, x^s)]}{\forall x[S(x) \rightarrow \forall x^{rb}[\neg sens(x) \rightarrow \neg caus(x^{rb}, x)]]}\ \text{Axiom 1}$$

$$\dfrac{\forall x[S(x) \rightarrow \forall x^{rb}[\neg sens(x) \rightarrow \neg caus(x^{rb}, x)]]}{S(x) \rightarrow \forall x^{rb}[\neg sens(x) \rightarrow \neg caus(x^{rb}, x)]}\ \forall E$$

$$\dfrac{\begin{matrix}[v]\\ \mathcal{D}_{S^p}\\ S(x)\end{matrix} \qquad S(x) \rightarrow \forall x^{rb}[\neg sens(x) \rightarrow \neg caus(x^{rb}, x)]}{\forall x^{rb}[\neg sens(x) \rightarrow \neg caus(x^{rb}, x)]}\ \rightarrow E$$

$$\dfrac{\forall x^{rb}[\neg sens(x) \rightarrow \neg caus(x^{rb}, x)]}{\forall y[RB(y) \rightarrow [\neg sens(x) \rightarrow \neg caus(y, x)]]}\ \text{Axiom 8}$$

$$\dfrac{\forall y[RB(y) \rightarrow [\neg sens(x) \rightarrow \neg caus(y, x)]]}{RB(y) \rightarrow [\neg sens(x) \rightarrow \neg caus(y, x)]}\ \forall E$$

$$\dfrac{\begin{matrix}[w]\\ \mathcal{D}_{HB}\\ RB(y)\end{matrix} \qquad RB(y) \rightarrow [\neg sens(x) \rightarrow \neg caus(y, x)]}{\neg sens(x) \rightarrow \neg caus(y, x)]}\ \rightarrow E$$

$$\dfrac{\neg sens(x) \rightarrow \neg caus(y, x)]}{HB(y) \rightarrow [\neg sens(x) \rightarrow \neg caus(y, x)]}\ \rightarrow I,w$$

$$\dfrac{HB(y) \rightarrow [\neg sens(x) \rightarrow \neg caus(y, x)]}{\forall y[HB(y) \rightarrow [\neg sens(x) \rightarrow \neg caus(y, x)]]}\ \forall I$$

$$\dfrac{\forall y[HB(y) \rightarrow [\neg sens(x) \rightarrow \neg caus(y, x)]]}{\forall x^{hb}[\neg sens(x) \rightarrow \neg caus(x^{hb}, x)]}\ \text{Axiom 8}$$

$$\dfrac{\forall x^{hb}[\neg sens(x) \rightarrow \neg caus(x^{hb}, x)]}{S^p(x) \rightarrow \forall x^{hb}[\neg sens(x) \rightarrow \neg caus(x^{hb}, x)]}\ \rightarrow I,v$$

$$\dfrac{S^p(x) \rightarrow \forall x^{hb}[\neg sens(x) \rightarrow \neg caus(x^{hb}, x)]}{\forall x[S^p(x) \rightarrow \forall x^{hb}[\neg sens(x) \rightarrow \neg caus(x^{hb}, x)]]}\ \forall I$$

$$\dfrac{\forall x[S^p(x) \rightarrow \forall x^{hb}[\neg sens(x) \rightarrow \neg caus(x^{hb}, x)]]}{\forall x^{s_p}\forall x^{hb}[\neg sens(x^{s_p}) \rightarrow \neg caus(x^{hb}, x^{s_p})]}\ \text{Axiom 2}$$

$$\dfrac{\forall x^{s_p}\forall x^{hb}[\neg sens(x^{s_p}) \rightarrow \neg caus(x^{hb}, x^{s_p})]}{\neg sens(x^{s_p}) \rightarrow \neg caus(x^{hb}, x^{s_p})}\ \forall E$$

$$\dfrac{\neg sens(x^{s_p}) \rightarrow \neg caus(x^{hb}, x^{s_p}) \qquad \neg sens(x^{s_p})}{\neg caus(x^{hb}, x^{s_p})}\ \rightarrow E$$

