Logics for Compact Hausdorff Spaces via de Vries Duality

MSc Thesis *(Afstudeerscriptie)*

written by

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under the supervision of **Dr Nick Bezhanishvili** and **Prof Yde Venema**,
and submitted to the Board of Examiners in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam.*

Date of the public defense:
June 3rd, 2016

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Abstract

In this thesis, we introduce a finitary logic which is sound and complete with respect to de Vries algebras, and hence by de Vries duality, this logic can be regarded as logic of compact Hausdorff spaces. In order to achieve this, we first introduce a system $\mathcal{S}$ which is sound and complete with respect to a wider class of algebras. We will also define Π_2-rules and establish a connection between Π_2-rules and inductive classes of algebras, and we provide a criterion for establishing when a given Π_2-rule is admissible in $\mathcal{S}$. Finally, by adding two particular rules to the system $\mathcal{S}$, we obtain a logic which is sound and complete with respect to de Vries algebras. We also show that these two rules are admissible in $\mathcal{S}$, hence $\mathcal{S}$ itself can be regarded as the logic of compact Hausdorff spaces. Moreover, we define Sahlqvist formulas and rules for our language, and we give Sahlqvist correspondence results with respect to semantics in pairs (X, R) where X is a Stone space and R a closed binary relation. We will compare this work with existing literature.

Acknowledgements

Most of all, I am grateful to my first supervisor, Nick Bezhanishvili, for making the work on my thesis such a unique experience. Thanks to the advice I received in our frequent meetings, I had very few chances to get stuck or lost. During this time, he would always come up with stimulating questions, opening doors for further investigation, and filling my mind with the right approaches for solving the problems. I could rely on him at any time or day of the week, for any question or discussion, and I think the greatest amount of what I achieved is due to the time he dedicated to me. Moreover, I thank him for taking me to conferences and inviting me to present my results in one of them, and finally I should thank him also for reasons which go beyond the scope of this thesis.

My gratitude goes also to my second supervisor, Yde Venema, for showing as much interest in my thesis as to join its supervision at a later stage. His careful reading of my drafts, followed by detailed comments, suggestions and questions, have substantially improved the quality of my work. It was a pleasure to work with Yde and to hear his interesting ideas about various aspects of logic.

I would also like to thank the members of my Master's thesis defense committee, Alexandru Baltag, Benno van den Berg and Sebastian Enqvist, for having enthusiastically read my thesis, and for asking interesting questions.

I want to thank Guram Bezhanishvili for providing crucial inputs, which allowed me to pursue the main goal of my thesis. Also, I want to thank Petr Cintula, George Metcalfe and Frederik Möllerström Lauridsen for being interested in my thesis' topic during the time I was working on it, and for engaging in interesting and useful discussions about it.

I am grateful to all staff members of the ILLC, for providing such a peaceful and stimulating environment. In particular, I want to thank all lecturers who taught the courses which I have taken, and Tanja Kassenaar for her speed in handling all bureaucratic issues, and for always being available for a chat.

I also want to thank Alessandro Berarducci who, during my undergraduate studies in Pisa, supervised my Bachelor's thesis and taught excellent courses which introduced me to the field of mathematical logic.

I would like to thank all my friends who shared nice time with me during these years, and in particular all those who I have been working with: Almudena, Dan, Evan, Francesco, Frank, Lorenzo and Michele. With them, it has been very hard to distinguish work from leisure.

Finally, I want to thank my family, who always provides me with plenty of love and support in all of my choices.

Contents

Chapter 1

Introduction

The work of this thesis belongs to the research area devoted to the study of the relations between logic and topology. The key tools of this study are the algebraization of logic and dualities between algebras and topological spaces.

The algebraization of logic has its roots in the nineteenth century in the work of Boole, followed by that of de Morgan, Peirce, Schröder and others. This field has been taken up in the twentieth century, in particular in the work of Birkhoff, Tarski, etc., who established a correspondence between equational axiomatizations of classes of algebras and deductive systems for propositional calculi. This correspondence is based on the construction of the Lindenbaum-Tarski algebra, which is a quotient algebra obtained from the algebra of all formulas. This, in particular, gives algebraic completeness of non-classical propositional logics, leading to the area of algebraic logic, which is nowadays a very active field of research. See e.g. [1, 12, 63].

The study of dualities between algebras and topological spaces has started with the work of Stone [51], who proved that Boolean algebras can be dually represented via compact Hausdorff zero-dimensional topological spaces. These spaces are nowadays called Stone spaces. This result allows to translate a problem about Boolean algebras into a problem about Stone spaces, and vice versa. Subsequently, other interesting classes of algebras have been connected via dualities to classes of topological spaces. Among the most famous examples are Priestley duality for distributive lattices, Esakia duality for Heyting algebras, and Jónsson-Tarski duality for modal algebras.

As well as a representation theorem for Boolean algebras, Stone's theorem can be regarded as a representation theorem for Stone spaces. This observation led to the development of Stone-like dualities, connecting interesting classes of topological spaces to appropriate classes of algebras. An example of this kind is de Vries duality [20], which is the one which this thesis is based on. De Vries duality connects the class of compact Hausdorff spaces to the class of de Vries algebras, which are particular Boolean algebras with a binary relation satisfying certain conditions. The main goal of this thesis consists in providing a finitary propositional deductive system which is sound and complete with respect to de Vries algebras. We show, via de Vries duality, that this system is sound and complete with respect to compact Hausdorff spaces.

De Vries' work [20] can also be seen as part of the research area of region-based theories of space. In this theory, introduced by de Laguna [18] and Whitehead [65], one replaces the primitive notion of point with that of a region. Many authors have been working on showing the equivalence of this approach to point-based theories of space. This is done via representation theorems for (pre)contact algebras of regions and adjacency spaces, see e.g. Dimov and Vakarelov [22, 23, 25], Vakarelov et al. [56, 55], Düntsch and Winter [27], Düntsch and Vakarelov [26], Roeper [47], Pratt and Schoop [44], Mormann [42], etc.

Contact algebras play a central role in this thesis. Before obtaining our completeness result for compact Hausdorff spaces, we prove a series of completeness results. First we introduce a system $\mathcal{S}$, and we show that it is sound and complete with respect to the class of contact algebras. Then we define Π_2-rules, and we prove that systems extending $\mathcal{S}$ with such rules are complete with respect to inductive classes of contact algebras.

In light of this completeness result, we develop the theory of what we call Π_2-rules. We show that for any inductive class K of contact algebras there exists a system extending $\mathcal{S}$ with Π_2-rules which is sound and complete with respect to K. Moreover, we prove a model-theoretic criterion for admissibility of Π_2-rules in $\mathcal{S}$.

Π_2-rules are a particular kind of non-standard rules, whose role is to mimic quantifiers in propositional logics. The most famous example of such rules is Gabbay's irreflexivity rule [29], see also Burgess [13] for earlier examples of such rules. Several other authors investigated rules of this sort, see e.g. Gabbay and Hodkinson [30], Kuhn [38], Venema [59, 58, 61, 60, 62], de Rijke [19], Roorda [48], Zanardo [66], Passy and Tinchev [43], Gargov and Goranko [31], Goranko [35], Balbiani et al. [3].

In [3], Balbiani et al. consider the system RCC (Region Connection Calculus)[1] introduced by Randel et al. [45]. They define propositional logics related to RCC, and they show completeness of these logics with respect to both relational and topological semantics, which are based on adjacency spaces and regular closed regions of topological spaces, respectively. One of the proofs of completeness concerns a propositional logic, which involves a rule similar to our Π_2-rules. The proof of completeness of this logic with respect to the relational semantics inspired our more general completeness result for Π_2-rules with respect to inductive classes of contact algebras.

The non-standard rules presented in [3] are two, namely (NOR) and (EXT), and these correspond to the Π_2-rules (ρ7) and (ρ8) which we define in this thesis. These rules correspond to $\forall\exists$-statements which are satisfied by contact algebras called compingent algebras. Thus, by the aforementioned completeness theorem, we derive that the system $\mathcal{S}+(\rho 7)+(\rho 8)$ is complete with respect to compingent algebras. Finally, using MacNeille completions of compingent algebras, we obtain completeness of this system with respect to de Vries algebras.

[1] RCC is one of the systems of region-based theory of space. It plays a central role in Qualitative Spacial Reasoning, see, e.g., [46].

The construction of the MacNeille completion of a poset, for embedding it into a complete lattice, is a generalization of Dedekind's extension of the rationals to the reals. As the latter construction involves also extending the operations to the reals, MacNeille completions of ordered algebras are generalized by also extending the operations. This can be done in two ways, via the so-called lower MacNeille completions and the upper MacNeille completions. Lower MacNeille completions have been introduced by Monk [41]. An investigation of the properties preserved by both the upper and lower constructions is given in Givant and Venema [32] and Harding and Bezhanishvili [36]. In [53], Theunissen and Venema discuss lower and upper MacNeille completions of lattices with additional operations. We define the MacNeille completion of a compingent algebra and show that it coincides with the lower MacNeille completions of those algebras. The fact that the class of compingent algebras is closed under this construction, is a key aspect which allows us to use MacNeille completions for obtaining completeness of $\mathcal{S} + (\rho 7) + (\rho 8)$ with respect to de Vries algebras, and hence, via de Vries duality, with respect to compact Hausdorff spaces. We notice that calculi whose algebraic models are closed under MacNeille completions are also those complete for classes of compact Hausdorff spaces. As a corollary, we obtain a calculus complete with respect to zero-dimensional compact Hausdorff spaces (equivalently, Stone spaces) and also a calculus complete with respect to connected compact Hausdorff spaces.

Finally, we investigate the expressiveness of our language in subordination spaces. Those spaces are particular topological spaces with a binary relation. They are obtained from Boolean algebras with subordinations via a duality which can be regarded, at least on objects, as a special case of the generalised Jónsson-Tarski duality (see e.g. [34])[2]. Following the work of Balbiani and Kikot [2], we define Sahlqvist formulas for our language, and we prove a Sahlqvist correspondence theorem with respect to semantics in subordination spaces. Moreover, we also define a new class of Sahlqvist Π_2-rules, and we give a correspondence theorem for them.

1.1 Outline of the thesis

In Chapter 2 we define all the structures involved in this thesis, such as contact algebras, compingent algebras, de Vries algebras and subordination spaces, which we use as semantics for our language in the following chapters. We also present dualities between classes of these structures. Based on these dualities we can regard our different semantics as equivalent.

In Chapter 3 we introduce the syntax of our language, and semantics with respect to Boolean algebras with a binary relation. Then we define the system $\mathcal{S}$, and we show a proof of strong completeness with respect to the class of contact algebras. Finally, we show that $\mathcal{S}$ has the finite model property, and

[2] Jónsson-Tarski duality is an extension of Stone duality, from Boolean algebras and Stone spaces to modal algebas and descriptive frames. This duality plays a central role in modal logic.

is decidable.

In Chapter 4 we define Π_2-rules, and we explain how to add them to the system $\mathcal{S}$, and we show that when added to $\mathcal{S}$ they form a sound and complete system with respect to the class defined by their associated $\forall\exists$-statements. We prove also that all $\forall\exists$-statements are equivalent to some $\forall\exists$-statement which is associated to some Π_2-rule, thus establishing a correspondence between sets of rules and inductive classes of contact algebras. Moreover, we give a semantic criterion for admissibility of these Π_2-rules in $\mathcal{S}$.

In Chapter 5 we define the Π_2-rules ($\rho 7$) and ($\rho 8$), which by the results of Chapter 4 make $\mathcal{S} + (\rho 7) + (\rho 8)$ sound and complete with respect to compingent algebras. Using MacNeille completions of a compingent algebras, we show that this system is also complete with respect to de Vries algebras. We define topological semantics for our language, and by de Vries duality we derive completeness of $\mathcal{S} + (\rho 7) + (\rho 8)$ with respect to compact Hausdorff spaces. We also prove that the rules ($\rho 7$) and ($\rho 8$) are admissible in $\mathcal{S}$. Finally, we define MacNeille canonical axioms and rules, which are those that express topological properties when added to $\mathcal{S} + (\rho 7) + (\rho 8)$, and we give two examples. In the last section, we compare our approach in Chapters 3, 4 and 5 with that of Balbiani, Tinchev and Vakarelov [3].

In Chapter 6 we consider interpretation of our formulas in subordination spaces. We define Sahlqvist formulas for our language, and we prove that a Sahlqvist formula φ is valid on a subordination space if and only if the latter satisfies a first-order formula which is effectively computable from φ. Moreover, we define Sahlqvist $\forall\exists$-statements, and we show a similar correspondence for such statements, and we observe that by the results of Chapter 4 this can be regarded as a Sahlqvist correspondence for Π_2-rules. Throughout the chapter, we compare our work with that of Balbiani and Kikot [2].

In Chapter 7 we summarize the content of this thesis. We also give ideas for future work, discussing some of them in detail.

1.2 Main results

- We prove a series of completeness results.

  In the first one (Theorem 3.2.9) we show completeness of our system $\mathcal{S}$ with respect to contact algebras, using standard techniques from algebraic logic. Then we prove that extensions of $\mathcal{S}$ with Π_2-rules are complete with respect to inductive classes of contact algebras (Theorem 4.1.5). The proof of the latter result has been obtained by adapting and generalising the results in [3, Section 7]. There, the authors present a specific rule of the kind of our Π_2-rules and show how to work with them in the setting of relational semantics. Instead, we give a more general completeness result for all Π_2-rules in an algebraic setting. We use a special case of Theorem 4.1.5 and MacNeille completions to obtain completeness of $\mathcal{S} + (\rho 7) + (\rho 8)$ with respect to de Vries algebras (Theorem 5.1.5), and finally via de Vries duality we derive completeness with respect to compact Hausdorff spaces (Corollary 5.2.2).

- We establish a correspondence between logics extending $\mathcal{S}$ with Π_2-rules and inductive classes of contact algebras.

  This correspondence is the result of Theorem 4.1.5 and Corollary 4.2.5, where the latter follows by Proposition 4.2.4.

- We give a criterion for establishing admissibility of Π_2-rules in the system $\mathcal{S}$ (Theorem 4.3.5).

  Moreover, in Propositions 5.1.8 and 5.1.11, we show that this criterion can be applied for showing admissibility of rules ($\rho 7$) and ($\rho 8$), respectively (Corollaries 5.1.9 and 5.1.12).

- We prove a Sahlqvist correspondence theorem for our Sahlqvist formulas (Theorem 6.1.15), which can be considered a variation of [2, Theorem 5.1]. We prove also a Sahlqvist correspondence theorem for our Sahlqvist statements (Theorem 6.2.5), and this can be regarded as a Sahlqvist correspondence for our Π_2-rules.

Chapter 2

Preliminaries

In this chapter, we introduce all the structures which we will use in this thesis. We also describe dualities connecting categories of these structures.

The following are the parts of this chapter which are required for understanding the rest of the thesis:

- All the content of Section 2.1 preceeding subsection 2.1.1.
  This is the most essential part of the preliminaries, as we repeatedly refer to it in all chapters. At the end of this part, we have put a table containing the definitions which we often refer to in this thesis.

- De Vries duality.
  We refer to this in Chapter 5. In order to understand this duality, one only needs to familiarize themselves with Definitions 2.2.1 and 2.2.3, the functor defined at the end of Section 2.2.1, and the contents of Section 2.2.2.

- Sections 2.1.1 and 2.1.2 and Lemma 2.1.12.
  These are required for understanding Chapter 6.

In order to make this chapter self-contained, we provide most of the proofs, and we point to specific references for those which are missing.

2.1 Subordinations and closed relations

Throughout this thesis, we will consider Boolean algebras enriched with a binary relation $\prec$, and we will require $\prec$ to satisfy certain properties. The simplest $\prec$ which we will study is called *subordination*.

Definition 2.1.1 (Subordination)**.** *A binary relation $\prec$ on a Boolean algebra B is called a* subordination *if it satisfies the following properties:*

(Q1) $0 \prec 0$ *and* $1 \prec 1$*;*

(Q2) $a \prec b, c$ *implies* $a \prec b \wedge c$;

(Q3) $a, b \prec c$ *implies* $a \vee b \prec c$;

(Q4) $a \leq b \prec c \leq d$ *implies* $a \prec d$.

A subordination $\prec$ on a Boolean algebra B could be equivalently described by an operation $\rightsquigarrow: B \times B \to \{0,1\} \subseteq B$ satisfying the following properties:

(Q0′) $a \rightsquigarrow b \in \{0,1\}$

(Q1′) $0 \rightsquigarrow 0 = 1 \rightsquigarrow 1 = 1$;

(Q2′) $a \rightsquigarrow b = a \rightsquigarrow c = 1$ implies $a \rightsquigarrow b \wedge c = 1$;

(Q3′) $a \rightsquigarrow c = b \rightsquigarrow c = 1$ implies $a \vee b \rightsquigarrow c = 1$;

(Q4′) $b \rightsquigarrow c = 1, a \leq b$ and $c \leq d$ implies $a \rightsquigarrow d = 1$.

Indeed, given a subordination $\prec$, we obtain an operation $\rightsquigarrow$ satisfying properties (Q0′)-(Q4′) by defining

$$a \rightsquigarrow b = \begin{cases} 1 & \text{if } a \prec b \\ \\ 0 & \text{otherwise} \end{cases}$$

and vice versa, given an operation $\rightsquigarrow$ satisfying properties (Q0′)-(Q4′), we obtain a subordination $\prec$ by defining $\prec := \{(a,b) \in B \times B \mid a \rightsquigarrow b = 1\}$.

Hence, we have a 1-1 correspondence between pairs $(B, \prec)$ satisfying (Q1)-(Q4) and algebras $(B, \wedge, \neg, 1, \rightsquigarrow)$ satisfying properties (Q0′)-(Q4′).

In Chapter 3, where we introduce logics and use algebras with subordinations as semantics, we will use the operation $\rightsquigarrow$ rather than the relation $\prec$.

Subordinations $\prec$ on a Boolean algebras are also in 1-1 correspondence with *proximities*, which have been introduced by Düntsch and Vakarelov [26]:

Definition 2.1.2 (Proximity)**.** *A binary relation δ on a Boolean algebra B is a* precontact relation*, or* proximity*, if it satisfies the following properties:*

(P1) $a\delta b$ *implies* $a, b \neq 0$;

(P2) $a\delta(b \vee c)$ *if and only if* $a\delta b$ *or* $a\delta c$;

(P3) $(a \vee b)\delta c$ *if and only if* $a\delta c$ *or* $b\delta c$.

Given a subordination $\prec$, the relation $a\delta_\prec b := a \not\prec \neg b$ is a proximity. Vice versa, given a proximity δ, the relation $a \prec_\delta b := a \not\delta \neg b$ is a subordination (see [8]).

Moreover, we have $\delta_{\prec_\delta} = \delta$ and $\prec_{\delta_\prec} = \prec$, so the map $\delta_{(-)}$ is a bijection from the set of subordinations to the set of proximities, and $\prec_{(-)}$ is its inverse.

As for subordinations, also proximities can be replaced by a binary operation $\diamond : B \times B \to \{0,1\}$ defined as:

$$a \diamond b = \begin{cases} 1 \text{ if } a\delta b \\ \\ 0 \text{ otherwise.} \end{cases}$$

In Chapter 6, it will be convenient for us to consider algebras with subordinations as algebras with the operation $\diamond$. Accordingly, we will also change the language of our logic, replacing the connective $\rightsquigarrow$ with the connective $\diamond$. This will give our Sahlqvist formulas a better shape.

Also, notice that $\diamond$ is monotone in both arguments, that is $a \leq a', b \leq b' \;\Rightarrow\; a \diamond b \leq a' \diamond b'$.

We are interested in subordinations $\prec$ satisfying more properties than those given in Definition 2.1.1.

Definition 2.1.3 (Contact algebra)**.** *Given a pair $(B, \prec)$, consisting of a Boolean algebra with a subordination, we call it a* contact algebra *if in addition it satisfies the following properties:*

(Q5) *$a \prec b$ implies $a \leq b$;*

(Q6) *$a \prec b$ implies $\neg b \prec \neg a$.*

The reason why we chose this name is the following. In the literature, a *contact relation* on a Boolean algebra is a precontact relation δ which in addition satisfies the following properties:

(P4) $a \neq 0$ implies $a\delta a$;

(P5) $a\delta b$ implies $b\delta a$.

It is easy to show that, given a subordination $\prec$, its corresponding precontact relation $\delta_\prec$ is a contact relation if and only if $\prec$ satisfies (Q5) and (Q6), and vice versa a precontact relation δ is a contact relation if and only if $\prec_\delta$ satisfies (Q5) and (Q6), see e.g., [9, 8]

This justifies that we call *contact algebras* pairs $(B, \prec)$ where B is a Boolean algebra and $\prec$ is a subordination satisfying (Q5) and (Q6).

Definition 2.1.4 (Compingent algebra)**.** *A contact algebra $(B, \prec)$ is a* compingent algebra *if in addition it satisfies the following properties:*

(Q7) *$a \prec b$ implies $\exists c : \; a \prec c \prec b$;*

(Q8) *$a \neq 0$ implies $\exists b \neq 0 : \; b \prec a$. Alternatively, we may say that $\prec$ is a* compingent relation *on B.*

Compingent relations on Boolean algebras were defined by de Vries [20], and the notion of subordination has been introduced in [9, 8] in order to generalise that of compingent relation.

As we will see in Section 2.2, the class of complete compingent algebras consists of objects of a category which is dual to **KHaus**, the category of compact Hausdorff spaces and continuous maps. This duality result is shown in [20], and for this reason complete compingent algebras are usually called *de Vries algebras*.

In this section, in Definitions 2.1.1, 2.1.3 and 2.1.4, we have defined conditions (Q1)-(Q8) of algebras $(B, \prec)$. Throughout this thesis, we will very frequently refer to those conditions. Thus, for convenience of the reader, we have collected them in the following table:

(Q1)	$0 \prec 0$ and $1 \prec 1$;
(Q2)	$a \prec b, c$ implies $a \prec b \wedge c$;
(Q3)	$a, b \prec c$ implies $a \vee b \prec c$;
(Q4)	$a \leq b \prec c \leq d$ implies $a \prec d$;
(Q5)	$a \prec b$ implies $a \leq b$;
(Q6)	$a \prec b$ implies $\neg b \prec \neg a$;
(Q7)	$a \prec b$ implies $\exists c : \; a \prec c \prec b$;
(Q8)	$a \neq 0$ implies $\exists b \neq 0 : \; b \prec a$.

2.1.1 Categories of Boolean algebras with subordinations, and subordination spaces

We denote by **Sub** the category whose objects are pairs $(B, \prec)$ where B is a Boolean algebra and $\prec$ as subordination on B, and whose arrows are Boolean homomorphisms $h : A \to B$ such that for all $a, b \in A$, if $a \prec b$ then $h(a) \prec h(b)$.

In the rest of this section, we will establish a dual equivalence between **Sub** and the category consisting of pairs (X, R) where X is a Stone space and R

is a binary relation which is closed as a subset $R \subseteq X \times X$. We will call such pairs *subordination spaces.*

First, we introduce closed relations on a topological space:

Definition 2.1.5 (Closed relation and subordination spaces)**.** *Let X be a topological space, and let R be a binary relation on X. We say that R is* closed *if $R \subseteq X \times X$ is a closed subset in the product topology.*

If X is a Stone space, and R is a closed relation on X, we call (X, R) a subordination space*.*

Notation 2.1.6. Given a set X, a binary relation $R \subseteq X \times X$, and a subset $A \subseteq X$, we denote by $R[A]$ and $R^{-1}[A]$ the following sets:

$$\begin{aligned} R[A] &:= \{y \in X \mid \exists x \in A : xRy\} \\ R^{-1}[A] &:= \{x \in X \mid \exists y \in A : xRy\}. \end{aligned}$$

The following lemma will be used throughout this thesis. Its proof can be found in [8].

Lemma 2.1.7. *Let X be a compact Hausdorff space, and R a binary relation on X. The following are equivalent:*

1. *R is a closed relation;*
2. *For each closed subset F of X, both $R[F]$ and $R^{-1}[F]$ are closed.*

Definition 2.1.8 (Stable map)**.** *Let X_1, X_2 be sets, and let R_1, R_2 be binary relations respectively on X_1 and X_2. A map $f : X_1 \to X_2$ is called* stable *if for all $x, y \in X_1$, if xR_1y then $f(x)R_2f(y)$.*

Let **StR** be the category whose objects are subordination spaces (X, R), and whose morphisms are continuous stable maps.

2.1.2 Duality of Sub and StR

In [25], Dimov and Vakarelov present a duality between the category of Boolean algebras with a proximity relation and the category of subordination spaces. As we mentioned above, proximities on Boolean algebras are equivalently described by subordinations[1], thus this leads to a duality between **Sub** and **StR**.

This duality is an extension of that of Celani [15], and it is a generalization of Stone duality. In fact, we use Stone duality to obtain a space X from a Boolean algebra B, and vice versa. Separately, we give a dual closed relation

[1] A direct proof of it can be found in [8].

R of a subordination $\prec$, and vice versa. So we split a pair $(B, \prec)$ in two parts, which will consist of the Stone dual X of B and the dual R of $\prec$.

Below, we will define contravariant functors $(-)_+ : \mathbf{Sub} \to \mathbf{StR}$ and $(-)^+ : \mathbf{StR} \to \mathbf{Sub}$ which will establish a dual equivalence.

The functor $(-)_+ : \mathbf{Sub} \to \mathbf{StR}$

Definition 2.1.9. *Given $(B, \prec)$ and a subset $S \subseteq B$, we define $\mathop{\uparrow}\limits^{\prec} S$ to be the upset of S with respect to the relation $\prec$, that is:*

$$\mathop{\uparrow}\limits^{\prec} S \ := \ \{b \in B \mid \exists s \in S : \ s \prec b\}.$$

Similarly, we define $\mathop{\downarrow}\limits^{\prec} S$ to be the downset of S with respect to $\prec$.

Given a pair $(B, \prec)$ consisting of a Boolean algebra with a subordination, we define $(B, \prec)_+ := (X, R)$ as follows:

$$\begin{aligned} X \ &:= \ \text{Stone dual of } B \ = \{ \text{ ultrafilters of } B\} \\ xRy \ &\Leftrightarrow \ \mathop{\uparrow}\limits^{\prec} x \subseteq y. \end{aligned}$$

Then R is a closed relation on X.

Given $(A, \prec), (B, \prec) \in \mathbf{Sub}$, and $h : A \to B$ a Boolean homomorphism satisfying $a \prec b$ implies $h(a) \prec h(b)$ for all $a, b \in A$, if $(A, \prec)_+ = (Y, S)$ and $(B, \prec)_+ = (X, R)$, we define $h_+ : X \to Y$ by $x \mapsto h^{-1}(x)$ as in Stone duality.

The functor $(-)^+ : \mathbf{StR} \to \mathbf{Sub}$

Given a subordination space (X, R), we define $(X, R)^+ := (\mathbf{Clop}(X), \prec)$, where for all $U, V \in \mathbf{Clop}(X)$ we let $U \prec V$ if and only if $R[U] \subseteq V$. Then we have that $\prec$ defined in this way is a subordination on the Boolean algebra $\mathbf{Clop}(X)$.

Given a continuous stable function $f : X \to Y$ between obejcts (X, R) and (Y, R) in $\mathbf{StR}$, we define $f^+ : \mathbf{Clop}(Y) \to \mathbf{Clop}(X)$ as $U \mapsto f^{-1}(U)$ as in Stone duality.

We now have two well-defined contravariant functors $(-)_+ : \mathbf{Sub} \to \mathbf{StR}$ and $(-)^+ : \mathbf{StR} \to \mathbf{Sub}$. As shown in [8], we have natural isomorphisms between $(B, \prec)$ and $((B, \prec)_+)^+$ in $\mathbf{Sub}$ and between (X, R) and $((X, R)^+)_+$ in $\mathbf{StR}$. Hence, we obtain the following result:

Theorem 2.1.10. *The categories* $\mathbf{Sub}$ *and* $\mathbf{StR}$ *are dually equivalent.*

2.1.3 Restriction of the duality

Now we are interested in restricting this duality to some full subcategories of **Sub** and **StR**. The next lemma shows that each of the conditions (Q5),(Q6) and (Q7), given in Definitions 2.1.3 and 2.1.4, correspond to elementary conditions on the dual subordination spaces. A proof can be found in [8] (cf. [15, 26]). For making the thesis self-contained, here we give an alternative proof:

Remark 2.1.11. Subordination spaces (X, R) satisfy the following properties, which we will often use without mentioning them explicitly:

- if $F, G \subseteq X$ are disjoint closed subsets, then there exists a clopen subset $U \subseteq X$ such that $F \subseteq U$ and $G \cap U = \emptyset$; [2]
- if $F \subseteq X$ is a closed subset, then $R[F]$ and $R^{-1}[F]$ are closed subsets of X (by Lemma 2.1.7, this is equivalent to R being closed).

Lemma 2.1.12. *Let (X, R) be a subordination space.*

1. *R is reflexive $\Leftrightarrow$ for all $U, V \in \mathbf{Clop}(X)$ we have $R[U] \subseteq V$ implies $U \subseteq V$. Hence, R is reflexive if and only if its dual algebra $(\mathbf{Clop}(X), \prec)$ satisfies* (Q5)*.*
2. *R is symmetric $\Leftrightarrow$ for all $U, V \in \mathbf{Clop}(X)$ we have $R[U] \subseteq V$ implies $R[X \setminus V] \subseteq X \setminus U$. Hence, R is symmetric if and only if its dual algebra $(\mathbf{Clop}(X), \prec)$ satisfies* (Q6)*.*
3. *R is transitive $\Leftrightarrow$ for all $U, V \in \mathbf{Clop}(X)$ we have $R[U] \subseteq V$ implies there exists $Z \in \mathbf{Clop}(X)$ such that $R[U] \subseteq Z$ and $R[Z] \subseteq V$. Hence, R is transitive if and only if its dual algebra $(\mathbf{Clop}(X), \prec)$ satisfies* (Q7)*.*

Proof. 1. ($\Rightarrow$) If R is reflexive, for all U we have $U \subseteq R[U]$, hence $R[U] \subseteq V$ implies $U \subseteq V$.

($\Leftarrow$) Suppose R is not reflexive, so there exists x such that $x \not R x$. This means in particular that $x \notin R^{-1}[x]$. So there is a clopen U such that $x \in U$ and $U \cap R^{-1}[x] = \emptyset$. The latter implies $x \notin R[U]$, so we can find a clopen V such that $x \notin V$ and $R[U] \subseteq V$. Since $x \in U$ and $x \notin V$, we have $U \nsubseteq V$. Hence $R[U] \subseteq V$ does not imply $U \subseteq V$.

2. ($\Rightarrow$) Suppose R is symmetric, and let U, V be such that $R[U] \subseteq V$. Assume we have $x \in R[X \setminus V]$, so there is $y \in X \setminus V$ such that yRx. Since R is symmetric, we have xRy. If $x \notin X \setminus U$, that is if $x \in U$, then since $R[U] \subseteq V$ and $y \in R[U]$ we have $y \in V$, which is a contradiction. Hence $x \in X \setminus U$. This shows $R[X \setminus V] \subseteq X \setminus U$.

[2] This is provable via standard arguments by total disconnectedness and compactness.

($\Leftarrow$) Suppose R is not symmetric, hence there exist $x, y \in X$ such that xRy and $y \not R x$, which means that $y \notin R^{-1}[x]$. Since $R^{-1}[x]$ is a closed set, there exists a clopen U such that $y \in U$ and $U \cap R^{-1}[x] = \emptyset$. By the latter condition, we have that the closed set $R[U]$ does not contain x. Hence, there is a clopen V such that $R[U] \subseteq V$ and $x \notin V$.

Since xRy and $x \in X \setminus V$, we have $y \in R[X \setminus V]$. But $y \in U$, that is $y \notin X \setminus U$. So we have found clopens U, V such that $R[U] \subseteq V$ but $R[X \setminus V] \nsubseteq X \setminus U$.

3. ($\Rightarrow$) Suppose R is transitive, and let U, V be clopens such that $R[U] \subseteq V$. Since $R[R[U]] \subseteq R[U] \subseteq V$ by transitivity, we have that $R[U]$ and $R^{-1}[X \setminus V]$ are disjoint. Hence, there is a clopen Z such that $R[U] \subseteq Z$ and $Z \cap R^{-1}[X \setminus V] = \emptyset$. The latter implies $R[Z] \subseteq V$. So, given U, V s.t. $R[U] \subseteq V$, there is always a clopen Z such that $R[U] \subseteq Z$ and $R[Z] \subseteq V$.

($\Leftarrow$) Suppose R is not transitive, so there exist $x, y, z \in X$ such that xRy, yRz and $x \not R z$. The latter means $x \notin R^{-1}[z]$, hence there is a clopen U such that $x \in U$ and $U \cap R^{1}[z] = \emptyset$. So we have $z \notin R[U]$, hence we can find V such that $R[U] \subseteq V$ and $z \notin V$. This means that, if Z is such that $R[U] \subseteq Z$, then by xRy we have $y \in Z$. So, by yRz, we have $z \in R[Z]$, which implies $R[Z] \nsubseteq V$ because by construction $z \notin V$.

So we have found U, V such that $R[U] \subseteq V$, but for all Z if $R[U] \subseteq Z$ then $R[Z] \nsubseteq V$.

$\square$

Lemma 2.1.12 states that properties (Q5),(Q6) and (Q7) of algebras $(B, \prec)$ correspond to elementary conditions on the dual subordination spaces (X, R), namely reflexivity, symmetry and transitivity. This lemma has been a starting point for our work in Chapter 6, where we will see more on Sahlqvist correspondence.

