Investigations into Linear Logic with Fixed-Point Operators

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

Linear logic [56] is a substructural logic [86, 87] that refines both classical and intuitionistic logic. In fact, linear logic is characterized by several dualities (which derive from the presence of a de Morgan negation), but at the same time has a strong constructive flavor. From a proof-theoretical perspective, classical (resp. intuitionistic) linear logic is obtained from classical (risp. intuitionistic) sequent calculus [55, 106] by dropping the structural rules of weakening and contraction [55, 106]. This makes the use of hypothesis in a proof *linear*, in the sense that each hypothesis must be used exactly once. Linear logic has two modalities, ! and ?, called exponential modalities, that allow to restore weakening and contraction in a controlled form. Having these modalities, both intuitionistic and classical logic can be encoded into linear logic.

Despite being interested *per se*, linear logic has several applications. In fact, linearity of hypothesis allows to look at formulas as resources or pieces of information, that cannot be neither freely duplicated nor deleted. Moreover, the absence of weakening and contraction leads to a finer distinction between classical (risp. intuitionistic) connectives, thus obtaining a new stock of connectives which capture in a natural way several operations between computational processes [7, 79].

Categorical Quantum Mechanics [6, 39] studies quantum processes as special computational processes. The underlying mathematical framework is given by (enrichments of) monoidal categories [72]. One of the main feature of monoidal categories is that the notion of categorical product [10, 11, 16, 72] is replaced with the weaker notion of tensor product. Tensor products allow to describe a rudimentary form of parallel composition and thus make monoidal categories suitable for an abstract description of physical and computational processes. It is well known [8, 15, 30, 77] that the underlying logic of monoidal categories is the multiplicative tensorial fragment of intuitionistic linear logic, so that the latter can be thought of as the logic describing the abstract structure of quantum processes.

For these reasons, it is useful to have a framework that allows to study and define processes (both physical and computational) that are characterized by infinite and iterative behaviors. This thesis deals with extensions of (specific enrichments of) monoidal categories with initial algebras and final coalgebras for a class of functors generalizing polynomial functors [68] over the monoidal signature, as well as their underlying logics. The latter are nothing but (fragments of) linear logic extended with least and greatest fixed point operators. Categories are mostly defined and studied equationally, according to Lambek's methodology [71]. This allows to easily design syntactical systems for such categories, which can then be made into

logical systems. We provide sequent calculi for all the logics investigated, and a deep inference [63] system for the extension of classical linear logic with least and greatest fixed point operators. We define exponential, relevant and affine modalities [86, 87, 109] as least and greatest fixed point of specific functors. This leads to a finer analysis of such modalities and their proof-theoretical properties, as well as their relationship.

Finally, some possible applications of the logics investigated are sketched, in particular in the direction of modal (especially epistemic) logics over a linear base.

Contents

Introduction

The aim of this thesis is to investigate extensions of (propositional) linear logic with fixed point operators, moving from basics reflections concerning its relationship with Category Theory [10, 11, 16, 72, 77]. The subject is vast and can, at least in principle, be approached from different perspectives (proof theory, type theory, category theory, game semantics and many others). The approach followed in this thesis is halfway between category theory and proof theory, and follows Lambek's methodology of categorical proof theory [71]. Nevertheless, the motivation that led the author to investigate this subject comes from Quantum Information Theory [83], and more exactly from the interplay between logic and the field of *Categorical Quantum Mechanics* [6, 39]. Categorical quantum mechanics employs monoidal categories [72] to describe the structure and the dynamics of quantum information, trough the concepts of computational and physical processes (see next section for informal details). Following the so-called *Curry-Howard-*

Lambek correspondence[1] we recognize linear logic[2] as the logic describing the abstract structure of quantum information. Adding structure to quantum information leads to enrichments of the monoidal framework, with consequent extensions of linear logic. In this thesis we investigate extensions of the basic categorical framework used in categorical quantum mechanics in order to be able to deal with processes (computational or physical) with controlled forms of infinite and iterative behavior, and their corresponding logics. Category theory provides a rich formal apparatus to study iterative and infinite computational phenomena, via the notions of algebra and coalgebra [68, 88]. As a consequence, we extend monoidal categories (and their variants) with initial algebras and final coalgebras for a specific class of endofunctors (which are de facto polynomial functors [68]). The resulting classes of categories are simple yet powerful, and allow to describe interesting systems and processes. Following Lambek's methodology, categories are viewed as deductive graphs (see next chapter) equipped with an equational theory for arrows. This approach provides syntax-oriented and equational definitions of categories, and leads to easily design logical systems for such categories. We will then obtain extensions of (fragments of) propositional linear logics with least and great-

[1]There is no agreement among researchers concerning the name of such correspondence. The original name was *Curry-Howard isomorphism*, since the correspondence between the natural deduction system for the implicational fragment of intuitionistic logic (defined as in e.g. [53, 55, 56, 84, 106]) and the simply typed λ-calculus (typed à la Church) [35, 56, 65, 96, 106] was observed by Howard [66] and was first recognized by Curry [41] in terms of Hilbert's systems and combinatory logic. Indeed, for these systems it is possible to define an isomorphism in a formal way. Moving to e.g. simply typed λ-calculus with Curry's typing [65, 96] breaks the isomorphism (although it can be recovered taking suitable equivalence classes of proofs and λ-terms). Nevertheless, there is a moral isomorphism between the two systems. For this reason researchers began to use the more informal term 'correspondence' in place of 'isomorphism'. Several people worked on such correspondence (for example, Martin-Löf introduced his intuitionistic type theory [75]), and the correspondence became de facto a paradigm: logical proofs carry out a computational content, viceversa programs are nothing but encodings of logical proofs. This led to call the Curry-Howard correspondence *Propositions-as-Types, Proofs-as-Programs correspondence*. In the same years Lambek [71] showed that the correspondence between intuitionistic proofs and λ-terms could be extended to arrows in cartesian closed categories [10, 11, 30, 71, 84, 106], so that people started to speak of the *Curry-Howard-Lambek correspondence*. Some researchers use the terminology *Propositions-as-Types-as-Objects, Proofs-as-Programs-as-Arrows correspondence*, but in general no agreement has been reached concerning such terminology. The reader can consult [19, 34, 65] for an historical account of this subject.

[2]Multiplicative tensorial linear logic [77, 78, 105] (i.e. the $\otimes$-fragment of linear logic), to be precise.

est fixed point operators. These logics are simple but extremely powerful. As we will see in Chapter 3, using fixed point operators we can recover (full) exponential modalities as well as relevant and affine ones [67, 76, 78, 109]. The analysis of exponential modalities as fixed points of specific functors reveals new aspects of their nature, concerning e.g. non-canonicity and some controversial aspects of their proof theory. Moreover, the algebraic-coalgebraic framework allows to recover a simple decomposition theorem in the spirit of [67] relating (full) exponential, affine and relevant modalities.

The analysis of these extensions of propositional linear logic raises several questions and at the same time opens the doors to new applications. In the last chapter some possible applications are sketched, in particular concerning non-categorical semantics and modal (especially epistemic) extensions of such logics.

Contributions and Summary of the Work

Contributions of this work are:

- The explicit design of a categorical framework, according to Lambek's methodology, of monoidal (and their extensions) categories extended with specific classes of initial algebras and final coalgebras. Although extensions of linear logic with fixed point operators seem to be folklore in the type theory community (via the notion of recursive and co-recursive linear types), the author was not able to find a formal exposition of the subject. The paper [14] investigates a higher-order linear logic (requiring typed variables, λ-abstractions and quantifiers) with fixed point operators. The approach is entirely syntactical and no semantics for the logic is proposed. Moreover, such syntactical and higher order approach hides several results concerning exponential modalities (which are in fact missing in that paper). Introducing the logic moving from its categorical counterpart seems to be much easier and more informative than other syntactical approaches and, to the best of the author's knowledge, entirely new. Moreover, such 'categorical' approach allows to deal with both intuitionistic and classical versions of linear logic, simply by changing the underlying base category.

- A finer proof-theoretical analysis of exponential modalities. Combining the proof-theoretical and categorical approach to exponential modalities we recover exponential, relevant and affine modalities as fixed points of specific functors. As a consequence, the extension of propositional linear logic (without exponentials) with fixed point operators subsumes full (propositional) linear logic, relevant linear logic and affine linear logic. This new categorical analysis of exponential modalities as fixed points of specific functors allows to see some correspondences between solutions to equations induced by such functors, and formulas satisfying specific sequent calculus rules. These correspondences shed new light on the nature of some sequent calculus modal rules and give semantics to possibly new exponential modalities. Finally, it is possible to formulate a decomposition theorem in the spirit of [67] that recovers functors associated with exponential modalities as sorts of compositions of functors associated with relevant and affine modalities.

- A coherent exposition of proof systems both in sequent calculus and deep inference style is given. These systems can be obtained in a straightforward way from the categorical formulation of the logic via the notion of deductive graph.

- In the last chapter a semantics for modal (especially epistemic) extension of linear logic (both with and without fixed point operators) is sketched. Epistemic linear logic has recently received attention due to its applicability to security problems [18, 42]. However, so far the treatment of such logic has been completely syntactical (moreover, although called 'epistemic linear logic' the modalities employed are essentially $S4$ modalities [28, 43], due to problems concerning well behaved sequent calculi for $S5$ modal logics [82, 97]). We sketch a possible semantics for such logic (and other modal extensions of propositional linear logic, both with and without fixed point operators) based on the notion of *pretopology* [89, 90]. Such semantics is introduced as a possible generalization of Aumann's structures (and that can be easily modified to give semantics to distributive epistemic linear logics). Finally, an explicit formulation of a deep inference system for epistemic linear logic is given.

The work is divided in four chapters starting from preliminaries about category theory and their relationship with (linear) logic (proof theory, actually), proceeding to the design of categories and categorical proof systems for dealing with fixed point operators, and ending with the study of syntactical systems for the logic obtained. More precisely, the work is divided as follows:

Introduction The rest of the introduction introduces monoidal categories and linear logic on an intuitive and informal level, focusing on motivations and applications. A brief section on categorical quantum mechanics gives a concrete example motivating the study of linear logic and monoidal categories.

Chapter 1. Chapter 1 gives the reader all the necessary background to read this thesis. Basic categorical notions are covered, recalling in particular the definitions of algebra and coalgebra. The approach followed is based on Lambek's notion of deductive graphs (see Chapter 1), and allows to give equational definitions of several categorical notions. Cartesian and monoidal categories are introduced, as well as their corresponding logics.

Chapter 2. In Chapter 2 we introduce ν-symmetric monoidal cartesian categories (νSMCCs for shorts). These are symmetric monoidal cartesian categories which have initial algebras and final coalgebras for the so-called polynomial functors. The latter are de facto functors built over the monoidal-cartesian signature. The underlying logic (called νLL) is a fragment of propositional liner logic (the $(\otimes, \&)$-fragment) enriched with least and greatest fixed point operators. Proof systems in Lambek's style are defined, and the equational theory associated with νSMCCs provides a notion of equality for proofs.

Chapter 3. In Chapter 3 a sequent calculus for νLL is defined. This system is equivalent to Lambek's style calculi given in Chapter 2, so that it is sound and complete with respect to the class of of νSMCCs. The logic νLL is enough powerful to encode the exponential modality !. In fact, $!A$ can be recovered as

$$\nu X.1 \,\&\, A \,\&\, (X \otimes X).$$

Other weaker *structural modalites* can be recovered, namely relevant and affine modalities (see Chapter 3 for references and definitions). As a consequence, νLL constitutes a powerful framework subsuming full linear, relevant linear and affine linear logic. These logics can then be studied and compared in a unique setting.

Finally, a categorical-proof theoretical analysis of exponential modalities is given. A correspondence between specific proof-theoretical properties of the exponential modality ! and properties of its associated defining functors is proved. These results generalize to weaker modalities and shed light on some specific unsatisfactory aspects of the 'standard' exponential !. The relationship between exponential, relevant and affine modalities is made formal via a decomposition theorem.

Chapter 4. Chapter 4 extends the calculus designed in Chapter 3 to full classical linear logic. This allows to exploit the duality between least and greatest fixed point operators. A sequent calculus (both one- and two-sided) for classical linear logic with fixed point operators is given. This easily leads to the design of a deep inference calculus.

Applications, Further Works and Conclusions. This chapter sketches some possible applications of the framework defined in previous chapters. These focus on the task of finding natural non-algebraic/categorical semantics for the logic investigated. These semantics should then be used to study epistemic and doxastic extension of linear logic, both with and without fixed point operators. In particular, a semantics based on the notion of *pretopology* (see the chapter for definitions and references) is proposed, arguing how pretopologies can be viewed as a possible generalization of Aumann's structures.

Finally, a list of open problems and enrichments that the author is aimed to investigate in future works is given.

Informal Introduction to the Subject

In this section we briefly (and informally) introduce monoidal categories and linear logic. First we recall some basic aspects concerning categorical quantum mechanics, especially regarding its methodology and goals. This justifies the choice of monoidal categories as basic mathematical framework, and of linear logic as basic logical system. We focus on informal ideas and intuitions, rather than on formal definitions and results (for which references are given).

Categorical Quantum Mechanics and Linear Logic

Categorical Quantum Mechanics (CQM) [6, 39] is a subfield of Quantum Information Theory and Quantum Computing (see [83] for a comprehensive introduction) that studies the abstract structure of quantum information. Primitive objects of CQM are physical systems and their transformations, physical processes. These are computational in nature, since they manipulate (quantum) information. A central notion is the one of interaction between systems, which produces non-local correlations (see [39]). Such interaction is described via *compounds systems*, and in order to formalize this latter notion, CQM takes definitions and ideas from computer science, specifically from concurrency theory [7, 79].

The standard formalism used in quantum mechanics is the one of *Hilbert spaces* (see [83]), although already in [27] Birkhoff and von Neumann introduced quantum logic as a more general foundation for quantum physics. Such formalism (and its variants) was not able to replace Hilbert spaces, since it does not take into account phenomena like *quantum entanglement*, which, as quantum information theory shows, can be explained as a form of interaction in compound systems [39].

The primary importance of compound systems and their interaction suggests to look at physical systems and processes as special computational systems and processes. Compound systems can then be described by means of the notion of parallel composition [7, 79, 91]. As already mentioned, computational phenomena are deeply connected to logical (and categorical) phenomena through the Curry-Howard correspondence. It is then natural to look at the underlying logic of physical systems and processes. Such logic turns out to be *linear logic* [56] (see next section for an informal introduction). A central feature of linear logic is the absence of the structural rules of weakening and contraction. The absence of these structural rules corresponds to the so-called *no-deleting* and *no-cloning* theorems [83], so that linear logic is a better candidate logic to describe the structure of quantum information than Birkhoof's and von Neumann's quantum logic.

From a mathematical perspective, linear logic can roughly be said to be the underlying logic of monoidal categories[3] [10, 72], so that the latter can

[3]Linear logic has a richer structure than the one given by monoidal categories (which, technically, correspond to the tensorial fragment of linear logic). Nevertheless, the tensorial fragment of linear logic (which gives a logical counterpart to the notion of parallel composition) is fundamental to linear logic, much in the same way as the implicational

be recognized as the basic framework to describe physical/computational systems and processes.

The above description can be 'reversed', in the following sense. As the next sections show, monoidal categories can be recognized to be a simple and powerful framework capable of describing physical systems and processes, as well as their interactions, according to the following desiderata:

1. The framework has to deal with the abstract notions of system and process in a resource-sensitive way. This means that we can think of systems as resources, and of processes as actions (or production rules) consuming resources to produce new ones. Thinking of systems as resources implies that systems cannot be neither duplicated nor deleted (see next section for intuitive examples). This is in line with the no-deleting and no-cloning theorems.

2. The framework has to provide an implicit notion of time, defined by means of sequential composition of processes.

3. The framework has to provide a notion of interaction, obtained via the possibility of forming compound systems/resources. We want to be able to run processes on compound systems as well as on specific components of compound systems. That is, we want a notion of parallel composition for processes (for a description of the informal desiderata that a good notion of parallel composition should satisfy, the reader can consult [7, 79, 91]).

As argued in next section, monoidal categories are a simple and elegant mathematical framework satisfying all these desiderata. Since the underlying logic of monoidal categories is essentially linear logic, it is then possible to recognize linear logic as the logic describing the structure of physical systems and processes.

The reader can consult [6, 39] for an overview of CQM, [37, 38] for an introduction to the categorical apparatus used in such discipline, and [36, 44] for a more logical-type theoretical overview of the subject. An excellent introduction to the interplay between physics, logic, topology and computer science is [15].

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fragment of intuitionistic (propositional) logic is a fundamental component of the logic (in fact, one usually refers to the connection between cartesian closed categories [10, 11, 16, 71] and intuitionistic (propositional logic), although the latter carries out a richer categorical structure). See below for details.

Why (Monoidal) Categories

Although usually regarded as a branch of abstract mathematics, (basic) category theory [10, 11, 16, 72] has a natural and useful 'operational' reading, providing a simple yet powerful formalism to deal with notions like systems, resources, formulas ... and their interaction (such as processes, measurements, transformations, proofs ...) on an abstract level. For example, let A, B, C be either physical systems or resources and suppose that system A evolves in system B by means of f, e.g. the measurement f makes A evolving into B, or the process f modifies the system from configuration A to configuration B, or the action (or production rule) f consumes the resource A to produce the resource B. In all these cases, we simply say that f is an arrow from A to B, notation $f : A \to B$, or, pictorially

$$A \xrightarrow{f} B$$

According to the system-processes analogy, we see that given another process $g : B \to C$, a natural requirement is to be able to run f and g sequentially. Such process exists, and is given by the arrow $g \circ f$. Similarly, it is natural to require the existence of a process that does nothing, and leave the system unchanged. This process is given by the identity arrow

$$\mathsf{id}_A : A \to A$$

Most importantly, we should have a notion of equality for processes. For example, it is a legitimate requirement that running a process $f : A \to B$ (on A), after having run the 'null' process id_A is essentially the same as just running f. This is captured by the equation

$$f \circ \mathsf{id}_A = f$$

(similarly we should require $\mathsf{id}_B \circ f = f$). Another desiderata is that sequential composition is associative, i.e. that for $f : A \to B$, $g : B \to C$ and $h : C \to D$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

holds. All these intuitive readings can be abstracted into the general notions of objects and arrows. Requiring identity arrows and the above equations then gives the notion of category.

Adding structure to categories we can then define new objects and new arrows, according to our intuition and the above informal reading. For example, the product of two objects A and B is an object[4] $A \,\&\, B$ together with two arrows (called projections) $p : A \,\&\, B \to A$ and $q : A \,\&\, B \to B$ that satisfy the following universal mapping property (UMP for shorts):

> *For any object C and pair of arrows $f : C \to A$, $g : C \to B$, there exists a unique arrow $\langle f, g\rangle : C \to A \,\&\, B$ such that the following equations hold*

$$\begin{aligned} p \circ \langle f, g\rangle &= f \\ p \circ \langle f, g\rangle &= g \end{aligned}$$

The above properties can be expressed via the following commutative diagram [10, 11, 16, 72]

$$\begin{tikzcd}
& C \arrow[dl, "f"'] \arrow[d, dotted, "\langle f, g\rangle"] \arrow[dr, "g"] & \\
A & A \,\&\, B \arrow[l, "p"] \arrow[r, "q"'] & B
\end{tikzcd}$$

where a dotted line denotes uniqueness of the arrow. Universal mapping properties define objects up to isomorphism (see next chapter) so that we can regard the product of two objects to be essentially unique. We can already observe how the notion of product is inadequate to capture some forms of interaction. First of all, note that we can construct an arrow

$$\Delta_A : A \to A \,\&\, A$$

called duplicator, simply by defining $\Delta_A = \langle \mathsf{id}_A, \mathsf{id}_A\rangle$. As a consequence, if we think of objects as physical or computational resources, we cannot think of $A \,\&\, B$ as the resource obtained combing A and B. In fact, if that would be the case, then resources would be duplicable, which is not realistic as the following example shows.

[4]The standard notation for product is $\times$. However, in order to avoid notational confusion, we use from the very beginning of this thesis the notation used in linear logic literature e.g. [54, 56, 61, 77].

Example (Beverage machine)**.** Consider a rudimentary beverage machine with the following actions (production rules):

$$\begin{aligned} \mathsf{get\_coffee} &: 1\$ \to \mathsf{coffee} \\ \mathsf{get\_tea} &: 1\$ \to \mathsf{tea} \end{aligned}$$

that allows to obtain a coffee, paying one dollar, or a tea, again paying one dollar. The universal mapping property of product then gives the action

$$\langle \mathsf{get\_coffee}, \mathsf{get\_tea} \rangle : 1\$ \to \mathsf{coffee} \,\&\, \mathsf{tea}$$

which is clearly unsatisfactory.

Even thinking of $A \,\&\, B$ as a proper interaction between A and B is problematic. If we think of objects as systems and to arrows as processes, it is natural to ask whether we can think of $A \,\&\, B$ as a parallel composition of A and B. The answer is negative, since projections always allow to 'separate' A and B from $A \& B$. This means that there is no proper interaction between A and B, which makes the product inadequate for modeling parallel composition (see e.g. [7, 79, 91] for some desiderata a model of parallel composition should satisfy).

A more satisfactory formalization of operations like parallel compositions is given through the notion of monoidal category [10, 16, 72]. Roughly, a monoidal category comes with a bifunctor $\otimes$ (see next chapter for formal definitions) that captures, among others, the informal idea of parallel composition. Given two objects A, B and two arrows f, g we have a new object $A \otimes B$ and a new arrow $f \otimes g$, which can be pictorially described as follows:

$$\begin{tikzcd} A \arrow[d, "f"'] & \otimes & B \arrow[d, "g"] & = & A \otimes B \arrow[d, "f \otimes g"] \\ C & & D & & C \otimes D \end{tikzcd}$$

An intuitive reading in terms of resources can be given as follows: given an action f that consumes the resource A to produce C, and an action g that consumes B to produce D, we have an action $f \otimes g$ that consumes both the resource A and B and produces both the resources C and D. Bifunctoriality gives specific equations for $\otimes$, notably

$$\begin{aligned} \mathsf{id}_A \otimes \mathsf{id}_B &= \mathsf{id}_{A \otimes B} \\ (f \otimes g) \circ (f' \otimes g') &= (f \circ f') \otimes (g \circ g') \end{aligned}$$

for f, f, g, g' of the right type. The second equation expresses some kind of sequentialization property. Let us clarify this idea reviewing previous example.

Example (Beverage Machine, continued)**.** Let us consider the beverage machine again, and let us assume the user has one pound ($1\pounds$) and one euro ($1€$). Suppose, for the sake of the example, that the currency exchange pounds-dollars and euros-dollars are both 1-1, and that the machine can accept both pounds and euros, but the user has to convert them into dollars in order to be able to get a beverage. We thus have the following actions (conv abbreviates $\mathsf{convert}$):

$$\begin{aligned}
\mathsf{conv}_{\pounds} &: 1\pounds \to 1\$ \\
\mathsf{conv}_{€} &: 1€ \to 1\$ \\
\mathsf{get_coffee} &: 1\$ \to \mathsf{coffee} \\
\mathsf{get_tea} &: 1\$ \to \mathsf{tea}
\end{aligned}$$

Bifunctoriality then gives

$$(\mathsf{get\_coffee} \otimes \mathsf{get\_tea}) \circ (\mathsf{conv}_{\pounds} \otimes \mathsf{conv}_{€})$$

$$=$$

$$(\mathsf{get\_coffee} \circ \mathsf{conv}_{\pounds}) \otimes (\mathsf{get\_tea} \circ \mathsf{conv}_{€})$$

which can be pictorially summarized as follows

$$\begin{tikzcd}
1\pounds \arrow[d, "\mathsf{conv}_{\pounds}"] & & 1€ \arrow[d, "\mathsf{conv}_{€}"] & & 1\pounds \otimes 1€ \arrow[d, "\mathsf{conv}_{\pounds} \otimes \mathsf{conv}_{€}"] \\
1\$ \arrow[d, "\mathsf{get_coffee}"] & \otimes & 1\$ \arrow[d, "\mathsf{get_tea}"] & = & 1\$ \otimes 1\$ \arrow[d, "\mathsf{get_coffee} \otimes \mathsf{get_tea}"] \\
\mathsf{coffee} & & \mathsf{tea} & & 1\mathsf{coffee} \otimes 1\mathsf{tea}
\end{tikzcd}$$

Previous examples show that monoidal categories provide a simple framework for studying systems and processes, and resources and actions. These categories provide operations to run processes in parallel and sequentially. The logical structure of monoidal categories is based on a tensor conjunction (tensor product), rather than on a standard conjunction (categorical product). This directly leads into the realm of linear logic.

Linear Logic

Linear Logic was introduced by J.Y. Girard in his seminal paper [56], moving from results and ideas obtained in the context of domain theory [3]. Linear logic can be introduced in several ways, moving from both mathematical considerations and informal intuitions. Here we 'justify' linear logic moving from simple intuitions and operational considerations, thus arriving to its 'proof-theoretical introduction'. Other approaches moving from more sophisticated mathematical theories can be found in e.g. [6, 15, 57, 81].

From a very intuitive and almost philosophical perspective, one can view classical logic as a system dealing with the notion of mathematical truth. Classical logic is concerned with those inference rules that preserve truth. This point of view leads to justify the validity of formulas like $A \vee \neg A$ (excluded middle). Intuitionistic logic deals with the concept of mathematical provability: given a proposition A, one is interested in establishing when A is provable. Therefore, in order to prove the validity of $A \vee B$, one has to produce either a proof of A or a proof of B. As a consequence, the validity of the excluded middle is rejected, since there are mathematical statements for which neither a proof nor a refutation can be produced. Both classical and intuitionistic logic manipulate mathematical entities, namely propositions (we work with propositional logics), and therefore have limitations concerning more concrete applications. Linear logic can be thought of a logic of resources. Rather than manipulating mathematical propositions, linear logic deals with resources and their manipulation. This gives rise to the informal reading of an implication $A \to B$ summarized in Figure 1.

As a consequence, given the linear implication $A \to B$ (which is usually written as $A \multimap B$) and the resource A, A is consumed to produce B. However, to do so he has to consume A, so that A is not available anymore. This phenomenon made the use of hypothesis (partially) linear, in the sense that an hypothesis in a proof cannot be used more than once. More elementary, a resource cannot be freely duplicated. It is customary in first introductions to linear logic to start with examples like the following: consider the proposition $\mathsf{having}\ 1\$$ meaning that a (fixed) user has one dollar. Then clearly having

Logic	Informal reading of $A \to B$
Classical	Whenever A is true, so is B.
Intuitionistic	Whenever a proof of A is given, it is possible to construct a proof of B.
Linear	The resource A can be consumed to produce B

Figure 1: Informal interpretation of implication.

one dollar does not imply having two dollars. Nevertheless, the following derivation is (classically and intuitionistically) correct

$$\frac{\textsf{having } 1\$ \vdash \textsf{having } 1\$ \qquad \textsf{having } 1\$ \vdash \textsf{having } 1\$}{\textsf{having } 1\$ \vdash \textsf{having } 1\$ \wedge \textsf{having } 1\$}$$

This shows that we cannot think of the conjunction $\wedge$ as a realistic way to put resources together. To fix such a problem a new conjunction is introduced, called tensor and denoted by $\otimes$. The informal meaning of $A \otimes B$ is that the resources A and B are both available. It is then natural to reject the implication $A \multimap A \otimes A$.

Having clarified the intuitions behind linear logic one has to face the problem of making these intuitions formal. Girard realized that to do so it is necessary to act on the so-called structural rules. A logic usually consists of a syntax and a semantics. The former specifies the objects the logic deals with (in our case propositions) and a formal calculus for such objects. There are several formalisms for formal calculi. Traditionally, the main three are the so-called Hilbert systems, natural deduction systems and sequent calculi (see e.g. [84, 96, 106] for an introduction). The latter were introduced by Gentzen [55] to provide a formal meta-theory for natural deduction proofs. Roughly, a sequent is an expression of the form $\Gamma \vdash \Delta$, where Γ and Δ are lists of formulas[5], usually called structures or contexts. Rules are divided into operational and structural. The former manipulates logical connectives, whereas the latter manipulates structures. Among structural rules, three are of major importance. These are given in Figure 2.

The first rule is called cut, the rules in second row are called left and right weakening, and the rules in the third line are called left and right contraction. If we think of formulas as resources, then contraction essentially states that

[5] Although one can consider other structures, like sets, multisets or trees, see [87, 106].

$$\frac{\Gamma \vdash A, \Delta \qquad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad\qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad\qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}$$

Figure 2: Structural rules.

resources are duplicable, whereas weakening states that resources can be deleted. These rules allow each hypothesis to be used any number of times. In linear logic none of these rules is allowed, so that one can obtain more control on resources. This leads to a specialization of logical connectives for conjunction and disjunction, as well as of the logical constants true and false[6]. Consider for example the following standard sequent calculus rule for introducing intuitionistic conjunction on the right:

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\,\mathrm{R}\wedge$$

According to the formulas-as-resources point of view, the rule says that if we can produce A consuming Γ and we can produce B consuming Γ, then we can produce $A \wedge B$ consuming Γ. This goes against our intuition, since we would need two copies of Γ in order to produce both A and B. We can then modify the rule as follows.

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\,\mathrm{R}\wedge'$$

The first rule is said to be *additive*, since the context Γ is copied from premises to conclusion. The second rule is said to be *multiplicative* since the contexts Γ and Δ are joined in the conclusion (or, equivalently, split from conclusion

[6]Usually, linear logic consider only one implication $\multimap$ and a negation $(\_)^\perp$, although these can be specialized too. See e.g. [105].

to premises). The two rules are equivalent in presence of weakening and contraction, as witnessed by the following derivations[7]:

$$\dfrac{\dfrac{\Gamma \vdash A}{\Gamma, \Delta \vdash A}\,\mathsf{weak} \qquad \dfrac{\Delta \vdash B}{\Gamma, \Delta \vdash B}\,\mathsf{weak}}{\Gamma, \Delta \vdash A \wedge B}\,\mathrm{R}\wedge \qquad\qquad \dfrac{\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma, \Gamma \vdash A \wedge B}\,\mathrm{R}\wedge'}{\Gamma \vdash A \wedge B}\,\mathsf{ctr}$$

The above rules become different if we drop weakening and contraction, and they give two distinct forms of conjunction: an additive one, denoted by & (read 'with'), and a multiplicative one, denoted by $\otimes$ (read 'tensor'). Rules governing these connectives are given in Figure 3.

$$\dfrac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \qquad \dfrac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Delta \vdash A \otimes B, \Delta, \Delta'}$$

$$\dfrac{\Gamma, A_i \vdash \Delta}{\Gamma, A_0 \,\&\, A_1 \vdash B} \qquad \dfrac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \,\&\, B, \Delta}$$

Figure 3: Sequent calculus for additive and multiplicative conjunction.