Definition 7.3

$$\dfrac{\forall x^s \forall x^{rb}[\neg caus(x^{rb}, x^s) \leftrightarrow NF(x^{rb}, x^s)]}{\forall x[S(x) \rightarrow \forall x^{rb}[\neg caus(x^{rb}, x) \leftrightarrow NF(x^{rb}, x)]]}\ \text{Axiom 1}$$

$$\dfrac{\forall x[S(x) \rightarrow \forall x^{rb}[\neg caus(x^{rb}, x) \leftrightarrow NF(x^{rb}, x)]]}{S(x) \rightarrow \forall x^{rb}[\neg caus(x^{rb}, x) \leftrightarrow NF(x^{rb}, x)]}\ \forall E$$

$$\dfrac{\begin{matrix}[v]\\ \mathcal{D}_{S^p}\\ S(x)\end{matrix} \qquad S(x) \rightarrow \forall x^{rb}[\neg caus(x^{rb}, x) \leftrightarrow NF(x^{rb}, x)]}{\forall x^{rb}[\neg caus(x^{rb}, x) \leftrightarrow NF(x^{rb}, x)]}\ \rightarrow E$$

$$\dfrac{\forall x^{rb}[\neg caus(x^{rb}, x) \leftrightarrow NF(x^{rb}, x)]}{\forall y[RB(y) \rightarrow [\neg caus(y, x) \leftrightarrow NF(y, x)]]}\ \text{Axiom 1}$$

$$\dfrac{\forall y[RB(y) \rightarrow [\neg caus(y, x) \leftrightarrow NF(y, x)]]}{RB(y) \rightarrow [\neg caus(y, x) \leftrightarrow NF(y, x)]}\ \forall E$$

$$\dfrac{\begin{matrix}[w]\\ \mathcal{D}_{HB}\\ RB(y)\end{matrix} \qquad RB(y) \rightarrow [\neg caus(y, x) \leftrightarrow NF(y, x)]}{\neg caus(y, x) \leftrightarrow NF(y, x)}\ \rightarrow E$$

$$\dfrac{\neg caus(y, x) \leftrightarrow NF(y, x)}{HB(y) \rightarrow [\neg caus(y, x) \leftrightarrow NF(y, x)]}\ \rightarrow I,w$$

$$\dfrac{HB(y) \rightarrow [\neg caus(y, x) \leftrightarrow NF(y, x)]}{\forall y[HB(y) \rightarrow [\neg caus(y, x) \leftrightarrow NF(y, x)]]}\ \forall I$$

$$\dfrac{\forall y[HB(y) \rightarrow [\neg caus(y, x) \leftrightarrow NF(y, x)]]}{\forall x^{hb}[\neg caus(x^{hb}, x) \leftrightarrow NF(x^{hb}, x)]}\ \text{Axiom 8}$$

$$\dfrac{\forall x^{hb}[\neg caus(x^{hb}, x) \leftrightarrow NF(x^{hb}, x)]}{S^p(x) \rightarrow \forall x^{hb}[\neg caus(x^{hb}, x) \leftrightarrow NF(x^{hb}, x)]}\ \rightarrow I,v$$

$$\dfrac{S^p(x) \rightarrow \forall x^{hb}[\neg caus(x^{hb}, x) \leftrightarrow NF(x^{hb}, x)]}{\forall x[S^p(x) \rightarrow \forall x^{hb}[\neg caus(x^{hb}, x) \leftrightarrow NF(x^{hb}, x)]]}\ \forall I$$

$$\dfrac{\forall x[S^p(x) \rightarrow \forall x^{hb}[\neg caus(x^{hb}, x) \leftrightarrow NF(x^{hb}, x)]]}{\forall x^{s_p}\forall x^{hb}[\neg caus(x^{hb}, x^{s_p}) \leftrightarrow NF(x^{hb}, x^{s_p})]}\ \text{Axiom 2}$$

$$\dfrac{\forall x^{s_p}\forall x^{hb}[\neg caus(x^{hb}, x^{s_p}) \leftrightarrow NF(x^{hb}, x^{s_p})]}{\neg caus(x^{hb}, x^{s_p}) \leftrightarrow NF(x^{hb}, x^{s_p})}\ \forall E$$

$$\dfrac{\neg caus(x^{hb}, x^{s_p}) \leftrightarrow NF(x^{hb}, x^{s_p})}{\neg caus(x^{hb}, x^{s_p}) \rightarrow NF(x^{hb}, x^{s_p})}\ \wedge E$$

$$\dfrac{\neg caus(x^{hb}, x^{s_p}) \qquad \neg caus(x^{hb}, x^{s_p}) \rightarrow NF(x^{hb}, x^{s_p})}{\mathcal{D}_{13}: \quad NF(x^{hb}, x^{s_p})}\ \rightarrow E$$