Definition 2.1.13. *Here we introduce some subcategories of* **Sub** *and* **StR***:*

- *Let* **SubK4** *be the full subcategory of* **Sub** *consisting of algebras* $(B, \prec)$ *satisfying* (Q7)*;*
- *Let* **SubS4** *be the full subcategory of* **Sub** *consisting of algebras* $(B, \prec)$ *satisfying* (Q5) *and* (Q7)*;*
- *Let* **SubS5** *be the full subcategory of* **Sub** *consisting of algebras* $(B, \prec)$ *satisfying* (Q5),(Q6) *and* (Q7)*;*
- *Let* **Com** *be the full subcategory of* **Sub** *consisting of compingent algebras* $(B, \prec)$*;*

- *Let* $\mathbf{StR}^{tr}$ *be the full subcategory of* $\mathbf{StR}$ *consisting of subordination spaces* (X, R) *such that* R *is transitive;*
- *Let* $\mathbf{StR}^{qo}$ *be the full subcategory of* $\mathbf{StR}$ *consisting of subordination spaces* (X, R) *such that* R *is a quasi-order (reflexive and transitive);*
- *Let* $\mathbf{StR}^{eq}$ *be the full subcategory of* $\mathbf{StR}$ *consisting of subordination spaces* (X, R) *such that* R *is an equivalence relation (reflexive,symmetric and transitive).*

By Lemma 2.1.12, we have the following result:

Theorem 2.1.14. *Restricting the duality presented in this section, we obtain the following:*

- *The categories* **SubK4** *and* $\mathbf{StR}^{tr}$ *are dually equivalent.*
- *The categories* **SubS4** *and* $\mathbf{StR}^{qo}$ *are dually equivalent.*
- *The categories* **SubS5** *and* $\mathbf{StR}^{eq}$ *are dually equivalent.*

Now, we restrict the duality to **Com**. In order to describe its dual full subcategory of **StR**, we need to define the notion of *irreducible* equivalence relation:

Definition 2.1.15 (Irreducible maps and irreducible equivalence relations)**.** *A surjective continuous map* $f : X \to Y$ *between compact Hausdorff spaces is called* irreducible *if for every proper closed subset* $F \subseteq X$*, we have that* $f[F] \subseteq Y$ *is a proper subset.*

A closed equivalence relation R *on a compact Hausdorff space* X *is said to be* irreducible *if the factor-map* $\pi : X \to X/R$ *is an irreducible map.*

In [9] it is shown that, for every $(B, \prec) \in \mathbf{SubS5}$, the equivalence relation R of its dual $(X, R) := (B, \prec)_+$ is irreducible if and only if $(B, \prec)$ satisfies (Q8). Hence, if we denote by $\mathbf{StR}^{\mathrm{ieq}}$ the category of subordination spaces (X, R) where R is an irreducible equivalence relations and continuous stable functions, we obtain the following:

Theorem 2.1.16. *The categories* **Com** *and* $\mathbf{StR}^{ieq}$ *are dually equivalent.*

2.2 De Vries algebras, compact Hausdorff spaces, and de Vries duality

In this section we describe de Vries duality [20], which is one of the key ingredients of the main completeness result of this thesis (Corollary 5.2.2).

Let **KHaus** be the category of compact Hausdorff spaces and continuous functions. We will consider the category **deV** of de Vries algebras and de Vries morphisms, which we define below, and then we will define contravariant functors $(-)_* : \mathbf{deV} \to \mathbf{KHaus}$ and $(-)^* : \mathbf{KHaus} \to \mathbf{deV}$ which will establish a dual equivalence between these two categories.

Definition 2.2.1 (De Vries algebra)**.** *If $(B, \prec)$ is a compingent algebra and B is a complete Boolean algebra, we say that $(B, \prec)$ is a* de Vries *algebra.*

Definition 2.2.2 (De Vries morphism)**.** *Let $(A, \prec)$ and $(B, \prec)$ be de Vries algebras. A map $h : A \to B$ is called a* de Vries morphism *if it satisfies the following properties:*

(V1) $h(0) = 0$;

(V2) $h(a \wedge b) = h(a) \wedge h(b)$;

(V3) $a \prec b$ *implies* $\neg h(\neg a) \prec h(b)$;

(V4) $h(a) = \bigvee\{h(b) \mid b \prec a\}$.

Given $(A, \prec), (B, \prec)$ and $(C, \prec)$ de Vries algebras and $h : A \to B$ and $k : B \to C$ de Vries morphisms, the composition $k * h : A \to C$ is defined as

$$k * h : a \mapsto \bigvee\{kh(b) \mid b \prec a\} .$$

In Sections 2.2.1 and 2.2.2, we present the duality given in [20]. In this duality, we do not split algebras $(B, \prec)$ in two parts B and $\prec$ as we did in the duality of the previous section, but we use B and $\prec$ together to build a compact Hausdorff space. Vice versa, a compact Hausdorff space X will give us a complete algebra B together with a binary relation $\prec$ which makes it a compingent algebra, and hence a de Vries algebra.

2.2.1 The functor $(-)_* : \mathbf{deV} \to \mathbf{KHaus}$

To define $(-)_* : \mathbf{deV} \to \mathbf{KHaus}$, we will need the notion of *maximal round filter* of an algebra $(B, \prec)$:

Definition 2.2.3 (Round filters and ends)**.** *Given an algebra $(B, \prec)$, and a subset $S \subseteq B$, let ${\uparrow\!\!\!\!\uparrow}S := \{b \in B \mid \exists a \in S : \ a \prec b\}$.*

A filter $F \subseteq B$ is a round filter *if $F = {\uparrow\!\!\!\!\uparrow}F$. If F is a proper round filter and it is not properly contained in any other proper round filter, we say that it is a* maximal round filter.

Remark 2.2.4. For all filters F, by (Q5) we have $\twoheaduparrow F \subseteq \uparrow F = F$. So a filter is round iff $F \subseteq \twoheaduparrow F$.

Maximal round filters can be defined in an alternative way:

Definition 2.2.5 (Ends)**.** *Given an algebra $(B, \prec)$, and a subset $F \subseteq B$, we call F an* end *if it safisfies the following properties:*

(E1) $a, b \in F \;\Rightarrow\; \exists c \in F \setminus \{0\} : \; c \prec a$ *and* $c \prec b$*;*

(E2) $a \prec b \;\Rightarrow\; \neg a \in F$ *or* $b \in F$*.*

In the literature, the name *end* is used as an alternative to *maximal round filter*, and Definition 2.2.5 is usually shown to be a characterization of maximal round filters. We decided to call *ends* those sets which satisfy Definition 2.2.5, and in the following lemma we show that this notion is equivalent to that of maximal round filter as defined in Definition 2.2.3.

Lemma 2.2.6. *Let $(B, \prec)$ be such that $\prec$ satisfies* (Q1)-(Q7)*, and let $F \subseteq B$. The following are equivalent:*

1. *F is a maximal round filter;*
2. *there exists an ultrafilter $\mathcal{U}$ such that $F = \twoheaduparrow \mathcal{U}$;*
3. *F is an end.*

Proof. First, we prove the following claim:

Claim 2.2.7. *If F is a proper filter, then $\twoheaduparrow F$ is a proper round filter.*

Proof of Claim. Let F be a proper filter.

- $\twoheaduparrow F$ is proper:
  Suppose for a contradiction that $\twoheaduparrow F$ is not proper. Hence $0 \in \twoheaduparrow F$. This means that there exist $b \in F$ such that $b \prec 0$. Then, by (Q5), we have $b \leq 0$, hence $b = 0$, so we have $0 \in F$, contradicting the fact that F is proper.

- $1 \in \twoheaduparrow F$:
  By (Q1) we have $1 \prec 1$, and since F is a filter we have $1 \in F$, hence $1 \in \twoheaduparrow F$.

- $a \in \twoheaduparrow F, a \leq b \;\Rightarrow\; b \in \twoheaduparrow F$:
  Let $a \in \twoheaduparrow F$ and $a \leq b$. By the former, there exists $c \in F$ such that $c \prec a$. So we have $c \leq c \prec a \leq b$, hence by (Q4) $c \prec b$. So $b \in \twoheaduparrow F$.

- $a, b \in {\uparrow}F \;\Rightarrow\; a \wedge b \in {\uparrow}F$:
  Let $a, b \in {\uparrow}F$. Then there exist $c, d \in F$ such that $c \prec a$ and $d \prec b$. So, we have $c \wedge d \leq c \prec a \leq a$ and $c \wedge d \leq d \prec b \leq b$, hence by (Q4) we have $c \wedge d \prec a, b$, so by (Q2) $c \wedge d \prec a \wedge b$. Since F is a filter and $c, d \in F$, we have $c \wedge d \in F$, and hence $a \wedge b \in {\uparrow}F$.

  So far, we have proved that ${\uparrow}F$ is a proper filter.

- $a \in {\uparrow}F \;\Rightarrow\; \exists c \in {\uparrow}F : \; c \prec a$:
  Let $a \in {\uparrow}F$. Then there exists $b \in F$ such that $b \prec a$. So, by (Q7), there exists c such that $b \prec c \prec a$. Since $b \prec c$, we have $c \in {\uparrow}F$, so we have found the c we were looking for.

This shows that ${\uparrow}F$ is a proper round filter. □

(1. $\Rightarrow$ 2.) Let F be a maximal round filter. Since it is a proper filter, there exists an ultrafilter $\mathcal{U}$ such that $F \subseteq \mathcal{U}$. Then we have $F = {\uparrow}F \subseteq {\uparrow}\mathcal{U}$. So, since by the claim we have that ${\uparrow}\mathcal{U}$ is a proper round filter, by maximality of F we have $F = {\uparrow}\mathcal{U}$.

(2. $\Rightarrow$ 3.) Let $F = {\uparrow}\mathcal{U}$, where $\mathcal{U}$ is an ultrafilter. We need to show that F satisfies properties (E1) and (E2).

(E1) Let $a, b \in F$. Then there exist $c, d \in \mathcal{U}$ such that $c \prec a$ and $d \prec b$. By (Q7), there exist e, f such that $c \prec e \prec a$ and $d \prec f \prec b$. Since $\mathcal{U}$ is a proper filter, we have $0 \neq c \wedge d \in \mathcal{U}$, and we have $c \wedge d \leq c \prec e \leq e$ and $c \wedge d \leq d \prec f \leq f$. So by (Q4) we have $c \wedge d \prec e, f$, hence by (Q2) we have $c \wedge d \prec e \wedge f$. So $e \wedge f \in F$, and by (Q5) $0 \neq c \wedge d \leq e \wedge f$, so $0 \neq e \wedge f$. Moreover, we have $e \wedge f \leq e \prec a \leq a$ and $e \wedge f \leq f \prec b \leq b$, so again by (Q4) we have $e \wedge f \prec a, b$.

(E2) Let $a \prec b$. By (Q7), there exists c such that $a \prec c \prec b$. So in particular we have $c \prec b$, and by (Q6) we have $\neg c \prec \neg a$. Since $\mathcal{U}$ is an ultrafilter, either $c \in \mathcal{U}$, and hence $b \in F$, or $\neg c \in \mathcal{U}$, and hence $\neg a \in F$.

(3. $\Rightarrow$ 1.) Suppose F is an end of B.

- $0 \notin F$:
  Suppose for a contradiction that $0 \in F$. Then, by (E1), there exist $0 \neq c \in F$ such that $c \prec 0$. But this by (Q5) implies $c \leq 0$, hence $c = 0$, which contradicts $c \neq 0$.
- $1 \in F$:
  By (Q1) and (Q4), we have $0 \prec 1$. Hence, by (E2), either $\neg 0 = 1 \in F$ or $1 \in F$, that is $1 \in F$.
- $a \in F, a \leq b \;\Rightarrow\; b \in F$:
  Let $a \in F$ and $a \leq b$. By (E1), there exists $0 \neq c \in F$ such that $c \prec a$. By $c \leq c \prec a \leq b$ and by (Q4) we have $c \prec b$. So, by (E2),

either $\neg c \in F$ or $b \in F$. If $\neg c \in F$, since also $c \in F$, by (E1) there exists $0 \neq d \in F$ such that $d \prec c, \neg c$, hence by (Q2) $d \prec c \wedge \neg c = 0$, so by (Q5) $d \leq 0$, contradicting the fact that $0 \neq d$. Hence we cannot have $\neg c \in F$, so necessarily $b \in F$.

- $a, b \in F \;\Rightarrow\; a \wedge b \in F$:
  If $a, b \in F$, by (E1) there exists $0 \neq c \in F$ such that $c \prec a, b$. So, by (Q2) we have $c \prec a \wedge b$. By (E2), we have either $\neg c \in F$ or $a \wedge b \in F$, and as we discussed in the previous item, we cannot have $\neg c \in F$ because already $c \in F$. Hence $a \wedge b \in F$.

The above items show that F is a proper filter. Then, trivially by property (E1), we have that F is a proper round filter.

It remains to show that it is maximal. Suppose for a contradiction that there exists a proper round filter G such that $F \subsetneq G$. Then there exists $a \in G \setminus F$. Since G is a round filter, there exists $b \in G$ such that $b \prec a$. Since F satisfies (E2), either $\neg b \in F$ or $a \in F$, and since the latter is not the case by assumption, we have $\neg b \in F$. But then, since $F \subseteq G$, we have $\neg b \in G$, hence $0 = b \wedge \neg b \in G$, contradicting the fact that G is proper.

Hence F is a maximal round filter.

□

Proposition 2.2.8 (Hausdorffness of the space of ends)**.** *Let* $(B, \prec)$ *be an algebra with* $\prec$ *satisfying* (Q1)-(Q7)*. Let* X *be the set of all its ends, with the topology generated by the basis* $\{U_a \mid a \in B\}$ *where* $U_a := \{x \in X \mid a \in x\}$*. Then* X *is an Hausdorff space.*

Proof. Let $x, y \in X$ be distinct. So there exists $a \in B$ such that $a \in x$ and $a \notin y$. Since x is a round filter, there exists $b \in x$ such that $b \prec a$. By Lemma 2.2.6, since y is an end, it satisfies property (E2). So, since $b \prec a$ and $a \notin y$, we must have $\neg b \in y$.

So we have $x \in U_b$ and $y \in U_{\neg b}$, and $U_b, U_{\neg b}$ are disjoint opens. This shows that X is Hausdorff. □

Proposition 2.2.9 (Compactness of the space of ends)**.** *Let* $(B, \prec)$ *be an algebra with* $\prec$ *satisfying* (Q1)-(Q7)*. Let* X *be the set of all its ends, with the topology generated by the basis* $\{U_a \mid a \in B\}$ *where* $U_a := \{x \in X \mid a \in x\}$*. Then* X *is a compact space.*

Proof. Let Y be the set of ultrafilter, with the Stone topology, that is the one generated by the basis $\{V_a \mid a \in B\}$ where $V_a := \{y \in Y \mid a \in y\}$.

Since Y is a Stone space, in particular it is compact. Then, consider the following map:

$$\begin{aligned} f : Y &\to X \\ y &\mapsto {\Uparrow} y \end{aligned}$$

By the above lemma, the elements of X are exactly those which are equal to ${\Uparrow} y$ for some ultrafilter $y \in Y$, hence f is well defined and surjective.

We now show that it is continuous. In fact, let $\bigcup_{a\in A} U_a$ be a generic open subset of X, where A is some subset of B. We have:

$$\begin{aligned} y \in f^{-1}\Big(\bigcup_{a\in A} U_a\Big) &\Leftrightarrow f(y) = {\Uparrow} y \in \bigcup_{a\in A} U_a \\ &\Leftrightarrow \exists a \in A : \ a \in {\Uparrow} y \\ &\Leftrightarrow \exists a \in A, \exists b : \ b \prec a \text{ and } b \in y \\ &\Leftrightarrow \exists b \in {\Downarrow} A : \ b \in y \\ &\Leftrightarrow \exists b \in {\Downarrow} A : \ y \in V_b \\ &\Leftrightarrow y \in \bigcup_{b \in {\Downarrow} A} V_b \end{aligned}$$

that is $f^{-1}\Big(\bigcup_{a\in A} U_a\Big) = \bigcup_{b\in {\Downarrow} A} V_b$ and the latter is an open of Y. This shows that f is continuous.

Since f is a continuous surjective function from the compact space Y to X, we have that X is compact. □

Now we can define the contravariant functor $(-)_* : \mathbf{deV} \to \mathbf{KHaus}$.

Given a de Vries algebra $(B, \prec)$, let $(B, \prec)_*$ be the space X of ends of $(B, \prec)$, with the topology defined as in Proposition 2.2.8. By Propositions 2.2.8 and 2.2.9, we have that X is a compact Hausdorff space.

Given $\alpha : A \to B$ de Vries morphism between de Vries algebras $(A, \prec)$ and $(B, \prec)$, we define

$$\begin{aligned} \alpha_* : (B, \prec)_* &\to (A, \prec)_* \\ F &\mapsto {\Uparrow} \alpha^{-1}(F) \end{aligned}$$

Then $\alpha_* : (B, \prec)_* \to (A, \prec)_*$ is a well-defined continuous function from the space of ends of $(B, \prec)$ to the space of ends of $(A, \prec)$.

2.2.2 The functor $(-)^* : \mathbf{KHaus} \to \mathbf{deV}$

For the other direction, we will map each compact Hausdorff space to the de Vries algebra of its regular opens subsets:

Definition 2.2.10 (Regular open)**.** *Let X be a topological space. A subset $U \subseteq X$ is a* regular open *if* $\mathbf{Int}(\mathbf{Cl}(U)) = U$*, where* $\mathbf{Int}$ *and* $\mathbf{Cl}$ *denote the interior and closure operators, respectively;*

Analogously, we say that a subset $F \subseteq X$ is regular closed *if* $\mathbf{Cl}(\mathbf{Int}(F)) = F$*.*

Given a topological space X, let $RO(X)$ be the set of its regular opens subsets. This forms a complete Boolean algebra with the operations:

$$\begin{aligned}
1 &:= X \\
0 &:= \emptyset \\
\neg U &:= \mathbf{Int}(X \setminus U) = X \setminus \mathbf{Cl}(U) \\
\bigvee_{i \in I} U_i &:= \mathbf{Int}(\mathbf{Cl}(\bigcup_{i \in I} U_i)) \\
\bigwedge_{i \in I} U_i &:= \mathbf{Int}(\bigcap_{i \in I} U_i) \ [= \bigcap_{i \in I} U_i \text{ if } I \text{ is finite}].
\end{aligned}$$

Moreover, if given $U, V \in RO(X)$ we define $U \prec V$ if and only if $\mathbf{Cl}(U) \subseteq V$, we have that $(RO(X), \prec)$ is a de Vries algebra.

We define the contravariant functor $(-)^* : \mathbf{KHaus} \to \mathbf{deV}$ as follows.

Given X compact Hausdorff spaces, let $X^* := (RO(X), \prec)$ with $\prec$ defined as above.

Given a continuous function $f : X \to Y$, let

$$\begin{aligned}
f^* : RO(Y) &\to RO(X) \\
U &\mapsto \mathbf{Int}(\mathbf{Cl}(f^{-1}(U))).
\end{aligned}$$

Then $f^* : RO(Y) \to RO(X)$ is a well defined de Vries morphism from the de Vries algebra $(RO(Y), \prec)$ to $(RO(X), \prec)$.

Thus, we can conclude the following:

Theorem 2.2.11 (De Vries duality, [20])**.** *The categories* $\mathbf{deV}$ *and* $\mathbf{KHaus}$ *are dually equivalent.*

2.2.3 Connection with de Vries duality

In the proof of Proposition 2.2.9, we have seen that the space X of ends of a compingent algebra $(B, \prec)$ is obtained by quotienting the Stone space Y of B under the closed equivalence relation $yRy' \iff {\Uparrow}y \subseteq y'$, which is the dual of the subordination $\prec$ on B according to the duality described in the previous section.

In particular, it follows that the dual compact Hausdorff space of a de Vries algebra $(B, \prec)$ (under de Vries duality) is homeomorphic to the quotient of the Stone space of B under the irreducible closed equivalence relation which is dual to $\prec$ by the duality described in the previous section.

This suggests a way to use the latter duality to construct a modal-like alternative to de Vries duality for the category **deV**.

First, we see which are the dual objects of de Vries algebras under the duality of the previous section:

Definition 2.2.12 (Gleason spaces)**.** *A subordination space (X, R) is called a* Gleason space *if the Stone space X is extremally disconnected*[3] *and R is an irreducible equivalence relation.*

Gleason spaces are introduced in [9, 8], and the choice of their name is motivated by the fact that there is a natural correspondence between Gleason spaces and Gleason covers [33] of compact Hausdorff spaces. We refer to [9, 8] for more information about this correspondence.

Recall that a Boolean algebra B is complete if and only if its Stone space X is extremally disconnected. Putting this observation together with Theorem 2.1.16, we obtain that the dual objects of de Vries algebras under the duality of the previous sections are exactly Gleason spaces.

This restriction yields a duality between the category of Gleason spaces and continuous stable functions and a category whose objects are de Vries algebras and arrows are Boolean homomorphisms preserving $\prec$. But we are interested in the category **deV**, where morphisms are de Vries morphisms. So, in order to obtain such a duality, we need to use a different notion of arrows between Gleason spaces. These will be particular binary relations, which we we call de Vries relations:

Definition 2.2.13 (de Vries relation, [9, 8])**.** *Let (X, R) and (Y, R) be Gleason spaces. A binary relation $r \subseteq Y \times X$ is a* de Vries relation *if the following are satisfied:*

- *for all $y \in Y$ there exists $x \in X$ such that yrx;*
- *for every $y \in Y$ and for every clopen $U \subseteq X$, we have that $r[y]$ and $r^{-1}[U]$ are respectively closed and clopen;*
- *for all $y, y' \in Y$ and $x, x' \in X$, if yRy', yrx and $y'rx'$, then xRx':*

$$\begin{tikzcd} y' \arrow[r, "r"] \arrow[d, dash, "R"'] & x' \arrow[d, dash, dotted, "R"] \\ y \arrow[r, "r"'] & x \end{tikzcd}$$

[3]Recall that a space X is called *extremally disconnected* if the closure of every open subset is open.

- *for every clopen $U \subseteq X$, we have $r^{-1}[U] = \mathbf{Int}(r^{-1}R^{-1}[U])$.*

Let **Gle** be the category of Gleason spaces and de Vries relations. The correspondence between de Vries algebras and Gleason spaces discussed above can be extended to a duality between **deV** and **Gle** as follows.

Given de Vries algebras $(A, \prec)$ and $(B, \prec)$, and a map $h : A \to B$, we can define a binary relation $r \subseteq Y \times X$ between their duals $(X, R) := (A, \prec)_+$ and $(Y, R) := (B, \prec)_+$ as:

$$yrx \quad \Leftrightarrow \quad \text{for all } a \in A \text{ , if } h(a) \in y \text{ then } a \in x.$$

As proved in [9], if h is a de Vries morphism, then the relation r defined as above is a de Vries relation.

Conversely, given a relation $r \subseteq Y \times X$ between Gleason spaces (X, R) and (Y, R), we define its corresponding $h : \mathbf{Clop}(X) \to \mathbf{Clop}(Y)$ as:

$$U \ \mapsto \ Y \setminus r^{-1}[X \setminus U] \ .$$

As proved again in [9], if r is a de Vries relation, then the map h defined as above is a de Vries morphism. Hence we obtain the following:

Theorem 2.2.14. **deV** *is dually equivalent to* **Gle**, *and hence* **Gle** *is equivalent to* **KHaus**.

Conclusion

In this chapter, we have introduced the structures which are involved in this thesis, namely Boolean algebras with subordinations, subordinations spaces and compact Hausdorff spaces. We recalled two dualities between categories of the aforementioned structures, and at the end we reviewed a connection between those dualities. The structures introduced in this chapter will be used in the rest of this thesis as semantics for the language which we will define in the next chapter.

Chapter 3

The logic of contact algebras

One of our main objectives is to define a finitary propositional calculus which is sound and complete with respect to compact Hausdorff spaces. We aim to do this by providing a calculus which is sound an complete with respect to de Vries algebras, and then by using de Vries duality (see Section 2.2) we will be able to show that this is the logic of compact Hausdorff spaces.

The first step towards this direction is made in this chapter, where we introduce a system $\mathcal{S}$ and we prove that it is sound and complete with respect to the class of contact algebras (see Definition 2.1.3). Then, in Chapter 4, we will see how to enhance $\mathcal{S}$ with a particular kind of non-standard rules, and in Chapter 5 we will show that there are specific rules which, once added to the system $\mathcal{S}$, give a system which is sound and complete with respect to the class of de Vries algebras.

Below we introduce formulas of our language, and we define semantics for these formulas. In the following section, we present the axioms and rules of the system $\mathcal{S}$ and we prove that it is sound and complete with respect to the class of contact algebras.

3.1 Syntax and semantics

Let $Prop$ be an countably infinite set of propositional variables. In what follows, we will consider formulas in the following language:

$$\varphi \ := \ p \mid \top \mid \varphi \wedge \varphi \mid \neg\varphi \mid \varphi \rightsquigarrow \varphi$$

where $p \in Prop$. We use standard abbreviations $\bot := \neg\top, \varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$ and $\varphi \to \psi := \neg\varphi \vee \psi$. Also, when we write formulas with missing brakets, our convention is that the connectives $\wedge, \vee, \neg$ have priority over $\rightsquigarrow$ and $\to$.

We interpret formulas in the above language in algebras $(B, \prec)$ with B a Boolean algebra and $\prec$ a binary relation on B. As we mentioned in Section 2.1, we regard pairs $(B \ \prec)$ as algebras $(B, 1, \wedge, \neg, \rightsquigarrow)$, where the operation $\rightsquigarrow: B \times B \to \{0, 1\} \subseteq B$ is defined as:

$$a \rightsquigarrow b \ := \begin{cases} 1 & \text{if } a \prec b \\ 0 & \text{otherwise.} \end{cases}$$

A *valuation* is a map $v : Prop \to B$, which is extended to all formulas in a standard way. We say that a valuation v on $(B, \prec)$ *satisfies* a formula φ if $v(\varphi) = 1$. If all valuations on $(B, \prec)$ satisfy φ, we say that $(B, \prec)$ *validates* φ, and we write $(B, \prec) \models \varphi$.

Let K be a class of algebras of the form $(B, \prec)$, let φ be a formula and let Γ be a set of formulas. Then, if for all $(B, \prec) \in K$ and for all valuations $v : Prop \to B$ we have $v(\varphi) = 1$ whenever $v(\psi) = 1$ for all $\psi \in \Gamma$, we write $\Gamma \models_K \varphi$.

Remark 3.1.1.
Formulas of the form $\top \rightsquigarrow \varphi$ have an important role in the proof of many results shown in this thesis. With any formula φ, our language allows us to associate the formula $\top \rightsquigarrow \varphi$, which is such that $v(\top \rightsquigarrow \varphi) \in \{0, 1\}$ under any valuation $v : Prop \to B$ into any algebra $(B, \prec)$. Moreover, if a class K consists of algebras satisfying (Q1) and (Q5) [1] it has the following property:

- for any formula φ, and for any valuation $v : Prop \to B$ into an algebra $(B, \prec) \in K$, we have

$$v(\varphi) = 1 \quad \Leftrightarrow \quad v(\top \rightsquigarrow \varphi) = 1.$$

It is easy to show that, for a class K, the above property is equivalent to the following:

- for any set Γ of formulas and for any formulas φ, ψ, we have

$$\Gamma \cup \{\varphi\} \models \psi \quad \Leftrightarrow \quad \Gamma \models (\top \rightsquigarrow \varphi) \to \psi.$$

In what follows, we will present a deductive system $\mathcal{S}$, and we will show that it is strongly sound and complete with respect to the class of contact algebras.

A key technical tool of our proof of completeness is given by Lemma 3.2.3. Such a lemma can be proven only if one aims to show strong completeness of a deductive system with respect to a class K which satisfies the property of Remark 3.1.1. For example, as stated in Remark 3.1.1, the class K of algebras which satisfy (Q1)-(Q5) does satisfy the property, but the class all algebras $(B, \prec)$ satisfying (Q1)-(Q4) does not. Indeed, it is possible to define a subsystem of $\mathcal{S}$ and show that it is sound and complete with respect to algebras satisfying (Q1)-(Q5), with a proof which would be virtually the same as the one which we present in this chapter. If we attempt to define an even

[1] See Definitions 2.1.1 and 2.1.3.

smaller subsystem and prove it to be sound and complete with respect to the class K of algebras satisfying (Q1)-(Q4), we would need a different proof. In fact, since K does not satisfy the property stated in Remark 3.1.1, Lemma 3.2.3 fails for any system strongly sound and complete with respect to it.

3.2 The system $\mathcal{S}$

Throughout this thesis, we assume we have fixed an arbitrary finite axiomatization for **CPC** (Classical Propositional Calculus).

Let $\mathcal{S}$ be the deductive system axiomatized by the following axioms and rules schemas:

- All axioms φ of **CPC**

(A1) $(\bot \rightsquigarrow \varphi) \wedge (\varphi \rightsquigarrow \top)$

(A2) $(\varphi \rightsquigarrow \psi) \wedge (\varphi \rightsquigarrow \chi) \to (\varphi \rightsquigarrow \psi \wedge \chi)$

(A3) $(\top \rightsquigarrow \neg\varphi \vee \psi) \wedge (\psi \rightsquigarrow \chi) \to (\varphi \rightsquigarrow \chi)$

(A4) $(\varphi \rightsquigarrow \psi) \to (\varphi \to \psi)$

(A5) $(\varphi \rightsquigarrow \psi) \to (\chi \rightsquigarrow (\varphi \rightsquigarrow \psi))$

(A6) $\neg(\varphi \rightsquigarrow \psi) \to (\chi \rightsquigarrow \neg(\varphi \rightsquigarrow \psi))$

(A7) $(\varphi \rightsquigarrow \psi) \leftrightarrow (\neg\psi \rightsquigarrow \neg\varphi)$

(MP) $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$

(R) $\dfrac{\varphi}{\top \rightsquigarrow \varphi}$

In the system $\mathcal{S}$, any finite list $\psi_1, \ldots, \psi_n$ of formulas can be regarded as a proof of some entailment of the form $\Gamma \vdash \varphi$, where Γ is a set of formulas and $\varphi = \psi_n$. As the following definition specifies, given a list $\psi_1, \ldots, \psi_n$, we distinguish formulas ψ_i which must be regarded as assumptions from those which are derived by the system, that is instances of axioms and formulas which follow by a rule from former ones. Then the list $\psi_1, \ldots, \psi_n$ will be defined as a proof of $\Gamma \vdash \psi_n$ for each Γ which contains all the assumptions of $\psi_1, \ldots, \psi_n$.

Definition 3.2.1 (Proofs)**.** *A* proof *is a finite list $\psi_1, \ldots, \psi_n$ of formulas. A formula ψ_i in the list is defined to be an* assumption *of the proof, unless it satisfies one of the following conditions:*

- *ψ_i is an instance of an axiom of* **CPC***, or*

- *ψ_i is an instance of an axiom among* (A1)-(A7), *or*
- *ψ_i follows from ψ_j, ψ_k for some $j, k < i$ by applying* (MP), *or*
- *ψ_i follows from ψ_j for some $j < i$ by applying the rule* (R),

If Γ_0 is the set of the assumptions of a proof $\psi_1, \ldots, \psi_n$, we say that the latter is a proof of $\Gamma \vdash \psi_n$ *for each set of formulas Γ such that $\Gamma_0 \subseteq \Gamma$.*

In particular, if $\psi_1, \ldots, \psi_n$ contains no assumption, then we say that $\psi_1, \ldots, \psi_n$ is a proof of $\vdash \psi_n$, *or more simply a proof of ψ_n.*

We say that Γ is *inconsistent* if $\Gamma \vdash \bot$. Otherwise, we say that Γ is *consistent.*

Lemma 3.2.2. *The following rules are derivable in $\mathcal{S}$:*

(D1) $$\frac{\varphi \to \psi}{(\psi \rightsquigarrow \chi) \to (\varphi \rightsquigarrow \chi)}$$

(D2) $$\frac{\varphi \to \psi}{(\chi \rightsquigarrow \varphi) \to (\chi \rightsquigarrow \psi)}$$

Then, since the system $\mathcal{S}$ includes **CPC**, *we obtain that also the following rules are derivable:*

(D3) $$\frac{\varphi \leftrightarrow \psi}{(\varphi \rightsquigarrow \chi) \leftrightarrow (\psi \rightsquigarrow \chi)}$$

(D4) $$\frac{\varphi \leftrightarrow \psi}{(\chi \rightsquigarrow \varphi) \leftrightarrow (\chi \rightsquigarrow \psi)}$$

Moreover, the following axiom schemas are provable in the system:

(A2′) $(\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi) \to (\varphi \vee \psi \rightsquigarrow \chi)$

(A3′) $(\varphi \rightsquigarrow \psi) \wedge (\top \rightsquigarrow \neg\psi \vee \chi) \to (\varphi \rightsquigarrow \chi)$

Proof. First, we show that rules (D1) and (D2) are derivable. Consider the following derivations:

(D1)

1. $\varphi \to \psi$
2. $\neg\varphi \vee \psi$
3. $\top \rightsquigarrow \neg\varphi \vee \psi$ follows by (R) from 2.
4. $(\top \rightsquigarrow \neg\varphi \vee \psi) \wedge (\psi \rightsquigarrow \chi) \to (\varphi \rightsquigarrow \chi)$ is an instance of (A3)
5. $(\top \rightsquigarrow \neg\varphi \vee \psi) \to \Big((\psi \rightsquigarrow \chi) \to (\varphi \rightsquigarrow \chi)\Big)$ follows by **CPC** from 4.
6. $(\psi \rightsquigarrow \chi) \to (\varphi \rightsquigarrow \chi)$ follows by (MP) from 3. and 5.

(D2)

1. $\varphi \to \psi$
2. $\neg\psi \to \neg\varphi$ follows by **CPC** from 1.
3. $(\neg\varphi \rightsquigarrow \neg\chi) \to (\neg\psi \rightsquigarrow \neg\chi)$ follows by (D1) from 2.
4. $(\chi \rightsquigarrow \varphi) \to (\neg\varphi \rightsquigarrow \neg\chi)$ follows by an instance of (A7)
5. $(\neg\psi \rightsquigarrow \neg\chi) \to (\chi \rightsquigarrow \psi)$ follows by an instance of (A7)
6. $(\chi \rightsquigarrow \varphi) \to (\chi \rightsquigarrow \psi)$ follows by **CPC** from 4. 3. and 5.

