One-Step Algebras and Frames for Modal and Intuitionistic Logics

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

This thesis is about one-step algebras and frames and their relation to the proof theory of non-classical logics. We show how to adapt the framework of modal one-step algebras and frames from [11] to intuitionistic logic. We prove that, as in the modal case, extension properties of one-step Heyting algebras can characterize a certain weak analytic subformula property (*the bounded proof property*) of hypersequent calculi. We apply our methods to a number of hypersequent calculi for well-known intermediate logics. In particular, we present a hypersequent calculus for the logic $\mathbf{BD}_3$ with the bounded proof property. Finally, we establish a connection between modal one-step algebras and filtrations [37].

Til Axel

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Contents

Chapter 1

Introduction

Having a well-behaved proof system for a given logic can help determine various desirable properties of this logic such as consistency and decidability. In many cases having a good proof theoretic presentation of a logic may be essential when it comes to applications. Gentzen-style sequent calculi have for a long time played a pivotal role in proof theory [52] and proving admissibility of the cut-rule has been one of the main tools for establishing good proof theoretic properties of sequent calculi. However, for various non-classical logics finding a cut-free sequent calculus can be a difficult task, even when the logic in question has a very simple semantics. In fact, in many cases no such calculus seems to exist. In the 1980's Pottinger [49] and Avron [3] independently introduced hypersequent calculi for handling certain modal and relevance logics. Hypersequents are nothing more than finite sets of sequents but they give rise to simple cut-free calculi for many logics for which no ordinary cut-free calculus has been found. Since then cut-free hypersequent calculi for various modal and intermediate logics have been given [4, 22, 21, 29, 41, 25, 45, 46]. However, establishing cut-elimination for Gentzen-style sequent or hypersequent calculi by syntactic means can be very cumbersome and errors are easily made. Although the basic idea behind syntactic proofs of cut-elimination is simple, each individual calculus will need its own proof of cut-elimination and proofs obtained for one calculus do not necessarily transfer easily to other – even very similar – calculi. Recently some steps to ameliorate this situation have been taken. For example [45, 46] provide general methods for obtaining cut-free calculi for larger classes of logics based on their semantics.

Semantic proofs of cut-elimination have been known since at least 1960 [51], but in recent years a general algebraic approach to proving cut-elimination for various substructural logics via McNeille completions has been developed [23, 24]. One of the attractive features of this approach is that it allows one to establish cut-elimination for large classes of logics in a uniform way. Moreover, [23, 24] also provide algebraic criteria determining when cut-free (hyper)sequent calculi for a given substructural logic can be obtained.[1] This algebraic approach suggests that algebraic semantics can be used to detect other desirable features of a proof system. It is this kind of *algebraic proof theory* that is the main subject of this thesis. However, we will take a somewhat different approach to connecting algebra and proof theory than the one outlined above. In particular, we will be focusing on characterizing a proof theoretic property weaker than – though in some ways similar to – cut-elimination.

The free algebra of a propositional logic encodes a lot of information about the logic. For instance it is well-known that the finitely generated free algebras constitute a powerful tool when it comes to establishing meta-theoretical properties for various propositional logics such as interpolation, definability, admissiblity of rules etc. In the early 1990's Ghilardi [35] showed that finitely generated free Heyting algebras are (chain) colimits of finite distributive lattices. A few years later he established a similar result for finitely generated free modal algebras; showing that these arise as colimits of finite Boolean algebras [36].[2] The intuition behind these constructions is that one constructs the finitely generated free algebra in stages by freely adding the Heyting implication (or in the case of modal algebras the modal operator) step by step. Lately this construction has received renewed attention in [18, 10] (for Heyting algebras) and in [14, 37, 39, 13] (for modal algebras). Finally, in [30] sufficient criteria are given for this construction to succeed for finitely generated free algebras in an arbitrary variety.

It was realized in [11] that the so-called *modal one-step algebras* arising as consecutive pairs of algebras in the colimit construction of finitely generates free modal algebras can be used to characterize a certain weak analytic subformula property of proof systems for modal logics. This property – called *the bounded proof property* – holds of an axiom system Ax if for every finite set of formulas $\Gamma \cup \{\varphi\}$ of modal depth[3] at most n such that Γ entails φ over Ax there exists a derivation in Ax witnessing this in which all the formulas have modal depth at most n. We write $\Gamma \vdash^n_{\mathrm{Ax}} \varphi$ if this is the case. With this notation the bounded proof property may be expressed as

$$\Gamma \vdash_{\mathrm{Ax}} \varphi \implies \Gamma \vdash^n_{\mathrm{Ax}} \varphi.$$

Even though this is a fairly weak property it does e.g. bound the search space when searching for proofs and thus it ensures decidability of logics with a finite axiomatization.

[1] However, these criteria only cover the lower levels ($\mathcal{N}_2$ and $\mathcal{P}_3$) of the substructural hierarchy of [23].

[2] The basic idea of constructing finitely generated free modal algebras in an incremental way is in some sense already present in [32] and [1]. Note that [1] is based on a talk given at the BCTCS in 1988.

[3] Recall that the *modal depth* of a formula φ is the maximal number of nested modalities occurring in φ.

Furthermore, having this property might serve as an indication of robustness of the axiom system in question. In this way it is like cut-elimination although in general it is much weaker.

In light of Ghilardi's original colimit construction of finitely generated free Heyting algebras it seems natural to ask if one can adapt the work of [11] to the setting of intuitionistic logic and its extensions. That is, we ask if it is possible to formulate the bounded proof property for intuitionistic logic and define a notion of one-step Heyting algebras which can characterize proof systems of intermediate logics with the bounded proof property. Answering this question will be the main focus of this thesis.

In order to do this one first needs to choose a proof theoretic framework for which to ask this question. In this respect there are two remarks to be made. First of all as any use of *modus ponens* will evidently make the bounded proof property with respect to implications fail, we will have to consider proof systems different from natural deduction or Hilbert-style proof systems. Therefore, a Gentzen-style sequent calculus might be a better option. In these systems modus ponens is replaced with the cut-rule which for good systems can be eliminated or at least restricted to a well-behaved fragment of the logic in question. Secondly, as mentioned in the beginning of the introduction, ordinary sequent calculi are often ill-suited when it comes to giving well-behaved calculi for concrete intermediate logics, in that they generally do not admit cut-elimination. Therefore, keeping up with the recent trend in proof theory of non-classical logics, we base our approach on hypersequent calculi. This makes our results more general and more importantly allows us to consider more interesting examples of proof systems for intermediate logics. This approach is also in line with [12] where the results of [11] are generalized to the framework of multi-conclusion rule systems for modal logics.

Using Ghilardi's colimit construction we define a notion of one-step Heyting algebras and develop a theory of these algebras parallel to the theory of one-step modal algebras [11]. We show that just like in the modal case the bounded proof property for intuitionistic hypersequent calculi can be characterized algebraically in terms of one-step Heyting algebras. We also develop a notion of intuitionistic one-step frames dual to that of one-step Heyting algebras and present a basic one-step correspondence theory for hypersequent rules enabling us to determine under which conditions a one-step frame validates a hypersequent calculus.

We test the obtained criterion of the bounded proof property on a fair number of examples of hypersequent calculi for intermediate logics: $\mathbf{LC}$, $\mathbf{KC}$, $\mathbf{BW}_n, \mathbf{BD}_2$ and $\mathbf{BD}_3$. For all but the last of these logics cut-free hypersequent calculi already exist. Using our methods we show that the naïvely constructed calculi for these logics do not have the bounded proof property. For $\mathbf{BW}_n$ we also give an alternative hypersequent calculus with the bounded proof property. In [25] a cut-free hypersequent calculus for $\mathbf{BD}_2$ was presented. For this logic the uniform semantic cut-elimination proof of [23] does not apply and therefore in [25] cut-elimination was established by purely syntactic means. Using our methods we show that this system has the bounded proof property. This of course already follows from [25]. However, using the one-step semantics of $\mathbf{BD}_2$ we construct a hypersequent calculus for $\mathbf{BD}_3$ with the bounded proof property. To the best of our knowledge no hypersequent calculus for the logic $\mathbf{BD}_3$ exists in the literature. Although we do not know whether or not our calculus for $\mathbf{BD}_3$ has cut-elimination, this result suggests that our methods might be useful when it comes to designing well-behaved proof systems for intermediate logics.

Finally, the discrepancy in the definition of one-step frames in the modal and intuitionistic sense inspires us to describe modal one-step frames in a way similar to the simpler intuitionistic one-step frames. In doing so we are able to shed some light on the connection between modal one-step frames and filtrations. To some degree this connection is already implicitly suggested in [37, 30, 39][4].

Outline of the thesis

In Chapter 2 we sketch the construction of finitely generated free Heyting and modal algebras and show how this in the modal case gives rise to the notion of modal one-step algebras and one-step frames. Finally, we briefly review the work of Bezhanishvili and Ghilardi [11] linking the modal one-step framework to proof theory.

In Chapter 3 we introduce one-step Heyting algebras. We work out the duality for finite one-step Heyting algebras and show how one-step Heyting algebras can interpret hypersequent rules.

In Chapter 4 we establish an algebraic characterization of the bounded proof property in terms of one-step Heyting algebras. Thus showing that the results of [11] may be transferred to the intuitionistic setting. Furthermore, we introduce a basic calculus for computing first-order one-step frame conditions corresponding to hypersequent rules.

In Chapter 5 we give a number of examples of hypersequent calculi having and lacking the bounded proof property. Most of the calculi that we consider will not have the bounded proof property. However, we present a hypersequent calculus for the logic $\mathbf{BD}_3$ which has the bounded proof property.

[4] Moreover, the connection between filtrations and modal one-step frames was explicitly suggested in a talk by Van Gool in connection with the presentation of the paper [39].

Finally, in Chapter 6, inspired by the surprisingly simple definition of a one-step intuitionistic one-step frame, we show how to describe the modal one-step frames of [13, 11] as pairs of standard Kripke frames with a certain kind of relation preserving maps between them. Moreover, we show that the duals of such pairs are in fact minimal filtrations in the algebraic sense.

In Chapter 7 we provide a brief summary of the thesis and suggest a few directions for further work.

Main results

We here briefly mention the original contributions of the thesis.

- We introduce a notion of one-step Heyting algebra and one-step intuitionistic Kripke frames and show how these can be used to characterize the bounded proof property with respect to implications for hypersequent calculi for intermediate logics.
- We introduce a hypersequent calculus for the logic $\mathbf{BD}_3$ and show that it enjoys the bounded proof property. We also give examples of calculi for well-known intermediate logics without the bounded proof property.
- We show that the modal one-step frames of [13, 11] can be realized as the duals of filtrations of modal algebras in the sense of [37].

Prerequisites

We assume some familiarity with the basics of modal logic [15, 19], intermediate logics [9, 19], universal algebra [5, 6] and category theory [47]. In particular, we assume that the reader is well acquainted with the most basic properties of (chain) colimits of algebras [20, 6]. Finally, it might be helpful if the reader has been exposed to Gentzen-style sequent calculi [52].

Chapter 2

Finitely generated free algebras as colimits

In this chapter we review the constructions of finitely generated free (Heyting and modal) algebras as a colimit of finite algebras. Moreover, we recall the basics of the theory of modal one-step frames and algebras from [11, 13]. The purpose of this chapter is two-fold: Firstly, it is to serve as a sketch of the development of the line of research of which this thesis is a continuation. Secondly, it is to serve as preliminaries, introducing well-known notions and techniques which will be used throughout this thesis.

2.1 Finitely generated free Heyting algebras as a colimit

The material in this section closely follows [35] and to a lesser extent [10, 18].

Let $\mathsf{bDist}_{<\omega}$ denote the category of finite bounded distributive lattices and bounded lattice homomorphisms. Moreover, let $\mathsf{Pos}_{<\omega}$ be the category of finite partially ordered sets and order-preserving maps between them.

By a *downset* we shall understand a subset U of poset $(P, \leq)$ such that if $p \in U$ and $q \leq p$ then $q \in U$. Let $\mathsf{Do}(P)$ denote the set of downsets of P[1].

If D is a distributive lattice and $a \in D$ is such that $a \neq \bot$, and

$$\forall b, c \in D\ (a = b \vee c \implies a = c \text{ or } a = c),$$

[1]When no confusion arises we will often refer to a poset – or any other structured set – by referring to its carrier set.

then we say that a is *join-irreducible*. Let $\mathcal{J}(D)$ denote the set of join-irreducible elements of D. Then $\mathcal{J}(D)$ is a poset with the order given by restricting the order on D to $\mathcal{J}(D)$.

For each order-preserving map $f\colon P \to Q$ between posets we obtain a bounded lattice homomorphism $\mathsf{Do}(f)\colon \mathsf{Do}(Q) \to \mathsf{Do}(P)$ by letting $\mathsf{Do}(f)(U) = f^{-1}(U)$. This determines a functor $\mathsf{Do}(-)\colon \mathsf{Pos}_{<\omega} \to \mathsf{bDist}^{\mathrm{op}}_{<\omega}$. Likewise, we obtain a functor $\mathcal{J}\colon \mathsf{bDist}_{<\omega} \to \mathsf{Pos}^{\mathrm{op}}_{<\omega}$ by letting $\mathcal{J}(h)\colon \mathcal{J}(D') \to \mathcal{J}(D)$ be the restriction to $\mathcal{J}(D')$ of the left adjoint $h^\flat\colon D' \to D$ of the bounded distributive lattice homomorphism $h\colon D \to D'$. Recall that the left adjoint $h^\flat\colon D' \to D$ is given by

$$h^\flat(a) = \bigwedge_{a \leq h(a')} a'.$$

The following is a well-known theorem originally due to Birkhoff.

Theorem 2.1 (Birkhoff 1933)**.** *The functors* Do *and* $\mathcal{J}$ *exhibit the categories* $\mathsf{bDist}_{<\omega}$ *and* $\mathsf{Pos}_{<\omega}$ *as dually equivalent.*

Recall that a *Heyting algebra* is a lattice $\mathfrak{A} = (A, \wedge, \vee, \to, \bot)$ with a least element $\bot$ and a binary operation $\to\colon A^2 \to A$, called a *(Heyting) implication*, satisfying the following residuation property

$$\forall a, b, c \in A\ (a \wedge c \leq b \iff c \leq a \to b).$$

It is an easy exercise to show that the underlying lattice of a Heyting algebra is in fact a bounded distributive lattice.

A *Heyting algebra homomorphism* $h\colon \mathfrak{A} \to \mathfrak{A}'$ between Heyting algebras will be a lattice homomorphism preserving $\bot$ and the implication. We thus obtain a category HA of Heyting algebras and Heyting algebra homomorphisms. Finally, recall that every finite distributive lattice D is a Heyting algebra with implication defined as

$$a \to b := \bigvee\{c \in D\colon a \wedge c \leq b\}.$$

We observe further that for P a finite poset the lattice $\mathsf{Do}(P)$ carries the structure of a Heyting algebra with the implication defined as

$$U \to V := \{a \in P\colon \forall b \leq a\ (b \in U \implies a \in V)\} = P \backslash {\uparrow}(U \backslash V).$$

In what follows we will show how Birkhoff duality can be used to describe the finitely generated free Heyting algebras. This however requires that we describe the subcategory of $\mathsf{Pos}_{<\omega}$ which is dually equivalent to the category of finite Heyting algebras. For this we need to determine the order-preserving maps corresponding to Heyting homomorphisms under Birkhoff duality.

Definition 2.2. We say that an order-preserving map $f\colon P \to Q$ between posets is *open* if the diagram

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\leq"'] & Q \arrow[d, "\leq"] \\ P \arrow[r, "f"] & Q \end{tikzcd}$$

commutes, as a square of relations, where for relations $R_1 \subseteq X \times Y$ and $R_2 \subseteq Y \times Z$ the composition $R_2 \circ R_1 \subseteq X \times Z$ is the relation given by

$$x(R_2 \circ R_1)z \iff \exists y \in Y \ (xR_1y \text{ and } yR_2z).$$

Proposition 2.3. *Let $f\colon P \to Q$ be an order-preserving map between posets. Then the following are equivalent:*

i) f is open;

ii) f^{-1} preserves all existing Heyting implications in $\mathsf{Do}(Q)$;

iii) $\forall a \in P\ \forall b \in Q\ (b \leq f(a) \implies \exists a' \in P\ (a' \leq a$ and $f(a') = b))$;

iv) f is an open map when considering P and Q as topological spaces with the topologies $\mathsf{Do}(P)$ and $\mathsf{Do}(Q)$, respectively.

Proof. Routine verification. □

Thus the Birkhoff duality restricts to a duality between the category $\mathsf{HA}_{<\omega}$ of finite Heyting algebras and Heyting algebra homomorphisms and the category $\mathsf{Pos}_{<\omega}^{\mathrm{open}}$ of finite poset and order-preserving open maps between them, see e.g. [38, Thm. 2.1] for a detailed proof.

It turns out that it is useful also to have a relative notion of open maps.

Definition 2.4. Let $f\colon P \to Q$ and $g\colon Q \to R$ be order-preserving maps between posets. Then we say that f is *g-open* if

$$\begin{tikzcd} P \arrow[r, "f\circ\leq_R"] \arrow[d, "\leq_Q\circ f"'] & Q \arrow[d, "g"] \\ Q \arrow[r, "g"] & R \end{tikzcd}$$

commutes, as a square of relations.

This notion will come up several times throughout this thesis.

Proposition 2.5. *Let $f\colon P \to Q$ and $g\colon Q \to R$ be an order-preserving maps. Then the following are equivalent:*

i) *f is g-open;*

ii) *f^{-1} preserves all existing Heyting implications of the form $g^{-1}(U) \to g^{-1}(U')$, with $U, U' \in \mathsf{Do}(R)$;*

iii) $\forall a \in P\ \forall b \in Q\ (b \leq f(a) \implies \exists a' \in P\ (a' \leq a \text{ and } g(f(a')) = g(b)))$.

Proof. Routine verification. □

If $g\colon Q \to R$ is order-preserving and $S \subseteq Q$ is such that the inclusion $\iota\colon S \to Q$ is g-open we say that S is a *g-open subset.*

Proposition 2.6. *Let $g\colon Q \to R$ be order-preserving and $S \subseteq Q$ a g-open subset. Then*

i) $\forall s \in S\ \forall b \in Q\ (b \leq s \implies \exists s' \in S(s' \leq s \text{ and } g(s') = g(b)))$;

ii) $\forall s \in S\ \forall r \in R\ ({\downarrow}s \cap g^{-1}(r) \neq \emptyset \implies (({\downarrow}s) \cap S) \cap g^{-1}(r) \neq \emptyset))$

iii) *The set $({\downarrow}s) \cap S$ is g-open for all $s \in S$.*

Proof. Item i) follows from proposition 2.5. Item ii) follows directly from item i) since $Q = \bigcup_{r \in R} g^{-1}(r)$. Finally to see that $({\downarrow}s) \cap S$ is g-open for all $s \in S$, let $s \in S$ be given and consider $s' \in ({\downarrow}s) \cap S$. Then if for some $b \in Q$ we have that $b \leq s'$, and since $s' \in S$ and S is an g-open subset of Q we know from item i) that there exists $s'' \in S$ such that $s'' \leq s'$ and such that $g(s'') = g(b)$. By transitivity $s'' \in ({\downarrow}s) \cap S$ and hence by proposition 2.5 that $({\downarrow}s) \cap S$ is g-open. □

Remark 2.7. Note that item i) is actually equivalent to S being a g-open subset.

Definition 2.8. For $g\colon Q \to R$ an order-preserving map we let

$$\mathscr{O}_g(Q) \coloneqq \{S \in \wp(Q)\colon\ S \text{ is } g\text{-open}\}.$$

By a *rooted subset* of Q we shall understand a subset with a *root*, i.e. a greatest element. We then let $\mathscr{O}^{\bullet}_g(Q)$ denote the subset of $\mathscr{O}_g(Q)$ consisting of rooted sets.

Proposition 2.9. *The map $r^g\colon \mathscr{O}^{\bullet}_g(Q) \to Q$ given by $S \mapsto \max(S)$ is a g-open surjection. Moreover, it has a right adjoint which is a section of r^g. That is, there exists an order-preserving map $r_g\colon Q \to \mathscr{O}^{\bullet}_g(Q)$ such that $r^g(S) \leq a$ iff $S \subseteq r_g(a)$ and such that $r^g \circ r_g = \mathrm{id}_Q$.*

Proof. Let $S \in \mathscr{O}_g^\bullet(Q)$ be given and let $b \in Q$ be such that $b \leq r^g(S) = \max(S)$. Because S is g-open it follows from proposition 2.6 i) that there exists $s' \in S$ such that $s' \leq \max(S)$ with $g(s') = g(b)$. Then by proposition 2.6 iii) $S' := ({\downarrow} s') \cap S \subseteq S$ is a rooted g-open subset satisfying $g(r^g(S')) = g(s') = g(b)$. This shows that $r^g \colon \mathscr{O}_g^\bullet(Q) \to Q$ is a g-open map.

For the last part of the statement of the proposition we claim that $r_g \colon Q \to \mathscr{O}_g^\bullet(Q)$ given by $a \mapsto {\downarrow} a$ is a right adjoint of r^g which is also a section. To see that r_g is indeed well-defined we first observe that ${\downarrow} a$ is clearly rooted, with root a. To see that ${\downarrow} a$ is also a g-open subset of Q note that if $b \leq a'$ for some $a' \in {\downarrow} a$ then $b \in {\downarrow} a$ and so b itself is a witness of g-openness.

Since $r^g(r_g(a)) = \max({\downarrow} a) = a$ we have that r_g is a section of r^g and consequently that r^g is a surjection.

Finally, to see that r_g is also a right adjoint of r^g we simply observe that for all rooted subsets S of Q and all $b \in Q$

$$r^g(S) \leq b \iff \max(S) \leq b \iff S \subseteq {\downarrow} b \iff S \leq r_g(b).$$

$\square$

The map $r^g \colon \mathscr{O}_g^\bullet(Q) \to Q$ induces a map $(r^g)^{-1} \colon \mathsf{Do}(Q) \to \mathsf{Do}(\mathscr{O}_g^\bullet(Q))$ of bounded distributive lattices with a useful universal property.

Lemma 2.10. *Let $g \colon Q \to R$ be an order-preserving map between finite posets. Then $(r^g)^{-1} \colon \mathsf{Do}(Q) \to \mathsf{Do}(\mathscr{O}_g^\bullet(Q))$ is a bounded lattice homomorphism with the following universal property: For any morphism $h \colon \mathsf{Do}(Q) \to D$ in the category* bDist *with the property that for all $U, U' \in \mathsf{Do}(Q)$ the Heyting implications $h(U) \to h(U')$ exists in D and h preserves Heyting implications of the form $g^{-1}(V) \to g^{-1}(V')$ for $V, V' \in \mathsf{Do}(R)$, there exists a unique factorisation*

$$\begin{tikzcd} \mathsf{Do}(Q) \arrow[r, "(r^g)^{-1}"] \arrow[dr, "h"'] & \mathsf{Do}(\mathscr{O}_g^\bullet(Q)) \arrow[d, dashed, "h'"] \\ & D \end{tikzcd}$$

of h in the category bDist. *Moreover, the map h' will preserve all Heyting implications of the form $(r^g)^{-1}(U) \to (r^g)^{-1}(U')$, for $U, U' \in \mathsf{Do}(Q)$.*

Proof. Because the variety of bounded distributive lattices is locally finite and $\mathsf{Do}(Q)$ is finite – as Q is – so is the bounded distributive lattice generated by the image of $\mathsf{Do}(Q)$

under h. Thus without loss of generality we may assume that D is a finite bounded distributive lattice. But then we may equally well prove the dual statement about the category $\mathsf{Pos}_{<\omega}$, which is the following: Every g-open order preserving map $f\colon P \to Q$ factors uniquely as

$$\begin{tikzcd} \mathscr{O}_g^{\bullet}(Q) \arrow[r, "r^g"] & Q \\ P \arrow[u, dashed, "f'"] \arrow[ur, "f"'] & \end{tikzcd}$$

with f' a r^g-open map.

We claim that $f'\colon P \to \mathscr{O}_g^{\bullet}(Q)$ given by $a \mapsto f({\downarrow}a)$ will be the required map. That this is a well-defined order-preserving r^g-open map making the above diagram commute is straightforward to check. To see that it is also unique suppose that $f''\colon P \to \mathscr{O}_g^{\bullet}(Q)$ is an order-preserving r^g-open map making the above diagram commute. Then we must have that $\max f''(a) = f(a)$ for all $a \in P$ and so by the assumption that f'' is order-preserving we may conclude that $f''(a) \supseteq f({\downarrow}a)$. Conversely if $b \in f''(a)$ then by Proposition 2.6 iii) we have that $S := {\downarrow}b \cap f''(a)$ is a g-open subset. Moreover as S is evidently rooted we have $S \in \mathscr{O}_g^{\bullet}(Q)$. Therefore as $S \subseteq f''(a)$ we obtain from the assumption that f'' is r^g-open that there exists $a' \leq a$ such that $r^g(f''(a')) = r^g(S)$. From this it follows that

$$f(a') = r^g(f''(a')) = r^g(S) = \max\{{\downarrow}b \cap f''(a)\} = b,$$

and thereby that $f''(a) \subseteq f({\downarrow}a)$, thus showing that f' is indeed the unique order preserving r^g-open map such that $f = f' \circ r^g$. □

2.1.1 The colimit construction

Now given a finite poset P we want to construct the Heyting algebra freely generated by the distributive lattice $D := \mathsf{Do}(P)$. That is, we want to characterise the Heyting algebra $\mathscr{F}(D)$ determined by the following universal property: There exists a bounded distributive lattice homomorphism $\iota\colon D \to \mathscr{F}(D)$ such that for any Heyting algebra $\mathfrak{A}'$ with bounded distributive lattice homomorphism $j\colon D \to \mathfrak{A}'$ there exists a unique Heyting algebra map $h\colon \mathscr{F}(D) \to \mathfrak{A}'$ such that

$$\begin{tikzcd} D \arrow[r, "\iota"] \arrow[d, "j"] & \mathscr{F}(D) \arrow[dl, dotted, "h"] \\ \mathfrak{A}' & \end{tikzcd}$$

commutes.

An equational description of $\mathscr{F}(D)$ can be obtained from an equational description of D as follows:

Proposition 2.11. *Given a presentation $\langle \vartheta, \kappa \rangle$ of D i.e. a cardinal κ and a congruence of the free κ-generated bounded distributive lattice $F_{\mathsf{bDist}}(\kappa)$ such that $D \cong F_{\mathsf{bDist}}(\kappa)/\vartheta$, then $\mathscr{F}(D) \cong F_{\mathsf{HA}}(\kappa)/\vartheta'$ where ϑ' is the congruence on the free κ-generated Heyting algebra $F_{\mathsf{HA}}(\kappa)$ generated by ϑ.*[2]

However, as freely generated Heyting algebras are notoriously complicated objects – only $F_{\mathsf{HA}}(1)$ can truly be said to be fully understood – this description of $\mathscr{F}(D)$ is not particularly informative.

In the following we give a description of $\mathscr{F}(D)$, for finite bounded distributive lattice D, as the colimit of a chain of finite bounded distributive lattices built from D. This construction can be seen as freely adding Heyting implication among the elements of D one step at a time at each stage making sure that the previously added implications will be preserved.

Given a finite poset P let $\mathbf{1}$ be the terminal object in the category of posets, i.e. the one-element poset $\{*\}$ consisting of one reflexive point. We then define a co-chain

$$\cdots \xrightarrow{r_{n+2}} P_{n+1} \xrightarrow{r_{n+1}} \cdots \xrightarrow{r_2} P_1 \xrightarrow{r_1} P_0 \xrightarrow{r_0} \mathbf{1}$$

in the category of finite posets by the following recursion: We let $P_0 \coloneqq P$ and we let $r_0 \colon P_0 \to \mathbf{1}$ be the obvious map. For $n \in \omega$ we then define

$$P_{n+1} \coloneqq \mathscr{O}^{\bullet}_{r_n}(P_n) \quad \text{and} \quad r_{n+1} \coloneqq r^{r_n}.$$

By Brikhoff duality this induces a chain in the category bDist. Moreover as the map r_n is surjective for all $n \in \omega$ we obtain that r_n^{-1} is injective for all $n \in \omega$.

Theorem 2.12 (Ghilardi [35])**.** *Let P be a finite poset. The free Heyting algebra generated by the distributive lattice $\mathsf{Do}(P)$ is realized as the colimit in the category of bounded distributive lattices of the chain*

$$\mathsf{Do}(\mathbf{1}) \xrightarrow{r_0^{-1}} \mathsf{Do}(P_0) \xrightarrow{r_1^{-1}} \mathsf{Do}(P_1) \xrightarrow{r_2^{-1}} \cdots \xrightarrow{r_n^{-1}} \mathsf{Do}(P_n) \xrightarrow{r_{n+1}^{-1}} \mathsf{Do}(P_{n+1}) \longrightarrow \cdots$$

Proof. Let $\mathfrak{A}$ be the colimit in bDist of the above diagram. Let $f_{m,n} \colon \mathsf{Do}(P_m) \to \mathsf{Do}(P_n)$ be the induced maps for $m \leq n$ and let $f_n \colon \mathsf{Do}(P_n) \to \mathfrak{A}$ be the maps given by $a \mapsto [a]$.

[2] This is well-defined as the language of bounded distributive lattices is a reduct of the language of the language of Heyting algebras.

First to see that $\mathfrak{A}$ is a Heyting algebra we claim that the following defines a Heyting implication

$$[a] \to [b] = [f_{m,k+1}(a) \to_{\mathsf{Do}(P_{k+1})} f_{n,k+1}(b)], \quad k = \max\{m, n\}$$

for $a \in \mathsf{Do}(P_m)$ and $b \in \mathsf{Do}(P_n)$. To see this let $a \in \mathsf{Do}(P_m)$ and $b \in \mathsf{Do}(P_n)$ and $c \in \mathsf{Do}(P_l)$ be given. We then have that $[c] \leq [a] \to [b]$ precisely when

$$f_{l,k'}(c) \leq f_{k+1,k'}(f_{m,k+1}(a) \to_{\mathsf{Do}(P_{k+1})} f_{n,k+1}(b)),$$

where $k = \max\{n, m\}$ and $k' = \max\{l, k\}$. Now since r_{j+1} is r_j-open for all $j \in \omega$ we must have that f_{k+1} preserves all implications in $\mathsf{Do}(P_{k+1})$ between elements in the images of $f_{j,k+1}$ for $j \in \omega$. It follows that

$$\begin{aligned}
[c] \leq [a] \to [b] &\iff f_{l,k'}(c) \leq f_{m,k'}(a) \to_{\mathsf{Do}(P_{k'})} f_{n,k'}(b) \\
&\iff f_{m,k'}(a) \wedge f_{l,k'}(c) \leq f_{n,k'}(b) \\
&\iff [a] \wedge [c] \leq [b].
\end{aligned}$$

Finally to see that it is in fact the free Heyting algebra generated by the bounded distributive lattice $\mathsf{Do}(P)$ we note that by the definition of a colimit we have an bounded distributive lattice homomorphism $\iota := f_1 \colon \mathsf{Do}(P) \to \mathfrak{A}$. This will be an injection since the maps r_k^{-1} are injective for all $k \geq 1$.

Now suppose that $\mathfrak{A}'$ is a Heyting algebra with the property that there exists a bounded lattice homomorphism $j \colon \mathsf{Do}(P) \to \mathfrak{A}'$. Then as $\mathfrak{A}$ is a Heyting algebra j clearly satisfies the conditions of Lemma 2.10 with respect to the map $r_0 \colon P \to \mathbf{1}$ hence we obtain a bounded lattice homomorphism $j' \colon \mathsf{Do}(P_1) \to \mathfrak{A}'$. Thus we may by recursion define a collection of r_n^{-1}-open maps $j_n \colon \mathsf{Do}(P_n) \to \mathfrak{A}'$ such that the triangle

$$\begin{tikzcd}
\mathsf{Do}(P_n) \arrow[r, "r_{n+1}^{-1}"] \arrow[d, "j_n"] & \mathsf{Do}(P_{n+1}) \arrow[dl, "j_{n+1}"] \\
\mathfrak{A}' &
\end{tikzcd}$$

commutes for all $n \in \omega$. We let $j_0 = j$ and $j_{n+1} = j_n'$ where j_n' is determined by Lemma 2.10. Since $\mathbf{1}$ is terminal in Pos the object $\mathsf{Do}(1)$ must be initial in the category bDist hence we have a unique map $j_{-1} \colon \mathsf{Do}(1) \to \mathfrak{A}'$ and so by the universal property of the colimit we obtain a unique bounded lattice homomorphism $h \colon \mathfrak{A} \to \mathfrak{A}'$ such that $j_n = h \circ f_n$, in particular unique such that $j = h \circ \iota$. Finally, we claim that h is a Heyting algebra homomorphism. Indeed for $a \in \mathsf{Do}(P_m)$ and $b \in \mathsf{Do}(P_n)$ we have with

$k = \max\{m, n\}$ that

$$\begin{aligned}
h([a] \to [b]) &= h([f_{m,k+1}(a) \to_{\mathsf{Do}(P_{k+1})} f_{n,k+1}(b)]) \\
&= hf_{k+1}(f_{m,k+1}(a) \to_{\mathsf{Do}(P_{k+1})} f_{n,k+1}(b)) \\
&= j_{k+1}(f_{m,k+1}(a) \to_{\mathsf{Do}(P_{k+1})} f_{n,k+1}(b)) \\
&= j_{k+1}(r_{k+1}^{-1}(f_{m,k}(a)) \to_{\mathsf{Do}(P_{k+1})} r_{k+1}^{-1}(f_{n,k}(b))) \\
&= j_{k+1}(r_{k+1}^{-1}(f_{m,k}(a))) \to_{H'} j_{k+1}(r_{k+1}^{-1}(f_{n,k}(b))) \\
&= j_{k+1}(f_{m,k+1}(a)) \to_{H'} j_{k+1}(f_{n,k+1}(b)) \\
&= hf_{k+1}(f_{m,k+1}(a)) \to_{H'} hf_{k+1}(f_{n,k+1}(b)) \\
&= h([a]) \to_{H'} h([b]),
\end{aligned}$$

showing that $\mathfrak{A}$ is indeed the free Heyting algebra generated by $\mathsf{Do}(P)$. □

This concludes the section on free finitely generated Heyting algebras. The colimit construction will serve as inspiration when defining one-step Heyting algebras in Chapter 3.