Proposition 21

$$\dfrac{\forall x^{hb}[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{hb}, x^{s_p})]^u \text{ (rep.)}}{PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{hb}, x^{s_p})}\,\forall\text{E}$$

$$\dfrac{PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{hb}, x^{s_p})}{PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner)}\,\wedge\text{E}$$

$$\dfrac{\forall\ulcorner\psi\urcorner[[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner)] \to \forall x^{rb}\forall x^{s}[DG(\ulcorner\psi\urcorner, x^{rb}, x^{s}) \to form(\ulcorner\psi\urcorner, x^{s})]]}{[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner)] \to \forall x^{rb}\forall x^{s}[DG(\ulcorner\psi\urcorner, x^{rb}, x^{s}) \to form(\ulcorner\psi\urcorner, x^{s})]}\,\forall\text{E}$$

$$\dfrac{PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \qquad [PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner)] \to \forall x^{rb}\forall x^{s}[DG(\ulcorner\psi\urcorner, x^{rb}, x^{s}) \to form(\ulcorner\psi\urcorner, x^{s})]}{\forall x^{rb}\forall x^{s}[DG(\ulcorner\psi\urcorner, x^{rb}, x^{s}) \to form(\ulcorner\psi\urcorner, x^{s})]}\,\to\text{E}$$

$$\dfrac{\forall x^{rb}\forall x^{s}[DG(\ulcorner\psi\urcorner, x^{rb}, x^{s}) \to form(\ulcorner\psi\urcorner, x^{s})]}{\forall y[RB(y) \to \forall x^{s}[DG(\ulcorner\psi\urcorner, y, x^{s}) \to form(\ulcorner\psi\urcorner, x^{s})]]}\,\text{Axiom 1}$$

$$\dfrac{\forall y[RB(y) \to \forall x^{s}[DG(\ulcorner\psi\urcorner, y, x^{s}) \to form(\ulcorner\psi\urcorner, x^{s})]]}{RB(y) \to \forall x^{s}[DG(\ulcorner\psi\urcorner, y, x^{s}) \to form(\ulcorner\psi\urcorner, x^{s})]}\,\forall\text{E}$$

$$\dfrac{RB(y) \to \forall x^{s}[DG(\ulcorner\psi\urcorner, y, x^{s}) \to form(\ulcorner\psi\urcorner, x^{s})] \qquad \begin{matrix}[w]\\ \mathcal{D}_{HB}\\ RB(y)\end{matrix}}{\forall x^{s}[DG(\ulcorner\psi\urcorner, y, x^{s}) \to form(\ulcorner\psi\urcorner, x^{s})]}\,\to\text{E}$$

$$\dfrac{\forall x^{s}[DG(\ulcorner\psi\urcorner, y, x^{s}) \to form(\ulcorner\psi\urcorner, x^{s})]}{\forall x[S(x) \to [DG(\ulcorner\psi\urcorner, y, x) \to form(\ulcorner\psi\urcorner, x)]]}$$

$$\dfrac{\forall x[S(x) \to [DG(\ulcorner\psi\urcorner, y, x) \to form(\ulcorner\psi\urcorner, x)]]}{S(x) \to [DG(\ulcorner\psi\urcorner, y, x) \to form(\ulcorner\psi\urcorner, x)]}\,\forall\text{E}$$

$$\dfrac{S(x) \to [DG(\ulcorner\psi\urcorner, y, x) \to form(\ulcorner\psi\urcorner, x)] \qquad \begin{matrix}[v]\\ \mathcal{D}_{S^p}\\ S(x)\end{matrix}}{DG(\ulcorner\psi\urcorner, y, x) \to form(\ulcorner\psi\urcorner, x)}\,\to\text{E}$$

$$\mathcal{D}_{11}\ \vdots$$

$$\dfrac{DG(\ulcorner\psi\urcorner, y, x) \to [PP(\ulcorner\psi\urcorner) \to \neg sens(x)]}{S^p(x) \to [DG(\ulcorner\psi\urcorner, y, x) \to [PP(\ulcorner\psi\urcorner) \to \neg sens(x)]]}\,\to\text{I},v$$