Next we show that (A2$'$) and (A3$'$) are theorems in our system:

(A2$'$)

1. $(\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi) \to (\neg\chi \rightsquigarrow \neg\varphi) \wedge (\neg\chi \rightsquigarrow \neg\psi)$ follows by instances of (A7) and **CPC**
2. $(\neg\chi \rightsquigarrow \neg\varphi) \wedge (\neg\chi \rightsquigarrow \neg\psi) \to (\neg\chi \rightsquigarrow \neg\varphi \wedge \neg\psi)$ is an instance of (A2)
3. $(\neg\chi \rightsquigarrow \neg\varphi \wedge \neg\psi) \to (\varphi \vee \psi \rightsquigarrow \neg\neg\chi)$ follows by an instance of (A7) and **CPC** [2]
4. $\neg\neg\chi \to \chi$ follows by **CPC**
5. $(\varphi \vee \psi \rightsquigarrow \neg\neg\chi) \to (\varphi \vee \psi \rightsquigarrow \chi)$ follows by (D2) from 4.
6. $(\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi) \to (\varphi \vee \psi \rightsquigarrow \chi)$ follows by **CPC** from 1. 2. 3. and 5.

(A3$'$)

1. $(\varphi \rightsquigarrow \psi) \wedge (\top \rightsquigarrow \neg\psi \vee \chi) \to (\neg\psi \rightsquigarrow \neg\varphi) \wedge (\top \rightsquigarrow \neg\psi \vee \chi)$ follows by an instance of (A7) and **CPC**
2. $(\neg\psi \rightsquigarrow \neg\varphi) \wedge (\top \rightsquigarrow \neg\psi \vee \chi) \to (\neg\chi \rightsquigarrow \neg\varphi)$ is an instance of (A3)
3. $(\neg\chi \rightsquigarrow \neg\varphi) \to (\varphi \rightsquigarrow \chi)$ follows by an instance of (A7)
4. $(\varphi \rightsquigarrow \psi) \wedge (\top \rightsquigarrow \neg\psi \vee \chi) \to (\varphi \rightsquigarrow \chi)$ follows by **CPC** from 1. 2. and 3.

□

In the rest of this chapter, we will show that $\mathcal{S}$ is sound and complete with respect to the class of contact algebras (see Definition 2.1.3).

3.2.1 Soundness

The aim of this section is to show that, if K is the class of contact algebras and $\models$ is $\models_K$, for any set of formulas Γ and any formula φ, we have

$$\Gamma \vdash \varphi \quad \Rightarrow \quad \Gamma \models \varphi.$$

[2] Recall that $\varphi \vee \psi$ is an abbreviation of $\neg(\neg\varphi \wedge \neg\psi)$.

This means that we have to show that, if we have a proof $\psi_1, \ldots, \psi_n = \varphi$ of $\Gamma \vdash \varphi$, then we have $v(\varphi) = 1$ for each valuation v into a contact algebra which satisfies all formulas in Γ.

In order to achieve this, we show that the axioms (A1)-(A7) are valid in any contact algebra $(B, \prec)$, and that for any valuation v into a contact algebra $(B, \prec)$, if the premise(s) of (R) or (MP) are satisfied, then also the conclusion is satisfied. This allows us to conclude the proof of soundness. In fact, if a valuation satisfies all formulas in Γ, then all the assumptions in the proof $\psi_1, \ldots, \psi_n$ would be satisfied. Moreover, by what we show below so would be all instances of axioms, as well as formulas which are derived from the rules (MP) and (R) by former ones which are satisfied. Thus, by the definition of proofs in the system $\mathcal{S}$, by induction on $\psi_1, \ldots, \psi_n$ we obtain that all formulas ψ_i are satisfied by the valuation v, hence so is $\varphi = \psi_n$.

- All axioms of **CPC** are valid because of the soundness of **CPC** with respect to Boolean algebras.

(A1) $(\bot \rightsquigarrow \varphi) \wedge (\varphi \rightsquigarrow \top)$:
Let v be any valuation. By (Q1), we have $v(\bot) = 0 \prec 0$, and $1 \prec 1 = v(\top)$. So, for any formula φ, we have $v(\bot) \leq 0 \prec 0 \leq v(\varphi)$ and $v(\varphi) \leq 1 \prec 1 \leq v(\top)$. Hence, by (Q4), we obtain $v(\bot) \prec v(\varphi)$ and $v(\varphi) \prec v(\top)$, that is we have respectively $v(\bot \rightsquigarrow \varphi) = 1$ and $v(\varphi \rightsquigarrow \top) = 1$. Hence we have $v(\bot \rightsquigarrow \varphi) \wedge (\varphi \rightsquigarrow \top) = v(\bot \rightsquigarrow \varphi) \wedge v(\varphi \rightsquigarrow \top) = 1$.

(A2) $(\varphi \rightsquigarrow \psi) \wedge (\varphi \rightsquigarrow \chi) \rightarrow (\varphi \rightsquigarrow \psi \wedge \chi)$:
Let v be any valuation. We have $v\Big((\varphi \rightsquigarrow \psi) \wedge (\varphi \rightsquigarrow \chi)\Big) = v(\varphi \rightsquigarrow \psi) \wedge v(\varphi \rightsquigarrow \chi) \in \{0, 1\}$. If it is 0, the axiom is satisfied. Suppose it is 1. Then we have $v(\varphi \rightsquigarrow \psi) = v(\varphi \rightsquigarrow \chi) = 1$. So $v(\varphi) \prec v(\psi), v(\chi)$, hence by (Q2) we get $v(\varphi) \prec v(\psi) \wedge v(\chi) = v(\psi \wedge \chi)$. Hence $v(\varphi \rightsquigarrow \psi \wedge \chi) = 1$, and this shows that the axiom is valid.

(A3) $(\top \rightsquigarrow \neg\varphi \vee \psi) \wedge (\psi \rightsquigarrow \chi) \rightarrow (\varphi \rightsquigarrow \chi)$:
Let v be any valuation. Suppose $v\Big((\top \rightsquigarrow \neg\varphi \vee \psi) \wedge (\psi \rightsquigarrow \chi)\Big) = 1$. Then we have $v(\top \rightsquigarrow \neg\varphi \vee \psi) = 1$ and $v(\psi \rightsquigarrow \chi) = 1$. So by the latter we have $v(\psi) \prec v(\chi)$, and by the former we have $1 = v(\top) \prec v(\neg\varphi \vee \psi)$, which by (Q5) implies $1 = v(\neg\varphi \vee \psi) = \neg v(\varphi) \vee v(\psi)$, that is $v(\varphi) \leq v(\psi)$. So we have $v(\varphi) \leq v(\psi) \prec v(\chi) \leq v(\chi)$, hence by (Q4) we obtain $v(\varphi) \prec v(\chi)$, that is $v(\varphi \rightsquigarrow \chi) = 1$. This shows that the axiom is valid.

(A4) $(\varphi \rightsquigarrow \psi) \rightarrow (\varphi \rightarrow \psi)$:
Let v be any valuation, and suppose $v(\varphi \rightsquigarrow \psi) = 1$. Then we have $v(\varphi) \prec v(\psi)$, hence by (Q5) we get $v(\varphi) \leq v(\psi)$, and so $v(\varphi \rightarrow \psi) = 1$. This shows that the axiom is valid.

(A5) $(\varphi \rightsquigarrow \psi) \rightarrow (\chi \rightsquigarrow (\varphi \rightsquigarrow \psi))$:
Let v be any valuation. If $v(\varphi \rightsquigarrow \psi) = 1$, then by (Q1) and (Q4) we

have that $a \prec 1 = v(\varphi \rightsquigarrow \psi)$. So in particular for any formula χ we have $v(\chi) \prec v(\varphi \rightsquigarrow \psi)$, hence $v(\chi \rightsquigarrow (\varphi \rightsquigarrow \psi)) = 1$. This shows that the axiom is valid.

(A6) $\neg(\varphi \rightsquigarrow \psi) \to (\chi \rightsquigarrow \neg(\varphi \rightsquigarrow \psi))$:
Let v be any valuation. If $v(\neg(\varphi \rightsquigarrow \psi)) = 1$, then by (Q1) and (Q4) we have that $a \prec 1 = v(\neg(\varphi \rightsquigarrow \psi))$. So in particular for any formula χ we have $v(\chi) \prec v(\neg(\varphi \rightsquigarrow \psi))$, hence $v(\chi \rightsquigarrow \neg(\varphi \rightsquigarrow \psi)) = 1$. This shows that the axiom is valid.

(A7) $(\varphi \rightsquigarrow \psi) \leftrightarrow (\neg\psi \rightsquigarrow \neg\varphi)$:
Let v be any valuation. Recall that we have $v(\varphi \rightsquigarrow \psi), v(\neg\psi \rightsquigarrow \neg\varphi) \in \{0,1\}$. If $v(\varphi \rightsquigarrow \psi) = 1$, then we have $v(\varphi) \prec v(\psi)$. Hence by (Q6) we have $v(\neg\psi) = \neg v(\psi) \prec \neg v(\varphi) = v(\neg\varphi)$. Conversely, if $v(\neg\psi \rightsquigarrow \neg\varphi) = 1$, we have $v(\neg\psi) \prec v(\neg\varphi)$, hence again by (Q6) we obtain $v(\varphi) = \neg\neg v(\varphi) = \neg v(\neg\varphi) \prec \neg v(\neg\psi) = \neg\neg v(\psi) = v(\psi)$. This shows that the axiom is valid.

(MP) $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$:
Let v be any valuation. Suppose $v(\varphi) = 1$ and $v(\varphi \to \psi) = 1$. Then we have $1 = v(\varphi) \leq v(\psi)$, so $v(\psi) = 1$. This shows that (MP) preserves satisfaction.

(R) $\dfrac{\varphi}{\top \rightsquigarrow \varphi}$:
Let v be any valuation, and suppose $v(\varphi) = 1$. Hence by (Q1) we have $v(\top) = 1 \prec 1 = v(\varphi)$, that is $v(\top \rightsquigarrow \varphi) = 1$. This shows that (R) preserves satisfaction.

3.2.2 Completeness

Since, for the system $\mathcal{S}$, the deduction theorem does not hold[3], in the following lemma we provide a weaker form of it:

Lemma 3.2.3 ($\rightsquigarrow$-deduction theorem)**.** *For any set Γ of formulas, and for any formulas φ, ψ, we have:*

$$\Gamma \cup \{\varphi\} \vdash \psi \quad \Leftrightarrow \quad \Gamma \vdash (\top \rightsquigarrow \varphi) \to \psi.$$

Proof. ($\Leftarrow$) Suppose $\Gamma \vdash (\top \rightsquigarrow \varphi) \to \psi$. Then there is a list of formulas ending with $(\top \rightsquigarrow \varphi) \to \psi$ in which the set of assumptions is some $\Gamma_0 \subseteq \Gamma$. We can extend it to a proof of $\Gamma \cup \{\varphi\} \vdash \psi$, with set of assumptions $\Gamma_0 \cup \{\varphi\}$, as follows:

1. $(\top \rightsquigarrow \varphi) \to \psi$

[3] The failure of the deduction theorem is caused by rule (R). For example, we have $p \vdash \top \rightsquigarrow p$, but $\nvdash p \to (\top \rightsquigarrow p)$. This follows by weak completeness (at the end of this section) and by the fact that $p \to (\top \rightsquigarrow p)$ is not a validity. In fact, if $(B, \prec)$ is such that there is $b \in B \backslash \{0,1\}$, then with the valuation $v : p \mapsto b$ we have $v(p \to (\top \rightsquigarrow p)) = b \to (1 \rightsquigarrow b) = b \to 0 = \neg b \neq 1$.

2. $\varphi \in \Gamma_0 \cup \{\varphi\}$
3. $\top \rightsquigarrow \varphi$ by (R) from 2.
4. ψ by **CPC** by (MP) from 1. and 3.

So we have $\Gamma \cup \{\varphi\} \vdash \psi$.

($\Rightarrow$) Suppose $\Gamma \cup \{\varphi\} \vdash \psi$. So we can assume wlog that we have a finite $\Gamma_0 \subseteq \Gamma$ and a proof $\psi_1, \ldots, \psi_n = \psi$ of $\Gamma \cup \{\varphi\} \vdash \psi$ with set of assumptions $\Gamma_0 \cup \{\varphi\}$. We show by induction on $i = 1 \ldots n$ that we can obtain a proof of $\Gamma \vdash (\top \rightsquigarrow \varphi) \to \psi_i$ with assumptions Γ_0, concluding that we have a proof of $\Gamma \vdash (\top \rightsquigarrow \varphi) \to \psi$.

- Suppose $\psi_i = \varphi$. Then we have $\Gamma \vdash (\top \rightsquigarrow \varphi) \to \varphi$, because $(\top \rightsquigarrow \varphi) \to \varphi$ is a theorem, in fact:
  1. $(\top \rightsquigarrow \varphi) \vee \neg(\top \rightsquigarrow \varphi)$ follows by **CPC**
  2. $(\top \rightsquigarrow \varphi) \to (\top \to \varphi)$ is an instance of (A4)
  3. $(\top \to \varphi) \to \varphi$ follows by **CPC**
  4. $(\top \rightsquigarrow \varphi) \to \varphi$ follows by **CPC** from 2. and 3.
- Suppose ψ_i is an instance of a theorem of **CPC** or an instance of one of the axioms (A1)-(A7). In that case ψ_i is a theorem, hence since $\psi_i \to ((\top \rightsquigarrow \varphi) \to \psi_i)$ is an instance of a theorem of **CPC**, by (MP) we obtain that also $(\top \rightsquigarrow \varphi) \to \psi_i$ is a theorem, hence $\Gamma \vdash (\top \rightsquigarrow \varphi) \to \psi_i$.
- Suppose a proof of $\Gamma \cup \{\varphi\} \vdash \psi_i$ is obtained applying (MP) to ψ_j and ψ_k, with $j, k < i$ and $\psi_k = \psi_j \to \psi_i$.

  By inductive hypothesis we have a proof of $\Gamma \vdash (\top \rightsquigarrow \varphi) \to \psi_j$ and a proof of $\Gamma \vdash (\top \rightsquigarrow \varphi) \to (\psi_j \to \psi_i)$. If we concatenate these proofs, we can extend the resulting list to a proof of $\Gamma \vdash (\top \rightsquigarrow \varphi) \to \psi_i$ as follows:
  1. $(\top \rightsquigarrow \varphi) \to \psi_j$
  2. $(\top \rightsquigarrow \varphi) \to (\psi_j \to \psi_i)$
  3. $\psi_j \to \psi_i \vee \neg(\top \rightsquigarrow \varphi)$ is equivalent to 2. by **CPC**
  4. $(\top \rightsquigarrow \varphi) \to \psi_i \vee \neg(\top \rightsquigarrow \varphi)$ follows by **CPC** from 1. and 3.
  5. $(\top \rightsquigarrow \varphi) \to \psi_i$ is equivalent to 4. by **CPC**
- Suppose $\psi_i = \top \rightsquigarrow \psi_j$, and that a proof of $\Gamma \cup \{\varphi\} \vdash \psi_i$ is obtained by applying (R) to ψ_j, with $j < i$.

  By inductive hypothesis we have a proof of $\Gamma \vdash (\top \rightsquigarrow \varphi) \to \psi_j$. So we can extend it as follows:
  1. $(\top \rightsquigarrow \varphi) \to \psi_j$
  2. $(\top \rightsquigarrow (\top \rightsquigarrow \varphi)) \to (\top \rightsquigarrow \psi_j)$ by (D2) from 1. (see Lemma 3.2.2)
  3. $(\top \rightsquigarrow \varphi) \to (\top \rightsquigarrow (\top \rightsquigarrow \varphi))$ is an instance of (A5)
  4. $(\top \rightsquigarrow \varphi) \to (\top \rightsquigarrow \psi_j)$ follows by **CPC** from 3. and 2.

which gives us a proof of $\Gamma \vdash (\top \rightsquigarrow \varphi) \to \psi_i$.

$\square$

Corollary 3.2.4. *For any set Γ of formulas, and for any formula φ, we have:*

(i)	$\Gamma \cup \{\varphi\} \vdash \bot$	$\Leftrightarrow$	$\Gamma \vdash \neg(\top \rightsquigarrow \varphi)$;
(ii)	$\Gamma \vdash \varphi$	$\Leftrightarrow$	$\Gamma \cup \{\neg(\top \rightsquigarrow \varphi)\} \vdash \bot$;
(iii)	$\Gamma \vdash \neg(\varphi \rightsquigarrow \psi)$	$\Leftrightarrow$	$\Gamma \cup \{\varphi \rightsquigarrow \psi\} \vdash \bot$.

Proof.

(i) This is a particular case of Lemma 3.2.3, with $\psi = \bot$.

(ii) ($\Rightarrow$) Let $\overline{\psi}$ be a list of formulas. If $\overline{\psi}, \varphi$ is a proof of $\Gamma \vdash \varphi$, then $\overline{\psi}, \varphi, \top \rightsquigarrow \varphi, \neg(\top \rightsquigarrow \varphi), \bot$ is a proof of $\Gamma \cup \{\neg(\top \rightsquigarrow \varphi)\} \vdash \bot$.

($\Leftarrow$) If $\Gamma \cup \{\neg(\top \rightsquigarrow \varphi)\} \vdash \bot$, then by the item (i) of this corollary we have $\Gamma \vdash \neg(\top \rightsquigarrow \neg(\top \rightsquigarrow \varphi))$. So, we prove $\Gamma \vdash \varphi$ extending a proof of the former as follows:

1. $\neg(\top \rightsquigarrow \neg(\top \rightsquigarrow \varphi))$
2. $\neg(\top \rightsquigarrow \neg(\top \rightsquigarrow \varphi)) \to \neg\neg(\top \rightsquigarrow \varphi)$ by contraposition from an instance of axiom (A6)
3. $\top \rightsquigarrow \varphi$ by (MP) and **CPC** from 1. and 2.
4. $(\top \rightsquigarrow \varphi) \to (\top \to \varphi)$ is an instance of axiom (A4)
5. $(\top \to \varphi)$ by (MP) from 3. and 4.
6. φ.

(iii) ($\Rightarrow$) Let $\overline{\psi}$ be a list of formulas. If $\overline{\psi}, \neg(\varphi \rightsquigarrow \psi)$ is a proof of $\Gamma \vdash \neg(\varphi \rightsquigarrow \psi)$, then $\overline{\psi}, \neg(\varphi \rightsquigarrow \psi), \varphi \rightsquigarrow \psi, \bot$ is a proof of $\Gamma \cup \{\varphi \rightsquigarrow \psi\} \vdash \bot$.

($\Leftarrow$) If $\Gamma \cup \{\varphi \rightsquigarrow \varphi\} \vdash \bot$, then by the item (i) of this corollary we have $\Gamma \vdash \neg(\top \rightsquigarrow (\varphi \rightsquigarrow \psi))$. So, we prove $\Gamma \vdash \neg(\varphi \rightsquigarrow \psi)$ extending a proof of the former as follows:

1. $\neg(\top \rightsquigarrow (\varphi \rightsquigarrow \psi))$
2. $\neg(\top \rightsquigarrow (\varphi \rightsquigarrow \psi)) \to \neg(\varphi \rightsquigarrow \psi)$ by contraposition from an instance of (A5)
3. $\neg(\varphi \rightsquigarrow \psi)$ by (MP) from 1. and 2.

$\square$

Lemma 3.2.3 is the syntactic analogue of the property stated in Remark 3.1.1, and it plays a crucial role in our proof of completeness. In fact, we will use it to prove Lemma 3.2.7, which allows us to extend consistent sets to maximally consistent sets. Then, we will use maximally consistent sets to construct a contact algebra with a valuation which satisfies all the formulas in the set.

In the next proposition, we will see that maximally consistent sets for the system $\mathcal{S}$ are those which satisfy properties (M1)-(M2) given in the following definition:

Definition 3.2.5 ($\rightsquigarrow$-maximal consistent set)**.** *A set S of formulas is called* $\rightsquigarrow$-maximal consistent *set if it satisfies the following properties:*

(M1) *S is a consistent set, and for all φ, if $S \vdash \varphi$ then $\varphi \in S$;*

(M2) *$\forall \varphi, \psi$, either $\varphi \rightsquigarrow \psi \in S$ or $\neg(\varphi \rightsquigarrow \psi) \in S$.*

Proposition 3.2.6. *Let S be a set of formulas. S is maximally consistent for the system $\mathcal{S}$ if and only if it is a $\rightsquigarrow$-maximal consistent set.*

Proof. ($\Rightarrow$) Suppose S is maximally consistent. We need to show that it satisfies properties (M1)-(M2):

(M1) S is in particular a consistent set. Let φ be such that $S \vdash \varphi$. Then $S \cup \{\varphi\}$ is consistent, hence by maximal consistency of S we have $\varphi \in S$.

This shows that S satisfies (M1).

(M2) Let φ, ψ be formulas. If $S \vdash \neg(\varphi \rightsquigarrow \psi)$, then by (M1) we get $\neg(\varphi \rightsquigarrow \psi) \in S$. Otherwise, if $S \nvdash \neg(\varphi \rightsquigarrow \psi)$, then by item (iii) of Corollary 3.2.4 we obtain $S \cup \{\varphi \rightsquigarrow \psi\} \nvdash \bot$. Hence, by maximal consistency of S we have $\varphi \rightsquigarrow \psi \in S$.

This shows that S satisfies (M2).

($\Leftarrow$) Let S be a $\rightsquigarrow$-maximal consistent set, and suppose for a contradiction that it is not maximally consistent. This means that there exists a formula φ such that $\varphi \notin S$ and $S \cup \{\varphi\} \nvdash \bot$.

By $\varphi \notin S$ and by (M1), we have $S \nvdash \varphi$. So, since $\{\top \rightsquigarrow \varphi\} \vdash \varphi$, we must have $\top \rightsquigarrow \varphi \notin S$.

By $S \cup \{\varphi\} \nvdash \bot$ and by item (i) of Corollary 3.2.4, we obtain $S \nvdash \neg(\top \rightsquigarrow \varphi)$, hence in particular $\neg(\top \rightsquigarrow \varphi) \notin S$.

So we have found formulas $\top, \varphi$ such that $\top \rightsquigarrow \varphi, \neg(\top \rightsquigarrow \varphi) \notin S$, which is a contradiction with property (M2).

□

Lemma 3.2.7 ($\rightsquigarrow$-Lindenbaum lemma)**.** *Let A be a consistent set of formulas. Then there exists a $\rightsquigarrow$-maximal consistent set S_A such that $\{\varphi \mid A \vdash \varphi\} \subseteq S_A$, hence in particular $A \subseteq S_A$.*

Proof. Starting from $A_0 := A$, we construct an increasing sequence $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$ of consistent sets of formulas.

Let $\{P_i\}_{i \in \omega}$ be an enumeration of all pairs $P_i = (\varphi, \psi)$ of formulas. We define A_{n+1} from A_n inductively as follows:

- If $A_n \vdash \varphi \rightsquigarrow \psi$, where $(\varphi, \psi) = (\varphi, \psi)_n$ define $A_{n+1} := A_n$.
- If $A_n \nvdash \varphi \rightsquigarrow \psi$, define $A_{n+1} := A_n \cup \{\neg(\varphi \rightsquigarrow \psi)\}$.

Then, by induction on n, we show that each A_n is consistent. By our assumption, $A_0 = A$ is consistent. Suppose A_n is consistent, and suppose for a contradiction that $A_{n+1} \vdash \bot$. If $(\varphi, \psi) = (\varphi, \psi)_n$ and $A_n \vdash \varphi \rightsquigarrow \psi$, then $A_{n+1} = A_n$, which contradicts the fact that A_n is consistent. So we must have $A_n \nvdash \varphi \rightsquigarrow \psi$ and $A_{n+1} = A_n \cup \{\neg(\varphi \rightsquigarrow \psi)\} \vdash \bot$. Then, by Corollary 3.2.4, we have a proof of $A_n \vdash \neg\Big(\top \rightsquigarrow \neg(\varphi \rightsquigarrow \psi)\Big)$. But then we can extend this proof as follows:

1. $\neg\Big(\top \rightsquigarrow \neg(\varphi \rightsquigarrow \psi)\Big)$
2. $\neg\Big(\top \rightsquigarrow \neg(\varphi \rightsquigarrow \psi)\Big) \to \neg\neg(\varphi \rightsquigarrow \psi)$ follows by **CPC** from an instance of (A6)
3. $\neg\neg(\varphi \rightsquigarrow \psi)$ follows by (MP) from 1. and 2.
4. $\varphi \rightsquigarrow \psi$ follows by **CPC** from 3.

Therefore, we have $A_n \vdash \varphi \rightsquigarrow \psi$, contradicting $A_n \nvdash \varphi \rightsquigarrow \psi$.

Thus, in all cases we arrived at a contradiction, hence A_{n+1} must be consistent.

Now define

$$S_A := \{\varphi \mid A_n \vdash \varphi \text{ for some } n \in \omega\} .$$

As $A = A_0$, by construction we have $\{\varphi \mid A \vdash \varphi\} \subseteq S_A$. Also, we can show that it is a $\rightsquigarrow$-maximal consistent set:

(M1) Suppose $\psi_1, \ldots, \psi_k$ is a proof of $S_A \vdash \varphi$, with set of assumptions $\Gamma_0 = \{\chi_1, \ldots, \chi_l\} \subseteq S_A$. By construction of S_A, for all $i = 1, \ldots, l$ there exists A_{h_i} such that $A_{h_i} \vdash \chi_i$. If $h = \max\{h_i \mid i = 1 \ldots l\}$, then we can turn the proof of $S_A \vdash \varphi$ into a proof of $A_h \vdash \varphi$, hence obtaining $\varphi \in S_A$. This shows that, for any formula φ, we have $S_A \vdash \varphi$ implies $\varphi \in S_A$.

Since each A_n is consistent, we have $\bot \notin S_A$. Hence, by what we have showed, we have $S_A \nvdash \bot$.

(M2) Given φ, ψ, there exists n such that $P_n = (\varphi, \psi)$. Hence, either $A_n \vdash \varphi \rightsquigarrow \psi$, and hence $\varphi \rightsquigarrow \psi \in S_A$, or by construction we have $\neg(\varphi \rightsquigarrow \psi) \in A_{n+1}$, so $A_{n+1} \vdash \neg(\varphi \rightsquigarrow \psi)$ and hence $\neg(\varphi \rightsquigarrow \psi) \in S_A$.

$\square$

In the following lemma, we show that we can use a $\rightsquigarrow$-maximal consistent set S_A to quotient the algebra of formulas in our language and obtain a contact algebra which satisfies all formulas in S_A. This will satisfy in particular all formulas in $A \subseteq S_A$, allowing us to prove strong completeness for our system.

Lemma 3.2.8. *Let A be a consistent set of formulas, and let S_A be a set obtained in Lemma 3.2.7.*

Consider the algebra $\mathcal{F} := (Form, \top, \wedge, \neg, \rightsquigarrow)$ of formulas of our language, and the following relation on $Form$:

$$\varphi \sim_{S_A} \psi \quad \Leftrightarrow \quad \varphi \leftrightarrow \psi \in S_A.$$

Then:

1. *$\sim_{S_A}$ is a congruence on $\mathcal{F}$.*
2. *Let $[\varphi]$ be the equivalence class of φ under $\sim_{S_A}$. We have*
$$[\varphi] = [\top] \quad \Leftrightarrow \quad \varphi \in S_A \ .$$
3. *Let B be the Boolean reduct of the quotient of $\mathcal{F}$ under $\sim_{S_A}$. There, for each φ, ψ, we have $[\varphi \rightsquigarrow \psi] \in \{[\top], [\bot]\}$, and the relation $\prec$ on B which results from $\rightsquigarrow$, that is*
$$[\varphi] \prec [\psi] \quad \Leftrightarrow \quad [\varphi \rightsquigarrow \psi] = [\top] \ ,$$
makes $(B, \prec)$ a contact relation.

Proof. 1. The fact that $\sim_{S_A}$ is a congruence on $\mathcal{F}$ follows by property (M1) of S_A, by Lemma 3.2.2 and by the fact that our system contains **CPC**.

2. If $[\varphi] = [\top]$, we have $\varphi \leftrightarrow \top \in S_A$, hence $\varphi \in S_A$. Conversely, if $\varphi \in S_A$, then $\varphi \leftrightarrow \top \in S_A$, so $\varphi \sim_{S_A} \top$, hence $[\varphi] = [\top]$.

3. By property (M2) of S_A, for each φ, ψ we have either $\varphi \rightsquigarrow \psi \in S_A$, hence by part 2. of this lemma we have $[\varphi \rightsquigarrow \psi] = [\top]$, or $\neg(\varphi \rightsquigarrow \psi) \in S_A$, hence $[\neg(\varphi \rightsquigarrow \psi)] = [\top]$ and so $[\varphi \rightsquigarrow \psi] = [\bot]$.

   It remains to show that $\prec$ satisfies (Q1)-(Q6):

   (Q1) $0 \prec 0$ and $1 \prec 1$
   By (A1) we have $(\bot \rightsquigarrow \bot) \wedge (\bot \rightsquigarrow \top), (\bot \rightsquigarrow \top) \wedge (\top \rightsquigarrow \top) \in S_A$, so in particular $(\bot \rightsquigarrow \bot), (\top \rightsquigarrow \top) \in S_A$. Hence $[\bot \rightsquigarrow \bot] = [\top \rightsquigarrow \top] = [\top]$, and so we have $[\bot] \prec [\bot]$ and $[\top] \prec [\top]$.

   (Q2) $a \prec b, c$ implies $a \prec b \wedge c$
   Suppose $[\varphi] \prec [\psi], [\chi]$. So we have $[\top] = [\varphi \rightsquigarrow \psi] = [\varphi \rightsquigarrow \chi]$, hence we have $(\varphi \rightsquigarrow \psi) \wedge (\varphi \rightsquigarrow \chi) \in S_A$. So, by (A2) and (MP) we have $(\varphi \rightsquigarrow \psi \wedge \chi) \in S_A$, so $[\varphi \rightsquigarrow \psi \wedge \chi] = [\top]$ and hence $[\varphi] \prec [\psi \wedge \chi] = [\psi] \wedge [\chi]$.

   (Q3) $a, b \prec c$ implies $a \vee b \prec c$
   Suppose $[\varphi], [\psi] \prec [\chi]$. So we have $[\top] = [\varphi \rightsquigarrow \chi] = [\psi \rightsquigarrow \chi]$, hence we have $(\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi) \in S_A$. So, by (A2′) and (MP) we have $(\varphi \vee \psi \rightsquigarrow \chi) \in S_A$, so $[\varphi \vee \psi \rightsquigarrow \chi] = [\top]$ and hence $[\varphi] \vee [\psi] = [\varphi \vee \psi] \prec [\chi]$.

(Q4) $a \leq b \prec c \leq d$ implies $a \prec d$

Suppose $[\varphi] \leq [\psi] \prec [\chi] \leq [\theta]$. By $[\varphi] \leq [\psi]$ and $[\psi] \prec [\chi]$ we obtain $\neg\varphi \vee \psi, \psi \rightsquigarrow \chi \in S_A$, and by (R) and **CPC** we have $(\top \rightsquigarrow \neg\varphi \vee \psi) \wedge (\psi \rightsquigarrow \chi) \in S_A$. Hence by (A3) and (MP) we have $\varphi \rightsquigarrow \chi \in S_A$. Then, by $[\chi] \leq [\theta]$ we obtain $\neg\chi \vee \theta \in S_A$, so again by (R) and **CPC** we get $(\varphi \rightsquigarrow \chi) \wedge (\top \rightsquigarrow \neg\chi \vee \theta) \in S_A$. So, by (A3′) and (MP), we have $\varphi \rightsquigarrow \theta \in S_A$, so $[\varphi \rightsquigarrow \theta] = [\top]$ and hence $[\varphi] \prec [\theta]$.

(Q5) $a \prec b$ implies $a \leq b$

Suppose $[\varphi] \prec [\psi]$, so $[\varphi \rightsquigarrow \psi] \in S_A$. By (A4) and (MP) we get $\varphi \to \psi \in S_A$, that is $\neg\varphi \vee \psi \in S_A$, hence we have $[\top] = [\neg\varphi \vee \psi] = \neg[\varphi] \vee [\psi]$, that is $[\varphi] \leq [\psi]$.

(Q6) $a \prec b$ implies $\neg b \prec \neg a$

Suppose we have $[\varphi] \prec [\psi]$. This means $[\varphi \rightsquigarrow \psi] = [\top]$, that is $\varphi \rightsquigarrow \psi \in S_A$. By axiom (A7), we have that $\neg\psi \rightsquigarrow \neg\varphi \in S_A$ as well. So we obtain $[\neg\psi \rightsquigarrow \neg\varphi] = \top$, that is $\neg[\psi] = [\neg\psi] \prec [\neg\varphi] = \neg[\varphi]$.

□

Theorem 3.2.9 (Strong completeness). *Let K be the class of contact algebras, and let $\models$ be $\models_K$. Then for any set of formulas Γ and for any formula φ, we have*

$$\Gamma \vdash \varphi \quad \Leftrightarrow \quad \Gamma \models \varphi.$$

Proof. ($\Rightarrow$) This is proved in Section 3.2.1.

($\Leftarrow$) We prove the contrapositive. Suppose $\Gamma \nvdash \varphi$. Then by Corollary 3.2.4 we have that $A := \Gamma \cup \{\neg(\top \rightsquigarrow \varphi)\}$ is consistent. Hence, by Lemma 3.2.7, we can extend it to a $\rightsquigarrow$-maximal consistent set S_A. So, we can consider the contact algebra $(B, \prec)$ constructed in Lemma 3.2.8, with the valuation $v : \psi \mapsto [\psi]$. Since this valuation satisfies all formulas in S_A, and since $A \subseteq S_A$, we have $v(\psi) = [\psi] = [\top] = 1_B$ for all $\psi \in A$.

This means that we have $v(\psi) = 1_B$ for all $\psi \in \Gamma$, and $v(\neg(\top \rightsquigarrow \varphi)) = 1_B$. By Remark 3.1.1, the latter is equivalent to $v(\varphi) \neq 1_B$. Hence what we have shown proves $\Gamma \not\models \varphi$.