Similarly, the conjunction $\vee$ is specialized in an additive one, denoted by $\oplus$ (read 'choice' or 'plus'), and a multiplicative one, denoted by $\parr$ (read 'par' or 'co-tensor'). A standard sequent calculus system for classical linear logic will be studied in Chapter 4, and is given in Figure 4.1. A sequent calculus for intuitionistic linear logic can be obtained from the classical system simply by restricting structures in the right-hand-side of $\vdash$ to single formulas.

The new connectives have a natural informal interpretation, according to the formulas-as-resources perspective. This is given in Figure 4.

Structural rules can be recovered in a controlled manner, by means of the so-called exponential modalities ! and ?. For example, the intuitive meaning of $!A$ is that the resource A is available ad libitum. That is, the user can use A once, twice, ... or even zero times (i.e. the user can delete A). Therefore,

[7] We use a generalized version of weakening and contraction that acts on structures rather than on formulas: these generalized versions can be easily proved to be admissible by induction on the length of the structures involved.

Proposition	Informal Interpretation
$A \otimes B$	Resources A and B are both available.
$A \& B$	Resources A and B are both potentially given, but we can use only one of them. We can *choose* which resource use.
$A \oplus B$	Either the resource A or the resource B is available. The choice of which one is external and nondeterministic.
$A \parr B$	Both A and B are available, but these cannot be used together.

Figure 4: Informal interpretation of connectives.

the intuitive meaning of $!A$ is approximated by the infinitary formula

$$1 \,\&\, A \,\&\, (A \otimes A) \,\&\, \cdots \,\&\, (\underbrace{A \otimes \cdots \otimes A}_{n}) \,\&\, \cdots$$

As we will see, having fixed point operators allows to make this intuition formal.

Exponential modalities allow to encode both classical and intuitionistic logic inside linear logic (see [57, 105] for details). Notably, the intuitionistic implication $A \to B$ is recovered as

$$!A \multimap B$$

From a categorical perspective, moving from e.g. intuitionistic logic to linear logic corresponds to moving from cartesian to monoidal categories. As we will see in Chapter 3, having weakening and contraction amounts to having an erasing arrow $e_A : A \to \top$ and a duplicator arrow $\Delta_A : A \to A \wedge A$, which are nothing but arrows that erase and duplicate the resource A, respectively (cf. previous section). Having clarified the basic intuitions and ideas behind linear logic, we can now start a formal treatment of linear logic and its categorical counterpart.

Chapter 1

Preliminaries

In this chapter we introduce the basic categorical machinery used in this thesis, which roughly amounts to basic categorical notions (up to natural transformations), algebras and coalgebras[1].

The approach followed will be proof-theoretical oriented, looking at categories as abstract semantical structures interpreting both formulas and proofs. Such an approach was introduced in [104] and later systematized by Lambek (the reader can consult [71] for a complete exposition of the results achieved, and [47, 48] for a more recent introduction to the subject).

From a (basic) semantical perspective, a logic can be abstractly though as a poset with the order given by the consequence relation. From a proof theoretical perspective such approach is rather unsatisfactory, since all proofs from say a formula A to a formula B are identified. It is then more perspicuous to think of a logic as a graph, whose (directed) arrows are proofs. Requiring the existence of identity arrows and of arrows' composition amounts to require the logic to be closed under the identity axiom and the cut rule, here simplified as

$$\frac{}{A \vdash A} \qquad\qquad \frac{A \vdash B \qquad B \vdash C}{A \vdash C}$$

The resulting structure is called a *deductive system* (or deductive graph) [71]. A deductive system can be presented as a graph, from which we obtain the

[1] I try to make this work self contained, although the reader probably needs some (really basic) background in Category Theory. Nevertheless, I explicitly introduce all definitions used (even those of categories and functor), so that (hopefully) nothing will be left implicit. The reader can consult [72] as standard reference. More accessible introductions are [10, 11, 16].

associated deductive system by closing the collection of arrows by rules giving identity arrows and compositions.

Informally, a category is nothing but a deductive graph with the usual equations for associativity and identity, which turned out to be closely related to so-called cut-transformations (see [10, 77]).

As a consequence, the notion of category is essentially equational over deductive systems (and hence graphs), much in the same way the notion of monoid is equational over sets (i.e. we can define a monoid as a set with extra structure plus equations). We can then give a more 'logical' presentation of categories as graphs with a collection of 'inference rules' for arrows (i.e. those rules under which the collection of arrows has to be closed) and an equational system for such arrows[2]. One of the advantages of such approach is that by defining a category in this way we exploit its underlying logical structure, which is given by its underlying deductive system. For example, given a cartesian category $\mathcal{C}$ presented as a deductive system, we can easily prove that such a deductive system is equivalent to a standard sequent calculus for the conjunctive fragment of intuitionistic propositional logic (IPL) (see e.g. [30, 71, 84]). This shows that we could regard the conjunctive fragment of IPL as a logic built over cartesian cateogories. In general, we refer to the logic defined by the underlying deductive system of a category as the underlying logic of the category.

Another major advantage of Lambek's approach is that we come up with equational definitions of some classes of categories and categorical constructions. Such definitions provide nice equational laws which are the base of an 'algebra of arrows'. This equational approach was very fruitful in the field of *programming algebra* [26], where one needs a point-free algebra of programs governed by simple equational laws.

One last remark. We are not interested in foundational questions. Therefore, to avoid size problems we assume we work inside the von Neumann-Bernays-Gödel set theory (NBG) and abstractly speak of *collections* of objects. For example, we define graphs consisting of collections of objects and arrows. Requiring these to be proper sets creates problems since there is no immediate way to consider the underlying graphs of a large category. For the relationship between category theory and set theory the reader can consult

[2]Actually, we should give an equational system for objects too. However, such task is usually trivial and based on 'syntactic-like' notions of equality. For example, let $A \,\&\, B$ denote the product of A and B, then we have the equality rule $A = C, B = D \Rightarrow A \,\&\, B = C \,\&\, D$.

[19, 62].

1.1 Categories, Algebras and Coalgebras

In this section we review basic notions concerning categories, algebras and coalgebras. We introduce categories as special equational deductive graphs. The reader can think of the latter as a collection of formulas $A, B, C \ldots$ together with a collection of proofs connecting them. The notation $f : A \to B$ is used for 'f is a proof of B from the assumption A'. Another useful informal reading is to think of objects $A, B, C \ldots$ as systems or resources, and to an arrow $f : A \to B$ as a process that makes system A evolving into system B, or as an action that consumes resource A to produce resource B.

The main reference for an introduction to category theory is [72]. Other more accessible introductions are [10, 11, 16].

Let us start by defining the notion of a graph and then specializes it to the notion of deductive graph (deductive system).

Definition 1 (Graph)**.** A (directed) graph consists of a collection $\mathcal{A}$ of arrows and a collection $\mathcal{O}$ of objects together with two mappings $src, tgt : \mathcal{A} \to \mathcal{O}$, called source and target, respectively. Diagrammatically,

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "src"] \arrow[r, shift right, "tgt"'] & \mathcal{O} \end{tikzcd}$$

We write $f : A \to B$ or $A \xrightarrow{f} B$ meaning that f is an arrow, A and B are objects and $src(f) = A$ and $tgt(f) = B$.

Definition 2 (Deductive System)**.** A deductive system is a graph such that for any object A there is an associated arrow $\mathsf{id}_A : A \to A$ and for any pair of arrows $f : A \to B$ and $g : B \to C$, there is an associated arrow $g \circ f : A \to C$. Equivalently, we say that the collection of arrows is closed under the rules

$$\frac{}{\mathsf{id}_A : A \to A}$$

$$\frac{f : A \to B \qquad g : B \to C}{g \circ f : A \to C}$$

Note that we can present a deductive system as a graph, and then close its collection of arrows under the above inference rules. From a logical perspective we can view the objects of a graph as formulas, and its arrows as (extra-logical) axioms. The deductive system obtained from that graph gives the logic obtained from the (extra-logical) axioms via the inference rules identity and cut.

We can now equip deductive graphs with an equational theory, and thus obtain the notion of category.

Definition 3 (Category)**.** A category is a deductive system in which the following equations[3] holds for any $f : A \to B$, $g : B \to C$ and $h : C \to D$

$$\begin{array}{rcl} f \circ (g \circ h) & = & (f \circ g) \circ h \\ f \circ \mathsf{id}_A & = & f \\ \mathsf{id}_B \circ f & = & f \end{array}$$

Remark. From a logical perspective we can think of a category as a deductive system together with a notion of equality for proofs. Such an equality is closely related to the so-called cut-transformations [77]. Associativity of composition gives associativiy of cut:

$$\dfrac{\dfrac{f : A \to B \qquad g : B \to C}{g \circ f : A \to C} \qquad h : C \to D}{h \circ (g \circ f) : A \to D}$$

$$=$$

$$\dfrac{f : A \to B \qquad \dfrac{g : B \to C \qquad h : C \to D}{h \circ g : B \to D}}{(h \circ g) \circ f : A \to D}$$

Identity equations give basic cut-elimination steps (see [57, 106] for details):

$$\dfrac{\mathsf{id}_A : A \to A \qquad f : A \to B}{f \circ \mathsf{id}_A : A \to B} \quad = \quad f : A \to B$$

and

[3] We are implicitly assuming $=$ to be an equality, i.e. a reflexive, symmetric and transitive relation on arrows, which is a congruence with respect to composition.

$$\frac{f : A \to B \qquad \mathsf{id}_B : B \to B}{\mathsf{id}_B \circ f \circ : A \to B} \quad = \quad f : A \to B$$

As already remarked, categories provide a notion of equality for proofs. Using such notion of equality, we can define new notions of equality for objects (which are weaker than syntactical equality). Among these notions, two deserve special attention for our purposes.

Definition 4. Given two objects A and B in a category $\mathcal{C}$, we say that A and B are equi-provable if there are arrows $f : A \to B$ and $g : B \to A$. Moreover, we say that A and B are isomorphic, and write $A \cong B$, if $f \circ g = \mathsf{id}_B$ and $g \circ f = \mathsf{id}_A$. In that case we say that f and g are each other inverses.

Having defined categories, it is then natural to define structure-preserving maps between them, which are known as *functors*.

Definition 5 (Functor)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is a mapping from objects to objects and arrows to arrows such that

1. If $f : A \to B$ in $\mathcal{C}$, then $F(f) : F(A) \to F(B)$ in $\mathcal{D}$;

2. The following equalities hold:

$$\begin{array}{rcl} F(g \circ f) & = & F(g) \circ F(f) \\ F(\mathsf{id}_A) & = & \mathsf{id}_{F(A)} \end{array}$$

Sometimes, the notation FA and Ff for $F(A)$ and $F(f)$ will be used.

Strongly connected with the notion of functor, there is the notion of *natural transformation*. Roughly, natural transformation can be though as maps between functors.

Definition 6. Given functors F and G from a category $\mathcal{C}$ to a category $\mathcal{D}$, a natural transformation is a family of arrows θ_A parametrized by objects in $\mathcal{C}$, such that

$$G(f) \circ \theta_A = \theta_B \circ F(f)$$

for any arrow $f : A \to B$. This means that the following diagram commutes.

$$\begin{tikzcd}
F(A) \arrow[r, "\theta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\theta_B"'] & G(B)
\end{tikzcd}$$

Functors and natural transformations organize themselves as a category. That is, given categories $\mathcal{C}$ and $\mathcal{D}$, one can define the functor category $\mathcal{D}^{\mathcal{C}}$ whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and whose arrows are natural transformations. The identity functor $1 : \mathcal{C} \to \mathcal{C}$ is defined by

$$\begin{aligned} 1(A) &= A \\ 1(f) &= f \end{aligned}$$

whereas the composition $G \circ F$ of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ is defined by

$$\begin{aligned} (G \circ F)(A) &= G(F(A)) \\ (G \circ F)(f) &= G(F(f)) \end{aligned}$$

We now introduce the notions of algebra and coalgebra. Algebras and coalgebras are well-known and deeply investigated notions, with applications in several fields such as computer science, logic, artificial intelligence and economics. Here I will recall only few basic definitions. The reader can consult the introductory textbook [68] for informal intuitions, examples and further results.

Definition 7 (F-Algebra/Coalgebra)**.** Given an endofunctor[4] $F : \mathcal{C} \to C$, an F-algebra is a pair (A, a) consisting of an object A in $\mathcal{C}$ together with an arrow $a : F(A) \to A$. An F-coalgebra is a pair (C, c) consisting of an object C of $\mathcal{C}$ together with an arrow $c : C \to F(C)$.

We can define F-algebra/coalgebra homomorphisms as arrows in C that preserve the F-structure.

[4] A functor from a category to itself is usually called an *endofunctor*.

Definition 8. An arrow $h : A \to B$ (in $\mathcal{C}$) is an F-algebra homomorphism between F-algebras (A, a) and (B, b) if

$$h \circ a = b \circ F(h)$$

holds. An arrow $h : C \to D$ (in $\mathcal{C}$) is an F-coalgebra homomorphism between F-coalgebras (C, c) and (D, d) if

$$F(h) \circ c = d \circ h$$

holds.

Among F-algebras special ones are the so-called initial F-algebras. These are, in a way, the smallest F-algebras.

Definition 9 (Initial F-algebra)**.** An F-algebra $(\mu F, \mathsf{in})$ is initial if for any F-algebra (A, a) there is an arrow $[\![a]\!] : \mu F \to A$, i.e. the collection of arrows is closed under the rule

$$\frac{a : F(A) \to A}{[\![a]\!] : \mu F \to A}$$

and the following equational law holds.

$$\frac{f \circ \mathsf{in} = a \circ F(f)}{\overline{\overline{}} \; f = [\![a]\!]}$$

where the double line read as an 'if and only if'.

The above definition shows that the notion of initial algebra is equational, and thus fits our approach to categories via deductive systems.

Remark. The last rule states that for an algebra $a : F(A) \to A$, there is a unique algebra homomorphism $[\![a]\!] : \mu F \to A$ such that

$$[\![a]\!] \circ \mathsf{in} = a \circ F([\![a]\!])$$

that is, there is a unique arrow $[\![a]\!]$ that makes the following diagram commutes

$$\begin{tikzcd}
F(\mu F) \arrow[r, "F({[\![a]\!]})"] \arrow[d, "\mathsf{in}"'] & F(A) \arrow[d, "a"] \\
\mu F \arrow[r, "{[\![a]\!]}"'] & A
\end{tikzcd}$$

We say that this rule gives universality of $[\![a]\!]$, since it gives part of a universal mapping property (see Introduction). Giving uniqueness of $[\![a]\!]$, the rule also gives uniqueness up to isomorphism[5] of initial F-algebras (see below). We will refer to rules stating uniqueness of specific arrows as 'universality rules'.

The notion of final coalgebra can be given in a similar fashion.

Definition 10 (Final F-colgebra)**.** An F-coalgebra $(\nu F, \mathsf{out})$ is final if for any F-coalgebra (C, c) there is an arrow $(\!|c|\!) : C \to \nu F$, i.e. the collection of arrows is closed under the rule

$$\frac{c : C \to F(C)}{(\!|c|\!) : C \to \nu F}$$

and the following equational law holds.

$$\frac{\mathsf{out} \circ f = F(f) \circ c}{\overline{} \; f = (\!|c|\!)}$$

Initial algebras and final coalgebras are unique up to isomorphism, which means e.g. that given two final coalgebras (C, c) and (D, d) we have $C \cong D$. As a consequence, we can regard final coalgebras and initial algebras to be unique, and refer to *the* initial algebra/final coalgebra of a functor F.

A fundamental result on initial algebras and final coalgebras is the so-called Lambek's Lemma (see e.g. [68]).

Lemma 1 (Lambek)**.** Let $(\mu F, \mathsf{in})$ and $(\nu F, \mathsf{out})$ be the initial algebra and final coalgebra of an endofunctor $F : \mathcal{C} \to \mathcal{C}$. Then

$$\begin{array}{rcl} F(\mu F) & \cong & \mu F \\ F(\nu F) & \cong & \nu F \end{array}$$

hold.

Proof. To prove $F(\mu F) \cong \mu F$ it is sufficient to find a pair of arrows which are each other inverses. These are given by in and $[\![F(\mathsf{in})]\!]$. Similarly for proving $F(\nu F) \cong \nu F$ we consider out and $(\!|F(\mathsf{out})|\!)$. ■

[5]The expression 'up to isomorphism' means that we are reasoning modulo $\cong$, that is we are identifying isomorphic objects.

Lambek's Lemma states that both μF and νF are fixed point of F (under the notion of equality given by $\cong$). Moreover, thinking of an arrow $f : A \to B$ as witnessing that B is 'bigger' than A (and thus defining a partial order on the collection of objects), then we see that νF is the greatest fixed point of F, whereas μF is the least one. The reader is invited to consult [68, 88] for more details.

We will say more about initial algebras and final coalgebras (especially in terms of deductive systems) in next chapters.

1.2 Cartesian Categories

In this section we give a first example of the interplay between logic, categories and deductive systems. We equip deductive systems with binary products and initial objects, thus obtaining the notion of *cartesian category*. At the same time, such deductive systems give a rudimentary calculus for the conjunctive fragment of intuitionistic propositional logic [96], thus exploiting the underlying logical structure of cartesian categories. There are several benefits from such correspondence: we can give to the conjunctive fragment of intuitionistic propositional logic a categorical semantics, and viceversa we have a syntactic calculus for cartesian categories. Moreover, categorical equations give a nice notion of equality between proofs, deeply linked to other notions of equality such as those based on cut elimination and normalization [10, 30, 57, 71, 106].

Let us start by defining the notion of binary product.

Definition 11. A deductive system $\mathcal{D}$ has binary products if for any two objects A and B of $\mathcal{D}$, there is an associated object $A \mathbin{\&} B$ (read 'A with B') which is an object of $\mathcal{D}$ too, and the collection of arrows is closed under the following rules[6] (the first two rules are axioms, so they state existence of special arrows)

$$\frac{}{p_{A,B} : A \mathbin{\&} B \to A}$$

$$\frac{}{q_{A,B} : A \mathbin{\&} B \to B}$$

[6] To be precise we should say rule schemes. In fact, these rules are parametrized by objects and arrows.

$$\frac{f : C \to A \qquad g : C \to B}{\langle f, g\rangle : C \to A \,\&\, B}$$

A category with binary product is a deductive systems $\mathcal{D}$ with binary product which, in addition to equations for identity and composition, satisfies the following equational law (which gives universality of $\langle f, g\rangle$). For $f : C \to A$, $g : B \to C$ and $h : C \to A \,\&\, B$ (we write p and q for $p_{A,B}$ and $q_{A,B}$ respectively)

$$\frac{p \circ h = f \qquad q \circ h = g}{\overline{h = \langle f, g\rangle}}$$

The above system allows us to prove simple equations between arrows, as well as to construct new arrows.

Example 1. 1. We can construct the following arrows

$$\begin{aligned} s_{A,B} &: A \,\&\, B \to B \,\&\, A \\ a_{A,B,C} &: (A \,\&\, B) \,\&\, C \to A \,\&\, (B \,\&\, C) \\ \Delta_A &: A \to A \,\&\, A \end{aligned}$$

called switching, associator and duplicator respectively, defining

$$\begin{aligned} s_{A,B} &= \langle q_{A,B}, p_{A,B}\rangle \\ a_{A,B,C} &= \langle p_{A,B} \circ p_{A\&B,C}, \langle q_{A,B} \circ p_{A\&B,C}, q_{A\&B,C}\rangle\rangle \\ \Delta_A &= \langle \mathsf{id}_A, \mathsf{id}_A\rangle \end{aligned}$$

2. We can prove the equational law

$$\langle f, g\rangle \circ h = \langle f \circ h, g \circ h\rangle$$

For, it is sufficient to prove

$$\begin{aligned} p \circ (\langle f, g\rangle \circ h) &= f \circ h \\ q \circ (\langle f, g\rangle \circ h) &= g \circ h \end{aligned}$$

These can be easily proved, once we know $p \circ \langle f, g\rangle = f$ and $q \circ \langle f, g\rangle = g$. The latter hold, since we can just instantiate h to be $\langle f, g\rangle$ itself in the rule for universality of $\langle f, g\rangle$.

3. We can prove that the following rule is admissible.

$$\frac{f = h \qquad g = k}{\langle f, g\rangle = \langle h, k\rangle}$$

For, it is sufficient to construct

$$\frac{\dfrac{f = h}{p \circ \langle f, g\rangle = h} \qquad \dfrac{g = k}{q \circ \langle f, g\rangle = k}}{\langle f, g\rangle = \langle h, k\rangle}$$

Note also that equational laws allow us to prove that the product of two objects is unique up to isomorphisms, hence we can correctly refer to it as *the* product. A product we will use later is the product of categories.

Definition 12. Given categories $\mathcal{C}$ and $\mathcal{D}$ we can define the product category $\mathcal{C} \times \mathcal{D}$ (in this specific case we use the notation $\times$ in place of &) as follows:

1. Objects are pairs of the form (C, D) for C object of $\mathcal{C}$ and D object of $\mathcal{D}$.

2. Given arrows $f : C \to C'$ in $\mathcal{C}$ and $g : D \to D'$ in $\mathcal{D}$, we have an arrow

$$(f, g) : (C, D) \to (C', D')$$

in $\mathcal{C} \times \mathcal{D}$.

The category Cat has (small) categories as objects (see [11]) and functors as arrows. The above definition equips Cat with binary products.

We can now define the notion of *bifunctor*. Given categories $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ a bifunctor is nothing but a functor

$$F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$$

In particular, we have that the following equations hold

$$\begin{array}{rcl} F(\mathsf{id}_A, \mathsf{id}_B) & = & \mathsf{id}_{F(A,B)} \\ F(g \circ f, g' \circ f') & = & F(g, g') \circ F(f, f') \end{array}$$

A useful lemma we will implicitly use, is the so-called bifunctor lemma [11].

Lemma 2 (Bifunctor)**.** Given categories $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, a map $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is a bifunctor if and only if

1. F is functorial in each argument. That is, for any A object in $\mathcal{A}$ and B object in B

$$\begin{array}{rcl} F(A, \_) & : & \mathcal{B} \to \mathcal{C} \\ F(\_, B) & : & \mathcal{A} \to \mathcal{C} \end{array}$$

   are functors[7].

2. Given $f : A \to A'$ and $g : B \to B'$, the following holds

$$F(A', g) \circ F(f, B) = F(f, B') \circ F(A, g).$$

Proof. See [11]. ■

In particular, if we define for arrows f and g in a category $\mathcal{C}$,

$$f \,\&\, g = \langle f \circ p, g \circ q \rangle$$

we obtain a bifunctor $\& : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ defined by

$$\begin{array}{rcl} (A, B) & \mapsto & A \,\&\, B \\ (f, g) & \mapsto & f \,\&\, g \end{array}$$

Simple calculations show that bifunctor equalities are indeed satisfied.

In order to define cartesian categories we need the notion of terminal object.

Definition 13. A category with a terminal object is a deductive system $\mathcal{D}$ with a distinguished object $\top$ and a family of arrows $!_A : A \to \top$, for each object A of $\mathcal{D}$. Moreover, in addition to equations for identity arrows and composition we require the following equational law to hold:

[7]Where e.g. $F(A, \_)$ is defined on objects as

$$F(A, \_)(B) = F(A, B)$$

and on arrows as

$$F(A, f) = F(\mathsf{id}_A, f).$$

$$\frac{f : A \to \top}{f = !_A}$$

Again, equational laws allow to prove that the terminal object is unique up to isomorphism. Finally, we can define cartesian categories.

Definition 14. A category $\mathcal{C}$ is cartesian if it has binary products and terminal objects.

We now summarize definitions given so far providing a definition of cartesian categories as deductive systems.

Definition 15. A cartesian deductive system is a deductive system $\mathcal{D}$ with binary products and terminal object. In particular, inference rules for a cartesian deductive system are given in Figure 1.1.

$$\frac{}{\mathsf{id}_A : A \to A}$$

$$\frac{f : A \to B \qquad g : B \to C}{g \circ f : A \to C}$$

$$\frac{}{!_A : A \to \top}$$

$$\frac{}{p_{A,B} : A \,\&\, B \to A}$$

$$\frac{}{q_{A,B} : A \,\&\, B \to B}$$

$$\frac{f : C \to A \qquad g : C \to B}{\langle f, g \rangle : C \to A \,\&\, B}$$

Figure 1.1: Inference rules for a cartesian deductive system.

Erasing arrows' names, we obtain the system given in Figure 1.2

The system can be further simplified by taking a 'single' rule for product, namely

$$\frac{C \to A \qquad C \to B}{C \to A \,\&\, B}$$

$$\frac{}{A \to A} \qquad \frac{A \to B \quad B \to C}{A \to C}$$

$$\frac{}{A \to \top}$$

$$\frac{}{A \,\&\, B \to A} \qquad \frac{}{A \,\&\, B \to B}$$

$$\frac{C \to A \quad C \to B}{C \to A \,\&\, B}$$

Figure 1.2: Arrow-free cartesian deductive system.

Projections are recovered from the axiom $A \,\&\, B \to A \,\&\, B$.

We now exploit the link between cartesian deductive systems and the $(\&, \top)$-fragment of intuitionistic propositional logic ($(\&, \top)$-IPL, for short). Given a set Prop of atomic propositions, the set of formulas of $(\&, \top)$-IPL is defined by the following grammar

$$A ::= a \mid \top \mid A \,\&\, A$$

where $a \in \mathsf{Prop}$. A sequent is an expression of the form $\Gamma \vdash A$, where A is a formula and Γ a multiset of formulas. A sequent calculus for $(\&, \top)$-IPL is given in Figure 1.3, where the rules in the second line are called left weakening and left contraction (see [57, 106] for details).

Given a multiset $\Gamma = A_1, \ldots, A_n$ we can define its 'logical' counterpart to be $A_1 \,\&\, \cdots \,\&\, A_n$, if $n > 0$, and $\top$ otherwise. An easy induction on derivations shows that if $A_1, \ldots, A_n \vdash A$ is provable in the sequent calculus, then $A_1 \,\&\, \cdots \,\&\, A_n \to A$ is provable in the system of Figure 1.2. Viceversa, if $A \to B$ is provable in such system, then the sequent $A \vdash B$ is provable too. This shows that we can give a presentation of $(\&, \top)$-IPL as a cartesian deductive system, whose objects are formulas. Arrows in the deductive system then give a formalism for derivations, and the equational theory given by the cartesian category induced by the deductive system gives a notion of equality between proofs. The translation between sequent calculus proofs to arrows in the deductive system is summarized in Figure 1.4.

Notice the presence of the duplicator arrow $\Delta_A : A \to A \,\&\, A$ for translating the contraction rule, and the presence of projections for translating the weakening rule.

$$\frac{}{A \vdash A} \qquad \frac{\Gamma \vdash A \qquad A, \Delta \vdash C}{\Gamma, \Delta \vdash C}$$

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B} \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}$$

$$\frac{}{\vdash \top}$$

$$\frac{\Gamma, A_i \vdash B}{\Gamma, A_0 \,\&\, A_1 \vdash B}\, i \in \{0,1\} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}$$

Figure 1.3: Sequent calculus for $(\&, \top)$-IPL.

$$\frac{}{A \vdash A} \qquad \frac{}{\mathsf{id}_A : A \to A}$$

$$\frac{\Gamma \vdash A \qquad \Delta, A \vdash C}{\Gamma, \Delta \vdash C} \qquad \frac{f : \Gamma \to A \qquad g : A \,\&\, \Delta \to B}{g \circ (f \,\&\, \mathsf{id}_\Delta) : \Gamma \,\&\, \Delta \to B}$$

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B} \qquad \frac{f : \Gamma \to B}{f \circ p_{\Gamma,A} : \Gamma \,\&\, A \to B}$$

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} \qquad \frac{f : \Gamma \,\&\, A \,\&\, A \to B}{f \circ (\mathsf{id}_\Gamma \,\&\, \Delta_A) : \Gamma \,\&\, A \to B}$$

$$\frac{}{\vdash \top} \qquad \frac{}{!_\top : \top \to \top}$$

$$\frac{\Gamma, A \vdash C}{\Gamma, A \,\&\, B \vdash C} \qquad \frac{\Gamma \,\&\, A \to C}{f \circ (\mathsf{id}_\Gamma \,\&\, p_{A,B}) : \Gamma \,\&\, A \,\&\, B \to C}$$

$$\frac{\Gamma, B \vdash C}{\Gamma, A \,\&\, B \vdash C} \qquad \frac{\Gamma \,\&\, B \to C}{f \circ (\mathsf{id}_\Gamma \,\&\, q_{A,B}) : \Gamma \,\&\, A \,\&\, B \to C}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B} \qquad \frac{f : \Gamma \to A \qquad g : \Gamma \to B}{\langle f, g \rangle : \Gamma \to A \,\&\, B}$$

Figure 1.4: Translation of sequent calculus proofs into arrows.

Remark. We worked with products modulo associativity and commutativity. This relies on the fact that the associator arrow

$$a_{A,B,C} : A \,\&\, (B \,\&\, C) \to (A \,\&\, B) \,\&\, C$$

and the switching arrow

$$s_{A,B} : A \,\&\, B \to B \,\&\, A$$

actually give isomorphisms, and thus we can work with products modulo associativity and commutativity.