$$\dfrac{S^p(x) \to [DG(\ulcorner\psi\urcorner, y, x) \to [PP(\ulcorner\psi\urcorner) \to \neg sens(x)]]}{\forall x[S^p(x) \to [DG(\ulcorner\psi\urcorner, y, x) \to [PP(\ulcorner\psi\urcorner) \to \neg sens(x)]]]}\,\forall\text{I}$$

$$\dfrac{\forall x[S^p(x) \to [DG(\ulcorner\psi\urcorner, y, x) \to [PP(\ulcorner\psi\urcorner) \to \neg sens(x)]]]}{\forall x^{s_p}[DG(\ulcorner\psi\urcorner, y, x^{s_p}) \to [PP(\ulcorner\psi\urcorner) \to \neg sens(x^{s_p})]]}\,\text{Axiom 2}$$

$$\dfrac{\forall x^{s_p}[DG(\ulcorner\psi\urcorner, y, x^{s_p}) \to [PP(\ulcorner\psi\urcorner) \to \neg sens(x^{s_p})]]}{HB(y) \to \forall x^{s_p}[DG(\ulcorner\psi\urcorner, y, x^{s_p}) \to [PP(\ulcorner\psi\urcorner) \to \neg sens(x^{s_p})]]}\,\to\text{I},w$$

$$\dfrac{HB(y) \to \forall x^{s_p}[DG(\ulcorner\psi\urcorner, y, x^{s_p}) \to [PP(\ulcorner\psi\urcorner) \to \neg sens(x^{s_p})]]}{\forall y[HB(y) \to \forall x^{s_p}[DG(\ulcorner\psi\urcorner, y, x^{s_p}) \to [PP(\ulcorner\psi\urcorner) \to \neg sens(x^{s_p})]]]}\,\forall\text{I}$$

$$\dfrac{\forall y[HB(y) \to \forall x^{s_p}[DG(\ulcorner\psi\urcorner, y, x^{s_p}) \to [PP(\ulcorner\psi\urcorner) \to \neg sens(x^{s_p})]]]}{\forall x^{hb}\forall x^{s_p}[DG(\ulcorner\psi\urcorner, x^{hb}, x^{s_p}) \to [PP(\ulcorner\psi\urcorner) \to \neg sens(x^{s_p})]]}\,\text{Axiom 8}$$

$$\dfrac{\forall x^{hb}\forall x^{s_p}[DG(\ulcorner\psi\urcorner, x^{hb}, x^{s_p}) \to [PP(\ulcorner\psi\urcorner) \to \neg sens(x^{s_p})]]}{DG(\ulcorner\psi\urcorner, x^{hb}, x^{s_p}) \to [PP(\ulcorner\psi\urcorner) \to \neg sens(x^{s_p})]}\,\forall\text{E}$$

$$\dfrac{\forall x^{hb}[PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{hb}, x^{s_p})]^u \text{ (rep.)}}{PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{hb}, x^{s_p})}\,\forall\text{E}$$

$$\dfrac{PP(\ulcorner\psi\urcorner) \wedge O_i(\ulcorner\psi\urcorner) \wedge DG(\ulcorner\psi\urcorner, x^{hb}, x^{s_p})}{DG(\ulcorner\psi\urcorner, x^{hb}, x^{s_p})}\,\wedge\text{E}$$

$$\dfrac{DG(\ulcorner\psi\urcorner, x^{hb}, x^{s_p}) \qquad DG(\ulcorner\psi\urcorner, x^{hb}, x^{s_p}) \to [PP(\ulcorner\psi\urcorner) \to \neg sens(x^{s_p})]}{\mathcal{D}_{12} \quad PP(\ulcorner\phi\urcorner) \to \neg sens(x^{s_p})}\,\to\text{E}$$

$$\dfrac{\vdots \qquad DG(\ulcorner\psi\urcorner, y, x) \rightarrow form(\ulcorner\psi\urcorner, x) \qquad DG(\ulcorner\psi\urcorner, y, x)^r}{form(\ulcorner\psi\urcorner, x)} \rightarrow\text{E}$$

$$\dfrac{\text{Axiom 5}}{\forall\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner) \rightarrow \forall x^s[form(\ulcorner\psi\urcorner, x^s) \rightarrow \neg matter(\ulcorner\psi\urcorner, x^s)]]}$$