□

Corollary 3.2.10 (Weak completeness). *Given a formula φ, we have that φ is a theorem of $\mathcal{S}$ if and only if it is valid on all contact algebras $(B, \prec)$.*

We prove also the alternative formulation of strong completeness:

Theorem 3.2.11 (Strong completeness, second formulation). *A set A of formulas is consistent if and only if there exists a contact algebra $(B, \prec)$ and a valuation v of formulas into B such that $v(\varphi) = 1$ for all $\varphi \in A$.*

Proof. ($\Leftarrow$) Suppose $A \vdash \bot$. Then, by soundness, if a valuation v on some $(B, \prec)$ satisfies all formulas $\varphi \in A$, it must satisfy also all the formulas φ such that $A \vdash \varphi$. So, since there is no valuation which satisfies $\bot$, there can be no $(B, \prec)$ and $v : Prop \to B$ such that $v(\varphi) = 1$ for all $\varphi \in A$.

($\Rightarrow$) Suppose A is consistent. Then, by Lemma 3.2.7, we can extend it to a $\rightsquigarrow$-maximal consistent set S_A. Hence, we can consider the algebra $(B, \prec)$ constructed in Lemma 3.2.8, with the valuation $v : \varphi \mapsto [\varphi]$. Since this valuation satisfies all formulas in S_A, and since $A \subseteq S_A$, we have $v(\varphi) = [\varphi] = [\top] = 1_B$ for all $\varphi \in A$.

□

Remark 3.2.12. The logic $\mathcal{S}$ introduced in this chapter is a conservative extension of **CPC**. In fact, suppose $\mathcal{S}$ proves a theorem $\vdash \varphi$ where φ is a $\rightsquigarrow$-free formula, and hence a formula within the language of **CPC**. Then, given any Boolean algebra B, there are ways to define a binary relation $\prec$ on B so that $(B, \prec)$ is a contact algebra. For example, we can define $\prec := \leq$, or $\prec := \{(0, a)\}_{a \in B} \cup \{(a, 1)\}_{a \in B}$. Hence, by Theorem 3.2.10, we obtain that φ is valid on $(B, \prec)$, and hence on B. This shows that φ is valid on all Boolean algebras, and by completeness of **CPC** with respect to Boolean algebras we have that φ is also a theorem of **CPC**.

Being a conservative extension of **CPC**, which is an algebraizable logic, also $\mathcal{S}$ is algebraizable. Indeed, our proof of completeness is made by standard reasoning in algebraic logic.

3.2.3 Finite model property

The completeness result of this chapter can be adapted to show that this logic has the finite model property.

Theorem 3.2.13 (Finite model property)**.** *Let A be a consistent set of formulas in which all proposition letters which occur in a formula of A belong to a finite set $\{p_1, \ldots, p_m\}$. Then there exists an algebra $(B, \prec)$ of size at most 2^{2^m} and a valuation $v : \{p_1, \ldots, p_m\} \to B$ such that $v(\varphi) = 1$ for all $\varphi \in A$.*

Proof. We can restrict the set of formulas to the formulas which are built up from the proposition letters $p_1, \ldots, p_m$ only. Then, the proofs of Lemmas 3.2.7 and 3.2.8 transfer to this case as well. So, by Lemma 3.2.8 we obtain the contact algebra $(B, \prec)$ with the valuation $v : p_i \mapsto [p_i]$ which is such that $v(\varphi) = 1$ for all $\varphi \in A$.

This algebra has size at most 2^{2^m}. In fact, if we consider the free Boolean algebra C_m generated by m elements $a_1, \ldots, a_m$ (which has size 2^{2^m}), and we define a Boolean algebra homomorphism $f : C_m \to B$ by $f(a_i) := [p_i]$, we can show by induction on formulas that f is surjective, that is, for all φ there exists $a \in C_m$ such that $f(a) = [\varphi]$:

- $\varphi = \top$:
  We have $f(1) = [\top]$.

- $\varphi = p_i$:
  We have $f(a_i) = [p_i]$.

- $\varphi = \psi \wedge \chi$:
  By inductive hypothesis, there exist $a, b \in C_m$ such that $f(a) = [\psi]$ and $f(b) = [\chi]$. So we have $f(a \wedge b) = f(a) \wedge f(b) = [\psi] \wedge [\chi] = [\psi \wedge \chi]$.

- $\varphi = \neg\psi$:
  By inductive hypothesis, there exist $a \in C_m$ such that $f(a) = [\psi]$. So we have $f(\neg a) = \neg f(a) = \neg[\psi] = [\neg\psi]$.

- $\varphi = \psi \rightsquigarrow \chi$:
  By Lemma 3.2.8, we have either $[\psi \rightsquigarrow \chi] = [\top]$ or $[\psi \rightsquigarrow \chi] = [\bot]$. Hence, we have either $f(1) = [\psi \rightsquigarrow \chi]$ or $f(0) = [\psi \rightsquigarrow \chi]$.

$\square$

By Theorem 3.2.13 and by finite axiomatizability of contact algebras we can conclude the following:

Corollary 3.2.14. *The system $\mathcal{S}$ is decidable.*

Proof. Suppose we are given a formula φ, and we need to establish whether it is a theorem of $\mathcal{S}$ or not. Let n be the length of φ. Then, by Theorem 3.2.13, φ is a theorem of $\mathcal{S}$ if and only if it is valid in all contact algebras of size at most 2^{2^m}. There are finitely many relational algebras $(B, \prec)$, and one can check whether such an algebra is a contact algebra by checking that B is a Boolean algebra and that $\prec$ satisfies (Q1)-(Q6). So, it is possible to check whether φ is valid in all contact algebras of size at most 2^{2^m}, and thus whether φ is a theorem of $\mathcal{S}$ or not. $\square$

Conclusion

In this chapter, we have introduced a finitary deductive system $\mathcal{S}$, and we showed that it is strongly sound and complete with respect to the class of contact algebras. The proof uses standard methods from algebraic logic, that is we, show how to extend a consistent set of formulas to a maximal consistent one, and then we use the latter to define a congruence on the algebra of formulas, and finally obtaining a contact algebras with a valuation which satisfies all formulas in the maximal consistent set. While showing completeness, we noticed that the deduction theorem does not hold for the system $\mathcal{S}$, thus we proved a weaker version of it in Lemma 3.2.3. There, we see that formulas of the form $\top \rightsquigarrow \varphi$ act as deduction terms. Finally, we showed decidability of $\mathcal{S}$ via the finite model property.

Chapter 4

Adding non-standard rules

In the previous chapter, we presented a deductive system $\mathcal{S}$, and we showed that it is strongly sound and complete with respect to the class of contact algebras.

In this chapter, we will consider extensions of $\mathcal{S}$ with a particular kind of non-standard rules, and we will see that there is a correspondence between these rules and $\forall\exists$-statements. This correspondence is the reason why we call them Π_2-*rules*. Namely, we show that the system $\mathcal{S}$ enriched with Π_2-rules is sound and complete with respect to the class of contact algebras satisfying the $\forall\exists$-statements which correspond to the added rules. Vice versa, given any inductive class K [1] of contact algebras, we can find a system extending $\mathcal{S}$ which is sound and complete with respect to K. This system is obtained by considering a $\forall\exists$-axiomatisation of the theory of K, and adding to $\mathcal{S}$ the set of Π_2-rules which correspond to this axiomatisation.

Notation 4.0.1. Starting from this chapter, we will often regard algebras $(B, \prec)$ as first-order structures over the signature $(\wedge, \neg, 1, \rightsquigarrow)$. Notice that first-order terms over this signature are syntactically the same as formulas of the language of $\mathcal{S}$. When we put these objects inside first-order formulas, we consider them as first-order terms. Otherwise, they represent formulas of the language of $\mathcal{S}$.

In order to not confuse connectives $\wedge, \vee, \neg, \rightarrow$ interpreted by Boolean algebras with connectives of first-order formulas, for the latter we will use the symbols $\underline{\wedge}, \underline{\vee}, \underline{\neg}$ and $\underline{\rightarrow}$.

Moreover, for formulas in our language, we chose to replace the constants $\top, \bot$ with $1, 0$, so that we can regard the former ones as top and bottom of first-order logic.

Notation 4.0.2. In what follows, $\bar{\varphi}$ and $\bar{p}$ denote respectively a tuple of formulas and a tuple of propositional variables.

[1]Recall that a class of first-order structures is called inductive when it is closed under directed limits. By a well known preservation theorem (see e.g. [16, Theorem 5.2.6.]), an elementary class in inductive if and only if its theory is axiomatised by $\forall\exists$-statements.

Let us first introduce our Π_2-rules, and their associated $\forall\exists$-statements:

Definition 4.0.3 (Π_2-rule)**.** *A* Π_2-rule *is one of the form:*

$$(\rho) \quad \frac{F(\bar{\varphi}, \bar{p}) \to \chi}{G(\bar{\varphi}) \to \chi}$$

where χ is a formula variable, and F, G are formulas, each involving formula variables $\bar{\varphi}$, and with F involving a tuple $\bar{p}$ of proposition letters.

With the rule (ρ), we associate the first order formula Φ_ρ, defined as:

$$\Phi_\rho \ := \quad \forall \bar{x}, z \left(G(\bar{x}) \not\leq z \quad \underline{\to} \quad \exists \bar{y} : \ F(\bar{x}, \bar{y}) \not\leq z \right) .$$

In this chapter, we consider deductive systems obtained by adding Π_2-rules to the system $\mathcal{S}$, which we presented in Section 3.2.

As the following definition clarifies, in such a system one can derive formulas using a Π_2-rule provided some conditions on the special proposition letters $\bar{p}$ are satisfied:

Definition 4.0.4 (Proofs in a system with Π_2-rules)**.** *Let (ρ_1), ..., (ρ_k) be Π_2-rules, where*

$$(\rho_i) \quad \frac{F_i(\bar{\varphi}, \bar{p}) \to \chi}{G_i(\bar{\varphi}) \to \chi}$$

and consider the system $\mathcal{S} + (\rho_1) + \ldots + (\rho_k)$ [2]*.*

As in Definition 3.2.1, any finite list $\psi_1, \ldots, \psi_n$ of formulas is considered as a proof *of some entailment. Formulas ψ_i are regarded as assumptions unless satisfy one of the following conditions:*

- *ψ_i is an instance of an axiom of* **CPC***, or*
- *ψ_i is an instance of an axiom among* (A1)-(A7)*, or*
- *ψ_i follows from ψ_j, ψ_k for some $j, k < i$ by applying* (MP)*, or*
- *ψ_i follows from ψ_j for some $j < i$ by applying the rule* (R)*, or*
- *$\psi_i = G_l(\bar{\varphi}) \to \chi$ for some formulas $\bar{\varphi}, \chi$, and there is $j < i$ such that $\psi_j = F_l(\bar{\varphi}, \bar{p}) \to \chi$, and such that the proposition letters $\bar{p}$ occur neither in the formulas $\bar{\varphi}, \chi$, nor in any of the assumptions within $\psi_1, \ldots, \psi_j$ (in this case we say that ψ_i* follows from ψ_j by the rule (ρ_l)*).*

[2]Here we are assuming wlog that the lengths of the tuples $\bar{\varphi}$ are the same in all F_i's and G_i's

If Γ_0 is the set of the assumptions of a proof $\psi_1, \ldots, \psi_n$, we say that the latter is a proof of $\Gamma \vdash \psi_n$ *for each set of formulas Γ such that $\Gamma_0 \subseteq \Gamma$. In particular, if $\psi_1, \ldots, \psi_n$ contains no assumption, then it is a* proof of $\vdash \psi_n$, *or more simply a proof of ψ_n.*

Definition 4.0.5 (($\rho_1, \ldots, \rho_k$)-algebra)**.** *Let $(B, \prec)$ be a contact algebra. We say that $(B, \prec)$ is a $(\rho_1, \ldots, \rho_k)$-algebra if and only if it satisfies $\Phi_{\rho_1} \wedge \cdots \wedge \Phi_{\rho_k}$, where we see $(B, \prec)$ as a first-order structure over the signature $(\wedge, \neg, 1, \rightsquigarrow)$.*

4.1 From logics to inductive classes

In this section we show that if we add a set of Π_2-rules to the system $\mathcal{S}$, then we obtain a logic which is sound and complete with respect to some inductive class of contact algebras.

For simplicity, we show in detail only the case in which we add a finite set of such rules. That is, we work in the system $\mathcal{S} + (\rho_1) + \ldots + (\rho_k)$ where (ρ_1), ..., (ρ_k) are some fixed Π_2-rules, and we show that this system is sound and complete with respect to the class of $(\rho_1, \ldots, \rho_k)$-algebras. This proof of completeness is built on top of the proof of completeness given in Section 3.2. At the end of the section, we will explain how to extend the obtained results to handle the case in which we add a countably infinite set of Π_2-rules.

4.1.1 Soundness

We show that if $\psi_1, \ldots, \psi_n$ is a proof, then if a valuation in a $(\rho_1, \ldots, \rho_k)$-algebra satisfies all the assumptions, it must also satisfy ψ_n. We argue by induction on the length n of the proof, and by the soundness results of the previous sections, all we need to handle is the case in which ψ_n follows from some ψ_i by a rule (ρ_l).

So, suppose this is the case, and suppose $v : Prop \to B$ is a valuation in a $(\rho_1, \ldots, \rho_k)$-algebra $(B, \prec)$ such that $v(\psi_n) \neq 1$. Since ψ_n follows by the rule (ρ_l), it must be of the form $G_l(\varphi) \to \chi$, and there must be $i < n$ such that $\psi_i = F_l(\varphi, \bar{p}) \to \chi$, with $\bar{p}$ proposition letters occurring neither in $\bar{\varphi}, \chi$, nor in any of the assumptions within $\psi_1, \ldots, \psi_i$. So we have $v(G_l(\varphi)) = G_l(v(\varphi)) \not\leq v(\chi)$. Hence, since $(B, \prec)$ satisfies Φ_{ρ_l}, there exists $\bar{c} \in B$ such that $F_l(v(\varphi), \bar{c}) \not\leq v(\chi)$. So, if we consider the valuation $v' := v[\bar{p} \mapsto \bar{c}]$, we have that v' coincides with v on all assumptions within $\psi_1, \ldots, \psi_i$, and also $v'(F_l(\varphi, \bar{p})) = F_l(v(\varphi), \bar{c}) \not\leq v(\chi) = v'(\chi)$, so $v'(F_l(\varphi, \bar{p}) \to \chi) \neq 1$. Hence, by inductive hypothesis, there must be some assumption within $\psi_1, \ldots, \psi_i$ which is not satisfied by v', and hence it is not satisfied by v.

4.1.2 Completeness

We start by proving the $\rightsquigarrow$-deduction theorem for the system $\mathcal{S} + (\rho_1) + \ldots + (\rho_k)$, and we do so by extending the proof of Lemma 3.2.3:

Lemma 4.1.1 ($\rightsquigarrow$-deduction theorem)**.** *For any set* Γ *of formulas, and for any formulas* φ, ψ*, we have:*

$$\Gamma \cup \{\varphi\} \vdash \psi \quad \Leftrightarrow \quad \Gamma \vdash (\top \rightsquigarrow \varphi) \rightarrow \psi.$$

Proof. ($\Leftarrow$) For this direction, we can use the same proof as in Lemma 3.2.3.

($\Rightarrow$) In Lemma 3.2.3, we proved this direction by induction on proofs. Here, we extend the proof of Lindembaum Lemma 4.1.3 by dealing with the case in which a step of the proof consisted of applying the rule (ρ_l), for some $l \in \{1 \ldots k\}$:

- Suppose $\psi_i = G_l(\bar{\varphi}) \rightarrow \chi$ for some formulas $\bar{\varphi}, \chi$, and there is $j < i$ such that $\psi_j = F_l(\bar{\varphi}, \bar{p}) \rightarrow \chi$, and such that the proposition letters $\bar{p}$ occur neither in the formulas $\bar{\varphi}, \chi$, nor in any of the assumptions within $\psi_1, \ldots, \psi_j$. By inductive hypothesis, we have a proof of $\Gamma \vdash (\top \rightsquigarrow \varphi) \rightarrow (F_l(\bar{\varphi}, \bar{p}) \rightarrow \chi)$, and by **CPC** we have a proof of $\Gamma \vdash F_l(\bar{\varphi}, \bar{p}) \rightarrow \chi \vee \neg(\top \rightsquigarrow \varphi)$. Hence, by applying rule (ρ_l) we can extend this proof to a proof of $\Gamma \vdash G_l(\bar{\varphi}) \rightarrow \chi \vee \neg(\top \rightsquigarrow \varphi)$. So, again by **CPC**, we obtain a proof of $\Gamma \vdash (\top \rightsquigarrow \varphi) \rightarrow (G_l(\bar{\varphi}) \rightarrow \chi)$, as desired.

□

Corollary 4.1.2. *For any set* Γ *of formulas, and for any formula* φ*, we have:*

(i)	$\Gamma \cup \{\varphi\} \vdash \bot$	$\Leftrightarrow$	$\Gamma \vdash \neg(\top \rightsquigarrow \varphi)$;
(ii)	$\Gamma \vdash \varphi$	$\Leftrightarrow$	$\Gamma \cup \{\neg(\top \rightsquigarrow \varphi)\} \vdash \bot$;
(iii)	$\Gamma \vdash \neg(\varphi \rightsquigarrow \psi)$	$\Leftrightarrow$	$\Gamma \cup \{\varphi \rightsquigarrow \psi\} \vdash \bot$.

Proof. The proof is the same as the proof of Corollary 3.2.4, using Lemma 4.1.1 instead of Lemma 3.2.3. □

As in the proof of completeness in Section 3.2, we prove that we can extend any consistent set of formulas to a $\rightsquigarrow$-maximal consistent set, which we then use to build an algebra with a valuation which satisfies all formulas in the set.

Since we aim to show completeness of $\mathcal{S} + (\rho_1) + \ldots + (\rho_k)$ with respect to the class of $(\rho_1), \ldots, (\rho_k)$-algebras, we want this algebra to be a $(\rho_1), \ldots, (\rho_k)$-algebra.

Lemma 4.1.3 (Lindenbaum lemma). *Let A be a consistent set of formulas, and suppose there are infinitely many propositional variables not occurring in formulas in A. Then there exists a $\rightsquigarrow$-maximal consistent set S_A such that $\{\varphi \mid A \vdash \varphi\} \subseteq S_A$, and which satisfies the following property:*

(Mρ) For each $l \in \{1 \dots k\}$, S_A is closed under the following infinitary rule:

($\rho_{l,\infty}$) if $G_l(\bar{\varphi}) \to \chi \notin S_A$, then there exists a tuple $\bar{p}$ of proposition letters such that $F_l(\bar{\varphi}, \bar{p}) \to \chi \notin S_A$.

Proof. This proof is very similar to that of Lemma 3.2.7. We build an increasing sequence (A_n) of consistent sets, each having infinitely many propositional variables not occuring in it. Then we define $S_A := \{\varphi \mid \exists n : \ A_n \vdash \varphi\}$, and we show that such S_A is a $\rightsquigarrow$-maximal consistent set which satisfies property (Mρ).

Let $A_0 := A$. We enumerate all the pairs $P_n = (\varphi, \psi)$ of formulas, all the tuples $(\bar{\varphi}, \chi)_n$ of formulas, and we define A_n inductively in stages as follows (below, $l \in \{1 \dots k\}$):

- $n = (k+1)i$:
  If $A_n \vdash \varphi \rightsquigarrow \psi$, define $A_{n+1} := A_n$. Otherwise define $A_{n+1} := A_n \cup \{\neg(\varphi \rightsquigarrow \psi)\}$.

- $n = (k+1)i + l$:
  Let $\bar{\varphi}, \chi = (\bar{\varphi}, \chi)_i$.

  If $A_n \vdash G_l(\bar{\varphi}) \to \chi$, define $A_{n+1} := A_n$.

  If $A_n \nvdash G_l(\bar{\varphi}) \to \chi$, pick variables $\bar{p}$ which do not occur in A_n, in $\bar{\varphi}$ and in χ, and define $A_{n+1} := A_n \cup \Big\{\neg\Big(\top \rightsquigarrow (F_l(\bar{\varphi}, \bar{p}) \to \chi)\Big)\Big\}$.

Then, by induction on n, we show that each A_n is consistent. By our assumption, $A_0 := A$ is consistent and has infinitely many propositional variables not occuring in it. Suppose A_n is consistent and has infinitely many propositional variables not occuring in it. We have the following cases:

- $n = (k+1)i$:
  If we defined $A_{n+1} := A_n$, we can directly apply the inductive hypothesis. Otherwise, if we defined $A_{n+1} := A_n \cup \{\neg(\varphi \rightsquigarrow \psi)\}$, A_{n+1} still has infinitely many propositional variables not occurring in it, and the proof that it is consistent is the same as in Lemma 3.2.7.

- $n = (k+1)i + l$:
  If we defined $A_{n+1} := A_n$, we can directly apply the inductive hypothesis. Otherwise, if we defined $A_{n+1} := A_n \cup \Big\{\neg\Big(\top \rightsquigarrow (F_l(\bar{\varphi}, \bar{p}) \to \chi)\Big)\Big\}$, then we have $A_n \nvdash G_l(\bar{\varphi}) \to \chi$.

  A_{n+1} still has infinitely many variables not occurring in it. If it were inconsistent, then we would have $A_n \vdash \neg\Big(\top \rightsquigarrow \neg\Big(\top \rightsquigarrow (F_l(\bar{\varphi}, \bar{p}) \to \chi)\Big)\Big)$, and we could extend a proof of this as follows:

1. $\neg\Big(\top \rightsquigarrow \neg\Big(\top \rightsquigarrow (F_l(\bar{\varphi},\bar{p}) \to \chi)\Big)\Big)$
2. $\neg\Big(\top \rightsquigarrow \neg\Big(\top \rightsquigarrow (F_l(\bar{\varphi},\bar{p}) \to \chi)\Big)\Big) \;\to\; \neg\neg\Big(\top \rightsquigarrow (F_l(\bar{\varphi},\bar{p}) \to \chi)\Big)$ by **CPC** from an instance of (A6)
3. $\neg\neg\Big(\top \rightsquigarrow (F_l(\bar{\varphi},\bar{p}) \to \chi)\Big)$ by (MP) from 1. and 2.
4. $\top \rightsquigarrow (F_l(\bar{\varphi},\bar{p}) \to \chi)$ by **CPC** from 3.
5. $\Big(\top \rightsquigarrow (F_l(\bar{\varphi},\bar{p}) \to \chi)\Big) \to \Big(\top \to (F_l(\bar{\varphi},\bar{p}) \to \chi)\Big)$ is an instance of (A4)
6. $\top \to (F_l(\bar{\varphi},\bar{p}) \to \chi)$ by (MP) from 4. and 5.
7. $F_l(\bar{\varphi},\bar{p}) \to \chi$ by **CPC** from 6.
8. $G_l(\bar{\varphi}) \to \chi$ by (ρ_l) from 7.

so we have $A_n \vdash G_l(\bar{\varphi}) \to \chi$, which is a contradiction. Hence A_{n+1} must be consistent.

Hence, in all cases we obtain that A_{n+1} is consistent and has infinitely many propositional variables not occuring in it.

Now, we need to show that $S_A := \{\varphi \mid \exists n : A_n \vdash \varphi\}$ satisfies the required properties. The proof of the fact that it is a $\rightsquigarrow$-maximal consistent set would go exaclty as in the proof of Lemma 3.2.7.

Concerning property (Mρ), suppose $G_l(\bar{\varphi}) \to \chi \notin S_A$, and let $\bar{\varphi}, \chi = (\bar{\varphi}, \chi)_i$. Then, at stage $n = (k+1)i + l$ we must have defined $A_{n+1} := A_n \cup \Big\{\neg\Big(\top \rightsquigarrow (F_l(\bar{\varphi},\bar{p}) \to \chi)\Big)\Big\}$. So we have $A_{n+1} \vdash \neg\Big(\top \rightsquigarrow (F_l(\bar{\varphi},\bar{p}) \to \chi)\Big)$, and hence $S_A \vdash \neg\Big(\top \rightsquigarrow (F_l(\bar{\varphi},\bar{p}) \to \chi)\Big)$. If we would have $F_l(\bar{\varphi},\bar{p}) \to \chi \in S_A$, then we would get $S_A \vdash \bot$, which would contradict to the fact that S_A is consistent (by property (M1)), in fact:

1. $F_l(\bar{\varphi},\bar{p}) \to \chi$ because it belongs to S_A
2. $\top \rightsquigarrow (F_l(\bar{\varphi},\bar{p}) \to \chi)$ by (R) from 1.
3. $\neg\Big(\top \rightsquigarrow (F_l(\bar{\varphi},\bar{p}) \to \chi)\Big)$ because it belongs to S_A
4. $\bot$ by (MP) from 2. and 3.

So necessarily $F_l(\bar{\varphi},\bar{p}) \to \chi \notin S_A$, and this shows that S_A is closed under the infinitary rule $(\rho_{l,\infty})$. $\square$

Our proof of Lemma 4.1.3 is analogous to that of [3, Lemma 7.10]. Balbiani et al. use this lemma to show that each consistent set can be extended to one which they call *maximal NOR_∞-theory*, which is a notion corresponding to our $\rightsquigarrow$-maximal consistent sets satisfying (Mρ).

Lemma 4.1.4. *Let A be a consistent set of formulas. Then, by adding countably infinite new propositional variables to the language, we can use Lemma 4.1.3 to extend it to a $\rightsquigarrow$-consistent set S_A which satisfies property $(M\rho)$ and such that $\{\varphi \mid A \vdash \varphi\} \subseteq S_A$. Then the algebra $(B, \prec)$ constructed as in Lemma 3.2.8 with respect to S_A is a $(\rho_1, \dots, \rho_k)$-algebra.*

Proof. We need to show that the algebra $(B, \prec)$ constructed as in Lemma 3.2.8 satisfies Φ_l for each $l \in \{1 \dots k\}$.

Suppose $\overline{[\varphi]}, [\chi] \in B$ are such that $G_l(\overline{[\varphi]}) = [G_l(\bar{\varphi})] \not\leq [\chi]$. This means that $G_l(\bar{\varphi}) \to \chi \notin S_A$. Hence, since by property (Mρ) S_A is closed under the infinitary rule $(\rho_{l,\infty})$, there exist variables $\bar{p}$ such that $F_l(\bar{\varphi}, \bar{p}) \to \chi \notin S_A$. Hence we have found $\overline{[p]} \in B$ such that $F_l(\overline{[\varphi]}, \overline{[p]}) = [F_l(\bar{\varphi}, \bar{p})] \not\leq [\chi]$, and this proves that $(B, \prec)$ satisfies Φ_{ρ_l}. □

So we can prove the completeness result:

Theorem 4.1.5 (Strong completeness)**.** *Let K be the class of $(\rho_1, \dots, \rho_k)$-algebras, and let $\models$ be $\models_K$. Then for any set of formulas Γ and for any formula φ, we have*

$$\Gamma \vdash \varphi \quad \Leftrightarrow \quad \Gamma \models \varphi.$$

where $\vdash$ is relative to the system $\mathcal{S} + (\rho_1) + \dots + (\rho_k)$.

Proof. ($\Rightarrow$) This is proved in Section 4.1.1.

($\Leftarrow$) We prove the contrapositive. Suppose $\Gamma \nvdash \varphi$. Then by Corollary 4.1.2 we have that $A := \Gamma \cup \{\neg(\top \rightsquigarrow \varphi)\}$ is consistent. Hence, by Lemma 4.1.3, we can extend it to a $\rightsquigarrow$-maximal consistent set S_A satisfying (Mρ). So, we can consider the $(\rho_1, \dots, \rho_k)$-algebra $(B, \prec)$ constructed in Lemma 4.1.4, with the valuation $v : \psi \mapsto [\psi]$. Since this valuation satisfies all formulas in S_A, and since $A \subseteq S_A$, we have $v(\psi) = [\psi] = [\top] = 1_B$ for all $\psi \in A$.

This means that we have $v(\psi) = 1_B$ for all $\psi \in \Gamma$, and $v(\neg(\top \rightsquigarrow \varphi)) = 1_B$. By Remark 3.1.1, the latter is equivalent to $v(\varphi) \neq 1_B$. Hence what we have shown proves $\Gamma \not\models \varphi$.

□

Corollary 4.1.6 (Weak completeness)**.** *Given a formula φ, we have that φ is a theorem of the system $\mathcal{S} + (\rho_1) + \dots + (\rho_k)$ if and only if it is valid on all $(\rho_1, \dots, \rho_k)$-algebras $(B, \prec)$.*

As we did in Section 3.2.2 for the system $\mathcal{S}$, we prove also the alternative formulation of strong completeness:

Theorem 4.1.7 (Strong completeness, second formulation)**.** *A set A of formulas is consistent if and only if there exists a $(\rho_1, \dots, \rho_k)$-algebra $(B \prec)$ and a valuation v of formulas into B such that $v(\varphi) = 1$ for all $\varphi \in A$.*

Proof. ($\Leftarrow$) Suppose there exists a $(\rho_1, \dots, \rho_k)$-algebra $(B, \prec)$ and a valuation $v : Form \to B$ such that $v(\varphi) = 1$ for all $\varphi \in A$. Suppose for a contradiction that we have a proof of $A \vdash \bot$. Then, by the proof in the soundness section we have $v(\bot) = 1$, which is a contradiction. So there can be no proof of $A \vdash \bot$, hence A is consistent.

($\Rightarrow$) Suppose A is consistent. Then, possibly adding countable infinite new propositional variables to the language, we may assume there are infinitly many variables which do not occur in A. Hence, by Lemma 4.1.3, we can build a $\rightsquigarrow$-maximal consistent set S_A satisfying (Mρ). Then, by Lemma 4.1.4, we obtain a $(\rho_1, \dots, \rho_k)$-algebra $(B, \prec)$ which satisfies all formulas in S_A under the valuation $v : \varphi \mapsto [\varphi]$. So, since $A \subseteq S_A$, this valuation satisfies all formulas of A.

□

Remark 4.1.8. The above results would hold also if we have a countably infinite set $\{\rho_l\}_{l<\omega}$ of Π_2-rules. Apart from the Lindenbaum lemma, the rest of the proof goes in the same way. To prove the Lindenbaum lemma, we fix a bijection $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, and we need to replace stages $n = (k + 1)i$ with stages $n = \pi(0, i)$, and we need to treat the case relative to the rule ρ_l in stages $n = \pi(l + 1, i)$. [3]

4.2 From inductive classes to logics

Inductive classes are defined as elementary classes which are closed under unions of chains [4]. It is a famous preservation theorem (see e.g. [16, Theorem 5.2.6.]), also known as Chang-Łoś-Suszko theorem, that inductive classes are exactly those which can be axiomatized by statements of the form $\forall \bar{x} \exists \bar{y} \Phi(\bar{x}, \bar{y})$, where $\Phi(\bar{x}, \bar{y})$ is a quantifier-free formula.

(Q1)-(Q6) are all universal statements, which are particular cases of $\forall\exists$-statements. Moreover, given Π_2-rules $\rho_1, \dots, \rho_k$, we have that $\Phi_{\rho_1} \underline{\triangle} \cdots \underline{\triangle} \Phi_{\rho_k}$ is

[3] In case we are dealing with infinitely many rules, for each rule we should have a distinct enumeration of the tuples $\bar{\varphi}, \chi$, because the length of these tuples may be different depending on the rule, and this length may be not globally bounded.

[4] The condition of closure under unions of chains can be shown be equivalent to closure under directed limits.

expressible as a $\forall\exists$-statement. Therefore, as the class of $(\rho_1, \ldots, \rho_k)$-algebras is axiomatized by $\forall\exists$-statements, it is an inductive class.

In this section we will see that, for every inductive class K of contact algebras $(B, \prec)$, there exists a set of Π_2-rules which, if added to the system $\mathcal{S}$, gives us a sound and complete logic with respect to K.

To obtain such a set of Π_2-rules, we will see that every $\forall\exists$ statement is equivalent to a statement of the form Φ_ρ for some Π_2-rule ρ. Hence, if we consider the set of Π_2-rules which correspond to a $\forall\exists$-axiomatization of K, by what we have shown in the previous section we will obtain the desired completeness result.

We assume that all atomic formulas $\Phi(\bar{x}, \bar{y})$ are of the form $t(\bar{x}, \bar{y}) \approx 1$ for some term t. In fact, an arbitrary atomic formula $t(\bar{x}, \bar{y}) \approx s(\bar{x}, \bar{y})$ is equivalent to $(t(\bar{x}, \bar{y}) \leftrightarrow s(\bar{x}, \bar{y})) \approx 1$. Thus, each atomic formula can be seen as containing exactly one term.

As we mentioned in Notation 4.0.1, a terms in the signature $(\wedge, \neg, 1, \rightsquigarrow)$ can be regarded as formulas of our language, and viceversa, because they are syntactically the same.

Definition 4.2.1. *Given a quantifier-free first order formula $\Phi(\bar{x}, \bar{y})$, define the formula $\tilde{\Phi}(\bar{p}, \bar{q})$ in the language of $\mathcal{S}$ as follows:*

$$\begin{array}{rcl} (t(\bar{x}, \bar{y}) \approx 1)^{\sim} & := & 1 \rightsquigarrow t(\bar{p}, \bar{q}) \\ (\underline{\neg}\Psi(\bar{x}, \bar{y}))^{\sim} & := & \neg\tilde{\Psi}(\bar{p}, \bar{q}) \\ (\Psi_1(\bar{x}, \bar{y}) \mathbin{\underline{\wedge}} \Psi_2(\bar{x}, \bar{y}))^{\sim} & := & \tilde{\Psi}_1(\bar{p}, \bar{q}) \wedge \tilde{\Psi}_2(\bar{p}, \bar{q}) \end{array}$$

In the base case of the definition of $(-)^{\sim}$ given above, we translate an atomic formula containing the term t into the formula $t' := 1 \rightsquigarrow t$ associated to t as a formula of our language. As we pointed out in Remark 3.1.1, this association is such that $v(t') \in \{0, 1\}$ and $v(t) = 1 \Leftrightarrow v(t') = 1$ for all valuations v. In Lemmas 4.2.2 and 4.2.3 we show that the same properties hold for the translation $(-)^{\sim}$ of quantifier-free formulas into formulas of our language. Then we use this fact in Proposition 4.2.4, where we show that a statement $\forall\bar{x}\exists\bar{y}\Phi(\bar{x}, \bar{y})$ is equivalent to another $\forall\exists$ statement which we build using $\tilde{\Phi}$ as a term.