Equational theories derived from categories are usually interesting ones, since they often provide a simpler and more elegant presentation of notion of equality coming from proof transformations like those of cut elimination and normalization (see [10, 71, 84, 106, 105]). Consider for instance the following (simplified) cut reduction:

$$\dfrac{\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B} \qquad \dfrac{A \vdash D}{A \,\&\, B \vdash D}}{\Gamma \vdash D} \quad = \quad \dfrac{\Gamma \vdash A \qquad A \vdash D}{\Gamma \vdash D}$$

This corresponds to

$$\dfrac{\dfrac{f : \Gamma \to A \qquad g : \Gamma \to B}{\langle f, g \rangle : \Gamma \to A \,\&\, B} \qquad \dfrac{h : A \to D}{h \circ p : A \,\&\, B \to D}}{h \circ p \circ \langle f, g \rangle : \Gamma \to D}$$

$$=$$

$$\dfrac{f : \Gamma \to A \qquad h : A \to D}{h \circ f : \Gamma \to D}$$

which indeed holds since

$$h \circ p \circ \langle f, g \rangle = h \circ f$$

For a complete exposition of the correspondence between cut reductions, normalization steps and equations in cartesian categories the reader can consult [10, 11, 84, 106], where such correspondence is extended to *cartesian closed categories* (see e.g. [10, 11, 16, 30, 71]), the $(\&, \top, \to)$-fragment of intuitionistic propositional logic and the simply typed λ-calculus (with unit, arrow and product types) [65, 66, 96, 106]. Such correspondence is known as *Proposition-as-Types Correspondence* (see [57, 96]) or *Curry-Howard-Lambek Correspondence* [15, 71, 84, 106] (see footnote 1 in the introduction).

1.3 Monoidal Categories

In previous section we reviwed cartesian categories and observed that their underlying logic is essentially the conjunctive fragment of intuitionistic propositional logic. Such categories, although mathematically attractive, do not provide the right structure for our purposes (see Introduction). Given two objects A and B, we want an operation for putting these objects together (e.g. in a parallel composition). If $\mathcal{C}$ is cartesian, with binary product & and terminal object $\top$, a natural candidate for the previous operation is $A \,\&\, B$. As we already argued in the intorduction, this choice is rather unsatisfactory. If we think of A and B as resources, and to $A \,\&\, B$ as the resource obtained by joining A and B, then the presence of the duplicator $\Delta_A : A \to A \,\&\, A$ simply states that resources are duplicable. Terminality of $\top$ gives an erasing arrow $!_A : A \to \top$, which allows to delete resources. Finally, having projections can be interpreted as having a too weak interaction between A and B in $A \,\&\, B$, since it is always possible to 'separate' them. To fix these problems we temporary abandon cartesian categories, and consider different structures, namely *monoidal categories*. These categories were introduced in [72], and since then were deeply investigated (see e.g. [10, 16, 77]).

Monoidal categories can be though as a generalization of the concept of monoid, and are characterized by the presence of a bifunctor $\otimes$ with unit 1. The former gives a way to make objects A and B interact as $A \otimes B$. More importantly, it gives a form of interaction between arrows. Given arrows f and g (recall that these are though as processes or actions), we can think of $f \otimes g$ as a parallel composition of f and g. Together with composition (which can be thought as a sequential composition), we have a simple framework for studying both parallel and sequential interactions.

Let us start by formally defining monoidal deductive systems and monoidal categories.

Definition 16 (Monoidal Deductive System)**.** A monoidal deductive system is a deductive system $\mathcal{D}$ with the addition of the following distinguished arrows (to be thought as zero-ary inference rules)

$$\begin{array}{rcl}
a_{A,B,C} & : & A \otimes (B \otimes C) \to (A \otimes B) \otimes C \\
a^{-1}_{A,B,C} & : & (A \otimes B) \otimes C \to A \otimes (B \otimes C) \\
l_A & : & 1 \otimes A \to A \\
l^{-1}_A & : & A \to 1 \otimes A \\
r_A & : & A \otimes 1 \to A \\
r^{-1}_A & : & A \to A \otimes 1
\end{array}$$

and the following two-premises rule

$$\frac{f : A \to C \qquad g : B \to D}{f \otimes g : A \otimes B \to C \otimes D}$$

Roughly, a monoidal category is a monoidal deductive system $\mathcal{D}$ in which the arrows a, l and r are natural isomorphisms, $\otimes$ is a bifunctor and the so-called coherence conditions [72] are satisfied. These coherence conditions can be summarized via the following commutative diagrams (called the pentagon and triangle identities).

$$\begin{tikzcd}
 & (A \otimes B) \otimes (C \otimes D) \arrow[dr, "a^{-1}_{A,B,C\otimes D}"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "a_{A\otimes B,C,D}"] \arrow[d, "a_{A,B,C} \otimes \mathsf{id}_D"'] & & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "a_{A,B\otimes C,D}"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "\mathsf{id}_A \otimes a_{B,C,D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes 1) \otimes B \arrow[rr, "a_{A,1,B}"] \arrow[dr, "r_A \otimes \mathsf{id}_B"'] & & A \otimes (1 \otimes B) \arrow[dl, "\mathsf{id}_A \otimes l_B"] \\
 & A \otimes B &
\end{tikzcd}$$

Definition 17 (Monoidal Category)**.** A monoidal category is a monoidal deductive system $\mathcal{D}$ equipped with categorical equations plus the following equations (in order to keep a light notation I considered arrows without objects subscripts; these can be understood from the context, or otherwise the reader can refer to the above commutative diagrams)[8]

$$\begin{array}{lll} a \circ a^{-1} = \mathsf{id} & l \circ l^{-1} = \mathsf{id} & r \circ r^{-1} = \mathsf{id} \\ a^{-1} \circ a = \mathsf{id} & l^{-1} \circ l = \mathsf{id} & r^{-1} \circ r = \mathsf{id} \end{array}$$

$$\begin{array}{rcl} (f \circ h) \otimes (g \circ k) & = & (f \otimes g) \circ (h \otimes k) \\ \mathsf{id}_A \otimes \mathsf{id}_B & = & \mathsf{id}_{A \otimes B} \end{array}$$

$$\begin{array}{rcl} (f \otimes (g \otimes h)) \circ a_{A,B,C} & = & a_{A',B',C'} \circ (f \otimes g) \otimes h \\ f \circ l_A & = & l_{A'} \circ (1 \otimes f) \\ f \circ r_A & = & r_{A'} \circ (f \otimes 1) \end{array}$$

$$\begin{array}{rcl} (a \otimes \mathsf{id}) \circ a \circ (\mathsf{id} \otimes a) & = & a \circ a \\ (r \otimes \mathsf{id}_1) \circ a & = & l \end{array}$$

plus the following equational law

$$\frac{f = f' \qquad g = g'}{f \otimes g = f' \otimes g'}$$

The first group of equations states that a, a^{-1}, l, l^{-1} and r, r^{-1} are indeed isomorphisms. The third group of equations gives naturality for them, where in virtue of equations in the first group we wrote e.g. a both for a and a^{-1}. The fourth group gives coherence conditions for the natural isomorphisms. The second group of equations gives bifunctoriality of $\otimes$. Finally, the equational law for $\otimes$ states that $\otimes$ is indeed a mapping (which is part of the definition of functor).

[8] In the third group of equations we assume $f : A \to A'$, $g : B \to B'$ and $h : C \to C'$.

Remark (Coherence Conditions)**.** The tensor $\otimes$ in a monoidal category does not need to be unique, in contrast with the cartesian product. Moreover, in general, $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$ are different objects. Suppose we want to form the length 3 tensor product $A \otimes B \otimes C$. Both $A \otimes (B \otimes C)$ and $(A \otimes B) \otimes C$ are natural candidates. The natural isomorphism a allows to identify them, reasoning up to isomorphism. As a consequence, we may write $A \otimes B \otimes C$, forgetting parenthesis. The question is whether we can do the same for longer tensor products, that is if we can define an object like

$$\bigotimes_i A_i$$

ignoring parenthesis in it. We have, for example, that

$$A \otimes (B \otimes (C \otimes D)) \cong ((A \otimes B) \otimes C) \otimes D.$$

Unfortunately, there is more than one isomorphism between them. The pentagon diagram states that such isomorphisms are all equals, i.e. that all possible ways to form $A \otimes B \otimes C \otimes D$ are the same. Mac Lane's theorem [72] generalizes this result proving that in any monoidal category, any two (natural) isomorphisms built out of a, l, r and id, by using $\otimes$ and composition, actually coincide[9]. For example,

$$A \otimes (B \otimes C) \otimes (A' \otimes B')$$

and

$$(A \otimes B) \otimes (C \otimes A' \otimes B')$$

are isomorphic in just one way. For more details see [72], or [105] for a logic-oriented proof of MacLane's theorem.

We are interested in categories in which the tensor product is commutative. This leads to symmetric monoidal categories (SMCs).

Definition 18. A symmetric monoidal deductive system is a deductive system $\mathcal{D}$ with the addition of the following distinguished arrows (to be thought as zero-ary inference rules)

$$\begin{aligned} s_{A,B} &: A \otimes B \to B \otimes A \\ s^{-1}_{A,B} &: B \otimes A \to A \otimes B \end{aligned}$$

[9]Theorems like this one are usually called *coherence theorems*.

A symmetric monoidal category is a symmetric monoidal deductive system equipped with the following additional equations,

$$\begin{array}{rcl} s \circ s^{-1} & = & \mathsf{id} \\ s^{-1} \circ s & = & \mathsf{id} \end{array}$$

equations to make s a natural isomorphism (see equations in previous definition), and the following equations (where we use previous notational conventions).

$$\begin{array}{rcl} l_A \circ s_{A,1} & = & r_A \\ (s \otimes \mathsf{id}) \circ a \circ (\mathsf{id} \otimes s) & = & a \circ s \circ a \end{array}$$

The above equations are summarized by the following commutative diagrams.

$$\begin{tikzcd} A \otimes 1 \arrow[r, "s_{A,1}"] \arrow[dr, "r_A"'] & 1 \otimes A \arrow[d, "l_A"] \\ & A \end{tikzcd}$$

$$\begin{tikzcd} A \otimes (B \otimes C) \arrow[r, "\mathsf{id}_A \otimes s_{B,C}"] \arrow[d, "a_{A,B,C}"'] & A \otimes (C \otimes B) \arrow[r, "a_{A,B,C}"] & (A \otimes C) \otimes B \arrow[d, "s_{A,C} \otimes \mathsf{id}_B"] \\ (A \otimes B) \otimes C \arrow[r, "s_{A\otimes B,C}"'] & C \otimes (A \otimes B) \arrow[r, "a_{A,B,C}"'] & (C \otimes A) \otimes B \end{tikzcd}$$

As already stressed, monoidal categories provide mathematical structures that allow to run processes in parallel, by means of $\otimes$, and sequentially, by means of $\circ$. Moreover, it is in general not possible to construct arrows

$$\begin{array}{rcl} \Delta_A & : & A \to A \otimes A \\ e_A & : & A \to 1 \end{array}$$

as one can observe that in general $\otimes$ is not a categorical product (whereas every product is a tensor).

We now examine more closely the underlying logic of SMCs. First of all let us summarize inference rules for generating arrows in symmetrical monoidal deductive system. These are given by zero-ary inference rules (we write them without the over bar denoting the absence of premises) in Figure 1.5 plus inference rules in Figure 1.6.

$$a_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C \qquad a^{-1}_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

$$l_A : 1 \otimes A \to A \qquad l'_A : A \to 1 \otimes A$$

$$r_A : A \otimes 1 \to A \qquad r^{-1}_A : A \to A \otimes 1$$

$$s_{A,B} : A \otimes B \to B \otimes A \qquad s^{-1}_{A,B} : B \otimes A \to A \otimes B$$

Figure 1.5: Arrows-generating rules for SMCs.

$$\frac{}{\mathsf{id}_A : A \to A} \qquad \frac{f : A \to C \qquad g : B \to D}{f \otimes g : A \otimes B \to C \otimes D} \qquad \frac{f : A \to B \qquad g : B \to C}{g \circ f : A \to C}$$

Figure 1.6: Arrows-generating rules for SMCs.

Erasing names for arrows in the above system we obtain a first 'logical calculus'. We can reduce the number of axioms by regrouping them as inference rules. For example, we can replace axioms

$$\begin{array}{rcl} A & \to & 1 \otimes A \\ 1 \otimes A & \to & A \end{array}$$

stating the equi-provability of A and $1 \otimes A$ with the following bidirectional rule

$$\frac{A \to 1 \otimes B}{\overline{A \to B}}$$

Note that the rule

$$\frac{1 \otimes A \to B}{A \to B}$$

works as well. The system obtained is given in Figure 1.7

$$\frac{}{A \to A} \qquad \frac{A \to B \qquad B \to C}{A \to C}$$

$$\frac{A \to C \qquad B \to D}{A \otimes B \to C \otimes D} \qquad \frac{D \to (A \otimes B) \otimes C}{D \to A \otimes (B \otimes C)}$$

$$\frac{A \to 1 \otimes B}{A \to B} \qquad \frac{A \to B \otimes 1}{A \to B}$$

$$\frac{A \to B \otimes C}{A \to C \otimes B}$$

Figure 1.7: Logical system for SMCs.

It is now easy to recognize that the logic we obtain is the $(\otimes, 1)$-fragment of (commutative) intuitionistic linear logic (also known as tensorial logic [105]). Formulas of this fragment are defined from a set Prop of atomic propositions by means of the following grammar

$$A ::= a \mid 1 \mid A \otimes A$$

where $a \in \mathsf{Prop}$. A sequent calculus for such fragment is given in Figure 1.8. Sequents are expressions of the form $\Gamma \vdash A$, where Γ is a multiset of formulas and A is a formula.

As for cartesian categories, the logical counterpart of a multiset $\Gamma = A_1, \ldots, A_n$ is defined as $A_1 \otimes \cdots \otimes A_n$, if $n > 0$, and 1 otherwise. An easy induction on derivations shows that if $A_1, \ldots, A_n \vdash A$ is provable in the sequent calculus, then $A_1 \otimes \cdots \otimes A_n \to A$ is provable in the system of Figure 1.7. Viceversa, if $A \to B$ is provable in such system, then the sequent $A \vdash B$ is provable too. This shows that we can give a presentation of tensorial logic as a symmetric monoidal deductive system, whose objects are formulas. Arrows in the deductive system give a formalism for derivations, and the equational theory

$$\frac{}{A \vdash A} \qquad \frac{\Gamma \vdash A \qquad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}$$

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$$

$$\frac{}{\vdash 1} \qquad \frac{\Gamma \vdash A}{\Gamma, 1 \vdash A}$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$$

Figure 1.8: Sequent calculus for tensorial intuitionistic logic.

given by the SMC induced by the deductive system gives a notion of equality between proofs. In Figure 1.9 we summarize how to associate to a provable sequent

$$A_1, \ldots, A_n \vdash A$$

an arrow

$$f : A_1 \otimes \cdots \otimes A_n \to A$$

(if $n > 0$, $f : 1 \to A$ otherwise).

As for cartesian categories, the equational theory associated to a symmetric monoidal deductive system gives a nice equational theory for proofs. For example, consider the bifunctoriality equation for $\otimes$, now written in proof-tree notation.

$$\frac{\dfrac{f : A \to A' \qquad g : B \to B'}{f \otimes g : A \otimes B \to A' \otimes B'} \qquad \dfrac{f' : A' \to C \qquad g' : B' \to D}{f' \otimes g' : A' \otimes B' \to C \otimes D}}{(f' \otimes g') \circ (f \otimes g) : A \otimes B \to C \otimes D}$$

$$=$$

$$\frac{\dfrac{f : A \to A' \qquad f' : A' \to C}{f' \circ f : A \to C} \qquad \dfrac{g : B \to B' \qquad g' : B' \to C}{g' \circ g : B \to C}}{(f' \circ f) \otimes (g' \circ g) : A \otimes B \to C \otimes D}$$

$$\frac{}{A \vdash A} \qquad \frac{}{\mathsf{id}_A : A \to A}$$

$$\frac{\Gamma \vdash A \qquad A, \Delta \vdash B}{\Gamma, \Delta \vdash B} \qquad \frac{f : \Gamma \to A \qquad g : A \otimes \Delta \to B}{g \circ (f \otimes \mathsf{id}_\Delta) : \Gamma \otimes \Delta \to B}$$

$$\frac{}{\vdash 1} \qquad \frac{}{\mathsf{id}_1 : 1 \to 1}$$

$$\frac{\Gamma \vdash A}{\Gamma, 1 \vdash A} \qquad \frac{f : \Gamma \to A}{f \circ r_\Gamma : \Gamma \otimes 1 \to A}$$

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad \frac{f : \Gamma \to A \qquad g : \Delta \to B}{f \otimes g : \Gamma \otimes \Delta \to A \otimes B}$$

$$\frac{\Gamma, A, B \otimes C}{\Gamma, A \otimes B \vdash C} \qquad \frac{f : (\Gamma \otimes A) \otimes B \to C}{f \circ a_{\Gamma,A,B} : \Gamma \otimes (A \otimes B) \to C}$$

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C} \qquad \frac{f : \Gamma \otimes A \otimes B \otimes \Delta \to C}{f \circ (\mathsf{id}_\Gamma \otimes s_{B,A} \otimes \mathsf{id}_\Delta) : \Gamma \otimes B \otimes A \otimes \Delta \to C}$$

Figure 1.9: Equivalence between the sequent and combinatorial calculus for SMCs.

It is easy (although a bit heavy) to prove that the equations obtained by the usual cut elimination transformations hold in the equational theory associated to symmetric monoidal deductive systems (see [77]). Moreover, such equational theory give other interesting notion of equality between proofs, namely those associated to the so-called identity expansions. Identity expansions are proofs' transformations between identity axioms for compound formulas, and specific derivations proving those very identities using only identity axioms for atomic formulas (see [57, 106, 105]). Here is an example

$$A \otimes B \to A \otimes B \quad = \quad \frac{A \to A \qquad B \to B}{A \otimes B \to A \otimes B}$$

The above equality is nothing but the equation

$$\mathsf{id}_{A \otimes B} = \mathsf{id}_A \otimes \mathsf{id}_B$$

given by bifunctoriality of $\otimes$.

We can endow SMCs with a cartesian structure, thus recovering the conjunctive fragment of intuitionistic linear logic (i.e. the one based on the connectives $\otimes$ and $\&$, multiplicative and additive conjunction, and constants 1 and $\top$, which are units for $\otimes$ and $\top$ respectively).

Definition 19. A SMC which is also cartesian is called a symmetric monoidal cartesian category (SMCC). It has tensor $\otimes$ with its unit 1, and binary product $\&$ with its unit (the terminal object) $\top$. The underlying logic is the $(\otimes, \&)$-fragment of linear logic.

Although basic, the $(\otimes, \&)$-fragment of intuitionistic linear logic is enough expressive to be enriched with fixed point operators in a non-trivial way. In fact, we can recover the exponential modality $!$ as a greatest fixed point of a specific functor. The mathematical structure needed for such an analysis is obtained by equipping SMCCs with initial algebras and final coalgebras for a specific class of functors. The categories thus obtained and their associated logic will be studied in the next chapter.

Chapter 2

Linear Logic and Fixed Points

In this chapter we introduce a first extension of propositional linear logic with least and greatest fixed point operators, moving from categorical considerations. We already remarked that monoidal categories allow to study both parallel and sequential composition of processes (formalized as arrows). It is then natural to be interested in infinite and iterative behaviors. For example one can be interested in studying a self-replicable resource A^ω, which gives an infinite amount of resources A. Its behavior can be easily described by the recursive equation

$$X \cong A \otimes X$$

and A^ω can be defined as the greatest fixed point of this equation, denoted by $\nu X.A \otimes X$. Studying processes emanated from A^ω (i.e. arrows of the form $f : A^\omega \to B$) or to A^ω (i.e. arrows of the form $f : B \to A^\omega$) requires to have precise mathematical notions to work with. Category theory provides such notions via the concepts of algebras and coalgebras [68, 88]. In particular, the latter has several applications in the study of systems with infinite behaviors ([68, 88]). We extend SMCCs with initial algebras and final coalgebras for a natural class of functors, namely the one of polynomial functors. The resulting categories, called νSMCCs are simple yet powerful systems for studying processes, systems, resources etc...equipped with infinite and iterative behaviors. Among notions definable in such categories there is the one of exponential storage modality !. The informal intuition behind $!A$ is that the resource A is available *ad libitum*. The framework of νSMCCs allows to make this intuition formal and, more importantly, it reveals important features of the operator !.

2.1 SMCC with Fixed Points

Here we introduce SMCCs with initial algebras and final coalgebras for the class of polynomial functors[1]. Roughly, given a SMCC $\mathcal{C}$ polynomial functors over $\mathcal{C}$ are endofunctors (i.e. functors from $\mathcal{C}$ into $\mathcal{C}$) inductively built from the constant and identity functors, using product and tensor operations (see Defintion 20).

We obtain an interesting class of categories with a powerful underlying logic that allows to introduce recursion and corecursion in the $(\otimes, \&)$-fragment of linear logic.

We start by briefly introducing the class of polynomial functors. Given a SMCC $\mathcal{C}$ with tensor $\otimes$, unit 1, product & and terminal object $\top$, we define some specific endofunctors over $\mathcal{C}$.

Definition 20. 1. Let C be an object of $\mathcal{C}$. Define the constant functor $C : \mathcal{C} \to \mathcal{C}$ as follows.

$$\begin{array}{rcl} C(A) & = & C \\ C(f) & = & \mathsf{id}_C \end{array}$$

2. Given two functors $F, G : \mathcal{C} \to \mathcal{C}$, define the functor $F \,\&\, G : \mathcal{C} \to \mathcal{C}$ as follows.

$$\begin{array}{rcl} (F \,\&\, G)(A) & = & FA \,\&\, GA \\ (F \,\&\, G)(f) & = & Ff \,\&\, Gf \end{array}$$

(recall that $f \,\&\, g = \langle f \circ p, g \circ q \rangle$, so that we have a bifunctor &).

3. Given two functors $F, G : \mathcal{C} \to \mathcal{C}$, define the functor $F \otimes G : \mathcal{C} \to \mathcal{C}$ as follows.

$$\begin{array}{rcl} (F \otimes G)(A) & = & FA \otimes GA \\ (F \otimes G)(f) & = & Ff \otimes Gf \end{array}$$

[1]Usually polynomial functors are defined differently from the definition given here, see e.g. [68]. The reason is that, especially in the 'coalgebraic literature', one is concerned with endofunctors over Set, the category of sets and functions. The category Set being cartesian closed, induces functor operations via products and exponentials. The same happens for coproducts. This essentially defines the class of polynomial functors, which can be extended with the addition of the powerset functor, giving the class of polynomial Kripke functors. Again, see [68].

These are indeed functors. For the constant functor this is trivial to see, for the tensor and product functors the result follows from bifunctoriality of $\otimes$ and &. E.g.

$$\begin{aligned} (F \otimes G)(g \circ f) &= F(g \circ f) \otimes G(g \circ f) \\ &= (Fg \circ Ff) \otimes (Gg \circ Gf) \\ &= (Fg \otimes Gg) \circ (Ff \otimes Gf) \\ &= (F \otimes G)(g) \circ (F \otimes G)(f) \end{aligned}$$

Definition 21. The collection of polynomial functors is the least class of functors from $\mathcal{C}$ to $\mathcal{C}$ satisfying the following clauses.

1. The identity functor $1 : \mathcal{C} \to \mathcal{C}$ is a polynomial functor.

2. For each object C, the constant functor $C : \mathcal{C} \to \mathcal{C}$ is a polynomial functor.

3. If F and G are polynomial functors, then so are $F \otimes G$ and $F \,\&\, G$.

Polynomial functors, although quite specific, are sufficient for several applications. Moreover, they allow inductive reasoning on their structure. This feature allows us to prove several results about them. Note that we did not close the class of polynomial functors under composition. That would be redundant.

Lemma 3. If F and G are polynomial functors, then so is $G \circ F$.

Proof. Since G is polynomial we can prove the lemma by induction on G.

Case 1. Suppose G is 1. Then $1 \circ F = F$, which is polynomial by hypothesis.

Case 2. Suppose G is C, for an object C. Then we have that $G \circ F = C$. In fact,

$$\begin{aligned} (G \circ F)(A) &= (C \circ F)(A) \\ &= CFA \\ &= C \\ &= C(A) \end{aligned} \qquad \begin{aligned} (G \circ F)(f) &= (C \circ F)(f) \\ &= CF(f) \\ &= \mathsf{id}_C \\ &= C(f) \end{aligned}$$

Since C is polynomial we are done.

Case 3. Suppose G is $H \otimes K$. Then we prove

$$(H \otimes K) \circ F = (H \circ F) \otimes (K \circ F).$$

By induction hypothesis both $K \circ F$ and $H \circ F$ are polynomials, hence $G \circ F$ is polynomial too.

$$\begin{array}{rcl}
(G \circ F)(A) & = & ((H \otimes K) \circ F)(A) \\
& = & (H \otimes K)(FA) \\
& = & HFA \otimes KFA \\
& = & (H \circ F)(A) \otimes (K \circ F)(A) \\
& = & ((H \circ F) \otimes (K \circ F))(A)
\end{array}$$

$$\begin{array}{rcl}
(G \circ F)(f) & = & ((H \otimes K) \circ F)(f) \\
& = & (H \otimes K)(Ff) \\
& = & HFf \otimes KFf \\
& = & (H \circ F)(f) \otimes (K \circ F)(f) \\
& = & ((H \circ F) \otimes (K \circ F))(f)
\end{array}$$

Case 4. Suppose G is $H \,\&\, K$. Then we prove

$$(H \,\&\, K) \circ F = (H \circ F) \,\&\, (K \circ F).$$

By induction hypothesis both $K \circ F$ and $H \circ F$ are polynomials, hence $G \circ F$ is polynomial too.

$$\begin{array}{rcl}
(G \circ F)(A) & = & ((H \,\&\, K) \circ F)(A) \\
& = & (H \,\&\, K)(FA) \\
& = & HFA \,\&\, KFA \\
& = & (H \circ F)(A) \,\&\, (K \circ F)(A) \\
& = & ((H \circ F) \,\&\, (K \circ F))(A)
\end{array}$$

$$\begin{array}{rcl}
(G \circ F)(f) & = & ((H \,\&\, K) \circ F)(f) \\
& = & (H \,\&\, K)(Ff) \\
& = & HFf \,\&\, KFf \\
& = & (H \circ F)(f) \,\&\, (K \circ F)(f) \\
& = & ((H \circ F) \,\&\, (K \circ F))(f)
\end{array}$$

■

Given a SMCC $(\mathcal{C}, \otimes 1, \&, \top)$ as above, the class of polynomial functors over $\mathcal{C}$ organizes itself as a category $\mathcal{C}[X]$. Moreover, $\mathcal{C}[X]$ inherits the symmetric monoidal and cartesian structure.

Proposition 1. Let $\mathcal{C}$ be a symmetric cartesian monoidal category, with unit 1, tensor $\otimes$, product $\&$ and terminal object $\top$. Then we have a category $\mathcal{C}[X]$ whose objects are polynomial functors over $\mathcal{C}$ and whose morphisms are natural transformations between them. Moreover, $\mathcal{C}[X]$ is a SMCC, with tensor and product defined pointwise, and unit and terminal object defined by their respective constant functor.

The proof of this proposition is a long but straightforward, and thus it is omitted.

We can now define the notion of νSMCC.

Definition 22. A νSMCC is a SMCC $\mathcal{C}$ that has initial algebras and final coalgebras for the class of polynomial (endo)functors $\mathcal{C}[X]$ over $\mathcal{C}$.

νSMCCs provide a natural operational framework for studying and defining new processes (and operations). Indeed, all equations of the form

$$X \cong F(X)$$

for F polynomial can be solved in νSMCCs, and it is possible to distinguish between minimal and maximal solutions. As a consequence, we can define systems and processes via equations describing their behavior.

Example 2. We come back to the example of a self-replicating system. Recall that we looked at a self-replicating system

$$A^\omega \cong A \otimes A \otimes \cdots \otimes A \otimes \cdots$$

meaning that we have an infinite amounts of resources A. A^ω can be described via the equation

$$X \cong A \otimes X$$

We thus obtain A^ω as the maximal fixed point of the functor[2] $F(X) \cong A \otimes X$. Now, given a resource B, what do we need in order to be able to consume

[2] $F(X)$ is indeed polynomial, since it is nothing but the functor $A \otimes 1$.

B for producing A^ω? Clearly we have to be able to produce A from B, i.e. $B \to A$. Moreover, we have to be able to do so infinitely often. Since from B we can produce A, it is sufficient that from B we can also produce B itself. That is, we need a process $f : B \to A \otimes B$. This means that (B, f) is an $A \otimes X$-coalgebra. Finality of A^ω then gives $(\!|f|\!) : B \to A^\omega$.

2.2 Examples

νSMCCs are very rich structures in which all equations expressible via polynomial functors have both minimal and maximal solutions. One could wonder whether there exist 'concrete' examples of such categories, that is if there are examples of well-known and used categories, which are νSMC. This is indeed the case. We give two examples (in the final chapter the notion of pretopology is introduced: the collection of saturated sets of a pretopology is a further example of a νSMCC). The first one is a trivial one, given by the category of sets and functions, whereas the second one is given by the category of games and strategies.

Sets

The category Set has sets as objects and (set-theoretic) functions as arrows. It is well known that Set is cartesian, with binary product given by the cartesian product and terminal object given by the one element set (we used the word 'the' because all one element sets are isomorphic). Moreover, one can prove that Set is trivially monoidal, with tensor and unit given by the cartesian product and terminal object. Actually, Set has much more structure; it has binary coporducts, given by disjoint union and initial object, given by the empty set, as well as exponentials, given by function spaces (see e.g. [11] for details).

In [68] it is proved that polynomial functors on Sets preserve both ω limits and colimits (see [10, 11, 16, 68] for definitions), and that they are continuous in a sense similar to continuity in CPOs [3]. In particular, such continuity allows to prove that all such functors have both initial algebras and final coalgebras, which can be constructed with an adaptation of Kleene's Approximation Theorem [3, 46, 91]. As a consequence, Set is a νSMCC.

Games

The category of games and strategies was introduced by Abramsky [4] as a refinement of Blass' framework [29], in order to provide a sound and complete semantics to linear logic [1]. The basic idea is that propositions are games played by two players, abstractly called *proponent* and *opponents* (but also *prover* and *refuter*, or *system* and *environment* are used). A proposition is provable if proponent has a winning strategy for the game associated with that proposition. We briefly recall basic definitions and define the category $\mathcal{G}$ of games and strategies (the reader can consult [4] for details). Such category is symmetric monoidal and cartesian. Moreover, as shown in [2, 73] a fixed point theorem with respect to a class of continuous functors for such category can be proved. In particular, polynomial functors are continuous in $\mathcal{G}$, so that we recognize $\mathcal{G}$ to be a νSMCC.