$$\dfrac{\forall\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner) \rightarrow \forall x^s[form(\ulcorner\psi\urcorner, x^s) \rightarrow \neg matter(\ulcorner\psi\urcorner, x^s)]]}{PP(\ulcorner\psi\urcorner) \rightarrow \forall x^s[form(\ulcorner\psi\urcorner, x^s) \rightarrow \neg matter(\ulcorner\psi\urcorner, x^s)]} \forall\text{E}$$

$$\dfrac{PP(\ulcorner\psi\urcorner)^z \qquad PP(\ulcorner\psi\urcorner) \rightarrow \forall x^s[form(\ulcorner\psi\urcorner, x^s) \rightarrow \neg matter(\ulcorner\psi\urcorner, x^s)]}{\forall x^s[form(\ulcorner\psi\urcorner, x^s) \rightarrow \neg matter(\ulcorner\psi\urcorner, x^s)]} \rightarrow\text{E}$$

$$\dfrac{\forall x^s[form(\ulcorner\psi\urcorner, x^s) \rightarrow \neg matter(\ulcorner\psi\urcorner, x^s)]}{\forall x[S(x) \rightarrow [form(\ulcorner\psi\urcorner, x) \rightarrow \neg matter(\ulcorner\psi\urcorner, x)]]} \text{Axiom 1}$$

$$\dfrac{\forall x[S(x) \rightarrow [form(\ulcorner\psi\urcorner, x) \rightarrow \neg matter(\ulcorner\psi\urcorner, x)]]}{S(x) \rightarrow [form(\ulcorner\psi\urcorner, x) \rightarrow \neg matter(\ulcorner\psi\urcorner, x)]} \forall\text{E}$$

$$\dfrac{S(x) \rightarrow [form(\ulcorner\psi\urcorner, x) \rightarrow \neg matter(\ulcorner\psi\urcorner, x)] \qquad \begin{matrix}[v]\ \text{(rep.)} \\ \mathcal{D}_{S^p} \\ S(x)\end{matrix}}{form(\ulcorner\psi\urcorner, x) \rightarrow \neg matter(\ulcorner\psi\urcorner, x)} \rightarrow\text{E}$$

$$\dfrac{form(\ulcorner\psi\urcorner, x) \rightarrow \neg matter(\ulcorner\psi\urcorner, x) \qquad form(\ulcorner\psi\urcorner, x)}{\neg matter(\ulcorner\psi\urcorner, x)} \rightarrow\text{E}$$

$$\dfrac{\text{Axiom 5}}{\forall\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner) \rightarrow \forall x^s[sens(x^s) \leftrightarrow matter(\ulcorner\psi\urcorner, x^s)]]}$$

$$\dfrac{\forall\ulcorner\psi\urcorner[PP(\ulcorner\psi\urcorner) \rightarrow \forall x^s[sens(x^s) \leftrightarrow matter(\ulcorner\psi\urcorner, x^s)]]}{PP(\ulcorner\psi\urcorner) \rightarrow \forall x^s[sens(x^s) \leftrightarrow matter(\ulcorner\psi\urcorner, x^s)]} \forall\text{E}$$

$$\dfrac{PP(\ulcorner\psi\urcorner)^z\ \text{(rep.)} \qquad PP(\ulcorner\psi\urcorner) \rightarrow \forall x^s[sens(x^s) \leftrightarrow matter(\ulcorner\psi\urcorner, x^s)]}{\forall x^s[sens(x^s) \leftrightarrow matter(\ulcorner\psi\urcorner, x^s)]} \rightarrow\text{E}$$

$$\dfrac{\forall x^s[sens(x^s) \leftrightarrow matter(\ulcorner\psi\urcorner, x^s)]}{\forall x[S(x) \rightarrow [sens(x) \leftrightarrow matter(\ulcorner\psi\urcorner, x)]]} \text{Axiom 1}$$

$$\dfrac{\forall x[S(x) \rightarrow [sens(x) \leftrightarrow matter(\ulcorner\psi\urcorner, x)]]}{S(x) \rightarrow [sens(x) \leftrightarrow matter(\ulcorner\psi\urcorner, x)]} \forall\text{E}$$