Lemma 4.2.2. *Let $(B, \prec)$ satisfy* (Q1)-(Q5). *Then for any quantifier-free formula $\Phi(\bar{x}, \bar{y})$, and for any $\bar{a}, \bar{b} \in B$, if $v : \bar{p}, \bar{q} \mapsto \bar{a}, \bar{b}$ we have $v(\tilde{\Phi}(\bar{p}, \bar{q})) \in \{0, 1\}$.*

Proof. Given $\bar{a}, \bar{b} \in B$ and a valuation $v : \bar{p}, \bar{q} \mapsto \bar{a}, \bar{b}$, we prove the statement by induction on Φ.

If $\Phi(\bar{x}, \bar{y}) = t(\bar{x}, \bar{y}) \approx 1$, then we have $v(\tilde{\Phi}(\bar{p}, \bar{q})) = 1 \rightsquigarrow t(\bar{a}, \bar{b}) \in \{0, 1\}$. If $\Phi(\bar{x}, \bar{y}) = \underline{\neg}\Psi(\bar{x}, \bar{y})$ or $\Phi(\bar{x}, \bar{y}) = \Psi_1(\bar{x}, \bar{y}) \mathbin{\underline{\wedge}} \Psi_2(\bar{x}, \bar{y})$ the same holds by inductive hypothesis. $\square$

Lemma 4.2.3. *Let* $(B, \prec)$ *satisfy* (Q1)-(Q5). *Then for any quantifier-free formula* $\Phi(\bar{x}, \bar{y})$*, and for any* $\bar{a}, \bar{b} \in B$*, if* $v : \bar{p}, \bar{q} \mapsto \bar{a}, \bar{b}$ *we have*

$$(B, \prec) \models \Phi(\bar{x}, \bar{y})[\bar{a}, \bar{b}] \qquad \Leftrightarrow \qquad v(\tilde{\Phi}(\bar{p}, \bar{q})) = \ 1.$$

Proof. The proof goes by induction on Φ:

- $\Phi(\bar{x}, \bar{y}) = t(\bar{x}, \bar{y}) \approx 1$:

$$\begin{aligned}(B, \prec) \models \Big(t(\bar{x}, \bar{y}) \approx 1\Big)[\bar{a}, \bar{b}] &\Leftrightarrow t(\bar{a}, \bar{b}) = 1 \\ &\Leftrightarrow 1 \prec t(\bar{a}, \bar{b}) \\ &\Leftrightarrow v(\tilde{\Phi}(\bar{p}, \bar{q})) = v(1 \rightsquigarrow t(\bar{p}, \bar{q})) = 1\end{aligned}$$

- $\Phi(\bar{x}, \bar{y}) = \underline{\neg}\Psi(\bar{x}, \bar{y})$:

$$\begin{aligned}(B, \prec) \models \underline{\neg}\Psi(\bar{x}, \bar{y})[\bar{a}, \bar{b}] &\Leftrightarrow (B, \prec) \not\models \Psi(\bar{x}, \bar{y})[\bar{a}, \bar{b}] \\ &\Leftrightarrow v(\tilde{\Psi}(\bar{p}, \bar{q})) \neq 1 \\ &\qquad \text{by inductive hypothesis} \\ &\Leftrightarrow v(\tilde{\Psi}(\bar{p}, \bar{q})) = 0 \\ &\qquad \text{by the previous lemma} \\ &\Leftrightarrow v(\tilde{\Phi}(\bar{p}, \bar{q})) = v(\neg\tilde{\Psi}(\bar{p}, \bar{q})) = \neg v(\tilde{\Psi}(\bar{p}, \bar{q})) = 1\end{aligned}$$

- $\Phi(\bar{x}, \bar{y}) = \Psi_1(\bar{x}, \bar{y}) \underline{\wedge} \Psi_2(\bar{x}, \bar{y})$:

$$(B, \prec) \models \Psi_1(\bar{x}, \bar{y})[\bar{a}, \bar{b}] \underline{\wedge} \Psi_2(\bar{x}, \bar{y})[\bar{a}, \bar{b}] \ \Leftrightarrow$$

$$(B, \prec) \models \Psi_1(\bar{x}, \bar{y})[\bar{a}, \bar{b}] \text{ and } (B, \prec) \models \Psi_2(\bar{x}, \bar{y})[\bar{a}, \bar{b}] \ \Leftrightarrow \quad \text{(by I.H.)}$$

$$v(\tilde{\Psi_1}(\bar{p}, \bar{q})) = 1 \text{ and } v(\tilde{\Psi_1}(\bar{p}, \bar{q})) = 1 \ \Leftrightarrow$$

$$v(\tilde{\Phi}(\bar{p}, \bar{q})) = v(\tilde{\Psi_1}(\bar{p}, \bar{q}) \wedge \tilde{\Psi_2}(\bar{p}, \bar{q})) = v(\tilde{\Psi_1}(\bar{p}, \bar{q})) \wedge v(\tilde{\Psi_2}(\bar{p}, \bar{q})) = 1.$$

□

Proposition 4.2.4. *Let* $(B, \prec)$ *satisfy* (Q1)-(Q5). *Then for any quantifier-free formula* $\Phi(\bar{x}, \bar{y})$ *we have* $(B, \prec) \models \forall\bar{x}\exists\bar{y}\Phi(\bar{x}, \bar{y})$ *if and only if* $(B, \prec) \models \forall\bar{x}, z\Big(1 \not\preceq z \underline{\rightarrow} \exists\bar{y} : \tilde{\Phi}(\bar{x}, \bar{y}) \not\preceq z\Big)$.

Proof. ($\Rightarrow$) Suppose $(B, \prec) \models \forall\bar{x}\exists\bar{y}\Phi(\bar{x}, \bar{y})$, and let $\bar{a}, c \in B$. By our assumption there exist $\bar{b} \in B$ such that $(B, \prec) \models \Phi(\bar{x}, \bar{y})[\bar{a}, \bar{b}]$. So, if $1 \not\preceq c$, by Lemma 4.2.3 we have $1 = \tilde{\Phi}(\bar{a}, \bar{b}) \not\preceq c$. This shows that $(B, \prec) \models \forall\bar{x}, z\Big(1 \not\preceq z \underline{\rightarrow} \exists\bar{y} : \tilde{\Phi}(\bar{x}, \bar{y}) \not\preceq z\Big)$.

(⇐) Suppose $(B, \prec) \models \forall \bar{x}, z\Big(1 \not\preceq z \rightarrow \exists \bar{y} : \tilde{\Phi}(\bar{x}, \bar{y}) \not\preceq z\Big)$, and let $\bar{a} \in B$. Since $1 \not\preceq 0$, there exist $\bar{b}$ such that $\tilde{\Phi}(\bar{a}, \bar{b}) \not\preceq 0$. So, by Lemma 4.2.2, we have $\tilde{\Phi}(\bar{a}, \bar{b}) = 1$, hence by Lemma 4.2.3 we have $(B, \prec) \models \Phi(\bar{x}, \bar{y})[\bar{a}, \bar{b}]$. This shows that $(B, \prec) \models \forall \bar{x} \exists \bar{y} \Phi(\bar{x}, \bar{y})$.

□

As a result of Proposition 4.2.4, given an arbitrary $\forall\exists$ statement $\forall \bar{x} \exists \bar{y} \Phi(\bar{x}, \bar{y})$, we have that it is equivalent to the $\forall\exists$ statement associated to the Π_2-rule

$$(\rho_\Phi) \qquad \frac{\tilde{\Phi}(\bar{\varphi}, \bar{p}) \to \chi}{\chi}$$

Thus, by the completeness result of Section 4.1.2, we obtain that the system $\mathcal{S} + (\rho_\Phi)$ is sound and complete with respect to the class of contact algebras $(B, \prec)$ satisfying $\forall \bar{x} \exists \bar{y} \Phi(\bar{x}, \bar{y})$.

More generally, we can conclude the following:

Corollary 4.2.5. *If T is a set of first-order statements of the form $\forall \bar{x} \exists \bar{y} \Phi(\bar{x}, \bar{y})$, then the system $\mathcal{S} + \{(\rho_\Phi) \mid \forall \bar{x} \exists \bar{y} \Phi(\bar{x}, \bar{y}) \in T\}$ is sound and complete with respect to the elementary class axiomatized by T and the theory of contact algebras.*

4.3 Admissible rules

We conclude this chapter with a section concerning admissibility of Π_2-rules, and in particular we give a semantic criterion[5] for establishing admissibility of a Π_2-rule in the system $\mathcal{S}$.

Definition 4.3.1. *Let (ρ) be a rule, either standard or possibly non-standard, such as a Π_2-rule. We say that (ρ) is* admissible *in $\mathcal{S}$ if whenever we can prove a theorem $\vdash \varphi$ in the system $\mathcal{S} + (\rho)$, we can prove $\vdash \varphi$ already in $\mathcal{S}$.*

In the rest of this paragraph, we will consider a generic Π_2-rule:

$$(\rho) \qquad \frac{F(\bar{\varphi}, \bar{p}) \to \chi}{G(\bar{\varphi}) \to \chi}$$

and we provide a criterion for establishing whether (ρ) is admissible or not in $\mathcal{S}$.

[5] This result is related to the work of Metcalfe [40], though it has been done independently. In [40] the author introduces a very general framework for admissibility via a model-theoretic approach.

Notation 4.3.2. We write $\bar{p} \ntrianglelefteq \varphi, \Gamma$ to mean that the proposition letters $\bar{p}$ do not occur in φ and do not occur in any $\psi \in \Gamma$.

Lemma 4.3.3. *(ρ) is admissible in $\mathcal{S}$ if and only if for any set of forulas Γ and for any tuple $\overline{\varphi}, \chi$ of formulas, if $\mathcal{S}$ proves $\Gamma \vdash F(\bar{\varphi}, \bar{p}) \to \chi$ with $\bar{p} \ntrianglelefteq \overline{\varphi}, \chi, \Gamma$ then $\mathcal{S}$ also proves $\Gamma \vdash G(\bar{\varphi}) \to \chi$.*

Proof. ($\Rightarrow$) Let (ρ) be admissible in $\mathcal{S}$. Suppose in $\mathcal{S}$ we have a proof $\psi_1, \ldots, \psi_n$, $F(\bar{\varphi}, \bar{p}) \to \chi$ of $\Gamma \vdash F(\bar{\varphi}, \bar{p}) \to \chi$ with $\bar{p} \ntrianglelefteq \overline{\varphi}, \chi, \Gamma$. Let $\psi_{i_1}, \ldots, \psi_{i_k} \in \Gamma$ be the assumptions in this proof. Then we have a proof of $\{\psi_{i_1}, \ldots, \psi_{i_k}\} \vdash F(\bar{\varphi}, \bar{p}) \to \chi$. Since our system includes **CPC**, having such a proof is equivalent to having a proof of $\{\psi\} \vdash F(\bar{\varphi}, \bar{p}) \to \chi$, where $\psi := \psi_{i_1} \wedge \cdots \wedge \psi_{i_k}$. So, by Lemma 3.2.3 there is a proof of $\vdash F(\bar{\varphi}, \bar{p}) \to \chi \vee \neg(\top \rightsquigarrow \psi)$ in $\mathcal{S}$. Since $\bar{p} \ntrianglelefteq \overline{\varphi}, \chi \vee \neg(\top \rightsquigarrow \psi)$ we have that $\mathcal{S} + (\rho)$ proves $\vdash G(\bar{\varphi}) \to \chi \vee \neg(\top \rightsquigarrow \psi)$, hence by admissibility of (ρ) also $\mathcal{S}$ does. So, again by Lemma 3.2.3, we have $\{\psi\} \vdash G(\bar{\varphi}) \to \chi$, which is equivalent to $\{\psi_{i_1}, \ldots, \psi_{i_k}\} \vdash G(\bar{\varphi}) \to \chi$, which gives us $\Gamma \vdash G(\bar{\varphi}) \to \chi$.

($\Leftarrow$) Let $\psi_1, \ldots, \psi_n = \psi$ be a proof of $\vdash \psi$ (hence with no assumptions) in $\mathcal{S} + (\rho)$. We show by induction on $i = 1 \ldots n$ that we can obtain a proof of ψ_i in $\mathcal{S}$, hence concluding that we can prove $\vdash \psi$ also in $\mathcal{S}$.

($i = 1$) ψ_1 is either an instance of an axiom of **CPC** or of one of the axioms (A1)-(A7), so also in $\mathcal{S}$ we have $\vdash \psi_1$.

($< i \Rightarrow i$) We have the following cases:

* If ψ_i is either an instance of an axiom of **CPC** or of one of the axioms (A1)-(A7), also in $\mathcal{S}$ we have $\vdash \psi_i$.
* If ψ_i follows by (MP) from ψ_j, ψ_k with $j, k < i$, by inductive hypothesis we have proofs of $\vdash \psi_j$ and of $\vdash \psi_k$ in $\mathcal{S}$, hence concatenating these proofs and applying (MP) to ψ_j and ψ_k we obtain a proof in $\mathcal{S}$ of $\vdash \psi_i$.
* If ψ_i following by (R) from ψ_j with $j < i$, by inductive hypothesis we have a proof of $\vdash \psi_j$ in $\mathcal{S}$, hence applying (R) to ψ_j in this proof gives as a proof of $\vdash \psi_i$ in $\mathcal{S}$.
* If $\psi_i = G(\overline{\varphi}) \to \chi$, and it follows from $\psi_j = F(\bar{\varphi}, \bar{p}) \to \chi$ with $j < i$ and $\bar{p} \ntrianglelefteq \overline{\varphi}, \chi$ by (ρ), then by inductive hypothesis there is a proof of $\vdash F(\bar{\varphi}, \bar{p}) \to \chi$ in $\mathcal{S}$. Since this can be seen as a proof of $\emptyset \vdash F(\bar{\varphi}, \bar{p}) \to \chi$, by assumption we have a proof of $\emptyset \vdash G(\overline{\varphi}) \to \chi$ in $\mathcal{S}$.

$\square$

Definition 4.3.4 (Atomic and elementary diagrams)**.** *With a structure $(B, \prec)$, we can associate the following sets of first-order formulas in the language of*

$(B, \prec)$ *extended with parameters* $\{a\}_{a\in B}$*:*

$$\begin{aligned}
\mathrm{Diag}_{\mathrm{at}}(B,\prec) &:= \{\Phi(\bar{a}) \mid (B,\prec) \models \Phi(\bar{x})[\bar{a}/\bar{x}] \text{ and } \Phi(\bar{x}) \text{ atomic } \}\cup \\
&\quad \cup \{\neg\Phi(\bar{a}) \mid (B,\prec) \models \neg\Phi(\bar{x})[\bar{a}/\bar{x}] \text{ and } \Phi(\bar{x}) \text{ atomic } \} \\
\mathrm{Diag}_{\mathrm{el}}(B,\prec) &:= \{\Phi(\bar{a}) \mid (B,\prec) \models \Phi(\bar{x})[\bar{a}/\bar{x}]\}
\end{aligned}$$

The first set is called the atomic diagram *of* $(B, \prec)$*, and the second one is called the* elementary diagram *of* $(B, \prec)$*.*

It is well known that a structure $(B, \prec)$ is a substructure (resp. elementary substructure) of another structure $(C, \prec)$ if and only if we can interpret the parameters $\{a\}_{a\in B}$ in $(C, \prec)$ so as to make it into a model of the atomic diagram (resp. elementary diagram) of $(B, \prec)$. See e.g. [39, Section 2.3].

Theorem 4.3.5 (Criterion for admissibility)**.** *(ρ) is admissible in $\mathcal{S}$ if and only if for any contact algebra $(B, \prec)$ there exists a (ρ)-algebra $(C, \prec)$ such that $(B, \prec)$ is a substructure of $(C, \prec)$.*

Proof. $(\Rightarrow)$ Suppose (ρ) is admissible in $\mathcal{S}$, and let $(B, \prec)$ be a contact algebra. We need to show that $(B, \prec)$ is a substructure of a (ρ)-algebra $(C, \prec)$.

Let $(B_0, \prec)$ be a countable elementary substructure of $(B, \prec)$.

Consider the set $\{p_a \mid a \in B_0\}$ of propositional letters for our formulas. Then we can consider the set of formulas:

$$A := \{\tilde{\Phi}(\overline{p_a}) \mid \Phi(\bar{a}) \in \mathrm{Diag}_{\mathrm{at}}(B_0, \prec)\},$$

where, as usual, we consider algebras $(B, \prec)$ as structures in the signature $(\wedge, \neg, 1, \rightsquigarrow)$, and where $\tilde{\Phi}$ is defined as in Definition 4.2.1.

Since A is satisfied in $(B_0, \prec)$ by the valuation $v : p_a \mapsto a$, by strong completeness A is a consistent set in $\mathcal{S}$.

So, if we add (countably) infinitely many new propositional letters to the language, by the Lindenbaum lemma (Lemma 4.1.3) we can extend A to a $\rightsquigarrow$-maximal consistent set S_A satisfying (Mρ) [6]. In fact, the proof of that lemma would hold in this case by admissibility of (ρ).

Let $(\mathcal{F}, \prec)$ be the algebra obtained by quotienting the algebra of formulas under the congruence given by S_A. As proved in Lemma 4.1.4, we have that $(\mathcal{F}, \prec)$ is a (ρ)-algebra. Moreover, since the interpretation $a^{\mathcal{F}} := [p_a]$ makes $(\mathcal{F}, \prec)$ into a model of $\mathrm{Diag}_{\mathrm{at}}(B_0, \prec)$, we have that $(B_0, \prec)$ is a substructure of $(\mathcal{F}, \prec)$, with the embedding $a \mapsto [p_a]$. Therefore, we have shown that the countable elementary substructure $(B_0, \prec)$ of $(B, \prec)$ can be embedded into a (ρ)-algebra $(\mathcal{F}, \prec)$.

Our aim is to show that we can embed $(B, \prec)$ itself into a (ρ)-algebra, and for doing this we prove the following:

[6] Recall that property (Mρ) is defined as closure under the infinitary version of the rule (ρ)

Claim 4.3.6. *There is an algebra $(C, \prec)$ such that $(\mathcal{F}, \prec)$ is an elementary substructure of $(C, \prec)$ and $(B, \prec)$ is a substructure of $(C, \prec)$.*

Proof of claim. Let T be the first-order theory of contact algebras (in terms of the signature $(\wedge, \neg, 1, \rightsquigarrow)$). A model $(C, \prec)$ of the theory

$$T \cup \mathrm{Diag}_{\mathrm{el}}(\mathcal{F}, \prec) \cup \mathrm{Diag}_{\mathrm{at}}(B, \prec)^{7}$$

would satisfy the conditions of the claim. Thus, it suffices to prove that this theory is consistent, and thus conclude that it has a model $(C, \prec)$.

Suppose for a contradiction that it is not consistent. Hence, by compactess, there exist $\bar{a} \in B_0, \bar{b} \in B \setminus B_0, \bar{c} \in \mathcal{F}$, a quantifier-free formula $\Psi(\bar{x}, \bar{y})$ and a formula $\Phi(\bar{x}, \bar{z})$ such that

$$(\mathcal{F}, \prec) \models \Phi(\bar{a}, \bar{c}) \tag{4.1}$$

$$(B, \prec) \models \Psi(\bar{a}, \bar{b}) \tag{4.2}$$

$$T \models \Phi(\bar{a}, \bar{c}) \underline{\rightarrow} \underline{\neg}\Psi(\bar{a}, \bar{b}). \tag{4.3}$$

Since the constants $\bar{a}, \bar{b}, \bar{c}$ do not occur in formulas in T, by 4.3 we have $T \models \exists \bar{z} \Phi(\bar{a}, \bar{z}) \underline{\rightarrow} \forall \bar{y} \underline{\neg} \Psi(\bar{a}, \bar{y})$.

So, since $(\mathcal{F}, \prec)$ is a model of T, by (4.1) we have $(\mathcal{F}, \prec) \models \forall \bar{y} \underline{\neg} \Psi(\bar{a}, \bar{y})$. Hence, as $\forall \bar{y} \underline{\neg} \Psi(\bar{a}, \bar{y})$ is a universal statement [8] and since $(B_0, \prec)$ is a substructure of $(\mathcal{F}, \prec)$, we obtain also $(B_0, \prec) \models \forall \bar{y} \underline{\neg} \Psi(\bar{a}, \bar{y})$.

But, since by (4.2) we have $(B, \prec) \models \exists \bar{y} \Psi(\bar{a}, \bar{y})$, and since $(B_0, \prec)$ is an elementary substructure of $(B, \prec)$, we also have $(B_0, \prec) \models \exists \bar{y} \Psi(\bar{a}, \bar{y})$, which is a contradiction. ☐

By the claim, we have that $(B, \prec)$ is a substructure of some (ρ)-algebra $(C, \prec)$. $(C, \prec)$ is a (ρ)-algebras because, since $(\mathcal{F}, \prec) \models \Phi_\rho$ and since $(\mathcal{F}, \prec)$ is an elementary substructure of $(C, \prec)$, it must be that also $(C, \prec) \models \Phi_\rho$. This concludes the proof of the direction $(\Rightarrow)$ of this theorem.

$(\Leftarrow)$ To show that (ρ) is admissible in $\mathcal{S}$, it suffices to show that whenever we have a proof of $\vdash F(\bar{\varphi}, \bar{p}) \to \chi$ in $\mathcal{S}$ with $\bar{p} \ntrianglelefteq \overline{\varphi}, \chi$, we also have a proof of $\vdash G(\bar{\varphi}) \to \chi$.

So suppose we have a proof of $\vdash F(\bar{\varphi}, \bar{p}) \to \chi$ in $\mathcal{S}$ with $\bar{p} \ntrianglelefteq \overline{\varphi}, \chi$. Let $(B, \prec)$ be any contact algebra. By our assumption, there exists a (ρ)-algebra $(C, \prec)$ such that $(B, \prec)$ is a substructure of $(C, \prec)$. Let $i : B \hookrightarrow C$ be the inclusion.

Let $q_1, \ldots, q_m$ be all the proposition letters occurring in $\overline{\varphi}, \chi$, and let $v : \mathit{Prop} \to B$ be any valuation. We can consider the valuation $v' :=$

[7] Where, in the diagrams, the constants coming from B_0 (under the respective embeddings) are regarded as the same.

[8] Here we are considering $(B_0, \prec), (\mathcal{F}, \prec), (B, \prec)$ as structures which interpret the constants $\bar{a}$ as themselves in B_0 and according to the embedding into $\mathcal{F}$ and B.

$i \circ v : Prop \to C$. Since $\mathcal{S}$ proves $\vdash F(\bar{\varphi}, \bar{p}) \to \chi$, for any $\bar{c} \in C$, if we define the valuation $v'' := (v')^{\bar{c}}_{\bar{p}}$, we have always $v'(F(\bar{\varphi}, \bar{p}) \to \chi) = 1_C$. This means that for all $\bar{c} \in C$ we have $F(\overline{v'(\varphi)}, \bar{c}) \leq v'(\chi)$, thus $(C, \prec) \models \forall \bar{y}\Big(F(\overline{v'(\varphi)}, \bar{y}) \leq v'(\chi)\Big)$, so since $(C, \prec) \models \Phi_\rho$ we have $(C, \prec) \models G(\overline{v'(\varphi)}) \leq v'(\chi)$. Hence, since $G(\overline{v'(\varphi)}) \leq v'(\chi)$ in C, we have $G(\overline{v(\varphi)}) \leq v(\chi)$ in B, that is $v(G(\overline{\varphi}) \to \chi) = 1_B$.

Since we have shown that given any algebra $(B, \prec)$ and any valuation $v : Prop \to B$ we have $v(G(\overline{\varphi}) \to \chi) = 1_B$, by Corollary 3.2.10 (weak completeness) we obtain that $\mathcal{S}$ proves $\vdash G(\overline{\varphi}) \to \chi$.

□

Given a class K of algebras, let $\mathbf{V}(K)$ be the variety generated by K. By Corollary 3.2.10, a rule (ρ) is admissible in $\mathcal{S}$ if and only if $\mathbf{V}(\mathrm{Mod}(T)) = \mathbf{V}(\mathrm{Mod}(T \cup \{\Phi_\rho\}))$, where T is the first-order theory of contact algebras.

Therefore, as a corollary of Theorem 4.3.5 we obtain the following proposition, of which we provide also a direct proof:

Proposition 4.3.7. $\mathbf{V}(\mathit{Mod}(T)) = \mathbf{V}(\mathit{Mod}(T \cup \{\Phi_\rho\}))$ *if and only if for all* $(B, \prec) \in \mathit{Mod}(T)$ *there exists* $(C, \prec) \in \mathit{Mod}(T + \Phi_\rho)$ *such that* $(B, \prec)$ *is a substructure of* $(C, \prec)$.

Proof. $(\Rightarrow)$ Suppose there exists $(B, \prec) \in \mathrm{Mod}(T)$ which cannot be embedded in any $(C, \prec) \in \mathrm{Mod}(T + \Phi_\rho)$. This means that $T \cup \{\Phi_\rho\} \cup \mathrm{Diag}_{\mathrm{at}}(B, \prec)$ does not have any model, hence it is inconsistent. By compactness, there exists a quantifier-free formula $\Psi(\bar{x})$ and a tuple $\bar{a}$ of elements of B such that

$$(B, \prec) \models \Psi(\bar{a}) \tag{4.4}$$

$$(C, \prec) \models \forall \bar{x} \neg \Psi(\bar{x}). \tag{4.5}$$

By (4.5), we have that $\neg\tilde{\Psi}(\bar{p})$ is a theorem of $\mathcal{S} + (\rho)$, hence by weak completeness of $\mathcal{S} + (\rho)$ with respect to $\mathrm{Mod}(T + \Phi_\rho)$ and by Lemma 4.2.3 we have that the equation $\neg\tilde{\Psi}(\bar{x}) \approx 1$ is satisfied by all algebras in $\mathrm{Mod}(T + \Phi_\rho)$. So since by (4.4) and by Lemma 4.2.3 this equation is not satisfied by $(B, \prec)$, we have $\mathbf{V}(\mathrm{Mod}(T)) \nsubseteq \mathbf{V}(\mathrm{Mod}(T + \Phi_\rho))$, hence in particular $\mathbf{V}(\mathrm{Mod}(T)) \neq \mathbf{V}(\mathrm{Mod}(T + \Phi_\rho))$.

$(\Leftarrow)$ Since $\mathrm{Mod}(T + \Phi_\rho) \subseteq \mathrm{Mod}(T)$, we have $\mathbf{V}(\mathrm{Mod}(T + \Phi_\rho)) \subseteq \mathbf{V}(\mathrm{Mod}(T))$.

Suppose for a contradiction that $\mathbf{V}(\mathrm{Mod}(T)) \nsubseteq \mathbf{V}(\mathrm{Mod}(T + \Phi_\rho))$. Therefore, as varieties are equational classes, there must exist an equation $t(\bar{x}) \approx 1$ such that $\mathrm{Mod}(T + \Phi_\rho) \models \forall \bar{x}(t(\bar{x}) \approx 1)$, but $\mathrm{Mod}(T) \not\models \forall \bar{x}(t(\bar{x}) \approx 1)$. So there exists an algebra $(B, \prec)$ in $\mathrm{Mod}(T)$ such that $(B, \prec) \models \exists \bar{x}(t(\bar{x}) \not\approx 1)$.

By our assumption, there exists an algebra $(C, \prec) \in \mathrm{Mod}(T + \Phi_\rho)$ such that $(B, \prec)$ is a substructure of $(C, \prec)$. Since we have $(C, \prec) \models \forall \bar{x}(t(\bar{x}) \approx 1)$, and since universal formulas are preserved under taking substructures, we also have $(B, \prec) \models \forall \bar{x}(t(\bar{x}) \approx 1)$, which is a contradiction.

□

Remark 4.3.8. Each class K which we are dealing with contain algebras in the signature $(\wedge, \neg, 1, \rightsquigarrow)$ which satisfy property (Q0′) stated right after Definition 2.1.1. Property (Q0′) expresses that $\rightsquigarrow$ is a map $B \times B \to \{0, 1\}$, which means that $\rightsquigarrow$ can be interchanged with a binary relation $\prec$ defined as $a \prec b \ \Leftrightarrow \ a \rightsquigarrow b = 1$. This allows us to refer to these algebras as $(B, \prec)$, with some abuse of notation. The same cannot be done with algebras belonging to the variety $\mathbf{V}(K)$, because in general they need not satisfy (Q0′). In fact, varieties are closed under products, but property (Q0′) is not preserved under products: if $a, b \in B$ are such that $a \rightsquigarrow b = 0$, then $(1, a), (1, b) \in B \times B$ are such that $(1, a) \rightsquigarrow (1, b) = (1 \rightsquigarrow 1, a \rightsquigarrow b) = (1, 0) \neq (0, 0), (1, 1)$. Thus, not every algebra in $\mathbf{V}(K)$ corresponds to some relational algebra of the form $(B, \prec)$.

Conclusion

In this chapter, we introduced Π_2-rules and their associated $\forall\exists$-statements, and we defined extensions of the system $\mathcal{S}$ obtained by adding Π_2-rules to it. First, by adapting the completeness proof of Chapter 3, we proved that such an extension is strongly sound and complete with respect to the class of contact algebras which satisfy the $\forall\exists$-statements associated to the added rules. Then, we showed that for any inductive class K of contact algebras there exists a deductive system which is strongly sound and complete with respect to K. This is possible thanks to formulas of the form $1 \rightsquigarrow \varphi$ [9], which allow us to encode all quantifier-free sentences into formulas of our language, and this allows us to prove that, up to equivalence, all $\forall\exists$-statements are associated to some Π_2-rule. Therefore, a sound and complete system with respect to K can be obtained by adding to $\mathcal{S}$ all the Π_2-rules which are associated to a $\forall\exists$-axiomatization of K. Finally, we discuss admissibility of rules, and we give a model-theoretic criterion for establishing whether a given Π_2-rule is admissible or not in $\mathcal{S}$. Also for proving this criterion, we needed to use the fact that we can encode all quantifier-free sentences into formulas of our language.

[9] Which in the previous chapter were denoted by $\top \rightsquigarrow \varphi$.

Chapter 5

Topological completeness via de Vries algebras

In this chapter, we add two particular rules ($\rho7$) and ($\rho8$) to the system $\mathcal{S}$, which correspond to properties (Q7) and (Q8) respectively. These properties, which are expressed by $\forall\exists$ statements, are satisfied by algebras which we defined as *compingent algebras*. By what we showed in the previous chapter, the system resulting from adding ($\rho7$) and ($\rho8$) to $\mathcal{S}$ is sound and complete with respect to the class of compingent algebras. Moreover, using the criterion of admissibility proved in Theorem 4.3.5, we show that both rules ($\rho7$) and ($\rho8$) are admissible in $\mathcal{S}$.

Then, using the fact that properties (Q7) an (Q8) are preserved under taking MacNeille completions we prove that this system is also complete with respect to the class of de Vries algebras. By this completeness result and by de Vries duality (see Section 2.2) we conclude that the system $\mathcal{S}$ enriched with the aforementioned rules is sound and complete with respect to the class of compact Hausdorff spaces. Moreover we argue that, whenever a rule or an axiom corresponds to a property which is preserved under MacNeille completions, such as (Q7) and (Q8), then adding these rules or axioms to $\mathcal{S}$ gives a system which is sound and complete with respect to a subclass of **KHaus**. As examples of this, we give a system which is complete with respect to Stone spaces and a system which is complete with respect to connected compact Hausdorff spaces.

We conclude this chapter by comparing our approach to that of Balbiani, Tinchev and Vakarelov [3], in which they provide completeness results very similar to ours, and which we used as inspiration for our proofs.

5.1 The logic of compingent algebras and de Vries algebras

In this chapter, we consider the following two Π_2-rules:

$$(\rho 7) \qquad \frac{(\varphi \rightsquigarrow p) \wedge (p \rightsquigarrow \psi) \to \chi}{(\varphi \rightsquigarrow \psi) \to \chi}$$

$$(\rho 8) \qquad \frac{p \wedge (p \rightsquigarrow \varphi) \to \chi}{\varphi \to \chi}$$

Proposition 5.1.1. *Let $(B, \prec)$ be a contact algebra*[1]*. Then we have:*

1. *$(B, \prec)$ satisfies* (Q7) *if and only if it satisfies $\Phi_{\rho 7}$, that is*

$$\forall a, b, d: \ \Big(a \rightsquigarrow b \not\leq d \underline{\to} \exists c: \ (a \rightsquigarrow c) \wedge (c \rightsquigarrow b) \not\leq d\Big).$$

2. *$(B, \prec)$ satisfies* (Q8) *if and only if it satisfies $\Phi_{\rho 8}$, that is*

$$\forall a, c: \ \Big(a \not\leq c \underline{\to} \exists b: \ b \wedge (b \rightsquigarrow a) \not\leq c\Big).$$

Proof.

1. ($\Rightarrow$) Suppose $(B, \prec)$ satisfies (Q7), and let a, b, d be such that $a \rightsquigarrow b \not\leq d$. This implies $d \neq 1$ and $a \rightsquigarrow b \neq 0$, so necessarily $a \rightsquigarrow b = 1$. This means that $a \prec b$, so by (Q7) there exists $c \in B$ such that $a \prec c \prec b$. So we have found c such that $a \rightsquigarrow c = c \rightsquigarrow b = 1$, and hence $1 = (a \rightsquigarrow c) \wedge (c \rightsquigarrow b) \not\leq d$.

   ($\Leftarrow$) Suppose $(B, \prec)$ satisfies $\Phi_{\rho 7}$, and let $a, b \in B$ be such that $a \prec b$. So we have $1 = (a \rightsquigarrow b) \not\leq 0$, hence by $\Phi_{\rho 7}$ there exists c such that $(a \rightsquigarrow c) \wedge (c \rightsquigarrow b) \not\leq 0$. This implies that $a \rightsquigarrow c = c \rightsquigarrow b = 1$, and so we have found $c \in B$ such that $a \prec c \prec b$.

2. ($\Rightarrow$) Suppose $(B, \prec)$ satisfies (Q8), and let $a, c \in B$ be such that $a \not\leq c$. Then we have $a \wedge \neg c \neq 0$, so by (Q8) there exists $b \neq 0$ such that $b \prec a \wedge \neg c$. Since $b \prec a \wedge \neg c \leq a, \neg c$, by (Q4) we have $b \prec a$ and $b \prec \neg c$. The former means that $b \prec a$, and by (Q5) the latter implies $b \leq \neg c$. So, since $b \neq 0$, we must have $b \not\leq c$, hence we have $b \wedge (b \rightsquigarrow a) = b \not\leq c$.