A game has two participants, P and O, called proponent and opponent respectively. A play of the game consists of a finite or infinite sequence of moves alternately by O and P. In the games we consider O always moves first. Given a set A we will use letters $s, t, u \ldots$ to range over finite sequences of elements of A (i.e. A^*), and reserve the symbol ε to denote the empty sequence. Given $a \in A$ and $s \in A^*$, $as \in A^*$ denotes the concatenation of a to s. Similarly for $s, t \in A^*$, we write st for their concatenation. As usual, $|s|$ denotes the length of s, and s_i is the i-th element of s. Given a set A, we write $s \upharpoonright A$ for the restriction of s to elements in A.

A game $G = (M_G, \lambda_G, P_G)$ consists of a set of moves M_G, a labeling function $\lambda_G : M_G \to \{P, O\}$ telling which player performed the last move, and a set P_G giving valid plays in G. In particular, G is a subset M_G^{Alt},the set of all $s \in M_G^*$ such that $\forall 1 \leq i \leq |s|$, $\lambda_G(s_i) = P$, if i is even, $\lambda_G(s_i) = O$, if i is odd.

A strategy for a game G is a sequence $\sigma \subseteq P_G^{even}$ (i.e. $\sigma \subseteq P_G$ and for any $s \in \sigma$, $|s|$ is even) such that:

1. $\varepsilon \in \sigma$;

2. If $sab \in \sigma$, then $s \in \sigma$ (i.e. σ is prefix closed);

3. If $sab \in \sigma$ and $sac \in \sigma$, then $b = c$ (i.e. σ is deterministic).

Given games $A = (M_A, \lambda_A, P_A)$ and $B = (M_B, \lambda_B, P_B)$ we can construct several new games:

1. The game $A \otimes B$ is defined as

$$\begin{array}{rcl} M_{A\otimes B} & = & M_{A+B} \\ \lambda_{A\otimes B} & = & [\lambda_A, \lambda_B] \\ P_{A\otimes B} & = & \{s \in M^{Alt}_{A\otimes B} \mid s \upharpoonright M_A \in P_A,\ s \upharpoonright M_B \in P_B\} \end{array}$$

2. The game $A \multimap B$ is defined as

$$\begin{array}{rcl} M_{A\multimap B} & = & M_{A+B} \\ \lambda_{A\multimap B} & = & [\overline{\lambda_A}, \lambda_B] \\ P_{A\multimap B} & = & \{s \in M^{Alt}_{A\multimap B} \mid s \upharpoonright M_A \in P_A,\ s \upharpoonright M_B \in P_B\} \end{array}$$

   where $\overline{\lambda_A}$ is defined by $\overline{\lambda_A}(m) = P$ if $\lambda_A(m) = O$ and $\overline{\lambda_A}(m) = O$ is $\lambda_A(m) = P$.

3. The game 1 is defined as

$$\begin{array}{rcl} M_1 & = & \emptyset \\ \lambda_1 & = & \emptyset \\ P_1 & = & \{\varepsilon\} \end{array}$$

4. The game $A \,\&\, B$ is defined as

$$\begin{array}{rcl} M_{A\&B} & = & M_{A+B} \\ \lambda_{A\&B} & = & [\lambda_A, \lambda_B] \\ P_{A\&B} & = & \{i_l(s) \mid s \in P_A\} \cup \{i_r(t) \mid t \in P_B\} \end{array}$$

   where i_l and i_r are the standard inclusion maps.

Games and strategies constitute a category, the category $\mathcal{G}$ of games and strategies. Objects are games, whereas an arrow $\sigma : A \to B$ is a strategy σ for $A \multimap B$. Given strategies/arrows $\sigma : A \to B$ and $\tau : B \to C$, we define $\tau \circ \sigma : A \to C$ by

$$\begin{array}{rcl} \tau \circ \sigma & = & \{s \upharpoonright A, C \mid s \in \sigma \mid \tau\} \\ \sigma \mid \tau & = & \{s \in (M_A + M_B + M_C)^* \mid s \upharpoonright A, B \in \sigma,\ s \upharpoonright B, C \in \tau\} \end{array}$$

The identity arrow $\mathsf{id}_A : A \to A$ is given by the so-called copy-cat strategy (A_1 and A_2 denote the different occurrences of A in $A \multimap A$)

$$\mathsf{id}_A = \{s \in P^{even}_{A_1 \multimap A_2} \mid \forall t \text{ even length prefix of } s, t \upharpoonright A_1 = t \upharpoonright A_2\}$$

The category $\mathcal{G}$ has more structure. In fact, it is monoidal with tensor product of A and B given by $A \otimes B$. Given strategies $\sigma : A \to B$ and $\tau : A' \to B'$ one has the tensor strategy $\sigma \otimes \tau : A \otimes A' \to B \otimes B'$ defined by

$$\sigma \otimes \tau = \{s \in P^{even}_{A \otimes A' \multimap B \otimes B'} \mid s \upharpoonright A, B \in \sigma,\ s \upharpoonright A', B' \in \tau\}$$

The tensor unit is given by the empty-game 1. It is also possible to prove that $A \& B$ gives the cartesian product of A and B, that 1 is also the terminal object of $\mathcal{G}$ and that $A \multimap B$ is an exponential with respect to $\otimes$.

Remark. A more interesting category than $\mathcal{G}$, is the category of games and winning strategies. These are defined by specifying which valid plays in P_G are winning for player P in the game G. All definitions given so far can be transposed to the category of games and winning strategies (see [4]). In [1] it is proved that games provide a sound and complete semantics for multiplicative linear logic, where formulas are interpreted as games, and proofs as winning strategies.

In [2] an order $\leq$ on games is defined by means of the full subgame relation. That is, given games A and B we say that A is smaller than B, written $A \leq B$, if

$$\begin{array}{rcl} M_A & \subseteq & M_B \\ \lambda_A & = & \lambda_B \upharpoonright M_A \\ P_A & = & P_B \cap M_A^{Alt} \end{array}$$

The order is then extended to strategies. Moreover, $\leq$ is a complete partial order (CPO) on games, with least element 1. By the fixed point theorem on CPOs [3], every equations of the form

$$X \cong G(X)$$

has a least solution, for G continuos. In particular, polynomial functors are continuos, so that we have games $\mu X.F(X)$ for any polynomial functor F.

Proving that $\mathcal{G}$ has maximal solutions to the above equations requires a bit more work, but the reader can consult [73] for a deeper analysis of the subject.

2.3 Logic

Having an equational theory for initial algebras, final coalgebras and the monoidal and cartesian constructors, it is easy to define νSMCCs as deductive systems. More interesting at this point, is to extract a syntactic calculus for νSMCCs, much in the same way as one can extract the $(\otimes, 1)$-fragment of linear logic from monoidal categories. In the remaining part of this section we will define such logic and a calculus with arrow terms for that. The design is simply based on providing a fully syntactic counterpart of νSMCCs. The resulting logic is simple, yet powerful and it can be studied independently from its categorical counterpart, in the sense that it is an interesting logic *per se* (as linear logic can be studied independently from monoidal categories). The calculus we give is clearly category theory-based, but it will be the base for an independent equivalent sequent calculus. This gives a standard syntactical presentation of the logic. Finally, the equational system for arrows provide a nice notion of equality between proofs.

Definition 23. Let us fixed a collection Prop whose elements are called propositional variables. Let X be a special symbol (the idea is that we have just one variable). Closed and open formulas (denoted by A and F respectively) are simultaneously generated by the following grammars

$$\begin{array}{rcl} A & ::= & p \mid 1 \mid \top \mid A \otimes A \mid A \,\&\, A \mid \nu X.F \mid \mu X.F \\ F & ::= & X \mid A \mid F \otimes F \mid F \,\&\, F \end{array}$$

where p is in Prop.

Roughly, a closed formula represents an object, whereas an open formula and endofunctor.

Open formulas are open in just one variable X, which is a syntactic device for the identity functor. The reason is that their intuitive meaning is that of an endofunctor $F : \mathcal{C} \to \mathcal{C}$. We could have considered formulas with arbitrary many free variables. The choice of working with a 'single-variable' grammar makes proofs and definitions much easier and natural then their more general counterpart. Moreover, the single variable fragment is already quite expressive. In any case, we will sketch the case for multiple variables at the end of this chapter. Although we have a single variable, in order to avoid syntactic problems we assume to have more variables so to be able to use renaming of bound variables. For example, we regard the formula

$$\mu X.X \,\&\, (A \otimes \nu X.X \otimes B)$$

to be

$$\mu X.X \mathbin{\&} (A \otimes \nu Y.Y \otimes B).$$

That is, we rename one of the bound variables X as Y. Indeed, each open subformula contains at most one free variable, and it does not really matter how we call it. This will be implicitly assumed from now on. We will also assume several forms of variable conventions, thus assuming that in a formula each operator μ and ν bounds exactly one variable, and that all variables bounded by different operators have different names.

Sometimes the notation μF and νF will be used in place of $\mu X.F$ and $\nu X.F$, provided this does not create confusion.

Since open formulas contain at most one free variable, we can define an operation that substitutes all occurrences of that variable with a given (closed) formula.

Definition 24. Given an open formula F and a closed formula A, we define $F(A)$ (abbreviation of $F[X := A]$) by recursion on F as follows:

$$\begin{array}{rcl} X(A) & = & A \\ C(A) & = & C \\ (F \otimes G)(A) & = & F(A) \otimes G(A) \\ (F \mathbin{\&} G)(A) & = & F(A) \mathbin{\&} G(A) \end{array}$$

As usual, we will often write FA for $F(A)$. Indeed, $F(A)$ is a closed formula.

We now give inference rules for the calculus. Such rules generate arrows (or combinators) i.e. expressions of the form $f : A \to B$, for (closed) formulas A and B. Rules are distinguished between axioms, i.e. zero-ary rules, and proper rules. Axioms are given in Figure 2.1 whereas proper inference rules are given in Figure 2.2.

Given an open formula F and an arrow $f : A \to B$, we can define a new arrow $F(f) : F(A) \to F(B)$. As for formulas, we will often write Ff for $F(f)$.

Definition 25. Let F be an open formula and $f : A \to B$. Define the arrow $F(f) : F(A) \to F(B)$ by recursion on F as follows:

$$\begin{array}{rcl} X(f) & = & f \\ C(f) & = & \mathsf{id}_C \\ (F \otimes G)(f) & = & Ff \otimes Gf \\ (F \mathbin{\&} G)(f) & = & Ff \mathbin{\&} Gf \end{array}$$

$$\mathsf{id}_A : A \to A \qquad !_A : A \to \top$$

$$p_{A,B} : A \,\&\, B \to A \qquad q_{A,B} : A \,\&\, B \to B$$

$$a_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C \qquad a^{-1}_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

$$l_A : 1 \otimes A \to A \qquad l^{-1}_A : A \to 1 \otimes A$$

$$r_A : A \otimes 1 \to A \qquad r^{-1}_A : A \to A \otimes 1$$

$$s_{A,B} : A \otimes B \to B \otimes A \qquad s^{-1}_{A,B} : B \otimes A \to A \otimes B$$

$$\mathsf{out} : \nu F \to F(\nu F) \qquad \mathsf{in} : F(\mu F) \to \mu F$$

Figure 2.1: Basic arrows for νSMCCs.

$$\frac{f : C \to A \qquad g : C \to B}{\langle f, g \rangle : C \to A \,\&\, B} \qquad \frac{f : A \to B \qquad g : B \to C}{g \circ f : A \to C} \qquad \frac{f : A \to C \qquad g : B \to D}{f \otimes g : A \otimes B \to C \otimes D}$$

$$\frac{f : A \to F(A)}{(\!| f |\!) : A \to \nu F} \qquad \frac{f : F(A) \to A}{[\![f]\!] : \mu F \to A}$$

Figure 2.2: Arrows-generating rules for νSMCCs.

The above definition allows us to prove the admissibility of the following rule.

Proposition 2. The rule

$$\frac{f : A \to B}{Ff : FA \to FB}$$

is admissible.

Proof. The proof is by induction on F. If F is X, then the thesis trivially follows. If F is the constant functor C, then Cf is $\mathsf{id}_C : C \to C$ which is an axiom. If F is $G \otimes H$, then we have

$$\begin{array}{rcl} (G \otimes H)(A) & = & GA \otimes HA \\ (G \otimes H)(f) & = & Gf \otimes Ff \end{array}$$

From $f : A \to B$, by induction hypothesis on G and H we obtain

$$\begin{array}{rcl} Gf & : & GA \to GB \\ Hf & : & HA \to HB \end{array}$$

And thus, by the bifunctoriality rule for $\otimes$,

$$Gf \otimes Hf : GA \otimes HA \to GB \otimes HB.$$

If F is $G \,\&\, H$ we proceed as above. ■

We can now define an equational theory for arrows[3]. This is defined by means of equations for SMCCs (defined as deductive systems) with in addition equational laws in Figure 2.3.

We now state and prove some useful syntactic lemmas. These will give functoriality of open formulas.

Lemma 4. The following rule is admissible.

$$\frac{f = g}{F(f) = F(g)}$$

[3]There is an overloading in the notation. The equality symbol $=$ is used to denote both standard syntactic/definitional equality, as e.g. in statements like 'consider the functor $F(X) = A \otimes X$', and the notion of equality for arrows defined in Figure 2.3. The context should avoid confusion between these two usage of $=$.

$$\frac{p \circ h = f \qquad g \circ h}{\overline{h = \langle f, g \rangle}} \qquad \frac{f = f' \qquad g = g'}{f \otimes f' = g \otimes g'} \qquad \frac{f : A \to \top}{\overline{f = !_A}}$$

$$\frac{F(g) \circ f = \mathsf{out} \circ g}{\overline{g = (\!| f |\!)}} \qquad \frac{F(g) \circ \mathsf{in} = f \circ g}{\overline{g = [\![f]\!]}}$$

Figure 2.3: Equational laws for νSMCCs.

Proof. The proof is an easy induction on F. For $F = X$ we are done by hypothesis. For F constant functor C we have that both Cf and Cg are equal id_C. Suppose $F = G \otimes H$. Then we have

$$\begin{aligned} (G \otimes H)(f) &= Gf \otimes Hf \\ &= Gg \otimes Hg \\ &= (G \otimes H)(g) \end{aligned}$$

where in the second line we used the induction hypothesis. The case for $F = G \,\&\, H$ is similar. ■

Remark. An examination of rules for deriving arrows/combinators in the above system, shows that we are essentially dealing with four constructors: $\otimes$, $\langle \_, \_ \rangle$, $(\!| \_ |\!)$ and $[\![\_]\!]$. Clearly, we would like our notion of equality $=$ for arrows to be a congruence with respect to these constructors. This amounts to require the admissibility of the following rules

$$\frac{f = f' \qquad g = g'}{\langle f, g \rangle = \langle f', g' \rangle} \qquad \frac{f = f' \qquad g = g'}{f \otimes g = f' \otimes g'} \qquad \frac{f = g}{(\!| f |\!) = (\!| g |\!)} \qquad \frac{f = g}{[\![f]\!] = [\![g]\!]}$$

We already proved the first one, whereas the second is explicitly given in the calculus. The remaining two rules can be easily proved, using the fact the $(\!| \_ |\!)$ and $[\![\_]\!]$ give universal arrows. As an example, we prove that if $f = g$, then $(\!| f |\!) = (\!| g |\!)$. Applying the universality rule and the hypothesis $f = g$ we obtain

$$\cfrac{\cfrac{F(\!| g |\!) \circ g = \mathsf{out} \circ (\!| g |\!)}{F(\!| g |\!) \circ f = \mathsf{out} \circ (\!| g |\!)}}{(\!| f |\!) = (\!| g |\!)}$$

The equation $F(\!|g|\!) \circ g = \mathsf{out} \circ (\!|g|\!)$ then follows from the universality rule for $(\!|\_|\!)$, with the axiom $(\!|g|\!) = (\!|g|\!)$.

Lemma 5. Given arrows $f : A \to C$ and $g : B \to D$, define

$$f \,\&\, g = \langle f \circ p_{A,B}, g \circ q_{A,B} \rangle : A \,\&\, B \to C \,\&\, D$$

Then, the following rule is admissible

$$\frac{f : A \to C \qquad g : B \to D}{f \,\&\, g : A \,\&\, B \to C \,\&\, D}$$

Moreover, we have the following equations

$$\begin{array}{rcl} (f \,\&\, g) \circ (f' \circ g') & = & (f \circ f') \,\&\, (g \,\&\, g') \\ \mathsf{id}_A \,\&\, \mathsf{id}_B & = & \mathsf{id}_{A\&B} \end{array}$$

Proof. Admissibility of the rule is straightforward. Equations can easily proved by iterated application of the rule for universality of $\langle f, g \rangle$. ∎

We now define a notion of composition for open formulas.

Definition 26. Given open formulas F and G we define the open formula $F \circ G$ by recursion on F as follows:

$$\begin{array}{rcl} X \circ G & = & G \\ C \circ G & = & C \\ (H \otimes K) \circ G & = & (H \circ G) \otimes (K \circ G) \\ (H \,\&\, K) \circ G & = & (H \circ G) \,\&\, (K \circ G) \end{array}$$

It is easy to see that $F \circ G$ is indeed an open formula and that for any formula A and arrow f we have

$$\begin{array}{rcl} (F \circ G)(A) & = & F(G(A)) \\ (F \circ G)(f) & = & F(G(f)) \end{array}$$

As a consequence, we see that $\circ$ is indeed a composition[4].

[4]In the sense that

$$F \circ (G \circ H) = (F \circ G) \circ H$$

and

$$F \circ X = F = X \circ F$$

.

Proposition 3. Open formulas are functorial. That is, let F be an open formula, A and object and $f : A \to B$ and $g : B \to A$ arrows. Then we have the following equations

$$\begin{aligned} F(\mathsf{id}_A) &= \mathsf{id}_{FA} \\ F(g \circ f) &= Fg \circ Ff \end{aligned}$$

Proof. The proof is by induction on F.

1. Suppose $F = X$. Then we have

$$\begin{aligned} X(\mathsf{id}_A) &= \mathsf{id}_A \\ &= \mathsf{id}_{X(A)} \end{aligned}$$

$$\begin{aligned} X(g \circ f) &= g \circ f \\ &= Xg \circ Xf \end{aligned}$$

2. Suppose $F = C$. Then we have

$$\begin{aligned} C(\mathsf{id}_A) &= \mathsf{id}_C \\ &= \mathsf{id}_{C(A)} \end{aligned}$$

$$\begin{aligned} C(g \circ f) &= \mathsf{id}_C \\ &= \mathsf{id}_C \circ \mathsf{id}_C \\ &= Cg \circ Cf \end{aligned}$$

3. Suppose $F = G \otimes H$. Then we have

$$\begin{aligned} (G \otimes H)(\mathsf{id}_A) &= G(\mathsf{id}_A) \otimes H(\mathsf{id}_A) \\ &= \mathsf{id}_{GA} \otimes \mathsf{id}_{HA} \\ &= \mathsf{id}_{GA \otimes HA} \\ &= \mathsf{id}_{(G \otimes H)(A)} \end{aligned}$$

$$\begin{aligned} (G \otimes H)(g \circ f) &= G(g \circ f) \otimes H(g \circ f) \\ &= (Gg \circ Gf) \otimes (Hg \circ Hf) \\ &= (Gg \otimes Hg) \circ (Gf \otimes Hf) \\ &= (G \otimes H)(g) \circ (G \otimes H)(f) \end{aligned}$$

where we used bifunctoriality equations.

4. If $F = G \,\&\, H$ we proceed as above, since we proved bifunctoriality equations for &.

■

The system given in Figure 2.1 and Figure 2.2 has few rules and several axioms (i.e. basic arrows). We can design an arrow-free system in which axioms are converted into proper inference rules. Such system gives an alternative presentation of the logic, and it will be the base of the sequent calculus we will study. The system is given in Figure 2.4.

$$\frac{}{A \to A} \qquad \frac{A \to B \quad B \to C}{A \to C}$$

$$\frac{A \to C \quad B \to D}{A \otimes B \to C \otimes D}$$

$$\begin{array}{c} A \to B \otimes C \\ \hline\hline A \to C \otimes B \end{array} \qquad \begin{array}{c} D \to (A \otimes B) \otimes C \\ \hline\hline D \to A \otimes (B \otimes C) \end{array}$$

$$\begin{array}{c} A \to 1 \otimes B \\ \hline\hline A \to B \end{array} \qquad \begin{array}{c} A \to B \otimes 1 \\ \hline\hline A \to B \end{array}$$

$$\frac{}{A \to \top} \qquad \begin{array}{c} C \to A \quad C \to B \\ \hline\hline C \to A \,\&\, B \end{array}$$

$$\frac{F(\nu F) \to A}{\nu F \to A} \qquad \frac{A \to FA}{A \to \nu F}$$

$$\frac{F(A) \to A}{\mu F \to A} \qquad \frac{A \to F(\mu F)}{A \to \mu F}$$

Figure 2.4: Logical system for νSMCCs.

In the rest of this chapter we refer to the proof system given in Figure 2.1 and Figure 2.2 as the 'combinatorial system', and to the proof system given in Figure 2.4 as the 'logical system'. The logical system is 'arrow-free':

however, we can recover arrows for it via suitable translations between the two systems. In fact, we have the following

Proposition 4. The logical and combinatorial systems are equivalent in the following sense. If $f : A \to B$ is provable in the combinatorial system, then $A \to B$ is provable in the logical system. Viceversa, if $A \to B$ is provable in the logical system, then there exists an arrow f such that $f : A \to B$ is provable in the combinatorial system.

Proof. We already discussed the equivalence of the two systems for the non-algebra/coalgebra part. Note that the two systems share two algebra/coalgebra rules, namely

$$\frac{f : A \to F(A)}{(\!| f |\!) : A \to \nu F} \qquad \frac{A \to F(A)}{A \to \nu F}$$

$$\frac{f : F(A) \to A}{[\![f]\!] : \mu F \to A} \qquad \frac{F(A) \to A}{\mu F \to A}$$

It remains to prove that the other two algebra/coalgebra rules of the logical system are admissible in the combinatorial system, and that the axioms $\nu F \to F(\nu F)$ and $F(\mu F) \to \mu F$ are derivable in the logical system. The former is schematically proved as follows.

$$\frac{F(\nu F) \to A}{\nu F \to A} \qquad \frac{\mathsf{out} : \nu F \to F(\nu F) \qquad f : F(\nu F) \to A}{f \circ \mathsf{out} : \nu F \to A}$$

$$\frac{A \to F(\mu F)}{A \to \mu F} \qquad \frac{f : A \to F(\mu F) \qquad \mathsf{in} : F(\mu F) \to \mu F}{\mathsf{in} \circ f : A \to \mu F}$$

For the latter, consider the following derivation.

$$\frac{F(\nu F) \to F(\nu F)}{\nu F \to F(\nu F)} \qquad \frac{F(\mu F) \to F(\mu F)}{F(\mu F) \to \mu F}$$

■

The above proof gives also a way to introduce new arrows, for example we could consider the rules

$$\frac{f : F(\nu F) \to A}{\mathsf{unfold}(f) : \nu F \to A} \qquad \frac{f : A \to F(\mu F)}{\mathsf{fold}(f) : A \to \mu F}$$

with

$$\begin{array}{rcl} \mathsf{unfold}(f) & = & f \circ \mathsf{out} \\ \mathsf{fold}(f) & = & \mathsf{in} \circ f \end{array}$$

Moreover, the equational theory associated with the combinatorial system gives an equational theory for proofs in the logical system. Consider, e.g. a principal cut for the new rules, i.e. a situation of the form (we look only at the 'ν case', the μ one is dual)

$$\frac{\dfrac{A \to F(A)}{A \to \nu F} \qquad \dfrac{F(\nu F) \to B}{\nu F \to B}}{A \to B}$$

It is natural to transform the above derivation as follows

$$\frac{A \to F(A) \qquad \dfrac{\begin{array}{c}\vdots \\ F(A) \to F(\nu F)\end{array} \qquad F(\nu F) \to B}{F(A) \to B}}{A \to B}$$

Clearly, a cut elimination procedure based on the above transformation in general will not terminate. The problem of finding normalizing cut elimination procedure for fixed point logics is an hard one, and most of the proposed solutions require to leave standard (i.e. finite) proof trees in favor of cyclic or infinite structures, see e.g. [80, 92, 102, 103]. Nevertheless, categorical equations give a natural notion of equality, although it does not seem to have nice algorithmic properties.

Assigning arrows to the above proofs we obtain

$$\frac{\dfrac{f : A \to F(A)}{(\!| f |\!) : A \to \nu F} \qquad \dfrac{g : F(\nu F) \to B}{\mathsf{unfold}(g) : \nu F \to B}}{\mathsf{unfold}(g) \circ (\!| f |\!) : A \to B}$$

and

$$\dfrac{f : A \to F(A) \qquad \dfrac{F(\!|f|\!) : F(A) \to F(\nu F) \qquad g : F(\nu F) \to B}{g \circ F(\!|f|\!) : F(A) \to B}}{g \circ F(\!|f|\!) \circ f : A \to B}$$

Therefore, we would like the equation

$$\mathsf{unfold}(f) \circ (\!|f|\!) = g \circ F(\!|f|\!) \circ f$$

to be provable in the categorical equational system. By definition of $\mathsf{unfold}(\_)$ the above equations becomes

$$g \circ \mathsf{out} \circ (\!|f|\!) = g \circ F(\!|f|\!) \circ f$$

which holds by very definition of final coalgebra.

Another interesting equality can be proved for the so-called identity expansion (which in general does not hold for fixed point logics [80]). In logical terms, identity expansions are proofs transformations expanding every instance of the identity axioms for compound formulas in a proof of the same sequent having only atomic instances of the identity axioms. Under the Curry-Howard correspondence identity expansions correspond to the so-called η-expansion rules [53, 57, 96, 106]. In the logical system above, for example we have the following transformation

$$A \,\&\, B \to A \,\&\, B \quad \mapsto \quad \dfrac{A \to A \qquad B \to B}{A \,\&\, B \to A \,\&\, B}$$

where we use bifunctoriality of product. Categorical equations identify the two above derivations. Indeed, such identification corresponds to the equation

$$\mathsf{id}_{A\&B} = \mathsf{id}_A \,\&\, \mathsf{id}_B$$

which is trivially given by bifunctoriality of &. In fixed point logics, such transformations are quite problematic, since in general identity axioms like $\nu F \to \nu F$ do not reduce to derivations with identity axiom on simpler formulas. For example, we have the expansion

$$\nu F \to \nu F \quad \mapsto \quad \dfrac{\dfrac{F(\nu F) \to F(\nu F)}{\nu F \to F(\nu F)}}{\nu F \to \nu F}$$

Still, we would like to identify the two above proofs. This is equivalent to prove the arrow equality

$$\mathsf{id}_{\nu F} = (\!|\mathsf{unfold}(\mathsf{id}_{F(\nu F)})|\!)$$

Such equality is provable according the categorical equations as follows

$$\begin{aligned} (\!|\mathsf{unfold}(\mathsf{id}_{F(\nu F)})|\!) &= (\!|\mathsf{id}_{F(\nu F)} \circ \mathsf{out}|\!) \\ &= (\!|\mathsf{out}|\!) \\ &= \mathsf{id}_{\nu F} \end{aligned}$$

where $(\!|\mathsf{out}|\!) = \mathsf{id}_{\nu F}$ holds by the universality rule for $(\!|\_|\!)$. Note that another possible expansion of $\nu F \to \nu F$ is

$$\cfrac{\cfrac{\cfrac{\cfrac{F(\nu F) \to F(\nu F)}{\nu F \to F(\nu F)}}{F(\nu F) \to F^2(\nu F)}}{F(\nu F) \to \nu F}}{\nu F \to \nu F}$$

This gives the arrow

$$\mathsf{unfold}((\!|F(\mathsf{unfold}(\mathsf{id}_{F(\nu F)}))|\!))$$

which can be rewritten as

$$\begin{aligned} \mathsf{unfold}((\!|F(\mathsf{unfold}(\mathsf{id}_{F(\nu F)}))|\!)) &= (\!|F(\mathsf{unfold}(\mathsf{id}_{F(\nu F)}))|\!) \circ \mathsf{out} \\ &= (\!|F(\mathsf{id}_{F(\nu F)} \circ \mathsf{out})|\!) \circ \mathsf{out} \\ &= (\!|F(\mathsf{id}_{F(\nu F)}) \circ F(\mathsf{out})|\!) \circ \mathsf{out} \\ &= (\!|\mathsf{id}_{F^2(\nu F)} \circ F(\mathsf{out})|\!) \circ \mathsf{out} \\ &= (\!|F(\mathsf{out})|\!) \circ \mathsf{out} \end{aligned}$$

We now ask whether or not the two expansions of $\nu F \to \nu F$ are equal. This amounts to prove the equation

$$\mathsf{id}_{\nu F} = (\!|F(\mathsf{out})|\!) \circ \mathsf{out}$$

This can be proved using the fact that $\mathsf{id}_{\nu F} = (\!|\mathsf{out}|\!)$, and thus we can use universality of $(\!|\_|\!)$. This requires to prove

$$\mathsf{out} \circ (\!|F(\mathsf{out})|\!) \circ \mathsf{out} = F((\!|F(\mathsf{out})|\!) \circ \mathsf{out}) \circ \mathsf{out}$$

i.e.

$$\mathsf{out} \circ (\!|F(\mathsf{out})|\!) \circ \mathsf{out} = F(\!|F(\mathsf{out})|\!) \circ F(\mathsf{out}) \circ \mathsf{out}$$

which holds since universality of $(\!|F(\mathsf{out})|\!)$ gives

$$F(\!|F(\mathsf{out})|\!) \circ F(\mathsf{out}) = \mathsf{out} \circ (\!|F(\mathsf{out})|\!).$$

Finally, we observe that our calculus is trivially complete with respect to the obvious categorical semantics. In fact, we have the following

Proposition 5. Let Comb the category of closed formulas and combinators. That is closed formulas are objects and arrows are equivalence classes of combinators $f : A \to B$ modulo $=$ (defined by the equational theory of combinators). Then Comb is a νSMCC.