2.2 Finitely generated free modal algebras as colmits

In [36] it was shown that one can also describe finitely generated free modal algebras as a colimits of finite Boolean algebras. In this sections we review this construction. This section is based on [36, 14, 13].

2.2.1 Modal algebras as algebras for the Vietoris functor

Recall that a *modal algebra* is an algebra $\mathfrak{A} = (A, \Diamond)$ such that A is a Boolean algebra and $\Diamond \colon A \to A$ is a hemimorphism, i.e a function satisfying

$$\Diamond\bot = \bot \quad \text{and} \quad \Diamond(a \vee b) = \Diamond a \vee \Diamond b.$$

By a *modal algebra homomorphism* from a modal algebra $\mathfrak{A}$ to a modal algebra $\mathfrak{B}$ we understand a Boolean algebra homomorphism $h \colon A \to B$ which satisfies $\Diamond \circ h = h \circ \Diamond$.

One easily verifies that modal algebras and modal algebra homomorphisms constitutes a category. We call this category MA.

We wish to construct an endo-functor $V : \mathsf{BA} \to \mathsf{BA}$ on the category of Boolean algebras and Boolean algebra homomorphisms such that MA is isomorphic to the category $\mathrm{Alg}(V)$ of algebras for the functor V.

Given a join-semilattice A we define the set $A_\Diamond = \{\Diamond a\colon a \in A\}$, where (for now) $\Diamond$ should just be seen as a formal symbol. Then we let $F_{\mathsf{BA}}\colon \mathsf{Set} \to \mathsf{BA}$ denote the free-object functor for BA. Given a join-semilattice A we define $V'(A)$ to be Boolean algebra $F_{\mathsf{BA}}(A_\Diamond)$ modulo the set of equations

$$\{\Diamond\bot = \bot, \Diamond(a \vee b) = \Diamond a \vee \Diamond b\colon a, b \in A\}.$$

Thus $V'(A)$ becomes a modal algebra with the obvious hemimorphism. Furthermore, $V'(A)$ has the following universal property:

Proposition 2.13. *Let A be a join-semilattice and B a Boolean algebra. Moreover, let $h\colon A \to B$ be a hemimorphism. Then there exists a* unique *Boolean algebra homomorphism $h^T\colon V'(A) \to B$ such that*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i_\Diamond"] & B \\ V'(A) \arrow[ur, "h^T"'] & \end{tikzcd}$$

commutes, where $i_\Diamond\colon A \to V'(A)$ is the join-semilattice homomorphism $a \mapsto [\Diamond a]$.

Proof. The map $h\colon A \to B$ induces a map $h'\colon A_\Diamond \to B$ by letting $h'(\Diamond a) = h(a)$. Hence by the universal property of the free-object functor we must have a unique Boolean algebra homomorphism $(h')^T\colon F_{\mathsf{BA}}(A_\Diamond) \to B$ such that $h'(\Diamond a) = (h')^T(\Diamond a)$ for all $a \in A$, hence $(h')^T(\Diamond a) = h(a)$, for all $a \in A$. Finally letting $\pi\colon F_{\mathsf{BA}}(A_\Diamond) \to V'(A)$ be the canonical projection we obtain from the Homomorphism Theorem ([5] Thm. 6.12) a unique homomorphism $h^T\colon F_{\mathsf{BA}}(A_\Diamond)/\ker(\pi) \cong V'(A) \to B$ such that $h^T \circ \pi = (h')^T$. We then see that

$$(h^T \circ i_\Diamond)(a) = (h^T \circ \pi)(\Diamond a) = (h')^T(\Diamond a) = h'(\Diamond a) = h(a),$$

as desired. $\square$

From this it follows that any homomorphism $h\colon A \to B$ between semilattices induces a Boolean algebra homomorphism $V'(h)\colon V'(A) \to V'(B)$ by letting $V'(h) = (i_\Diamond^B \circ h)^T$. In this way we obtain a functor $V'\colon \vee\mathsf{SemLat} \to \mathsf{BA}$.

Observe that by the above proposition we have that for A a join-semilattice and B a Boolean algebra, then the function

$$(-)^T\colon \operatorname{Hom}_{\vee\mathsf{SemLat}}(A, U(B)) \to \operatorname{Hom}_{\mathsf{BA}}(V'(A), B) \tag{\dagger}$$

is a bijection, where $U\colon \mathsf{BA} \to \vee\mathsf{SemLat}$ is the forgetfull functor. Moreover, one may verify that the isomorphism (†) is in fact natural in A and B, whence we obtain that $V'\colon \vee\mathsf{SemLat} \to \mathsf{BA}$ is the left adjoint to the forgetfull functor $U\colon \mathsf{BA} \to \vee\mathsf{SemLat}$.

We may now define an endofunctor $V\colon \mathsf{BA} \to \mathsf{BA}$ by letting $V = V' \circ U$. Then given a Boolean algebra homomorphism $f\colon A \to B$ we see that $V(f)\colon V(A) \to V(B)$ is the map $(i^B_\Diamond \circ f)^T$, i.e. $V(f)$ is the unique map making the following diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i^A_\Diamond"] & B \arrow[r, "i^B_\Diamond"] & V(B) \\
V(A) \arrow[urr, dotted] & &
\end{tikzcd}$$

commute. By the isomorphism (†) we indeed obtain that MA is isomorphic to the category $\mathrm{Alg}(V)$ of algebras for the functor $V\colon \mathsf{BA} \to \mathsf{BA}$.

Finally, as $V = V' \circ U$ and since forgetfull functors preserve filtered[3] colimits [17, Prop. 3.4.2] and left adjoins preserve all colimits [47, Thm. V.5.1.] we obtain that $V\colon \mathsf{BA} \to \mathsf{BA}$ preserves all filtered colimits. In particular V will preserve all chain colimits. This is essential for the construction of finitely generated free algebras.

2.2.2 The colimit construction

We want to show that the free modal algebra on n generators can be obtained as a colimit of finite Boolean algebras.

Therefore let n be a fix natural number and let A_0 be the free Boolean algebra on n generators, and define by recursion on $k \in \omega$

$$A_{k+1} = A_0 + V(A_k)$$

Now define maps $\Diamond^T_k\colon V(A_k) \to A_{k+1}$ as the second coproduct injection and define maps $i_k\colon A_k \to A_{k+1}$ by recursion on $k \in \omega$ as follows: $i_0\colon A_0 \to A_1$ is defined to be the first coproduct injection and set $i_k\colon A_k \to A_{k+1}$ to be $\mathrm{id} + V(i_k)$.

[3]Recall that a colimit of a diagram $F\colon \mathsf{D} \to \mathsf{C}$ is *filtered* if the category D is filtered i.e. if every diagram in D has a co-cone. We are ignoring some cardinality issues in this definition, in what follows we will only need to consider $\aleph_1$-filtered cateories and $\aleph_1$-filtered colimits.

As the variety of Boolean algebras is locally finite the algebra A_k will be finite for all $k \in \omega$.

Now since BA is a variety and as such co-complete [6, Thm. 8.4.13] we may define A_∞ to be the colimit, in the category BA, of the diagram

$$A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} \dots \xrightarrow{i_{k-1}} A_k \xrightarrow{i_k} A_{k+1} \xrightarrow{i_{k+1}} \dots$$

Now we easily see that

$$i_{k+1} \circ \Diamond_k^T = (\mathrm{id} + V(i_k)) \circ \Diamond_k^T = \Diamond_k^T + (V(i_k) \circ \Diamond_k^T) = \Diamond_{k+1}^T \circ V(i_k),$$

as maps from $V(A_k)$ to A_{k+2}. It follows that the diagram

$$\begin{tikzcd}
V(A_0) \arrow[r, "V(i_0)"] \arrow[dr, "\Diamond_0^T"] & V(A_1) \arrow[r, "V(i_1)"] \arrow[dr, "\Diamond_1^T"] & \dots \arrow[r, "V(i_{k-1})"] & V(A_k) \arrow[r, "V(i_k)"] \arrow[dr, "\Diamond_k^T"] & V(A_{k+1}) \arrow[r, "V(i_{k+1})"] \arrow[dr, "\Diamond_{k+1}^T"] & \dots \\
A_0 \arrow[r, "i_0"] & A_1 \arrow[r, "i_1"] & \dots \arrow[r, "i_{k-1}"] & A_k \arrow[r, "i_k"] & A_{k+1} \arrow[r, "i_{k+1}"] & \dots
\end{tikzcd}$$

commutes.

Consequently as $V \colon \mathsf{BA} \to \mathsf{BA}$ preserves filtered colimits and hence in particular chain colimits we obtain a homomorphism $\Diamond_\infty^T \colon V(A_\infty) \to A_\infty$ of Boolean algebras. Now letting $\Diamond_\infty \colon A_\infty \to A_\infty$ be the corresponding join-semilattice homomorphism we obtain a modal algebra $\mathfrak{A}_\infty = (A_\infty, \Diamond_\infty)$. In fact $\mathfrak{A}_\infty$ will be the free modal algebra on n generators. For a detailed proof see [36]. Slightly different proofs of this fact can also be found in [14, 1].

If one wishes to construct free L-algebra for some normal modal logic L, this turns out to be a bit more complicated than above. In particular if L is not axiomatizable by formulas of rank 1, i.e. formulas in which every propositional letter is in the scope of precisely 1 modal operator. This seems to be due to the fact that modal algebras determined by equations which are not of rank 1 will not be algebras for some endo-functor on BA cf. [43, 44].

To describe the free L-algebra as colimit of finite algebras it will be helpful to introduce the notion of a modal one-step algebra as done in [13]. We will indeed introduce modal one-step algebras in the next section. We will then return to the construction of free L-algebras in section 2.4.

2.3 Modal one-step algebras and frames

In this section we briefly sketch the basics of the theory of modal one-step algebras and frames developed in [13, 11].[4] As we will be developing a similar theory of one-step Heyting algebras in Chapter 3 we will try to be rather brief.

Definition 2.14. A *one-step modal algebra* is a quadruple $\mathcal{A} = (A_0, A_1, i_0, \Diamond_0)$ consisting of two Boolean algebras A_0 and A_1; a Boolean algebra homomorphism $i_0 \colon A_0 \to A_1$ and a hemimorphism $\Diamond_0 \colon A_0 \to A_1$. A one-step modal algebra $\mathcal{A} = (A_0, A_1, i_0, \Diamond_0)$ is *conservative* if $i_0 \colon A_0 \to A_1$ is injective and the set $\Diamond_0 A_0 \cup i_0(A_0)$ generates the Boolean algebra A_1.

Definition 2.15. A *modal one-step frame* is a quadruple $\mathcal{S} = (W_1, W_0, f, R)$ consisting of a function $f \colon W_1 \to W_0$ between sets W_1 and W_0 and a relation $R \subseteq W_1 \times W_0$. A one-step frame $\mathcal{S} = (W_1, W_0, f, R)$ is *conservative* if $f \colon W_1 \to W_0$ is surjective and satisfies:

$$\forall w, w' \in W_1 \ ((f(w) = f(w') \text{ and } R[w'] = R[w]) \implies w' = w),$$

where as usual $R[w] = \{v \in W_0 \colon wRv\}$ denotes the set of R-successors of w.

Note that a one-step modal algebra of the form $(A, A, \mathrm{id}, \Diamond)$ may be identified with a modal algebra, and that a one-step frame of the form (W, W, id, R) may likewise be identified with a modal Kripke frame. We will calls such algebras and frames *standard*.

As we will see in subsection 2.3.1 modal one-step frames and algebras can interpret modal formulas of modal depth at most 1.

Definition 2.16. Let $\mathcal{A} = (A_0, A_1, i_0, \Diamond_0)$ and $\mathcal{A}' = (A_0', A_1', i_0', \Diamond_0')$ be modal one-step algebras. By a *one-step homomorphism* from $\mathcal{A}$ to $\mathcal{A}'$ we shall understand a pair of Boolean algebra homomorphism $h \colon A_0 \to A_0'$ and $k \colon A_1 \to A_1'$, such that the following diagrams

$$\begin{array}{ccc} A_0 & \xrightarrow{h} & A_0' \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle i_0'} \\ A_1 & \xrightarrow{k} & A_1' \end{array} \qquad\qquad \begin{array}{ccc} A_0 & \xrightarrow{h} & A_0' \\ {\scriptstyle \Diamond_0}\downarrow & & \downarrow{\scriptstyle \Diamond_0'} \\ A_1 & \xrightarrow{k} & A_1' \end{array}$$

commute. We write $(k, h) \colon \mathcal{A} \to \mathcal{A}'$ for a one-step homomorphism. We say that a one-step homomorphism $(k, h) \colon \mathcal{A} \to \mathcal{A}'$ is a *one-step embedding* if both k and h are injective.

[4] A similar but essential equivalent approach to analysing the colimit construction of finitely generated free modal algebras using *partial modal algebras* is taking in [33, 30]. There, however, the connections to proof theory is not investigated.

Definition 2.17. A *one-step extension* of a modal one-step algebra $\mathcal{A} = (A_0, A_1, i_0, \Diamond_0)$ is a modal one-step algebra of the form $\mathcal{A}' = (A_1, A_2, i_1, \Diamond_1)$ such that $(i_0, i_1) \colon \mathcal{A} \to \mathcal{A}'$ is a one-step homomorphism.

The one-step extensions will play an important role towards the end of this chapter, and again in the intuitionistic case in Chapters 3 and 4.

Note that if both $\mathcal{A}$ and $\mathcal{A}'$ are standard modal algebras then one-step homomorphisms between $\mathcal{A}$ and $\mathcal{A}'$ may be identified with standard modal algebra homomorphism. Furthermore, if $\mathfrak{A}' = (A', \Diamond)$ is a standard modal algebra, then a one-step homomorphism into $\mathfrak{A}$ will be determined by a Boolean algebra homomorphism $k \colon A_1 \to A'$ satisfying

$$k \circ \Diamond_0 = \Diamond \circ k \circ i_0,$$

in the sense that $(k \circ i_0, k)$ will then be a one-step homomorphism from $\mathcal{A}$ to $\mathfrak{A}'$.

We let MOSAlg denote the category whose objects are modal one-step algebras and whose morphisms are one-step homomorphism.

Similarly we may define maps between one-step frames.

Definition 2.18. Let $\mathcal{S}' = (W_1', W_0', f', R')$ and $\mathcal{S} = (W_1, W_0, f, R)$ be modal one-step frames. A *one-step p-morphism* from $\mathcal{S}'$ to $\mathcal{S}$ is a pair of functions $\mu \colon W_1' \to W_1$ and $\nu \colon W_0' \to W_0$ such that the following diagrams

$$\begin{tikzcd} W_1' \arrow[r, "\mu"] \arrow[d, "f'"'] & W_1 \arrow[d, "f"] \\ W_0' \arrow[r, "\nu"] & W_0 \end{tikzcd} \qquad \begin{tikzcd} W_1' \arrow[r, "\mu"] \arrow[d, "R'"'] & W_1 \arrow[d, "R"] \\ W_0' \arrow[r, "\nu"] & W_0 \end{tikzcd}$$

commute. We write $(\mu, \nu) \colon \mathcal{S}' \to \mathcal{S}$ for one-step p-morphism. Moreover, we say that a one-step p-morphism $(\mu, \nu) \colon \mathcal{S}' \to \mathcal{S}$ is surjective if both μ and ν are surjective.

We may then let MOSFrm denote the category whose objects are modal one-step frames and whose morphisms are one-step p-morphisms.

Finally, we define a one-step extension of a modal one-step frame $\mathcal{S} = (W_1, W_0, f, R)$ to be a one-step frame of the form $\mathcal{S} = (W_2, W_1, g, R')$ such that $(g, f) \colon \mathcal{S}' \to \mathcal{S}$ is a one-step p-morphism.

2.3.1 One-step semantics for modal logic

In just the same way that modal algebras provide semantics for modal logic, modal one-step algebras provide a semantics for the depth 1 fragment of modal logic. Moreover, as every modal formula may be transformed into an equivalent rule only consisting of formula of depth at most 1 [11, Prop. 3] we see that one-step modal algebras may in fact interpret modal formulas of arbitrary depth. We briefly describe how this works. We refer to [11, Sec. 4.2] for more details.

Given a one-step algebra $\mathcal{A} = (A_0, A_1, i, \Diamond)$ a *one-step valuation* on $\mathcal{A}$ will be a function v assigning to each propositional variable an elements of the Boolean algebra A_0. Given such a valuation we may define for each formula φ of modal depth 0 an element φ^{v_0} of A_0, as follows: For p a propositional variable we let $p^{\mathrm{v}_0} = v(p)$, otherwise we let

$$(\neg\varphi)^{\mathrm{v}_0} = \neg(\varphi^{\mathrm{v}_0}), \quad (\varphi * \psi)^{\mathrm{v}_0} = \varphi^{\mathrm{v}_0} * \psi^{\mathrm{v}_0}, \; * \in \{\wedge, \vee\}.$$

Similarly for each formula φ of depth at most 1 we define an element $\varphi^{\mathrm{v}_1} \in A_1$ as follows: If φ is of depth 0 we let $\varphi^{\mathrm{v}_1} = i(\varphi^{\mathrm{v}_0})$ and if φ is of depth 1 we let

$$(\Diamond\varphi)^{\mathrm{v}_1} = \Diamond(\varphi^{\mathrm{v}_0}), \quad (\neg\varphi)^{\mathrm{v}_1} = \neg(\varphi^{\mathrm{v}_1}), \quad (\varphi * \psi)^{\mathrm{v}_1} = \varphi^{\mathrm{v}_1} * \psi^{\mathrm{v}_1}, \; * \in \{\wedge, \vee\}.$$

Recall from [11, Def. 1] that a modal rule r is *reduced* if all the formulas occurring in r are of depth at most 1 and if every proposition variable occurring in r has an occurrence within the scope of a modal operator.

Definition 2.19. Let $\mathcal{A}$ be a one-step algebra and let

$$\frac{\varphi_1, \ldots, \varphi_n}{\psi}\,(r)$$

be a reduced modal rule. We say that $\mathcal{A}$ *validates* the rule (r) if for all valuations v on $\mathcal{A}$ we have that

$$(\varphi_1^{\mathrm{v}_1} = \top \text{ and } \ldots \text{ and } \varphi_1^{\mathrm{v}_1} = \top) \implies \psi^{\mathrm{v}_1} = \top.$$

We say $\mathcal{A}$ validates a reduced axiom system Ax if it validates all the rules of Ax.

In this way we may speak about a modal one-step frame validating an arbitrary logic L. Note that if Ax and Ax$'$ are two reduced axiom system for a logic L it may be that a one-step modal algebra $\mathcal{A}$ validates one of these axioms systems but not the other. Thus the one-step algebras can not only distinguish between different logics but also between their axiomatizations.

2.3.2 Duality for finite one-step frames and algebras

As the reader might have expected the Jónsson-Tarski duality between finite modal algebras and finite Kripke frames induces a dual equivalence between the categories $\mathsf{MOSAlg}_{<\omega}$ and $\mathsf{MOSFrm}_{<\omega}$. We sketch the details below.

If $\mathcal{S} = (W_1, W_0, f, R)$ is a *finite* one-step algebra then we define a one-step algebra $\mathcal{S}^* = (\wp(W_0), \wp(W_1), f^*, \Diamond_R)$, where $f^* \colon \wp(W_0) \to \wp(W_1)$ is the inverse image function $U \mapsto f^{-1}(U)$ and $\Diamond_R \colon \wp(W_0) \to \wp(W_1)$ is given by

$$\Diamond_R(U) = \{w \in W_1 \colon R[w] \cap U \neq \emptyset\}.$$

Conversely if $\mathcal{A} = (A, B, i, \Diamond)$ is a *finite* one-step algebra we may define its dual one-step frame, as $\mathcal{A}_* = (\mathrm{At}(B), \mathrm{At}(A), \mathrm{At}(i), R_\Diamond)$, where $\mathrm{At}(-) \colon \mathsf{BA} \to \mathsf{Set}^{\mathrm{op}}$ is the functor taking a Boolean algebra to its set of atoms[5]. Since i is order preserving and A is finite and hence complete it follows that $i \colon A \to B$ has a left adjoint $i^\flat \colon B \to A$, i.e.

$$b \leq i(a) \iff i^\flat(b) \leq a,$$

given by $i^\flat(b) = \bigwedge\{a \in A \colon b \leq i(a)\}$. One may then show that $i^\flat(b)$ is an atom whenever b is. From this is will follow, that letting $\mathrm{At}(i) \colon \mathrm{At}(B) \to \mathrm{At}(A)$ be $i^\flat \restriction \mathrm{At}(B)$ is well-defined.

Finally $R_\Diamond \subseteq \mathrm{At}(B) \times \mathrm{At}(A)$ is given by

$$bR_\Diamond a \iff b \leq \Diamond a.$$

Since every element of a Boolean algebra is determined by the set of atoms below it we see that the maps $h \colon A \to \wp(\mathrm{At}(A))$ and $k \colon B \to \wp(\mathrm{At}(B))$ given by $x \mapsto {\downarrow}x$, are Boolean algebra isomorphisms. Finally, one may readily check that $(h, k) \colon \mathcal{A} \to (\mathcal{A}_*)^*$ is a one-step map, i.e. that

$${\downarrow}i(a) = (i^\flat)^{-1}({\downarrow}a) \quad \text{and} \quad {\downarrow}\Diamond a = \Diamond_{R_\Diamond}({\downarrow}a),$$

whence we obtain that $\mathcal{A}$ and $(\mathcal{A}_*)^*$ are isomorphic as one-step algebras. Conversely, one may readily check that $(\mathcal{S}^*)_*$ and $\mathcal{S}$ are isomorphic as one-step frames.

Finally, we have that the property of being conservative is both preserved and reflected by the operation $(-)^*$ on finite one-step frames [11, Prop. 5]. More precisely, we have the following:

[5] A element a of boolean algebra is called an atom if $a > 0$ and no non-zero elements are strictly below a.

Proposition 2.20. *The categories* $\mathsf{MOSAlg}_{<\omega}$ *and* $\mathsf{MOSFrm}_{<\omega}$ *are dually equivalent. Furthermore, this dual equivalence restricts to a dual equivalence between the categories* $\mathsf{MOSAlg}^{cons}_{<\omega}$ *and* $\mathsf{MOSFrm}^{cons}_{<\omega}$ *of finite conservative modal one-step algebras and of finite conservative modal one-step frames, respectively.*

2.4 The colimit construction revisited: Adding equations

We now have the tools to properly describe how the colimit construction can be extended to modal algebras for normal modal logics extending basic modal logic **K**.

As before let B_0 be the free Boolean algebra on n generators and $B_1 = B_0 + V(B_0)$. Furthermore, let $i_0 \colon B_0 \to B_1$ and $\Diamond^T_0 \colon V(B_0) \to B_1$ be the co-product injections. Now define, by recursion on $k \in \omega$, Boolean algebras B_k and maps i_k and $\Diamond^T_k$ by the following pushout

$$\begin{array}{ccc} V(B_{k-1}) & \xrightarrow{V(i_{k-1})} & V(B_k) \\ \downarrow{\scriptstyle \Diamond^T_{k-1}} & & \downarrow{\scriptstyle \Diamond^T_k} \\ B_k & \xrightarrow{i_k} & B_{k+1} \end{array}$$

That is, $B_{k+1} = B_k +_{V(B_{k-1})} V(B_k)$. Note that in the original construction the algebra A_{k+1} was built using both A_0 and A_k. However, with this approach we now only need the algebra B_k to construct the algebra B_{k+1}. Moreover, note that $(B_k, B_{k+1}, i_k, \Diamond_k)$ is a modal one-step algebra and $(B_{k+1}, B_{k+2}, i_{k+1}, \Diamond_{k+1})$ is a one-step extension[6].

Now if we let B_∞ be the colimit in BA of the diagram:

$$B_0 \xrightarrow{i_0} B_1 \xrightarrow{i_1} \ldots \xrightarrow{i_{k-1}} B_k \xrightarrow{i_k} B_{k+1} \xrightarrow{i_{k+1}} \ldots$$

Then as before because the functor V commutes with filtered colimits and $i_{k+1} \circ \Diamond^T_k = \Diamond_{k+1} \circ V(i_k)$ by construction, the homomorphisms $\Diamond^T_k \colon V(B_k) \to B_{k+1}$ may be extended to a homomorphism $\Diamond^T_\infty \colon V(B_\infty) \to B_\infty$, with the property that $\mathfrak{B}_\infty = (B_\infty, \Diamond_\infty)$ is the free modal algebra on n generators, see [13, Prop. 6].

Now suppose that L is a normal modal logic axiomatized by a set Ax of formulas. Some of these formulas may be of modal depth greater than 1, and so considering the quotient modulo these equations will be harmful to the step-by-step approach for constructing the finitely generated free L-algebra. However, by [11, Prop. 3] we may equivalently axiomatize L by a set Ax′ of *reduced rules*. Thus, for each rule r of Ax′ we obtain a

[6] As the reader might have expected this particular one-step extension is determined by a universal property. However, we shall not concern ourselves with this property here.

quasi-equation

$$(t_1^r(\underline{x}) = \top \text{ and } \dots \text{ and } t_n^r(\underline{x}) = \top) \implies u^r(\underline{x}) = \top, \tag{‡}$$

where t_k^r and u^r are terms in the two-sorted language of one-step algebras of modal depth at most 1.

Given a finite[7] modal one-step algebra $\mathcal{A} = (A_0, A_1, i, \Diamond)$ and a reduced modal axiom system Ax we would like to quotient out by the set of quasi-equations determined by Ax, to obtain a one-step algebra validating Ax. This is done by letting ϑ_1 be the congruence on A_1 generated by the set

$$\{(u^r(\underline{x}), \top) \colon r \in \text{Ax},\ \forall k \leq n\ (t_k^r(\underline{x}) = \top)\}$$

and letting A_1^1 be A_1/ϑ_1. Then iterating this construction we obtain a sequence of algebras $A_1^1, A_1^2, \dots$, and as A_1 is finite there must exist $k \in \omega$ such that $A_1^k = A_1^{k+1}$. Let A_1' denote A_1^k for $k \in \omega$ minimal with this property. We then have a homomorphism $\pi \colon A_1 \to A_1'$ such that the one-step algebra $(A_0, A_1', \pi \circ i, \pi \circ \Diamond)$ validates all the rules of Ax.

Let C_0 be the free Boolean algebra on n generators and $C_1 = (C_0 + V(C_0))'$. Furthermore, let $i_0 \colon C_0 \to C_1$ and $\Diamond_0^T \colon V(C_0) \to C_1$ be the co-product injections post-composed with the canonical projections. Now define by recursion on $k \in \omega$ the algebra $C_{k+1} = (C_k +_{V(C_{k-1})} V(C_k))'$ and the maps i_k and $\Diamond_k^T$ as the pushout maps post-composed with the canonical projections. We thus proceed as before only at each stages of the construction we ensure that the one-step algebra $(C_k, C_{k+1}, i_k, \Diamond_k)$ validates all the rules from Ax.

We then let C_∞ be the colimit in BA of the diagram:

$$C_0 \xrightarrow{i_0} C_1 \xrightarrow{i_1} \dots \xrightarrow{i_{k-1}} C_k \xrightarrow{i_k} C_{k+1} \xrightarrow{i_{k+1}} \dots$$

Then as before because the functor V commutes with filtered colimits and by construction $i_{k+1} \circ \Diamond_k^T = \Diamond_{k+1}^T \circ V(i_k)$ the homomorphisms $\Diamond_k^T \colon V(C_k) \to C_{k+1}$ may be extended to a homomorphism $\Diamond_\infty^T \colon V(C_\infty) \to C_\infty$, from which we obtain a hemimorphism $\Diamond_\infty \colon C_\infty \to c_\infty$. Moreover, since by construction each of the algebras $(C_k, C_{k+1}, i_k, \Diamond_k)$ are one-step algebras validating Ax we obtain that $\mathfrak{C}_\infty = (C_\infty, \Diamond_\infty)$ will be an L-algebra. In fact $\mathfrak{C}_\infty$ will be the free L-algebra on n generators [13, Prop. 7].

[7] This assumption is not essential for what follows. However, it simplifies matters a bit.

Given the duality between finite modal one-step algebras and finite modal one-step frames the dual spaces of a finitely generated free modal algebra can be described as a limit of finite sets, see [13, Sec. 2.2, 3.2] for details.

An important observation is that *a priori* nothing ensures that the maps $i_k \colon C_k \to C_{k+1}$ will be injective. In fact we have

Proposition 2.21 ([13, Prop. 8], [30])**.** *If $i_k \colon C_k \to C_{k+1}$ is an injection for all $k \in \omega$, then the logic L is decidable and the algebra C_k is isomorphic to a Boolean subalgebra of the free L-algebra $F_L(n)$, consisting of terms of depth k.*

Intuitively the above proposition shows that if the injectivity of the maps $i_k \colon B_k \to B_{k+1}$ is preserved when moding out by the set of quasi-equations determined by the axiom system Ax, then Ax has the property that whenever $\vdash_{\mathrm{Ax}} \varphi \leftrightarrow \psi$ then this is witnessed by a derivation only containing formulas of modal depth not exceeding that of $\varphi \leftrightarrow \psi$. Thus, Proposition 2.21 seems to suggest that the one-step algebras can be used to describe the behaviour of proof systems for modal logics. This will be made precise in the next section.

2.5 The bounded proof property for modal axiom systems

In this section we will briefly review the work of Bezhanishvili and Ghilardi on the bounded proof property for modal proof systems [11, 12]. Again we will be fairly brief in our presentation as the work done in Chapter 3 and Chapter 4 will resemble that of [11, 12] rather closely.

We say that an axiom system Ax has the *bounded proof property* if for all sets $\Gamma \cup \{\varphi\}$ of formulas of modal depth at most n, such that φ can be derived in Ax from Γ there exists a derivation witnessing this in which only formulas whose modal depth does not exceed n occurs. We refer to [11, Sec. 3] for the precise definition of a derivation in an axiom system.

In [11] it is shown that the bounded proof property can be characterized in terms of a rather nice property of the finite conservative one-step algebras validating Ax.

We say that a class K of modal one-step algebras has the *extension property* if all of its members have a one-step extension which also belongs to K. This definition also applies *mutatis mutandis* to a class of modal one-step frames.

Theorem 2.22 ([11, Thm. 1])**.** *Let* Ax *be a reduced axiom system. Then the following are equivalent:*

i) Ax *has the bounded proof property;*

ii) *The class of finite one-step modal algebras validating* Ax *has the extension property;*

iii) *The class of finite one-step frames validating* Ax *has the extension property.*

For applications it can be somewhat difficult to work with one-step extensions. It turns out that in the presence of the finite model property a version of the above theorem which avoids the concept of one-step extensions can be obtained.

Theorem 2.23 ([11, Thm. 2])**.** *Let* Ax *be a reduced axiom system. Then the following are equivalent:*

i) Ax *has the bounded proof property and the finite model property;*

ii) *Every finite conservative one-step modal algebra validating* Ax *embeds into some finite modal algebra validating* Ax*;*

iii) *Every finite conservative one-step frame validating* Ax *is the p-morphic image of some finite frame validating* Ax*.*

In [11, Sec. 8] it is shown that the following well-known logics:

$$\mathbf{K},\ \mathbf{T},\ \mathbf{K4},\ \mathbf{S4},\ \mathbf{S4.3},\ \mathbf{S5},\ \mathbf{GL},$$

all have reduced axiom systems with the finite model property and the bounded proof property.

Summary of Chapter 2: In this chapter we have shown how to construct finitely generated free Heyting and modal algebras as a colimit of finite bounded distributive lattices and finite Boolean algebras, respectively. We have seen how this construction in the modal case gives rise to the notion of a one-step algebra – and by duality to one-step frames. Finally, we have see that the modal one-step algebras can be used to characterize the bounded proof property establishing a connection between proof theory and one-step algebras and frames. In the following two chapters we will, starting from Ghilardi's colimit construction of finite generated free Heyting algebras, develop a theory of one-step Heyting algebras and show how to obtain a characterization of the bounded proof property for hypersequent calculi in the case of intuitionistic logic.