   ($\Leftarrow$) Suppose $(B, \prec)$ satisfies $\Phi_{\rho 8}$, and let $a \in B$ be such that $a \neq 0$. Then in particular $a \not\leq 0$, so by $\Phi_{\rho 8}$ there exists $b \in B$ such that $b \wedge (b \rightsquigarrow a) \not\leq 0$. This implies $(b \rightsquigarrow a) = 1$, that is $b \prec a$, and $b \neq 0$, so we have found the $b \in B$ we were looking for.

□

[1] For proving this proposition, it would suffice for $(B, \prec)$ to satisfy (Q1)-(Q5)

By Proposition 5.1.1, we have that a $(\rho 7, \rho 8)$-algebra is exactly the same as a compingent algebras (see Definition 2.1.4). Hence, by the results of the previous chapter, the system $\mathcal{S}+(\rho 7)+(\rho 8)$ is sound and complete with respect to compingent algebras.

Using the next lemma, we will show that this system is also sound and complete with respect to de Vries algebras. Recall that de Vries algebras are compingent algebras $(B, \prec)$ where B is a complete Boolean algebra.

Definition 5.1.2 (MacNeille completion of a compingent algebra)**.** *The* MacNeille completion *of a compingent algebra $(B, \prec)$ is the algebra $(\overline{B}, \prec)$ where $\overline{B}$ is the MacNeille completion*[2] *of B, and $\prec$ is defined as:*

$$\alpha \prec \beta \;\Leftrightarrow\; \text{there exist } a, b \in B \text{ such that } \alpha \leq a \prec b \leq \beta.$$

When we deal with MacNeille completions, Latin letters always denote elements of $\overline{B}$ which are in the range of the embedding $\eta : B \hookrightarrow \overline{B}$, while Greek letters denote generic elements of $\overline{B}$.

If we consider $(B, \prec)$ and $(\overline{B}, \prec)$ as algebras in the signature $(\wedge, \neg, 1, \rightsquigarrow)$, the extension of the operation $\rightsquigarrow$ to $\overline{B}$ can be described in terms of the *lower MacNeille extension* of an order preserving map:

Definition 5.1.3. *Let L, M be lattices, let $\overline{L}, \overline{M}$ be their respective MacNeille completions, and let $f : L \to M$ be an order preserving map. The* lower MacNeille extension *of f is the map $f^\circ : \overline{L} \to \overline{M}$ defined as follows:*

$$f^\circ(\alpha) \;:=\; \bigvee\{f(a) \mid L \ni a \leq \alpha\}.$$

If we let B' be the Boolean algebra obtained from B by reversing the order, then $\rightsquigarrow : B' \times B \to \{0, 1\}$ is an order preserving map. If we consider its lower MacNeille extension, we obtain a map $\rightsquigarrow^\circ : \overline{B'} \times \overline{B} \to \{0, 1\}$ [3] which coincides with the operation on $\overline{B}$ which replaces the relation $\prec$ defined as in Definition

[2] See e.g. [17, Chapter 7.] for a definition of the MacNeille completion of an ordered set.

[3] As proved in [53, Proposition 2.5], the MacNeille completion of a product of lattices is isomorphic to the product of the individual MacNeille completions of the lattices, thus the MacNeille completion of $B' \times B$ is isomorphic to $\overline{B'} \times \overline{B}$. Moreover, notice that $\overline{B'}$ coincides with $\overline{B}$ with the reversed order.

5.1.2. In fact, for all $\alpha, \beta \in \overline{B}$, we have:

$$\begin{aligned}
\alpha \rightsquigarrow^{\circ} \beta &= \bigvee\{a \rightsquigarrow b \mid (a,b) \in B' \times B \text{ and } (a,b) \leq (\alpha,\beta) \text{ in } \overline{B'} \times \overline{B}\} \\
&= \bigvee\{a \rightsquigarrow b \mid a,b \in B, \alpha \leq a \text{ and } b \leq \beta\} \\
&= \begin{cases} 1 & \text{if there exist } a,b \in B \text{ such that } \alpha \leq a, b \leq \beta \text{ and } a \rightsquigarrow b = 1 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if there exist } a,b \in B \text{ such that } \alpha \leq a \prec b \leq \beta \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \alpha \prec \beta \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Lemma 5.1.4. *Given a compingent algebra $(B, \prec)$, its MacNeille completion $(\overline{B}, \prec)$ is a de Vries algebra.*

Proof. Since the MacNeille completion $\overline{B}$ is a complete algebra, we have to show that $(\overline{B}, \prec)$ is compingent, that is that it satisfies (Q1)-(Q8).

(Q1) $0 \prec 0$ and $1 \prec 1$:
This is because $0 \leq 0 \prec 0 \leq 0$ and $1 \leq 1 \prec 1 \leq 1$.

(Q2) $\alpha \prec \beta, \delta$ implies $\alpha \prec \beta \wedge \delta$:
Suppose $\alpha \prec \beta, \delta$. So there exist a, a', b, c such that $\alpha \leq a \prec b \leq \beta$ and $\alpha \leq a' \prec c \leq \delta$. This gives us $a \wedge a' \leq a \prec b \leq b$ and $a \wedge a' \leq a' \prec c \leq c$, so by (Q4) we have $a \wedge a' \prec b, c$. Hence by (Q2) we have $a \wedge a' \prec b \wedge c$, so $\alpha \leq a \wedge a' \prec b \wedge c \leq \beta \wedge \delta$, so $\alpha \prec \beta \wedge \delta$.

(Q3) $\alpha, \beta \prec \delta$ implies $\alpha \vee \beta \prec \delta$:
This property becomes redundant once we prove (Q6). In fact, if both (Q2) and (Q6) hold, we have $\alpha, \beta \prec \delta$ implies $\neg\delta \prec \neg\alpha, \neg\beta$ by (Q6), and by (Q2) this gives us $\neg\delta \prec \neg\alpha \wedge \neg\beta$, and again by (Q6) we have $\alpha \vee \beta \prec \delta$.

(Q4) $\alpha \leq \beta \prec \delta \leq \gamma$ implies $\alpha \prec \gamma$:
Since $\beta \prec \delta$, there exist b, c such that $\beta \leq b \prec c \leq \delta$. So in particular we have $\alpha \leq b \prec c \leq \gamma$, and hence $\alpha \prec \gamma$.

(Q5) $\alpha \prec \beta$ implies $\alpha \leq \beta$:
If $\alpha \prec \beta$, there exist a, b such that $\alpha \leq a \prec b \leq \beta$. By (Q5) we have $a \leq b$, hence $\alpha \leq \beta$.

(Q6) $\alpha \prec \beta$ implies $\neg\beta \prec \neg\alpha$:
If $\alpha \prec \beta$, there exist a, b such that $\alpha \leq a \prec b \leq \beta$, so by (Q6) we have $\neg\beta \leq \neg b \prec \neg a \leq \neg\alpha$, and hence $\neg\beta \prec \neg\alpha$.

(Q7) $\alpha \prec \beta$ implies $\exists\delta:\ \alpha \prec \delta \prec \beta$:
If $\alpha \prec \beta$, there exist a, b such that $\alpha \leq a \prec b \leq \beta$. Since $a \prec b$, by (Q7) there exists c such that $a \prec c \prec b$. So we have $\alpha \leq a \prec c \leq c$ and $c \leq c \prec b \leq \beta$, hence $\alpha \prec c \prec \beta$.

(Q8) $\alpha \neq 0$ implies $\exists \beta \neq 0 : \beta \prec \alpha$:
If $\alpha \neq 0$, there exists $a \neq 0$ such that $a \leq \alpha$. By (Q8), there exists $b \neq 0$ such that $b \prec a$. So we have $b \leq b \prec a \leq \alpha$, and hence we have found $b \neq 0$ such that $b \prec \alpha$.

□

So we can conclude:

Theorem 5.1.5 (Strong completeness). *Let K be the class of de Vries algebras, and let $\models$ be $\models_K$. Then for any set of formulas Γ and for any formula φ, we have*

$$\Gamma \vdash \varphi \quad \Leftrightarrow \quad \Gamma \models \varphi.$$

where $\vdash$ is relative to the system $\mathcal{S} + (\rho_7) + (\rho_8)$.

Proof. ($\Rightarrow$) This direction follows by soundness of $\mathcal{S}+(\rho_7)+(\rho_8)$ with respect to the class of compingent algebras, which contains the class of de Vries algebras.

($\Leftarrow$) We show the contrapositive. Suppose Γ and φ are such that $\Gamma \nvdash \varphi$. Then, by Theorem 4.1.5, there exists a compingent algebra $(B, \prec)$ and a valuation $v : Prop \to B$ such that $v(\psi) = 1_B$ for all $\psi \in \Gamma$ and $v(\varphi) \neq 1_B$. Then, by Lemma 5.1.4, we have that the MacNeille completion $(\overline{B}, \prec)$ of $(B, \prec)$ is a de Vries algebra, and the valuation $\bar{v} = \eta \circ v : Var \to B \hookrightarrow \overline{B}$ is such that $\bar{v}(\psi) = \eta(v(\psi)) = 1_{\overline{B}}$ for all $\psi \in \Gamma$ and $\bar{v}(\varphi) = \eta(v(\varphi)) \neq 1_{\overline{B}}$. This shows that $\Gamma \not\models \varphi$

□

Corollary 5.1.6 (Weak completeness). *Let φ be a formula. Then φ is a theorem of $\mathcal{S} + (\rho 7) + (\rho 8)$ if and only if it is valid on all de Vries algebras $(B, \prec)$.*

Admissibility of ($\rho 7$)

In this subsection, we use the criterion for admissibility proved in Theorem 4.3.5 to show that the rule ($\rho 7$) is admissible in the system $\mathcal{S}$.

Lemma 5.1.7. *Let X be a set and let R be an equivalence relation on X. Then the algebra $(\mathcal{P}(X), \prec)$, where $U \prec V$ is defined*[4] *as $R[U] \subseteq V$, satisfies* (Q1)-(Q7).

[4]Note that we are defining $\prec$ on $\mathcal{P}(X)$ in the same way as we defined $\prec$ on $\mathbf{Clop}(X)$ in Section 2.1.2

Proof.

(Q1) $R[\emptyset] \subseteq \emptyset$ and $R[X] \subseteq X$.

(Q2) If $U \prec V, W$, that is if $R[U] \subseteq V, W$, then $R[U] \subseteq V \cap W$, that is $U \prec V \cap W$.

(Q3) If $U, V \prec W$, that is if $R[U], R[V] \subseteq W$, then $R[U \cup V] \subseteq W$, that is $U \cup V \prec W$.

(Q4) If $U \subseteq V \prec W \subseteq Z$, that is if $U \subseteq V$ and $R[V] \subseteq W \subseteq Z$, then $R[U] \subseteq R[V] \subseteq Z$, that is $U \prec Z$.

(Q5) By reflexivity of R, for all $U \subseteq X$ we have $U \subseteq R[U]$. So, if $U \prec V$, that is if $R[U] \subseteq V$, we have $U \subseteq V$.

(Q6) Let $U \prec V$, that is $R[U] \subseteq V$, and let $x, y \in X$ be such that $x \in X \setminus V$ and xRy. Then, by symmetry of R, we have yRx, so since $x \notin V$ we have $y \notin U$. So we have $R[X \setminus V] \subseteq X \setminus U$.

(Q7) Suppose $U \prec V$, that is $R[U] \subseteq V$, and let us define $W := R[U]$. If x, y are such that $x \in W$ and xRy, there exists $z \in U$ such that zRx, and by transitivity of R we have zRy. So $y \in V$. This shows that we have found W such that $R[U] \subseteq W$ and $R[W] \subseteq V$, that is $U \prec W \prec V$. $\square$

Proposition 5.1.8. *Every contact algebra* $(B_0, \prec)$ *can be embedded into a contact algebra* $(B, \prec)$ *satisfying* (Q7).

Proof. Let $(X, R) = (B_0, \prec)_+$. Let $Y := \{ \{x, y\} \mid x, y \in X, xRy\}$, and consider the set

$$X' := \{(x, \alpha) \in X \times Y \mid x \in \alpha\}$$

and the equivalence relation R' on X' defined as $(x, \alpha)R'(y, \beta) \Leftrightarrow \alpha = \beta$. If we consider the surjective map $f : X' \to X$ defined as $(x, \alpha) \mapsto x$, we obtain that $f^{-1} : (\mathbf{Clop}(X), \prec) \hookrightarrow (\mathcal{P}(X'), \prec')$ is an embedding of first-order structures. In fact, by Stone duality it is a Boolean algebra embedding, and moreover we have:

- If $U, V \in \mathbf{Clop}(X)$ are such that $U \prec V$, that is $R[U] \subseteq V$, then $f^{-1}(U) \prec' f^{-1}(V)$, that is $R'[f^{-1}(U)] \subseteq f^{-1}(V)$.

  In fact, let $(x, \alpha) \in f^{-1}(U)$, that is $x \in U$, and let (y, β) be such that $(x, \alpha)R'(y, \beta)$. Then we have $\alpha = \beta$, and by reflexivity and symmetry of R, we have xRy. So, we obtain that $y \in V$. Hence $(y, \beta) \in f^{-1}(V)$.

- If $U, V \in \mathbf{Clop}(X)$ are such that $U \not\prec V$, that is $R[U] \not\subseteq V$, then $f^{-1}(U) \not\prec' f^{-1}(V)$, that is $R'[f^{-1}(U)] \not\subseteq f^{-1}(V)$.

  In fact, let $x \in U$ and $y \notin V$ be such that xRy. Then, if we take $\alpha := \{x, y\}$, we obtain $(x, \alpha)R'(y, \alpha)$, with $(x, \alpha) \in f^{-1}(U)$ and $(y, \alpha) \notin f^{-1}(V)$.

Therefore, if we take $(B, \prec) := (\mathcal{P}(X'), \prec')$, by Lemma 5.1.7 we have found a contact algebra satisfying (Q7) into which we have embedded $(B_0, \prec)$. $\square$

By Proposition 5.1.8, by the fact that (Q7) is equivalent to $\Phi_{\rho 7}$, and by the criterion for admissibility, we can conclude the following:

Corollary 5.1.9. *$(\rho 7)$ is admissible in $\mathcal{S}$.*

Remark 5.1.10. The idea of the construction of the structure (X', R') in the proof of Proposition 5.1.8 comes from the proof of [3, Lemma 2.5].

Admissibility of $(\rho 8)$

In this subsection, we use the criterion for admissibility proved in Theorem 4.3.5 to show that the rule $(\rho 8)$ is admissible in the system $\mathcal{S}$.

Proposition 5.1.11. *Every contact algebra $(B_0, \prec)$ can be embedded into a contact algebra $(B, \prec)$ satisfying* (Q8).

Proof. Starting from $(B_0, \prec)$, we inductively build a chain of structures and embeddings $(B_0, \prec) \hookrightarrow (B_1, \prec) \hookrightarrow (B_2, \prec) \hookrightarrow (B_3, \prec) \hookrightarrow \cdots$ of contact algebras, which will be such that the structure $(B, \prec)$, where $B := \bigcup_{n \in \omega} B_n$ [5], satisfies (Q8), and hence the proposition would be proved.

Suppose we have $(B_n, \prec)$. Define $(B_{n+1}, \prec)$ as:

$$\begin{aligned} B_{n+1} &:= B_n \times B_n \quad \text{(product of Boolean algebras)} \\ \forall a_1, a_2, b_1, b_2 \in B_n : \quad (a_1, a_2) \prec (b_1, b_2) &\Leftrightarrow a_1 \prec b_1 \text{ and } a_2 \leq b_2. \end{aligned}$$

That is, we define $(B_{n+1}, \prec) := (B_n, \prec) \times (B_n, \leq)$ as a product of contact algebras.

We have that $(B_{n+1}, \prec)$ defined in this way is a contact algebra:

(Q1) Since $(B_n, \prec)$ satisfies (Q1), we have $0 \prec 0$ and $1 \prec 1$, hence $(0, 0) \prec (0, 0)$ and $(1, 1) \prec (1, 1)$.

(Q2) Suppose $(a_1, a_2) \prec (b_1, b_2), (c_1, c_2)$. Then we have $a_1 \prec b_1, c_1$ and $a_2 \leq b_2, c_2$, so $a_2 \leq b_2 \wedge c_2$ and since $(B_n, \prec)$ satisfies (Q2) also $a_1 \prec b_1 \wedge c_1$, so we have $(a_1, a_2) \prec (b_1 \wedge c_1, b_2 \wedge c_2) = (b_1, b_2) \wedge (c_1, c_2)$.

(Q3) Suppose $(a_1, a_2), (b_1, b_2) \prec (c_1, c_2)$. Then we have $a_1, b_1 \prec c_1$ and $a_2, b_2 \leq c_2$, so $a_2 \vee b_2 \leq c_2$ and since $(B_n, \prec)$ satisfies (Q3) also $a_1 \vee b_1 \prec c_1$, so we have $(a_1, a_2) \vee (b_1, b_2) = (a_1 \vee b_1, a_2 \vee b_2) \prec (c_1, c_2)$.

[5] Here we are identifying B_n with its image in B_{n+1} under the embedding

(Q4) Suppose $(a_1, a_2) \leq (b_1, b_2) \prec (c_1, c_2) \leq (d_1, d_2)$. Then we have $a_1 \leq b_1 \prec c_1 \leq d_1$ and $a_2 \leq b_2 \leq c_2 \leq d_2$, so $a_2 \leq d_2$ and since $(B_n, \prec)$ satisfies (Q4) also $a_1 \prec d_1$, so we have $(a_1, a_2) \prec (d_1, d_2)$.

(Q5) Suppose $(a_1, a_2) \prec (b_1, b_2)$, that is $a_1 \prec b_1$ and $a_2 \leq b_2$. Since $(B_n, \prec)$ satisfies (Q5), we have also $a_1 \leq b_1$, hence $(a_1, a_2) \leq (b_1, b_2)$.

(Q6) Suppose $(a_1, a_2) \prec (b_1, b_2)$, that is $a_1 \prec b_1$ and $a_2 \leq b_2$. So we have $\neg b_2 \leq \neg a_2$, and since $(B_n, \prec)$ satisfies (Q6) we also have $\neg b_1 \prec \neg a_1$. Hence we have $\neg(b_1, b_2) = (\neg b_1, \neg b_2) \prec (\neg a_1, \neg a_2) = \neg(a_1, a_2)$.

Moreover, we have that $(B_n, \prec)$ is a subalgebra of $(B_{n+1}, \prec)$, under the embedding $a \mapsto (a, a)$.

Now, we need to show that the union $(B, \prec)$ of the chain $\{(B_n, \prec)\}_{n \leq \omega}$ satisfies (Q8).

Let $0 \neq a \in B$. Then, there exists n such that $a \in B_n$. In B, such an element a is the same as the element $(a, a) \in B_{n+1}$. Then, if we take $b := (0, a) \in B_{n+1}$, we have that $b \neq 0$ and $b \prec (a, a)$, because $0 \prec a$ and $a \leq a$. This shows that $(B, \prec)$ satisfies (Q8). $\square$

By Proposition 5.1.11, by the fact that (Q8) is equivalent to $\Phi_{\rho 8}$, and by the criterion for admissibility, we can conclude the following:

Corollary 5.1.12. *$(\rho 8)$ is admissible in $\mathcal{S}$.*

In conclusion, by our proofs of admissibility of the rules $(\rho 7)$ and $(\rho 8)$ in $\mathcal{S}$, we have shown that any contact algebra $(B, \prec)$ can be embedded into a compingent algebra. Therefore, by embedding the latter into its MacNeille completion, we can conclude that any contact algebra $(B, \prec)$ can be embedded into a de Vries algebra.

5.2 Topological completeness

5.2.1 Compact Hausdorff spaces

We can use compact Hausdorff spaces as semantics for our formulas:

Definition 5.2.1. *A* topological model *is a pair (X, v) where X is a compact Hausdorff space, and $v : Prop \to RO(X)$ is a valuation. Then the valuation is extended to all formulas, as usual, into the algebra $(RO(X), \prec)$ (which is defined below Definition 2.2.10).*

By de Vries duality[6] and by Theorem 5.1.5 we obtain one of the main results of this thesis:

[6]See Section 2.2

Corollary 5.2.2. *The system $\mathcal{S} + (\rho 7) + (\rho 8)$ is strongly sound and complete with respect to compact Hausdorff spaces.*

Proof. By de Vries duality, any de Vries algebra $(B, \prec)$ is isomorphic to one of the form $(RO(X), \prec)$ for some compact Hausdorff space X. Therefore, a valuation $v : Prop \to B$ into a de Vries algebra $(B, \prec)$ can be seen as a topological model (X, v). Vice versa, a topological model (X, v) gives us a valuation v into the de Vries algebra $(RO(X), \prec)$. In this correspondence, the formulas which are satisfied by v are the same, thus semantics with respect to topological models is equivalent to semantics with respect to de Vries algebras. Thus, since by Theorem 5.1.5 the system $\mathcal{S} + (\rho 7) + (\rho 8)$ is strongly sound and complete with respect to de Vries algebras, we have that it is also strongly complete with respect to topological models, that is compact Hausdorff spaces. □

The dual of a de Vries algebra $(A, \prec)$ is given by the space X of its ends, with the topology which has as basis the set $\{U_a \mid a \in A\}$, where $U_a := \{x \in X \mid a \in x\}$.

If we start with a compingent algebra $(B, \prec)$ and we do the same construction, the set $\{U_a \mid a \in B\}$ would still form a basis for a compact Hausdorff topology on X. Moreover, we have that this is the same space which is dual to the MacNeille completion $(\overline{B}, \prec)$ of $(B, \prec)$:

Lemma 5.2.3. *Let $(B, \prec)$ be a compingent algebra, and let $(\overline{B}, \prec)$ be its MacNeille completion (see Definition 5.1.2).*

Then, if X_B is the compact Hausdorff space of ends of $(B, \prec)$ and $X_{\overline{B}}$ is the de Vries dual of $(\overline{B}, \prec)$, we have $X_B \cong X_{\overline{B}}$. That is, the two spaces are homeomorphic.

Proof. Consider the following maps:

$$\begin{aligned} f : X_{\overline{B}} &\to X_B \\ x &\mapsto x \cap B \end{aligned}$$

and

$$\begin{aligned} g : X_B &\to X_{\overline{B}} \\ x &\mapsto {\uparrow} x \end{aligned}$$

We have:

- f is well defined:
  We need to show that if x is an end of $(\overline{B}, \prec)$, then $x \cap B$ is an end of $(B, \prec)$.

- $a, b \in x \cap B \;\Rightarrow\; \exists\, c \in x \cap B \setminus \{0\}$ such that $c \prec a, b$:
  Let $a, b \in x \cap B$. Then, since $a, b \in x$ and x is an end of $(\overline{B}, \prec)$, there exists $0 \neq \gamma \in x$ such that $\gamma \prec a, b$. This means that there exist $c, d \in B$ such that $\gamma \leq c \prec a$ and $\gamma \leq d \prec b$. So we have $0 \neq \gamma \leq c \wedge d \in x$. Since B is a subalgebra of $\overline{B}$, we also have $c \wedge d \in B$, hence $c \wedge d \in x \cap B$, and by $c \wedge d \leq c \prec a$ and $c \wedge d \leq d \prec b$ we obtain respectively $c \wedge d \prec a$ and $c \wedge d \prec b$, as desired.
- $a \prec b \;\Rightarrow\; \neg a \in x \cap B$ or $b \in x \cap B$:
  Suppose $a \prec b$. Then, since also $a \prec b$ in $(\overline{B}, \prec)$, and since x is an end, we have either $\neg a \in x$ or $b \in x$, that is either $\neg a \in x \cap B$ or $b \in x \cap B$.

- g is well defined:
  We need to show that if x is an end of $(B, \prec)$, then $\uparrow x$ is an end of $(\overline{B}, \prec)$.
  - $\alpha, \beta \in \uparrow x \;\Rightarrow\; \exists\, \gamma \in \uparrow x \setminus \{0\}$ such that $\gamma \prec \alpha, \beta$:
    Let $\alpha, \beta \in \uparrow x$. This means there exist $a, b \in x$ such that $a \leq \alpha$ and $b \leq \beta$. Since x is an end of $(B, \prec)$, there exists $0 \neq c \in x \subseteq \uparrow x$ such that $c \prec a, b$. So by $c \prec a \leq \alpha$ and $c \prec b \leq \beta$ we obtain $c \prec \alpha, \beta$.
  - $\alpha \prec \beta \;\Rightarrow\; \neg\alpha \in \uparrow x$ or $\beta \in \uparrow x$:
    Suppose $\alpha \prec \beta$, which means there exist $a, b \in B$ such that $\alpha \leq a \prec b \leq \beta$. By $a \prec b$, since x is an end of B we have either $x \ni \neg a \leq \neg\alpha$ or $x \ni b \leq \beta$, so either $\neg\alpha \in \uparrow x$ or $\beta \in \uparrow x$.

- $f \circ g = \mathrm{id}_{X_B}$:
  In fact for all ends x of $(B, \prec)$ we have $x = \uparrow_{\overline{B}} x \cap B$.

- $g \circ f = \mathrm{id}_{X_{\overline{B}}}$:
  We need to show that for all ends x of $(\overline{B}, \prec)$ we have $x = \uparrow(x \cap B)$.
  (⊇) Since $x \cap B \subseteq x$, we have $\uparrow(x \cap B) \subseteq \uparrow x = x$.
  (⊆) If $\alpha \in x$, since x is an end of $(\overline{B}, \prec)$ there exists $\beta \in x$ such that $\beta \prec \alpha$. This means there exist $a, b \in B$ such that $\beta \leq b \prec a \leq \alpha$. So, since $\beta \leq b$ we have $b \in x \cap B$, and since $b \prec c \leq \alpha$ we have $b \leq \alpha$, hence $\alpha \in \uparrow(x \cap B)$.

- f is continuous:
  Let $\{U_a \mid a \in B\}$ and $\{V_a \mid a \in \overline{B}\}$ be the respective basis of X_B and $X_{X_{\overline{B}}}$. It suffices to show that for each U_a from the basis of X_B we have that $f^{-1}(U_a)$ is open. Indeed, given $x \in X_{\overline{B}}$ and $a \in B$, we have $a \in x \cap B$ if and only if $a \in x$, hence $f^{-1}(U_a) = V_a$, which is open.

So $f : X_{\overline{B}} \to X_B$ is a bijective continuous function. Since a continuous function from a compact space to a Hausdorff space is a homeomorphism if and only if it is bijective[7], we can conclude that $X_{\overline{B}}$ and X_B are homeomorphic.

□

[7] See, e.g., [37].

Remark 5.2.4. The completeness result of this section can be seen as a result about compact Hausdorff metrizable spaces. Soundness would follow just because the class of compact Hausdorff metrizable spaces is a subclass of compact Hausdorff spaces. Concerning completeness, note that in the proof of completeness with respect to compingent algebras, the algebra we build is countable. This means that the corresponding space (which by Lemma 5.2.3 is the dual of the MacNeille completion of the algebra) has a countable basis for its topology, which means that it is second countable. Therefore, since a compact Hausdorff space is metrizable if and only if it is second countable, we have also completeness with respect to metrizable compact Hausdorff spaces.

In the proof of completeness of $\mathcal{S} + (\rho 7) + (\rho 8)$ with respect to de Vries algebras, we have used the fact that properties (Q1)-(Q8), when valid on some $(B, \prec)$, are carried on to its MacNeille completion $(\overline{B}, \prec)$ (see Definition 5.1.2). That is we used the fact that, according to the following definition, being a compingent algebra is a property of which is preserved under taking MacNeille completions:

Definition 5.2.5 (MacNeille canonical properties). *A property*[8] P *is* MacNeille canonical*, or* preserved under MacNeille completion *if whenever a compingent algebra* $(B, \prec)$ *satisfies it, also its MacNeille completion* $(\overline{B}, \prec)$ *does.*

By de Vries duality, a de Vries algebra is isomorphic to the algebra of regular open subsets of its dual compact Hausdorff space. Hence, a first-order statement in the language of Boolean algebras plus a binary relation, when satisfied by a de Vries algebra, can be regarded as expressing a property which is satisfied by the regular open subsets of its dual space.

This fact can be used to obtain deductive systems complete with respect to particular classes of topological spaces. More precisely, if rules $\rho_1, \dots, \rho_k$ are such that their respective first-order statements $\Phi_{\rho_1}, \dots, \Phi_{\rho_k}$ are MacNeille canonical (thus so is their conjunction), then the logic $\mathcal{S} + (\rho 7) + (\rho 8) + (\rho_1) + \cdots + (\rho_k)$ is sound and complete with respect to compact Hausdorff spaces satisfying the property $\Phi_{\rho_1} \mathbin{\underline{\wedge}} \cdots \mathbin{\underline{\wedge}} \Phi_{\rho_k}$. In fact, by the results of the previous chapter this logic is complete with respect to $(\rho_1, \dots, \rho_k)$-algebras, and in the same way as we proved completeness of $\mathcal{S} + (\rho 7) + (\rho 8)$ with respect to de Vries algebras, using the fact that $\Phi_{\rho_1} \mathbin{\underline{\wedge}} \cdots \mathbin{\underline{\wedge}} \Phi_{\rho_k}$ is MacNeille canonical we obtain that $\mathcal{S} + (\rho 7) + (\rho 8) + (\rho_1) + \cdots + (\rho_k)$ is complete with respect to de Vries algebras satisfying $\Phi_{\rho_1} \mathbin{\underline{\wedge}} \cdots \mathbin{\underline{\wedge}} \Phi_{\rho_k}$.

By de Vries duality, this can be seen as a completeness result with respect to compact Hausdorff spaces which satisfy the topological property expressed by $\Phi_{\rho_1} \mathbin{\underline{\wedge}} \cdots \mathbin{\underline{\wedge}} \Phi_{\rho_k}$.

Other than extensions with Π_2-rules, one can also consider axiomatic extensions. Formulas in our language naturally corresponds to a universal first-order statement. If the statement which correspond to a formula φ is MacNeille

[8]Here we mean a property which can be satisfied by an algebra of the form $(B, \prec)$.

canonical, then we have that $\mathcal{S}+(\rho 7)+(\rho 8)+(\varphi)$ is complete with respect to compact Hausdorff spaces of which regular open subsets satisfy that statement.

In conclusion, we have:

Corollary 5.2.6. *If the formulas $\varphi_1,\ldots,\varphi_n$ and the Π_2-rules $(\rho_1),\ldots,(\rho_k)$ correspond to MacNeille canonical statements, then the logic $\mathcal{S}+(\rho 7)+(\rho 8)+(\varphi_1)+\cdots+(\varphi_n)+(\rho_1)+\cdots+(\rho_k)$ is complete with respect to a subclass of compact Hausdorff spaces.*

5.2.2 Example: Stone spaces

Consider the following property:

(Q9) $a \prec b$ implies $\exists c : a \prec c \prec b$ and $c \prec c$.

Subordinations[9] satisfying (Q9) are introduced in [5], and are called *lattice subordinations.*

Proposition 5.2.7. *An algebra*[10] $(B,\prec)$ *satisfies* (Q9) *if and only if it satisfies*

$$\forall a,b,d: \Big(a \rightsquigarrow b \nleq d \mathrel{\underline{\to}} \exists c:\ (a \rightsquigarrow c)\wedge(c \rightsquigarrow b)\wedge(c \rightsquigarrow c) \nleq d\Big)$$

Proof. $(\Rightarrow)$ Suppose $a \rightsquigarrow b \nleq d$. Then $d \neq 1$ and $a \rightsquigarrow b \neq 0$, so $a \rightsquigarrow b = 1$. Hence $a \prec b$, so there exists c such that $a \prec c \prec b$ and $c \prec c$, that is $1 = (a \rightsquigarrow c)\wedge(c \rightsquigarrow b)\wedge(c \rightsquigarrow c) \nleq d$.

$(\Leftarrow)$ Suppose $a \prec b$. Then $1 = a \rightsquigarrow b \nleq 0$, hence there exists c such that $(a \rightsquigarrow c)\wedge(c \rightsquigarrow b)\wedge(c \rightsquigarrow c) \nleq 0$, which implies $(a \rightsquigarrow c)\wedge(c \rightsquigarrow b)\wedge(c \rightsquigarrow c) = 1$, which means $a \prec c \prec b$ and $c \prec c$.

$\square$

Therefore, if we consider the Π_2-rule

$$(\rho 9) \qquad \frac{(\varphi \rightsquigarrow p)\wedge(p \rightsquigarrow \psi)\wedge(p \rightsquigarrow p) \to \chi}{(\varphi \rightsquigarrow \psi) \to \chi}$$

we have that $\mathcal{S}+(\rho 7)+(\rho 8)+(\rho 9)$ is sound and complete with respect to compingent algebras satisfying (Q9), because in Proposition 5.2.7 we have shown that (Q9) is equivalent to $\Phi_{\rho 9}$.

[9]See Definition 2.1.1

[10]Here we mean any Boolean algebra with a binary relation $\prec$, which as usual is replaced with the operation $\rightsquigarrow$ defined as

$$a \rightsquigarrow b := \begin{cases} 1 & \text{if } a \prec b \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 5.2.8. (Q9) *is MacNeille canonical.*

Proof. Let $(B, \prec)$ be a compingent algebra satisfying (Q9), and let $(\overline{B}, \prec)$ be its MacNeille completion. Let $\alpha, \beta \in \overline{B}$ be such that $\alpha \prec \beta$. This means that there exist $a, b \in B$ such that $\alpha \leq a \prec b \leq \beta$, so by (Q9) there exists $c \in B \subseteq \overline{B}$ such that $a \prec c \prec b$ and $c \prec c$, and by (Q4) this c is also such that $\alpha \prec c \prec \beta$. This shows that $(\overline{B}, \prec)$ satisfies (Q9). □

So we have that $\mathcal{S} + (\rho 7) + (\rho 8) + (\rho 9)$ is also complete with respect to de Vries algebras satisfying (Q9). These are exactly those such that their dual space is a Stone space:

Proposition 5.2.9. *A de Vries algebra* $(B, \prec)$ *satisfies* (Q9) *if and only if its dual space* X *is a Stone space.*

Proof. (⇒) Let $x \neq y$ be points of X, which are maximal round filters. Since they are different, there exists $b \in B$ such that $b \in x$ and $b \notin y$. Since x is a round filter, there must exist $a \in x$ such that $a \prec b$. So, by (Q9), there exists c such that $a \prec c \prec b$ and $c \prec c$. By $a \prec c$ and $a \in x$ we obtain $c \in x$, so $x \in U_c$, which is a clopen subset of X. By $c \prec b$ and $b \notin y$ we obtain $c \notin y$, hence $y \notin U_c$. Thus we have found a clopen subset U_c which separates x and y, and this shows that X is totally disconnected. Since X is also compact, it is a Stone space.