Proof. Standard calculations show that Comb is a SMCC, with tensor $\otimes$, unit 1, product & and terminal object $\top$. Moreover, it is easy to prove that polynomial functors over Comb are open formulas (we already proved their functoriality), and for a polynomial functor F, $\mu X.F$ and $\nu X.F$ give the initial algebra of F and the final coalgebra of F respectively. ■

Clearly, the combinatorial calculus can be interpreted in νSMCC, with open formulas interpreted as polynomial functors.

The framework defined in this chapter allows to give new explanations and results concerning exponential modalities. These will be formulated in the poset category[5] given by the logical system, which provides a proof-theoretical framework expressed in categorical terms. In the next chapter we introduce a sequent calculus equivalent to the logical system defined above. We use the former to introduce exponentials and to prove that we can recover exponential modalities as specific fixed points. The relationship between these fixed points will be then studied further, using ideas from category theory.

We conclude this chapter with a brief digression concerning open formulas with arbitrary many free variables.

[5]Given a category $\mathcal{C}$, the poset category associated to $\mathcal{C}$ is the category whose objects are those of $\mathcal{C}$ and, given two objects A and B, there is at most one arrow from A to B, depending whether there exists an arrow $f : A \to B$ in $\mathcal{C}$.

Digression: Non-Restricted Fixed Point Calculus

We defined open formulas with the goal of obtaining a syntactic definition of polynomial functors. These are, in particular, endofunctors, i.e. functors from a category $\mathcal{C}$ in itself. As a consequence, we had to require open formulas to contain at most one free variable. However, from a syntactical perspective it seems more natural to consider a countable set Var of variables, and to define open formulas via the grammar

$$F ::= A \mid X \mid F \otimes F \mid F \,\&\, F \mid \mu X.F \mid \nu X.F$$

where A is a closed formula, and $X \in \mathsf{Var}$. Therefore, expressions like

$$\mu X.(X \,\&\, (\nu Y.X \otimes Y))$$

are well-formed formulas.

Allowing open formulas to contain arbitrary many free variables generalizes the class of polynomial functors to the class of n-ary polynomial functors. These are functors $F : \mathcal{C}^n \to \mathcal{C}$, for arbitrary $n \geq 0$, which are polynomial in each component (much in the same way as a bifunctor is functorial in each component[6]. That is, given $F(X_1, \ldots, X_n) : \mathcal{C}^n \to \mathcal{C}$ we have that for any $i \leq n$ and $C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_n$ objects of $\mathcal{C}$,

$$F(C_1, \ldots, C_{i-1}, X, C_{i+1}, \ldots, C_n) : \mathcal{C} \to \mathcal{C}$$

is a polynomial functor.

However, it is not clear whether, given a functor $F(X, X_1, \ldots, X_n) : \mathcal{C}^{n+1} \to \mathcal{C}$, both

$$\begin{aligned} \mu X.F(X, X_1, \ldots, X_n) &: \mathcal{C}^n \to \mathcal{C} \\ \nu X.F(X, X_1, \ldots, X_n) &: \mathcal{C}^n \to \mathcal{C} \end{aligned}$$

are indeed functors. Actually, it is not even clear how to define their action on arrows. This is problematic, since we introduced formulas of the form $\eta X.F(X, X_1, \ldots, X_n)$ (for $\eta \in \{\mu, \nu\}$) but we do not have a semantics for

[6] In fact, taking $n = 1$ we obtain polynomial functors, taking $n = 2$ we have polynomial bifunctros etc...).

them. We now sketch those results and definitions necessary to prove that we indeed have the above functors.

Recall that given categories $\mathcal{C}$ and $\mathcal{D}$, we can define the functor category $\mathcal{D}^{\mathcal{C}}$ whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$, and whose arrows are natural transformations between such functors. The functor category $\mathcal{D}^{\mathcal{C}}$ is the exponential object of $\mathcal{C}$ and $\mathcal{D}$ (see e.g. [11] for definitions) in the category Cat of (small) categories and functors. This makes Cat cartesian closed [10, 11, 16, 71], with binary products given by the product $\times$ of categories (see Definition 12), terminal object given by the one-object category[7], and exponentials given by functor categories. As a consequence, we can *currying* a functor, i.e. we can uniquely associate to a functor

$$F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \times \mathcal{C} \to \mathcal{D}$$

its transpose

$$\lambda(F) : \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathcal{D}^{\mathcal{C}}$$

In particular, given a functor

$$F(X_1, \ldots, X_n, X) : \mathcal{C}_1 \times \cdots \times C_n \times C \to \mathcal{D}$$

we can transpose F to (we write F for $\lambda(F)$)

$$F(X_1, \ldots, X_n, X) : \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathcal{D}^{\mathcal{C}}$$

Given objects C_i in $\mathcal{C}_i$ and arrows $f_i : C_i \to C'_i$ in $\mathcal{C}_i$ (for $i \leq n$), we have functors

$$\begin{array}{c} F(C_1, \ldots, C_n, X) : \mathcal{C} \to \mathcal{D} \\ F(C'_1, \ldots, C'_n, X) : \mathcal{C} \to \mathcal{D} \end{array}$$

defined by parametrization, that is e.g.

$$\begin{array}{rcl} F(C_1, \ldots, C_n, X)(C) & = & F(C_1, \ldots, C_n, C) \\ F(C_1, \ldots, C_n, X)(f) & = & F(\mathsf{id}_{C_1}, \ldots, \mathsf{id}_{C_n}, f) \end{array}$$

Moreover, we have a natural transformation

$$F(f_1, \ldots, f_n, X) : F(C_1, \ldots, C_n, X) \Rightarrow F(C'_1, \ldots, C'_n, X)$$

[7]The one-object category 1 has only one object, say $*$, and a unique arrow $\mathsf{id} : * \to *$.

defined for an object C in $\mathcal{C}$ by the arrow

$$F(f_1, \ldots, f_n, \mathsf{id}_C) : F(C_1, \ldots, C_n, C) \to F(C'_1, \ldots, C'_n, C)$$

In what follows it is useful to have a parametrized version of Lambek's Lemma (Lemma 1)

Lemma 6. Let $F : \mathcal{C}^{n+1} \to \mathcal{C}$ be a functor. Currying we have $F : \mathcal{C}^n \to \mathcal{C}^{\mathcal{C}}$. Thus, given $\bar{C} = (C_1, \ldots, C_n)$ object in $\mathcal{C}^n$, we have a functor $F_{\bar{C}} : \mathcal{C} \to \mathcal{C}$. Let $\mu F_{\bar{C}}$ and $\nu F_{\bar{C}}$ the initial algebra and final coalgebra for that functor, respectively (assume it exists). Then

$$\begin{array}{rcl} F_{\bar{C}}(\mu F_{\bar{C}}) & \cong & \mu F_{\bar{C}} \\ F_{\bar{C}}(\nu F_{\bar{C}}) & \cong & \nu F_{\bar{C}}. \end{array}$$

We can now prove the main lemma needed in this section, which essentially proves semantics to open formulas of the form $\mu X.F(X, X_1, \ldots, X_n)$ and $\nu X.F(X, X_1, \ldots, X_n)$.

Lemma 7. Let $F : \mathcal{C}^{n+1} \to \mathcal{C}$ be a functor. Then, currying, we obtain $F : \mathcal{C}^n \to \mathcal{C}^{\mathcal{C}}$, so that for any tuples $\bar{C} = (C_1, \ldots, C_n)$, $\bar{D} = (D_1, \ldots, D_n)$ and $\bar{f} = (C_1 \xrightarrow{f_1} D_1, \ldots, C_n \xrightarrow{f_n} D_n)$ in $\mathcal{C}^n$, we have functors $F_{\bar{C}}, F_{\bar{D}} : \mathcal{C} \to \mathcal{C}$ and a natural transformation $F_{\bar{f}} : F_{\bar{C}} \Rightarrow F_{\bar{D}}$ defined as above (i.e. by parametrization).
Moreover, suppose $\mathcal{C}$ has initial algebras for $F_{\bar{C}}$ and $F_{\bar{D}}$. Then, $\bar{f}$ induces a unique arrow $\mu\bar{f} : \mu F_{\bar{C}} \to \mu F_{\bar{D}}$ such that we can define a functor $\mu X.F : \mathcal{C}^n \to \mathcal{C}$ such that

$$\begin{array}{rcl} \mu X.F(\bar{C}) & = & \mu F_{\bar{C}} \\ \mu X.F(\bar{f}) & = & \mu\bar{f} \end{array}$$

Proof. Consider the initial algebras

$$\begin{array}{rcl} c & : & F_{\bar{C}}(\mu F_{\bar{C}}) \to \mu F_{\bar{C}} \\ d & : & F_{\bar{D}}(\mu F_{\bar{D}}) \to \mu F_{\bar{D}} \end{array}$$

given by hypothesis. Look at the natural transformation $F_{\bar{f}} : F_{\bar{C}} \Rightarrow F_{\bar{D}}$ and consider

$$F_{\bar{f}}(\mu F_{\bar{D}}) : F_{\bar{C}}(\mu F_{\bar{D}}) \to F_{\bar{D}}(\mu F_{\bar{D}}).$$

Post-composing with d we obtain an $F_{\bar{C}}$ algebra

$$d \circ F_{\bar{f}}(\mu F_{\bar{D}}) : F_{\bar{C}}(\mu F_{\bar{D}}) \to \mu F_{\bar{D}}.$$

Using initiality we obtain a (unique) arrow $\mu \bar{f}$ such that the above diagram commutes

$$\begin{tikzcd}
F_{\bar{C}}(\mu F_{\bar{C}}) \arrow[r, dotted, "F_{\bar{C}}(\mu \bar{f})"] \arrow[d, "c"'] & F_{\bar{C}}(\mu F_{\bar{D}}) \arrow[d, "d \circ F_{\bar{f}}(\mu F_{\bar{D}})"] \\
\mu F_{\bar{C}} \arrow[r, dotted, "\mu \bar{f}"'] & \mu F_{\bar{D}}
\end{tikzcd}$$

We have now to prove that we indeed have a functor. Suppose $\bar{C} \xrightarrow{\bar{f}} \bar{D} \xrightarrow{\bar{g}} \bar{E}$. We prove $\mu(\bar{g} \circ \bar{f}) = \mu\bar{g} \circ \mu\bar{f}$. The following are their 'defining' diagrams:

$$\begin{tikzcd}
F_{\bar{C}}(\mu F_{\bar{C}}) \arrow[r, "F_{\bar{C}}(\mu \bar{f})"] \arrow[d, "c"'] & F_{\bar{C}}(\mu F_{\bar{D}}) \arrow[d, "d \circ F_{\bar{f}}(\mu F_{\bar{D}})"] \\
\mu F_{\bar{C}} \arrow[r, "\mu \bar{f}"'] & \mu F_{\bar{D}}
\end{tikzcd}
\qquad
\begin{tikzcd}
F_{\bar{D}}(\mu F_{\bar{D}}) \arrow[r, "F_{\bar{D}}(\mu \bar{g})"] \arrow[d, "d"'] & F_{\bar{D}}(\mu F_{\bar{E}}) \arrow[d, "e \circ F_{\bar{g}}(\mu F_{\bar{E}})"] \\
\mu F_{\bar{D}} \arrow[r, "\mu \bar{g}"'] & \mu F_{\bar{E}}
\end{tikzcd}$$

$$\begin{tikzcd}
F_{\bar{C}}(\mu F_{\bar{C}}) \arrow[r, "F_{\bar{C}}(\mu(\bar{g} \circ \bar{f}))"] \arrow[d, "c"'] & F_{\bar{C}}(\mu F_{\bar{E}}) \arrow[d, "e \circ F_{\bar{g} \circ \bar{f}}(\mu F_{\bar{E}})"] \\
\mu F_{\bar{C}} \arrow[r, "\mu(\bar{g} \circ \bar{f})"'] & \mu F_{\bar{E}}
\end{tikzcd}$$

We prove that $\mu\bar{g} \circ \mu\bar{f}$ makes the latter diagram commute. By uniqueness of $\mu(\bar{g} \circ \bar{f})$ we will have the thesis. We have to prove

$$e \circ F_{\bar{g} \circ \bar{f}}(\mu F_{\bar{E}}) \circ F_{\bar{C}}(\mu\bar{g} \circ \mu\bar{f}) = \mu\bar{g} \circ \mu\bar{f} \circ c.$$

Now, $F_{\bar{g}\circ\bar{f}} : F_{\bar{C}} \Rightarrow F_{\bar{E}}$ is a natural transformation. Moreover, its very definition is

$$F_{\bar{g}\circ\bar{f}} = F(\bar{g}\circ\bar{f}).$$

Since $F : \mathcal{C}^n \to \mathcal{C}^{\mathcal{C}}$ is a functor, we have functoriality in each arguments. This gives

$$F_{\bar{g}\circ\bar{f}} = F_{\bar{g}} \circ F_{\bar{f}}$$

which is the composition of two natural transformations. The latter is defined componentwise[8] so that we have

$$F_{\bar{g}\circ\bar{f}}(\mu F_{\bar{E}}) = F_{\bar{g}}(\mu F_{\bar{E}}) \circ F_{\bar{f}}(\mu F_{\bar{E}}).$$

Similarly, $F_{\bar{C}} : \mathcal{C} \to \mathcal{C}$ is a functor, hence

$$F_{\bar{C}}(\mu\bar{g}\circ\mu\bar{f}) = F_{\bar{C}}(\mu\bar{g}) \circ F_{\bar{C}}(\mu\bar{f}).$$

Putting these equalities together we see that to prove the thesis it is sufficient to prove

$$e \circ F_{\bar{g}}(\mu F_{\bar{E}}) \circ F_{\bar{f}}(\mu F_{\bar{E}}) \circ F_{\bar{C}}(\mu\bar{g}) \circ F_{\bar{C}}(\mu\bar{f}) = \mu\bar{g}\circ\mu\bar{f}\circ c.$$

which follows by chasing the following diagram:

$$\begin{tikzcd}
F_{\bar{C}}(\mu F_{\bar{C}}) \arrow[r, "F_{\bar{C}}(\mu\bar{f})"] \arrow[dd, "c"'] & F_{\bar{C}}(\mu F_{\bar{D}}) \arrow[r, "F_{\bar{C}}(\mu\bar{g})"] \arrow[d, "F_{\bar{f}}(\mu F_{\bar{D}})"'] & F_{\bar{C}}(\mu F_{\bar{E}}) \arrow[d, "F_{\bar{f}}(\mu F_{\bar{E}})"] \\
& F_{\bar{D}}(\mu F_{\bar{D}}) \arrow[r, "F_{\bar{D}}(\mu\bar{g})"] \arrow[d, "d"'] & F_{\bar{D}}(\mu F_{\bar{E}}) \arrow[d, "e\circ F_{\bar{g}}(\mu F_{\bar{E}})"] \\
\mu F_{\bar{C}} \arrow[r, "\mu\bar{f}"'] & \mu F_{\bar{D}} \arrow[r, "\mu\bar{g}"'] & \mu F_{\bar{E}}
\end{tikzcd}$$

[8]Recall that for $\alpha : F \Rightarrow G$, $\eta : G \to H$ natural transformations, their composite $\eta\circ\alpha$ is defined for any object C by

$$(\eta\circ\alpha)(C) = \eta(C)\circ\alpha(C).$$

We now prove that

$$\mu \mathsf{id}_{\bar{C}} = \mathsf{id}_{\mu F_{\bar{C}}}$$

Again, $\mu \mathsf{id}_{\bar{C}}$ is the unique arrow that makes the following diagram commute

$$\begin{tikzcd}
F_{\bar{C}}(\mu F_{\bar{C}}) \arrow[r, "F_{\bar{C}}(\mu \mathsf{id}_{\bar{C}})"] \arrow[d, "c"'] & F_{\bar{C}}(\mu F_{\bar{C}}) \arrow[d, "c \circ F_{\mathsf{id}_{\bar{C}}}(\mu F_{\bar{C}})"] \\
\mu F_{\bar{C}} \arrow[r, "\mu \mathsf{id}_{\bar{C}}"'] & \mu F_{\bar{C}}
\end{tikzcd}$$

so it is enough to prove that $\mathsf{id}_{\mu F_{\bar{C}}}$ makes the diagram commute as well.

Recall that $F : \mathcal{C}^n \to \mathcal{C}^{\mathcal{C}}$. Thus, $F_{\mathsf{id}_{\bar{C}}} = 1_{F_{\bar{C}}}$, where $1_F : F \Rightarrow F$ is the identity natural transformation defined by $1_F(X) = \mathsf{id}_{F(X)}$. This gives

$$F_{\mathsf{id}_{\bar{C}}}(\mu F_{\bar{C}}) = \mathsf{id}_{F_{\bar{C}}(\mu F_{\bar{C}})}.$$

Similarly, by functoriality

$$F_{\bar{C}}(\mathsf{id}_{\mu F_{\bar{C}}}) = \mathsf{id}_{F_{\bar{C}}(\mu F_{\bar{C}})}.$$

We thus obtain

$$\begin{aligned}
c \circ F_{\mathsf{id}_{\bar{C}}}(\mu F_{\bar{C}}) \circ F_{\bar{C}}(\mathsf{id}_{\mu F_{\bar{C}}}) &= c \circ \mathsf{id}_{F_{\bar{C}}(\mu F_{\bar{C}})} \circ \mathsf{id}_{F_{\bar{C}}(\mu F_{\bar{C}})} \\
&= c \\
&= \mathsf{id}_{\mu F_{\bar{C}}} \circ c
\end{aligned}$$

i.e. the desired result.

■

A similar result can be proved for $\nu X.F$.

We can now interpret a formula F with $FV(F) = \{X_1, \ldots, X_n\}$ as a functor $[\![F]\!] : \mathcal{C}^n \to \mathcal{C}$. We do that by recursion on F.

1. $[\![X]\!] : \mathcal{C} \to \mathcal{C}$ is the identity functor.

2. $[\![C]\!]$ is the constant unit functor.

3. $[\![F \otimes G]\!]$ is defined as follows. Let

$$\begin{array}{rcl} FV(F) & = & \{X_1, \ldots, X_n\} \\ FV(G) & = & \{Y_1, \ldots, Y_m\} \end{array}$$

so that

$$FV(F \otimes G) = \{X_1, \ldots, X_n, Y_1, \ldots, Y_m\}$$

(this is the most general cases; cases for $X_i = Y_j$ for some i, j are special cases of the above one). Then $[\![F \otimes G]\!] : \mathcal{C}^{n+m} \to \mathcal{C}$ is defined as follows (in order to have a lighter exposition, we overload the notation and write $F \otimes G$ for $[\![F \otimes G]\!]$):

- On objects $(C_1, \ldots, C_n, D_1, \ldots, D_m)$ we have

$$(F \otimes G)(C_1, \ldots, C_n, D_1, \ldots, D_m) = F(C_1, \ldots, C_n) \otimes G(D_1, \ldots, D_m).$$

- On arrows $f_i : C_i \to C'_i$ and $g_j : D_j \to D'_j$, for $i \leq n$, $j \leq m$, we have

$$(F \otimes G)(f_i, \ldots, f_n, g_i, \ldots, g_n) = F(f_1, \ldots, f_n) \otimes G(g_1, \ldots, g_n).$$

It is not hard to see that $F \otimes G$ is a functor. We prove that

$$(F \otimes G)(f'_1 \circ f_1, \ldots, f'_n \circ f_n, g'_1 \circ g_1, \ldots, g'_m \circ g_m)$$

$$=$$

$$(F \otimes G)(f'_1, \ldots, f'_n, g'_1, \ldots, g'_m) \circ (F \otimes G)(f_1, \ldots, f_n, g_1, \ldots, g_m)$$

for suitable arrows f_i, f'_i and g_j, g'_j.

Let us write $\bar{f}$ and $\bar{g}$ for $f_1, \ldots, f_n$ and $g_1, \ldots, g_m$. Similarly, write $\bar{f}'$ and $\bar{g}'$ for $f'_1, \ldots, f'_n$ and $g'_1, \ldots, g'_m$. Finally write, $\overline{f' \circ f}$ and $\overline{g' \circ g}$ for $f'_1 \circ f_1, \ldots, f'_n \circ f_n$ and $g'_1 \circ g_1, \ldots, g'_m \circ g_m$.

We then have to prove

$$(F \otimes G)(\overline{f' \circ f}, \overline{g' \circ g}) = (F \otimes G)(\bar{f}', \bar{g}') \circ (F \otimes G)(\bar{f}, \bar{g})$$

Indeed we have

$$
\begin{aligned}
(F \otimes G)(\overline{f' \circ f}, \overline{g' \circ g}) &= F(\overline{f' \circ f}) \otimes G(\overline{g' \circ g}) \\
&= (F\bar{f}' \circ F\bar{f}) \otimes (G\bar{g}' \circ G\bar{g}) \\
&= (F\bar{f}' \otimes G\bar{g}') \circ (F\bar{f} \otimes G\bar{g}) \\
&= (F \otimes G)(\bar{f}', \bar{g}') \circ (F \otimes G)(\bar{f}, \bar{g})
\end{aligned}
$$

4. The case for $F \,\&\, G$ is essentially the same as $F \otimes G$.

5. Consider $\mu X.F$. Suppose $FV(F) = \{X_1, \ldots, X_n, X\}$ so that, by induction hypothesis $[\![F]\!] : \mathcal{C}^{n+1} \to \mathcal{C}$ functor. Then define

$$[\![\mu X.F]\!] = \mu X.[\![F]\!]$$

according to Lemma 7.

6. The case for $\nu X.F$ is as the one for $\mu X.F$, with the corresponding lemma formulated for final coalgebras instead of initial algebras.

This shows that we can generalize open formulas to have more than a single free variable. The corresponding categorical structure is not conceptually harder than νSMCCs, but proofs are longer and more technical. For this reason we keep working with a single-variable fragment of the logic or, semantically, with polynomial functors. As already stressed, this fragment is simple, has a natural semantical interpretation and it is quite expressive. In particular, it is enough expressive to encode exponential modalities, as we will see in the next chapter.

Chapter 3

Sequent Calculus and Weaker Modalities

In previous chapter we defined νSMCCs and designed a syntactic calculus and a logical system for them. We call the resulting logic νLL (for ν-linear logic). We gave a sound and complete calculus for νLL in Lambek's style, which we refer to as the 'logical calculus'. In the first part of this chapter we define a sequent calculus equivalent to the logical calculus given in previous chapter. As a consequence, the sequent calculus is sound and complete with respect to the class of νSMCCs. We then give a syntactic analysis of the exponential modality !. An exponential formula $!A$ is encoded in the system as

$$\nu X.1 \,\&\, A \,\&\, (X \otimes X)$$

This observation allows to recognize how exponential(s)[1] over A are linked with solutions to the equation

$$X \cong 1 \,\&\, A \,\&\, (X \otimes X)$$

which can be obtained in a surprisingly natural and intuitive way. With the same naturality one obtains two other equations, namely

$$\begin{array}{rcl} X & \cong & 1 \,\&\, A \\ X & \cong & A \,\&\, (X \otimes X) \end{array}$$

[1]The exponential ! is in fact non-canonical. This will be investigated in the second part of this chapter.

These are clearly related to the exponential equation, and we will show that they are the defining equations for relevant and affine modalities [86, 87, 109]. Roughly, relevant modalities are weaker forms of exponentials that allow to restore weakening in a controlled form, but not contraction. Contraction can be explicitly added to the system, and the resulting logic is the so-called *non-distributive relevance logic* [86, 87]. Adding distributivity gives usual systems of *relevance logic* [9, 74, 87]. Non-distributive relevant systems without contraction are also known as *relevant linear logics* [109]. The same can be done with contraction. *Affine linear logic* [8, 69] is nothing but linear logic with an exponential modality (called affine modality) that allows to restore contraction, but not weakening[2]. To avoid terminological confusion, we speak of weakening modality for the modality $!_W$ that restores weakening in a controlled form (i.e. relevant modality), of contraction modality for the modality $!_C$ that restores contraction in a controlled form (i.e. affine modality) and of exponential modality for the usual (linear logic) exponential modality ! (in this chapter we deal only with the modality !, and not with its dual ?, so that referring to ! as the'exponential modality' should not create any confusion).

The link between exponential, weakening and contraction modalities should be evident at this point: the former can be though has a kind of composition of the latter two. It is then natural to ask whether this intuition can be made formal. In [67] exponential, weakening and contraction modalities are deeply investigated from a categorical perspective. These modalities are studied via the notion of comonad (see e.g. [10, 11]), and trough the requirement of specific conditions, a decomposition theorem stating that the exponential modality can the obtained as composition of weakening and contraction modality is proved. Moreover, such composition commutes. Although mathematically elegant, the framework used is quite technical and seems to be too far from intuitions (some conditions required are justified only from a technical perspective). The analysis we propose in this thesis seems to be simpler but still informative. In fact, as already said, we can recover exponential, weakening and contraction modalities as final coalgebras of specific functors. More importantly, we prove a decomposition theorem stating that the functor associated to the exponential modality is the result

[2] The reader can consult [86, 87] for an introduction to substrucutural logics, covering specifically linear and relevant logic. In [109] these logics are studied and compared by means of their associated type systems.

of a composition-like operation between functors associated to the weakening and contraction modalities. Moreover, such result continues to hold for weaker versions of these modalities. These results cannot be observed in a comonadic approach, since this has essentially no modularity. The functor associated with the exponential modality can be divided in three parts: one for weakening, one for contraction and one for dereliction (see below). Such modularity allows us to 'decompose' it, so that we can have a finer analysis of its properties. In particular, one can observe that apparently there is no link between the functor associated to the exponential modality and the *promotion rule*[3]

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}$$

This rule imposes the strong restriction that the context Γ has to be exponentially bound (i.e. any formula in Γ has to be under the scope of the exponential !). Promotion, together with dereliction

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}$$

make the exponential modality an $S4$ modality [28].

As already said, dereliction can be recognized as a part of the functor associated to the exponential modality. Promotion turns out to be satisfied if one takes the final coalgebra for that functor, but the author was not able to recognize any structural link between such rule and the functor associated to the exponential modality. This observation suggests that one could consider non-$S4$ modalities for recovering weakening and contraction, and still have a natural mathematical interpretation for them. All these results will be proved in the second part of the chapter.

3.1 Sequent Calculus

We now introduce a sequent system for νLL, and encode exponential, weakening and contraction modalities in the system, thus showing how exponential, relevant and affine $(\otimes, \&)$-fragment of linear logic are subsumed by νLL.

[3] The same holds for weakening and contraction modalities, and their associated functors.

Sequents are defined as usual as expressions of the form $\Gamma \vdash A$. We assume the context Γ to be a list, that is we assume its constructor comma ',' to be associative. We can in fact consider rules in the calculus giving associativity. This assumption will make proofs lighter, and it is in line with Mac Lane's Theorem (see Chapter 1). De facto, although we will explicitly state an exchange rule (which gives commutativity of the comma constructor) we will use it implicitly, regarding contexts to be multisets of formulas. Again, this is in line with Mac Lane's Theorem.

The proof system is given in Figure 3.1, and we refer to it as the 'sequent calculus' (so that we have a combinatorial, logical and sequent calculus). A long but easy proof by induction on the structure of the derivations gives the following

Proposition 6. The sequent and logical calculi are equivalent.

As a consequence, the sequent and combinatorial calculi are equivalent too. Therefore, the sequent calculus is sound and complete with respect to the class of νSMCCs.

We now introduce the standard exponential modality !. To do so we first enrich the $(\otimes, \&)$-fragment of linear logic with an $S4$ modality $\Box$. The $S4$ rules are known in the linear logic literature as dereliction and promotion.

Formulas of the $(\otimes, \&, \Box)$-linear logic are constructed by adding the construction $\Box A$ to the grammar of $(\otimes, \&)$-linear logic. That is, formulas are generated by the following grammar:

$$A ::= a \mid 1 \mid \top \mid A \otimes A \mid A \,\&\, A \mid \Box A$$

where $a \in \mathsf{Prop}$, for a given set Prop of atomic propositions.

Moreover, we enrich the sequent calculus with the rules given in Figure 3.2. These rules are called *promoition* and *dereliction*, respectively. In this case we say that $\Box$ is an $S4$ modality.

The exponential modality ! is introduced to recover weakening and contraction in a controlled manner. Rules governing ! are obtained by adding to promotion and dereliction the rules given in Figure 4.2, which we call modal weakening and modal contraction respectively.

Adding to promotion and dereliction only the modal weakening rule gives rise to a weakening (i.e. relevant) modality, whereas adding only the modal contraction rule defines a contraction (i.e. affine) modality. As we will see both these modalities can be encoded in νLL. Moreover, as it should be clear

Structural Rules

$$\frac{\Gamma, A, B \vdash C}{\Gamma, B, A \vdash C}$$

$$\frac{}{A \vdash A} \qquad \frac{\Gamma \vdash A \qquad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}$$

Multiplicative Rules

$$\frac{\Gamma \vdash A}{\Gamma, 1 \vdash A} \qquad \frac{}{\vdash 1}$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$$

Additive Rules

$$\frac{A_i, \Gamma \vdash B}{A_1 \,\&\, A_2, \Gamma \vdash B} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}$$

$$\frac{}{\Gamma \vdash \top}$$

Fixed Point Rules

$$\frac{F(A) \vdash A}{\mu X.F \vdash A} \qquad \frac{\Gamma \vdash F(\mu X.F)}{\Gamma \vdash \mu X.F}$$

$$\frac{\Gamma, F(\nu X.F) \vdash A}{\Gamma, \nu X.F \vdash A} \qquad \frac{A \vdash F(A)}{A \vdash \nu X.F}$$

Figure 3.1: Sequent calculus for νLL.

$$\frac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A} \qquad \frac{\Gamma, A \vdash B}{\Gamma, \Box A \vdash B}$$

Figure 3.2: Sequent calculus for νSMCCs.

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B} \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$$

Figure 3.3: Modal weakening and modal contraction.

now, exponential, weakening and contraction modalities (which we refer to as *structural modalities*) are strongly related, since the former, in a way, subsumes the latter two. From a proof-theoretical perspective, their relationship can be pictorially summarized as follows

$$\begin{tikzcd}
& \Box & \\
!_W \arrow[ur] & & !_C \arrow[ul] \\
& ! \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

where $\Box$ stands for an $S4$ modality, and an arrow $X \to Y$ means that X satisfies the rules governing Y.