$$\dfrac{\begin{matrix}[v]\ \text{(rep.)} \\ \mathcal{D}_{S^p} \\ S(x)\end{matrix} \qquad S(x) \rightarrow [sens(x) \leftrightarrow matter(\ulcorner\psi\urcorner, x)]}{sens(x) \leftrightarrow matter(\ulcorner\psi\urcorner, x)} \rightarrow\text{E}$$

$$\dfrac{sens(x) \leftrightarrow matter(\ulcorner\psi\urcorner, x)}{sens(x) \rightarrow matter(\ulcorner\psi\urcorner, x)} \wedge\text{E}$$

$$\dfrac{sens(x) \rightarrow matter(\ulcorner\psi\urcorner, x)}{\neg matter(\ulcorner\psi\urcorner, x) \rightarrow \neg sens(x)} \text{IL.2}$$

$$\dfrac{\neg matter(\ulcorner\psi\urcorner, x) \qquad \neg matter(\ulcorner\psi\urcorner, x) \rightarrow \neg sens(x)}{\neg sens(x)} \rightarrow\text{E}$$

$$\dfrac{\neg sens(x)}{PP(\ulcorner\psi\urcorner) \rightarrow \neg sens(x)} \rightarrow\text{I},z$$

$$\dfrac{PP(\ulcorner\psi\urcorner) \rightarrow \neg sens(x)}{\mathcal{D}_{11} : \quad DG(\ulcorner\psi\urcorner, y, x) \rightarrow [PP(\ulcorner\psi\urcorner) \rightarrow \neg sens(x)]} \rightarrow\text{I},r$$

B7.

(Section 7.4) The formal deductions $\mathcal{D}_{16}$ and $\mathcal{D}_{17}$ as belonging to the proof of Proposition 31.

$\theta = PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner, x^{hb}, x^{s_p})$
$\chi = PP(\ulcorner ML\urcorner) \wedge O_i(\ulcorner ML\urcorner) \wedge DG(\ulcorner ML\urcorner, x^{hb}, x^{s_p})$:

Definition 7.5

$$\forall x^{rb}\forall x^{s}[FR(x^{rb}, x^{s}) \leftrightarrow [NF(x^{rb}, x^{s}) \wedge PF(x^{rb}, x^{s})]]$$

Similar to right branch $\mathcal{D}_9$

$$\dfrac{\dfrac{\dfrac{O_p(\ulcorner FR\urcorner)^v \qquad O_p(\ulcorner FR\urcorner) \rightarrow \forall x^{hb}\forall x^{s_p} FR(x^{hb}, x^{s_p})^u}{\forall x^{hb}\forall x^{s_p} FR(x^{hb}, x^{s_p})}\rightarrow\text{E}}{FR(x^{hb}, x^{s_p})}\forall\text{E} \qquad FR(x^{hb}, x^{s_p}) \rightarrow PF(x^{hb}, x^{s_p})}{}\rightarrow\text{E}$$

Definition 7.4

$$\forall x^{rb}\forall x^{s}[PF(x^{rb}, x^{s}) \leftrightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner, x^{rb}, x^{s})]]$$

Similar to left branch $\mathcal{D}_9$

$$\dfrac{\dfrac{PF(x^{hb}, x^{s_p}) \qquad PF(x^{hb}, x^{s_p}) \rightarrow \exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner, x^{hb}, x^{s_p})]}{\exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner, x^{hb}, x^{s_p})]}\rightarrow\text{E}}{\mathcal{D}_{16}: \quad \forall x^{hb}\forall x^{s_p}\exists\ulcorner\phi\urcorner[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner) \wedge DG(\ulcorner\phi\urcorner, x^{hb}, x^{s_p})]}\forall\text{I}$$

$$\dfrac{\dfrac{O_p(\ulcorner FR \urcorner)^v \qquad O_p(\ulcorner FR \urcorner) \rightarrow \forall x^{hb}\forall x^{s_p} FR(x^{hb}, x^{s_p})^u}{\forall x^{hb}\forall x^{s_p} FR(x^{hb}, x^{s_p})} \rightarrow\mathrm{E}}{FR(x^{hb}, x^{s_p})} \forall\mathrm{E}$$