(⇐) If X is a Stone space, then it is zero-dimensional, that is it has a basis of clopen subsets. By de Vries duality, an element $c \in B$ is such that $c \prec c$ if and only if $\mathbf{Cl}(U_c) \subseteq U_c$, that is if and only if $\mathbf{Cl}(U_c) = U_c$, which means that U_c is clopen. Hence, the clopen subsets of a compact Hausdorff space X are exactly the regular open subsets U_c such that $c \prec c$. Thus, by zero-dimensionality, each open subset of X is a union of a subfamily of $\{U_c \mid c \prec c\}$. This holds in particular for every regular open subset.

Let $a \prec b$. By de Vries duality, this means that $\mathbf{Cl}(U_a) \subseteq U_b$. Let $U_b = \bigcup_{i \in I} U_{c_i}$ where $c_i \prec c_i$ for all $i \in I$. Since $\mathbf{Cl}(U_a)$ is compact, and since the clopens $\{U_{c_i}\}_{i \in I}$ cover it, there are finitely many $c_1, \ldots, c_n$ such that $\mathbf{Cl}(U_a) \subseteq U_{c_1} \cup \cdots \cup U_{c_n}$. Since the U_{c_i}'s are clopen subsets, also $U_{c_1} \cup \cdots \cup U_{c_n}$ is clopen, and so

$$U_{c_1} \cup \cdots \cup U_{c_n} = \mathbf{Int}(\mathbf{Cl}(U_{c_1} \cup \cdots \cup U_{c_n})) = U_{c_1 \vee \cdots \vee c_n}$$

Hence, if $c := c_1 \vee \cdots \vee c_n$, we have $\mathbf{Cl}(U_a) \subseteq U_c$ and $\mathbf{Cl}(U_c) = U_c \subseteq U_b$, that is $a \prec c \prec b$, and $c \prec c$ because U_c is clopen.

This shows that $(B, \prec)$ satisfies (Q9)

□

Hence, we obtained the following:

Corollary 5.2.10. *The system $\mathcal{S} + (\rho7) + (\rho8) + (\rho9)$ is sound and complete with respect to Stone spaces.*

5.2.3 Example: Connectedness

Consider the following axiom schema:

(C) $(\varphi \rightsquigarrow \varphi) \to (\top \rightsquigarrow \varphi) \vee (\top \rightsquigarrow \neg\varphi)$.

We have that the logic $\mathcal{S} + (\rho7) + (\rho8) +$ (C) is sound and complete with respect to compingent algebras which validate the formula $(p \rightsquigarrow p) \to (\top \rightsquigarrow p) \vee (\top \rightsquigarrow \neg p)$.

Proposition 5.2.11. *Validating the formula $(p \rightsquigarrow p) \to (\top \rightsquigarrow p) \vee (\top \rightsquigarrow \neg p)$ is a MacNeille canonical property.*

Proof. We show that if $(\overline{B}, \prec)$ does not validate $(p \rightsquigarrow p) \to (\top \rightsquigarrow p) \vee (\top \rightsquigarrow \neg p)$, then also $(B, \prec)$ does not.

Let $\alpha \in \overline{B}$ be such that the valuation $v : p \mapsto \alpha$ does not satisfy $(p \rightsquigarrow p) \to (\top \rightsquigarrow p) \vee (\top \rightsquigarrow \neg p)$. This means that $1 \not\prec \alpha, 1 \not\prec \neg\alpha$ and $\alpha \prec \alpha$. So $\alpha \neq 0, 1$, and there exist $a, b \in B$ such that $\alpha \leq a \prec b \leq \alpha$. By (Q5), we have $a = \alpha$. Thus, with the valuation $v : p \mapsto a \in B$ we have $v((p \rightsquigarrow p) \to (\top \rightsquigarrow p) \vee (\top \rightsquigarrow \neg p)) = 0$. Hence $(B, \prec)$ does not validate $(p \rightsquigarrow p) \to (\top \rightsquigarrow p) \vee (\top \rightsquigarrow \neg p)$. □

Proposition 5.2.12. *A de Vries algebra $(B, \prec)$ validates* (C) *if and only if its dual space X is connected.*

Proof. A valuation $v : p \mapsto c \in B$ satisfies $(p \rightsquigarrow p) \to (\top \rightsquigarrow p) \vee (\top \rightsquigarrow \neg p)$ unless c is such that $c \prec c$ and $c \neq 0, 1$. So, $(B, \prec) \cong (RO(X), \prec)$ validates $(p \rightsquigarrow p) \to (\top \rightsquigarrow p) \vee (\top \rightsquigarrow \neg p)$ if and only if there exist no U_c such that $U_c \neq X, \emptyset$ and $\mathbf{Cl}(U_c) \nsubseteq U_c$, that is if and only if X has no clopen subset different from $X, \emptyset$, that is if and only if X is connected. □

Hence, we obtained the following:

Corollary 5.2.13. *The system $\mathcal{S} + (\rho7) + (\rho8) +$ (C) is sound and complete with respect to connected compact Hausdorff spaces.*

5.3 Related work

The completeness results of this thesis are strongly inspired by the work of Balbiani, Tinchev and Vakarelov [3], where they provide propositional logics for reasoning about regions in region-based theories of space, which they call *Region-Based Propositional Modal Logics of Space* (RPMLS).

The language $\mathbf{L}(\mathbf{C}, \leq)$ of RPMLS is defined as follows: complex formulas are built from atomic ones using standard propositional connectives, where atomic formulas are those of the form $a\mathbf{C}b$ and $a \leq b$ where a, b are Boolean terms, and Boolean terms are built from Boolean variables using Boolean operations. Boolean terms are meant to be interpreted as regions, and formulas $a\mathbf{C}b$ and $a \leq b$ mean that "regions a and b are in contact" and "region a is contained in region b" respectively.

Such a construction of formulas, using terms and atomic formulas expressing relations between terms, makes the language $\mathbf{L}(\mathbf{C}, \leq)$ resemble a first-order language without quantifiers. Though, as the authors point out in [3], they decided to call this language *modal* because the tools and techniques which they apply to it are typical of more ordinary modal languages. Indeed, the kind of semantics they mostly focus on is a Kripke-type semantics, which is based on some classes of Kripke frames (W, R), where W is a set and R a binary relation. They regard these frames as *adjacency spaces*. Here, regions are interpreted as subsets of W, and they use the relation R to interpret the contact relation $\mathbf{C}$ between regions: a and b are in contact if $R[a] \cap b$ is non-empty, that is if there is $x \in a$ and $y \in b$ such that xRy. The authors investigate modal definability and undefinability of classes of such frames, where they define and use a notion of p-morphism, and they show completeness of the minimal logic $\mathbb{L}_{\min}$ for their language with respect to these frames using a canonical model construction. Also, using the method of filtration, they show that axiomatic extensions of $\mathbb{L}_{\min}$ have the finite model property. Moreover, as we will discuss more in detail in next chapter, in [2] Balbiani and Kikot provide Sahlqvist correspondence and canonicity results for $\mathbf{L}(\mathbf{C}, \leq)$ with respect to the Kripke-type semantics. Hence, the modal nature of their approach is evident.

In [3], Balbiani et al. introduce propositional logics which correspond to some systems which are related to RCC, the *Region Connection Calculus* introduced in [45]. For doing this, they extend their minimal logic $\mathbb{L}_{\min}$ using axioms and rules expressed in the language $\mathbf{L}(\mathbf{C}, \leq)$, which are based on the first-order reformulation of RCC given in [50]. This reformulation is based on contact algebras $(B, \mathbf{C})$ [11], and it contains:

- a set of axioms which, once translated in terms of the corresponding subordination $\prec_{\mathbf{C}}$, are equivalent to our axioms (Q1)-(Q6).

  The logic presented in [3] which corresponds to this set of axioms, is called PWRCC. Our logic $\mathcal{S}$ can be regarded as the analogous of PWRCC in

[11] Where $\mathbf{C}$ is a proximity relation rather than a subordination (see Chapter 2).

our language, which is simpler than $\mathbf{L}(\mathbf{C}, \leq)$ yet equally expressive.

- an axiom which is meant to express connectedness.

  This axiom is analogous to the axiom schema (C) which we introduced in Section 5.2.3.

- two axioms, which Balbiani et al. call *normality* and *extensionality*, that are equivalently expressible by $\forall\exists$-statements. These, in terms of $\prec_{\mathbf{C}}$, correspond to properties (Q7) and (Q8).

To mimic the normality and extensionality axioms in the language $\mathbf{L}(\mathbf{C}, \leq)$, in [3, Section 7] Balbiani et al. illustrate how to use non-standard rules such as (NOR) and (EXT) to their system PWRCC. They do so by proving completeness of PWRCC + (NOR) as an example. As they show completeness of PWRCC via canonical model construction, then they show how to modify the proof in presence of the rule (NOR). They need to modify the notion of maximal consistent set which they use to make the canonical model construction, thus they prove a different Lindenbaum lemma for showing that every consistent set can be extended to a maximal consistent one. The maximal consist sets which they work with are those which are closed with respect to the infinitary version (NOR_∞) of (NOR). This is very similar to what we did in Section 4.1.2, where we adapt our more general proof of completeness for extensions of $\mathcal{S}$ with sets of arbitrary Π_2-rules. In fact, in Lemma 4.1.3 we show that consistent sets can be extended to $\rightsquigarrow$-maximal consistent sets which are closed under the infinitary version of the rules which we added. For the rest, our proof of completeness is different from that of Balbiani et al., as we work with algebraic semantics rather than relational semantics.

The Π_2-rules ($\rho 7$) and ($\rho 8$) which we considered in this chapter are the analogous of rules (NOR) and (EXT). In Theorem 4.3.5 we give an admissibility criterion for Π_2-rules, and we apply it to show that rules ($\rho 7$) and ($\rho 8$) are admissible in $\mathcal{S}$. Similarly, [3, Section 6] the authors prove that rules (NOR) and (EXT) are admissible in the system PWRCC, though they use different techniques and our admissibility criterion is not related to their work.

In Secton 5.2.1 we define compact Hausdorff spaces semantics, and we prove our completeness result via de Vries duality and MacNeille completions. Also Balbiani et al., in [3, Section 9], consider an equivalent topological semantics, though their approach is different. There they make a different proof for showing completeness of the list of their propositional logics[12] with respect to topological semantics in the respective class of topological spaces. Similarly, in Sections 5.2.2 and 5.2.3, we give logics sound and complete with respect to the class of Stone spaces and connected compact Hausdorff spaces, respectively. We provide these as examples of our more general investigation of logics for classes of topological spaces, which is unrelated to the approach of Balbiani et al.. In fact, our proof of completeness has lead us to the notion and in-

[12]Those related to RCC which they presented.

vestigation of MacNeille canonical axioms and rules, which we have shown be complete with respect to classes of compact Hausdorff spaces.

Conclusion

In this chapter, we specified two Π_2-rules (ρ7) and (ρ8) of which associated $\forall\exists$ statements are equivalent to (Q7) and (Q8), respectively. Hence, by the completeness result of the previous chapter, we obtained that the system $\mathcal{S} + (\rho 7) + (\rho 8)$ is sound and complete with respect to the class of compingent algebras. Moreover, using the criterion of admissibility proved in Section 4.3, we showed that rules $(\rho 7) + (\rho 8)$ are admissible in $\mathcal{S}$.

Then we defined and considered MacNeille completions of compingent algebras, and so we derived completeness of $\mathcal{S} + (\rho 7) + (\rho 8)$ with respect to the class of de Vries algebras. Using this, de Vries duality, and defining topological semantics, we could conclude topological completeness with respect to the class of compact Hausdorff spaces. We also defined the notion of *MacNeille canonical* property, and we explained how this notion can be used to obtain completeness results with respect to classes of topological spaces. As example, we provide a logic for Stone spaces and a logic for connected compact Hausdorff spaces.

In the last section, we made an overview of the work of Balbiani et al. [3], and a comparison with the work done in this thesis.

Chapter 6

Sahlqvist correspondence

In the previous chapters, we have been mainly focusing on the algebraic semantics of our language, and in Chapter 5 we introduced topological semantics for compact Hausdorff spaces.

In this chapter, we consider a third kind of semantics. Namely, we will interpret our formulas subordination spaces (see Definition 2.1.5). Those are pairs (X, R) which, according to the duality described in Section 2.1, are dual to algebras $(B, \prec)$, where B is a Boolean algebra and $\prec$ a subordination. The aim of this chapter is to provide a version of the Sahlqvist correspondence theorem with respect to the semantics of subordination spaces.

More precisely, in Section 6.1 we establish a fragment of our language whose formulas φ have the following property: φ is valid on a subordination space (X, R) if and only if (X, R) satisfies a specific first-order formula which is effectively computable from φ.

In Section 6.2, we identify particular $\forall\exists$-statements in the signature of our algebras $(B, \prec)$ which are satisfied by an algebra $(B, \prec)$ if and only if its dual $(X, R) := (B, \prec)_+$ satisfies some other first-order statement, which is again effectively computable from the starting one.

The work of this chapter has been inspired by two sources. The first one is Lemma 2.1.12, which states that each of the conditions (Q5),(Q6),(Q7) is satisfied by an algebra $(B, \prec)$ if and only if its dual (X, R) satisfies some respective first-order condition. This result encouraged us to find further correspondences between conditions on algebras and elementary conditions on the dual subordination spaces. The second source of inspiration has been the work by Balbiani and Kikot [2], where the authors develop a Sahlqvist-like theory in the setting of region-based propositional modal logics of space (RPMLS)[1]. Here, we mimic their approach, and we adapt it to the case of our formulas, thus extending the result of Lemma 2.1.12 with further correspondences.

Notation 6.0.1 ($\diamond$ and $\Box$)**.** For the scope of this chapter, it is convenient to replace the connective $\rightsquigarrow$ with the connective $\diamond$, which is definable from $\rightsquigarrow$ by $\varphi \diamond \psi := \neg(\varphi \rightsquigarrow \psi)$. As we have seen at the beginning of Chapter 2, in algebras

[1] We briefly introduced the language of RPMLS in Section 5.3.

$(B, \prec)$ the connective $\diamond$ is interpreted by the function $\diamond : B \times B \to \{0,1\}$ which replaces the proximity relation $\delta_\prec$ associated to the subordination $\prec$.

Moreover, we introduce the connective $\square$, defined as $\varphi \square \psi := \neg(\neg\varphi \diamond \neg\psi)$.

Remark 6.0.2. As we noticed above, in this chapter we consider Boolean algebras with subordinations $(B, \prec)$ as $\diamond$-expanded Boolean algebras (see e.g. [63]). Then, their dual subordination spaces can be seen as obtained via the ultrafilter frame construction (see e.g. [63, 11]).

This is not evident on the first sight. In fact, in the ultrafilter frame construction, to an n-ary operation $f : B^n \to B$ corresponds an $(n+1)$-ary relation $R_f \subseteq X^{n+1}$, defined as:

$$R_f x x_1, \dots, x_n \;\Leftrightarrow\; f(a_1, \dots, a_n) \in x \;\text{ for all } a_i \in x_i, i = 1, \dots, n.$$

Instead, we make the binary operation $\diamond$ correspond to a binary relation R on $X = Ult(B)$. Despite this, what we do can be regarded as the same as the above construction, because our binary relation R completely describes the ternary relation $R_\diamond$. In fact, for all $x, y, z \in X$, we have:

$$\begin{aligned} R_\diamond zxy \;&\Leftrightarrow\; a \diamond b \in z \;\text{ for all } a \in x, b \in y \\ &\Leftrightarrow\; a \diamond b = 1 \;\text{ for all } a \in x, b \in y \\ &\Leftrightarrow xRy. \end{aligned}$$

That is, our binary relation R is virtually a ternary relation which, given $x, y \in X$, either relates (z, x, y) for all $z \in X$ or it does not relate (z, x, y) for any $z \in X$.

Here we define the semantics of our formulas with respect to subordination spaces (X, R).

Definition 6.0.3. *Given a subordination space (X, R), we call a* valuation *any map $V : Prop \to \mathcal{P}(X)$. In particular, we call* finite valuation *a map $V : Prop \to \mathbf{Fin}(X)$, and we call* clopen valuation *a map $V : Prop \to \mathbf{Clop}(X)$.*

A valuation $V : Prop \to \mathcal{P}(X)$ can then be extended to all formulas in the following way:

$$\begin{aligned} V(1) \;&:=\; X \\ V(\varphi \wedge \psi) \;&:=\; V(\varphi) \cap V(\psi) \\ V(\neg\varphi) \;&:=\; X \setminus V(\varphi) \\ V(\varphi \diamond \psi) \;&:=\; \begin{cases} X & \text{if } R[V(\varphi)] \cap V(\psi) \neq \emptyset \\ & \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

Then, given a formula φ, we write $(X, R, V) \Vdash \varphi$ if and only if $V(\varphi) = X$. Note that the interpretation of the connective $\square$ results in

$$V(\varphi \square \psi) := \begin{cases} X & \text{if } R[X \setminus V(\varphi)] \cap (X \setminus V(\psi)) = \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

It is easy to show that, if V is a clopen valuation, then for each formula φ we have that $V(\varphi)$ is a clopen subset of X.

Indeed, clopen valuations can be seen as valuations on the dual algebra $(B, \prec) := (X, R)^+ = (\mathbf{Clop}(X), \prec)$, and vice versa a valuation $v : Prop \to B$ into an algebra $(B, \prec)$ can be seen as a clopen valuation in its dual $(X, R) := (B, \prec)_+$.

Hence, if we define the notion of validity of a formula φ in a subordination space (X, R) as:

$$(X, R) \Vdash \varphi \quad \text{if} \quad (X, R, V) \Vdash \varphi \text{ for all clopen valuations } V : Prop \to \mathbf{Clop}(X)$$

we have that this coincides with validity in its dual algebra. Therefore, we decide to work with this notion of validity.

In Theorem 6.1.19 we will show that, for the Sahlqvist formulas which we define in Definition 6.1.1, validity on a subordination space (X, R) in the above sense coincides with satisfaction under all valuations.

Notation 6.0.4. Given two valuations V, V', we write $V \leq V'$ to indicate that $V(p) \subseteq V'(p)$ for all $p \in Prop$.

Moreover, given a valuation V, a tuple $\bar{p} = p_1, \ldots, p_n$ of proposition letters and a tuple $\bar{A} = A_1, \ldots, A_n \subseteq X$ of subsets, we denote by $V_{\bar{p}}^{\bar{A}}$ the valuation which maps $p_i \mapsto A_i$ and $q \mapsto V(q)$ for all $q \neq p_1, \ldots, p_n$.

6.1 Sahlqvist formulas and correspondence

First, we need to define the Sahlqvist fragment, and to do so, we need to define some particular kinds of formulas. These are given in the following definition.

Definition 6.1.1.

- *A formula φ is a* positive $\diamond$-free *formula if it is built from* 1 *and proposition letters by using $\wedge, \vee$ only.*
- *A formula θ is a* Sahlqvist antecedent *if it is built from* 1 *and formulas $\varphi \diamond \psi$ using $\wedge, \vee$, where φ, ψ are positive $\diamond$-free formulas.*

- *A formula χ is* positive *if it is built from* 1 *and formulas* $\varphi \diamond \psi$ *and* $\varphi \,\Box\, \psi$ *using* $\wedge, \vee$*, where* φ, ψ *are positive* $\diamond$*-free formulas.*
  Note that all Sahlqvist antecedent are also positive formulas.

- *A* non-separating *formula* $S(p)$ *is one of the form* $F(p) \vee G(\neg p)$*, where there are positive* $\diamond$*-free formulas* φ, ψ *such that* $F(p)$ *is equal to either* $\varphi \diamond p$ *or* $p \diamond \varphi$ *and* $G(p)$ *is equal to either* $\psi \diamond p$ *or* $p \diamond \psi$*;*

- *A* general positive *formula is a formula* $\chi(\bar{p})$ *which is a conjunction of non-separating formulas* $S(p_1), \ldots, S(p_n)$ *and positive formulas, where* $\bar{p} = p_1, \ldots, p_n$ *are proposition letters.*

- *A* Sahlqvist formula *is a formula* $\theta \to \chi(\bar{p})$ *where* θ *is a Sahlqvist antecedent,* $\chi(\bar{p})$ *is a general positive formula, and the proposition letters* $\bar{p} = p_1, \ldots, p_n$ *do not occur in* θ*.*

Example 6.1.2. The following are examples of Sahlqvist formulas:

1. $q \diamond r \;\to\; q \diamond p \;\vee\; \neg p \diamond r$;
2. $q \diamond r \;\to\; q \diamond p \;\vee\; r \diamond \neg p$;
3. $q \diamond r \;\to\; p \diamond q \;\vee\; r \diamond \neg p$;
4. $q \diamond r \;\to\; p \diamond q \;\vee\; \neg p \diamond r$.

At the end of this section, we will see formulas such as those above it is possible to compute a first-order correspondent. In order to convey the basic idea, here we show directly that the formula number 2. corresponds to the first order statement $\Phi := \forall x, y \;:\; \Big(xRy \underline{\to} \exists w \;:\; (xRw \underline{\wedge} yRw)\Big)$. That is, we have $(X, R) \Vdash q \diamond r \;\to\; q \diamond p \;\vee\; r \diamond \neg p$ if and only if $(X, R) \models \Phi$.

Proof. ($\Rightarrow$) We show the contrapositive. Suppose $(X, R) \not\models \Phi$. This means that there exist $a, b \in X$ such that there is no w such that aRw and bRw, that is $R[a] \cap R[b] = \emptyset$. Since $R[a]$ and $R[b]$ are disjoint closed subsets of X, there exists a clopen subset $U \subseteq X$ such that $R[a] \cap U = \emptyset$ and $R[b] \subseteq U$. By $R[a] \cap U = \emptyset$ we obtain $a \notin R^{-1}[U]$, and since $R^{-1}[U]$ is a closed subset, there exists a clopen subset $A \subseteq X$ such that $a \in A$ and $A \cap R^{-1}[U] = \emptyset$. The latter can be rewritten as $R[A] \cap U = \emptyset$. By $R[b] \subseteq U$ we obtain $b \notin R^{-1}[X \setminus U]$, thus we can find a clopen subset $B \subseteq X$ such that $b \in B$ and $B \cap R^{-1}[X \setminus U] = \emptyset$. The latter is equivalent to $R[B] \cap (X \setminus U) = \emptyset$.

Now, we can consider a clopen valuation V such that $V(p) = U, V(q) = A$ and $V(r) = B$. This valuation is such that $R[V(q)] \cap R[V(r)] \neq \emptyset$, because $a \in V(q) = A$, $b \in V(r) = B$ and aRb. Hence $(X, R, V) \Vdash q \diamond r$. But we also have $R[V(q)] \cap V(p) = R[A] \cap U = \emptyset$ and $R[V(q)] \cap V(\neg p) = R[B] \cap (X \setminus U) = \emptyset$. Therefore we obtain $(X, R, V) \not\Vdash q \diamond p$ and $(X, R, V) \not\Vdash q \diamond \neg p$, respectively. Hence we have $(X, R, V) \not\Vdash q \diamond p \vee r \diamond \neg p$.

This shows $(X, R) \not\Vdash q \diamond r \;\to\; q \diamond p \vee \diamond\neg p$.

($\Leftarrow$) Suppose $(X, R) \models \Phi$. Let V be any clopen valuation. We need to show that, if $(X, R, V) \Vdash q \diamond r$, then $(X, R, V) \Vdash q \diamond p \vee r \diamond \neg p$.

Suppose $(X, R, V) \Vdash q \diamond r$, which means that $R[V(q)] \cap V(r) \neq \emptyset$. So there exist $a \in V(q), b \in V(r)$ such that aRb. Since $(X, R) \models \Phi$, this implies that there exists $d \in X$ such that aRd and bRd. Let $U := V(p)$. If $d \in U$, then we have $d \in R[V(q)] \cap V(p)$, and so $(X, R, V) \Vdash q \diamond p$. Otherwise, if $d \in X \setminus U$, we have $d \in R[V(r)] \cap V(\neg p)$, and hence $(X, R, V) \Vdash r \diamond \neg p$. So in any case we obtain $(X, R, V) \Vdash q \diamond p \vee r \diamond \neg p$.

□

The use of positive formulas on the right-hand side of Sahlqvist implications is common practice in standard developments of Sahlqvist-like theories. In fact, it is usually easy to prove that such formulas are monotone with respect to valuations, and this is a key ingredient when proving a Sahlqvist correspondence theorem.

Our Sahlqvist fragment has some non-standard features. One particular aspect is the use of non-separating formulas, of which we have not found an analogue in the literature. These formulas are particular because they require their special variables to not occur anywhere else in the Sahlqvist formulas in which the non-separating formulas is used. This requirement will allow us to prove, in Lemma 6.1.9, that non-separating formulas satisfy a weaker form of monotonicity, despite having a negated term in them.

Another difference of our approach to more standard ones, is that in our definition we do not allow for nested diamonds and boxes.

Apart from non-separating formulas, the rest of our definition of Sahlqvist formulas is very similar to that given in [2, Section 2]. In particular, our Sahlqvist antecedents and positive formulas can be regarded as special cases of the non-negative and positive formulas defined in [2], respectively.

In the following lemma we show that positive $\diamond$-free formulas and Sahlqvist antecedents can be equivalently written in a shape which will be convenient for technical purposes. When we say that a formula is *equivalent* to another one, we mean that they have the same interpretation under any valuation V in any subordination space (X, R).

Remark 6.1.3. Given any formulas $\varphi_1, \varphi_2, \psi$, we have that $(\varphi_1 \vee \varphi_2) \diamond \psi$ is equivalent to $(\varphi_1 \diamond \psi) \vee (\varphi_2 \diamond \psi)$. This follows directly from how we defined the extension of valuations V to all formulas, right after Definition 6.0.3.

Lemma 6.1.4. *1. Any positive $\diamond$-free formula φ is equivalent to either* 1 *or to a disjunction of proper conjunctions of proposition letters;*

2. *Any Sahlqvist antecedent θ is equivalent to either* 1 *or to a disjunction of proper conjunctions of formulas $\varphi \diamond \psi$ where each φ, ψ is either* 1 *or a proper conjunction of proposition letters.*

Proof. The proof is a routine induction on the complexity of the formulas. □

Note that the formula 1 is equivalent to a disjunction of empty conjunctions, thus in the previous lemma could have simply stated that positive $\diamond$-free formulas and Sahlqvist antecedents are equivalent to disjunctions of conjunctions of formulas of the respective required form. Though, in Lemma 6.1.4 we decided to distinguish the two cases, because this will be useful for technical purposes.

Lemma 6.1.5. 1. *If φ is a positive $\diamond$-free formula, and V, V' are valuations such that $V \leq V'$, then $V(\varphi) \subseteq V'(\varphi)$.*

2. *Let χ be a positive formula, and let V, V' be valuations such that $V \leq V'$. Then $(X, R, V) \Vdash \chi$ implies $(X, R, V') \Vdash \chi$.*

Proof. Item 1. is a routine induction on the formulas. Also item 2. follows easily by induction on the formulas, using item 1. and the fact that the connectives $\diamond$, $\square$ are monotone with respect to valuations. □

By the previous lemma, positive formulas are monotone with respect to valuations. As usual in standard developments of Sahlqvist theories, one seeks to define a fragment which consists of implications $\theta \to \chi$ such that the minimal valuations satisfying θ are first-order definable and such that χ is monotone with respect to valuations. In this setting, one can then prove a Sahlqvist correspondence result in a standard fashion.

In the proof of the following proposition, we will see that our non-separating formulas $S(p)$ satisfy a weaker form of monotonicity. This extends to all general positive formulas.

Proposition 6.1.6. *Given a general positive formula $\chi(\bar{p})$, and a finite valuation V_0 such that $V_0(p_i) = \emptyset$ for all $i = 1 \dots n$, we have*

$$(X, R, (V_0)^{\bar{A}}_{\bar{p}}) \Vdash \chi(\bar{p}) \textit{ for any tuple of subsets } \bar{A} \subseteq X \quad \Leftrightarrow$$
$$(X, R, V) \Vdash \chi(\bar{p}) \textit{ for all valuations } V \textit{ s.t. } V_0 \leq V.$$

Proof. We argue by induction on the complexity of the general positive formula $\chi(\bar{p})$. The case in which $\chi(\bar{p})$ is a positive formula follows directly by Lemma 6.1.5. If the statement is true also for all non-separating formulas, then it is easily follows that it holds for all general positive formulas, which we recall be defined as (finite) conjuctions of positive and/or non-separating formulas. Therefore, we prove the case in which $\chi(\bar{p})$ is a non-separating formula.

We only treat the case in which the shape of $S(p)$ is $(\varphi \diamond p) \vee (\neg p \diamond \psi)$, because the proofs of the other three cases are very similar.

($\Rightarrow$) Let $V \geq V_0$ be any valuation, and let $U := V(p)$. By hypothesis, we have $(X, R, (V_0)^U_p) \Vdash S(p)$, which means that either $R[V_0(\varphi)] \cap U \neq \emptyset$ or $R[X \setminus U] \cap V_0(\psi) \neq \emptyset$. Hence, by Lemma 6.1.5, since $V_0(\varphi) \subseteq V(\varphi)$ and $V_0(\psi) \subseteq V(\psi)$, we have either $R[V(\varphi)] \cap U \neq \emptyset$ or $R[X \setminus U] \cap V(\psi) \neq \emptyset$, that is $(X, R, V) \Vdash (\varphi \diamond p) \vee (\neg p \diamond \psi)$.

This shows that $(X, R, V) \Vdash S(p)$ for all valuations V s.t. $V_0 \leq V$.

($\Leftarrow$) This direction is trivial, because for any $A \subseteq X$ we have $V_0 \leq (V_0)^A_p$.

□

If we would have defined validity on a subordination space (X, R) as satisfaction under *all* valuations, rather than under all *clopen* valuations, then Proposition 6.1.6 would allow us to prove a Sahlqvist correspondence result for our formulas with respect to this semantics. In fact, as we will see in the proof of Theorem 6.1.15, our Sahlqvist formulas $\theta \to \chi(\bar{p})$ are defined so that the minimal valuations V_0 satisfying θ are first-order definable and such that $V_0(p_i) = \emptyset$ for all p_i associated with the non-separating formulas.

Since we are interested in the notion of validity defined as satisfaction under all clopen valuations[2] only, in order to prove Theorem 6.1.15 we first need to show the analogue of Proposition 6.1.6 with respect to this semantics.

Remark 6.1.7. In the following proofs, we will very often make use of the fact that subordination spaces (X, R) satisfy the following properties:

- if $F, G \subseteq X$ are disjoint closed subsets, then there exists a clopen subset $U \subseteq X$ such that $F \subseteq U$ and $G \cap U = \emptyset$; [3]
- if $F \subseteq X$ is a closed subset, then $R[F]$ and $R^{-1}[F]$ are closed subsets of X. [4]

Lemma 6.1.8. *Let χ be a positive formula, and let V_0 be a finite valuation*[5]*. Then we have*

$$(X, R, V_0) \Vdash \chi \quad \Leftrightarrow \quad (X, R, V) \Vdash \chi \text{ for all clopen valuations } V \text{ s.t. } V_0 \leq V.$$

Proof. The direction $\Rightarrow$ follows by Lemma 6.1.5. For the other direction, we argue by induction on χ:

[2] Because, as we already mentioned, this semantics corresponds to our algebraic semantics by duality.

[3] This is provable via standard arguments by total disconnectedness and compactness.

[4] We mentioned this in Lemma 2.1.7.

[5] Valuations, and in particular finite valuations, are defined in Definition 6.0.3

- $\chi = 1$:
  In this case we always have $(X, R, V) \Vdash \chi$ for any valuation V, so in particular this holds for $V = V_0$ and for any clopen valuation V such that $V_0 \leq V$.

- $\chi = \varphi \diamond \psi$:
  We consider four cases:

  - φ and ψ are both equivalent to 1:
    In this case, for any valuation V we have $V(\chi) = V(1 \diamond 1) = X \diamond X$, so the equivalence holds.
  - φ is equivalent to 1 and ψ is a $\wedge, \vee$-combination of proposition letters:
    Since $V_0(\psi)$ is a $\cap, \cup$-combination of valuations of proposition letters, which are finite sets, it is itself a finite set, say, $V_0(\psi) = \{x_1, \ldots, x_k\}$.
    Suppose $(X, R, V) \Vdash \chi$ for all clopen valuations V, and suppose for a contradiction $(X, R, V_0) \nVdash \chi$, that is $(X, R, V) \nVdash \varphi \diamond \psi$.
    Then we have that

    $$R[X] \cap \{x_1, \ldots, x_k\} = R[V_0(\varphi)] \cap V_0(\psi) = \emptyset.$$

    Let $y_1, \ldots, y_h$ be all points different from the x_i's which belong to $V_0(p)$ for some proposition letter p occurring in ψ. Since $R[X] \cup \{y_1, \ldots, y_h\}$ and $\{x_1, \ldots, x_k\}$ are disjoint closed sets, there exists a clopen subset U such that

    $$\{x_1, \ldots, x_k\} \subseteq U \text{ and } (R[X] \cup \{y_1, \ldots, y_h\}) \cap U = \emptyset.$$

    So, if we take pairwise disjoint clopen subsets $U_1, \ldots, U_h$ such that $y_j \in U_j$ and $U_j \cap U = \emptyset$ for each $j = 1 \ldots h$, we can define the following clopen valuation:

    $$V : p \mapsto U \cup \bigcup \{U_j \mid y_j \in V_0(p)\}.$$

    We claim that $V(\psi) = U$. To see this, first observe that each $V(p)$ contains U and is contained in $U \cup \bigcup_j U_j$, thus $U \subseteq V(\psi) \subseteq U \cup \bigcup_j U_j$. Second, observe that for each p and each j we have $y_j \in V(p) \Leftrightarrow y_j \in V_0(p)$, and that if $y_j \notin V(p)$ then $U_j \cap V(p) = \emptyset$. Thus, as none of the y_j's belong to the $\cap, \cup$-combination of the $V(p)$'s which results in $V(\psi)$, we have $V(\psi) \cap \bigcup_j U_j = \emptyset$. So we obtain that $V(\psi) \subseteq U$, and therefore we can conclude that $V(\psi) = U$.
    Hence, since $V(\psi) = U$ and $R[X] \cap U = \emptyset$, we have found a clopen valuation V such that $V_0 \leq V$ and $(X, R, V) \nVdash 1 \diamond \psi$, that is $(X, R, V) \nVdash \chi$. This contradicts our assumption.