We first show that we can recover the exponential modality $!A$ as $\nu X.1\&A\&(X\otimes X)$. We then recover both weakening and contraction modalities.

Lemma 8. Define $!A$ by

$$\nu X.1 \,\&\, A \,\&\, (X \otimes X).$$

Then we have that weakening, dereliction and contraction

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B} \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$$

are admissible

Proof. We show admissibility of the above three rules.

1. Suppose we have $\Gamma \vdash B$. Construct

$$\cfrac{\cfrac{\cfrac{\cfrac{\Gamma \vdash B}{\Gamma, 1 \vdash B}}{\Gamma, 1 \,\&\, A \vdash B}}{\Gamma, 1 \,\&\, A \,\&\, (!A \otimes !A) \vdash B}}{\Gamma, \nu X.1 \,\&\, A \,\&\, (X \otimes X) \vdash B}$$

2. Suppose we have $\Gamma, A \vdash B$. Construct

$$\cfrac{\cfrac{\cfrac{\Gamma, A \vdash B}{\Gamma, 1 \,\&\, A \vdash B}}{\Gamma, 1 \,\&\, A \,\&\, (!A \otimes !A) \vdash B}}{\Gamma, \nu X.1 \,\&\, A \,\&\, (X \otimes X) \vdash B}$$

3. Suppose to have $\Gamma, !A, !A \vdash B$. Construct

$$\cfrac{\cfrac{\cfrac{\cfrac{\Gamma, !A, !A \vdash B}{\Gamma, !A \otimes !A \vdash B}}{\Gamma, A \,\&\, (!A \otimes !A) \vdash B}}{\Gamma, 1 \,\&\, A \,\&\, (!A \otimes !A) \vdash B}}{\Gamma, \nu X.1 \,\&\, A \,\&\, (X \otimes X) \vdash B}$$

■

Remark. Note that in the above derivations we just used the fact that $\nu X.F$ is a fixed point of F, and not that $\nu X.F$ is the greatest fixed point of F. This means that to recover weakening, contraction and dereliction, any fixed point of $1 \,\&\, A \,\&\, (X \otimes X)$ is sufficient. The fact that $\nu X.F$ is the greatest fixed point is needed to recover promotion. We will make this point more precise in the next section.

Lemma 9. The sequents $!A \otimes !B \vdash !(A \,\&\, B)$ is provable

Proof. First construct

$$\dfrac{\dfrac{!A \otimes !B \vdash 1 \qquad !A \otimes !B \vdash A \,\&\, B \qquad !A \otimes !B \vdash ((!A \otimes !B) \otimes (!A \otimes !B))}{!A \otimes !B \vdash 1 \,\&\, (A \,\&\, B) \,\&\, ((!A \otimes !B) \otimes (!A \otimes !B))}}{!A \otimes !B \vdash \nu X.1 \,\&\, (A \,\&\, B) \,\&\, (X \otimes X)}$$

We prove the leaf sequents.

1. Construct

$$\dfrac{\dfrac{\dfrac{\vdash 1}{!A \vdash 1}}{!A, !B \vdash 1}}{!A \otimes !B \vdash 1}$$

   where we used previous lemma (weakening).

2. Construct

$$\dfrac{\dfrac{\dfrac{!A \vdash A}{!A, !B \vdash A}}{!A \otimes !B \vdash A} \qquad \dfrac{\dfrac{!B \vdash B}{!A, !B \vdash B}}{!A \otimes !B \vdash B}}{!A \otimes !B \vdash A \,\&\, B}$$

   where, again, we used previous lemma (weakening). The sequent $!A \vdash A$ is provable too:

$$\dfrac{\dfrac{\dfrac{A \vdash A}{A \,\&\, (!A \otimes !A) \vdash A}}{1 \,\&\, A \,\&\, (!A \otimes !A) \vdash A}}{\nu X.1 \,\&\, A \,\&\, (X \otimes X) \vdash A}$$

3. Construct

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{!A \vdash !A \qquad !B \vdash !B}{!A, !B \vdash !A \otimes !B} \qquad \dfrac{!A \vdash !A \qquad !B \vdash !B}{!A, !B \vdash !A \otimes !B}}{!A, !A, !B, !B \vdash (!A \otimes !B) \otimes (!A \otimes !B)}}{!A, !A, !B \vdash (!A \otimes !B) \otimes (!A \otimes !B)}}{!A, !B \vdash (!A \otimes !B) \otimes (!A \otimes !B)}}{!A \otimes !B \vdash (!A \otimes !B) \otimes (!A \otimes !B)}$$

where we used previous lemma (contraction).

■

Lemma 10. The sequent $!(A \,\&\, B) \vdash\, !A \otimes\, !B$ is provable.

Proof. First of all we show that the sequent

$$!A \vdash\, !A \otimes\, !A$$

is provable. This is an important sequent since from a categorical perspective it witnesses the existence of the duplicator arrow

$$\Delta_A :\, !A \to\, !A \otimes\, !A$$

which, in general, does not exist in monoidal categories. Construct

$$\dfrac{\dfrac{!A \vdash\, !A \qquad !A \vdash\, !A}{!A,\, !A \vdash\, !A \otimes\, !A}}{!A \vdash\, !A \otimes\, !A}$$

where, again, we used previous lemma (contraction).
Now for the main proof. Construct

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{!(A \,\&\, B) \vdash\, !A \qquad !(A \,\&\, B) \vdash\, !B}{!(A \,\&\, B),\, !(A \,\&\, B) \vdash\, !A \otimes\, !B}}{!(A \,\&\, B) \otimes\, !(A \,\&\, B) \vdash\, !A \otimes\, !B}}{A \,\&\, B,\, !(A \,\&\, B) \otimes\, !(A \,\&\, B) \vdash\, !A \otimes\, !B}}{(A \,\&\, B) \,\&\, (!(A \,\&\, B) \otimes\, !(A \,\&\, B)) \vdash\, !A \otimes\, !B}}{1 \,\&\, (A \,\&\, B) \,\&\, (!(A \,\&\, B) \otimes\, !(A \,\&\, B)) \vdash\, !A \otimes\, !B}}{\nu X.1 \,\&\, (A \,\&\, B) \,\&\, (X \otimes X) \vdash\, !A \otimes\, !B}$$

We show that $!(A \,\&\, B) \vdash\, !A$ is provable. Consider

$$\dfrac{\dfrac{!(A \,\&\, B) \vdash 1 \qquad !(A \,\&\, B) \vdash A \qquad !(A \,\&\, B) \vdash\, !(A \,\&\, B) \otimes\, !(A \,\&\, B)}{!(A \,\&\, B) \vdash 1 \,\&\, A \,\&\, (!(A \,\&\, B) \otimes\, !(A \,\&\, B))}}{!(A \,\&\, B) \vdash \nu X.1 \,\&\, A \,\&\, (X \otimes X)}$$

It is easy to see that the leaf sequents are all provable (the first two starting from the left are essentially of the same form of other sequents we already showed to be provable, whereas the third one is an instance of the duplicator arrow, and we observed that the corresponding sequent is provable).

■

One final

Lemma 11. The rule of promotion

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}$$

is admissible.

Proof. First we prove that the following rule is admissible:

$$\frac{!A \vdash B}{!A \vdash !B}$$

Indeed we have

$$\cfrac{\cfrac{\cfrac{\vdash 1}{!A \vdash 1} \qquad !A \vdash B \qquad !A \vdash !A \otimes !A}{!A \vdash 1 \mathbin{\&} B \mathbin{\&} (!A \otimes !A)}}{!A \vdash \nu X.1 \mathbin{\&} B \mathbin{\&} (X \otimes X)}$$

Using this rule we first construct

$$\cfrac{\cfrac{!(A_1 \mathbin{\&} \cdots \mathbin{\&} A_n) \vdash !A_1 \otimes \cdots \otimes !A_n \qquad \begin{array}{c} !A_1, \ldots, !A_n \vdash B \\ \vdots \\ !A_1 \otimes \cdots \otimes !A_n \vdash B \end{array}}{!(A_1 \mathbin{\&} \cdots \mathbin{\&} A_n) \vdash B}}{!(A_1 \mathbin{\&} \cdots \mathbin{\&} A_n) \vdash !B}$$

and then

$$\cfrac{!A_1, \ldots, !A_n \vdash !A_1 \otimes \cdots \otimes !A_n \qquad \cfrac{!A_1 \otimes \cdots \otimes !A_n \vdash !(A_1 \mathbin{\&} \cdots \mathbin{\&} A_n) \qquad !(A_1 \mathbin{\&} \cdots \mathbin{\&} A_n) \vdash !B}{!A_1 \otimes \cdots \otimes !A_n \vdash B}}{!A_1, \ldots, !A_n \vdash !B}$$

thus concluding the proof. ∎

Putting all these lemmas together we obtain the following

Proposition 7. Define $!A$ by

$$\nu X.1 \mathbin{\&} A \mathbin{\&} (X \otimes X)$$

Then $!A$ behaves like the usual exponential modality, in the sense that it satisfies the rules of weakening, dereliction, contraction and promotion:

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B} \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B} \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}$$

The above analysis shows that the exponential $!A$, usually taken as primitive in linear logic, is actually the result of a non-trivial interaction between additive and multiplicative connectives. In [67, 76, 78] it is observed that $!A$ is the result of the interaction between the weakening and contraction modalities $!_W$ and $!_C$. Figure 3.4 gives rules for these modalities.

Contraction Modality

$$\frac{\Gamma, !_C A, !_C A \vdash B}{\Gamma, !_C A \vdash B} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !_C A \vdash B} \qquad \frac{!_C \Gamma \vdash A}{!_C \Gamma \vdash !_C A}$$

Weakening Modality

$$\frac{\Gamma \vdash B}{\Gamma, !_W A \vdash B} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !_W A \vdash B} \qquad \frac{!_W \Gamma \vdash A}{!_W \Gamma \vdash !_W A}$$

Figure 3.4: Rules for weakening and contraction modalities.

Our analysis of $!A$ as

$$\nu X.1 \mathbin{\&} A \mathbin{\&} (X \otimes X)$$

allows us to have a better understanding of structural modalities. In fact, dropping the unit 1 in the definition of $!A$ we obtain a new formula

$$\Box A = \nu X.A \mathbin{\&} (X \otimes X)$$

which we claim to behave like a contraction modality. In the next section we will see how to achieve this definition of contraction modality in a natural way. Moreover, as we will see, our analysis of structural modalities can be done independently from the promotion rule, thus producing canonical structural modalities for weaker logics.

Proposition 8. Define $\Box A$ as

$$\nu X.A \,\&\, (X \otimes X)$$

Then the rules of dereliction, contraction and promotion

$$\frac{\Gamma, A \vdash B}{\Gamma, \Box A \vdash B} \qquad \frac{\Gamma, \Box A, \Box A \vdash B}{\Gamma, \Box A \vdash B} \qquad \frac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A}$$

are admissible.

Proof. We immediately note that the first two rules (starting from the left) are indeed admissible, as the following show

$$\dfrac{\dfrac{\Gamma, A \vdash B}{\Gamma, A \,\&\, (\Box A \otimes \Box A) \vdash B}}{\Gamma, \nu X.A \,\&\, (X \otimes X) \vdash B} \qquad \dfrac{\dfrac{\dfrac{\Gamma, \Box A, \Box A \vdash B}{\Gamma, \Box A \otimes \Box A \vdash B}}{\Gamma, A \,\&\, (\Box A \otimes \Box A) \vdash B}}{\Gamma, \nu X.A \,\&\, (X \otimes X) \vdash B}$$

We also note that we can prove the 'duplicator' sequent $\Box A \vdash \Box A \otimes \Box A$ as follows

$$\dfrac{\dfrac{\Box A \otimes \Box A \vdash \Box A \otimes \Box A}{A \,\&\, (\Box A \otimes \Box A) \vdash \Box A \otimes \Box A}}{\nu X.A \,\&\, (X \otimes X) \vdash \Box A \otimes \Box A}$$

We finally prove the admissibility of the last rule.

$$\begin{array}{c}
\Box A_1 \vdash \Box A_1 \otimes \Box A_1 \qquad \cdots \qquad \Box A_n \vdash \Box A_n \otimes \Box A_n \\
\ddots \;\; \iddots \\
\Box A_1, \ldots, \Box A_n \vdash (\Box A_1 \otimes \Box A_1) \otimes \cdots \otimes (\Box A_n \otimes \Box A_n) \\
\vdots \\
\Box A_1 \otimes \cdots \otimes \Box A_n \vdash (\Box A_1 \otimes \Box A_1) \otimes \cdots \otimes (\Box A_n \otimes \Box A_n) \\
\vdots
\end{array}$$

$$\dfrac{\dfrac{\Box A_1 \otimes \cdots \otimes \Box A_n \vdash A \qquad \Box A_1 \otimes \cdots \otimes \Box A_n \vdash (\Box A_1 \otimes \cdots \otimes \Box A_n) \otimes (\Box A_1 \otimes \cdots \otimes \Box A_n)}{\Box A_1 \otimes \cdots \otimes \Box A_n \vdash A \,\&\, ((\Box A_1 \otimes \cdots \otimes \Box A_n) \otimes (\Box A_1 \otimes \cdots \otimes \Box A_n))}}{\Box A_1 \otimes \cdots \otimes \Box A_n \vdash \nu X.A \,\&\, (X \otimes X)}$$

The above derivation allows us to conclude with an instance of the cut rule with

$$\begin{array}{ccc} \Box A_1 \vdash \Box A_1 & \cdots & \Box A_n \vdash \Box A_n \\ & \ddots \;\; \cdot \;\; \cdot^{\cdot^{\cdot}} & \\ & \Box A_1, \ldots, \Box A_n \vdash \Box A_1 \otimes \cdots \otimes \Box A_n & \end{array}$$

■

Remark. We observed that, on an intuitive level, having a contraction modality is enough to guarantee the existence of the duplicator arrow

$$\Delta_A : !_C A \to !_C A \otimes !_C A.$$

Such arrow is clearly strongly linked with the admissibility of contraction, since it duplicates the information $!_C A$

In the proof of the next proposition, we observe that a weakening modality is enough to guarantee the existence of the erasing arrow

$$e_A : !_W A \to 1$$

which is strongly connected with weakening, since it erases the information $!_W A$. In the next section we will see that having duplicator and erasing arrows is equivalent to having weakening and contraction.

Proposition 9. Define $\Box A$ as

$$\nu X.1 \,\&\, A$$

which is nothing but $1 \,\&\, A$. Then $\Box A$ behaves like a weakening modality.

Proof. The proof is close to the one for the relevant modality. First we observe that the 'erasing' sequent $\Box A \vdash 1$ is provable:

$$\dfrac{\dfrac{1 \vdash 1}{1 \,\&\, A \vdash 1}}{\Box A \vdash 1}$$

Proving the lemma is straightforward (follows the pattern of previous proofs). The only real difference is that proving the admissibility of dereliction requires us to prove

$$\Box A_1 \otimes \cdots \otimes \Box A_n \vdash 1.$$

This is done by observing that for all $i \leq n$

$$\Box A_i \vdash 1$$

is provable (it is an instance of the 'erasing' sequent), so that we can prove

$$\Box A_1 \otimes \cdots \otimes \Box A_n \vdash \underbrace{1 \otimes \cdots \otimes 1}_{n}$$

But

$$\underbrace{1 \otimes \cdots \otimes 1}_{n} \vdash 1$$

is provable too, and thus, by cut, we can conclude the proof.

■

We showed how to encode exponential !, weakening $!_W$ and contraction $!_C$ modalities in νLL, thus proving that the exponential, relevant and affine fragment of $(\otimes, \&)$-linear logic are subsumed by νLL.

The next step is to analyze the relationship between the proposed encodings and the rules they are supposed to satisfy. The results proved establish interesting and (to the best of the author's knowledge) new links between structural modalities, their underlying equations and their associated inference rules. The analysis also pointed out several problems related to the rule of promotion.

3.2 Structural Modalities and Decompositions

In this section we analyze the above encodings of structural modalities and their relationship. Let us first remark the fact that we proposed *encodings* of structural modalities. In fact, what is meant is that e.g. $\nu X.1 \& A \& (X \otimes X)$ (which we abbreviate as νA in this brief discussion) behaves, from a proof-theoretical perspective, like the 'original' $!A$. This means that νA satisfies the same rules governing $!A$, namely weakening, contraction, dereliction and promotion. From that we cannot conclude that $!A$ and νA are 'the same' formula. We cannot even prove their equi-provability, i.e. the provability of the sequents $!A \vdash \nu A$ and $\nu A \vdash !A$. This is because $!A$ is non-canonical. This means that if we introduce another formula, say $!'A$, and add to the sequent calculus weakening, contraction, promotion and dereliction for $!'A$, there is no way to prove that $!A$ and $!'A$ are equi-provable. However, our encoding of $!A$ as νA has a form of canonicity, since it is the 'biggest' exponential over A. As we will see if $!'A$ satisfies weakening, contraction and dereliction (we do not even need promotion), then $!'A$ is a coalgebra of the functor $E(X) = 1 \,\&\, A \,\&\, (X \otimes X)$, and thus it is smaller than the final coalgebra for $E(X)$, which is νA.

As a consequence, it is not entirely correct to say that we are encoding $!A$, since the proposed encoding νA, is something 'stronger' than $!A$. What we did was actually to define a canonical exponential in νLL. Using canonicity and modularity of νA we can now propose a finer analysis of structural modalities.

We use ideas and terminology from category theory (as typically done in categorical proof theory), although a large part of our analysis is essentially done in poset categories.

In fact, our interests do not concern proofs of/from exponentials, but rather exponentials themselves. For this reason we work in the poset category Comb given by the logical calculus in Figure 2.4. As we know, the latter is an νSMCC. This framework allows us to work on a proof-theoretic level using ideas and tools from category theory. Isomorphic objects/formulas in Comb are equi-prorvable formulas. That is if $A \cong B$, then both $A \to B$ and $B \to A$ are provable.

It is also worth mentioning that our definitions of (canonical) structural modalities are close to informal intuition and operationally justified. As far as the author knows, no analysis of structural modalities like the one we give here has ever been proposed. Moreover, this analysis allows us to state and prove a new decomposition theorem for the exponential modality, and thus gives canonical affine (i.e. contraction) and relevant (i.e. weakening) modalities, as well as a formal analysis of their canonicity.

Let us start with a brief analysis of the combinatorial counterpart of the structural rules of weakening and contraction. These are

$$\frac{A \to B}{A \otimes C \to B} \qquad \frac{C \otimes (A \otimes A) \to B}{C \otimes A \to B}$$

Weakening can also be rewritten as

$$\frac{A \otimes 1 \to B}{A \otimes C \to B}$$

The connection with duplicator and erasing arrows

$$\begin{array}{rcl} \Delta_A & : & A \to A \otimes A \\ e_A & : & A \to 1 \end{array}$$

is evident. The connection is even stronger if expressed on a formal level.

Lemma 12. Let $\mathcal{C}$ be a SMC. Then the rule

$$\frac{C \otimes (A \otimes A) \to B}{C \otimes A \to B}$$

is admissible if and only if for any object A there is a duplicator

$$\Delta_A : A \to A \otimes A.$$

Similarly, The rule

$$\frac{C \otimes 1 \to B}{C \otimes A \to B}$$

is admissible if and only if for any object A there exists an erasing arrow $e_A : A \to 1$.

Proof. Given $f : C \otimes (A \otimes A) \to B$, we take $f \circ (\mathsf{id}_C \otimes \Delta_A) : C \otimes A \to B$. Viceversa, we take $I = C$, and use the natural isomorphisms l and r. ∎

As a consequence, we see that weakening and contraction essentially amount to the presence of arrows Δ_A and e_A (which can be abstracted to natural transformations Δ and e). Having clarified this aspect, we can recover $!A$ in a easy operational way. What we want is a formula X (unknown for now) such that it allows to perform weakening on it (erase it), or to perform contraction on it (duplicate it), or just do/use A (which corresponds to use A once). Moreover, we can choose between these alternatives. So, we can do weakening on X, thus we have an arrow

$$X \to 1$$

We can do contraction on X, thus we have an arrow

$$X \to X \otimes X$$

We can use A, thus we have an arrow

$$X \to A$$

We have all these arrows, and we can choose which one. It is then natural to use the choice operator &. Thus, from the three above arrows we obtain

$$\frac{X \to 1 \qquad X \to A \qquad X \to X \otimes X}{X \to 1 \,\&\, A \,\&\, (X \otimes X)}$$

As a consequence, we are looking for a coalgebra of the functor

$$E(X) = 1 \,\&\, A \,\&\, (X \otimes X)$$

In previous chapter we took the maximal solution to this equation, which seems the natural one since, intuitively, greatest fixed points capture the idea of performing infinitely many iterations (recall that our informal goal is to be able to duplicate $!A$ infinitely many times). We thus obtained the formula

$$\nu X.1 \,\&\, A \,\&\, (X \otimes X).$$

More formally, we can define the functor

$$E : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

by

$$\begin{array}{rcl} E(A, Y) & = & 1 \,\&\, A \,\&\, (Y \otimes Y) \\ E(f, y) & = & \mathsf{id}_1 \,\&\, f \,\&\, (y \otimes y) \end{array}$$

Let also denote by $E_A : \mathcal{C} \to \mathcal{C}$ the functor defined by

$$\begin{array}{rcl} E_A(Y) & = & E(A, Y) \\ E_A(y) & = & E(\mathsf{id}_A, y) \end{array}$$

Indeed E is a functor, and for any A we have the polynomial functor E_A, called the exponential functor over A. Let us fixed a formula A and write $E(X)$ for $E_A(X)$. The functor $E(X)$ can be divided in three parts:

$$E(X) = \underbrace{1}_{\text{weakening}} \;\&\; \underbrace{A}_{\text{dereliction on } A} \;\&\; \underbrace{(X \otimes X)}_{\text{contraction}}$$

We want to make formal the relationship between coalgebras $C \to E(C)$ and formulas satisfying weakening, contraction and dereliction on A.

First we observe an important property of the functor $E(X)$, namely that any for any coalgebra $C \to E(C)$, C is a solution to the equation (which we refer to as the 'exponential equation')

$$X \cong E(X)$$

(recall that we are working in the poset category Comb, and thus $X \cong E(X)$ amounts to prove both $X \to E(X)$ and $E(X) \to X$).

Lemma 13. If $C \to E(C)$, then $C \cong E(C)$.

Proof. We have to prove $E(C) \to C$, i.e.

$$1 \,\&\, A \,\&\, (C \otimes C) \to C$$

We reduce the proof to proving $C \otimes C \to C$, which follows from $C \otimes C \to C \otimes 1$. This requires to prove $C \to C$, on one hand, and $C \to 1$ on the other. The former clearly holds. For the latter we use $C \to E(C)$ and prove $E(C) \to 1$, i.e.

$$1 \,\&\, A \,\&\, (C \otimes C) \to 1$$

This follows from $1 \to 1$. ∎

Remark. From a sequent calculus perspective, the above lemma states the admissibility of the rule

$$\dfrac{\Gamma, C \vdash E(C)}{\Gamma, E(C) \vdash C}$$

This is proved as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, C \vdash 1 \,\&\, A \,\&\, (C \otimes C) \qquad \dfrac{\dfrac{1 \vdash 1}{1 \,\&\, A \vdash 1}}{1 \,\&\, A \,\&\, (C \otimes C) \vdash 1}}{\Gamma, C \vdash 1} \qquad C \vdash C}{\Gamma, C, C \vdash 1 \otimes C}}{\Gamma, C \otimes C \vdash 1 \otimes C} \qquad \dfrac{C \vdash C}{1 \otimes C \vdash C}}{\Gamma, C \otimes C \vdash C}}{\Gamma, A \,\&\, (C \otimes C) \vdash C}}{\Gamma, 1 \,\&\, A \,\&\, (C \otimes C) \vdash C}$$

As a consequence, in order to prove that C is a solution to $X \cong E(X)$ it is actually sufficient to prove $C \to E(C)$. We can now prove the following

Proposition 10. Let C be a formula. Then C satisfies the following rules (dereliction on A, weakening and contraction)

$$\frac{A \to B}{C \to B} \qquad \frac{D \otimes 1 \to B}{D \otimes C \to B} \qquad \frac{C \otimes C \to B}{C \to B}$$

iff

$$C \cong E(C).$$

Proof. We first prove that if C satisfies the above rules, then C is a solution to $X \cong E(X)$. By previous lemma, it is sufficient to prove that C is an E-coalgebra, i.e. $C \to E(C)$. For, we have to prove $C \to 1$, $C \to A$ and $C \to C \otimes C$. The following summarizes the necessary derivations.

$$\cfrac{\cfrac{\cfrac{1 \to 1}{1 \otimes 1 \to 1}}{1 \otimes C \to 1}}{C \to 1} \qquad \frac{A \to A}{C \to A} \qquad \frac{C \otimes C \to C \otimes C}{C \to C \otimes C}$$

Suppose now C is a solution to $X \cong E(X)$, so that we can freely replace $1 \otimes A \otimes (C \otimes C)$ for C. Note that we can prove $C \to 1$, since

$$\cfrac{\cfrac{1 \to 1}{1 \,\&\, A \to 1}}{1 \,\&\, A \,\&\, (C \otimes C) \to 1}$$

We can then prove dereliction on A and weakening as follows:

$$\cfrac{\cfrac{A \to B}{1 \,\&\, A \to B}}{1 \,\&\, A \,\&\, (C \otimes C) \to B} \qquad \cfrac{\cfrac{\cfrac{D \otimes 1 \to B}{D \to B} \quad C \to 1}{D \otimes C \to B \otimes 1} \quad \cfrac{B \to B}{B \otimes 1 \to B}}{D \otimes C \to B}$$

Finally, we prove contraction

$$\cfrac{\cfrac{C \otimes C \to C}{A \,\&\, (C \otimes C) \to C}}{1 \,\&\, A \,\&\, (C \otimes C) \to A}$$

$\blacksquare$

The above proposition shows that formulas defined via the three above rules are nothing but solutions to the exponential equations, and that solutions to the exponential equation satisfy the three above rules. It clearly follows that the connective $\Box$ defined by '$\Box A \cong E(\Box A)$' i.e.'$\Box A$ satisfies the three above rules' do not define a canonical connective. Indeed, the equation $X \cong E(X)$ in general admits more than one solution. This raises the question of the role of the rule promotion. Such rule is often associated to functoriality (although the equivalence between the two rules does not hold). Functoriality is given by the rule

$$\frac{A \to B}{\Box A \to \Box B}$$

and we can reformulate it by requiring a form of monotonicity for solutions. That is, we require that if $C \cong E_A(C)$, $D \cong E_B(D)$ and $A \to B$, then $C \to D$. Such rule however has huge consequences. For example, it identifies all solutions of an exponential equation. Indeed, let $C \cong E_A(C)$ and $D \cong E_A(D)$. Then, since $A \to A$ holds, we have both $C \to D$ and $D \to C$, and thus $C \cong D$.

This discussions shows that looking at $!A$ as a solution to the equation $X \cong E(X)$ is not as straightforward as it looks like. The correspondence between solutions to $X \cong E(X)$ and formulas satisfying weakening, contraction and dereliction seems to be lost in presence of promotion.

Nonetheless, our intuition behind $!A$ is essentially correct, since $!A$ gives solutions to $X \cong E(X)$ that satisfy promotion. Non-canonicity can be seen as the fact that $!A$ does not give a canonical solution to $X \cong E(X)$. More interesting is the observation that if we just look at solutions to $X \cong E(X)$ we have two canonical solutions, namely the greatest and least solutions, i.e.

$$\mu X.E(X) \quad \text{and} \quad \nu X.E(X).$$

Moreover, these canonical solutions satisfy a weak promotion rule, as the following proposition shows.

Proposition 11. For both $\mu X.E(X)$ and $\nu X.E(X)$ the weak promotion rules

$$\frac{\mu X.E_A(X) \to B}{\mu X.E_A(X) \to \mu X.E_B(X)} \qquad \frac{\nu X.E_A(X) \to B}{\nu X.E_A(X) \to \nu X.E_B(X)}$$

hold.

Proof. We already proved the promotion rule for the ν-case. Let us write $\Box A$ and $\Box B$ for $\mu X.E_A$ and $\mu X.E_B$ respectively, and assume $\Box A \to B$. We prove $\Box A \to \Box B$. First construct

$$
\dfrac{\dfrac{\dfrac{1 \to 1}{\vdots}}{\dfrac{1 \,\&\, A \,\&\, (1 \otimes 1) \to 1}{\Box A \to 1}} \qquad \Box A \to B \qquad \Box A \to \Box B \otimes \Box B}{\dfrac{\Box A \to 1 \,\&\, B \,\&\, (\Box B \otimes \Box B)}{\Box A \to \Box B}}
$$

To conclude we have to prove $\Box A \to \Box B \otimes \Box B$, which follows from $1 \,\&\, A \,\&\, ((\Box B \otimes \Box B) \otimes (\Box B \otimes \Box B)) \to \Box B \otimes \Box B$. To prove this it is sufficient to prove

$$((\Box B \otimes \Box B) \otimes (\Box B \otimes \Box B)) \to \Box B \otimes \Box B$$

This amounts to prove $\Box B \otimes \Box B \to \Box B$. This requires to prove

$$
\begin{array}{lcl}
\Box B \otimes \Box B & \to & 1 \\
\Box B \otimes \Box B & \to & B \\
\Box B \otimes \Box B & \to & \Box B \otimes \Box B
\end{array}
$$

The first holds, since we can just prove $\Box B \otimes \Box B \to 1 \otimes 1$, and thus $\Box B \to 1$, which we know to hold. The third one clearly holds. We prove the second.

$$
\dfrac{\dfrac{\dfrac{B \to B}{\vdots}}{\dfrac{1 \,\&\, B \,\&\, (B \otimes B) \to B}{\Box B \to B}} \qquad \dfrac{\dfrac{1 \to 1}{\vdots}}{\dfrac{1 \,\&\, B \,\&\, (1 \otimes 1) \to 1}{\Box B \to 1}}}{\Box B \otimes \Box B \to B \otimes 1}
$$

which gives the desired result since, $B \otimes 1$ and B are equivalent. ∎

Remark. Recall the the promotion rule for $!A$ (in a sequent calculus formalism) is

$$\dfrac{!\Gamma \vdash A}{!\Gamma \vdash !A}$$

The weak promotion rule considered above, i.e.

$$\frac{!A \to B}{!A \to !B}$$

corresponds to the sequent calculus rule

$$\frac{!A \vdash B}{!A \vdash !B}$$

which is much weaker than its general counterpart. This can be better observed in a categorical setting. The weak promotion rule is essentially given by the notion of comonad [10, 11, 77], which can be thought (specifically for this setting[4]) as a triple $(!, \varepsilon, \delta)$ consisting of an endofunctor $! : \mathcal{C} \to \mathcal{C}$ and natural transformations $\varepsilon : ! \to 1$ (here 1 denotes the identity functor) and $\delta : ! \to !^2$. Using δ and functoriality of ! it is possible to recover the weak promotion rule as follows

$$\frac{\delta_A : !A \to !!A \qquad \dfrac{f : !A \to B}{!f : !!A \to B}}{!f \circ \delta_A : !A \to !B}$$

Full promotion, categorically written as

$$\frac{!A_1 \otimes \cdots \otimes !A_n \to B}{!A_1 \otimes \cdots \otimes !A_n \to !B}$$

demands more than simple comonadicity. In fact, it requires the comonad to be linear [10, 30, 94]. In particular it requires the existence of a natural transformation

$$\lambda_{A,B} : !A \otimes !B \to !(A \otimes B).$$

[4]The definition of comonad is more general than the one given here, since it requires some specific equations to hold (it has specific coherence conditions). The reader can consult [10, 11].