Chapter 3

One-step Heyting algebras

In this chapter we develop the basic theory of one-step Heyting algebras along the lines of the theory of one-step modal algebras of [11] described in Chapter 2.

3.1 One-step Heyting algebras

We take our inspiration from Ghilardi's construction of finitely generated free Heyting algebras as chain colimits in the category $\mathsf{bDist}_{<\omega}$ of finite bounded distributive lattices and bounded lattice homomorphisms [35, 10, 18], as described in Chapter 2.

Definition 3.1. A *one-step Heyting algebra* is a triple $\mathcal{H} = (D_0, D_1, i)$ consisting of a pair of bounded distributive lattices D_0, D_1 together with a bounded lattice homomorphism $i\colon D_0 \to D_1$, such that for all $a, b \in D_0$ the Heyting implication $i(a) \to i(b)$ exists in D_1.

Note that a one-step Heyting algebra is an object in the arrow category $\mathsf{bDist}^{\to}$ of the category bounded distributive lattices. Moreover, the finite one-step Heyting algebras are precisely the objects in the category $\mathsf{bDist}^{\to}_{<\omega}$, where we take a one-step Heyting algebra $\mathcal{H} = (D_0, D_1, i)$ to be finite if both D_0 and D_1 are finite.

We say that a one-step Heyting algebra is *standard* if it is of the form (D, D, id_D) for some distributive lattice D. Thus if (D, D, id_D) is standard then D is in fact a Heyting algebra in the usual sense.

Definition 3.2. We say that a one-step Heyting algebra (D_0, D_1, i) is *conservative* if i is an injection and the set $\{i(a) \to i(b)\colon a, b \in D_0\}$ generates D_1 as a bounded distributive lattice.

Definition 3.3. A *one-step homomorphism* between two one-step Heyting algebras $\mathcal{H} = (D_0, D_1, i)$ and $\mathcal{H}' = (D'_0, D'_1, i')$ is a pair (g_0, g_1) of bounded lattice homomorphisms $g_0 \colon D_0 \to D'_0$ and $g_1 \colon D_1 \to D'_1$ making the diagram

$$\begin{tikzcd} D_0 \arrow[r, "g_0"] \arrow[d, "i"] & D'_0 \arrow[d, "i'"] \\ D_1 \arrow[r, "g_1"] & D'_1 \end{tikzcd}$$

commute, and such that for all $a, b \in D_0$

$$g_1(i(a) \to i(b)) = g_1(i(a)) \to g_1(i(b)).$$

The above definitions determines a category OSHA of one-step Heyting algebras and one-step homomorphisms between them. This is a non-full subcategory of the category $\mathsf{bDist}^{\to}$. We let $\mathsf{OSHA}_{<\omega}$ denote the full subcategory of OSHA consisting of finite one-step Heyting algebras.

Definition 3.4. A *one-step extension* of a one-step Heyting algebra $\mathcal{H} = (D_0, D_1, i)$ is a one-step Heyting algebra of the form $\mathcal{H}' = (D_1, D_2, j)$ such that $(i, j) \colon \mathcal{H} \to \mathcal{H}'$ is a one-step homomorphism.

A one-step extension of (D_0, D_1, i) is thus nothing more that a bounded distributive lattice D_2 together with a bounded lattice homomorphism $j \colon D_1 \to D_2$ such that j preserves all Heyting implications between elements in the image of i, i.e. for all $a, b \in D_0$

$$j(i(a) \to i(b)) = j(i(a)) \to j(i(b)).$$

In what follows we shall only be interested in finite one-step Heyting algebras. Therefore, if not explicitly stated otherwise all one-step Heyting algebras will be finite. In fact, in this chapter the category $\mathsf{OSHA}^{\mathrm{cons}}_{<\omega}$ of finite conservative one-step Heyting algebras will be the main focus of investigation.

3.2 Intuitionistic one-step frames

Using the Birkhoff duality between the categories $\mathsf{bDist}_{<\omega}$ and $\mathsf{Pos}_{<\omega}$ we define a notion of intuitionistic one-step frame.

Definition 3.5. By an *intuitionistic one-step frame* we shall understand a triple (P_1, P_0, f) such that P_1 and P_0 are posets and $f \colon P_1 \to P_0$ is an order-preserving map.

A *standard* intuitionistic one-step frame will be an intuitionistic one-step frame of the form (P, P, id_P).

Definition 3.6. We say that an intuitionistic one-step frame (P_1, P_0, f) is *conservative* if f is a surjection satisfying

$$f({\downarrow}a) \subseteq f({\downarrow}b) \implies a \leq b,$$

for all $a, b \in P_1$.

Definition 3.7. A one-step map from an intuitionistic one-step frame $\mathcal{S}' = (P_1', P_0', f')$ to an intuitionistic one-step frame $\mathcal{S} = (P_1, P_0, f)$ is a pair (μ_1, μ_0) of order-preserving maps $\mu_1 \colon P_1' \to P_1$ and $\mu_0 \colon P_0' \to P_0$ making the diagram

$$\begin{tikzcd} P_1' \arrow[r, "\mu_1"] \arrow[d, "f'"] & P_1 \arrow[d, "f"] \\ P_0' \arrow[r, "\mu_0"] & P_0 \end{tikzcd}$$

commute. Moreover, we require that μ_1 is f-open[1].

We then let $\mathsf{IOSFrm}_{<\omega}$ denote the category of finite intuitionistic one-step frames and one-step maps between them.

Finally, we define a *one-step extension* of an intuitionistic one-step frame $\mathcal{S} = (P_1, P_0, f)$ to be an intuitionistic one-step frame $\mathcal{S}' = (P_2, P_1, g)$ such that $(g, f) \colon \mathcal{S}' \to \mathcal{S}$ is a map of one-step frames.

3.3 Duality

In this section we show that the duality between the categories $\mathsf{bDist}_{<\omega}$ and $\mathsf{Pos}_{<\omega}$ can be extended to a duality between the categories $\mathsf{OSHA}_{<\omega}$ and $\mathsf{IOSFrm}_{<\omega}$.

Proposition 3.8. *The categories* $\mathsf{OSHA}_{<\omega}$ *and* $\mathsf{IOSFrm}_{<\omega}$ *are dually equivalent.*

Proof. The functors $\mathsf{Do} \colon \mathsf{Pos}_{<\omega} \to \mathsf{HA}^{\mathrm{op}}_{<\omega}$ and $\mathcal{J} \colon \mathsf{HA}_{<\omega} \to \mathsf{Pos}^{\mathrm{op}}_{<\omega}$, constituting the Birkhoff duality, induce functors $\mathsf{Do}^{\to} \colon \mathsf{Pos}^{\to}_{<\omega} \to (\mathsf{HA}^{\to}_{<\omega})^{\mathrm{op}}$ and $\mathcal{J}^{\to} \colon \mathsf{HA}^{\to}_{<\omega} \to (\mathsf{Pos}^{\to}_{<\omega})^{\mathrm{op}}$ on the arrow categories. We show that $\mathsf{Do}^{\to}$ restricts to a functor from $\mathsf{IOSFrm}_{<\omega}$ to $\mathsf{OSHA}^{\mathrm{op}}_{<\omega}$ and $\mathcal{J}^{\to}$ restricts to a functor from $\mathsf{OSHA}_{<\omega}$ to $\mathsf{IOSFrm}^{\mathrm{op}}_{<\omega}$.

[1] Recall the definition of relative open maps for Definition 2.2 of Chapter 2.

To see this, let first $(\mu_1, \mu_0) \colon (P_1', P_0', f') \to (P_1, P_0, f)$ be a one-step map between intuitionistic one-step frames. Then we have that

$$\mathsf{Do}^{\to}(\mu_1, \mu_0) = (\mu_0^*, \mu_1^*) \colon (\mathsf{Do}(P_0), \mathsf{Do}(P_1), f^*) \to (\mathsf{Do}(P_0'), \mathsf{Do}(P_1'), (f')^*).$$

That $\mu_1^* \circ f^* = (f')^* \circ \mu_0^*$ is evident. Now as μ_1 is f-open we obtain that μ_1^* preserves all implications of the form $f^*(U) \to f^*(V)$. Consequently we must have that

$$\mu_1^*(f^*(U) \to f^*(V)) = \mu_1^*(f^*(U)) \to \mu_1^*(f^*(V)) = (f')^*(\mu_0(U)) \to (f')^*(\mu_0(V)).$$

That $\mathcal{J}^{\to}(g_0, g_1)$ is a one-step map of one-step frames when (g_0, g_1) is a one-step homomorphism of algebras is now evident.

Finally, by Birkhoff duality it follows that

$$\mathsf{Do}^{\to} \circ \mathcal{J}^{\to} \cong \mathrm{id}_{\mathsf{OSHA}_{<\omega}} \quad \text{and} \quad \mathcal{J}^{\to} \circ \mathsf{Do}^{\to} \cong \mathrm{id}_{\mathsf{IOSFrm}_{<\omega}},$$

which concludes the proof of the proposition. □

Next we show that the above duality restricts to a duality between the categories $\mathsf{OSHA}^{\text{cons}}_{<\omega}$ and $\mathsf{IOSFrm}^{\text{cons}}_{<\omega}$. To this end the following lemma is essential.

Lemma 3.9. *Let $(P, \leq)$ be a finite poset and let $\mathscr{G} \subseteq \mathsf{Do}(P)$. Then $\mathscr{G}$ generates $L := \mathsf{Do}(P)$ as a bounded distributive lattice iff for all $a, b \in P$*

$$\forall G \in \mathscr{G}(b \in G \implies a \in G) \implies a \leq b.$$

Proof. Define an equivalence relation $\sim_{\mathscr{G}}$ on P by

$$a \sim_{\mathscr{G}} b \iff \forall G \in \mathscr{G}(a \in G \iff b \in G).$$

We may then define a partial order $\leq_{\mathscr{G}}$ on the quotient $P' := P/{\sim_{\mathscr{G}}}$ by

$$[a] \leq_{\mathscr{G}} [b] \iff \forall G \in \mathscr{G}(b \in G \implies a \in G).$$

It is straightforward to check that this is indeed well-defined. We then claim that $\mathsf{Do}(P')$ is (isomorphic) to the sublattice L of $\mathsf{Do}(P)$ generated by $\mathscr{G}$.

To see this notice that the canonical projection $\pi \colon P \to P'$ given by $a \mapsto [a]$ is an order preserving surjection. Whence we obtain an injection $\pi^* \colon \mathsf{Do}(P') \to \mathsf{Do}(P)$ of bounded distributive lattices. Let $L' = \mathrm{Im}(\pi^*)$. We then show that L' is the least subalgebra of $\mathsf{Do}(P)$ containing $\mathscr{G}$.

That L' is a bounded distributive lattice which is a sublattice of $\mathsf{Do}(P)$ is immediate. To see that $G \subseteq L'$ it suffices to show that for each $G \in \mathscr{G}$ there exists a set $U_G \in \mathsf{Do}(P')$, such that $\pi^*(U_G) = G$. We claim that

$$U_G \coloneqq \pi(G) = \{[a] \colon a \in G\},$$

is such a set. To see that U_G is indeed a downset of P' assume that $[a] \in \pi(G)$. Then $a \sim_{\mathscr{G}} a'$ for some $a' \in G$, whence $a \in G$ so if $[b] \leq_{\mathscr{G}} [a]$ we have that $b \in G$ as well, and thereby $[b] \in \pi(G)$. Next we claim that $\pi^*(U_G) = G$. To see this we first observe that $G \subseteq \pi^{-1}(\pi(G)) = \pi^*(U_G)$. Moreover, as shown above if $a \in G$ and $a \sim_{\mathscr{G}} a'$ then $a' \in G$, whence

$$\pi^*(U_G) = \bigcup_{u \in U_G} \pi^*(u) = \bigcup_{a \in G} \pi^{-1}([a]) = \bigcup_{a \in G} \{a' \in P \colon a' \sim_{\mathscr{G}} a\} \subseteq G.$$

So we indeed have that $\pi^*(U_G) = G$ and thereby that $\mathscr{G} \subseteq L'$, as desired.

To see that L' is the least bounded distributive sublattice of $\mathsf{Do}(P)$ with this property we simply observe that if $V \in \mathrm{Im}(\pi^*)$ then $V = \pi^{-1}(U)$ for some downset of equivalence classes $U \in \mathsf{Do}(P')$. Now we claim that

$$\pi^*(U) = \bigcup_{[a] \in U} \pi^{-1}([a]) = \bigcup_{[a] \in U} [a] = \bigcup_{[a] \in U} \bigcap_{a \in G \in \mathscr{G}} G.$$

All but the last of the above equalities are straightforward to verify. For the last equality we have that if $a' \in \bigcup_{[a] \in U} [a]$ then $a' \sim_{\mathscr{G}} a$ for some $a \in P$ with $[a] \in U$. Then $a \in G$ implies that $a' \in G$ for all $G \in \mathscr{G}$. Whence $a' \in \bigcap_{a \in G \in \mathscr{G}} G$. Note that by convention $\cap \emptyset = P$. However, if $\{G \in \mathscr{G} \colon a \in G\} = \emptyset$ then since U is a downset of P' we must have $[a] \in U$ implies that $U = P'$, and thereby that $\pi^*(U) = P$.

Conversely if $a' \in \bigcap_{a \in G \in \mathscr{G}} G$ for some $a \in P$ with $[a] \in U$, it follows

$$\forall G \in \mathscr{G} (a \in G \implies a' \in G)$$

and therefore that $[a'] \leq_{\mathscr{G}} [a]$. From the assumption that U is a $\leq_{\mathscr{G}}$-downset it follows that $[a'] \in U$ whence as $\sim_{\mathscr{G}}$ is an equivalence relation we have that $a' \in [a']$ and thereby that $a' \in \bigcup_{[a] \in U} [a]$.

So as P is finite we have that for each $U \in \mathsf{Do}(P')$ that $\pi^*(U)$ is a finite join of finite meets of elements of $\mathscr{G}$ so if L'' is a sublattice containing $\mathscr{G}$ it must also contain L'.

To conclude the proof observe that $\mathscr{G}$ generates $\mathsf{Do}(P)$ iff $L' = \mathsf{Do}(P)$, i.e. iff π^* is an isomorphism. Because P is finite this in turn happens precisely when $\sim_{\mathscr{G}}$ is the equality relation on P and $\leq_{\mathscr{G}}=\leq$ is the order on P. The proposition now follows as

$$\forall a, b \in P(\forall G \in \mathscr{G}(b \in G \implies a \in G) \implies a \leq b)$$

is evidently equivalent to the statement that $\sim_{\mathscr{G}}$ is the equality relation and $\leq_{\mathscr{G}}=\leq$. $\square$

The above proposition can also be obtained as an immediate consequence of the correspondence between sublattices of a bounded distributive lattice D and *compatible quasi-orders* on the dual space X_D of D, cf. e.g. [33, Sec. 6]. However, we have found it helpful to spell out the details of this special case.

Proposition 3.10. *The duality between* $\mathsf{OSHA}_{<\omega}$ *and* $\mathsf{IOSFrm}_{<\omega}$ *restricts to a duality between the categories* $\mathsf{OSHA}^{cons}_{<\omega}$ *and* $\mathsf{ISOFrm}^{cons}_{<\omega}$.

Proof. By Proposition 3.8 and Birkhoff duality, to establish the proposition it suffices to show that $(\mathsf{Do}(P_0), \mathsf{Do}(P_1), f^*)$ is a conservative one-step Heyting algebra iff (P_1, P_0, f) is a conservative intuitionistic one-step frame.

We first note that f^* is an injection iff f is a surjection.

Now $(\mathsf{Do}(P_0), \mathsf{Do}(P_1), f^*)$ is conservative iff f^* is injective and the bounded distributive lattice $\mathsf{Do}(P_1)$ is generated by the set

$$\{f^*(U) \to f^*(V) \colon U, V \in \mathsf{Do}(P_0)\} = \{P \backslash {\uparrow} f^{-1}(U \backslash V) \colon U, V \in \mathsf{Do}(P_0)\}.$$

By Lemma 3.9 this happens precisely when for all $a, b \in P_1$,

$$\forall U, V \in \mathsf{Do}(P_0)(a \in {\uparrow} f^{-1}(U \backslash V) \implies b \in {\uparrow} f^{-1}(U \backslash V)) \implies a \leq b. \qquad (\dagger)$$

Now since for all $c \in P_0$ the set ${\downarrow} c \backslash \{c\}$ is a downset we see that all singletons are of the form $U \backslash V$ with $U, V \in \mathsf{Do}(P_0)$. More precisely

$$\{c\} = {\downarrow} c \backslash ({\downarrow} c \backslash \{c\}).$$

Therefore if we for all $a, b \in P_1$ have that

$$a \in {\uparrow} f^{-1}(U \backslash V) \implies b \in {\uparrow} f^{-1}(U \backslash V),$$

for all $U, V \in \mathsf{Do}(P_0)$. Then in particular we must have that for all $a, b \in P_1$

$$\forall c \in P_0 \ (a \in {\uparrow} f^{-1}(c) \implies b \in {\uparrow} f^{-1}(c)).$$

This is easily seen to be equivalent to $f(\downarrow a) \subseteq f(\downarrow b)$. It therefore follows that conservativity of (P_0, P_1, f) implies the conservativity of $(\mathsf{Do}(P_1), \mathsf{Do}(P_0), f^*)$.

Conversely, given $a, b \in P_1$, we see that $a \in \uparrow f^{-1}(U \backslash V)$ iff $f(c) \in U \backslash V$ for some $c \leq a$. Hence if $f(\downarrow a) \subseteq f(\downarrow b)$ it follows that there exists $c' \leq b$ such that $f(c') = f(c)$ whence $b \in \uparrow f^{-1}(U \backslash V)$. Consequently if (†) obtains we may conclude that $a \leq b$, and so we have shown that if $(\mathsf{Do}(P_0), \mathsf{Do}(P_1), f^*)$ is conservative so is (P_1, P_0, f). □

3.4 One-step semantics

In this section we show how to interpret a hypersequent calculus for **IPC** in one-step Heyting algebras.

Recall that a *sequent* is a pair of finite sets of formulas represented as $\Gamma \Rightarrow \Delta$. We read the formulas on the left-hand side of the arrow conjunctively and the formulas on the right-hand side disjunctively. We say that a sequent $\Gamma \Rightarrow \Delta$ is a single-succedent sequent if $|\Delta| \leq 1$, other we say that $\Gamma \Rightarrow \Delta$ is a multi-succedent sequent. By a *hypersequent* we will understand a finite set of sequents. We will represent a hypersequent as

$$S \coloneqq \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n.$$

We call a sequent $\Gamma_k \Rightarrow \Delta_k$ a *component* of S. Thus a single component hypersequent is nothing but a sequent. The symbol | can be thought of as a meta-level disjunction.

We will use small letters $s, s_1, s_2 \ldots$ to denote sequents and capital letter $S, S_1, S_2 \ldots$ to denote hypersequents. Finally we will use $\mathscr{S}, \mathscr{S}', \ldots$ to denote sets of hypersequents.

A *hypersequent rule* will be a $(n+1)$-tuple of hypersequents represented as

$$\frac{S_1, \ldots, S_n}{S}$$

In Appendix A we present two hypersequent calculi for **IPC**, viz. the single-succedent hypersequent calculus $HInt$ and the multi-succedent hypersequent calculus HJL'. The theory developed in this chapter and in Chapter 4 works equally well for both $HInt$ and HJL'.

Given a set P of propositional variables we define $Form(\mathsf{P})$ to be the set of formulas in P the language of intuitionistic logic. Now given a formula $\varphi \in Form(\mathsf{P})$ we define the *implicational degree of* φ, denoted $d(\varphi)$ by the following recursion.

$$d(p) = 0, \quad \text{and} \quad d(\varphi * \psi) = \max\{d(\varphi), d(\psi)\},$$

for $p \in \mathsf{P} \cup \{\bot\}$ and $* \in \{\wedge, \vee\}$. Finally,

$$d(\varphi \to \psi) = \max\{d(\varphi), d(\psi)\} + 1.$$

For $n \in \omega$ we let $Form_n(\mathsf{P}) := \{\varphi \in Form(\mathsf{P}) \colon d(\varphi) \leq n\}$.

We may extend d to sequents and hypersequents as follows

$$d(\Gamma \Rightarrow \Delta) = \max\{d(\varphi) \colon \varphi \in \Gamma \cup \Delta\} \quad \text{and} \quad d(S) = \max\{d(s) \colon s \in S\},$$

where S is a hypersequent. By the implicational degree of a hypersequent rule r we will understand the maximal degree of hypersequents occurring in r.

Definition 3.11. Given two disjoint finite sets P_0 and P_1 of propositional variables, a *valuation* on a one-step algebra $\mathcal{H} = (D_0, D_1, i)$ is a pair of functions $v = (v_0, v_1)$ such that $v_0 \colon \mathsf{P}_0 \to D_0$ and $v_1 \colon \mathsf{P}_1 \to D_1$.

Given a one-step algebra $\mathcal{H}$ together with a valuation $v = (v_0, v_1)$ for every formula $\varphi(\underline{p}) \in Form_0(\mathsf{P}_0)$ we define an element $\varphi^{\mathtt{v}_0} \in D_0$ as follows:

$$\bot^{\mathtt{v}_0} = \bot \quad \text{and} \quad p_i^{\mathtt{v}_0} = v_0(p_i) \quad \text{for } p_i \in \underline{p},$$

and

$$(\varphi_1 * \varphi_2)^{\mathtt{v}_0} = \varphi_1^{\mathtt{v}_0} * \varphi_2^{\mathtt{v}_0}, \qquad * \in \{\wedge, \vee\}.$$

Moreover, for every formula $\psi(\underline{p}, \underline{q}) \in Form_1(\mathsf{P}_0 \cup \mathsf{P}_1)$, where the elements of $\underline{q} \subseteq \mathsf{P}_1$ do not have any occurrence in the scope of an implication, we define an element $\psi^{\mathtt{v}_1} \in D_1$ as follows:

$$\bot^{\mathtt{v}_1} = \bot \quad \text{and} \quad q^{\mathtt{v}_1} = v_1(q) \quad \text{and} \quad p^{\mathtt{v}_1} = i(v_0(p)) \quad \text{for } q \in \underline{q} \text{ and } p \in \underline{p},$$

and

$$(\psi_1 * \psi_2)^{\mathtt{v}_1} = \psi_1^{\mathtt{v}_1} * \psi_2^{\mathtt{v}_1} \qquad * \in \{\wedge, \vee\}$$

Finally, for $\varphi_1, \varphi_2 \in Form_0(\mathsf{P}_0)$ we let,

$$(\varphi_1 \to \varphi_2)^{\mathtt{v}_1} = i(\varphi_1^{\mathtt{v}_0}) \to i(\varphi_2^{\mathtt{v}_0}),$$

Recall that by the definition of a one-step Heyting algebra the implications of the form $i(a) \to i(a)$ exist in D_1 and so the above is indeed well-defined.

Definition 3.12. Let $\Gamma \Rightarrow \Delta$ be a sequent of degree at most 1. We say that a one-step algebra $\mathcal{H}$ *validates* the sequent $\Gamma \Rightarrow \Delta$ *under a valuation* $v = (v_0, v_1)$ if

$$(\bigwedge \Gamma)^{\mathtt{v}_1} \leq (\bigvee \Delta)^{\mathtt{v}_1},$$

with the convention that $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$. A one-step algebra $\mathcal{H}$ validates a hypersequent $S = \Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$ under a valuation v if it validates at least one of the sequents $\Gamma_k \Rightarrow \Delta_k$ under v. We write $(\mathcal{H}, v) \vDash S$, if this is the case.

Finally, we say that a one-step algebra $\mathcal{H}$ validates a hypersequent S if it validates it under all possible valuations v on $\mathcal{H}$, in which case we write $\mathcal{H} \vDash S$. Moreover, if $r = (S_1, \dots, S_n)/S$ is a hypersequent rule of degree at most 1 we say that $\mathcal{H}$ validates r if for all valuations v on $\mathcal{H}$ we have that if $(\mathcal{H}, v) \vDash S_i$ for all $i \in \{1, \dots, n\}$ then $(\mathcal{H}, v) \vDash S$.

When speaking of validating a set HC of hypersequent rules of degree at most 1 we will only consider valuations (v_0, v_1) such that the domain of v_0 is the set of propositional variables occurring under the scope of an implication in some rule $r \in$ HC and the domain of v_1 is the set of propositional variables which does not have any occurrence under the scope of an implication in any of the rules belonging to HC.

We say that an intuitionistic one-step frame $\mathcal{S} = (P_1, P_0, f)$ validates a sequent, hypersequent or hypersequent rule if its dual one-step Heyting algebra $\mathcal{S}^* = (\mathsf{Do}(P_0), \mathsf{Do}(P_1), f^*)$ does.

Proposition 3.13. *Any finite one-step Heyting algebra validates all the rules of degree at most* 1 *of the hypersequent calculus* $HInt$ *as well as all the rules of the hypersequent calculus* HJL' *of Appendix A.*

Proof. To establish the proposition one must simply check that all the rules of $HInt$ and of HJL' are validated in any one-step Heyting algebra under any valuation. In most cases this is immediately seen to be the case by direct inspection. Only the left and right rules for introducing the implication might call for a closer inspection in both calculi.

The right rule is a direct consequence of the residuation property of the implication in D_1.

For the left rule it suffices to show that for any valuation v on a Heyting algebra $\mathcal{H}$ we have for all finite $\Gamma \cup \{\chi\} \subseteq Form_1(\mathsf{P})$ and all $\varphi, \psi \in Form_0(\mathsf{P})$ that

$$((\bigwedge \Gamma)^{\mathtt{v}_1} \leq \varphi^{\mathtt{v}_1} \text{ and } \psi^{\mathtt{v}_1} \wedge (\bigwedge \Gamma)^{\mathtt{v}_1} \leq \chi^{\mathtt{v}_1}) \implies (\bigwedge \Gamma)^{\mathtt{v}_1} \wedge \varphi^{\mathtt{v}_1} \to \psi^{\mathtt{v}_1} \leq \chi^{\mathtt{v}_1}.$$

Now since the equation $a \wedge (a \to b) = a \wedge b$ is true in any Heyting algebra it is in particular true in D_1, whence given that the above inequalities hold

$$\begin{aligned}(\bigwedge \Gamma)^{\mathrm{v}_1} \wedge (\varphi^{\mathrm{v}_1} \to \psi^{\mathrm{v}_1}) &= (\bigwedge \Gamma)^{\mathrm{v}_1} \wedge \varphi^{\mathrm{v}_1} \wedge (\varphi^{\mathrm{v}_1} \to \psi^{\mathrm{v}_1}) \\ &= (\bigwedge \Gamma)^{\mathrm{v}_1} \wedge \varphi^{\mathrm{v}_1} \wedge \psi^{\mathrm{v}_1} \\ &= (\bigwedge \Gamma)^{\mathrm{v}_1} \wedge \psi^{\mathrm{v}_1} \leq \chi^{\mathrm{v}_1}.\end{aligned}$$

□

Thus all the rules of the hypersequent calculi for **IPC** are sound with respect to the one-step semantics.

3.4.1 A one-step frame semantics for IPC

Given a finite intuitionistic one-step frame $\mathcal{S} = (P_1, P_0, f)$ and a step valuation $v = (v_0, v_1)$ on the dual one-step Heyting algebra, i.e. $v_i \colon \mathsf{P}_i \to \mathsf{Do}(P_i)$, we obtain a semantic relation $\Vdash$ for the degree 1 fragment of intuitionistic logic, as follows.

For $c \in P_0$ and $p \in \mathsf{P}_0$ we have

$$(\mathcal{S}, v), c \Vdash p \iff c \in v_0(p),$$

and $(\mathcal{S}, v), c \nVdash \bot$. For $\varphi_1, \varphi_2 \in Form_0(\mathsf{P}_0)$ we have that

$$\begin{aligned}(\mathcal{S}, v), c \Vdash \varphi_1 \wedge \varphi_2 &\iff (\mathcal{S}, v), c \Vdash \varphi_1 \text{ and } (\mathcal{S}, v), c \Vdash \varphi_2 \\ (\mathcal{S}, v), c \Vdash \varphi_1 \vee \varphi_2 &\iff (\mathcal{S}, v), c \Vdash \varphi_1 \text{ or } (\mathcal{S}, v), c \Vdash \varphi_2.\end{aligned}$$

For $a \in P_1$ and $p \in \mathsf{P}_0$, $q \in \mathsf{P}_1$ we let

$$(\mathcal{S}, v), a \Vdash p \iff f(a) \in v_0(p) \quad \text{and} \quad (\mathcal{S}, v), a \Vdash q \iff a \in v_1(q).$$

For $\psi_1, \psi_2 \in Form_1(\mathsf{P}_0 \cup \mathsf{P}_1)$ we let

$$\begin{aligned}(\mathcal{S}, v), a \Vdash \psi_1 \wedge \psi_2 &\iff (\mathcal{S}, v), a \Vdash \psi_1 \text{ and } (\mathcal{S}, v), a \Vdash \psi_2 \\ (\mathcal{S}, v), a \Vdash \psi_1 \vee \psi_2 &\iff (\mathcal{S}, v), a \Vdash \psi_1 \text{ or } (\mathcal{S}, v), a \Vdash \psi_2.\end{aligned}$$

Finally we for $\varphi_1, \varphi_2 \in Form_0(\mathsf{P}_0)$ let

$$(\mathcal{S}, v), a \Vdash \varphi_1 \to \varphi_2 \iff \forall b \leq a \; ((\mathcal{S}, v), f(b) \Vdash \varphi_1 \implies (\mathcal{S}, v), f(b) \Vdash \varphi_2).$$

Thus with this definition we have that

$$(S, v), c \Vdash \varphi \iff c \in \varphi^{\mathrm{v}_1}.$$

For a formula $\psi \in Form_1(\mathsf{P})$ we write $(\mathcal{S}, v) \Vdash \psi$ if $(\mathcal{S}, v), a \Vdash \psi$ for all $a \in P_1$ and we write $\mathcal{S} \Vdash \psi$, if $(\mathcal{S}, v) \Vdash \psi$ for all one-step valuations v on $\mathcal{S}$.

By the duality between finite intuitionistic one-step frames and finite one-step Heyting algebras we have that

$$\mathcal{S} \Vdash \psi \iff \mathcal{H} \vDash \psi,$$

for any formula $\psi \in Form_1(\mathsf{P})$ and any finite intuitionistic one-step frame $\mathcal{S}$ with dual one-step Heyting algebra $\mathcal{H}$.

3.5 Adding rules

By a *hypersequent calculus* we shall understand a set HC of hypersequent rules extending either the calculus $HInt$ or the calculus HJL'.

If $\mathscr{S} \cup \{S\}$ is a set of hypersequents and HC a hypersequent calculus we write $\mathscr{S} \vdash_{\mathrm{HC}} S$ if there is a derivation of the hypersequent S in the hypersequent calculus possibly using the hypersquents from $\mathscr{S}$ as (global) assumptions, i.e. as leaves in the derivation tree. In particular, substitutions may not be applied to the hypersequents in $\mathscr{S}$. In appendix A we give a rigorous definition of derivability in the hypersequent calculus. Most importantly we we still allow internal-cut and external contraction in all hypersequent calculi HC even though adding rules to the base calculi might not preserve the eliminability of these rules. Moreover, as we always apply rules in their contextual form external weakening will hold in all hypersequent calculi.

A problem with adding hypersequent rules to one of the basic hypersequent calculus $HInt$ or HJL' for **IPC**, is that we cannot speak of a one-step Heyting algebra validating an arbitrary rule since its degree might exceed one. However, as it turns out we can always find an equivalent hypersequent rule of degree at most one. Following the terminology of [11, 12] we shall call such a rule *reduced*.

To make sense of the above statement we first need to define a notion of equivalence of hypersequent rules.

Definition 3.14. We say that a hypersequent rule $(S_1 \dots S_n)/S$ is *derivable* in a hypersequent calculus HC, if

$$\{S_1, \dots, S_n\} \vdash_{\mathrm{HC}} S.$$

We then say that two hypersequent calculi HC and HC′ are *equivalent* if every rule of HC is derivable in HC′ and vice versa.

By a straightforward inductive argument we see that equivalent hypersequent calculi have the same set of derivable hypersequents. If HC and HC′ are equivalent hypersequent calculi and S is a hypersequent then

$$\vdash_{\mathrm{HC}} S \iff \vdash_{\mathrm{HC}'} S.$$

However, equivalent hypersequent calculi might not share all proof-theoretic properties.

Given a hypersequent calculus we want to construction a canonical Heyting algebra validating HC. This is done by modifying the well-known Lindenbaum-Tarski construction.[2] To describe this construction we will need to make use of the following lemma.