Definition 7.5

$$\dfrac{\dfrac{\forall x^{rb}\forall x^{s}[FR(x^{rb}, x^{s}) \leftrightarrow [NF(x^{rb}, x^{s}) \wedge PF(x^{rb}, x^{s})]}{\vdots \ \text{Similar to } \mathcal{D}_{10}}}{FR(x^{hb}, x^{s_p}) \rightarrow NF(x^{hb}, x^{s_p})}$$

$$\dfrac{FR(x^{hb}, x^{s_p}) \rightarrow NF(x^{hb}, x^{s_p}) \qquad FR(x^{hb}, x^{s_p})}{NF(x^{hb}, x^{s_p})} \rightarrow\mathrm{E}$$

Proposition 25

$$\dfrac{\forall x^{rb}\forall x^{s}[NF(x^{rb}, x^{s}) \rightarrow \forall \ulcorner\phi\urcorner[[PP(\ulcorner\phi\urcorner) \wedge O_i(\ulcorner\phi\urcorner)] \rightarrow [DB(x^{rb}, \ulcorner\phi\urcorner, x^{s}) \leftrightarrow [DG(\ulcorner\phi\urcorner, x^{rb}, x^{s}) \wedge form(\ulcorner\phi\urcorner, x^{s})]]]]}{\vdots \ \text{Similar to } \mathcal{D}_{10}}$$

$$\dfrac{\forall x^{hb}\forall x^{s_p}[NF(x^{hb}, x^{s_p}) \rightarrow \forall \ulcorner\phi\urcorner[\ \theta \rightarrow DB(x^{hb}, \ulcorner\phi\urcorner, x^{s_p})]]}{NF(x^{hb}, x^{s_p}) \rightarrow \forall \ulcorner\phi\urcorner[\ \theta \rightarrow DB(x^{hb}, \ulcorner\phi\urcorner, x^{s_p})]} \forall\mathrm{E}$$

$$\dfrac{NF(x^{hb}, x^{s_p}) \qquad NF(x^{hb}, x^{s_p}) \rightarrow \forall \ulcorner\phi\urcorner[\ \theta \rightarrow DB(x^{hb}, \ulcorner\phi\urcorner, x^{s_p})]}{\forall \ulcorner\phi\urcorner[\ \theta \rightarrow DB(x^{hb}, \ulcorner\phi\urcorner, x^{s_p})]} \rightarrow\mathrm{E}$$

$$\dfrac{\forall \ulcorner\phi\urcorner[\ \theta \rightarrow DB(x^{hb}, \ulcorner\phi\urcorner, x^{s_p})]}{\theta \rightarrow DB(x^{hb}, \ulcorner\phi\urcorner, x^{s_p})} \forall\mathrm{E}$$

Proposition 26

$$\dfrac{\forall x^{hb}\forall x^{s}\forall \ulcorner\phi\urcorner[[\theta \rightarrow DB(x^{hb}, \ulcorner\phi\urcorner, x^{s})] \rightarrow [\theta \rightarrow DB(x^{hb}, \ulcorner\phi\urcorner, x^{s})][\ulcorner\phi\urcorner / \ulcorner ML \urcorner]]}{\vdots \ \text{Similar to } \mathcal{D}_{10}}$$

$$\dfrac{\theta \rightarrow DB(x^{hb}, \ulcorner\phi\urcorner, x^{s_p}) \qquad [\theta \rightarrow DB(x^{hb}, \ulcorner\phi\urcorner, x^{s_p})] \rightarrow [\chi \rightarrow DB(x^{hb}, \ulcorner ML \urcorner, x^{s_p})]}{\chi \rightarrow DB(x^{hb}, \ulcorner ML \urcorner, x^{s_p})} \rightarrow\mathrm{E}$$

Proposition 26

$$\dfrac{\forall x^{hb}\forall x^{s}\forall \ulcorner\phi\urcorner[\theta \rightarrow \theta[\ulcorner\phi\urcorner / \ulcorner ML \urcorner]]}{\vdots \ \text{Similar to } \mathcal{D}_{10}}$$

$$\dfrac{\theta^w \qquad \theta \rightarrow \chi}{\chi} \rightarrow\mathrm{E}$$

$$\dfrac{\chi \qquad \chi \rightarrow DB(x^{hb}, \ulcorner ML \urcorner, x^{s_p})}{\mathcal{D}_{17}: \quad DB(x^{hb}, \ulcorner ML \urcorner, x^{s_p})} \rightarrow\mathrm{E}$$

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