Full promotion can be then schematically recovered as follows[5], given $f : !A_1 \otimes \cdots \otimes !A_n \to B$

$$
\begin{tikzcd}
!A_1 \otimes \cdots \otimes !A_n \arrow[d, "\delta_{A_1} \otimes \cdots \otimes \delta_{A_n}"] \\
!!A_1 \otimes \cdots \otimes !!A_n \arrow[d, "\lambda_{A_1,\ldots,A_n}"] \\
!(!A_1 \otimes \cdots \otimes !A_n) \arrow[d, "!f"] \\
!B
\end{tikzcd}
$$

What we achieved is that the exponential functor over A,

$$E(X) = 1 \mathbin{\&} A \mathbin{\&} (X \otimes X)$$

captures weakening, dereliction and contraction. Any coalgebra for this functor is a solution to the equation

$$X \cong E(X)$$

so that one is led to think of $!A$ simply as a solution to the above equation (and thus to explain non-canonicity of $!A$ as the fact that $!A$ is just *a* solution to the equation, rather than a canonical one). This intuition is not completely correct, since, in general, solutions to $X \cong E(X)$ do not satisfy promotion. This turned out to be the case for the (canonical) greatest solution, namely $\nu X.E(X)$. The least solution $\mu X.E(X)$ satisfies a weaker form of promotion, which is considerably different than its general counterpart. The link between properties of solutions to $X \cong E(X)$ and formulas satisfying promotion is not evident. Nevertheless, it is possible to consider exponential modalities that do not satisfy promotion (thus non-$S4$ modalities). These, from a mathematical perspective, are as natural as the standard exponential !.

We observed that the the exponential functor $E(X)$ is (informally) divided in three parts, one for weakening, one for promotion and one for dereliction. We now make such division formal, and thus prove a decomposition theorem for the exponential functor.

[5]We write $\lambda_{A_1,\cdots,A_n}$ for the generalization of λ to n-ary tensor products.

Let us first modify the informal argument we used to design the exponential functor to formulate analogous weakening and contraction functors.

For the weakening modality $!_W A$, we look for a formula X such that we can do/use A, hence

$$X \to A$$

or do weakening on it, i.e.

$$X \to 1$$

and we can choose which of the two alternatives follows. This leads to coalgebras

$$X \to 1 \,\&\, A$$

and thus to the (constant) functor

$$W(X) = 1 \,\&\, A.$$

For the contraction modality $!_C A$ we look for a formula X such that we can do/use A, hence

$$X \to A$$

or do contraction on it, i.e.

$$X \to X \otimes X$$

and we can choose which of the two alternatives follows. This leads to coalgebras

$$X \to A \,\&\, (X \otimes X)$$

and thus to the functor

$$C(X) = A \,\&\, (X \otimes X)$$

Again, we have canonical modalities, namely $\nu X.W(X)$ (which is trivially $1 \,\&\, A$) and $\nu X.C(X)$. Formally, we designed three functors

$$\begin{array}{rcl} E, C & : & \mathcal{C} \times \mathcal{C} \to \mathcal{C} \\ W & : & \mathcal{C} \to \mathcal{C} \end{array}$$

defined by

$$
\begin{array}{rcl}
E(A,Y) & = & 1 \mathbin{\&} A \mathbin{\&} (Y \otimes Y) \\
E(f,y) & = & \mathsf{id}_1 \mathbin{\&} f \mathbin{\&} (y \otimes y) \\
\\
C(A,Y) & = & A \mathbin{\&} (Y \otimes Y) \\
C(f,y) & = & f \mathbin{\&} (y \otimes y) \\
\\
W(A) & = & 1 \mathbin{\&} A \\
W(f) & = & \mathsf{id}_1 \mathbin{\&} f
\end{array}
$$

In particular, for each object/formula A, there are the associated functors $E(A)$, $C(A)$ and $W(A)$ called the exponential, contraction and weakening functors over A, respectively. We can then define the formulas $!A$, $!_C A$ and $!_W A$ as the final coalgebra of $E(A)$, $C(A)$ and $W(A)$:

$$
\begin{array}{rcl}
!_C A & = & \nu X.C(A) \\
!A & = & \nu X.E(A) \\
!_W A & = & \nu X.W(A) \quad = W(A)
\end{array}
$$

Note that $W(A)$ is a constant functor, thus taking its final coalgebra is a trivial operation (but we do that for uniformity with other cases). This means that the above modalities are essentially given by their corresponding functors.

The exponential modality recovers both weakening and contraction, the weakening modality only weakening and the contraction modality only contraction. Moreover, all of them allow to recover A, which amounts to have dereliction (over A). It seems then natural to see the exponential modality as a combination of the weakening and the contraction modality. This can be made completely formal at the level of functors.

Lemma 14. Given a bifunctor $C : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a functor $W : \mathcal{C} \to \mathcal{C}$ define the mappings $C(W), W(C) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ as follows:

$$
\begin{array}{rcl}
C(W)(A,Y) & = & C(W(A),Y) \\
C(W)(f,y) & = & C(W(f),y) \\
\\
W(C) & = & W \circ C
\end{array}
$$

Then $C(W)$ and $W(C)$ are bifunctors.

Proof. The result follows from bifunctoriality of C and functoriality of W. Let us prove

$$\begin{array}{rcl} C(W)(g \circ g', f \circ f') & = & C(W)(g, f) \circ C(W)(g', f') \\ W(C)(g \circ g', f \circ f') & = & W(C)(g, f) \circ W(C)(g', f') \end{array}$$

For the first one, we have

$$\begin{array}{rcl} C(W)(g \circ g', f \circ f') & = & C(W(g \circ g'), f \circ f') \\ & = & C(W(g) \circ W(g'), f \circ f') \\ & = & C(W(g), f) \circ C(W(g'), f') \\ & = & C(W)(g, f) \circ C(W)(g', f') \end{array}$$

$$\begin{array}{rcl} W(C)(g \circ g', f \circ f') & = & (W \circ C)(g \circ g', f \circ f') \\ & = & W(C(g \circ g', f \circ f')) \\ & = & W(C(g, f) \circ C(g', f')) \\ & = & W(C(g, f)) \circ W(C(g', f')) \\ & = & (W \circ C)(g, f) \circ (W \circ C)(g', f') \\ & = & W(C)(g, f) \circ W(C)(g', f') \end{array}$$

Cases for identities are proved similarly. We show

$$C(W)(\mathsf{id}_A, \mathsf{id}_B) = \mathsf{id}_{C(W)(A,B)}$$

as an example. We have

$$\begin{array}{rcl} C(W)(\mathsf{id}_A, \mathsf{id}_B) & = & C(W(\mathsf{id}_A), \mathsf{id}_B) \\ & = & C(\mathsf{id}_{W(A)}, \mathsf{id}_B) \\ & = & \mathsf{id}_{C(W(A),B)} \\ & = & \mathsf{id}_{C(W)(A,B)} \end{array}$$

■

We want to use the above lemma to decompose the exponential functor via the weakening and contraction functor. We know that the weakening functor is indeed a functor (is polynomial). Therefore, in order to apply the above lemma we need to prove that the contraction functor is a bifunctor. This is indeed the case.

Lemma 15. The contraction functor $C : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor.

Proof. This follows from bifunctoriality of & and $\otimes$. Let us prove

$$\begin{array}{rcl} C(g \circ g', f \circ f') & = & C(g, f) \circ C(g', f') \\ C(\mathsf{id}_A, \mathsf{id}_B) & = & \mathsf{id}_{C(A,B)} \end{array}$$

We have

$$\begin{array}{rcl} C(g \circ g', f \circ f') & = & (g \circ g') \,\&\, ((f \circ f') \otimes (f \circ f')) \\ & = & (g \circ g') \,\&\, ((f \otimes f) \circ (f' \otimes f')) \\ & = & (g \,\&\, (f \otimes f)) \circ (g' \,\&\, (f' \otimes f')) \\ & = & C(g, f) \circ C(g', f') \end{array}$$

$$\begin{array}{rcl} C(\mathsf{id}_A, \mathsf{id}_B) & = & \mathsf{id}_A \,\&\, (\mathsf{id}_B \otimes \mathsf{id}_B) \\ & = & \mathsf{id}_A \,\&\, \mathsf{id}_{B \otimes B} \\ & = & \mathsf{id}_{A \& (B \otimes B)} \\ & = & \mathsf{id}_{C(A,B)} \end{array}$$

∎

We can finally prove a decomposition theorem.

Proposition 12. We have the following decomposition:

$$E \cong W(C) \cong C(W)$$

Proof. Formally, we should give natural isomorphisms between these functors. However, since we have

$$
\begin{array}{rcl}
W(C)(A, Z) & = & W(C(A, Z)) \\
& = & 1 \,\&\, C(A, Z) \\
& = & 1 \,\&\, (A \,\&\, (Z \otimes Z)) \\
& \cong & (1 \,\&\, A) \,\&\, (Z \otimes Z) \\
& = & W(A) \,\&\, (Z \otimes Z) \\
& = & C(W(A), Z) \\
& = & C(W)(A, Z)
\end{array}
$$

we see that the only step in which we need an isomorphisms rather than definitional equality is for associativity of product &. We already know that we have a natural isomorphism for that (moreover, recall that any binary product is also a tensor product), so that we can conclude that the above functors are indeed isomorphic. ■

This result completes our analysis of the exponential modality, its weaker variants and, in light of the above proposition, its components, namely the weakening and contraction modalities.

In the next chapter we extend νLL to full classical linear logic. Working on a linear classical base allows to exploit the duality between the fixed point operators μ and ν and to introduce interesting new formalisms, such as the *Calculus of Structures* [63, 64]. The duality between μ and ν, formalized as

$$(\mu X.F)^{\perp} = \nu X.F^{\perp}$$

(but see how we define negation on variables), allows to recover the modality $?B$ (read 'why not B') as

$$\mu X.\perp \oplus B \oplus (X \parr X)$$

which gives that ? is indeed dual to our encoding of !.

Chapter 4

Classical Linear Logic

In this chapter we introduce calculi for full classical propositional linear logic [54, 56, 57, 105] and its extension with fixed point operators. Classical linear logic is characterized by the presence of an involution $(\_)^{\perp}$, called linear negation, which satisfies the so-called de Morgan's duality (see below). This gives to linear logic a classical flavor since each formula is equivalent to its double negation (that is, for any formula A, A and $A^{\perp\perp}$ are equivalent).

In this chapter we look at classical linear logic from a proof-theoretical perspective, focusing on calculi rather than on semantics. In fact, from a categorical perspective, everything works as in previous chapters, but with a $*$-autonomous, rather than monoidal, base (see [10, 15, 30, 77, 95]). $*$-autonomous categories allow to model linear negation $(\_)^{\perp}$ as a contravariant endofunctor $(\_)^{\perp}$ that induces a natural isomorphism between A and $A^{\perp\perp}$, for any object A, thus making $(\_)^{\perp}$ an involution.

Working on a classical base allows us to design deep inference calculi in the Calculus of Structures formalism [63, 64] in a straightforward way. These calculi are based on simple algebraic-like manipulations of formulas, thus providing an usable machinery for making calculations.

4.1 Classical Linear Logic

We start introducing a sequent calculus for classical (propositional) linear logic.

Definition 27. Given a collection of atomic propositions Prop, formulas of classical propositional linear logic (CLL) are defined by the following grammar

$$A ::= a \mid \top \mid \bot \mid 1 \mid 0 \mid A^\bot \mid A \otimes A \mid A \,\&\, A \mid A \parr A \mid A \oplus A \mid\, !A \mid ?A$$

We immediately reduce the syntax using de Morgan dualities:

Definition 28. Given a set Prop as above we equip it with a bijection $(\_)^\bot : \mathsf{Prop} \to \mathsf{Prop}$, such that for every $a \in \mathsf{Prop}$, we have $a^{\bot\bot} = a$ and $a^\bot \neq a$. Moreover, we require Prop to contain four special elements, called constants, which are denoted by $\bot, 1, 0$ and $\top$ (called bottom, one, zero, and top, respectively). The function $(\_)^\bot$ is defined on them as follows:

$$\begin{array}{rcl} 1^\bot & = & \bot \\ \bot^\bot & = & 1 \\ \top^\bot & = & 0 \\ 0^\bot & = & \top \end{array}$$

Atomic propositions are elements of $\mathsf{Prop} \cup \mathsf{Prop}^\bot$, i.e. if $a \in \mathsf{Prop}$, then both a and $a^\bot$ are atomic propositions. Formulas are then built from atomic propositions as above, by means of the connectives $\otimes, \&, \oplus, \parr, !$ and $?$. Linear negation is the extension of the function $(\_)^\bot$ to all formulas by de Morgan equations:

$$\begin{array}{rcl} (A \otimes B)^\bot & = & A^\bot \parr B^\bot \\ (A \parr B)^\bot & = & A^\bot \otimes B^\bot \\ (A \,\&\, B)^\bot & = & A^\bot \oplus B^\bot \\ (A \oplus B)^\bot & = & A^\bot \,\&\, B^\bot \\ (!A)^\bot & = & ?A^\bot \\ (?A)^\bot & = & !A^\bot \end{array}$$

Linear implication $\multimap$ is defined by

$$A \multimap B = A^\bot \parr B$$

It directly follows from this definition (with an easy induction of formulas) that

$$A^{\bot\bot} = A$$

holds for any formula A.

Figure 4.1 gives a sequent calculus for CLL. Sequents are expressions of the form $\Gamma \vdash \Delta$, where Γ and Δ are multisets of formulas. As for classical logic, a sequent

$$A_1, \ldots, A_n \vdash B_1, \ldots, B_m$$

has its operational counterpart as the formula

$$A_1 \otimes \cdots \otimes A_n \multimap B_1 \parr \cdots \parr B_m.$$

Rules for the exponential modalities ! and ? are given in Figure 4.2

We can use properties of linear negation to come with a one-sided calculus. In fact, it is easy to prove that a sequent $\Gamma \vdash \Delta$ is equivalent to $\vdash \Gamma^\perp, \Delta$, where for $\Gamma = A_1, \ldots, A_n$, $\Gamma^\perp$ is $A_1^\perp, \ldots, A_n^\perp$. Using this fact, we come up with a calculus in which sequents are expressions of the form $\vdash \Gamma$ (hence the name 'one-sided'). Such system is given in Figure 4.3.

An easy induction on the structure of derivations gives the following

Proposition 13. The system given in Figure 4.1 and Figure 4.2, is equivalent to the system in Figure 4.3.

We now add fixed point operators to the above logic. Again, we work with the one-variable fragment, although all syntactic results given here generalize to the case of arbitrary many variables. We extend the syntax of open formulas with constructors $\parr$ and $\oplus$. Note that we do not define the negation $F^\perp$ of an open formula F as primitive. In fact, we can define $F^\perp$ inductively as follows:

$$\begin{array}{rcl}
X^\perp &=& X \\
A^\perp &=& A^\perp \\
(F \otimes G)^\perp &=& F^\perp \parr G^\perp \\
(F \parr G)^\perp &=& F^\perp \otimes G^\perp \\
(F \mathbin{\&} G)^\perp &=& F^\perp \oplus G^\perp \\
(F \oplus G)^\perp &=& F^\perp \mathbin{\&} G^\perp
\end{array}$$

Again, we can prove $F^{\perp\perp} = F$ for any open formula F. The fact that $X^\perp = X$ can be justified as follows: the variable X is a syntactic device for the identity functor, which is self-dual. Moreover, in standard mathematics, if one consider a function $f(X)$ in the variable X, then its dual does not modify the place-holder X.

Identity Group

$$\frac{}{A \vdash A} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

Negation

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, A^{\perp} \vdash \Delta} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^{\perp}, \Delta}$$

Multiplicative Rules

$$\frac{\Gamma \vdash A}{\Gamma, 1 \vdash A} \qquad \frac{}{\vdash 1}$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'}$$

$$\frac{}{\perp \vdash} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \perp, \Delta}$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \parr B \vdash \Delta, \Delta'} \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta}$$

Additive Rules

$$\frac{}{\Gamma \vdash \top}$$

$$\frac{A_i, \Gamma \vdash \Delta}{A_1 \,\&\, A_2, \Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \,\&\, B, \Delta}$$

$$\frac{}{\Gamma, 0 \vdash \Delta}$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta} \qquad \frac{\Gamma \vdash A_i, \Delta}{\Gamma \vdash A_1 \oplus A_2, \Delta}$$

Figure 4.1: Sequent Calculus for classical linear logic.

$$\frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta} \quad \frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta} \quad \frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta} \quad \frac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash ?A, \Delta} \quad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash ?A, \Delta} \quad \frac{\Gamma \vdash ?A, ?A, \Delta}{\Gamma \vdash ?A, \Delta} \quad \frac{!\Gamma, A \vdash ?\Delta}{!\Gamma, ?A \vdash ?\Delta}$$

Figure 4.2: Sequent calculus rules for exponentials.

We can now extend negation to fixed point formulas as follows

$$\begin{aligned} (\mu X.F)^\perp &= \nu X.F^\perp \\ (\nu X.F)^\perp &= \mu X.F^\perp \end{aligned}$$

which makes $(\_)^\perp$ an involution.

Remark. If one takes $X^\perp$ as primitive (like $a^\perp$, for $a \in \mathsf{Prop}$), then we have to modify negation of fixed point formulas as in e.g.

$$(\mu X.F)^\perp = \nu X.F^\perp[X^\perp := X].$$

Rules for fixed points are as usual. The main difference is that we can now use duality to give a one-side presentation of these rules. One-sided rules are given in Figure 4.4.

Again, we can recover $!A$ as $\nu X.1 \,\&\, A \,\&\, (X \otimes X)$. Dually, it is easy to prove (dualizing proofs given for $!A$) that we can recover $?A$ as

$$\mu X.\perp \oplus A \oplus (X \parr X)$$

Moreover, these encodings give the desired duality between ! and ?. Indeed we have

$$\begin{aligned} (!A)^\perp &= (\nu X.1 \,\&\, A \,\&\, (X \otimes X))^\perp \\ &= \mu X.\perp \oplus A^\perp \oplus (X \parr X) \\ &= ?A^\perp \end{aligned}$$

Similarly we can recover the weaker structural modalities $?_C$ and $?_W$ as

$$\begin{aligned} ?_W A &= \perp \oplus A \\ ?_C A &= \mu X.A \oplus (X \parr X) \end{aligned}$$

Identity Group

$$\frac{}{\vdash A, A^{\perp}} \qquad\qquad \frac{\vdash A, \Gamma \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}$$

Multiplicative Rules

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta} \qquad\qquad \frac{\vdash A, B, \Gamma}{\vdash A \mathbin{⅋} B, \Gamma}$$

$$\frac{}{\vdash 1} \qquad\qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}$$

Additive Rules

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Gamma}{\vdash A \mathbin{\&} B, \Gamma} \qquad\qquad \frac{\vdash A_i, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}$$

$$\frac{}{\vdash \top, \Gamma}$$

Exponentials

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A} \qquad\qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}$$

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} \qquad\qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}$$

Figure 4.3: One-sided sequent calculus for classical linear logic.

$$\frac{\vdash \Gamma, F(\mu X.F)}{\vdash \Gamma, \mu X.F} \qquad \frac{\vdash A^{\perp}, F(A)}{\vdash A^{\perp}, \nu X.F}$$

Figure 4.4: One-sided sequent calculus rules for fixed point operators.

having the desired dualities

$$\begin{array}{rcl}(!_W A)^\perp & = & ?_W A^\perp \\ (!_C A)^\perp & = & ?_C A^\perp\end{array}$$

Unfortunately, the rule for introducing $\nu X.F$ has a bad feature: it imposes a strong restriction on its context, namely that there has to be no context at all. The reason is simple: the rule states nothing but finality of $\nu X.F$, and thus states that if A is an F-coalgebra (which means $A \vdash F(A)$, and thus $\vdash A^\perp, F(A)$), then it is 'smaller' than the final F-coalgebra $\nu X.F$ (that is $A \vdash \nu X.F$, and thus $\vdash A^\perp, \nu X.F$). If we have a premise like $\vdash B^\perp, A^\perp, F(A)$ we cannot know whether A is an F-coalgebra or simply $F(A)$ is related to B.

That rule can be replaced with the equivalent

$$\frac{\vdash \Gamma, A \qquad \vdash A^\perp, F(A)}{\vdash \Gamma, \nu X.F}$$

which unfortunately does not solve the problem.

A promising way to deal with such problems seems to move to richer calculi, like display calculi [20] or deep sequents [33], that have enough structure to 'package' assumptions together. For example, suppose we allow sequents like $\Gamma[\Delta]$ with the informal meaning that Δ has no dependency from Γ. Then we could consider the rule

$$\frac{\Gamma[A^\perp, F(A)]}{\Gamma[A, \nu X.F]}$$

These kinds of proposals were extensively studied in the context of deep sequents [33], with the goal of designing usable proof systems for several modal logics. Another interesting solution was introduced in [45] dealing with a deep inference formulation of a higher-order coinduction axiom. Such a solution is based on the introduction of a semantically null modality, which gives a syntactic device for grouping assumptions together. We will adopt that idea for the deep inference calculus given in the next section.

Digression: Cut Elimination

Proving that the above sequent calculi for classical linear logic with fixed point operators (overloading terminology, we refer to such logic as νLL) enjoys cut-elimination is not easy and, as far as the author knows, there is no direct proof of such theorem. In [14] an indirect cut elimination theorem for an higher-order linear logic with fixed point operators is proved. Although the logic studied in that paper is different from those considered in this thesis (it requires typed variables, quantifiers and λ abstractions), the proof of the cut elimination theorem can be slightly modified to work for the above calculi. The proof is in fact based of an encoding of the logic in the second-order propositional linear logic with exponentials [58, 70], for which a cut elimination theorem holds. We do not give details here and address the reader to the original paper [14]. The intuition behind the encoding comes from Knaster-Tarski Theorem [3, 46, 91]

Proposition 14 (Knaster-Tarski)**.** Let $\langle L, \leq \rangle$ be a complete lattice and $f : L \to L$ be a monotene map. Then f has both least and greatest fixed points, denoted by μf and νf respectively, which are given as follows:

$$\begin{aligned} \mu f &= \bigwedge \{x \in L \mid f(x) \leq x\} \\ \nu f &= \bigvee \{x \in L \mid x \leq f(x)\} \end{aligned}$$

As a consequence, in a lattice-based logic with connectives capturing arbitrary joints and meets, and the order relation, we can use the Knaster-Tarski Theorem to define least and greatest fixed points of specific functions. Usually, second order universal and existential quantifiers are interpreted as arbitrary joints and meets, whereas implication is interpreted as the order relation. For example, in second order linear logic we can encode $\mu X.F$ as

$$\forall \alpha.!(F\alpha \multimap \alpha) \multimap \alpha$$

where we assume we have already encoded F (the presence of ! is due, to the best of the author's knowledge, to technical motivations rather than to some specific intuitions).

Finding a direct proof of the cut-elimination theorem is still an open problem, and a promising approach to such problem seems to move to calculi based on richer formalisms and better structural properties, such as display calculi [20]. In [21] a display-like calculus for full propositional linear logic is proposed. Such calculus is subject to specific technical constraints due to the restrictions imposed by rules for exponential modalities. In [50, 51, 52] the display-calculus formalism is extended to the so-called multi-type calculi. Such calculi allow to deal with objects living in different domains, and seems to be the appropriate formalism for giving a well-behaved proof theory both for exponential modalities (for example introducing a type for additive formulas and connectives, and a type for multiplicative formulas and connectives) and fixed point operators.

4.2 A Deep Inference Calculus

We now introduce deep inference calculi for the above logics, following the *Calculus of Structures* formalism [63, 64, 99]. The main feature of this formalism is that rules can be applied deep inside formulas (hence the name 'deep inference'). In fact, the calculus of structures drops the usual distinction between the object-level and meta-level (which is one of the main features of sequent calculi[1]). This leads to the introduction of new syntactic objects, called *structures*. These define *contexts*, structures with a hole $\xi\{\_\}$, which can be made into a proper structures simply by filling the hole, like in $\xi\{A\}$. Rules are figure of the form

$$\frac{\xi\{A\}}{\xi\{B\}}$$

The methodology is in line with functoriality requirements: given a proof π for A to B (written $\pi : A \to B$), a context $\xi\{\_\}$ is functorial, in the sense we automatically have a proof $\xi\{\pi\}$ of $\xi\{B\}$ from $\xi\{A\}$ (i.e. $\xi\{\pi\} : \xi\{A\} \to \xi\{B\}$).

Let us now give a deep system for classical linear logic (without fixed point operators). Structures are defined as formulas, and are considered modulo the equational theory generated by equations in Figure 4.5, the De Morgan

[1]E.g. Consider the rule

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}$$

The formula $A \otimes B$ is introduced at the object-level, whereas the context Γ, Δ is introduce at the meta-level of sequents.

equations and the following two equations:

$$\begin{array}{rcl} 1 \,\&\, 1 & = & 1 \\ \bot \oplus \bot & = & \bot \end{array}$$

Associativity	Commutativity	Units
$A \otimes (B \otimes C) = (A \otimes B) \otimes C$	$A \otimes B = B \otimes A$	$1 \otimes A = A$
$A \parr (B \parr C) = (A \parr B) \parr C$	$A \parr B = B \parr A$	$\bot \parr A = A$
$A \oplus (B \oplus C) = (A \oplus B) \oplus C$	$A \oplus B = B \oplus A$	$0 \oplus A = A$
$A \,\&\, (B \,\&\, C) = (A \,\&\, B) \,\&\, C$	$A \,\&\, B = B \,\&\, A$	$\top \,\&\, A = A$

Figure 4.5: Equations for structures.

Definition 29. An inference rule is a scheme of the kind

$$\frac{A}{B}$$

for structures A and B. As usual A is called the premise and B the conclusion. An inference rule is called an axiom if its premise is empty.

Actually, a typical rule has shape

$$\frac{\xi\{A\}}{\xi\{B\}}$$

and specifies a step of rewriting, by the implication[2] $A \Rightarrow B$, inside a generic context $\xi\{\_\}$. Rules with empty contexts correspond to the case of the sequent calculus.

Definition 30. A (formal) system $\mathcal{S}$ is given by a set of inference rules. A derivation Δ in $\mathcal{S}$ is a finite chain of instances of inference rules in $\mathcal{S}$. A derivation can consist of just one structure. The topmost structure in a derivation, if present, is called the premise of the derivation, and the bottommost structure is called its conclusion. A proof π is a finite derivation whose topmost inference rule is an axiom.

[2]With $\Rightarrow$ we refer to the notion of implication modelled by the system (we are in fact assuming to deal with logics with a notion of consequence and implication). For linear logic we take $\Rightarrow$ to be linear implication $\multimap$.

Usually, in the calculus of structures rules come in pairs, a down-version

$$\frac{\xi\{A\}}{\xi\{B\}}$$

and an up-version

$$\frac{\xi\{B^\perp\}}{\xi\{A^\perp\}}$$

This duality derives from the duality between $A \Rightarrow B$ and $B^\perp \Rightarrow A^\perp$, where $\Rightarrow$ is the implication modelled in the system, and $(\_)^\perp$ is the negation modelled in the system (which we usually require to be an involution[3]). In our case the latter are linear implication and negation.

The core rules of systems we will consider are the interaction and co-interaction rules, which correspond to identity and cut, respectively. These are

$$\frac{\xi\{1\}}{\xi\{A \parr A^\perp\}}\,\mathsf{id} \qquad\qquad \frac{\xi\{A \otimes A^\perp\}}{\xi\{\bot\}}\,\mathsf{cut}$$

Note how the calculus allows to exploit the duality between identity and cut. The other fundamental rule is the switching rule

$$\frac{\xi\{(A \parr B) \otimes C\}}{\xi\{(A \otimes C) \parr B\}}\,\mathsf{s}$$

That the rule is sound is essentially witnessed by the following derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A \parr B \vdash A, B} \qquad C \vdash C}{A \parr B, C \vdash A \otimes C, B}}{(A \parr B) \otimes C \vdash A \otimes C, B}}{(A \parr B) \otimes S \vdash (A \otimes C) \parr B}$$

Having interaction, co-interaction and switching, we can prove an important duality result.

Proposition 15. Let $\mathcal{S}$ be a system with identity id, cut cut and switch s. Then, for every rule ρ

[3] Being interested in linear logic only, we do not care too much about these generalities.

$$\frac{\xi\{A\}}{\xi\{B\}}$$

its dual $\rho^\perp$

$$\frac{\xi\{B^\perp\}}{\xi\{A^\perp\}}$$

is admissible.