Lemma 3.15. *Let $\mathscr{S}$ be a set of hypersequents, let s be a sequent and let S be a hypersequent. Then for every hypersequent calculus* HC *we have that*

$$(\mathscr{S} \cup \{s\} \vdash_{\mathrm{HC}} S \quad \textit{and} \quad \mathscr{S} \vdash_{\mathrm{HC}} s \mid S) \quad \textit{implies} \quad \mathscr{S} \vdash_{\mathrm{HC}} S.$$

Proof. Assuming that $\mathscr{S} \vdash_{\mathrm{HC}} s \mid S$ we see that for any hypersequent S' if $\mathscr{S} \cup \{s\} \vdash_{\mathrm{HC}} S'$, then, by an induction on a derivation witnessing this, we must have that $\mathscr{S} \vdash_{\mathrm{HC}} S' \mid S$. Therefore, if $\mathscr{S} \vdash_{\mathrm{HC}} s \mid S$ and $\mathscr{S} \cup \{s\} \vdash_{\mathrm{HC}} S$ we may conclude that $\mathscr{S} \vdash_{\mathrm{HC}} S \mid S$, whence by applying external contraction we obtain that $\mathscr{S} \vdash_{\mathrm{HC}} S$, as desired. □

Proposition 3.16. *For every hypersequent calculus HC and every set of hypersequents $\mathscr{S} \cup \{S\}$ such that $\mathscr{S} \nvdash_{HC} S$ there exists a Heyting algebra $\mathfrak{LT}_{HC}(\mathscr{S}, S)$ validating HC and a valuation on $\mathfrak{LT}_{HC}(\mathscr{S}, S)$ under which $\mathfrak{LT}_{HC}(\mathscr{S}, S)$ validates $\mathscr{S}$ but not S.*

Proof. Let $\mathscr{S}'$ be a maximal set of hypersequents extending $\mathscr{S}$ such that $\mathscr{S}' \nvdash_{HC} S$. Assuming Zorn's Lemma such a set always exists. Then define an equivalence relation $\approx$ on the formula algebra $Form(\mathsf{P})$ by

$$\varphi \approx \psi \iff \mathscr{S}' \vdash_{HC} \varphi \Rightarrow \psi \text{ and } \mathscr{S}' \vdash_{HC} \psi \Rightarrow \varphi.$$

Note that by maximailty of $\mathscr{S}'$ we have that $\varphi \approx \psi$ precisely when $\mathscr{S}'$ contains both of the sequents $\varphi \Rightarrow \psi$ and $\psi \Rightarrow \varphi$. Since HC extends a hypersequent calculus of **IPC** one may readily verify that $\mathfrak{LT}_{HC}(\mathscr{S}, S) = Form(\mathsf{P})/{\approx}$ is a Heyting algebra.

[2]Note that an arbitrary class of algebras will in general not have free algebras. The class of Abelian groups of order 2 or 3 is a standard example of a universal class of algebras not having free algebras.

We observe that by the maximality of $\mathscr{S}'$, Lemma 3.15 together with the assumption that $\mathscr{S}' \nvdash_{HC} S$ yields that

$$\mathscr{S}' \vdash_{HC} s_1 \mid \ldots \mid s_m \mid S \implies \mathscr{S}' \vdash_{HC} s_i \text{ for some } 1 \leq i \leq m\ , \tag{$\dagger$}$$

for all sequents $s_1, \ldots s_m$. For suppose not, then in particular $\mathscr{S}' \cup \{s_1\} \vdash_{HC} S$ by maximality $\mathscr{S}'$. So by Lemma 3.15 we must have that $\mathscr{S} \vdash_{HC} s_2 \mid \ldots \mid s_m \mid S$. Thus after repeating this argument m times we obtain $\mathscr{S} \vdash_{HC} S$, in direct contradiction with the initial assumption.

Observe that from ($\dagger$) and external weakening it follows that if $\mathscr{S} \vdash_{\mathrm{HC}} s_1 \mid \ldots \mid s_m$ then $\mathscr{S} \vdash_{\mathrm{HC}} s_i$ for some $1 \leq i \leq m$. From this it is easy to verify that $\mathfrak{LT}_{HC}(\mathscr{S}, S)$ validates all the rules of HC.

Finally we claim that under the valuation determined by sending propositional variables to their equivalence class of the equivalence relation $\approx$, the algebra $\mathfrak{LT}_{HC}(\mathscr{S}, S)$ validates all the hypersequents from $\mathscr{S}$ but does not validate the hypersequent S. This, however, is evident. □

With the help of the Lindenbaum-Tarski algebras for hypersequent calculus we can easily establish an algebraic criterion for derivability of rules.

Proposition 3.17. *Let HC be a hypersequent calculus and let $r = (S_1 \ldots S_n)/S$ be a hypersequent rule. Then the following are equivalent.*

i) $\{S_1, \ldots, S_n\} \vdash_{HC} S$;

ii) Every Heyting algebra validating HC validates the rule r.

Proof. That item i) implies item ii) follows by an easy induction on the length of a proof witnessing that $\{S_1, \ldots, S_n\} \vdash_{HC} S$.

By considering the contrapositive it immediately follows from Proposition 3.16 that item ii) implies item i). □

We say that a formula *occurs* in a hypersequent S if it is either a formula on the right or the left hand side of the sequent arrow of some sequent belonging to S.

Proposition 3.18. *Any hypersequent rule is equivalent of a reduced hypersequent rule.*

Proof. Given a hypersequent rule $r = (S_1, \ldots, S_m)/S_{m+1}$ of depth $n+1$ with $n \geq 1$ and an *occurrence* of a formula α of depth $n+1$ in r, we produce an equivalent rule with one less occurrence of the formula α.

Let S_i be the hypersequent with the given occurrence of α and let $\Gamma \Rightarrow \Delta$ be the sequent in S_i with the given occurrence of α. As the formula α is of depth $n+1$ it must be of the form $\varphi \to \psi$ with $\max\{d(\varphi), d(\psi)\} = n$. We introduce a fresh variable p and replace the given occurrence of α in S_i with $p \to \psi$ or $\varphi \to p$, depending on whether $d(\varphi) = n$ or $d(\psi) = n$. If both $d(\varphi)$ and $d(\psi) = n$ we introduce two fresh variables. Let S_i' be the hypersequent resulting form such a replacement. Evidently S_i' has one less occurrence of the formula α than S_i . Moreover if $i \leq m$ let S_i'' be the hypersequent obtained by replacing the sequent $\Gamma \Rightarrow \Delta$ in S_i with the sequent $p \Rightarrow \psi$ or $\varphi \Rightarrow p$ depending on whether we replace $\varphi \to \psi$ with $\varphi \to p$ or with $p \to \psi$ in S_i. If $i = m+1$ let S_i'' be the hypersequent consisting of the single component hypersequent $p \Rightarrow \psi$ or $\varphi \Rightarrow p$ again depending on whether we replace $\varphi \to \psi$ with $\varphi \to p$ or with $p \to \psi$ in S_i.

In this way we obtain a rule

$$\frac{S_1, \ldots, S_{i-1}, S_i', S_{i+1}, \ldots, S_m, S_i''}{S_{m+1}}\ (r')$$

Which by Proposition 3.17 must be equivalent to the rule r.

Continuing this procedure for each occurrence of a formula of depth $n+1$ in r we obtain a rule r_1 of depth n which is equivalent to r. In this way we obtain a sequence $r_{n+1}, r_n \ldots, r_1$ of equivalent rule such that $r_{n+1} = r$ and $d(r_k) = k$, for all $1 \leq k \leq n+1$. □

Note as the above procedure abstracts away one occurrence of a formula of the form $\varphi \to \psi$ at a time, and since we first abstract away outermost occurrences, it is always clear whether to replace the formula occurring negatively or positively in the formula $\varphi \to \psi$.

By the above proposition we know that for every hypersequent calculus there exists an equivalent calculus which may be interpreted in one-step Heyting algebras. Thus the one-step semantics is in fact not as restrictive as it may seem at first.

We say that $\mathcal{H}$ validates a reduced hypersequent calculus HC if it validates all the rules in HC. In light of Proposition 3.18 we may without loss of generality assume that all hypersequent calculi are reduced, i.e. only consisting of reduced rules.

Let HC be a reduced hypersequent calculus and let $\mathscr{S} \cup \{S\}$ be a set of hypersequents of implicational degree at most n. We write $\mathscr{S} \vdash^n_{\mathrm{HC}} S$ if $\mathscr{S} \vdash_{\mathrm{HC}} S$ and this is witnessed by a derivation not containing any hypersequents of implicational degree exceeding n. We may then establish the following proposition showing that one-step semantics is sound with respect to the derivability relation $\vdash^1_{\mathrm{HC}}$.

Proposition 3.19. *Let $\mathcal{H}$ be a one-step algebra,* HC *a reduced hypersequent calculus, and $\mathscr{S} \cup \{S\}$ a set of hypersequents of degree at most* 1*. If $\mathcal{H}$ validates both $\mathscr{S}$ and HC and $\mathscr{S} \vdash^1_{\mathrm{HC}} S$ then $\mathcal{H}$ validates S as well.*

Proof. By induction on the length of a derivation witnessing $\mathscr{S} \vdash^1_{\mathrm{HC}} S$. □

We conclude this chapter by proving a lemma showing that our notion of one-step embedding is indeed the right one in the sense that one-step embeddings preserve validity.

Lemma 3.20. *Let $(g_0, g_1)\colon \mathcal{H} \to \mathcal{H}'$ be an embedding of one-step algebras. If $v = (v_0, v_1)$ and $v' = (v'_0, v'_1)$ are valuations on $\mathcal{H}$ and $\mathcal{H}'$, respectively, such that $v'_0(p) = g_0(v_0(p))$ and $v'_1(q) = g_1(v_1(p))$, for all $p \in \mathsf{P}_0$ and $q \in \mathsf{P}_1$, then for any hypersequent calculus* HC *we have that $(\mathcal{H}, v)$ validates* HC *iff $(\mathcal{H}', v')$ validates* HC.

Proof. It suffices to show that for all formulas $\varphi, \psi \in Form_1(\mathsf{P}_0 \cup \mathsf{P}_1)$

$$\varphi^{\mathrm{v}_1} \leq \psi^{\mathrm{v}_1} \iff \varphi^{\mathrm{v}'_1} \leq \psi^{\mathrm{v}'_1} \tag{†}$$

Since (g_0, g_1) is a map of one-step algebras an easy inductive argument shows that the assumption $v'_0(p) = g(v_0(p))$ and $v'_1(q) = g_1(v_1(p))$ for all $p \in \mathsf{P}_0, q \in \mathsf{P}_1$ implies that $\varphi^{\mathrm{v}'_1} = g_1(\varphi^{\mathrm{v}_1})$ for all $\varphi \in Form_1(\mathsf{P}_0 \cup \mathsf{P}_1)$. From this (†) readily follows as any injective lattice homomorphism will necessarily be both order-preserving and order-reflecting. □

In particular, we have that if $\mathcal{H}'$ is a one-step algebra validating HC and $\mathcal{H}$ embeds into $\mathcal{H}'$ then $\mathcal{H}$ validates HC as well. This fact will be used several times in the next chapter.

Summary of Chapter 3: In this chapter we have set up of the basic theory of one-step Heyting algebras and shown how it gives rise to a semantics for hypersequent calculi for intuitionistic logic and its extensions. Furthermore, we have shown that the one-step semantics is in fact sufficient for dealing with hypersequent calculi for extensions of intuitionistic logic as every hypersequent rule is equivalent to a rule of implicational degree at most 1. We are now ready to investigate the bounded proof property for hypersequent calculi in the next chapter.

Chapter 4

The bounded proof property for hypersequent calculi

In this chapter we introduce the bounded proof property for hypersequent calculi, and give an algebraic characterisation of this property analogous to the one found in [11]. Finally, we present a basic calculus for one-step correspondence for hypersequent rules.

In this chapter we will assume that all hypersequents calculi are reduced.

4.1 The bounded proof property

We say that a hypersequent calculus HC has the *bounded proof property* if whenever $\mathscr{S} \cup \{S\}$ is a set of hypersequents of implicational depth at most n such that $\mathscr{S} \vdash_{\mathrm{HC}} S$ then $\mathscr{S} \vdash^n_{\mathrm{HC}} S$, i.e. there exists a proof witnessing $\mathscr{S} \vdash_{\mathrm{HC}} S$ consisting only of hypersequents of degree at most n. The bounded proof property is thus a very weak form of analyticity. However, having this property will indicate some kind of robustness of the hypersequent calculus in question. For instance the subformula property will entail the bounded proof property. Therefore if a hypersequent calculus enjoys cut-elimination it will also, under mild additional assumptions, have the subformula property and hence the bounded proof property. Finally, as in the modal case, having the bounded proof property will ensure that the derivability relation $\vdash_{\mathrm{HC}}$ is decidable, given that membership of HC is decidable.

The next proposition shows that the bounded proof property is completely determined by the degree 1 case.

Proposition 4.1. *A hypersequent calculus* HC *has the bounded proof property iff for each set* $\mathscr{S} \cup \{S\}$ *consisting of hypersequents of degree at most* 1*, we have*

$$\mathscr{S} \vdash_{\mathrm{HC}} S \iff \mathscr{S} \vdash^1_{\mathrm{HC}} S.$$

Proof. The implication from left to right is evident.

For the converse implication let $\mathscr{S} \cup \{S\}$ be a set of hypersequents of degree at most n. We define a sequence of triples $(\mathscr{S}_i, S_i, \sigma_i)_{i=0}^{n-1}$ such that

i) $\mathscr{S}_i \cup \{S_i\}$ is a set of hypersequents of degree at most $n-i$ and σ_i is a substitution such that $d(\chi\sigma_i) \leq d(\chi)+1$ for all formulas χ occurring in $\mathscr{S}_i \cup \{S_i\}$;

ii) $S_{i+1}\sigma_{i+1} = S_i$;

iii) $\mathscr{S}_{i+1}\sigma_{i+1}$ equals $\mathscr{S}_i$ union some set of sequents of the form $\chi \Rightarrow \chi$;

iv) $\mathscr{S}_{i+1} \vdash_{\mathrm{HC}} S_{i+1} \iff \mathscr{S}_i \vdash_{\mathrm{HC}} S_i$.

Let $\mathscr{S}_0$ be $\mathscr{S}$, S_0 be S and let σ_0 be the identity substitution. Now assume that the triple $(\mathscr{S}_i, S_i, \sigma_i)$ has been defined. Then for each subformula of the form $\varphi \to \psi$ with $d(\varphi) = d(\psi) = 0$ occurring in some formula of some sequent of some hypersequent in $\mathscr{S}_i \cup \{S_i\}$ introduce a fresh variable $p_{\varphi\psi}$ and replace $\varphi \to \psi$ with $p_{\varphi\psi}$ everywhere. Let $\mathscr{S}'_i$ and S_{i+1} be the result of such replacements. Finally let

$$\mathscr{S}_{i+1} = \mathscr{S}'_i \cup \{p_{\varphi\psi} \Rightarrow \varphi \to \psi,\ \varphi \to \psi \Rightarrow p_{\varphi\psi}\}_{\varphi\to\psi}.$$

The substitution σ_{i+1} is then defined as $\sigma_{i+1}(p_{\varphi\psi}) = \varphi \to \psi$.

With this definition i)-iii) are easily seen to hold. For item iv) it suffices to observe that the derivability relation is structural, i.e. preserved by substitutions.

Now if $\mathscr{S} \vdash_{\mathrm{HC}} S$ then by construction we must have that $\mathscr{S}_{n-1} \vdash_{\mathrm{HC}} S_{n-1}$. Moreover, as per item i) the degree of $\mathscr{S}_{n-1} \cup \{S_{n-1}\}$ is at most 1, hence the initial hypothesis yields $\mathscr{S}_{n-1} \vdash^1_{HC} S_{n-1}$. This suffices to establish the proposition as soon as we observed that if $\mathscr{S}_{n-k} \vdash^k_{\mathrm{HC}} S_{n-k}$ then $\mathscr{S}_{n-(k+1)} \vdash^{k+1}_{\mathrm{HC}} S_{n-(k+1)}$. However, this is an immediate consequence of item ii) and iii) together with the fact that for any hypersequent S we have that $d(S\sigma_{i+1}) \leq d(S)+1$. □

We then need a Lindenbaum-Tarski construction for one-step algebras analogous to Proposition 3.16.

Proposition 4.2. *For every hypersequent calculus* HC *and every set of hypersequents* $\mathscr{S} \cup \{S\}$ *of degree at most* 1 *such that* $\mathscr{S} \nvdash^1_{\mathrm{HC}} S$ *there exists a finite conservative one-step Heyting algebra* $\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ *validating* HC *and a one-step valuation on* $\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ *under which* $\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ *validates* $\mathscr{S}$ *but not* S*.*

Proof. Similar to the proof of Proposition 3.16. One needs to verify that a bounded version of Lemma 3.15 obtains. Then simply let

$$D_0 := \{[\varphi] \in \mathfrak{LT}_{\mathrm{HC}}(\mathscr{S}, S) \colon d(\varphi) = 0\} \quad \text{and} \quad D_1 := \{[\varphi] \in \mathfrak{LT}_{\mathrm{HC}}(\mathscr{S}, S) \colon d(\varphi) \leq 1\},$$

with $i \colon D_0 \to D_1$ the obvious map. □

4.2 Diagrams

Given a finite conservative one-step Heyting algebra $\mathcal{H} = (D_0, D_1, i)$ we will define the *diagram* associate to $\mathcal{H}$. This construction is analogous to the diagrams of a finite conservative one-step modal algebra from [11]. IN fact they are a two-sorted version of the diagrams know from model theory [20].

We introduce a set of propositional variables $\mathsf{P}_{\mathcal{H}} = \{p_a \colon a \in D_0\}$. Then by the conservativity of $\mathcal{H}$ it follows that for each $a \in D_1$ there exists a formula $\vartheta_a \in Form_1(\mathsf{P}_{\mathcal{H}})$ such that $\vartheta_b^{\mathrm{v}_1} = b$, where v is the *natural* valuation on $\mathcal{H}$ given by $v_0(p_a) = a$. In particular, we have that $\vartheta_{i(a)} = p_a$ for all $a \in D_0$.

Now let

$$\begin{aligned}
\mathscr{S}^0_{\mathcal{H}} :=& \{p_{a\wedge b} \Rightarrow p_a \wedge p_b,\ p_a \wedge p_b \Rightarrow p_{a\wedge b} \colon a, b \in D_0\} \\
& \cup \{p_{a\vee b} \Rightarrow p_a \vee p_b,\ p_a \vee p_b \Rightarrow p_{a\vee b} \colon a, b \in D_0\} \\
& \cup \{p_\perp \Rightarrow \perp,\ \perp \Rightarrow p_\perp\},
\end{aligned}$$

and

$$\begin{aligned}
\mathscr{S}^1_{\mathcal{H}} :=& \{\vartheta_{a\wedge b} \Rightarrow \vartheta_a \wedge \vartheta_b,\ \vartheta_a \wedge \vartheta_b \Rightarrow \vartheta_{a\wedge b} \colon a, b \in D_1\} \\
& \cup \{\vartheta_{a\vee b} \Rightarrow \vartheta_a \vee \vartheta_b,\ \vartheta_a \vee \vartheta_b \Rightarrow \vartheta_{a\vee b} \colon a, b \in D_1\} \\
& \cup \{\vartheta_{i(a)\to i(b)} \Rightarrow \vartheta_{i(a)} \to \vartheta_{i(b)},\ \vartheta_{i(a)} \to \vartheta_{i(b)} \Rightarrow \vartheta_{i(a)\to i(b)} \colon a, b \in D_0\}
\end{aligned}$$

We then define the *positive diagram of* $\mathcal{H}$ to be $\mathscr{S}_{\mathcal{H}} := \mathscr{S}^0_{\mathcal{H}} \cup \mathscr{S}^1_{\mathcal{H}}$.

For each $a, b \in D_1$ we let s_{ab} be the sequent $\vartheta_a \Rightarrow \vartheta_b$ if $a \not\leq b$ and the empty sequent if $a \leq b$. We then define the *negative diagram of* $\mathcal{H}$ to be the hypersequent

$$S_{\mathcal{H}} \coloneqq \{s_{ab} \colon a, b \in D_1\}.$$

Definition 4.3. By the *diagram* of a finite conservative one-step Heyting algebra we will understand the hypersequent rule $\mathscr{S}_{\mathcal{H}}/S_{\mathcal{H}}$.

Note that in a diagram $\mathscr{S}_{\mathcal{H}}/S_{\mathcal{H}}$ all propositional variables have an occurrence under the scope of an implication.

We say that a step algebra $\mathcal{H}'$ *refutes* a diagram $\mathscr{S}_{\mathcal{H}}/S_{\mathcal{H}}$ under a valuation v if $(\mathcal{H}', v) \vDash \mathscr{S}_{\mathcal{H}}$ but $(\mathcal{H}', v) \nvDash S_{\mathcal{H}}$.

The following proposition shows why we are interested in diagrams.

Proposition 4.4. *Let $\mathcal{H} = (D_0, D_1, i)$ and $\mathcal{H}' = (D'_0, D'_1, i')$ be conservative one-step Heyting algebras with $\mathcal{H}$ finite. Then the following are equivalent:*

i) There exists a one-step embedding from $\mathcal{H}$ into $\mathcal{H}'$;

ii) There exists a valuation v on $\mathcal{H}'$ such that $(\mathcal{H}', v')$ refutes the diagram of $\mathcal{H}$.

Proof. First assume that there exists a one-step embedding $(g_0, g_1) \colon \mathcal{H} \to \mathcal{H}'$. We then define a valuation $v' = (v'_0, v'_1)$ on $\mathcal{H}'$ by $v_0(p_a) = g_0(a)$, (since all propositional variables in $\mathscr{S}_{\mathcal{H}}/S_{\mathcal{H}}$ have an occurrence under the scope of an implication we do not need to define v_1). Then as $\mathcal{H}$ evidently refutes its own diagram under the natural valuation $v_0(p_a) = a$ it immediately follows from Lemma 3.20 that $(\mathcal{H}', v')$ refutes $\mathscr{S}_{\mathcal{H}}/S_{\mathcal{H}}$ as well.

Conversely if there exists a valuation $v' = (v'_0, v'_1)$ on $\mathcal{H}'$ such that $(\mathcal{H}', v')$ refutes the diagram of $\mathcal{H}$. Then we claim that defining $(g_0, g_1) \colon \mathcal{H} \to \mathcal{H}'$ by

$$g_0(a) = v'_0(p_a) \quad \text{and} \quad g_1(b) = \vartheta_b^{\mathrm{v}'_1},$$

yields an embedding of one-step algebras.

First of all since $\mathcal{H}$ is conservative the function g_1 is well-defined. Because i' is an injection and $(\mathcal{H}', v') \vDash \mathscr{S}^0_{\mathcal{H}}$ we see that g_0 must be a bounded lattice homomorphism. As $(\mathcal{H}', v')$ also validates $\mathscr{S}^1_{\mathcal{H}}$ we see that g_1 is a bounded lattice homomorphism.

Now to see that $i' \circ g_0 = g_1 \circ i$ we simply observe that for all $a \in D_0$

$$i(g_0(a)) = i(v'_0(p_a)) = p_a^{\mathrm{v}'_1} = \vartheta_{i(a)}^{\mathrm{v}'_1} = g_1(i(a)).$$

From the assumption that $(\mathcal{H}', v')$ does not validate any of the sequents $\vartheta_a \Rightarrow \vartheta_b$ when $a \not\leq b$ it immediately follows that g_1 is an injection. So as i is an injection we must have that g_0, being the first component of the injection $g_1 \circ i$, is an injection as well.

Finally because $(\mathcal{H}', v')$ validates all sequents of the form $\vartheta_{i(a)\to i(b)} \Rightarrow \vartheta_{i(a)} \to \vartheta_{i(b)}$ and $\vartheta_{i(a)} \to \vartheta_{i(b)} \Rightarrow \vartheta_{i(a)\to i(b)}$ we have that

$$g_1(i(a) \to i(a)) = i'(g_0(a)) \to i'(g_0(b)),$$

and so we can conclude that (g_0, g_1) is indeed an embedding of one-step algebras. $\square$

4.3 An algebraic characterisation of the bounded proof property

We say that a class K of one-step Heyting algebras (respectively one-step intuitionisitic Kripke frames) has the *extension property* if every member of K has a one-step extension belonging to K, cf. Section 2.5 of Chapter 2.

In this section we prove that the extension property for finite conservative one-step Heyting algebras validating a hypersequent calculus HC obtains precisely when HC enjoys the bounded proof property. To this end the following sufficient criterion for the extension property will be useful.

Lemma 4.5. *Let* HC *be a hypersequent calculus and let* $Con^{Alg}_{<\omega}(\mathrm{HC})$ *be the class of finite conservative one-step Heyting algebras validating* HC*. If every* $\mathcal{H} \in Con^{Alg}_{<\omega}(\mathrm{HC})$ *embeds into some standard Heyting algebra validating* HC *then the class* $Con^{Alg}_{<\omega}(\mathrm{HC})$ *has the extension property.*

Proof. Let $\mathcal{H}$ be a finite (conservative) one-step Heyting algebra and suppose that there exists an embedding $(g_0, g_1)\colon \mathcal{H} \to \mathfrak{A}$ into some Heyting algebra $\mathfrak{A}$ validating HC. Letting A be the bounded lattice reduct of $\mathfrak{A}$, we see that $\mathcal{H}'' = (D_1, A, g_1)$ is a one-step algebra validating HC and extending $\mathcal{H}$.

To obtain a finite conservative one-step Heyting algebra validating HC and extending $\mathcal{H}$ let D_2 be the bounded distributive sublattice of A generated by the set $\{g_1(a) \to g_1(b)\colon a, b \in D_1\}$. As the variety of bounded distributive lattices is locally finite D_2 is finite. Moreover, we have $\mathrm{Im}(g_1) \subseteq D_2$. Therefore, $\mathcal{H}' = (D_1, D_2, g_1)$ will be a finite conservative one-step algebra validating HC and extending $\mathcal{H}$. $\square$

Theorem 4.6. *Let* HC *be a hypersequent calculus. Then the following are equivalent.*

i) HC *has the bounded proof property;*

ii) The class $Con^{Alg}_{<\omega}(\mathrm{HC})$ *of finite conservative one-step algebras validating HC has the extension property;*

iii) The class $Con^{Frm}_{<\omega}(\mathrm{HC})$ *of finite conservative one-step frames validating HC has the extension property.*

Proof. Item ii) and item iii) are easily seen to be equivalent by elementary considerations on the duality between finite conservative one-step algebras and finite conservative one-step frames.

To see that ii) implies i) let $\mathscr{S} \cup \{S\}$ be a set of hypersequents of degree at most 1. We prove that if the class of finite conservative one-step algebras validating HC has the extension property then $\mathscr{S} \nvdash^1_{\mathrm{HC}} S$ implies that $\mathscr{S} \nvdash_{\mathrm{HC}} S$. By Proposition 4.1 this suffices to establish the bounded proof property for HC. We proceed by constructing a standard Heyting algebra $\mathfrak{A}$ together with a valuation v such that $(\mathfrak{A}, v)$ validates HC and $\mathscr{S}$ but not the hypersequent S. From Lemma 3.17 it then follows that $\mathscr{S} \nvdash_{\mathrm{HC}} S$.

We use the modified Lindenbaum-Tarski construction from Proposition 4.2. Let P be the set of propositional variables occurring in $\mathscr{S} \cup \{S\}$. By Proposition 4.2 there exists a finite conservative one-step Heyting algebra $\mathcal{H}_0 := \mathcal{LT}_{HC}(\mathscr{S}, S) = (D_0, D_1, i_0)$ validating HC and a valuation v^0 such that $(\mathcal{H}_0, v^0)$ validates $\mathscr{S}$ but not S. Therefore, by the assumption that $Con^{Alg}_{<\omega}(\mathrm{Ax})$ has the extension property, there exists a one-step extension $\mathcal{H}_1 = (D_1, D_2, i_1)$ of $\mathcal{H}_0$ also validating HC. Moreover, letting $v^1 = (v^1_0, v^1_1)$ be the valuation on $\mathcal{H}_1$ given by $v^1_0(p) = g_0(v^0_0(p))$ and $v^1_1(q) = g_1(v^0_1(q))$, Lemma 3.20 ensures that $(\mathcal{H}_1, v^1) \vDash \mathscr{S}$ and $(\mathcal{H}_1, v^1) \nvDash S$. In this way we may recursively define a chain

$$D_0 \xrightarrow{i_0} D_1 \xrightarrow{i_1} \dots \xrightarrow{i_{n-1}} D_n \xrightarrow{i_n} D_{n+1} \xrightarrow{i_{n+1}} \dots \qquad (\dagger)$$

in the category $\mathsf{bDist}_{<\omega}$ with the property that for all $n \in \omega$ and all $a, b \in D_n$

$$i_{n+1}(i_n(a) \to_{n+1} i_n(b)) = i_{n+1}(i_n(a)) \to_{n+2} i_{n+1}(i_n(b)). \qquad (\ddagger)$$

Letting $\mathfrak{A}$ be the chain colimit of diagram $(\dagger)$ in the category $\mathsf{bDist}_{<\omega}$ we see by a similar argument to the one used in the proof of Theorem 2.12 that $\mathfrak{A}$ is in fact a Heyting algebra with Heyting implication defined by

$$[a] \to [b] = [i_{n,k+1}(a) \to_{k+2} i_{m,k+1}], \quad k = \max\{m, n\}$$

for $a \in D_n$ and $b \in D_m$ and $i_{j,l} \colon D_j \to D_l$ is the obvious map for $j \leq l$.

As all the step algebras $\mathcal{H}_n = (D_n, D_{n+1}, i_n)$ validate HC we must also have that $\mathfrak{A}$ validates HC. Finally if we let v' be the valuation on $\mathfrak{A}$ be given by $v'_0(p) = [v^0_0(p)]$ and $v'_1(q) = [v^0_1(q)]$ then because the maps $i_k \colon D_k \to D_{k+1}$ are all injective lattice homomorphisms we must have that $(\mathfrak{A}, v)$ validates $\mathscr{S}$ but not S.

Conversely, to see that item i) implies item ii) assume that HC is a hypersequent calculus with the bounded proof property and let $\mathcal{H} = (D_0, D_1, i)$ be a finite conservative one-step Heyting algebra validating HC. As $\mathcal{H}$ refutes the diagram $\mathscr{S}_{\mathcal{H}}/S_{\mathcal{H}}$ it follows from Proposition 3.19 that $\mathscr{S}_{\mathcal{H}} \not\vdash^1_{\mathrm{HC}} S_{\mathcal{H}}$. Therefore, since by assumption HC has the bounded proof property, $\mathscr{S}_{\mathcal{H}} \not\vdash_{\mathrm{HC}} S_{\mathcal{H}}$. Hence by Proposition 3.17 there exists a Heyting algebra $\mathfrak{A}$ which refutes $\mathscr{S}_{\mathcal{H}}/S_{\mathcal{H}}$. Consequently by Proposition 4.4 there must exist an embedding $(g_0, g_1) \colon \mathcal{H} \to \mathfrak{A}$ and so by Lemma 4.5 it follows that the class $Con^{Alg}_{<\omega}(\mathrm{HC})$ has the extension property. □

In practice it can be somewhat cumbersome to work with one-step extensions of one-step algebras and frames. However, in what follows we show that under the assumption that the hypersequent calculus has the finite model property we obtain a version of Theorem 4.6 which avoids the notion of one-step extensions.

Definition 4.7. We say that a hypersequent calculus HC has the *(global) finite model property* if for each set $\mathscr{S} \cup \{S\}$ of hypersequents, $\mathscr{S} \not\vdash_{\mathrm{HC}} S$ iff there exists a finite Heyting algebra $\mathfrak{A}$ validating HC and a valuation v on $\mathfrak{A}$ such that $(\mathfrak{A}, v)$ validates all the hypersequents from $\mathscr{S}$ but not the hypersequent S.

Proposition 4.8. *A hypersequent calculus* HC *has the finite model property iff for each set $\mathscr{S} \cup \{S\}$ of hypersequents, $\mathscr{S} \not\vdash_{\mathrm{HC}} S$ iff there exists a finite intuitionistic Kripke frame $\mathfrak{F}$ validating* HC *and a valuation v on $\mathfrak{F}$ such that $(\mathfrak{F}, v)$ validates all the hypersequents from $\mathscr{S}$ but not the hypersequent S.*

Proof. Immediate by the duality between finite Heyting algebras and finite intuitionistic Kripke frames. □

Lemma 4.9. *Let* HC *be a hypersequent calculus. Then HC has the finite model property iff if for each set $\mathscr{S} \cup \{S\}$ of hypersequents of degree at most* 1*, $\mathscr{S} \not\vdash_{\mathrm{HC}} S$ iff there exists a finite Heyting algebra $\mathfrak{A}$ together with a valuation v such $(\mathfrak{A}, v)$ validates HC and all the hypersequents from $\mathscr{S}$ but not the hypersequent S.*

Proof. Similar to the proof of Proposition 4.1. □

If $\mathfrak{F} = (P, \leq)$ is an intuitionistic Kripke frame and $\mathcal{S} = (P_1, P_0, f)$ is a one-step frame we say that $\mathcal{S}$ is the *relative open image* of $\mathfrak{F}$ if there exists a surjective one-step map from

$\mathfrak{F}$ viewed as a standard one-step frame into $\mathcal{S}$. Evidently this is equivalent to requiring that there exist an f-open order preserving surjection $g\colon P \to P_1$.

In the presence of the finite model property we may prove a strengthened version of Theorem 4.6.

Theorem 4.10. *Let* HC *be a hypersequent calculus. Then the following are equivalent.*

i) HC *has the bounded proof property and the finite model property;*

ii) Every finite conservative one-step algebra validating HC *embeds into some finite (standard) Heyting algebra validating* HC*;*

iii) Every finite conservative one-step frame validating HC *is the relative open image of some finite (standard) intuitionistic Kripke frame validating* HC.

Proof. As before the equivalence between item ii) and iii) is an easy exercise in duality.

To see that item i) implies item ii) assume that HC is a hypersequent calculus with the bounded proof property and the finite model property and let $\mathcal{H}$ be a finite conservative one-step Heyting algebra validating HC. Then as $\mathcal{H}$ refutes the diagram $\mathscr{S}_{\mathcal{H}}/S_{\mathcal{H}}$ with the natural valuation we have by Proposition 3.19 that $\mathscr{S}_{\mathcal{H}} \not\vdash^1_{\mathrm{HC}} S_{\mathcal{H}}$. So as HC has the bounded proof property we obtain that $\mathscr{S}_{\mathcal{H}} \not\vdash_{\mathrm{HC}} S_{\mathcal{H}}$. Consequently by the assumption that HC has the finite model property there exists a finite Heyting algebra $\mathfrak{A}$ which validates HC and refutes the diagram $\mathscr{S}_{\mathcal{H}}/S_{\mathcal{H}}$. Therefore by Proposition 4.4 the one-step algebra $\mathcal{H}$ is embeddable into $\mathfrak{A}$.

To see that item ii) implies item i) we first note that if item ii) obtains then by Lemma 4.5 and Theorem 4.6 the calculus HC has the bounded proof property. To see that HC also has the finite model property we know by Lemma 4.9 that it suffices to consider sets $\mathscr{S} \cup \{S\}$ of hypersequents of degree at most 1 such that $\mathscr{S} \not\vdash_{\mathrm{HC}} S$.

Given such a set we consider the one-step Lindenbaum-Tarski algebra $\mathcal{H} = \mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S)$. By Lemma 4.2 this is a finite conservative one-step algebra validating HC, such that under the natural valuation $\mathcal{H}$ validates $\mathscr{S}$ but not S. Now by hypothesis there exists a finite Heyting algebra $\mathfrak{A}$ validating HC together with an embedding $(g_0, g_1)\colon \mathcal{H} \to \mathfrak{A}$ and so letting $v_0(p) = g_0(v_0'(p))$ and $v_1(q) = g_1(v_1'(q))$, where v' is the natural valuation on the one-step Lindenbaum-Tarski algebra $\mathcal{H}$, we obtain a valuation on $\mathfrak{A}$ witnessing that $\mathscr{S} \not\vdash_{\mathrm{HC}} S$. □

4.4 Hypersequent calculi, logics and universal classes

Since the bounded proof property is a property of calculi and not of logics we will define what it means for a hypersequent calculus to be a calculus *for* a logic.

Definition 4.11. A *universal sentence* is a first-order sentence of the form $\forall \underline{x}(\Phi(\underline{x}))$ with $\Phi(\underline{x})$ a quantifier free formula.

Let $\mathcal{F}$ be a language of algebras and let K be a class of algebras of type $\mathcal{F}$. Recall [5, Def. V.2.19] that K is a *universal class* is if is an elementary class of algebras which can be axiomatized by a set of universal sentences in the first-order language of $\mathcal{F}$.

Proposition 4.12. *Let* HC *be a hypersequent calculus. The class of Heyting algebras validating* HC *is a universal class.*

Proof. Let $r = (S_1, \dots, S_n)/S_{n+1}$ be a hypersequent rule of HC, with $S_i = s_{i1} \mid \dots \mid s_{im_i}$. Moreover, let $\mathfrak{A}$ be a Heyting algebra and let v be a valuation on $\mathfrak{A}$. Then $\mathfrak{A}$ validates r under v iff

$$((\mathfrak{A}, v) \vDash S_1 \text{ and } \dots \text{ and } (\mathfrak{A}, v) \vDash S_n) \implies (\mathfrak{A}, v) \vDash S_{n+1}. \tag{\dagger}$$

Given a sequent $s = \Gamma \Rightarrow \Delta$ we let s^I be the formula $\bigwedge \Gamma \to \bigvee \Delta$. We then have that $(\mathfrak{A}, v) \vDash S_i$ iff

$$(\mathfrak{A}, v) \vDash s_{i1}^I = \top \text{ or } \dots \text{ or } (\mathfrak{A}, v) \vDash s_{im_i}^I = \top.$$

Thus letting $\varphi_{ij}(\underline{x})$ be the Heyting algebra term corresponding to the formula $(s_{ij})^I$ and letting $\Phi_r(\underline{x})$ be the formula in the first-order language of Heyting algebras[1]

$$(\varphi_{11} = \top \textsf{ or } \dots \textsf{ or } \varphi_{1m_1} = \top) \textsf{ and } \dots \textsf{ and } (\varphi_{n1} = \top \textsf{ or } \dots \textsf{ or } \varphi_{nm_n} = \top))$$

and $\Psi_r(\underline{x})$ be the formula

$$(\varphi_{n+11} = \top \textsf{ or } \dots \textsf{ or } \varphi_{n+1m_{n+1}} = \top),$$

we see that (†) obtains iff

$$\mathfrak{A} \vDash \forall \underline{x}(\Phi_r(\underline{x}) \implies \Psi_r(\underline{x})).$$

showing that the class of Heyting algebras validating HC is a universal class. □

[1] We will be using "$\textsf{and}$" and "$\textsf{or}$" for conjunction and disjunction, respectively to avoid confusion with the meets and joins of the Heyting algebra signature.

In what follows we will need a weaker version of equivalence of hypersequent calculi namely,

Definition 4.13. We say that two hypersequent calculi HC and HC$'$ are *weakly equivalent* if for any set $\{\varphi_1, \dots, \varphi_n, \varphi\}$ of formulas we have that

$$\{\Rightarrow \varphi_1,\ \dots,\ \Rightarrow \varphi_n\} \vdash_{HC} \Rightarrow \varphi \text{ iff } \{\Rightarrow \varphi_1,\ \dots,\ \Rightarrow \varphi_n\} \vdash_{HC'} \Rightarrow \varphi.$$

If one is primarily interested in the theorems of a given intermediate logic, i.e. derivable sequents of the form $\Rightarrow \varphi$ then the notion of weak equivalence is the proper notion of sameness of hypersequent calculi.

Definition 4.14. Let L be an intermediate logic, i.e. **IPC** $\subseteq L \subseteq$ **CPC** and let HC be a hypersequent calculus. We say that HC is a *hypersequent calculus for the logic L* if

$$\varphi \in L \quad \text{iff} \quad \vdash_{\mathrm{HC}} \Rightarrow \varphi.$$

We provide the following algebraic characterization of weakly equivalent hypersequent calculi.

Proposition 4.15. *Let HC and HC$'$ be two hypersequent calculi. Then HC and HC$'$ are weakly equivalent iff for every subdirectly irreducible Heyting algebra $\mathfrak{A}$ we have*

$$\mathfrak{A} \models HC \iff \mathfrak{A} \models HC'.$$

Proof. Let $\mathcal{U}(\mathrm{HC})$ and $\mathcal{U}(\mathrm{HC}')$ be the universal classes of Heyting algebras validating the hypersequent calculi HC and HC$'$, respectively. Now by definition HC and HC$'$ are weakly equivalent if and only if the classes $\mathcal{U}(\mathrm{HC})$ and $\mathcal{U}(HC')$ determine the same equational theory. That is HC and HC$'$ are weakly equivalent if and only if the classes $\mathcal{U}(\mathrm{HC})$ and $\mathcal{U}(HC')$ generate the same variety. However as each variety is generated by its subdirectly irreducible members [5, Thm. 8.6] the proposition follows. $\square$

Given this it is easy to verify that if L is an intermediate logic axiomatized by a set of formulas $\{\varphi_i\}_{i\in I}$ then adding the rules

$$\frac{}{\Rightarrow \varphi_i}\,(r_i)$$

to $HInt$ or HJL' will determine a hypersequent calculus for L. However, as we will see in the following chapter, calculi obtained in this way will often not be particularly well-behaved.

4.5 (Preliminary) Algorithmic one-step correspondence

For applications of Theorem 4.6 and 4.10 it is essential that one can determine when a one-step frame validates a given rule. In general this will be a second-order condition. However, we may adapt the well-known ALBA-algorithm [29, 26, 27] to the framework of intuitionistic one-step frames analogous to its adaptation to the framework of modal one-step in [11, Sec. 7].

We note that if $\mathcal{H}$ is a one-step Heyting algebra and $r = (S_1, \dots, S_n)/S_{n+1}$ is a reduced hypersequent rule then, letting $\widetilde{\Phi}_r$ and $\widetilde{\Psi}_r$ be as in the proof of Proposition 4.12 only with every variable x having an occurrence under the scope of a Heyting implication replaced everywhere with $i(x)$, we have that $\mathcal{H}$ validates the rule r iff

$$\mathcal{H} \vDash \forall \underline{x}(\widetilde{\Phi}_r(\underline{x}) \implies \widetilde{\Psi}_r(\underline{x})),$$

as a two-sorted structure of the two-sorted language of one-step Heyting algebras.

Thus we may enrich the two-sorted algebraic language $\mathcal{L}_{Alg}$ of one-step Heyting algebras to obtain a sound[2] algorithmic procedure – in the form of an ALBA-style calculus – for computing the first-order correspondent of a hypersequent rule on one-step algebras.

The enrichment of $\mathcal{L}_{Alg}$ is justified by the fact that every finite one-step Heyting algebra is of the form $(\mathsf{Do}(P_0), \mathsf{Do}(P_1), f^*)$ for some finite one-step frame (P_1, P_0, f) as well as the following fact:

Proposition 4.16. *If P is a finite poset then $\mathsf{Do}(P)$ is a* perfect[3] *lattice. Furthermore, if $f\colon P_1 \to P_0$ is an order-preserving surjection between finite posets, then $i := f^*\colon \mathsf{Do}(P_0) \to \mathsf{Do}(P_1)$ has a left adjoint $i^\flat\colon \mathsf{Do}(P_1) \to \mathsf{Do}(P_0)$ and a right adjoint $i_!\colon \mathsf{Do}(P_1) \to \mathsf{Do}(P_0)$ given by*

$$i^\flat(U) = {\downarrow} f(U) \quad \textit{and} \quad i_!(U) = P_0 \backslash {\uparrow}(f(U)^c).$$

Proof. If P a poset we have that

$$U = \bigcup_{a \in U} {\downarrow} a \quad \text{and} \quad U = \bigcap_{a \notin U} P \backslash {\uparrow} a,$$

for all $U \in \mathsf{Do}(P)$. Moreover, it is easy to verify that if P is finite then the downsets of the form ${\downarrow} a$ are precisely the completely join-irreducible elements of $\mathsf{Do}(P)$ and that

[2] We do not make any claim of completeness.

[3] Recall that a complete bounded distributive lattice is *perfect* if all its elements are both a join of completely join-irreducible elements and a meet of completely meet-irreducible elements, where an element a of a bounded distributive lattice is *completely join-irreducible* if whenever $a = \bigvee S$ then $a \in S$. Similarly a is *completely meet-irreducible* if whenever $a = \bigwedge S$ then $a \in S$.

the downsets of the form $P\backslash{\uparrow}a$ are precisely the completely meet-irreducible elements of $\mathsf{Do}(P)$.

For the latter part of the proposition simply note that as P_0 is finite $\mathsf{Do}(P_0)$ must be finite as well and hence $i\colon \mathsf{Do}(P_0) \to \mathsf{Do}(P_1)$ will have left adjoint $i^\flat\colon \mathsf{Do}(P_1) \to \mathsf{Do}(P_0)$ given by

$$i^\flat(U) = \bigcap_{\substack{U \subseteq i(V)\\ V \in \mathsf{Do}(P_0)}} V = \bigcap_{\substack{U \subseteq f^{-1}(V)\\ V \in \mathsf{Do}(P_0)}} V = \bigcap_{\substack{f(U) \subseteq V\\ V \in \mathsf{Do}(P_0)}} V = {\downarrow} f(U),$$

and right adjoint $i_!\colon \mathsf{Do}(P_1) \to \mathsf{Do}(P_0)$ given by

$$i_!(U) = \bigcup_{\substack{i(V) \subseteq U\\ V \in \mathsf{Do}(P_0)}} V = \bigcup_{\substack{f^{-1}(V) \subseteq U\\ V \in \mathsf{Do}(P_0)}} V = \bigcup_{\substack{V \subseteq f(U)\\ V \in \mathsf{Do}(P_0)}} V = P_0 \backslash (\bigcap_{\substack{f(U)^c \subseteq V\\ V \in \mathsf{Up}(P_0)}} V) = P_0 \backslash {\uparrow}(f(U)^c).$$

□

Therefore given a finite one-step Heyting algebra (D_0, D_1, i) we may enrich the language $\mathcal{L}_{Alg}$ to obtain the language $\mathcal{L}^+_{Alg}$ defined as follows:

The level 0 language $\mathcal{L}^{+,0}_{Alg}$ and the level 1 language $\mathcal{L}^{+,1}_{Alg}$ are defined by the following mutual recursion

$$\varphi^0 ::= \bot \mid \top \mid x^0 \mid \varphi^0 \wedge \psi^0 \mid \varphi^0 \vee \psi^0 \mid i^\flat(\varphi^1) \mid i_!(\varphi^1)$$

where x^0 ranges over D_0, and φ^1 over $\mathcal{L}^{+,1}_{Alg}$, and

$$\varphi^1 ::= \bot \mid \top \mid x^1 \mid \mathbf{i} \mid \mathbf{m} \mid i(\varphi^0) \mid \varphi^1 \wedge \psi^1 \mid \varphi^1 \vee \psi^1 \mid i(\varphi^0) \to i(\psi^0)$$

where x^1 ranges over D_1 and φ^0, ψ^0 over $\mathcal{L}^{+,1}_{Alg}$ and $\mathbf{i}$ over *nominals*, i.e. the completely join-irreducible elements of D_1 and $\mathbf{m}$ ranges over *co-nominals* of D_1, i.e. the completely meet-irreducible elements of D_1. We will write $\mathcal{L}^+_{Alg}$ for $\mathcal{L}^{+,0}_{Alg} \cup \mathcal{L}^{+,1}_{Alg}$.

Since we will need to consider not only single inequalities $\varphi \leq \psi$ with $\varphi, \psi \in \mathcal{L}^{+,0}_{Alg}$ or $\varphi, \psi \in \mathcal{L}^{+,1}_{Alg}$ but expressions of the form

$$\begin{aligned}&((\varphi_{11} \leq \psi_{11} \text{ or } \dots \text{or } \varphi_{1m_1} \leq \psi_{1m_1}) \text{ and } \dots \text{ and } (\varphi_{n1} \leq \psi_{n1} \text{ or } \dots \text{or } \varphi_{nm_n} \leq \psi_{nm_n}))\\ &\implies (\varphi_{n+11} \leq \psi_{n+11} \text{ or } \dots \text{ or } \varphi_{n+11} \leq \psi_{n+1m_{n+1}}),\end{aligned} \tag{$\bigstar$}$$

we will need to incorporate symbols & and $\oplus$ for Boolean conjunction and disjunction on the meta-level. Moreover, we will need symbols for the algebraic inequality $\leq$. This is done, using the so-called **n**-trick from [34] which in [54] is adapted to the algorithmic context. Thus, following [54], we expand the language $\mathcal{L}^+_{Alg}$ to a language $\mathcal{L}^{++}_{Alg}$ which also allows terms[4] of the form

$$\mathsf{l}(\varphi, \psi) \quad \text{and} \quad \mathsf{k}(\varphi, \psi).$$

and terms of the form

$$\varphi \& \psi \quad \text{and} \quad \varphi \oplus \psi,$$

where we restrict & and $\oplus$ to *truth-values* i.e. terms of the form $\top$, $\bot$ or $\mathsf{l}(\varphi_1, \psi_1)$. The intended interpretation of l and k on a bounded distributive lattice D is as follows:

$$\mathsf{l}^D(x, y) := \begin{cases} \top \text{ if } x \leq_D y, \\ \bot \text{ otherwise.} \end{cases}$$

$$\mathsf{k}^D(x, y) := \begin{cases} x \text{ if } x = \bot, \\ y \text{ otherwise.} \end{cases}$$

The interpretation of & and $\oplus$ will then be min and max, respectively.

Then ($\bigstar$) may be encoded in the language $\mathcal{L}^{++}_{Alg}$ as the inequality

$$\mathop{\&}_{i+1}^{n} \left(\bigoplus_{j=1}^{m_i} (\mathsf{l}\,(\varphi_{ij}, \psi_{ij})) \right) \leq \bigoplus_{j=1}^{m_{n+1}} \left(\mathsf{l}(\varphi_{n+1j}, \psi_{n+1j})\right).$$

to which we may apply an ALBA-style procedure. In Appendix B we give the rules of a basic calculus for correspondence which may easily be checked to be sound under the interpretation given above. However, we must stress that these rules are by no means complete as in general frame validity is a second-order condition.

By duality we may turn the above condition into a formula in the two-sorted first-order language $\mathcal{L}^{++}_{Frm}$ for one-step intuitionistic frames. The nominals will be interpreted as completely join-irreducible elements, i.e. elements of the form $\downarrow a$ and the co-nominals will be interpreted as completely meet-irreducible elements i.e. elements of the form $P \backslash \uparrow a$. Furthermore, l and k will be given their intended interpretations as the binary function $\mathsf{l}, \mathsf{k} \colon D_1^2 \to D_1$. Finally, i will be interpreted as $f^* \colon \mathsf{Do}(P_0) \to \mathsf{Do}(P_1)$ and $i^\flat$ and $i_!$ the left and right adjoint of i, respectively.

We note that the rules given in Appendix B only work well for hypersequent rules consisting of only single-component hypersequents in the premisses. However, in practice this does not seem to be a real restriction as all the rules encountered will be equivalent to such rules due to the admissibility of external weakening. More precisely: The rule

[4] We will of course need a level 0 and a level 1 of all of these terms. However, as the type can always be inferred from the context we will not introduce it explicitly.

$$\frac{G \mid S_1, \ldots, G \mid S_n}{G \mid S} \, (r)$$

is equivalent to the rule

$$\frac{S_1, \ldots, S_n}{S} \, (r')$$

Thus if all of the hypersequents $S_1, \ldots, S_n$ are single-component hypersequents then the rule (r') will be of the appropriate type and so we may apply algorithmic one-step correspondence to the rule (r') rather than to the rule (r).

Summary of Chapter 4: In this chapter we have introduced the bounded proof property for hypersequent calculi and given an algebraic characterisation of this property. Finally, have described a basic algorithmic procedure which in some cases can determine the first-order one-step frame condition under which a finite one-step frame validates a hypersequent rule. We are thus ready to look at some examples of hypersequent calculi with and without the bounded proof property. This will be the content of the following chapter.

Chapter 5

Some calculi with and without the bounded proof property

In this chapter we apply the results obtained in Chapter 4 to give examples of hypersequent calculi with and without the bounded proof property and the finite model property. We will first, however, review the work of Ciabattoni, Galatos and Terui [23], which already ensures the bounded proof property for a large class of hypersequent calculi.

We want to stress that in developing the theory of one-step Heyting algebras and intuitionistic one-step frames we have – following [35, 18, 10] – based the duality on downsets rather than on upsets. In particular on frames propositional letters are evaluated as downsets. The difference is of course immaterial. However, the reader used to the upset approach will find that all of our examples are "upside down".

5.1 The Ciabattoni-Galatos-Terui Theorem

In [23] a procedure for constructing cut-free hypersequent calculi with the subformula property for certain classes of substructural logics is given. We here outline a version of the result in the setting of intermediate logics.

Definition 5.1 (*The substructural hierarchy of intuitionistic logic*)**.** Given a set P of propositional letters we define for each $n \in \omega$ the sets $\mathcal{P}_n$ and $\mathcal{N}_n$ of positive and negative formulas by the following recursion:

(0) $\mathcal{P}_0 = \mathcal{N}_0 = \mathsf{P}$;

(P1) $\top, \bot \in \mathcal{P}_{n+1}$ and $\mathcal{N}_n \subseteq \mathcal{P}_{n+1}$;

(P2) If $\varphi, \psi \in \mathcal{P}_{n+1}$ then $\varphi \vee \psi, \varphi \wedge \psi \in \mathcal{P}_{n+1}$;

(N1) $\top, \bot \in \mathcal{N}_{n+1}$ and $\mathcal{P}_n \subseteq \mathcal{N}_{n+1}$;

(N2) If $\varphi, \psi \in \mathcal{N}_{n+1}$ then $\varphi \wedge \psi \in \mathcal{N}_{n+1}$;

(N3) If $\varphi \in \mathcal{P}_{n+1}$ and $\psi \in \mathcal{N}_{n+1}$ then $\varphi \to \psi \in \mathcal{N}_{n+1}$.

The intuition is that $\mathcal{P}_n$ (respectively $\mathcal{N}_n$) contains formulas with a positive (and negative, respectively) leading connective.

The following proposition is easily established, justifying the name hierarchy for this stratification of the set of formulas.

Proposition 5.2. *For all $n \in \omega$ we have that*

$$\mathcal{P}_n \subseteq \mathcal{P}_{n+1} \quad \text{and} \quad \mathcal{N}_n \subseteq \mathcal{N}_{n+1}.$$

Moreover,

$$Form(\mathsf{P}) = \bigcup_{n\in\omega} \mathcal{P}_n = \bigcup_{n\in\omega} \mathcal{N}_n.$$

We give a few examples of axioms for well-known intermediate logics and the different levels of the hierarchy at which they occur for the first time.

The Jankov axiom $\neg p \vee \neg\neg p$ belongs to $\mathcal{P}_3$ and the Gödel-Dumment axiom

$$(p \to q) \vee (q \to p)$$

belongs to $\mathcal{P}_2$. Moreover, for each $n \in \omega$ the axioms $\mathbf{bw}_n$ and $\mathbf{bc}_n$ given by

$$\mathbf{bw}_n\colon \bigvee_{i=0}^{n} (p_i \to \bigvee_{j\neq i} p_j)$$

and

$$\mathbf{bc}_n\colon \bigvee_{i=0}^{n} (\bigwedge_{j=0}^{i} p_j \to p_i),$$

belong to $\mathcal{P}_3$.

The axiom $\mathbf{bd}_n$ determined by the following recursion:

$$\mathbf{bd}_0 = \bot \quad \text{and} \quad \mathbf{bd}_{n+1} = p_{n+1} \vee (p_{n+1} \to \mathbf{bd}_n),$$

belongs to $\mathcal{P}_{n+2}$ for all $n \geq 2$.

Let $\mathfrak{A}$ be a finite subdirectly irreducible Heyting algebra with second greatest element s, and let $D \subseteq A^2$. Then introducing for each $a \in A$ a propositional letter p_a we may define the *canonical formula* $\alpha(\mathfrak{A}, D, \bot)$ *associated with* $\mathfrak{A}$ *and* D to be

$$(\bigwedge_{a,b\in A}(p_{a\wedge b} \leftrightarrow p_a \wedge p_b) \wedge \bigwedge_{a,b\in A}(p_{a\to b} \leftrightarrow p_a \to p_b) \wedge \bigwedge_{a\in A}(p_{\neg a} \leftrightarrow \neg p_a) \wedge \bigwedge_{(a,b)\in D}(p_{a\vee b} \leftrightarrow p_a \vee p_b)) \to p_s.$$

It may readily be checked that all such canonical formulas belong to level $\mathcal{N}_3$. From [53] we know that every intermediate logic can be axiomatized by a – not necessarily finite – set of canonical formulas. Consequently, over **IPC** the hierarchy collapses at level $\mathcal{N}_3$.

In [23] the following surprising theorem was established.

Theorem 5.3 (The Ciabattoni-Galatos-Terui Theorem for intermediate logics)**.** *Any intermediate logic axiomatized by formulas in* $\mathcal{P}_3$ *has a cut-free hypersequent calculus.*

In particular, Theorem 5.3 ensures that there already exists cut-free hypersequent culculi with the subformula property for the logics **LC** and **KC** (see [4, 22] for the rules) and for $\mathbf{BW}_n$, $\mathbf{BC}_n$ and $\mathbf{G}_n$ (see [21] for the rules).

Note that the original theorem was stated more generally for Full Lambek calculus with exchange. For our purposes this version, which can also be found in [25], will suffice.

An important fact about the hypersequent calculi whose existence is guaranteed by Theorem 5.3 is that they are obtained by extending a standard hypersequent calculus for **IPC** with so-called *completed rules*. Such rules have a number of very nice properties. For us the most important of these is the *subformula property*. A rule has the subformula property if all the formulas occurring in the premiss of the rule also occur in the conclusion.

Proposition 5.4. *Let* HC *be a hypersequent calculus such that all of the rules of* HC *have the subformula property. If* HC *enjoys cut-elimination then* HC *has the subformula property, i.e. if* $\mathscr{S} \vdash_{\mathrm{HC}} S$ *then this is witnessed by a proof containing only formulas occurring in* $\mathscr{S} \cup \{S\}$*.*

Proof. By induction on a cut-free derivation of $\mathscr{S} \vdash_{\mathrm{HC}} S$. □

In particular, such hypersequent calculi will have the bounded proof property. Therefore, for non-trivial applications of Theorem 4.6 and Theorem 4.10 we need to consider either sequent calculi for which cut-elimination does not obtain or hypersequent calculi for intermediate logics axiomatized by formulas above $\mathcal{P}_3$ in the hierarchy.

5.2 Calculi for the Gödel-Dumment logic

We first consider the sequent calculus for Gödel-Dumment logic **LC** given by adding the rule

$$\frac{}{\Rightarrow (\varphi \to \psi) \vee (\psi \to \varphi)} \, (r_{LC})$$

This rule is different from the rule obtained from the Ciabattoni-Galatos-Terui Theorem.

Proposition 5.5. *A one-step frame (P_1, P_0, f) validates the rule (r_{LC}) iff*

$$\forall a, b, b' \in P_1 \ (b \leq a \text{ and } b' \leq a \implies f(b) \leq f(b') \text{ or } f(b') \leq f(b)).$$

Proof. We show how to apply the basic one-step correspondence calculus. We spell everything out very carefully, something we will not do for the following examples.

First of all we see that $\top \leq \mathsf{l}(\top, i(a) \to i(b) \vee i(b) \to i(a))$ is equivalent to

$$\mathsf{k}(\top, \top) \leq i(a) \to i(b) \vee i(b) \to i(a),$$

which in turn will be equivalent to

$$\top \leq i(a) \to i(b) \vee i(b) \to i(a).$$

Now using the first approximation rule we obtain

$$\forall \mathbf{i} \forall \mathbf{m} \ (i(a) \to i(b) \vee i(b) \to i(a) \leq \mathbf{m} \implies \mathbf{i} \leq \mathbf{m}).$$

Then applying the rule $(\mathrm{LA}_\vee)$ yields

$$\forall \mathbf{i} \forall \mathbf{m} \ (i(a) \to i(b) \leq \mathbf{m} \text{ and } i(b) \to i(a) \leq \mathbf{m}) \implies \mathbf{i} \leq \mathbf{m}),$$

and applying the rule $(\mathrm{LA}_\to)$ yields,

$$\forall \mathbf{i} \forall \mathbf{m} \, (\exists \mathbf{j}_1 \, (\mathbf{j}_1 \to i(b) \leq \mathbf{m} \text{ and } \mathbf{j}_1 \leq i(a)) \text{ and } \exists \mathbf{j}_2 \, (\mathbf{j}_2 \to i(a) \leq \mathbf{m} \text{ and } \mathbf{j}_2 \leq i(b))) \implies \mathbf{i} \leq \mathbf{m}).$$

Now by some basic first-order logic this is equivalent to

$$\forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}_1 \forall \mathbf{j}_2 \, ((\mathbf{j}_1 \to i(b) \leq \mathbf{m} \text{ and } \mathbf{j}_1 \leq i(a)) \text{ and } (\mathbf{j}_2 \to i(a) \leq \mathbf{m} \text{ and } \mathbf{j}_2 \leq i(b))) \implies \mathbf{i} \leq \mathbf{m}).$$

By applying the rule (RA_i) twice we obtain

$$\forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}_1 \forall \mathbf{j}_2 \, ((\mathbf{j}_1 \to i(b) \leq \mathbf{m} \text{ and } i^\flat(\mathbf{j}_1) \leq a) \text{ and } (\mathbf{j}_2 \to i(a) \leq \mathbf{m} \text{ and } i^\flat(\mathbf{j}_2) \leq b)) \implies \mathbf{i} \leq \mathbf{m}),$$

and finally applying the right Ackermann rule two times we obtain

$$\forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}_1 \forall \mathbf{j}_2 \ (\mathbf{j}_1 \to i(i^\flat(\mathbf{j}_2)) \leq \mathbf{m} \text{ and } (\mathbf{j}_2 \to i(i^\flat(\mathbf{j}_1)) \leq \mathbf{m}) \implies \mathbf{i} \leq \mathbf{m}).$$

As the last condition only involves variables for nominals and co-nomimals we have that validating (r_{LC}) is a condition on one-step frames which is first-order definable in the one-step frame language. Recall that the nomimals are interpreted as $\downarrow a$ and the co-nomimals as $P_1 \backslash \uparrow a$. Moreover, i will be intrepreted as f^{-1} and $i^\flat$ as the left adjoint of $f^{-1} \colon \mathsf{Do}(P_0) \to \mathsf{Do}(P_1)$ viz. the downset of the direct-image. Given this, after some elementary rewriting using that $\downarrow f(\downarrow c)) = \downarrow f(c)$, we see that (P_1, P_0, f) validates the rule (r_{LC}) iff

$$\forall a, b, c_1, c_2 \in P_1 \ ((\uparrow b \subseteq \uparrow(\downarrow c_1 \backslash f^{-1}(\downarrow f(c_2))) \text{ and } \uparrow b \subseteq \uparrow(\downarrow c_2 \backslash f^{-1}(f(\downarrow c_1))) \implies \downarrow a \subseteq P_1 \backslash \uparrow b)$$

Now as $\downarrow b \not\subseteq P_1 \backslash \uparrow b$ for any $b \in P_1$ we obtain that the above is equivalent to

$$\forall b, c_1, c_2 \in P_1 \ ((\uparrow b \subseteq \uparrow(\downarrow c_1 \backslash f^{-1}(\downarrow(f(c_2)))) \implies \uparrow b \not\subseteq \uparrow(\downarrow c_2 \backslash f^{-1}(\downarrow f(c_1))).$$

Spelling this out we obtain

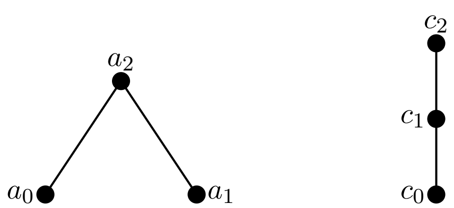
$$\forall b, c_1, c_2 \in P_1 \ (\exists d_1 \leq c_1, b \ (f(d_1) \notin \downarrow f(c_2)) \implies \forall d_2 \leq c_2, b \ (f(d_2) \in \downarrow f(c_1))),$$

which can readily be shown to be equivalent to

$$\forall a, b, b' \in P_1 (b, b' \leq a \implies f(b) \leq f(b') \text{ or } f(b') \leq f(b)),$$

as desired. □

To see that adding the rule (r_{LC}) does not yield a sequent calculus with the bounded proof property, consider the one-step frame $\mathcal{S} = (P_1, P_0, f)$ presented as:



That is P_1 is a 2-fork and P_0 is a 3-chain. The function f is the obvious map given by $a_i \mapsto c_i$ for $i \in \{0, 1, 2\}$. This is easily seen to be a finite conservative one-step frame validating the rule (r_{LC}). Now suppose towards a contradiction that $\mathcal{S}$ has a one-step extension, say $\mathcal{S}' = (P_2, P_1, g)$. As f is in fact bijective it follows from the assumption that g is f-open that g must be an open map. Therefore, we must have $z_0, z_1, z_2 \in P_2$ with $z_0, z_1 \leq z_2$, such $g(z_i) = a_i$ for $i \in \{0, 1, 2\}$. But this shows that $\mathcal{S}'$ fails to validate the rule (r_{LC}) and consequently that $\mathcal{S}$ does not have any one-step extensions validating (r_{LC}).

Thus by Theorem 4.6 we see that the hypersequent calculus obtained by adding the rule (r_{LC}) does not have the bounded proof property and so in particular it cannot enjoy cut-elimination.

5.3 Calculi for the logic of weak excluded middle

Recall that the logic **KC** is obtained by adding the axiom $\neg p \vee \neg\neg p$ to **IPC**. It is well-known that this is the logic of directed frames.

Now consider the rule

$$\frac{\varphi \wedge \psi \Rightarrow \bot}{\Rightarrow \neg\varphi \vee \neg\psi}\ (r_{KC})$$

We show that adding this rule gives a calculus for **KC** without the bounded proof property. Therefore, this is a different rule from the one given by the Ciabattoni-Galatos-Terui Theorem.