Proof. Construct

$$\cfrac{\cfrac{\cfrac{\cfrac{\xi\{B^\perp\}}{\xi\{1 \otimes B^\perp\}}=}{\xi\{(A \parr A^\perp) \otimes B^\perp\}}\mathsf{id}}{\xi\{(A \otimes B^\perp) \parr A^\perp\}}\mathsf{s}}{\cdots}$$

$$\xi\{(A \otimes B^\perp) \parr A^\perp\} \;\Big\Vert\rho\; \xi\{(B \otimes B^\perp) \parr A^\perp\}$$

$$\cfrac{\cfrac{\xi\{(B \otimes B^\perp) \parr A^\perp\}}{\xi\{\bot \parr A^\perp\}}\mathsf{cut}}{\xi\{A^\perp\}}=$$

■

The notation

$$\begin{array}{c} \xi\{A\} \\ \pi \Big\Vert \\ \xi\{B\} \end{array}$$

means that we have a derivation π for $\xi\{B\}$ from $\xi\{A\}$.

Remark. Note that the above result strongly relies on the de Morgan duality. If the latter is dropped, the above proposition would not hold anymore.

The deep inference calculus for classical linear logic is given in Figure 4.6, where we also give for each rule ρ its dual $\rho^\perp$ (although, according to previous proposition, that is redundant, except for the pair $(\mathsf{id}, \mathsf{cut})$). Note also that the switch rule s is self-dual.

This system gives a deduction theorem, in the following sense:

$$\frac{\xi\{1\}}{\xi\{A \parr A^{\perp}\}}\ \mathsf{id} \qquad\qquad \frac{\xi\{A \otimes A^{\perp}\}}{\xi\{\perp\}}\ \mathsf{cut}$$

$$\frac{\xi\{(A \parr B) \otimes C\}}{\xi\{(A \otimes C) \parr B\}}\ \mathsf{s}$$

$$\frac{\xi\{(A \parr B) \,\&\, (C \parr D)\}}{\xi\{(A \,\&\, C) \parr (B \oplus C)\}}\ \mathsf{d} \qquad\qquad \frac{\xi\{(A \oplus B) \otimes (C \,\&\, D)\}}{\xi\{(A \otimes C) \oplus (B \otimes D)\}}\ \mathsf{d}^{\perp}$$

$$\frac{\xi\{0\}}{\xi\{A\}} \qquad\qquad \frac{\xi\{A\}}{\xi\{\top\}}$$

$$\frac{\xi\{A \oplus A\}}{\xi\{A\}} \qquad\qquad \frac{\xi\{A\}}{\xi\{A \,\&\, A\}}$$

Figure 4.6: Deep system for classical linear logic.

Proposition 16. There is a derivation $\begin{array}{c} A \\ \Delta \Big\| \\ B \end{array}$ if and only if there is a derivation $\begin{array}{c} 1 \\ \Delta' \Big\| \\ A^{\perp} \parr B \end{array}$.

Proof. Consider the following derivations, where we use the equation $A = 1 \otimes A$.

$$\begin{array}{c} \mathsf{id}\,\dfrac{1}{A \parr A^{\perp}} \\ \Delta \Big\| \\ B \parr A^{\perp} \end{array} \qquad\qquad \begin{array}{c} 1 \otimes A \\ \Delta' \Big\| \\ \mathsf{cut}\,\dfrac{\mathsf{s}\,\dfrac{(A^{\perp} \parr B) \otimes A}{(A^{\perp} \otimes A) \parr B}}{\dfrac{\dfrac{[(T, \bar{T}), R]}{\perp \parr B}}{B}\,=} \end{array}$$

$\blacksquare$

The system in Figure 4.6 does not have axioms, since we think of a provable formula A as provable from 1. Nonetheless, we can add the axiom-rule

$$\frac{}{1}$$

which states that the unit 1 is provable. Note that this rule breaks the symmetry of the calculus.

It is possible to restrict the identity rule to its atomic version

$$\frac{\xi\{1\}}{\xi\{a \parr a^\perp\}}$$

as it is often done in sequent calculi. A specific feature of deep systems, which is usually not available in sequent calculi, is that we can restrict the cut rule to its atomic version as well.

Proposition 17. If we modify the system in Figure 4.6 by substituting the rule cut with

$$\frac{\xi\{a \otimes a^\perp\}}{\xi\{\perp\}}$$

then the resulting system is equivalent to the original one. That is, the rule cut is admissible in the new system.

Proof. We proceed by induction on A. The basic case is given by hypothesis. We essentially have two further cases to consider.

1. Consider the case for $A \,\&\, B$. We have

$$\begin{array}{c}
\dfrac{\dfrac{\xi\{(A \,\&\, B) \otimes (A^\perp \oplus B^\perp)\}}{\xi\{(A^\perp \oplus B^\perp) \otimes (A \,\&\, B)\}} =}{\xi\{(A^\perp \otimes A) \oplus (B^\perp \otimes B)\}} \mathsf{d}^\perp \\
\Big\Vert \, \text{i.h.} \\
\xi\{\perp \oplus \perp\}
\end{array}$$

which gives the result since $\bot \oplus \bot = \bot$. Note that the case for the structure $A \oplus B$ essentially reduces to this case.

2. Consider the case for $A \otimes B$. We have

$$
\cfrac{\cfrac{\xi\{(A \otimes B) \otimes (A^{\bot} \parr B^{\bot})\}}{\xi\{(A^{\bot} \parr B^{\bot}) \otimes A \otimes B\}}\, =}{\xi\{((A^{\bot} \otimes A) \parr B^{\bot}) \otimes B\}}\, \mathsf{s}
$$

$$
\Big\Vert\ \text{i.h.}
$$

$$
\cfrac{\xi\{(\bot \parr B^{\bot}) \otimes B\}}{\xi\{B^{\bot} \otimes B\}}\, =
$$

$$
\Big\Vert\ \text{i.h.}
$$

$$
\xi\{\bot\}
$$

Note that the case for the structure $A \parr B$ essentially reduces to this case.

■

The calculus obtained is simple and easy to use. It is indeed meant to be a calculus to be used on an operational level, where one needs to *use* linear logic, rather than to *study* linear logic (and thus to formulate and prove meta-theoretic properties, for which the sequent calculus formalism seems to be better). This is more or less the same difference one can observe between natural deduction systems and sequent calculi.

It is possible to prove that the deep system in Figure 4.6 is equivalent to the sequent calculus given in Figure 4.3 (see e.g. [100, 101] for details). The result is proved by means of translations from deep inference proofs to sequent calculus proofs. Moreover, such translations send cut-free proofs (in one system) to cut-free proofs (in the other system), so that we have an indirect proof of cut-elimination for the deep inference calculus: given a deep inference derivation Δ, we map it to a sequent calculus derivation Δ'. We run the cut-elimination process on Δ', thus obtaining Δ'_0, and then map it back to a cut-free deep inference derivation Δ_0. It is also possible to give a direct proof of the cut elimination theorem (as done in [99]). This requires a technique called *splitting* [63, 64]. The proof is rather technical and long, and the reader is invited to consult [99] for details.

Having one-sided sequent calculus rules for exponentials allows to extend the deep inference calculus to full linear logic in a straightforward way. However, the resulting system is not very satisfactory (see [101]), and a better system can be obtained by considering rules given in Figure 4.7.

$$\frac{\xi\{!(A \parr B)\}}{\xi\{!A \parr ?B\}} \qquad \frac{\xi\{!A\}}{\xi\{1\}} \qquad \frac{\xi\{!A\}}{\xi\{!A \otimes A\}}$$

$$\frac{\xi\{?A \otimes !B\}}{\xi\{?(A \otimes B)\}} \qquad \frac{\xi\{\bot\}}{\xi\{?A\}} \qquad \frac{\xi\{?A \parr A\}}{\xi\{?A\}}$$

Figure 4.7: Deep inference rules for exponentials.

Contrary to our analysis of exponential modalities, these rules are not modular and make exponentials something hard to deal with.

We can extend the deep system with rules for fixed point operators. It seems natural to add the rules

$$\frac{\xi\{F(\mu X.F)\}}{\xi\{\mu X.F\}} \qquad \frac{\xi\{F(\nu X.F)\}}{\xi\{\nu X.F\}}$$

Translating the rule

$$\frac{A \vdash F(A)}{A \vdash \nu X.F}$$

is more problematic, since it cannot be simply wrapped with contexts, as in

$$\frac{\xi\{A^\perp \parr F(A)\}}{\xi\{A^\perp \parr \nu X.F\}}$$

In fact, this rule is not sound. We adopt the solution used in [45], where it was faced the problem of finding a deep inference rule corresponding to a first order coinduction axiom. The solution consists in introducing a semantically empty modality †, so to be able to deal with 'fixed contexts'. Intuitively, in a structure $\xi\{\dagger A\}$, we can only operate either on A, or on $\xi\{\_\}$, but there cannnot be any interaction between these two. The only rule governing † is

$$\frac{\xi\{1\}}{\xi\{\dagger 1\}}\,\mathsf{i}$$

Proceeding bottom-up, the rule states e.g. that in order to prove $\xi\{\dagger A\}$, we have to be able to prove A in isolation. In fact, this means that A can be derived from 1, so that we obtain $\xi\{\dagger 1\}$. We can now eliminate $\dagger$. In particular, $\dagger A$ and A are equi-provable, so that, semantically, $\dagger A$ is interpreted simply as A. This makes the syntactic device of $\dagger$ sound.

We can now formulate the rule for finality of $\nu X.F$ as

$$\frac{\xi\{\dagger(A^\perp \parr F(A))\}}{\xi\{A^\perp \parr \nu X.F\}}$$

Having $A^\perp \parr F(A)$ in isolation indeed requires A to be an F-coalgebra, and thus makes the rule sound.

Remark. The modality $\dagger$ does not have a de Morgan dual, so that the above rule does not have a dual version. In fact, the dual of that rule (even without dagger and contexts) is unsound.

We thus come up with a stock of (deep inference) rules for fixed point operators, as summarized in Figure 4.8.

$$\frac{\xi\{F(\mu X.F)\}}{\xi\{\mu X.F\}}\,\mu \qquad \frac{\xi\{1\}}{\xi\{\dagger 1\}}\,\mathsf{i} \qquad \frac{\xi\{\dagger(A^\perp \parr FA)\}}{\xi\{A^\perp \parr \nu X.F\}}\,\nu$$

Figure 4.8: Deep inference rules for fixed point operators.

Note that we left out the rule

$$\frac{\xi\{F(\nu X.F)\}}{\xi\{\nu X.F\}}$$

In fact, we can recover this rule as the axiom $(F(\nu X.F))^\perp \parr \nu X.F$, as shown by the following derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\xi\{1\}}{\xi\{\dagger 1\}}\,\mathsf{i}}{\xi\{\dagger((F(F(\nu X.F)))^{\perp} \parr F(F(\nu X.F)))\}}\,\mathsf{id}}{\xi\{\dagger(F^{\perp}(F^{\perp}(\mu X.F^{\perp})) \parr F(F(\nu X.F)))\}}\,=}{\xi\{\dagger(F^{\perp}(\mu X.F^{\perp}) \parr F(F(\nu X.F)))\}}\,\mu}{\xi\{\dagger((F(\nu X.F))^{\perp} \parr F(F(\nu X.F)))\}}\,=}{\xi\{\dagger((F(\nu X.F))^{\perp} \parr \nu X.F)\}}\,\nu
$$

We designed a sequent calculus (both two- and one-sided) and a deep inference calculus for classical linear logic enriched with least and greatest fixed point operators νLL. The study of the meta-theoretical properties of these calculi is at a preliminary stage, and the author believes that a better proof-theoretical account to νLL can be achieved by moving to calculi based on richer formalisms, like e.g. display calculi [20] or deep sequents [33] (see next chapter for more details).

We conclude this work sketching some possible applications of the framework designed so far.

Applications, Further Works and Conclusions

In this chapter we briefly outline some possible applications of the framework studied so far, which will be investigated in future works. As a consequence, the treatment will not be completely formal, focused more on ideas than concrete results. Here we focus on some specific applications, thus leaving out some other interesting ones. Among these, it is worth mentioning possible applications in type theory [109] (inductive and coinductive linear types), game semantics [4] (for example concerning infinite and iterative games) and, clearly, those fields mentioned in the introduction, notably categorical quantum mechanics (e.g. infinite and iterative protocols).

Epistemic and doxastic logics built over a linear base have been recently investigated, due to their applications in computer science [18, 42]. Results are mostly syntactical and, although claimed to be 'epistemic', only $S4$-like modal linear logics were considered. The reason is the lack of well-behaved sequent calculi for $S5$-modal logics [82, 97]. This problem can be fixed simply moving to richer formalisms, like display calculi (see [111] for an introduction to display calculi for several modal logics) or deep inference calculi [98]. In the previous chapter we studied a deep inference proof system for classical linear logic with fixed point operators. Adding to that system rules for an $S5$ modality, gives a proof system for a standard epistemic logic built over a linear base. Finally, adding fixed point operators allows to recover both common knowledge and exponential modalities. The same can be done with other modalities.

A more interesting yet hard question regards semantics for combinations of modal and linear logics. In fact, the linear base of the logic makes Kripke models less intuitive than their 'classical counterpart' (see [87] for a general introduction to relational semantics for substructural logics). It is possible to give sound and complete algebraic semantics to several modal linear logic (via e.g. CL algebras [54, 105]), but these are rather syntactical-oriented and ad hoc. A survey of relational and algebraic semantics for linear logic is given in [108].

Here we sketch possible modifications of Aumann structures [49, 85] in order to be able to deal with non-classical bases. Aumann structures provide intuitive set-theoretic models for epistemic logics, and are usually equivalent to Kripke models (see [49] for details). However, their set-theoretic nature makes them suitable for extensions, preserving at the same time their intuitive character. We propose some possible extensions that make such models closer to the notion of *pretopology* [89, 90, 17], a notion which was successfully employed to give sound and complete semantics to several fragments of linear logic. These models have the major advantage of being simple and intuitive, and thus seem to constitute a good starting point for building new semantics for modal logics built over a linear base.

Towards Epistemic Linear Logic

Let us start by reviewing basic ideas behind Aumann structures (the reader can consult [85] for details, and [49] for a more logic-oriented introduction). Given a collection Ω of states (which can be thought as complete descriptions of the world), an Aumann structure is obtained by equipping Ω with an information function $P : \Omega \to 2^{\Omega}$ that associates with every state $\omega \in \Omega$ a nonempty subset $P(\omega)$ of Ω. The intended (informal) meaning is that when the state is ω the agent knows only that the state is in the set $P(\omega)$. That is, the agent considers possible that the true state could be any state in $P(\omega)$. Information function can be used to model several concepts, like knowledge and belief, depending on which conditions we impose on P. In order to model knowledge we require an information function P to satisfy the following two conditions:

1. $\omega \in P(\omega)$ for every $\omega \in \Omega$.

2. If $\omega' \in P(\omega)$, then $P(\omega) = P(\omega')$.

These two conditions make P partitional, that is we require the existence of a partition of Ω such that for any $\omega \in \Omega$ the set $P(\omega)$ is the element of the partition that contains ω.

An event is a subset of Ω, which means that we are taking 2^Ω to be the event space. This set has a boolean algebra structure, so that we have the classical operations of conjunction, disjunction and negation for events, given by intersection, union and complementation, respectively. Given an event $E \in 2^\Omega$ and a state $\omega \in \Omega$, we say that the agent knows E if $P(\omega) \subseteq E$. This induces a knowledge operator $K : 2^\Omega \to 2^\Omega$ as

$$K(E) = \{\omega \in \Omega \mid P(\omega) \subseteq E\}$$

This knowledge operator satisfies the following properties, which make it an $S5$-operator (see [49] for details).

1. $K(\Omega) = \Omega$.
2. If $E \subseteq F$, then $K(E) \subseteq K(F)$.
3. $K(E) \cap K(F) \subseteq K(E \cap F)$.
4. $K(E) \subseteq E$.
5. $K(E) \subseteq K(K(E))$.
6. $\Omega \setminus K(E) \subseteq K(\Omega \setminus K(E))$.

A specific feature of Aumann structures is that the event space is simply 2^Ω, and thus has a classical nature. This makes the framework suitable for extensions and improvements. For example, it is possible to equip Ω with a topology and consider events to be open sets, thus moving from a classical to an intuitionistic base. In [12, 13] topological extensions of Aumann structures were considered, in order to formalize a notion of distance between events and to define the notion of *limit knowledge*.

Another possible way to extend Aumann structure is to consider structured sets of states. In the specific case of linear logic, we could take as basic states an ordered monoid of partial descriptions of the world. More formally, we consider a monoid $(\Omega, \cdot, 1, \leq)$ where $(\Omega, \cdot, 1)$ is a monoid (either commutative or not), and $\leq$ is a partial order on Ω. Intuitively, elements of Ω are partial descriptions of the world. Given two such descriptions ω, ω' we read $\omega \leq \omega'$ as the fact that the description ω' is more informative than ω. Moreover, we have the binary operation $\cdot$ to join descriptions together. The result is a new description/observation, say $\omega \cdot \omega'$, which is in general different from just having both the description ω and the description ω'. Taking $\cdot$

non-commutativity we could think ok $\omega \cdot \omega'$ as the fact that the observation ω' is made after observing ω (thus introducing some form of causality). For example if we let ω stand for the observation "Alice sees a lighting" and ω' for "Alice hears a thunder", then the observation $\omega \cdot \omega'$ relates the observations ω and ω' temporally, in the sense that whenever Alice sees a lighting, then she also hears a thunder. Taking the event space to be the whole 2^{Ω} we can define, together with classical operations, a tensor products $\otimes$ over events as follows:

$$E \otimes F = \{\omega \cdot \omega' \mid \omega \in E,\ \omega' \in F\}$$

Taking a monoid of partial observations, rather than a set of complete descriptions of the world gives an interesting structure.Unfortunately, such structure carries out a distributive base, since e.g. $\cap$ distributes over $\cup$, and thus we cannot use them to model the additive connectives $\&$ and $\oplus$[4]. To obtain a model for propositional linear logic (both classical and intuitionistic), we have to add structure to the state space, and at the same time to restrict the event space.

For what concern the state space we consider a commutative monoid $(\Omega, \cdot, 1)$ together with a cover relation $\preceq$ between states and sets of states, and a special set $\bot \subseteq \Omega$. The intuition behind Ω is as above. Given a state ω and a set of states $E \in 2^{\Omega}$, we read $\omega \preceq E$ as "the information/observation ω is subsumed by the collection of observations E". Finally, the set $\bot$ is the set of absurd observations. For example we could have two observations ω and ω' which are incompatible. This means that we have $\omega \cdot \omega' \in \bot$. We require $\bot$ to be $\preceq$-closed (i.e. if $\omega \preceq \bot$, then $\omega \in \bot$). The cover relation is required to satisfy the following properties:

$$\frac{\omega \in E}{\omega \preceq E} \qquad \frac{\omega \preceq E \qquad E \preceq F}{\omega \preceq F} \qquad \frac{\omega \preceq E \qquad \omega' \preceq E'}{\omega \cdot \omega' \preceq E \cdot E'}$$

[4]In linear logic neither additives nor multiplicatives distributes over each other, in the sense that none of these equivalences is provable:

$$\begin{array}{rcl}
A \,\&\, (B \oplus C) & \dashv\vdash & (A \,\&\, B) \oplus (A \,\&\, C) \\
A \oplus (B \,\&\, C) & \dashv\vdash & (A \oplus B) \,\&\, (A \oplus C) \\
A \otimes (B \parr C) & \dashv\vdash & (A \otimes B) \parr (A \otimes C) \\
A \parr (B \otimes C) & \dashv\vdash & (A \parr B) \otimes (A \parr C)
\end{array}$$

Nevertheless, we have distribution of additives over multiplicatives. That is, $\otimes$ distributes over $\oplus$ (and viceversa), and $\parr$ over $\&$ (and viceversa). These distributive properties justify the notation used for connectives.

where $E \preceq F$ abbreviates $\forall \omega \in E. \omega \preceq F$ and $E \cdot E' = \{\omega \cdot \omega' \mid \omega \in E,\ \omega' \in E'\}$.

The event-space is the set of *saturated sets*. A set $E \in 2^\Omega$ is saturated if it is closed under $\preceq$. That is, if $\omega \preceq E$ implies $\omega \in E$. This means that events (i.e. saturated sets) are some kind of logically complete collections of observations.

The cover relation $\preceq$ induces a closure operator $\mathcal{C} : 2^\Omega \to 2^\Omega$ given by

$$\mathcal{C}(E) = \{\omega \in \Omega \mid \omega \preceq E\}$$

satisfying the so-called stability properties, i.e.

$$\mathcal{C}(E) \cdot \mathcal{C}(F) \subseteq \mathcal{C}(E \cdot F).$$

Viceversa, a closure operator $\mathcal{C} : 2^\Omega \to 2^\Omega$ satisfying stability induces a cover relation $\preceq$ defined by

$$\omega \preceq E \quad \text{iff} \quad \omega \in \mathcal{C}(E)$$

(see [90] for details). As a consequence, we can define pretopologies as structures $(\Omega, \cdot, 1, \perp)$ as above, together with a closure operator $\mathcal{C}$ satisfying stability, instead of a cover relation $\preceq$. Saturated sets are nothing but fixed points of $\mathcal{C}$. We write $\mathcal{C}(\Omega)$ for the collection of saturated sets over Ω.

The collection $\mathcal{C}(\Omega)$ carries out a complete lattice structure (with respect to the inclusion order $\subseteq$), given by the following operations (see [89, 90] for proofs):

$$\begin{aligned} \bigwedge_{i \in I} E_i &= \bigcap_{i \in I} E_i \\ \bigvee_{i \in I} E_i &= \mathcal{C}(\bigcup_{i \in I} E_i) \end{aligned}$$

Moreover, we can define a tensorial product over $\mathcal{C}(\Omega)$ as

$$E \otimes F = \mathcal{C}(\{\omega \cdot \omega' \mid \omega \in E,\ \omega' \in F\})$$

with unit $\mathcal{C}(\{1\})$. This gives to $\mathcal{C}(\Omega)$ a quantale structure.

Finally, we can define a linear implication as

$$E \multimap F = \{\omega \in \Omega \mid \omega \cdot E \preceq F\}$$

This allows to define negation as

$$E^{\perp} = E \multimap \perp$$

We thus have a sound and complete model for intuitionistic propositional linear logic (see [89] for details). We can even obtain a model for classical linear logic by taking the closure operator $\mathcal{C}(\_)$ defined by

$$\mathcal{C}(E) = (E \multimap \perp) \multimap \perp$$

To obtain a model for epistemic or doxastic linear logic it is sufficient to consider an operator $K : \mathcal{C}(\Omega) \to \mathcal{C}(\Omega)$ satisfying the desired conditions (like e.g. those for the knowledge operator above).

Proof systems for modal linear logics can be obtained as combinations of the proof systems for modal and linear logics. For example, we obtain a deep inference proof system for an $S5$ linear logic (which we could call epistemic linear logic) by enriching the deep inference system for propositional linear logic in Section 4.2 with formulas of the form $\Box A$ and $\Diamond A$, for a formula A, equations

$$\begin{array}{rcl} (\Box A)^{\perp} & = & \Diamond A^{\perp} \\ (\Diamond A)^{\perp} & = & \Box A^{\perp} \\ \Box 1 & = & 1 \\ \Diamond \perp & = & \perp \end{array}$$

and rules in Figure 4.9 (together with their duals)

$$\frac{\xi\{\Box(A \parr B)\}}{\xi\{\Box A \parr \Diamond B\}}\,\mathsf{k} \qquad \frac{\xi\{\Box A\}}{\xi\{A\}}\,\mathsf{t} \qquad \frac{\xi\{\Diamond\Diamond A\}}{\xi\{\Diamond A\}}\,4 \qquad \frac{\xi\{\Diamond\Box A\}}{\xi\{\Box A\}}\,5$$

Figure 4.9: Deep inference rules for $S5$ modalities.

We can finally add fixed point operators to the above language to obtain modal linear logics with fixed point operators. We can give semantics to the logics thus obtained using pretopologies. Let us consider the classical case. We have a closure operator $\mathcal{C}(\_)$ given by $(\_)^{\perp\perp}$, where $E^{\perp}$ is defined as above, that satisfy stability. We also have an interpretation for additive conjunction and disjunction (via the lattice operator), and thus for additive units (in fact, we have a complete lattice). We can interpret multiplicative connectives too. We have already defined the tensor product over saturated sets. We can define the cotensor $\parr$ by

$$E \parr F = (E^{\perp} \cdot F^{\perp})^{\perp}$$

so that we indeed have $(E \otimes F)^{\perp} = E^{\perp} \parr F^{\perp}$. Multiplicative units are given as $\{1\}^{\perp\perp}$ and $\emptyset^{\perp\perp}$. The interpretation of linear negation is defined as above, whereas modalities are interpreted as suitable operators over $\mathcal{C}(\Omega)$.

Each operation used to interpret connectives is monotone with respect to the inclusion order $\subseteq$ in all its arguments (with the obvious exception of linear negation, which is antitone: however, recall that the grammars we used to define extensions of linear logic with fixed point operators do not allow variables to occur under the scope of a linear negation). Operators used to interpret modalities are monotone, provided the modalities are normal. As a consequence, an open formulas $F(X)$ is interpreted in a monotone function $[\![F]\!] : \mathcal{C}(\Omega) \to \mathcal{C}(\Omega)$. Since $\langle \mathcal{C}(\Omega), \subseteq \rangle$ forms a complete lattice, Knaster-Tarski Thorem (Proposition 14 in previous chapter) gives the existence of both least and greatest fixed points of $[\![F]\!]$. We use them to interpret $\mu X.F(X)$ and $\nu X.F(X)$, respectively.

Further Works

Here we list some further research directions and open problems that the author wants to investigate in future works.

Better Proof Systems. We designed several proof systems, either in the sequent calculus formalism or in the Calculus of Structures formalism. The latter are easy to use, closer to algebraic calculations than to 'standard' proof theory. However, investigating meta-theoretic properties of such calculi is in general harder than investigating the meta-theory of sequent calculi. These, unfortunately, are known to be inadequate for some families of modal logics (like $S5$, see [82, 97]). An interesting research direction is the design of proof systems for linear logics, enriched with both fixed point operators and modalities, in new formalisms suitable for meta-theoretic investigations. In particular, display calculi [20] were successfully employed to design well-behaved proof systems for several modal logics [111], and were recently extended [50, 51, 52] to the so-called *multi-type display calculi* in order to deal with dynamic logics. The strength of such calculi is a finer distinction between the object level of formulas and the meta-level of the so-called *structures*. Such distinction allows to formulate eight conditions concerning the shape of the rules that, together, guarantees a cut-elimination theorem.

As already mentioned, display calculi for full propositional linear logic already exist [21]. However, such calculi are rather *ad hoc* and do not provide a satisfactory treatment of exponential modalities. Some preliminary work has already been done to employ the multi-type formalism in order to come up with better proof systems for full linear logic.

Completeness Results. We mostly focused on categorical semantics, just sketching a possible 'non-algebraic' semantics in this last chapter. Such semantics is based on the notion of pretopology, and a soundness theorem can be proved. Working out proofs in [90], it is also possible to prove completeness theorems for several modal linear logics without fixed point operators. Proving a completeness theorem for e.g. linear logic with fixed point operators is not easy, and the author was not able to come up with such a proof so far. The main strategy adopted was to use techniques from duality theory. For example, a standard proof via the Lindenbaum algebra construction allows to prove a completeness theorem for νLL with respect to an extension of CL-algebras [40, 105] obtained as poset νSMCCs. However, moving from CL-algebras to pretopologies requires to perform a completion procedure (like e.g. Dedeking-MacNeille completion [54, 46]), which unfortunately does not preserve least and greatest fixed points (in fact, the completion, in general, creates new pre- and post-fixed points). This problem is studied in [93] and the author hopes that some of the results proved there can help in finding completeness' proofs for the 'pretopological semantics'.

More Intuitive Semantics. We sketched a semantics for several logics studied in this thesis based on the notion of pretopology, with the informal motivation of having more intuitive semantics than categorical and algebraic ones[5]. The author believes that pretopologies are, at least in

[5]Relational semantics for substructural logics exist, and were deeply investigated (see

principle, quite intuitive structures that could help to achieve an intuitive comprehension of linear logic's connectives, maintaining at the same time mathematical rigor. This is especially true for what concern the intuitionistic propositional fragment of linear logic. Intuition is partially lost when moving to classical linear logic. For this reason finding new and more intuitive semantics seems to be particularly relevant.

Conclusions

In this thesis we introduced and studied some extensions of propositional linear logic with least and greatest fixed point operators. We started by adding structure to symmetric cartesian monoidal categories. Namely, we equipped such categories with initial algebras and final coalgebras for the so-called polynomial functors, thus obtaining a new class of categories. We defined these categories equationally, following Lambek's methodology [71]. Such approach allowed to easily recognize the underlying logic of such categories, which is an extension of the $(\otimes, \&)$-fragment of (propositional) intuitionistic logic with (least and greatest) fixed point operators. This logic is powerful, and allows to recover the exponential modality ! as well as its relevant and affine versions. Looking at the exponential $!A$ as final coalgebra of the exponential functor

$$E(X) = 1 \,\&\, A \,\&\, (X \otimes X)$$

allowed to achieve a finer analysis of the proof-theoretical properties of !. In particular, we recognized such functor to be decomposable in the so-called relevant and affine functors, which are nothing but the defining functors of relevant and affine modalities. Such decomposition was made formal via a new decomposition theorem in the spirit of [67].

Studying the properties of the exponential functor, it was possible to recognize some correspondences between its structure and specific proof-theoretical properties of the exponential modality !. This analysis gives new information regarding non-canonicity of the exponential modality as well as

e.g. [86, 87] for a comprehensive introduction to the subject). These are essentially based on the notion of *information frame* [110], which provides an intuitive (informal) interpretation of several substructural distributive logics. Moving to non-distributive logics usually require to equip the information frame with some closure operators, thus loosing part of its intuitive character. For this reason, relational semantics for linear logic do not seem to add any intuition compared to pretopologies.

its relationship with specific sequent calculus rules it has to satisfy, notably the promotion rule.

These results can be extended to relevant and affine modalities, as well as !'s dual modality, ?, and its relevant and affine variants. We gave a sequent calculus for classical (propositional) linear logic enriched with fixed point operators, as well as a deep inference calculus.

Finally, some possible applications of the logics investigated were sketched, as well as a non-categorical/algebraic semantics, based on the notion of pretopology [89].

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