Proposition 5.6. *A step frame (P_1, P_0, f) validates the rule (r_{KC}) iff*

$$\forall a, b_1, b_2 \in P_1\ (b_1 \leq a \textit{ and } b_2 \leq a \implies \exists c \in P_0\ (c \leq f(b_1) \textit{ and } c \leq f(b_2))$$

Proof. Applying algorithmic correspondence to the inequality

$$\mathsf{I}(i(x) \wedge i(y), \bot) \leq \mathsf{I}(\top, \neg i(x) \wedge \neg i(y))$$

yields

$$\forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}_1 \forall \mathbf{n} \forall \mathbf{j}_2\ ((i^\flat(\mathbf{j}_1) \wedge i^\flat(\mathbf{j}_2) \leq \bot \textsf{ and } \neg\mathbf{j}_1 \leq \mathbf{n} \textsf{ and } \neg\mathbf{j}_2 \leq \mathbf{n}) \implies \mathbf{i} \leq \mathbf{m}).$$

This translates to

$$\forall a', b', a, b_1, b_2 \in P_1\ ((\downarrow f(b_1) \cap \downarrow f(b_2) = \emptyset \textsf{ and } P_1 \backslash \uparrow\downarrow b_1 \subseteq P_1 \backslash \uparrow a \textsf{ and } P_1 \backslash \uparrow\downarrow b_2 \subseteq P_1 \backslash \uparrow a) \\ \implies \downarrow a' \subseteq P_1 \backslash \uparrow b').$$

As the consequent of this implication will be false for $a' = b'$ we must have that the antecedent is false for all a, b_1, b_2. Thus we may rewrite the above as

$$\forall a, b_1, b_2 \in P_1 \ ((P_1 \backslash {\uparrow}{\downarrow} b_1 \subseteq P_1 \backslash {\uparrow} a \text{ and } P_1 \backslash {\uparrow}{\downarrow} b_2 \subseteq P_1 \backslash {\uparrow} a) \implies {\downarrow} f(b_1) \cap {\downarrow} f(b_2) \neq \emptyset),$$

which again may be rewritten as

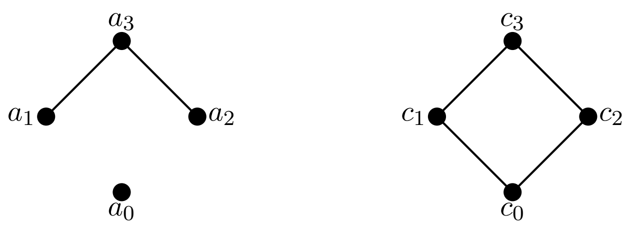
$$\forall a, b_1, b_2 \in P_1 \ (a \in {\uparrow}{\downarrow} b_1 \text{ and } a \in {\uparrow}{\downarrow} b_2 \implies {\downarrow} f(b_1) \cap {\downarrow} f(b_2) \neq \emptyset).$$

Evidently this implies

$$\forall a, b_1, b_2 \in P_1 \ (b_1 \leq a \text{ and } b_2 \leq a \implies {\downarrow} f(b_1) \cap {\downarrow} f(b_2) \neq \emptyset). \tag{†}$$

To see that the converse implication holds as well, assume that (†) obtains and that $a, b_1, b_2 \in P_1$ are such that $a \in {\uparrow}{\downarrow} b_1$ and $a \in {\uparrow}{\downarrow} b_2$. Then $a \geq b_1'$ and $a \geq b_2'$ for some $b_1' \leq b_1$ and $b_2' \leq b_2$. Then from (†) it follows that ${\downarrow} f(b_1') \cap {\downarrow} f(b_2') \neq \emptyset$. Therefore, as ${\downarrow} f(b_1') \subseteq {\downarrow} f(b_1)$ and ${\downarrow} f(b_2') \subseteq {\downarrow} f(b_2)$ we obtain that ${\downarrow} f(b_1) \cap {\downarrow} f(b_2) \neq \emptyset$. □

Consider the one-step frame $\mathcal{S} = (P_1, P_0, f)$ presented as



with f given by $a_i \mapsto c_i$. Then $\mathcal{S}$ is a finite conservative one-step frame validating the rule (r_{KC}). If P_2 is a finite poset and $g\colon P_2 \to P_1$ is a f-open map, then as f is a bijection the f-openness condition implies that g is an open map and therefore, that for $a \in f^{-1}(a_3)$ we have $b, b' \leq a$ such that $g(b) = a_1$ and $g(b') = a_2$. But as ${\downarrow} a_1$ and ${\downarrow} a_2$ are disjoint we see that (P_2, P_1, g) will not validate the rule (r_{KC}), and thus $\mathcal{S}$ does not have any one-step extensions validating (r_{KC}).

By Theorem 4.6 it then immediately follows that the calculus obtained by adding the rule (r_{KC}) does not have the bounded proof property.

5.4 Calculi for logics of bounded width

Consider the logic $\mathbf{BW}_n$ obtained by adding the axiom

$$\bigvee_{i=0}^{n}(p_i \to \bigvee_{j\neq j} p_i) \qquad (\mathbf{bw}_n)$$

to **IPC**. It is well-known that a Kripke frame $\mathfrak{F} = (W, \leq)$ validates $\mathbf{bw}_n$ iff

$$\forall w, w_0, \ldots, w_n \ (w_0 \leq w \text{ and } \ldots \text{ and } w_n \leq w \implies \exists i, j \leq n \ (i \neq j \text{ and } w_i \leq w_j)) .$$

It follows that $\mathbf{BW}_n$ is the logic of frames $\mathfrak{F}$ such that every rooted subframe of $\mathfrak{F}$ does not contain any anti-chains of more that n nodes.

Proposition 5.7. *An intuitionistic one-step frame* (P_1, P_0, f) *validates the rule*

$$\frac{}{\Rightarrow \mathbf{bw}_n}\,(r_{bw_n})$$

iff

$$\forall a, a_0, \ldots, a_n \ (a_0 \leq a \textit{ and } \ldots \textit{ and } a_n \leq a \implies \exists i, j \leq n \ (i \neq j \textit{ and } f(a_i) \leq f(a_j))) .$$

Proof. Applying one-step correspondence to

$$\top \leq \mathsf{I}(\top, \bigvee_{i=0} (i(a_i) \to \bigvee_{j\neq i} i(a_j)))$$

we obtain

$$\forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}_0 \ldots \forall \mathbf{j}_n \ ((\mathbf{j}_1 \to \bigvee_{j\neq 1} i(i^\flat(\mathbf{j}_j)) \leq \mathbf{m} \text{ and } \ldots \text{ and } \mathbf{j}_n \to \bigvee_{j\neq n} i(i^\flat(\mathbf{j}_j)) \leq \mathbf{m}) \implies \mathbf{i} \leq \mathbf{m}).$$

This translates to the one-step frame condition

$$\forall a, a_0, \ldots, a_n \in P_1 \ \neg \left(\forall i \in \{0, \ldots, n\} \ (a \in \uparrow(\downarrow a_i \backslash \bigcup_{j\neq i} f^{-1}(\downarrow f(a_j))) \right),$$

which in turn may be rewritten as

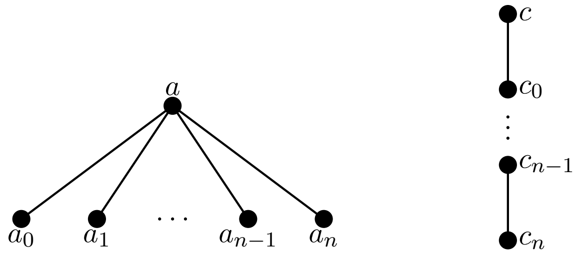
$$\forall a, a_0, \ldots, a_n \in P_1 \ \exists i \in \{0, \ldots, n\} \ a \notin \uparrow \left(\downarrow a_i \backslash \bigcup_{j\neq i} f^{-1}(\downarrow f(a_j)) \right).$$

Finally, this may readily be checked to be equivalent to

$$\forall a, a_0, \ldots, a_n \ (a_0 \leq a \text{ and } \ldots \text{ and } a_n \leq a \implies \exists i, j \leq n \ (i \neq j \text{ and } f(a_i) \leq f(a_j)))\,.$$

□

We show that adding the rule (r_{bw_n}) does not yield a calculus with the bounded proof property. To see this let $\mathcal{S} = (P_1, P_0, f)$ be the intuitionistic one-step frame presented as



with $f(a) = c$ and $f(a_i) = f(c_i)$. This is evidently a finite conservative one-step frame and by the above proposition $\mathcal{S}$ validates (r_{bw_n}). Now if $g \colon P_2 \to P_1$ is such that (P_2, P_1, g) is a finite conservative one-step frame we must have that g is open since f is an injection. Thus taking $b \in g^{-1}(a)$ since $a_i \leq g(b)$ for all $i \in \{0, \ldots, n\}$, we must have that there exists $b_i \leq b$, for $i \in \{0, \ldots, n\}$ such that $g(b_i) = a_i$. But then we have that $g(b_i) \not\leq g(b_j)$ when $i \neq j$ and so (P_2, P_1, f) does not validate (r_{bw_n}). We therefore conclude that the class of finite conservative one-step frames validating (r_{bw_n}) does not have the extension property and therefore by Theorem 4.6 adding the axiom (r_{bw_n}) does not yield a calculus with the bounded proof property.

However, we may also consider the hypersequent rule

$$\frac{}{p_0 \Rightarrow \bigvee_{j \neq 0} p_j \mid \ldots \mid p_i \Rightarrow \bigvee_{j \neq i} p_j \mid \ldots \mid p_n \Rightarrow \bigvee_{j \neq n} p_j}\ (HBw_n)$$

Despite not being a structural rule no implications occur in this rule, and so it might be a good candidate for a rule with the bounded proof property.

Proposition 5.8. *An intuitionistic one-step frame (P_1, P_0, f) validates the rule (HBw_n) iff P_1 does not contain an anti-chain of more that n elements.*

Proof. We proceed by applying algorithmic correspondence to the set of quasi-equations:

$$\top \leq \bigoplus_{i=0}^{n} \mathsf{I}(x_j, \bigvee_{j<i} x_j)$$

and obtain

$$\forall \mathbf{i}\forall \mathbf{m}\forall \mathbf{j}_0 \ldots \mathbf{j}_n \ (\mathsf{I}(\mathbf{j}_0, \bigvee_{j<0} x_j) \leq \mathbf{m} \text{ and } \mathbf{j}_0 \leq x_0 \text{ and } \ldots \text{ and } \mathsf{I}(\mathbf{j}_n, \bigvee_{j<n} x_j) \leq \mathbf{m} \text{ and } \mathbf{j}_n \leq x_n) \\ \implies \mathbf{i} \leq \mathbf{m})$$

As $\mathsf{I}(-,-)$ is positive in the second coordinate we may apply the left Ackermann rule to obtain

$$\forall \mathbf{i}\forall \mathbf{m}\forall \mathbf{j}_0 \ldots \mathbf{j}_n \ ((\mathsf{I}(\mathbf{j}_0, \bigvee_{j<0} \mathbf{j}_j) \leq \mathbf{m}) \text{ and } \ldots \text{ and } \mathsf{I}(\mathbf{j}_n, \bigvee_{j<n} \mathbf{j}_j)) \implies \mathbf{i} \leq \mathbf{m}).$$

Finally, using the elimination rule for I we obtain

$$\forall \mathbf{i}\forall \mathbf{m}\forall \mathbf{j}_0 \ldots \mathbf{j}_n \ ((\mathbf{j}_0 \not\leq \bigvee_{j<0} \mathbf{j}_j \text{ and } \ldots \text{ and } \mathbf{j}_n \not\leq \bigvee_{j<n} \mathbf{j}_j) \implies \mathbf{i} \leq \mathbf{m}),$$

which translates to

$$\forall a, b, a_0, \ldots, a_n \in P_1 \ \left(\forall i \leq n \ ({\downarrow} a_i \not\subseteq \bigcup_{j<i} {\downarrow} a_j) \implies {\downarrow} a \subseteq P_1 \backslash {\uparrow} b \right).$$

Since the consequent of this implication is false for $a = b$ the antecedent must be false for all $a_0, \ldots, a_n$ and so we may rewrite the above as

$$\forall a_0, \ldots, a_n \in P_1 \ \exists i \leq n \ ({\downarrow} a_i \subseteq \bigcup_{j<i} {\downarrow} a_j),$$

which in turn is evidently equivalently to

$$\forall a_0, \ldots, a_n \in P_1 \ \exists i, j \leq n \ (i \neq j \text{ and } a_i \leq a_j).$$

We may therefore conclude that a finite conservative one-step frame validates the hypersequent rule (HBw_n) iff P_1 does not contain an anti-chain of more than n-elements. □

As one would have expected it now immediately follows from Theorem 4.10 that adding the rule (HBw_n) yields a calculus with the bounded proof property and the finite model property.

5.5 Calculi for logics of bounded depth

The logic $\mathbf{BD}_2$, consisting of formulas valid precisely on frames of depth at most 2, is axiomatized by the axiom $p_2 \vee (p_2 \to (\neg p_1 \vee p_1))$ which belongs to the class $\mathcal{P}_4$ and so the Ciabattoni-Galatos-Terui Theorem does not apply. However, in [25] a cut-free hypersequent calculus which is sound and complete with respect to $\mathbf{BD}_2$, is presented. This calculus is obtained by adding the rule

$$\frac{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_2 \qquad G \mid \Gamma_2, \varphi_1 \Rightarrow \varphi_0, \Delta_1}{G \mid \Gamma_1 \Rightarrow \Delta_2 \mid \Gamma_2 \Rightarrow \varphi_1 \to \varphi_0, \Delta_1} \,(HBd_2)$$

to the multi-succedent hypersequent calculus HJL' for **Int**.

Proposition 5.9. *An intuitionistic one-step frame (P_1, P_0, f) validates the rule (HBd_2) iff every maximal element of a 2-chain in P_1 is the root of P_1.*

Proof. A straightforward – but rather involved – application of the basic calculus for one-step correspondence to the inequality

$$\mathsf{I}(x_1 \wedge x_2, y_2) \;\&\; \mathsf{I}(x_2 \wedge i(z_1), i(z_0) \vee y_1) \leq \mathsf{I}(x_1, y_2) \oplus \mathsf{I}(x_2, (i(z_1) \to i(z_0)) \vee y_1),$$

yields

$$(\mathbf{j}_3 \wedge i(i^\flat(\mathbf{j}_1)) \leq i(i_!(\mathbf{n}_2)) \vee \mathbf{n}_1 \text{ and } \mathbf{j}_1 \to \mathbf{n}_2 \leq \mathbf{n}_1 \text{ and } \mathbf{j}_2 \not\leq \mathbf{j}_2 \wedge \mathbf{j}_3 \text{ and } \mathbf{j}_3 \not\leq \mathbf{n}_1)) \implies \mathbf{i} \leq \mathbf{m},$$

for all nominals $\mathbf{i}, \mathbf{j}_1, \mathbf{j}_2, \mathbf{j}_3$ and all co-nominals $\mathbf{m}, \mathbf{n}_1, \mathbf{n}_2$.

By the now standard argument this is equivalent to the statement that

$$(\mathbf{j}_3 \wedge i(i^\flat(\mathbf{j}_1)) \leq i(i_!(\mathbf{n}_2)) \vee \mathbf{n}_1 \text{ and } \mathbf{j}_1 \to \mathbf{n}_2 \leq \mathbf{n}_1 \text{ and } \mathbf{j}_3 \not\leq \mathbf{n}_1) \implies \mathbf{j}_2 \leq \mathbf{j}_2 \wedge \mathbf{j}_3,$$

for all $\mathbf{j}_1, \mathbf{j}_2, \mathbf{j}_3$ and all $\mathbf{n}_1, \mathbf{n}_2$. This will translate to the first-order one-step frame condition on (P_1, P_0, f) that for all $a_1, a_2, a_3, b_1, b_2 \in P_1$ if

$${\downarrow} a_3 \cap f^{-1}({\downarrow} f({\downarrow} a_1)) \subseteq f^{-1}(P_0 \backslash {\uparrow}[f(P_1 \backslash {\uparrow} b_2)^{\mathrm{c}}]) \cup P_1 \backslash {\uparrow} b_1$$

and

$$P_1 \backslash {\uparrow}({\downarrow} a_1 \backslash (P_1 \backslash {\uparrow} b_2)) \subseteq P_1 \backslash {\uparrow} b_1 \quad \text{and} \quad {\downarrow} a_3 \not\subseteq P \backslash {\uparrow} b_1,$$

then

$${\downarrow} a_2 \subseteq {\downarrow} a_2 \cap {\downarrow} a_3.$$

Some basic rewriting will show that this is equivalent to for all $a_1, a_2, a_3, b_1, b_2 \in P_1$:

$$(\downarrow a_3 \cap f^{-1}(\downarrow f(a_1)) \subseteq P_1 \backslash (f^{-1}(\uparrow f(b_2)) \cap \uparrow b_1) \text{ and } b_2 \leq a_1, b_1 \text{ and } b_1 \leq a_3)) \implies a_2 \leq a_3,$$

where we have used the fact that $f(P_1 \backslash \uparrow b_2)^{\mathrm{c}} = f(\uparrow b_2)$, as f is surjective, as well as the fact that $\uparrow f(\uparrow b_2) = \uparrow f(b_2)$ and $\downarrow f(\downarrow a_1) = \downarrow f(a_1)$.

Observe that a_2 does not occur in the antecedent of the above implication. Thus we see that the above is equivalent to requiring that for all $a_1, a_2, b_1, b_2 \in P_1$, such that $b_2 \leq b_1, a_2$ and $b_1 \leq a_1$ if

$$\forall c \leq a_1 \ (f(c) \leq f(a_2) \implies b_1 \not\leq c \text{ or } f(b_2) \not\leq f(c)) \tag{$\dagger$}$$

then $\max\{P_1\} = a_1$, i.e. that a_1 is a root.

We claim that this is equivalent to the condition that every maximal element of a 2-chain is a root. To see this we assume first that $a_1 \in P_1$ is a maximal element of a 2-chain i.e. that there exists an element strictly below a_1. Now assume towards a contradiction that for every $a < a_1$ there exists $c \leq a_1$ such that $f(c) \leq f(a)$ implies $a_1 \leq c$ and $f(a) \leq f(c)$. Then we must have that $f(a) = f(a_1)$ for all $a \leq a_1$ and thus that $f(\downarrow a_1) = \{f(a_1)\}$. So as (P_1, P_0, f) is conservative it follows that $\downarrow a_1 = \{a_1\}$ in direct contradiction with the assumption that a_1 had an element strictly below it. Therefore, let $a_2 < a_1$ be such that

$$\forall c \leq a_1 (f(c) \leq f(a_2) \implies a_1 \not\leq c \text{ or } f(a_2) \not\leq f(c)).$$

We have that ($\dagger$) is satisfied with $a_1 = b_1$ and $a_2 = b_2$ and hence we must conclude that a_1 is a root.

Conversely, suppose that there exists $a_1, a_2, b_1, b_2 \in P_1$ with $b_2 \leq b_1, a_2$ and $b_1 \leq a_1$ such that ($\dagger$) obtains, but a_1 is not a root.

Now assume contrary to the desired conclusion that the maximal element of every 2-chain is a root. This entails that $a_1 = b_1 = b_2 = a_2$, but then ($\dagger$) cannot obtain for these elements in direct contradiction of our initial assumption about a_1, a_2, b_1 and b_2.

We have thus shown that a finite conservative one-step frame (P_1, P_0, f) validates the rule (HBd_2) iff every maximal element of a 2-chain is a root. □

Now consider the logic $\mathbf{BD}_3$, of formulas valid on frames of depth at most 3, axiomatized by the formula

$$p_3 \vee (p_3 \to (p_2 \vee (p_2 \to (p_1 \vee \neg p_1)))). \tag{$\mathbf{bd}_3$}$$

This is a formula of implicational degree 3 which belongs to level $\mathcal{P}_5$ of the hierarchy.

Consider the rule

$$\frac{G \mid \Gamma_1, \Gamma_3 \Rightarrow \Delta_3 \qquad G \mid \Gamma_2, \Gamma_3, \varphi_1 \Rightarrow \varphi_0, \Delta_1 \qquad G \mid \Gamma_2, \Gamma_3 \Rightarrow \Delta_2}{G \mid \ \Gamma_1 \Rightarrow \Delta_3 \mid \Gamma_2, \Gamma_3 \Rightarrow \varphi_1 \to \varphi_0, \Delta_1 \mid \Gamma_3 \Rightarrow \Delta_2} (HBd_3)$$

We prove that the hypersequent calculus $HLJ' + (HBd_3)$ is a calculus for the logic $\mathbf{BD}_3$ and moreover that this calculus has the bounded proof property as well as the finite model property.

Unlike in the case for $\mathbf{BD}_2$ we will work out the one-step correspondence manually as the algorithmic one-step correspondence becomes a bit to involved to manage.

Proposition 5.10. *A one-step frame (P_1, P_0, f) validates the rule (HBd_3) iff P_1 does not contain a 3-chain, whose maximal element is not a root.*

Proof. We first note that a step frame (P_1, P_0, f) validates the rule HBd_3 iff for all $U_1, U_2, U_3, V_1, V_2, V_3 \in \mathsf{Do}(P_1)$ and all $W_0, W_1 \in \mathsf{Do}(P_0)$ we have that if

$$U_1 \cap U_3 \subseteq V_3 \quad \text{and} \quad U_2 \cap U_3 \cap f^{-1}(W_1) \subseteq f^{-1}(W_0) \cup V_1 \quad \text{and} \quad U_2 \cap U_3 \subseteq V_2,$$

then

$$U_1 \subseteq V_3 \quad \text{or} \quad U_2 \cap U_3 \subseteq P_1 \backslash {\uparrow} f^{-1}(W_1 \backslash W_0) \cup V_1 \quad \text{or} \quad U_3 \subseteq V_2.$$

Evidently this is equivalent to the statement where we restrict the downsets U_1, U_2 and U_3 to downsets of the form ${\downarrow}x_1, {\downarrow}x_2$ and ${\downarrow}x_3$.

Thus if (P_1, P_0, f) fails to validate the rule HBd_3 then we have $x_1, x_2, x_3 \in P_1$ as well as $V_1, V_2, V_3 \in \mathsf{Do}(P_1)$ and $W_1, W_0 \in \mathsf{Do}(P_0)$ such that

$${\downarrow}x_1 \cap {\downarrow}x_3 \subseteq V_3 \quad \text{and} \quad {\downarrow}x_2 \cap {\downarrow}x_3 \cap f^{-1}(W_1) \subseteq f^{-1}(W_0) \cup V_1 \quad \text{and} \quad {\downarrow}x_2 \cap {\downarrow}x_3 \subseteq V_2,$$

but

$${\downarrow}x_1 \not\subseteq V_3 \quad \text{and} \quad {\downarrow}x_2 \cap {\downarrow}x_3 \not\subseteq P_1 \backslash {\uparrow} f^{-1}(W_1 \backslash W_0) \cup V_1 \quad \text{and} \quad {\downarrow}x_3 \not\subseteq V_2.$$

It follows that there exists $y \leq x_2, x_3$ such that $y \notin V_1$ but for some $y' \leq y$ we have that $f(y') \in W_1 \backslash W_0$ whence y' must belong to ${\downarrow}x_2 \cap {\downarrow}x_3 \cap f^{-1}(W_1)$ and consequently we must have $y' \in V_1$. We thus obtain that $y' < y$. Now as ${\downarrow}x_2 \cap {\downarrow}x_3 \subseteq V_2$ but ${\downarrow}x_3 \not\subseteq V_2$, we have that $x_3 \not\leq x_2$. So if $y = x_2$ then we have that $y < x_3$.

We have thus shown that P_1 contains a 3-chain $y' < y < x$, where x is either x_2 or x_3. If $y < x_2$ then evidently there exists a 3-chain in P_1 whose maximal element is not a root. On the other hand if $x_2 = y$ then $y' < y < x_3$. We claim that x_3 is not a root. For suppose that x_3 is a root. Then in particular $x_1 \leq x_3$ but then $V_3 \subseteq {\downarrow}x_1 \cap {\downarrow}x_3 = {\downarrow}x_1$ in direct contradiction with the assumption that ${\downarrow}x_1 \not\subseteq V_3$.

Conversely, if P_1 contains a 3-chain $x_1 < x_2 < x_3$ such that x_3 is not a root then if we let

$$U_1 := P_1, \quad U_2 := \downarrow x_2, \quad U_3 := \downarrow x_3,$$

$$V_1 := f^{-1}(\downarrow f(x_1)), \quad V_2 := \downarrow x_2, \quad V_3 := \downarrow x_3,$$

$$W_0 := \emptyset, \quad W_1 := \downarrow f(x_1).$$

we immediately see that

$$U_1 \cap U_3 \subseteq V_3 \quad \text{and} \quad U_2 \cap U_3 \cap f^{-1}(W_1) \subseteq f^{-1}(W_0) \cup V_1 \quad \text{and} \quad U_2 \cap U_3 \subseteq V_2.$$

However, as x_3 is not a root $U_1 \nsubseteq V_3$ and as $x_3 \nleq x_2$ we have that $U_3 \nsubseteq V_2$. Finally if

$$U_2 \cap U_3 \subseteq P_1 \backslash \uparrow f^{-1}(W_1 \backslash W_0) \cup V_1$$

then since $x_2 \in U_2 \cap U_3$ we must have that x_2 belongs to $P_1 \backslash \uparrow f^{-1}(\downarrow f(x_1)) \cup f^{-1}(\downarrow f(x_1))$. Now as $x_1 < x_2$ and $x_1 \in f^{-1}(\downarrow f(x_1))$ it follows that $x_2 \in \uparrow f^{-1}(\downarrow f(x_1))$ whence we must have that $x_2 \in f^{-1}(\downarrow f(x_1))$. Hence $f(x_2) \leq f(x_1)$. It then follows from the fact that f is order preserving that $f(x_2) = f(x_1)$. However as $x_2 \nleq x_1$ and (P_1, P_0, f) is conservative we must have that $f(\downarrow x_2) \nsubseteq f(\downarrow x_1)$. Consequently we must have $x \leq x_2$ such that $f(x) \notin f(\downarrow x_1)$. In particular, we must have that $x < x_2$ and that $f(x_2)(= f(x_1)) \neq f(x)$. But then we have that $x_2 \in \uparrow f^{-1}(\downarrow f(x))$ and $x_2 \notin f^{-1}(\downarrow f(x))$. Whence $x_2 \notin P_1 \backslash \uparrow f^{-1}(\downarrow f(x)) \cup f^{-1}(\downarrow f(x))$.

Therefore letting $V_1' := f^{-1}(\downarrow f(x))$ and $W_1' := \downarrow f(x)$, we see that

$$U_1 \cap U_3 \subseteq V_3 \quad \text{and} \quad U_2 \cap U_3 \cap f^{-1}(W_1') \subseteq f^{-1}(W_0) \cup V_1' \quad \text{and} \quad U_2 \cap U_3 \subseteq V_2.$$

But

$$U_1 \nsubseteq V_3 \quad \text{and} \quad U_2 \cap U_3 \nsubseteq P_1 \backslash \uparrow f^{-1}(W_1' \backslash W_0) \cup V_1' \quad \text{and} \quad U_3 \nsubseteq V_2.$$

Whence (P_1, P_0, f) fails to validate the hypersequent rule (HBd_3). □

From the above proposition it is clear that Theorem 4.10 applies to finite conservative step frames validating HBd_3 whence we obtain the following:

Corollary 5.11. *The hypersequent calculus $HJL' + (HBd_3)$ has the bounded proof property and the finite model property.*

Now since $HJL' + (HBd_3)$ has the finite model property, the universal class $\mathcal{U}(HBd_3)$ of Heyting algebras validating HBd_3 will be generated by its finite subdirectly irreducible elements. A finite Heyting algebra is subdirectly irreducible iff its dual is rooted. Consequently it follows from Proposition 5.10 that the universal class of Heyting algebras validating (HBd_3) will be generated by the class of finite Heyting algebras the duals of which are finite rooted frames not containing a 4-chain. Since the logic $\mathbf{BD}_3$ has the finite model property we know that the variety of Heyting algebras validating $\mathbf{bd}_3$ is generated by the the class of finite subdirectly Heyting algebras validating $\mathbf{bd}_3$. From this we may conclude that the universal classes $\mathcal{U}(HBd_3)$ and $\mathcal{U}(\mathbf{BD}_3)$ generates the same variety and thereby that $HJL' + (HBd_3)$ is a hypersequent calculus for the logic $\mathbf{BD}_3$.

We have thus found a calculus with the bounded proof property for a logic above $\mathcal{P}_3$. It of course still remains to be shown whether or not the calculus $HJL' + (HBd_3)$ has cut-elimination.

5.6 Stable canonical calculi

In [8] a *canonical rule* $\rho(\mathfrak{A}, D)$ *associated to* a finite Heyting algebra $\mathfrak{A}$ and a $D^2 \subseteq A^2$ is introduced.[1] Namely, $\rho(A, D)$ is the multi-conclusion rule Γ/Δ where

$$\begin{aligned}\Gamma :=& \{p_\bot \leftrightarrow \bot,\ p_\top \leftrightarrow \top\} \cup \{p_{a\wedge b} \leftrightarrow p_a \wedge p_b \colon a, b \in A\} \cup \\ & \{p_{a\vee b} \leftrightarrow p_a \vee p_b \colon a, b \in A\} \cup \{p_{a\to b} \leftrightarrow (p_a \to p_b) \colon (a, b) \in D\}\end{aligned}$$

and

$$\Delta := \{p_a \leftrightarrow p_b \colon a, b \in A,\ \ a \neq b\}.$$

We then have

Proposition 5.12 ([8, Prop. 3.2])**.** *For any pair $\mathfrak{A}$ and $\mathfrak{B}$ of Heyting algebras, with $\mathfrak{A}$ finite and any $D \subseteq A^2$ we have that $\mathfrak{B} \not\models \rho(\mathfrak{A}, D)$ iff there exists an embedding $h \colon \mathfrak{A} \to \mathfrak{B}$ of bounded distributive lattices such that for all $(a, b) \in D$,*

$$h(a \to b) = h(a) \to h(b).$$

We say that a canonical rule $\rho(\mathfrak{A}, D)$ is *stable* if $D = \emptyset$. It is known that multi-conclusion calculi given by stable canonical rules have the finite model property [8, Prop. 4.4].

In what follows we define a hypersequent analogue of a stable canonical rule associated with a finite Heyting algebra and show that all hypersequent calculi consisting of such

[1] Of course this builds on the notion of canonical formulas introduced in [53] as well as the notion of canonical rules introduced in [42].

stable canonical rules have the bounded proof property and the finite model property, thus obtaining an analogous result of [12, Thm. 5.3] stating that modal stable rules has the bounded proof property and finite model property.

Definition 5.13. By the *stable canonical hypersequent rule* $\eta(\mathfrak{A})$ *associated to* a finite Heyting algebra $\mathfrak{A}$ we shall understand the hypersequent rule $\mathscr{S}/S$ with

$$\begin{aligned}\mathscr{S} :=& \{p_\bot \Rightarrow \bot,\ \bot \Rightarrow p_\bot,\ p_\top \Rightarrow \top,\ \top \Rightarrow p_\top\} \cup \\ & \{p_{a\wedge b} \Rightarrow p_a \wedge p_b,\ p_a \wedge p_b \Rightarrow p_{a\wedge b} \colon a, b \in A\} \cup \\ & \{p_{a\vee b} \Rightarrow p_a \vee p_b,\ p_a \vee p_b \Rightarrow p_{a\vee b} \colon a, b \in A\} \cup\end{aligned}$$

and S the hypersequent

$$\ldots \mid p_a \Rightarrow p_b \mid \ldots,$$

with a and b ranging over all $a, b \in A$ such that $a \not\leq b$.

Lemma 5.14. *Let $\mathfrak{A}$ be a finite Heyting algebra. Then for every Heyting algebra $\mathfrak{B}$ we have that $\mathfrak{B} \not\models \eta(\mathfrak{A})$ iff there exists an embedding $h\colon \mathfrak{A} \to \mathfrak{B}$ of bounded distributive lattices.*

Proof. Similar to the proof of Proposition 4.4. □

The following theorem is now an easy consequence of Theorem 4.10.

Proposition 5.15. *Let K be a class of finite Heyting algebras. Then the hypersequent calculus consisting of the hypersequent rules $(\eta(\mathfrak{A}))_{\mathfrak{A}\in\mathsf{K}}$ has the bounded proof property and the finite model property.*

Proof. We show that if $\mathcal{H} = (D_0, D_1, i)$ is a finite conservative one-step Heyting algebra and $\mathfrak{A}$ is a finite Heyting algebra then if $\mathcal{H}$ validates $\eta(\mathfrak{A})$ so does D_1. Given this the statement of the theorem follows immediately from Theorem 4.6.

We argue by establishing the contrapositive. If D_1 does not validate the hypsersequent rule $\eta(\mathfrak{A})$ then by Lemma 5.14 there exists a bounded lattice embedding $h\colon \mathfrak{A} \to D_1$. Now Let v_1 be the one-step valuation on $\mathcal{H}$ given by $v_1(p_a) = h(a)$. This is well given as none of the variables p_a has an occurrence under the scope of a Heyting implication.

Since h is a bounded lattice homomorphism it is straightforward to verify that v_1 validates all the premisses of $\eta(\mathfrak{A})$ and as h is also injective, being a lattice homomorphism, it must also be order-reflecting whence we see that $v_1(p_a) = h(a) \not\leq h(b) = v_1(p_b)$ for $a \not\leq b$ and therefore v_1 does not validate any of the sequents $p_a \Rightarrow p_b$ in the conclusion. □

Given the definition of a stable canonical hypersequent rule it is of course not at all surprising that Proposition 5.15 holds and could also easily be obtained by purely syntactic methods.

Summary of Chapter 5: In this chapter we have considered a number of different hypersequent rules for different intermediate logics. In most cases it turned out that turning an axiom into a rule in the naïve way did not yield a calculus with the bounded proof property. However, for the logic $\mathbf{BD}_3$ we managed to find a calculus with the bounded proof property.

Chapter 6

Modal one-step frames and filtrations

We observe that the definition of an intuitionistic one-step frame differs quite a lot from the definition of a modal one-step frame. Knowing the definition of a modal one-step frame one might have expected the definition of an intuitionistic one-step frame to be a quadruple (X, Y, f, R) such that $f\colon X \to Y$ is function between two sets and $R \subseteq X \times Y$ a relation satisfying some conditions similar to the conditions satisfied by relations coming from modal one-step algebras validating **S4**, i.e. *step-reflexivity* and *step-transitivity*. However, as we think that it is more natural to work with Kripke frames and functions between them, we take our inspiration from the definition of intuitionistic one-step frames and ask if its possible to describe modal one-step frames as triples $(\mathfrak{F}_1, \mathfrak{F}_0, f)$ consisting of (certain kinds of) Kripke frames $\mathfrak{F}_1, \mathfrak{F}_0$ and a relation-preserving function between them.

In this chapter we answer this question in the affirmative, by exhibiting a category of so-called *minimal filtration frames* – whose objects are determined by a pair of Kripke frames and relation-preserving maps between them – which will be isomorphic to the category of conservative modal one-step frames. Moreover, this enables us to establish a connection between modal one-step algebras and the algebraic approach to filtrations found in [37]. This also addresses the question raised at the end of the presentation[1] of [39] at AiML 2014 in Groningen about the connection between modal one-step frames and filtrations.

[1] The slides from the talk are available at http://www.samvangool.net/talks/vangool-groningen-08082014.pdf. The question appears on the very last slide.

6.1 Minimal filtration frames

Definition 6.1. By a *minimal filtration frame* we shall understand a triple $(\mathfrak{F}_1, \mathfrak{F}_0, f)$ such that $\mathfrak{F}_1 = (W_1, R_0)$ and $\mathfrak{F}_0 = (W_0, R_0)$ are Kripke frames and $f\colon W_1 \to W_0$ is a function satisfying

i) $\forall w, w' \in W_1\ (wR_1w' \implies f(w)R_0f(w'))$;

ii) $\forall v, v' \in W_0\ (vR_0v' \implies \exists w, w' \in W_1\ (wR_1w' \text{ and } f(w) = v \text{ and } f(w') = v'))$;

iii) $\forall w, w', w'' \in W_1\ ((wR_1w' \ \&\ f(w') = f(w'')) \implies wR_1w'')$.

Thus a minimal filtration frame is determined by a relation-preserving function $f\colon \mathfrak{F}_1 \to \mathfrak{F}_0$ such that the relation R_0 on W_0 is minimal among relations on W_0 making f order-preserving. The last condition is reminiscent of the quotient condition on q-frames from [30]. We will return to the relationship between minimal filtration frames and q-frames later in this chapter.

A map from a minimal filtration frame $(\mathfrak{F}_1', \mathfrak{F}_0', f')$ to a minimal filtration frame $(\mathfrak{F}_1, \mathfrak{F}_0, f)$ will be a pair of functions $g_1\colon W_1' \to W_1$ and $g_0\colon W_0' \to W_0$ such that

i(a) $\forall w, w' \in W_1'\ (wR_1'w' \implies g_1(w)R_1g_1(w'))$

i(b) $\forall v, v' \in W_0'\ (vR_0'v' \implies g_0(v)R_0g_0(v'))$

ii) $f \circ g_1 = g_0 \circ f'$;

iii) $\forall w' \in W_1'\ \forall w \in W_1\ (g_1(w')R_1w \implies \exists w'' \in W_1'\ (w'R_1'w'' \text{ and } f(g_1(w'')) = f(w)))$;

Thus a map between minimal filtration frames is a pair of relation-preserving functions with g_1 a f-open map making the diagram

$$\begin{array}{ccc} W_1' & \xrightarrow{g_1} & W_1 \\ \Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle f} \\ W_0' & \xrightarrow{g_0} & W_0 \end{array}$$

commute.

Let MFFrm be the category whose objects are minimal filtration frames and whose morphisms are maps between minimal filtration frames.

We say that a minimal filtration frame $(\mathfrak{F}_1, \mathfrak{F}_0, f)$ is *conservative* if $f\colon W_1 \to W_0$ is surjective and

$$\forall w, w' \in W_1\ (R_1[w] = R_1[w'] \text{ and } f(w) = f(w')) \implies w = w')$$

Let $\mathsf{MFFrm}^{\text{cons}}$ denote the full-subcategory of conservative minimal filtration frames and let $\mathsf{MFFrm}^{\text{cons}}_{<\omega}$ denote the full subcategory of finite conservative minimal filtration frames, where we call a minimal filtration frame $(\mathfrak{F}_1, \mathfrak{F}_0, f)$ finite if both $\mathfrak{F}_1$ and $\mathfrak{F}_0$ are finite Kripke frames.

Recall form Definition 2.15 that a modal one-step frame is a quadruple (W_1, W_0, f, R) such that $f\colon W_1 \to W_0$ is a function and $R \subseteq W_1 \times W_0$ is a relation. Moreover, recall that a p-morphism from a modal one-step frame (W_1', W_0', f', R') to a modal one-step frame (W_1, W_0, f, R) is a pair of functions $\mu\colon W_1' \to W_1$ and $\nu\colon W_0' \to W_0$ such that

$$f \circ \mu = \nu \circ f' \quad \text{and} \quad R \circ \mu = \nu \circ R'.$$

Finally, recall that a modal one-step frame (W_1, W_0, f, R) is conservative if f is surjective and

$$\forall w, w' \in W_1((R[w] = R[w'] \text{ and } f(w) = f(w')) \implies w = w').$$

We then have the following theorem.

Theorem 6.2. *The category* MFFrm^{cons} *is isomorphic to the category of* MOSFrm^{cons} *of conservative modal one-step frames and p-morphisms between them.*

Proof. We define functors $F\colon \mathsf{MFFrm}^{cons} \to \mathsf{MOSFrm}^{cons}$ and $G\colon \mathsf{MOSFrm}^{cons} \to \mathsf{MFFrm}^{cons}$ as follows:

Given a minimal filtration frame $\mathcal{M} = (\mathfrak{F}_1, \mathfrak{F}_0, f)$ we let $F(\mathcal{M}) = (W_1, W_0, R_F, f)$ where $R_F \subseteq W_1 \times W_0$ is the relation given by

$$wR_Fv \iff \exists w' \in W_1 \ (wR_1w' \text{ and } f(w') = v).$$

Given a modal one-step frame $\mathcal{S} = (W_1, W_0, f, R)$ we let $G(\mathcal{S}) = (\mathfrak{F}_1, \mathfrak{F}_0, f)$ with $\mathfrak{F}_1 = (W_1, R_1^G)$ and $\mathfrak{F}_0 = (W_0, R_0^G)$, where the relations $R_1^G \subseteq W_1^2$ and $R_0^G \subseteq W_0^2$, are given by

$$wR_1^Gw' \iff wRf(w') \quad \text{and} \quad vR_0^Gv' \iff \exists w \in W_1 \ (wRv' \text{ and } f(w) = v).$$

Finally, we let both F and G act as the identity functor on maps.

We first show that the above indeed defines functors into the categories MOSFrm^{cons} and MFFrm^{cons}, respectively.

To show that the functor F is well-defined on objects it suffices to show that $F(\mathcal{M})$ is conservative. This is an immediate consequence of the fact that $\mathcal{M}$ is conservative, since by definition $R_F[w] = f(R_1[w])$ and so since wR_1w' and $f(w') = f(w'')$ we must have that

$$R_F[w] = R_F[w'] \iff f(R_1[w]) = f(R_1[w']) \iff R_1[w] = R_1[w'].$$

To see that G is also well-defined on objects we must show that if $\mathcal{S} = (W_1, W_0, f, R)$ is a modal one-step frame then $G(\mathcal{S})$ is a minimal filtration frame. It is straightforward to verify that $G(\mathcal{S})$ satisfies item i) and iii) of the definition of a minimal filtration frame. To see that item ii) obtains as well, let $v, v' \in W_0$ be such that $vR_0^G v'$. Then by definition of the relation R_0^G we have $w \in W_1$ such that wRv' and $f(w) = v$. As $\mathcal{S}$ is conservative the function f is surjective and therefore there exists $w' \in W_1$, such that $f(w') = v'$ and so by the definition of R_1^G we have that $wR_1^G w'$.

Now to see that $G(\mathcal{S})$ is conservative when $\mathcal{S}$ is, we simply note that $f(R_1^G[w]) = R[w]$ as f is surjective, and hence that $G(\mathcal{S})$ is conservative is an immediate consequence of the assumption that $\mathcal{S}$ is.

This shows that the functors F and G are well-defined on objects. We now show that they are also well-defined on maps.

If $(g_1, g_0)\colon \mathcal{M}' \to \mathcal{M}$ is a map between conservative minimal filtration frames $\mathcal{M}' = (\mathfrak{F}_1', \mathfrak{F}_0', f')$ and $\mathcal{M} = (\mathfrak{F}_1, \mathfrak{F}_0, f)$ then we must show that $F(g_1, g_0)\colon F(\mathcal{M}') \to F(\mathcal{M})$ is a p-morphism between modal one-step frames. For this it suffices to show that

$$R_F \circ g_1 = g_0 \circ R_F'.$$

Therefore let $w' \in W_1'$ and $v \in W_0$ be such that $w'(R_F \circ g_1)v$, i.e. $g_1(w')R_F v$. Then by the definition of R_F we have that there exists $w \in W_1$ such that $g_1(w')R_1 w$ and $f(w) = v$, so from item iii) of the definition of maps between minimal filtration frames $g_1(w')R_1 w$ implies that there exists $w'' \in W_1'$ such that $w'R_1w''$ and $f(g_1(w'')) = f(w)$. Taking $v' := f'(w'')$ we see that

$$g_0(v') = g_0(f'(w'')) = f(g_1(w'')) = f(w) = v.$$

So since $w'R_1'w''$ and $f(w'') = v'$ we see that $w'R_F'v'$ whence we may conclude that $w'(g_0 \circ R_F')v$.

Conversely, if $w' \in W_1'$ and $v \in W_0$ is such that $w'(g_0 \circ R_F')v$ then this must be because there exists $v' \in W_0'$ such that $w'R_F'v'$ and $g_0(v') = v$. From the definition of R_F' it follows that there exists $w'' \in W_1'$ such that $w'R_1'w''$ and $f'(w'') = v'$. Now as g_1 preserves the relations R_1' we obtain that $g_1(w')R_1g_1(w'')$ and since $f(g_1(w'')) = g_0(f'(w'')) = g_0(v') = v$ we may conclude that $g_1(w')R_F v$, i.e. that $w'(R_F \circ g_1)v$.

This shows that F indeed defines a functor from MFFrm^{cons} to MOSFrm^{cons}.

To see that G is is well-defined on maps let $(\mu, \nu)\colon \mathcal{S}' \to \mathcal{S}$ be a p-morphism between modal one-step frames $\mathcal{S}' = (W_1', W_0', f', R')$ and $\mathcal{S} = (W_1, W_0, f, R)$. We then show that $G(\mu, \nu)\colon G(\mathcal{S}') \to G(\mathcal{S})$ is a map of minimal filtration frames.

First to see that $\mu\colon W_1' \to W_1$ preserves the relation $(R_1')^G$ we observe that $\mu(w) R_1^G \mu(w')$ precisely when $w(R \circ \mu) f(\mu(w'))$. As (μ, ν) is p-morphism between step frames we have that $(R \circ \mu) = (\nu \circ R')$, whence we may conclude that $\mu(w) R_1^G \mu(w')$ precisely when there exists $v' \in W_0'$ such that $wR'v'$ and $\nu(v') = f(\mu(w'))$. Now $w(R_1')^G w'$ precisely when $wR'f'(w')$ and so as $f \circ \mu = \nu \circ f'$ we may take $v' := f(w')$ to witness that $\mu(w) R_1^G \mu(w')$. Likewise, to see that $\nu\colon W_0' \to W_0$ preserves the relation $(R_0')^G$, assume that $v(R_0')^G v'$. As we have already shown that $G(\mathcal{S})$ is a minimal filtration frame it follows that there exists $w, w' \in W_1'$ such that $w(R_1')^G w'$ and $f'(w) = v$ and $f'(w') = v'$. Therefore since μ preserves the relation $(R_1')^G$ we obtain that $\mu(w)(R_1)^G \mu(w')$, and so as f preserves the relation $(R_1)^G$ we must have that $f(\mu(w))(R_0)^G f(\mu(w'))$. But then $\nu(v)(R_0)^G \nu(v')$, as $f(\mu(w)) = \nu(f'(w)) = \nu(v)$ and similarly $f(\mu(w')) = \nu(v')$.

Lastly to see that $G(\mu, \nu)$ satisfies item iii) of the definition of maps between minimal filtration frames we must show that if $\mu(w') R_1^G w$ then there exists $w'' \in W_1'$ such that $w'(R_1')^G w''$ and $f(\mu(w'')) = f(w)$. To see this, observe that by definition $\mu(w') R_1^G w$ implies that $\mu(w') R f(w)$, i.e. that $w'(R \circ \mu) f(w)$ whence as (μ, ν) is a p-morphism between one-step frame we must have $w'(\nu \circ R') f(w)$. This means that there exists $v' \in W_0'$ such that $w'R'v'$ and $\nu(v') = f(w)$. Now as f is surjective we may take $w'' \in W_1'$ such that $f'(w'') = v'$. But then we have that $w'(R_1')^G w''$ and that

$$f(\mu(w'')) = \nu(f'(w'')) = \nu(v') = f(w),$$

and so we have that w'' is an element of W_1' with the desired properties.

Finally, we must show that $F(G(\mathcal{S})) = \mathcal{S}$ and $G(F(\mathcal{M}))$ for all conservative modal one-step frames $\mathcal{M}$ and all conservative minimal filtration frames $\mathcal{M}$. For this it suffices to show that $R = (R_1^G)_F$ and that $R_1 = (R_F)_1^G$ and $R_0 = (R_F)_0^G$.

We first show that $R = (R_1^G)_F$. Unravelling the definitions we see that

$$\begin{aligned} w(R_1^G)_F v &\iff \exists w' \in W_1 \ wR_1^G w' \text{ and } f(w') = v \\ &\iff \exists w' \in W_1 \ wRf(w') \text{ and } f(w') = v \\ &\iff wRv. \end{aligned}$$

To see that $(R_F)_1^G = R_1$ we observe that

$$\begin{aligned} w(R_F)_1^G w' &\iff wR_F f(w') \\ &\iff \exists w'' \in W_1 \ wR_1w'' \text{ and } f(w'') = f(w') \\ &\iff wR_1w', \end{aligned}$$

where the last equivalence is a consequence of item iii) of the definition of a minimal filtration frame.

Finally we have that

$$\begin{aligned} v(R_F)_0^G v' &\iff \exists w \in W_1 \ (wR_Fv' \text{ and } f(w) = v) \\ &\iff \exists w, w' \in W_1 \ (wR_1w' \text{ and } f(w) = v \text{ and } f(w') = v') \\ &\iff vR_0v', \end{aligned}$$

where the last equality follows from item i) and ii) in the definition of a minimal filtration frame. $\square$

Note that the above isomorphism restricts to an isomorphism between the categories $\mathsf{MFFrm}^{\mathrm{cons}}_{<\omega}$ and $\mathsf{MOSFrm}^{\mathrm{cons}}_{<\omega}$.

6.2 Duality for minimal filtration frames

In this section we consider the duals of finite conservative minimal filtration frames induced by the standard Jónsson-Tarski duality between finite Kripke frames and finite modal algebras.

Recall the language-free algebraic definition of a filtration from [37].

Definition 6.3. Given a modal algebra $\mathfrak{A} = (A, \Diamond_A)$ and a Boolean algebra B together with an embedding of Boolean algebras $i\colon B \to A$, we say that a hemimorphism $\Diamond_B\colon B \to B$ is a *filtration of* $\mathfrak{A}$ *over* i if

i) $\forall b \in B \ (\Diamond_A i(b) \leq i(\Diamond_B b))$;

ii) $\forall b, b' \in B \ (\Diamond_A i(b) = i(b') \implies \Diamond_B b \leq b')$.

The condition in item i) is known in the literature as *continuity* [37] or *stability* [7].

Note that from item i) it follows that if $\Diamond_A i(b) = i(b')$ then $i(b') \leq i(\Diamond_B b)$ and so as i is an injective Boolean algebra homomorphism we obtain that $b' \leq \Diamond_B b$. Hence item ii) is equivalent to ii') $\forall b, b' \in B\ (\Diamond_A i(b) = i(b') \implies \Diamond_B b = b')$,

in the presence of item i). This observation is in fact Ghilardi's Filtration Lemma [37, Lem. 3.2].

Now given a conservative minimal filtration frame $(\mathfrak{F}_1, \mathfrak{F}_0, f)$ we obtain via finite Jónsson-Tarski duality a triple $(\mathfrak{A}_0, \mathfrak{A}_1, i)$ consisting of a pair of modal algebras with an embedding $i\colon A_0 \to A_1$ of Boolean algebras.

Proposition 6.4. *If $(\mathfrak{F}_1, \mathfrak{F}_0, f)$ is a finite minimal filtration frame its dual $(\mathfrak{A}_0, \mathfrak{A}_1, i)$ is such that $\Diamond_0\colon A_0 \to A_0$ is a filtration of $\mathfrak{A}_1 = (A_1, \Diamond_1)$ over $i\colon A_0 \to A_1$, satisfying*

$$\Diamond_1 a = \Diamond_1 i i^\flat(a),$$

where $i^\flat\colon A_1 \to A_0$ is the left adjoint of i.

Moreover, $\mathfrak{A}_0 = (A_0, \Diamond_0)$ is the minimal filtration of $\mathfrak{A}_1$ over $i\colon A_0 \to A_1$, in the sense that if $\Diamond\colon A_0 \to A_0$ is any other filtration of A_1 over i then $\Diamond_0 a \leq \Diamond a$ for all $a \in A_0$.

Proof. Let $\mathfrak{F}_1 = (W_1, R_1)$ and $\mathfrak{F}_0 = (W_0, R_0)$. By duality we then have that $\mathfrak{A}_j = (\wp(W_j), \Diamond_{R_j})$ and $i\colon \wp(W_0) \to \wp(W_1)$ is the map f^* given by $f^*(U) = f^{-1}(U)$. Moreover, it is easy to verify that $i^\flat\colon \wp(W_1) \to \wp(W_0)$ is the direct-image along f.

That i satisfies item i) of the definition of filtration is a consequence of the fact that continuous maps between algebras are dual to relation-preserving maps between frames. To see that item ii) is also satisfied we must show that if

$$\{w \in W_1 \colon R_1[w] \cap f^{-1}(U) \neq \emptyset\} = f^{-1}(U'),$$

then

$$\{v \in W_0 \colon R_0[v] \cap U \neq \emptyset\} \subseteq U',$$

for all $U, U' \subseteq W_0$.

Therefore, assume that the antecedent obtains and let $v \in W_0$ be such that there exists $v' \in U$ with vR_0v'. Then there exists $w, w' \in W_1$ such that wR_1w', $f(w) = v$ and $f(w') = v'$. It follows that $R_1[w] \cap f^{-1}(U)$ is non-empty and so by our initial assumption we have that w belongs to $f^{-1}(U')$ and thereby that $v = f(w) \in U'$, as desired.

This shows that $\Diamond_0\colon A_0 \to A_0$ is a filtration of $\mathfrak{A}_1$ over $i\colon A_0 \to A_1$.

To see that $\Diamond_1 a = \Diamond_1 ii^\flat a$ for all $a \in A_1$ we note that since the left adjoint $i^b\colon \wp(W_1) \to \wp(W_0)$ is the direct-image along f this amounts to showing that

$$\Diamond_{R_1} U = \Diamond_{R_1} f^{-1} f(U),$$

for all $U \in \wp(W_1)$. This, however, is an immediate consequence of item iii) in the definition of a minimal filtration frame, i.e. that wR_1w' and $f(w') = f(w'')$ implies wR_1w''.

Finally, to see that $\Diamond_0$ is indeed the minimal filtration of $\mathfrak{A}_1$ over $i\colon A_0 \to A_1$ we observe that item i) of the definition of a filtration of $\mathfrak{A}_1$ over i implies that if $\Diamond\colon A_0 \to A_0$ is a filtration of $\mathfrak{A}_1$ over i then for all $a \in A_0$

$$i^\flat(\Diamond_1 i(a)) \leq \Diamond a.$$

We show that,

$$\Diamond_0 a = i^\flat(\Diamond_1 i(a)),$$

for all $a \in A_0$, from which it follows that $\Diamond_0$ is the least filtration of $\mathfrak{A}_1$ over i.

As we have already observed $i^\flat(\Diamond_1 i(a)) \leq \Diamond_0 a$, for all $a \in A_0$, as $\Diamond_0\colon A_0 \to A_0$ is a filtration of $\mathfrak{A}_0$ over i. For the converse inequality we must show that $\Diamond_0 a \leq i^\flat(\Diamond_1 i(a))$ for all $a \in A_0$. By duality this means that

$$\Diamond_{R_0} U \subseteq f(\Diamond_{R_1} f^{-1}(U))$$

for all $U \in \wp(W_0)$. Therefore let $U \in \wp(W_0)$ and let $v \in \Diamond_{R_0} U$ be given. Then vR_0u for some $u \in U$. By assumption that $(\mathfrak{F}_1, \mathfrak{F}_0, f)$ is a minimal filtration frame it follows that there exists $w, w' \in W_1$ such that wR_1w' and $f(w) = v$ and $f(w') = u$. But then $w' \in f^{-1}(U)$ whence $w \in \Diamond_{R_1} f^{-1}(U)$ and so $u \in f(\Diamond_{R_1} f^{-1}(U))$ as desired. $\square$

The above proposition explains the name minimal filtration frames.

Using the above proposition we now wish to determine a category dually equivalent to the category of finite conservative minimal filtration frames.

By a *localization* of a Boolean algebra A we will understand a Boolean algebra embedding $i\colon A' \to A$ with a left-adjoint $i^\flat\colon A \to A'$.

Definition 6.5. By a *minimal filtration algebra* we will understand a triple $(\mathfrak{A}_0, \mathfrak{A}_1, i)$ consisting of modal algebras $\mathfrak{A}_0 = (A_0, \Diamond_0)$ and $\mathfrak{A}_1 = (A_1, \Diamond_1)$ together with a localization $i\colon A_0 \to A_1$ satisfying

$$\Diamond_1 ii^\flat(a) = \Diamond_1 a,$$

such that $\mathfrak{A}_0 = (A_0, \Diamond_0)$ is the minimal filtration of $\mathfrak{A}_1 = (A_1, \Diamond_1)$ over $i\colon A_0 \to A_1$.

We call a minimal filtration algebra $(\mathfrak{A}_0, \mathfrak{A}_1, i)$ *conservative* if A_1 is generated as a Boolean algebra by the set $i(A_0) \cup \Diamond_1(i(A_0))$.

Definition 6.6. Let $\mathscr{A} = (\mathfrak{A}_0, \mathfrak{A}_1, i)$ and $\mathscr{A}' = (\mathfrak{A}'_0, \mathfrak{A}'_1, i')$ be minimal filtration algebras. Then a homomorphism from $\mathscr{A}$ to $\mathscr{A}'$ will be a pair of continuous Boolean algebra homomorphisms $h_1\colon A_1 \to A'_1$ and $h_0\colon A_0 \to A'_0$, such that $h_1(\Diamond_1 i(a_0)) = \Diamond'_1 h_1(i(a_0))$, making the diagram

$$\begin{array}{ccc} A_0 & \xrightarrow{h_0} & A'_0 \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} \\ A_1 & \xrightarrow{h_1} & A'_1 \end{array}$$

commute.

In this way we obtain a category MFAlg of *minimal filtration algebras* and homomorphisms between them.

As the reader might have expected the Jónsson-Tarski duality extends to a dual equivalence between the categories $\mathsf{MFFrm}^{\mathrm{cons}}_{<\omega}$ and $\mathsf{MFAlg}^{\mathrm{cons}}_{<\omega}$. This in turn by Theorem 6.2 and Proposition 2.20 implies that the categories $\mathsf{MFAlg}^{\mathrm{cons}}_{<\omega}$ and $\mathsf{MOSAlg}^{\mathrm{cons}}_{<\omega}$ are equivalent. In fact as we will show below these categories are isomorphic.

Theorem 6.7. *The category of* $\mathsf{MFAlg}^{cons}_{<\omega}$ *of finite conservative minimal filtration algebras is isomorphic to the category* $\mathsf{MOSAlg}^{cons}_{<\omega}$ *of finite conservative one-step modal algebras.*

Proof. Given a finite conservative modal one-step algebra $\mathcal{A} = (A_0, A_1, i, \Diamond)$ define hemimorphism $\Diamond^J_0\colon A_0 \to A_0$ and $\Diamond^J_1\colon A_0 \to A_0$ by

$$\Diamond^J_1 a_1 := \Diamond i^\flat(a_1) \quad \text{and} \quad \Diamond^J_0 a_0 := i^\flat(\Diamond^J_1 i(a_0)).$$

We then let $\mathfrak{A}_j = (A_j, \Diamond^J_j)$ for $j \in \{0, 1\}$ and define $J(\mathcal{A})$ to be the triple $(\mathfrak{A}_0, \mathfrak{A}_1, i)$. We then let J act as the identity on maps.

We claim that this determines a functor $J\colon \mathsf{MOSAlg}^{\mathrm{cons}}_{<\omega} \to \mathsf{MFAlg}^{cons}_{<\omega}$. To see this we must first check that $J(\mathcal{A})$ is a finite conservative minimal filtration algebra.

That $J(\mathcal{A})$ is finite is evident from the assumption that $\mathcal{A}$ is finite. Moreover, as i is injective we must have that $i^\flat(i(a_0)) = a_0$ for all $a_0 \in A_0$, and thereby that $\Diamond^J_1(i(A_0)) = \Diamond(A_0)$ and so it follows that $J(\mathcal{A})$ is conservative as $\mathcal{A}$ is conservative. Moreover, from the fact that $i^\flat \circ i = \mathrm{id}_{A_0}$, we also see that

$$\Diamond^J_1 i(i^\flat(a_1)) = \Diamond i^\flat(i(i^\flat(a_1))) = \Diamond i^\flat(a_1) = \Diamond^J_1(a_1).$$

By definition we have that $\Diamond_0^J$ is the minimal filtration of $\Diamond_1^J$ over $i\colon A_0 \to A_1$, since the minimal filtration is indeed a filtration. This shows that J is well-defined on objects. To see that it is also well-defined on morphisms let $(h,k)\colon \mathcal{A} \to \mathcal{A}'$ be a map between modal one-step algebras. Then we must show that $k(\Diamond_1^J i(a)) = (\Diamond_1^J)' k i(a)$. This, however, is an easy consequence of the fact that (h,k) is an homomorphism of one-step algebras. Finally, we must check that h and k are continuous with respect to $\Diamond_0^J$ and $\Diamond_1^J$, respectively. For this one may readily check that $(i')^\flat(k(a)) \leq k i^\flat(a)$ from which it follows that

$$(\Diamond_1^J)' k(a) = \Diamond' (i')^\flat k(a) \leq \Diamond' k(i^\flat(a)) = k(\Diamond i^\flat(a)) = k(\Diamond_1^J(a)).$$

A similar computation shows that $(\Diamond_0^J)' h(a) \leq h(\Diamond_0^J a)$.

Conversely, given a minimal filtration algebra $\mathscr{A} = (\mathfrak{A}_0, \mathfrak{A}_1, i)$ with $\mathfrak{A}_j = (A_j, \Diamond_j)$ we let $I(\mathscr{A})$ be the quadruple $(A_0, A_1, i, \Diamond^I)$, where

$$\Diamond^I a_0 \coloneqq \Diamond_1 i(a_0).$$

It is then straightforward to check that letting I act as the identity on morphisms we obtain a functor $I\colon \mathsf{MFAlg}_{<\omega}^{\mathrm{cons}} \to \mathsf{MOAlg}_{<\omega}^{cons}$.

Finally, we claim that $IJ(\mathcal{A}) = \mathcal{A}$ for all finite conservative modal one-step algebras $\mathcal{A}$ and that $JI(\mathscr{A}) = \mathscr{A}$ for all minimal filtration algebras $\mathscr{A}$. To see this it suffices to show that

$$\Diamond^{IJ(\mathcal{A})} = \Diamond \quad \text{and} \quad \Diamond_1^{JI(\mathscr{A})} = \Diamond_1$$

for all $\mathcal{A} = (A_0, A_1, i, \Diamond)$ and for all $\mathscr{A} = (\mathfrak{A}_0, \mathfrak{A}_1, i)$.

A simple computation shows that

$$\Diamond^{IJ(\mathcal{A})} a_0 = \Diamond_1^{J(\mathcal{A})} i(a_0) = \Diamond i^\flat(i(a_0)) = \Diamond a_0,$$

since $i^\flat(i(a_0)) = a_0$.

Similarly we have that

$$\Diamond_1^{JI(\mathscr{A})} a_1 = \Diamond^{I(\mathcal{A})} i^\flat(a_1) = \Diamond_1 i(i^\flat(a_1)) = \Diamond_1 a_1,$$

where the last equality is one of the defining properties of a minimal filtration algebra.

□

As a corollary we obtain that the categories $\mathsf{MFAlg}_{<\omega}^{\mathrm{cons}}$ and $\mathsf{MFFrm}_{<\omega}^{\mathrm{cons}}$ are dually equivalent.

6.3 q-frames and partial modal algebras

In [30] a different approach to the one-step framework is taken. In the section we compare the q-frames and the partial modal algebras of [30] to our minimal filtration frames and algebras.

Definition 6.8. A *q*-frame is a triple $(X, \sim, R)$, such that $\sim$ is an equivalence relation on X and R is a relation on X satisfying

$$\forall x, y, y' \in X \ (xRy \text{ and } y \sim y' \implies xRy').$$

A *bounded map* from a q-frame $(X, \sim, R)$ to a q-frame $(X', \sim', R')$ is a function $g\colon X \to X'$ which preserves both of the relations $\sim$ and R and which moreover satisfies

$$\forall x, y \in X \ (g(x)R'y \implies \exists x' \in X \ (xRx' \text{ and } g(x') \sim' y)).$$

We may then define *conservative* q-frames as q-frames $(X, \sim, R)$, satisfying

$$\forall x, x' \in X (R[x] = R[x'] \text{ and } x \sim x' \implies x = x').$$

It is then not difficult to see that we have a functor F from the $\mathsf{qFrm}^{\mathrm{cons}}$ of conservative q-frames and bounded maps to the category MFFrm^{cons} given by $F(X, \sim, R) = (\mathfrak{F}_1, \mathfrak{F}_0, \pi)$ with $\mathfrak{F}_1 = (X, R)$ and $\mathfrak{F}_0 = (X/\sim, R_\sim)$ and $\pi\colon X \to X/\sim$ the canonical projection, where

$$[x]R_\sim[y] \iff \exists x', y' \in X \ (x'Ry' \text{ and } x' \sim x \text{ and } y' \sim y).$$

For bounded maps $g\colon (X, \sim, R) \to (X', \sim', R')$ we let $F(g) = (g, \bar{g})$ where the function $\bar{g}\colon X/\sim \to X'/\sim'$ is given by $\bar{g}([x]) = [g(x)]$. That this is well-defined follows from the fact that $x \sim y$ implies $g(x) \sim' g(y')$ when g is a bounded map between q-frames.

Note that the functor $F\colon \mathsf{qFrm}^{\mathrm{cons}} \to \mathsf{MFFrm}^{\mathrm{cons}}$ described above restricts to a functor from $\mathsf{qFrm}^{\mathrm{cons}}_{<\omega}$ to $\mathsf{MFFrm}^{\mathrm{cons}}_{<\omega}$.

Finally, recall from [30] that a *partial modal algebra* is a pair $(A, \Diamond^A)$ such that A is a Boolean algebra and $\Diamond^A\colon A \rightharpoonup A$ is a partial function, the domain of which is a Boolean subalgebra of A, which satisfies

$$\Diamond^A \bot = \bot \quad \text{and} \quad \Diamond^A(b \vee b') = \Diamond^A b \vee \Diamond^A b' \quad \text{for all } b, b' \in \mathrm{dom}(\Diamond^A).$$

A *partial modal algebra homomorphism* from a partial modal algebra $(A, \Diamond^A)$ to a partial modal algebra $(A', \Diamond^{A'})$ is a Boolean algebra homomorphism $h\colon A \to A'$ with the property that

$$h(\mathrm{dom}(\Diamond^A)) \subseteq \mathrm{dom}(\Diamond^{A'}) \quad \text{and} \quad h(\Diamond^A b) = \Diamond^{A'} h(b) \quad \text{for all } b \in \mathrm{dom}(\Diamond^A).$$

Let pMA denote the category of partial modal algebras and partial modal algebra homomorphisms between them.

Definition 6.9. We say that a partial modal algebra $(A, \Diamond^A)$ is *conservative* if A is generated as a Boolean algebra by the set $\mathrm{dom}(\Diamond^A) \cup \mathrm{Im}(\Diamond^A)$.

As the reader might have expected we have

Theorem 6.10 ([30, Thm. 4.3])**.** *The categories $\mathsf{qFrm}_{<\omega}$ and $\mathsf{pMA}_{<\omega}$ are dually equivalent.*

Inspecting the proof of Theorem 6.10 one may verify that the dual equivalence between $\mathsf{qFrm}_{<\omega}$ and $\mathsf{pMA}_{<\omega}$ restricts to dual equivalence between the categories $\mathsf{qFrm}^{\mathrm{cons}}_{<\omega}$ and $\mathsf{pMA}^{cons}_{<\omega}$.

Again it is not difficult to see that we obtain a functor from $F\colon \mathsf{pMA}_{<\omega} \to \mathsf{MFAlg}_{\omega}$ by letting $F(A, \Diamond^A) = (\mathfrak{A}_0, \mathfrak{A}_1, i)$ where $i\colon \mathrm{dom}(\Diamond^A) \to A$ is the canonical inclusion of Boolean algebras and $\mathfrak{A}_0 = (\mathrm{dom}(\Diamond^A), \Diamond_0)$ and $\mathfrak{A}_1 = (A, \Diamond_1)$ with

$$\Diamond_1 a = \Diamond^A i^\flat(a) \quad \text{and} \quad \Diamond_0 a = i^\flat(\Diamond_1 i(a)).$$

For maps $h\colon (A, \Diamond^A) \to (A', \Diamond^{A'})$ we let $F(h) = (h{\restriction}\mathrm{dom}(\Diamond^A), h)$.

We may thus represent the relationship between the one-step framework of [13, 11], the partial algebra framework of [30, 39] and our minimal filtration frames and algebras in a diagrammatic form:

$$\begin{tikzcd}
\mathsf{MOSFrm}^{\mathrm{cons}}_{<\omega} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathsf{MFFrm}^{\mathrm{cons}}_{<\omega} \arrow[d, leftrightarrow] & \mathsf{qFrm}^{\mathrm{cons}}_{<\omega} \arrow[l] \arrow[d, leftrightarrow] \\
\mathsf{MOSAlg}^{\mathrm{cons}}_{<\omega} \arrow[r, leftrightarrow] & \mathsf{MFAlg}^{\mathrm{cons}}_{<\omega} & \mathsf{pMA}^{\mathrm{cons}}_{<\omega} \arrow[l]
\end{tikzcd}$$

The vertical arrows are dual equivalences of categories and the horizontal arrows are isomorphisms of categories except for the arrows out of $\mathsf{qFrm}^{\mathrm{cons}}_{<\omega}$ and $\mathsf{pMA}^{\mathrm{cons}}_{<\omega}$ which are faithfull functors[2].

[2] It is of course also possible to define functors in the opposite directions. However, these will not be part of an isomorphism between categories.

6.4 The bounded proof property and filtrations

We here briefly consider how to characterize the bounded proof property in terms of minimal filtration algebras and frames. We believe that this provides an interesting perspective on the results obtained by Bezhanishvili and Ghilardi in [11]. Moreover, we hope that this perspective might be useful when it comes to deciding whether or not a given axiomatization of a modal logic has the bounded proof property.

We say that a minimal filtration frame $\mathcal{M} = (\mathfrak{F}_1, \mathfrak{F}_0, f)$ validates a reduced modal axiom system Ax if the corresponding modal one-step frame validates Ax. Similarly we say that a minimal filtration algebra $\mathscr{A}$ validates Ax if the corresponding modal one-step algebra validates Ax.

Then for every reduced modal axiom system Ax we let $\mathsf{MFFrm}(\mathrm{Ax})$ denote the full subcategory of $\mathsf{MFFrm}^{\mathrm{cons}}_{<\omega}$ consisting of finite conservative minimal filtration frames validating Ax. Similarly we let $\mathsf{MFAlg}(\mathrm{Ax})$ denote the full subcategory of $\mathsf{MFAlg}^{\mathrm{cons}}_{<\omega}$ consisting of finite conservative minimal filtration algebras validating Ax.

We can then make the following easy observation

Lemma 6.11. *Let* Ax *be a reduced modal axiom system.*

i) The class $Con^{Alg}_{<\omega}(\mathrm{Ax})$ of finite conservative modal one-step algebras validating Ax *has the extension property iff for all $(\mathfrak{A}_0, \mathfrak{A}_1, i)$ belonging to $\mathsf{MFAlg}(\mathrm{Ax})$ there exists a modal algebras $\mathfrak{A}_2 = (A_2, \Diamond_2)$ and a continuous embedding of Boolean algebras $j\colon A_1 \to A_2$ satisfying $j(\Diamond_1 i(a)) = \Diamond_2 j(i(a))$ such that $(\mathfrak{A}_1, \mathfrak{A}_2, j)$ belongs to $\mathsf{MFAlg}(\mathrm{Ax})$;*

ii) The class $Con^{Frm}_{<\omega}(\mathrm{Ax})$ of finite conservative modal one-step frames validating Ax *has the extension property iff for all $(\mathfrak{F}_1, \mathfrak{F}_0, f)$ belonging to $\mathsf{MFFrm}(\mathrm{Ax})$ there exists a Kripke frame $\mathfrak{F}_2$ and a relation-preserving surjection $g\colon W_2 \to W_1$ which is a p-morphism relative to f such that $(\mathfrak{F}_2, \mathfrak{F}_1, g)$ belongs to $\mathsf{MFAlg}(\mathrm{Ax})$.*

We say that $\mathsf{MFAlg}(\mathrm{Ax})$ (respectively, $\mathsf{MFFrm}(\mathrm{Ax})$) has the extension property if the condition of item i) (respectively, item ii)) of Lemma 6.11 is met. Given this we obtain the following version of [11, Thm. 1].

Theorem 6.12. *Let* Ax *be a reduced modal axiom system. Then the following are equivalent*

i) Ax *has the bounded proof property;*

ii) $\mathsf{MFAlg}(\mathrm{Ax})$ *has the extension property;*

iii) $\mathsf{MFFrm}(\mathrm{Ax})$ *has the extension property.*

Similarly we obtain a version of [11, Thm. 2].

Theorem 6.13. *Let* Ax *be a reduced modal axiom system. Then the following are equivalent*

i) Ax *has the bounded proof property and the finite model property;*

ii) For every finite conservative minimal filtration algebra $(\mathfrak{A}_0, \mathfrak{A}_1, i)$ *validating* Ax*, there exits a finite modal algebra* $\mathfrak{A}_2$ *validating* Ax *and a continuous embedding* $j\colon A_1 \to A_2$ *of Boolean algebras such that*

$$j(\Diamond_1 i(a)) = \Diamond_2 j(i(a));$$

iii) For every finite conservative minimal filtration frame $(\mathfrak{F}_1, \mathfrak{F}_0, f)$ *validating* Ax *there exists a finite Kripke frame* $\mathfrak{F}_2$ *validating* Ax *together with a relation-preserving surjection* $g\colon W_2 \to W_1$ *such that* g *is a p-morphism relative to* f*.*

Of course it is also possible to characterize the bounded proof property in terms of partial modal algebras and q-frames in a similar way.

6.5 Minimal filtration frames for S4

An important remark to make is that if $\mathscr{A} = (\mathfrak{A}_0, \mathfrak{A}_1, i)$ is a minimal filtration algebra validating a reduced axiom system Ax then this does not necessarily imply that the algebras $\mathfrak{A}_0$ and $\mathfrak{A}_1$ validate Ax separately. A similar remark applies to minimal filtration frames.

In this sections we illustrate this by an example. We look at the categories $\mathsf{MFFrm}(\mathrm{Ax})$ and $\mathsf{MFAlg}(\mathrm{Ax})$ when Ax is a reduced axiom systems for the modal logic **S4**.

The logic **T** is the normal extension of basic modal logic **K** obtain by adding the axiom $p \to \Diamond p$ to a standard Hilbert-style presentation of **K**. As this axiom is already reduced we may consider it a rule

$$\frac{}{p \to \Diamond p}\,(r_T)$$

with no premisses. From [11] it is known that a finite conservative modal one-step frame (W_1, W_0, f, R) validates the rule (r_T) precisely when $wRf(w)$ for all $w \in W_1$. It follows that a finite conservative minimal filtration frame $(\mathfrak{F}_1, \mathfrak{F}_0, f)$ validates precisely when

$$\forall w \in W_1 \; \exists w' \in W_1 \; (wR_1w' \text{ and } f(w') = f(w))$$

And since wR_1w' and $f(w') = f(w)$ implies wR_1w by item iii) of the definition of minimal filtration frames we may conclude that $(\mathfrak{F}_1, \mathfrak{F}_0, f)$ validates the rule (r_T) precisely when R_1 is reflexive.

Finally, since reflexivity is preserved under take minimal filtrations we obtain that the finite conservative minimal filtrations frames validating (r_T) are precisely those consisting of two reflexive Kripke frames.

The logic **K4** is the normal extension of **K** obtained by adding the axiom $\Diamond\Diamond p \to \Diamond p$. This axiom is equivalent to the rule

$$\frac{x \to \Diamond y}{\Diamond x \to \Diamond y} \; (r_4)$$

From [11] we know that a finite conservative modal one-step frame (W_1, W_0, f, R) validates this rule precisely when

$$\forall w \in W_1 \forall v \in W_0 \; (wRv \implies \exists w' \in W_1 \; (f(w') = v \text{ and } R[w'] \subseteq R[w])).$$

We call relations satisfying this *step-transitive*.

Proposition 6.14. *Let $\mathcal{S} = (W_1, W_0, f, R)$ be a conservative modal one-step frame. Then $\mathcal{S}$ validates the rules (r_T) and (r_4) iff the corresponding minimal filtration frame $\mathcal{M} = (\mathfrak{F}_1, \mathfrak{F}_0, f)$ satisfies*

$$\forall w, w' \in W_1 \; (wR_1w' \iff \exists w'' \in W_1 \; (f(w') = f(w'') \textit{ and } R_1[w''] \subseteq R_1[w])).$$

Proof. It is easy to see that $\mathcal{S}$ validates (r_4) iff $\mathcal{M}$ satisfies

$$\forall w, w' \in W_1 \; (wR_1w' \implies \exists w'' \in W_1 \; (f(w') = f(w'') \text{ and } f(R_1[w'']) \subseteq f(R_1[w])).$$

Moreover we see that $\mathcal{S}$ validates (r_4) and (r_T) iff

$$\forall w, w' \in W_1 \; (wR_1w' \iff \exists w'' \in W_1 \; (f(w') = f(w'') \text{ and } f(R_1[w'']) \subseteq f(R_1[w])).$$

Thus to establish the proposition it suffices to show that for all $w, w' \in W_1$ we have

$$f(R_1[w']) \subseteq f(R_1[w])) \iff R_1[w'] \subseteq R_1[w].$$

The implication from right to left is immediate. For the implication from left to right let $w, w' \in W_1$ be given and assume that $f(R_1[w']) \subseteq f(R_1[w]))$. Then if $w'R_1w''$ it follows that $f(w'') \in f(R_1[w]))$ whence for some $w''' \in W_1$ such that wR_1w''' we have that $f(w'') = f(w''')$. But then by item iii) of the definition of a minimal filtration frame, we must have that $w'' \in R_1[w]$. We may thus conclude that $R_1[w'] \subseteq R_1[w]$, as desired. $\square$

Compare this with the definition of a *q-qoset* from [30], where a q-qoset is defined to be a q-frame $(X, \sim, R)$ such that

i) The relation R is reflexive:

ii) $\forall x, y \in X \; (xRy \implies \exists y' \in X \; (y \sim y' \text{ and } R[y'] \subseteq R[x])$.

These are the duals of the finite partial **S4**-algebras which in [30] are defined as partial modal algebras $(A, \Diamond^A)$ satisfying

$$a \leq \Diamond^A a \quad \text{and} \quad a \leq \Diamond^A a' \implies \Diamond^A a \leq \Diamond^A a',$$

for all $a, a' \in \mathrm{dom}(\Diamond^A)$.

Summary of Chapter 6: In this chapter we have tried to align the definition of a one-step modal algebra with that of a one-step Heyting algebra by showing that the category of finite conservative modal one-step frames is isomorphic to the category of finite conservative minimal filtration frames. Moreover, we have shown that the duals of minimal filtration frames are algebraic filtrations in the sense of [37]. Finally, we have shown how to use the results of [11] to characterize the bounded proof property for modal axiom systems in terms of minimal filtration frames and algebras.

Chapter 7

Conclusion and future work

In this chapter we briefly summarize the main results of the thesis and outline a few directions for future work.

7.1 Conclusion

In this thesis we have shown how to extend the one-step framework of [11] to the intuitionistic setting. We have proven that a hypersequent calculus has the bounded proof property iff the class of finite conservative one-step algebras/frames validating it has the extension property. Thus showing that the theory of modal one-step algebras and frames can successfully be transferred to the realm of intuitionistic logic.

We have also considered a fair number of examples of hypersequent calculi having and lacking the bounded proof property. Most notably among these examples is the hypersequent calculus for the logic $\mathbf{BD}_3$ which was shown to have the bounded proof property. The hypersequent calculus for $\mathbf{BD}_3$ presented here is, as far as we are aware, new. In fact, no hypersequent calculus for $\mathbf{BD}_3$ seems to exist in the literature.

Finally, we have made explicit the connection between filtrations and modal one-step algebras by showing that the category $\mathsf{MOSFrm}^{\mathrm{cons}}_{<\omega}$ of finite conservative modal one-step frames is isomorphic to the category $\mathsf{MFFrm}^{cons}_{<\omega}$ of finite conservative minimal filtration frames. That the one-step framework was somehow related to filtrations will probably not come as a surprise to the experts in the field, however to the best of our knowledge this connection has not been made precise anywhere before.

7.2 Future Work

We here list questions which we think would be interesting to pursue in light of the aforementioned results.

- *Generalization of the one-step framework:* The fact that the one-step approach relatively easily transfers to the setting of intuitionistic logic might indicate that this method can be extended to other non-classical logics. It would therefore be interesting to investigate whether or not this is indeed the case, e.g. by trying to establish similar results for other logics such as coalgebraic modal logics or certain substrutural logics. We conjecture that the one-step framework can be extended to characterize a bounded proof property of any algebraizable logic L as long as the connective c to be bound is such that the c-free reduct of the logic is locally tabular and that the finite algebras of the c-free reduct carry the structure of an L-algebra. Even though this situation does not obtain for Full Lambek Calculus we believe that stronger substructural logics such as the logics of k-potent residuated lattices might be amenable to the one-step framework. In particular, the variety of k-potent residuated lattices has the finite embeddability property [16], which is similar to the one-step extension property and so it might be a good test case for this hypothesis.

- *One-step correspondence:* We have developed a preliminary algorithmic one-step correspondence for intuitionistic hypersequent rules. It would be interesting to isolate a syntactically defined Sahlqvist-like class of rules for which our procedure would always yield a first-order correspondent on the two-sorted language of one-step frames.

- *Climbing the hierarchy:* As we have seen it is possible to find hypersequent rules with the bounded proof property for logics axiomatized by formulas above $\mathcal{P}_3$, in the substructural hierarchy of [23]. We conjecture that the case for $\mathbf{BD}_3$ can be generalized to arbitrary n showing that the bounded proof property is not limited to hypersequent calculi for logics axiomatized by formulas below a certain level of the substructural hierarchy. However, as working out the correspondence becomes exceedingly complicated when moving up the hierarchy, testing this conjecture might require some ingenuity.

- *Cut-elimination:* We have shown that the hypersequent calculus for $\mathbf{BD}_3$ obtained by adding the rule (HBd_3) to HJL' enjoys the bounded proof property. However, it would be interesting to know whether or not this calculus also enjoys cut-elimination.

- *Filtrations:* We still feel that more can be said about the relationship between modal one-step algebras and filtrations. For example, we need to compare the one-step framework with the *closed domain condition* of [7] and with [28] which has a different approach to algebraic filtration. Furthermore, we would also need to compare our approach to that of [48] which studies canonical formulas and the normal form construction due to [32] in terms of minimal filtrations of the canonical model. Finally, we believe that more can be said about the relationship between finitely generated free modal algebras and filtrations. For example is $F_{\mathsf{MA}}(n)$ the colimit, in the category MA_c of modal algebras and continuous maps, of a chain of finite modal algebras

$$\mathfrak{A}_0 \xrightarrow{i_0} \mathfrak{A}_1 \xrightarrow{i_1} \dots \xrightarrow{r_n} \mathfrak{A}_{n+1} \longrightarrow \dots$$

  such that $(\mathfrak{A}_m, \mathfrak{A}_{m+1}, i_m)$ is a minimal filtration algebra for all $m \in \omega$? Answering this question in the affirmative would shed some light on how to generalize the construction of [37] from the variety of **S4**-algebras to arbitrary varieties of modal algebras.

- *Finitely generated free Heyting algebras:* Finally drawing on [13, 30, 39] it would be interesting to see if the framework of one-step Heyting algebras and intuitionistic one-step frames could be used to describe finitely generated free algebras – and their dual Esakia spaces – in subvarieties of HA. Of course from [30] we already know how this works in principle. However, we would like to see whether the duality between finite one-step Heyting algebras and finite intuitionistic one-step frames can be used to obtain an interesting description of the dual space of $F_{\mathsf{V}}(n)$ for concrete subvarieties V of HA. A first test case could be the finitely generated free Gödel-algebras for which a nice description of the dual spaces already exists [2].

Appendix A

Hypersequent calculi for IPC

See also [50, Chap. 4] for an introduction to hypersequent calculi for intuitionistic propositional logic.

A.1 A single-succedent hypersequent calculus for IPC

We here present the rule for a variant of single-succedent hypersequent calculus $HInt$ for **IPC** found in [21].

Each of the rules below is a rule schema and as such represents infinitely many rules. Thus in the following φ and ψ range over formulas, Γ and Δ over finite sets of formulas and G over hypersequents.

Axioms:

$$\frac{}{G \mid \varphi \Rightarrow \varphi}\,(\mathrm{Init}) \qquad \frac{}{G \mid \bot \Rightarrow}\,(\mathrm{L}\bot)$$

External structural rules:

$$\frac{G \mid \Gamma \Rightarrow \varphi \mid \Gamma \Rightarrow \varphi}{G \mid \Gamma \Rightarrow \varphi}\,(\mathrm{EC}) \qquad \frac{G}{G \mid \Gamma \Rightarrow \varphi}\,(\mathrm{EW})$$

Internal structural rules:

$$\frac{G \mid \Gamma, \psi, \psi \Rightarrow \varphi}{G \mid \Gamma, \psi \Rightarrow \varphi}\,(\mathrm{IC}) \qquad \frac{G \mid \Gamma \Rightarrow \varphi}{G \mid \Gamma, \psi \Rightarrow \Delta}\,(\mathrm{LIW}) \qquad \frac{G \mid \Gamma \Rightarrow}{G \mid \Gamma \Rightarrow \varphi}\,(\mathrm{RIW})$$

Logical rules:

$$\frac{G \mid \Gamma \Rightarrow \varphi \qquad G' \mid \Gamma, \psi \Rightarrow \chi}{G' \mid G' \mid \Gamma, \varphi \to \psi \Rightarrow \chi} \text{L}\to \qquad \frac{G \mid \Gamma, \psi \Rightarrow \varphi}{G \mid \Gamma \Rightarrow \psi \to \varphi} \text{R}\to$$

$$\frac{G \mid \Gamma, \psi_i \Rightarrow \varphi}{G \mid \Gamma, \psi_1 \wedge \psi_2 \Rightarrow \varphi} \text{L}\wedge_i \text{ for } i \in \{1,2\} \qquad \frac{G \mid \Gamma \Rightarrow \varphi \qquad G' \mid \Gamma \Rightarrow \chi}{G \mid G' \mid \Gamma \Rightarrow \varphi \wedge \chi} \text{R}\wedge$$

$$\frac{G \mid \Gamma, \psi_1 \Rightarrow \varphi \qquad G' \mid \Gamma, \psi_2 \Rightarrow \varphi}{G \mid G' \mid \Gamma, \psi_1 \vee \psi_2 \Rightarrow \varphi} \text{L}\vee \qquad \frac{G \mid \Gamma \Rightarrow \psi_i}{G \mid \Gamma \Rightarrow \psi_1 \vee \psi_2} \text{R}\vee_i \text{ for } i \in \{1,2\}$$

The cut rule:

$$\frac{G \mid \Gamma \Rightarrow \psi \qquad G' \mid \Gamma, \psi \Rightarrow \varphi}{G \mid G' \mid \Gamma \Rightarrow \varphi} \text{(Cut)}$$

A.2 A multi-succedent hypersequent calculus for IPC

We here present the rules for the multi-succedent sequent hypersequent calculus HJL' for **IPC** found in [25].

Each of the rules below is a rule schema and as such represents infinitely many rules. Thus in the following φ and ψ range over formulas, Γ and Δ over finite sets of formulas and G over hypersequents.

Axioms:

$$\frac{}{G \mid \varphi \Rightarrow \varphi} \text{(Init)} \qquad \frac{}{G \mid \bot \Rightarrow} \text{(L}\bot\text{)}$$

External structural rules:

$$\frac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta} \text{(EC)} \qquad \frac{G}{G \mid \Gamma \Rightarrow \Delta} \text{(EW)}$$

Internal structural rules:

$$\frac{G \mid \Gamma \Rightarrow \varphi, \varphi, \Delta}{G \mid \Gamma \Rightarrow \varphi, \Delta} \text{(RIC)} \qquad \frac{G \mid \Gamma, \varphi, \varphi \Rightarrow \Delta}{G \mid \Gamma, \varphi \Rightarrow \Delta} \text{(LIC)}$$

$$\frac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma, \varphi \Rightarrow \Delta} \text{(LIW)} \qquad \frac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \varphi, \Delta} \text{(RIW)}$$

Logical rules:

$$\frac{G \mid \Gamma \Rightarrow \varphi, \Delta \qquad G \mid \Gamma, \psi \Rightarrow \Delta}{G \mid \Gamma, \varphi \to \psi \Rightarrow \Delta} \text{L}\to \qquad \frac{G \mid \Gamma, \varphi \Rightarrow \psi}{G \mid \Gamma \Rightarrow \varphi \to \psi} \text{R}\to$$

$$\frac{G \mid \Gamma, \varphi, \psi \Rightarrow \Delta}{G \mid \Gamma, \varphi \wedge \psi \Rightarrow \Delta}\,\mathrm{L}\wedge \qquad \frac{G \mid \Gamma \Rightarrow \varphi, \Delta \qquad G \mid \Gamma \Rightarrow \psi \Delta}{G \mid \Gamma \Rightarrow \varphi \wedge \psi, \Delta}\,\mathrm{R}\wedge$$

$$\frac{G \mid \Gamma, \varphi \Rightarrow \Delta \qquad G \mid \Gamma, \psi \Rightarrow \Delta}{G \mid \Gamma, \varphi \vee \psi \Rightarrow \Delta}\,\mathrm{L}\vee \qquad \frac{G \mid \Gamma \Rightarrow \varphi, \psi, \Delta}{G \mid \Gamma \Rightarrow \varphi \vee \psi, \Delta}\,\mathrm{R}\vee$$

The cut rule:

$$\frac{G \mid \Gamma \Rightarrow \varphi, \Delta \qquad G' \mid \varphi, \Sigma \Rightarrow \Pi}{G \mid G' \mid \Gamma, \Sigma \Rightarrow \Pi, \Delta}\,(\mathrm{Cut})$$

A.3 Hyperproofs

In this section we define for each hypersequent calculus HC the derivability relation $\vdash_{HC}$ between a set of hypersequents and hypersequents.

Definition A.1. Let $S, S_1, \ldots, S_n$ be a set of hypersequents and let

$$\frac{S'_1, \ldots, S'_n}{S'}\,(r)$$

be a hypersequent rule. We say that S is *obtained from* $S_1, \ldots, S_n$ *by an application of the rule* (r), if there exists a hypersequent G such that S is $G \mid S'$ and S_i is $G \mid S'_i$ for $i \in \{1, \ldots, n\}$.

If $\mathscr{S} \cup \{S\}$ is a set of hypersequents and HC is a hypersequent calculus we say that S is *derivable* (or *provable*) *from* $\mathscr{S}$ *over* HC, written $\mathscr{S} \vdash_{HC} S$, if there exists a finite sequence of hypersequents $S_1, \ldots S_n$ such that S_n is the hypersequent S and for all $1 \leq k < n$ either S_k belongs to $\mathscr{S}_k$ or S_k is obtained by applying a rule from HC to some subset of $\{S_1, \ldots, S_{k-1}\}$.

Note that it is not allowed to apply substitutions to hypersequents in $\mathscr{S}$. Thus $\vdash_{HC}$ denotes the global consequence relation, in the sense that the members of $\mathscr{S}$ will be taken as axioms, i.e. leaves in a derivation tree.

Note also that because we apply rules in their contextual form the use of dummy contexts is strictly speaking unnecessary when presenting rules as in A.1 and A.2. However, we have chosen to keep them in order to remind the reader of this convention and to adhere to the common way of presenting hypersequent calculi.

The following proposition can be established by an easy inductive arguments.

Proposition A.2. *If HC is either $HInt$ or HJL' then*

$$\vdash_{HC} \Gamma_1 \Rightarrow \varphi_1 \mid \ldots \mid \Gamma_n \Rightarrow \varphi_n$$

implies that $\vdash_{HC} \Gamma_k \Rightarrow \varphi_k$ *for some* $k \in \{1, \ldots, n\}$.

From this is it easy to verify that both $HInt$ and HJL' are sound and complete with respect to **IPC**.

Appendix B

A basic calculus for one-step correspondence

In what follows we will make use of the typing convention that variables $\mathbf{i}$ and $\mathbf{j}$ range over nomimals and variables $\mathbf{m}$ and $\mathbf{n}$ range over co-nomimals.

First approximation rule:

$$\frac{\varphi \leq \psi}{\forall \mathbf{i} \forall \mathbf{m}\ ((\mathbf{i} \leq \varphi \text{ and } \psi \leq \mathbf{m}) \implies \mathbf{i} \leq \mathbf{m})}\ (\mathrm{FA})$$

Approximation rules for implication:

$$\frac{\varphi \to \psi \leq \mathbf{m}}{\exists \mathbf{j}\ (\mathbf{j} \to \psi \leq \mathbf{m} \text{ and } \mathbf{j} \leq \varphi)}\ (\mathrm{LA}_{\to}) \qquad \frac{\varphi \to \psi \leq \mathbf{m}}{\exists \mathbf{n}\ (\varphi \to \mathbf{n} \leq \mathbf{m} \text{ and } \psi \leq \mathbf{n})}\ (\mathrm{RA}_{\to})$$

Approximation rules for I*:*

$$\frac{\mathsf{I}(\varphi, \psi) \leq \mathbf{m}}{\exists \mathbf{j}\ (\mathsf{I}(\mathbf{j}, \psi) \leq \mathbf{m} \text{ and } \mathbf{j} \leq \varphi)}\ (\mathrm{LA}_{\mathsf{I}}) \qquad \frac{\mathsf{I}(\varphi, \psi) \leq \mathbf{m}}{\exists \mathbf{n}\ (\mathsf{I}(\varphi, \mathbf{n}) \leq \mathbf{m} \text{ and } \psi \leq \mathbf{n})}\ (\mathrm{RA}_{\mathsf{I}})$$

The approximation rules are subject to the side-condition that the variables for nominals and co-nominals which are quantified over in the conclusion are fresh, i.e. that they do not occur in the formula φ and ψ.

Adjunction and residuation rules:

$$\frac{\varphi_1 \wedge \varphi_2 \leq \psi}{\varphi_2 \leq \varphi_1 \to \psi}\ (\mathrm{LR}_{\wedge}) \qquad \frac{\psi \leq \varphi_1 \to \varphi_2}{\psi \wedge \varphi_1 \leq \varphi_2}\ (\mathrm{RR}_{\to})$$

$$\frac{\varphi_1 \vee \varphi_2 \leq \psi}{\varphi_1 \leq \psi \text{ and } \varphi_2 \leq \psi}\,(\mathrm{LA}_\vee) \qquad \frac{\psi \leq \varphi_1 \wedge \varphi_2}{\psi \leq \varphi_1 \text{ and } \psi \leq \varphi_2}\,(\mathrm{RA}_\wedge)$$

$$\frac{\varphi \leq \mathsf{l}(\psi,\chi)}{\mathsf{k}(\varphi,\psi) \leq \chi}\,(\mathrm{RR}_\mathsf{l}) \qquad \frac{\mathsf{k}(\varphi,\psi) \leq \chi}{\varphi \leq \mathsf{l}(\psi,\chi)}\,(\mathrm{LR}_\mathsf{k})$$

$$\frac{\varphi \leq i(\psi)}{i^\flat(\varphi) \leq \psi}\,(\mathrm{LA}_i) \qquad \frac{i(\varphi) \leq \psi}{\varphi \leq i_!(\psi)}\,(\mathrm{RA}_i)$$

If one also wants the right residuation rules for $\vee$ it is necesarry to introduce the Heyting co-implication. However, as we are working with finite distributive lattices such a co-implication always exists. But, as it is not necessary for any of the examples we leave it out.

Rules for & and $\oplus$:

$$\frac{\varphi_1 \oplus \varphi_2 \leq \psi}{\varphi_1 \leq \psi \text{ and } \varphi_2 \leq \psi}\,(\mathrm{LA}_\oplus) \qquad \frac{\psi \leq \varphi_1 \mathbin{\&} \varphi_2}{\psi \leq \varphi_1 \text{ and } \psi \leq \varphi_2}\,(\mathrm{RA}_\&)$$

Where the rules $(\mathrm{LA}_\oplus)$ and $(\mathrm{RA}_\&)$ are subject to the side condition that φ_1 and φ_2 are truth-values, i.e. terms of the form $\mathsf{l}(\chi_1,\chi_2)$ or $\bot,\top$.

Elimination rules for l *and* k*:*

$$\frac{\mathsf{l}(\varphi,\psi) \leq \mathbf{m}}{\varphi \nleq \psi}\,(\mathrm{LE}_\mathsf{l}) \qquad \frac{\mathsf{k}(\mathbf{i},\varphi) \leq \psi}{\varphi \leq \psi}\,(\mathrm{LE}_\mathsf{k}) \qquad \frac{\top \leq \mathsf{l}(\varphi,\psi)}{\varphi \leq \psi}\,\mathrm{RE}_\mathsf{l}$$

Ackermann rules:

$$\frac{(\varphi_1(x) \leq \psi_1(x) \text{ and } \ldots \text{ and } \varphi_n(x) \leq \psi_n(x) \text{ and } \alpha \leq x) \implies \psi(x) \leq \varphi(x)}{(\varphi_1(\alpha/x) \leq \psi_1(\alpha/x) \text{ and } \ldots \text{ and } \varphi_n(\alpha/x) \leq \psi_n(\alpha/x)) \implies \psi(\alpha/x) \leq \varphi(\alpha/x)}\,(\mathrm{LAck})$$

and

$$\frac{(\varphi_1(x) \leq \psi_1(x) \text{ and } \ldots \text{ and } \varphi_n(x) \leq \psi_n(x) \text{ and } x \leq \alpha) \implies \psi(x) \leq \varphi(x)}{(\varphi_1(\alpha/x) \leq \psi_1(\alpha/x) \text{ and } \ldots \text{ and } \varphi_n(\alpha/x) \leq \psi_n(\alpha/x)) \implies \psi(\alpha/x) \leq \varphi(\alpha/x)}\,(\mathrm{LAck})$$

The left Ackermann rule (LAck) is subject to the side-condition that x does not occur in α and that x occurs positively[1] in all φ_i and in φ and that x occurs negatively in all ψ_i and in ψ. Similarly the right Ackermann rule (RAck) is subject to the side-condition

[1]Recall that the polarity (positive or negative) of an occurrence of a subformula in a formula φ is defined by the following recursion: All propositional letters and constants occurs positively in φ and all the connectives preserve the polarity with the exception of $\to$ and l which reverses it in the first coordinate.

that x does not occur in α and that x occurs negatively in all φ_i and in φ and that x occurs positively in all ψ_i and in ψ.

For a proof of the soundness of the Ackermann rules see e.g. [27, Lem. 1].

Note that unlike all the other rules the two Ackermann rules only apply globally to the entire quasi-equation in the language $\mathcal{L}^{++}_{Alg}$.

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