- φ is a $\wedge, \vee$-combination of proposition letters and ψ is equivalent to 1:
  In this case, $V_0(\varphi)$ is a finite set: $V_0(\psi) = \{x_1, \ldots, x_k\}$.
  Suppose $(X, R, V) \Vdash \chi$ for all clopen valuations V, and suppose for a contradiction $(X, R, V_0) \nVdash \chi$, that is $(X, R, V) \Vdash \varphi \diamond \psi$.
  Then we have $R[\{x_1, \ldots, x_k\}] \cap X = R[V_0(\varphi)] \cap V_0(\psi) = \emptyset$, that is, $\{x_1, \ldots, x_k\} \cap R^{-1}[X] = \emptyset$. So, arguing in the previous case, we can find a clopen valuation V such that $V_0 \leq V$ and $(X, R, V) \nVdash \varphi \diamond 1$, that is $(X, R, V) \nVdash \chi$.
- φ and ψ are both $\wedge, \vee$-combinations of proposition letters:
  In this case, there are $x_1, \ldots, x_k$ and $y_1, \ldots, y_h$ such that $V_0(\varphi) = \{x_1, \ldots, x_k\}$ and $V_0(\psi) = \{y_1, \ldots, y_h\}$.
  Suppose that $(X, R, V) \Vdash \chi$ for all clopen valuations V, and suppose for a contradiction that $(X, R, V_0) \nVdash \varphi \diamond \psi$, which implies that for all i, j we have $x_i \not R\, y_j$. Let $z_1, \ldots, z_l$ be all the points different from the x_i's and the y_j's which belong to $V_0(p)$ for some proposition letter p occurring in φ or ψ.
  For each i we have $x_i \notin R^{-1}[\{y_1, \ldots, y_h\}] \cup \{z_1, \ldots, z_l\}$, so we can find disjoint clopen subsets $W_1, \ldots, W_k$ such that
  $$W_i \cap (R^{-1}[\{y_1, \ldots, y_h\}] \cup \{z_1, \ldots, z_l\}) = \emptyset \text{ and } W_i \cap \{x_1, \ldots, x_k\} = \{x_i\}.$$
  This implies that $R[W_i] \cap \{y_1, \ldots, y_h\} = \emptyset$. Then, if we define $W := \bigcup_i W_i$, we have $W \cap \{z_1, \ldots, z_l\} = \emptyset$, and for each j we have $y_j \notin R[W] \cup \{z_1, \ldots, z_l\}$. By the latter, we can find disjoint clopen subsets $W'_1, \ldots, W'_h$ such that
  $$W'_j \cap (R^[W] \cup \{z_1, \ldots, z_l\}) = \emptyset \text{ and } W'_j \cap \{y_1, \ldots, y_h\} = \{y_j\}.$$
  If we define $W' := \bigcup_j W'_j$, we have $W' \cap \{z_1, \ldots, z_l\} = \emptyset$ and $R[W] \cap W' = \emptyset$.
  Then, consider the clopen subsets $U_1, \ldots, U_k, U'_1, \ldots, U'_h$ defined as
  $$U_i := \begin{cases} W_i \cap W'_j & \text{if } x_i = y_j \text{ for some } j \\ W_i & \text{otherwise} \end{cases}$$
  $$U'_j := \begin{cases} W'_j \cap W_i & \text{if } y_j = x_i \text{ for some } i \\ W'_j & \text{otherwise} \end{cases}$$
  and take $U := \bigcup_i U_i$ and $U' := \bigcup_j U'_j$. Notice that $U \subseteq W$ and $U' \subseteq W'$. Since U and U' are both disjoint from $\{z_1, \ldots, z_l\}$, we can find disjoint clopen subsets $U''_1, \ldots, U''_l$ such that $z_s \in U''_s$ and $U''_s \cap U = U''_s \cap U' = \emptyset$ for all $s = 1 \ldots l$.
  Finally, consider the following clopen valuation:
  $$\begin{aligned} V : p \mapsto & \bigcup \{U_i \mid x_i \in V_0(p) \cap \{x_1, \ldots, x_k\}\} \\ & \cup \bigcup \{U'_j \mid y_j \in V_0(p) \cap \{y_1, \ldots, y_h\}\} \\ & \cup \bigcup \{U''_s \mid z_s \in V_0(p) \cap \{z_1, \ldots, z_l\}\}. \end{aligned}$$

As $V_0(\varphi)$ and $V_0(\psi)$ are $\cap, \cup$-combinations of the $V(p)$'s which result in $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_h\}$ respectively, we obtain that the valuation V is such that $V(\varphi) = \bigcup_i U_i = U$ and $V(\psi) = \bigcup_j U'_j = U'$.

Thus, since by $R[W] \cap W' = \emptyset, U \subseteq W$ and $U' \subseteq W'$ we obtain $R[U] \cap U' = \emptyset$, we have found a clopen valuation $V \geq V_0$ such that $(X, R, V) \nVdash \varphi \diamond \psi$, that is $(X, R, V) \nVdash \chi$. This contradicts our assumption.

- $\chi = \varphi \,\Box\, \psi$:
  There are two cases:

  - Either φ or ψ is equivalent to 1:
    In this case, for any valuation V, we have either $V(\neg\varphi) = \emptyset$ or $V(\neg\psi) = \emptyset$, so $R[V(\neg\varphi)] \cap V(\neg\psi) = \emptyset$. Hence for all valuations we have $(X, R, V) \Vdash \neg(\neg\varphi \diamond \neg\psi)$, and this holds in particular for $V = V_0$ and for all clopen valuations V such that $V_0 \leq V$.

  - Neither φ nor ψ is equivalent to 1:
    In this case, both $V_0(\varphi)$ and $V_0(\psi)$ are finite, let them be $V_0(\varphi) = \{x_1, \ldots, x_k\}$ and $V_0(\psi) = \{y_1, \ldots, y_h\}$.

    Suppose $(X, R, V) \Vdash \chi$ for all clopen valuations V, and suppose for a contradiction $(X, R, V_0) \nVdash \varphi \,\Box\, \psi$. This means that $R[X \setminus \{x_1, \ldots, x_k\}] \cap (X \setminus \{y_1, \ldots, y_h\}) \neq \emptyset$, thus there are $u, v \in X$ such that uRv and $u \neq x_i, v \neq y_j$ for all i, j. Let $z_1, \ldots, z_l$ be all points different from the x_i's and the y_j's which belong to $V_0(p)$ for some proposition letter p occurring in φ or ψ. Then we can find disjoint clopen subsets $W_1, \ldots, W_k$ such that for each $i = 1 \ldots k$

    $$W_i \cap \{u, z_1, \ldots, z_l\} = \emptyset \quad \text{and} \quad W_i \cap \{x_1, \ldots, x_k\} = \{x_i\}.$$

    Analogously, we can find disjoint clopen subsets $W'_1, \ldots, W'_h$ such that for each $j = 1 \ldots h$

    $$W'_j \cap \{v, z_1, \ldots, z_l\} = \emptyset \quad \text{and} \quad W'_j \cap \{y_1, \ldots, y_h\} = \{y_j\}.$$

    Then, consider the clopen subsets $U_1, \ldots, U_k, U'_1, \ldots, U'_h$ defined as

    $$U_i := \begin{cases} W_i \cap W'_j & \text{if } x_i = y_j \text{ for some } j \\ W_i & \text{otherwise} \end{cases}$$
    $$U'_j := \begin{cases} W'_j \cap W_i & \text{if } y_j = x_i \text{ for some } i \\ W'_j & \text{otherwise} \end{cases}$$

    and take $U := \bigcup_i U_i$ and $U' := \bigcup_j U'_j$. These are such that

    $$U \cap \{u, z_1, \ldots, z_l\} = U' \cap \{v, z_1, \ldots, z_l\} = \emptyset,$$

    thus we can find disjoint clopen subsets $U''_1, \ldots, U''_l$ such that $z_s \in U''_s$ and $U''_s \cap U = U''_s \cap U' = \emptyset$ for all $s = 1 \ldots l$.

Finally, if we consider the following clopen valuation:

$$\begin{aligned} V : p \mapsto & \bigcup\{U_i \mid x_i \in V_0(p) \cap \{x_1, \ldots, x_k\}\} \\ & \cup \bigcup\{U'_j \mid y_j \in V_0(p) \cap \{y_1, \ldots, y_h\}\} \\ & \cup \bigcup\{U''_s \mid z_s \in V_0(p) \cap \{z_1, \ldots, z_l\}\}. \end{aligned}$$

we obtain $V(\varphi) = U$ and $V(\psi) = U'$.
So, since $u \notin V(\varphi)$ and $v \notin V(\psi)$ and uRv, we have $(X, R, V) \not\Vdash \neg(\neg\varphi \diamond \neg\psi)$, so we have found a clopen valuation V such that $V_0 \leq V$ and $(X, R, V) \not\Vdash \chi$. This contradicts our assumption.

- $\chi = \chi_1 \vee \chi_2$:
  Suppose $(X, R, V) \Vdash \chi$ for all clopen valuations V, and suppose for a contradiction $(X, R, V_0) \not\Vdash \chi_1 \vee \chi_2$. This means $(X, R, V_0) \not\Vdash \chi_1$ and $(X, R, V_0) \not\Vdash \chi_2$. So by inductive hypothesis there exist clopen valuations $V_1, V_2 \geq V_0$ such that $(X, R, V_1) \not\Vdash \chi_1$ and $(X, R, V_2) \not\Vdash \chi_2$. Hence, by Lemma 6.1.5, if we consider the (clopen) valuation $V : p \mapsto V_1(p) \cap V_2(p)$ we have $V_0 \leq V$ and $(X, R, V) \not\Vdash \chi_1$ and $(X, R, V) \not\Vdash \chi_2$, that is $(X, R, V) \not\Vdash \chi_1 \vee \chi_2$. Thus, we obtained a contradiction.

- $\chi = \chi_1 \wedge \chi_2$:
  If for all clopen valuations $V \geq V_0$ we have $(X, R, V) \Vdash \chi_1 \wedge \chi_2$, that is $(X, R, V) \Vdash \chi_1$ and $(X, R, V) \Vdash \chi_2$, then by inductive hypothesis we have $(X, R, V_0) \Vdash \chi_1$ and $(X, R, V_0) \Vdash \chi_2$, that is $(X, R, V_0) \Vdash \chi_1 \wedge \chi_2$.

□

Lemma 6.1.9. *Let $S(p)$ be a non-separating formula, and let V_0 be a finite valuation such that $V_0(p) = \emptyset$. Then we have*

$$(X, R, (V_0)^A_p) \Vdash S(p) \text{ for any subset } A \subseteq X \quad \Leftrightarrow$$
$$(X, R, V) \Vdash S(p) \text{ for all clopen valuations } V \text{ s.t. } V_0 \leq V.$$

Proof. Here we treat the case in which the shape of $S(p)$ is $(\varphi \diamond p) \vee (\neg p \diamond \psi)$, because the proofs of the other three cases are very similar.

($\Rightarrow$) This direction follows directly from Proposition 6.1.6.

($\Leftarrow$) We show the contrapositive.

Suppose there is a subset $A \subseteq X$ such that $(X, R, (V_0)^A_p) \not\Vdash S(p)$. This means that

- $R[V_0(\varphi)] \cap A = \emptyset$, that is $R[V_0(\varphi)] \subseteq X \setminus A$, and
- $R[X \setminus A] \cap V_0(\psi) = \emptyset$, that is $R^{-1}[V_0(\psi)] \subseteq A$.

Hence we have $R[V_0(\varphi)] \cap R^{-1}[V_0(\psi)] = \emptyset$.

This implies $V_0(\varphi) \cap R^{-1}[R^{-1}[V_0(\psi)]] = \emptyset$. Since the last two non-intersecting sets are closed, there exists a clopen subset U such that $V_0(\varphi) \subseteq U$ and $U \cap R^{-1}[R^{-1}[V_0(\psi)]] = \emptyset$. The latter equality implies $R[R[U]] \cap V_0(\psi) = \emptyset$, and since these are again non-intersecting closed subsets of X, there exists a clopen subset U' such that $V_0(\psi) \subseteq U'$ and $R[R[U]] \cap U' = \emptyset$, that is $R[U] \cap R^{-1}[U'] = \emptyset$. So, again, there exists a clopen subset W such that $R[U] \cap W = \emptyset$ and $R^{-1}[U'] \subseteq W$, that is $R[X \setminus W] \cap U' = \emptyset$.

Hence, if we build a clopen valuation V such that $V(\varphi) = U$, $V(\psi) = U'$ and $V(p) = W$ [6] then we obtain

- $R[V(\varphi)] \cap V(p) = R[U] \cap W = \emptyset$, and
- $R[V(\neg p)] \cap V(\psi) = R[X \setminus W] \cap U' = \emptyset$,

that is $V_0 \leq V$ and $(X, R, V) \not\Vdash S(p)$.

$\square$

By Lemmas 6.1.8 and 6.1.9, we can conclude the following:

Corollary 6.1.10. *Given a general positive formula $\chi(\bar{p})$, and a finite valuation V_0 such that $V_0(p_i) = \emptyset$ for all $i = 1 \dots n$, we have*

$$(X, R, (V_0)^{\bar{A}}_{\bar{p}}) \Vdash \chi(\bar{p}) \text{ for any tuple of subsets } \bar{A} \subseteq X \quad \Leftrightarrow$$
$$(X, R, V) \Vdash \chi(\bar{p}) \text{ for all clopen valuations } V \text{ s.t. } V_0 \leq V.$$

Lemmas 6.1.8 and 6.1.9 and Corollary 6.1.10 are very similar to Esakia's lemma [28], which has been used by Sambin and Vaccaro [49] for giving an elegant proof of Sahlqvist's theorem. There this lemma is used to show that, once a canonical model satisfies a Sahlqvist formula under all admissible valuations[7], then it satisfies it under all valuations. This leads to a completeness result. It is common, among Sahlqvist-like results (see e.g. [10, 2]), to prove a version of Esakia's lemma in order to obtain a completeness result for Sahlqvist formulas via canonical models.

Here, we are not working with canonical models, but as we already mentioned, we needed to show an analogue result because in our correspondence result we address semantics with respect to clopen valuations.

When working with formulas of a language which has a frame-like semantics, it is customary to develop a correspondence theory. Usually, one of the

[6] Which we can do similarly as we did in the proof Lemma 6.1.8.

[7] Which can be seen as our clopen valuations.

starting points of such a development consists of defining the Standard Translation of formulas. This translates formulas of the given language into first-order formulas in the relational first-order language of frames extended with unary predicates, where each predicate is associated with a proposition letter[8]. Then a model $M = (F, V)$, where F is a frame and $V : Prop \to \mathcal{P}(F)$ is a valuation, can be seen as a first-order structure in the relational language extended with unary predicates. Each predicate P is interpreted as $V(p)$, where p is its associated proposition letter. The Standard Translation is defined so that, given a model M and a formula φ, M satisfies φ if and only if M satisfies, as a first-order structure, the translation of φ. Therefore, this provides model-theoretic correspondence.

Sahlqvist correspondence is part, instead, of frame-theoretic correspondence. In fact, a Sahlqvist formula φ corresponds to a first-order sentence (in the relational language only) which is satisfied exactly by those frames which validate φ. Usually, when proving a Sahlqvist correspondence result, one uses the Standard Translation while working with valuations which make the interpretation of the unary predicates definable in the relational language. As we also do so, in Definition 6.1.12 we give the Standard Translation of our formulas.

In our definition, instead of using unary predicates, we have decided to define the Standard Translation of formulas relatively to a fixed valuation, which we require to interpret proposition letters as first-order definable subsets of subordination spaces. Moreover, instead of defining the translation of all formulas of our language, we restrict ourselves to general positive formulas[9]. The reason is that, for our non-separating formulas $S(p)$, we need to define a translation which treats the special proposition letter p differently from the others, and this cannot be achieved with a general inductive translation of all formulas.

Notation 6.1.11. We often regard algebras $(B, \prec)$ as first-order structures over the signature $(\wedge, \neg, 1, \diamond)$. Notice that first-order terms over this signature are syntactically the same as formulas of our language. When we put these objects inside first-order formulas, we consider them as first-order terms. Otherwise, they represent formulas of our language.

As we did in the previous chapters, in order to not confuse connectives $\wedge, \vee, \neg, \to$ interpreted by Boolean algebras with connectives of first-order formulas, for the latter we will use the symbols $\underline{\wedge}, \underline{\vee}, \underline{\neg}$ and $\underline{\to}$.

Moreover, for formulas in our language, we chose to replace the constants $\top, \bot$ with $1, 0$, so that we can regard the former ones as top and bottom of

[8] Or, in case the language in consideration does not have proposition letters, unary predicates are associated with some other basic syntactic elements which has the same semantic interpretation as our propositional letters. For example, in their definition of Standard Translation, in [2, Section 4] the authors use unary predicates for translating Boolean terms of their language, and the interpretation of these terms in analogous to how we interpret proposition letters.

[9] Which are the only ones to which we will apply the Standard Translation, in Theorems 6.1.15 and 6.2.5.

first-order logic.

Definition 6.1.12. *Let V be a valuation for which there exist a tuple $\bar{a} \in X$ such that, for each proposition letter q, the subset $V(q)$ is definable in (X, R) by a first-order formula $Q(u, \bar{a})$ with parameters from $\bar{a}$ in the language with one binary predicate R.*

For each positive $\diamond$-free formula φ, we define the formula $ST(u, \bar{a}, \varphi)$ inductively as follows:

$$\begin{aligned}
ST(u, \bar{a}, 1) &:= (u \approx u) \\
ST(u, \bar{a}, q) &:= Q(u, \bar{a}) \\
ST(u, \bar{a}, \varphi_1 \wedge \varphi_2) &:= ST(u, \bar{a}, \varphi_1) \,\underline{\wedge}\, ST(u, \bar{a}, \varphi_2) \\
ST(u, \bar{a}, \varphi_1 \vee \varphi_2) &:= ST(u, \bar{a}, \varphi_1) \,\underline{\vee}\, ST(u, \bar{a}, \varphi_2).
\end{aligned}$$

For each positive formula χ, we define the formula $ST(\bar{a}, \chi)$ inductively as follows:

$$\begin{aligned}
ST(\bar{a}, 1) &:= \top \\
ST(\bar{a}, \varphi \diamond \psi) &:= \exists u, v : (uRv \,\underline{\wedge}\, ST(u, \bar{a}, \varphi) \,\underline{\wedge}\, ST(v, \bar{a}, \psi)) \\
ST(\bar{a}, \varphi \,\square\, \psi) &:= \underline{\neg}\exists u, v : (uRv \,\underline{\wedge}\, \underline{\neg}ST(u, \bar{a}, \varphi) \,\underline{\wedge}\, \underline{\neg}ST(v, \bar{a}, \psi)) \\
ST(\bar{a}, \chi_1 \wedge \chi_2) &:= ST(\bar{a}, \chi_1) \,\underline{\wedge}\, ST(\bar{a}, \chi_2) \\
ST(\bar{a}, \chi_1 \vee \chi_2) &:= ST(\bar{a}, \chi_1) \,\underline{\vee}\, ST(\bar{a}, \chi_2).
\end{aligned}$$

Finally, for each general positive formula $\chi(\bar{p})$, we extend the definition of $ST(\bar{a}, \chi(\bar{p}))$ inductively as follows:

$$\begin{aligned}
ST(\bar{a}, (\varphi \diamond p) \vee (\psi \diamond \neg p)) &:= \exists u, v, w : (uRw \,\underline{\wedge}\, vRw \,\underline{\wedge}\, ST(u, \bar{a}, \varphi) \,\underline{\wedge}\, ST(v, \bar{a}, \psi)) \\
ST(\bar{a}, (\varphi \diamond p) \vee (\neg p \diamond \psi)) &:= \exists u, v, w : (uRw \,\underline{\wedge}\, wRv \,\underline{\wedge}\, ST(u, \bar{a}, \varphi) \,\underline{\wedge}\, ST(v, \bar{a}, \psi)) \\
ST(\bar{a}, (p \diamond \varphi) \vee (\psi \diamond \neg p)) &:= \exists u, v, w : (wRu \,\underline{\wedge}\, vRw \,\underline{\wedge}\, ST(u, \bar{a}, \varphi) \,\underline{\wedge}\, ST(v, \bar{a}, \psi)) \\
ST(\bar{a}, (p \diamond \varphi) \vee (\neg p \diamond \psi)) &:= \exists u, v, w : (wRu \,\underline{\wedge}\, wRv \,\underline{\wedge}\, ST(u, \bar{a}, \varphi) \,\underline{\wedge}\, ST(v, \bar{a}, \psi)) \\
ST(\bar{a}, \chi_1(\bar{p}) \wedge \chi_2(\bar{p})) &:= ST(\bar{a}, \chi_1(\bar{p})) \,\underline{\wedge}\, ST(\bar{a}, \chi_2(\bar{p})).
\end{aligned}$$

Proposition 6.1.13. *Let V be a valuation for which there exist a tuple $\bar{a} \in X$ such that, for each proposition letter q, the subset $V(q)$ is definable in (X, R) by a first-order formula $Q(u, \bar{a})$ with parameters from $\bar{a}$ in the language with one binary predicate R.*

Then we have:

(i) For all positive $\diamond$-free formulas φ and $b \in X$:

$$b \in V(\varphi) \;\Leftrightarrow\; (X, R) \models ST(u, \bar{x}, \varphi)[b/u, \bar{a}/\bar{x}].$$

(ii) For all general positive formulas $\chi(\bar{p})$, we have:

$$(X, R, (V)^{\bar{A}}_{\bar{p}}) \Vdash \chi(\bar{p}) \text{ for any tuple of subsets } \bar{A} \subseteq X \quad \Leftrightarrow \quad (X, R) \models ST(\bar{x}, \chi(\bar{p}))[\bar{a}/\bar{x}].$$

Proof. The proof of item (i) is a routine induction on the complexity of formulas. Concerning item (ii), its proof is also a routine induction, except for the case in which $\chi(\bar{p})$ is a non-separating formula. Here we prove the case in which $\chi(\bar{p}) = S(p) = \varphi \diamond p \vee \neg p \diamond \psi$. The other cases are analogous.

Let

$$\begin{aligned}\Phi &:= ST(\bar{x}, \varphi \diamond p \vee \neg p \diamond \psi)[\bar{a}/\bar{x}] \\ &= \exists u, v, w : (uRw \triangle wRv \triangle ST(u, \bar{a}, \varphi) \triangle ST(v, \bar{a}, \psi)).\end{aligned}$$

We need to show that $(X, R, (V)^{\bar{A}}_{\bar{p}}) \Vdash \varphi \diamond p \vee \neg p \diamond \psi$ for all subsets $A \subseteq X$ if and only if $(X, R) \models \Phi$.

($\Rightarrow$) We show the contrapositive. Suppose $(X, R) \not\models \Phi$. This means that, for all $b, c \in X$, if $(X, R) \models ST(u, \bar{a}, \varphi)[b/u]$ and $(X, R) \models ST(v, \bar{a}, \psi)[c/v]$, then $\nexists d$ such that bRd and dRc. By item (i), this is equivalent to saying that, for all $b \in V(\varphi)$ and $c \in V(\psi)$ there is no $d \in X$ such that bRd and dRc. That is, $R[V(\varphi)] \cap R^{-1}[V(\psi)] = \emptyset$.

If we consider $A := R^{-1}[V(\psi)]$, we have $R[V(\varphi)] \cap A = \emptyset$, and since in particular $R^{-1}[V(\psi)] \subseteq A$, we obtain $R[X \setminus A] \subseteq X \setminus V(\psi)$, that is $R[X \setminus A] \cap V(\psi) = \emptyset$. Therefore, if we consider the valuation V^A_p, we have obtained $(X, R, V^A_p) \not\Vdash \varphi \diamond p \vee \neg p \diamond \psi$.

($\Leftarrow$) Suppose $(X, R) \models \Phi$. This means that there exist $b, c, d \in X$ such that $(X, R) \models ST(u, \bar{a}, \varphi)[b/u], (X, R) \models ST(v, \bar{a}, \psi)[c/v], bRd$ and dRc. By item (i), this means that there exist $b \in V(\varphi)$ and $c \in V(\psi)$ such that bRd and dRc. This is equivalent to saying that $R[V(\varphi)] \cap R^{-1}[V(\psi)] \neq \emptyset$ [10].

Let $A \subseteq X$ be any subset. If $(X, R, V^A_p) \Vdash \varphi \diamond p$, then $(X, R, V^A_p) \Vdash \varphi \diamond p \vee \neg p \diamond \psi$. Otherwise, we have $R[V(\varphi)] \cap A = \emptyset$, and hence $R[V(\varphi)] \subseteq X \setminus A$. Then, since $R[V(\varphi)] \cap R^{-1}[V(\psi)] \neq \emptyset$, we obtain $(X \setminus A) \cap R^{-1}[V(\psi)] \neq \emptyset$, that is $R[X \setminus A] \cap V(\psi) \neq \emptyset$. So we have $(X, R, V^A_p) \Vdash \neg p \diamond \psi$, and hence $(X, R, V^A_p) \Vdash \varphi \diamond p \vee \neg p \diamond \psi$. This shows that in any case we obtain $(X, R, V^A_p) \Vdash \varphi \diamond p \vee \neg p \diamond \psi$.

□

Notation 6.1.14. Recall that, given a subordination space (X, R) and a formula ϕ, with $(X, R) \Vdash \phi$ we mean $(X, R, V) \Vdash \phi$ for all clopen valuations V. Moreover, recall from Chapter 4 that the notation $p \trianglelefteq \varphi$ expresses "p occurs in φ".

[10] As we see here, and as we saw in the proof of the other direction, the non-separating formula $\varphi \diamond p \vee \neg p \diamond \psi$ is satisfied by V^A_p for any $A \subseteq X$ if and only if $R[V(\varphi)] \cap R^{-1}[V(\psi)] \neq \emptyset$. That is, the formula expresses that $R[V(\varphi)]$ and $R^{-1}[V(\psi)]$ are not separated. Likewise, the other non-separating formulas express, that $R[V(\varphi)]$ and $R[V(\psi)]$, or $R^{-1}[V(\varphi)]$ and $R^{-1}[V(\psi)]$, or $R^{-1}[V(\varphi)]$ and $R[V(\psi)]$, are not separated. This motivated us to call these formulas *non-separating*.

Finally, in Theorem 6.1.15 we prove the correspondence result for our Sahlqvist formulas. The proof follows the lines of the proof of [2, Theorem 5.1], and the adaptation to our case relies on Corollary 6.1.10.

Theorem 6.1.15 (Sahlqvist correspondence)**.** *Let $\phi = \theta \to \chi(\bar{p})$ be a Sahlqvist formula. Then there exists a first-order sentence $\alpha(\phi)$ in the language with one binary predicate R such that $\alpha(\phi)$ is effectively computable from ϕ and such that for any (X, R) we have*

$$(X, R) \Vdash \phi \ \Leftrightarrow \ (X, R) \models \alpha(\phi).$$

Proof. In this proof we see how to compute a first-order sentence $\alpha(\phi)$ which is satisfied by a subordination space (X, R) if and only if ϕ is satisfied in (X, R) by all clopen valuations. We make a case distinction as to the nature of the Sahlqvist antecedent θ:

- Case $\theta = 1$:
  In this case, we have $(X, R) \Vdash \phi$ if and only if $(X, R) \Vdash \chi(\bar{p})$, that is if for al clopen valuations V we have $(X, R, V) \Vdash \chi(\bar{p})$. By Corollary 6.1.10, this is equivalent to $(X, R, (V_0)^{\bar{A}}_{\bar{p}}) \Vdash \chi(\bar{p})$ for all $A \subseteq X$, where V_0 is the empty valuation, that is the one such that $V_0(q) = \emptyset$ for all proposition letters q. Since in this valuation every $V_0(q)$ is definable by the formula $Q(u) := (u \not\approx u)$ (with no parameters), then by Proposition 6.1.13 we can conclude that $(X, R) \Vdash \phi$ if and only if $(X, R) \models ST(\chi(\bar{p}))$.

  So we can define $\alpha(\phi) := ST(\chi(\bar{p}))$.

- Case $\theta = \bigwedge\limits_{i=1}^{n} (\varphi_i \diamond \psi_i)$, where the φ_i's and the ψ_i's are either 1 or conjunctions of proposition letters:
  We have that $(X, R) \Vdash \theta \to \chi(\bar{p})$ if and only if

  $$\text{for all clopen valuations } V : \Big((X, R, V) \Vdash \theta \ \Rightarrow \ (X, R, V) \Vdash \chi(\bar{p})\Big). \tag{6.1}$$

  Given a valuation V, we have $(X, R, V) \Vdash \theta$ if and only if there exist elements $\bar{a} = a_1, \ldots, a_n$ and $\bar{b} = b_1, \ldots, b_n$ in X such that $a_i R b_i$, $a_i \in V(\varphi_i)$ and $b_i \in V(\psi_i)$ for $i = 1 \ldots n$. Hence, if given elements $\bar{a}, \bar{b}$ we define the finite valuation $V_{\bar{a},\bar{b},\theta}$ by:

  $$V_{\bar{a},\bar{b},\theta} : q \mapsto \{a_i \mid q \trianglelefteq \varphi_i\} \cup \{b_i \mid q \trianglelefteq \psi_i\}$$

  we have that $(X, R, V) \Vdash \theta$ if and only if there exist $\bar{a}$ and $\bar{b}$ such that $\bar{a} R \bar{b}$ and $V_{\bar{a},\bar{b},\theta} \leq V$. So (6.1) is equivalent to:

  $$\text{for all clopen valuations } V : \Big(\exists \bar{a}, \bar{b} : \bar{a} R \bar{b} \text{ and } V_{\bar{a},\bar{b},\theta} \leq V \ \Rightarrow \ (X, R, V) \Vdash \chi(\bar{p})\Big) \tag{6.2}$$

which can be rephrased as

$$\text{for all } \bar{a}, \bar{b} \text{ s.t. } \bar{a}R\bar{b} : \Big((X, R, V) \Vdash \chi(\bar{p}) \text{ for all clopen valuations } V \geq V_{\bar{a},\bar{b},\theta}\Big). \tag{6.3}$$

By Corollary 6.1.10, (6.3) is equivalent to

$$\text{for all } \bar{a}, \bar{b} \text{ s.t. } \bar{a}R\bar{b} : \Big((X, R, (V_{\bar{a},\bar{b},\theta})^{\bar{A}}_{\bar{p}}) \Vdash \chi(\bar{p}) \text{ for all subsets } \bar{A} \subseteq X\Big). \tag{6.4}$$

Since for each proposition letter q we have that $V_{\bar{a},\bar{b},\theta}(q)$ is definable with parameters $\bar{a}, \bar{b}$ by the formula

$$Q(u, \bar{a}, \bar{b}) := \underline{\bigvee}\{u \approx a_i \mid q \trianglelefteq \varphi_i\} \mathrel{\underline{\vee}} \underline{\bigvee}\{u \approx b_i \mid q \trianglelefteq \psi_i\},$$

by Proposition 6.1.13 we have that (6.4) is equivalent to

$$\text{for all } \bar{a}, \bar{b} \text{ s.t. } \bar{a}R\bar{b} : \Big((X, R) \models ST(\bar{x}, \bar{y}, \chi(\bar{p}))[\bar{a}/\bar{x}, \bar{b}/\bar{y}]\Big) \tag{6.5}$$

and (6.5) can be rewritten as

$$(X, R) \models \forall \bar{x}, \bar{y} : \Big(\underline{\bigwedge_{i=1}^{n}} x_i R y_i \mathrel{\underline{\rightarrow}} ST(\bar{x}, \bar{y}, \chi(\bar{p}))\Big). \tag{6.6}$$

So we have obtained the first-order correspondent

$$\alpha(\phi) := \forall \bar{x}, \bar{y} : \Big(\underline{\bigwedge_{i=1}^{n}} x_i R y_i \mathrel{\underline{\rightarrow}} ST(\bar{x}, \bar{y}, \chi(\bar{p}))\Big).$$

- Case $\theta = \bigvee_{j=1}^{k} \theta_j$, where each θ_j is as in the previous case:

  Let $\alpha_j := \alpha(\theta_j \to \chi(\bar{p}))$. Then we have $(X, R) \Vdash \left(\bigvee_{j=1}^{k} \theta_j\right) \to \chi(p)$ if and only if $(X, R) \Vdash \theta_j \to \chi(\bar{p})$ for all $j = 1 \ldots k$. By the previous case, this holds if and only if $(X, R) \models \alpha_j$ for all $j = 1 \ldots k$, that is if and only if $(X, R) \models \underline{\bigwedge}_{j=1}^{k} \alpha_j$.

  So we have obtained the first-order correspondent

$$\alpha(\phi) := \underline{\bigwedge_{j=1}^{k}} \alpha_j.$$

□

By the duality presented in Section 2.1.2, the above theorem can be restated in terms of algebras their dual subordination spaces:

Corollary 6.1.16. *Let $\phi = \theta \to \chi(\bar{p})$ be a Sahlqvist formula. Then there exists a first-order sentence $\alpha(\phi)$ in the language with one binary predicate R such that $\alpha(\phi)$ is effectively computable from ϕ and such that for any Boolean algebra $(B, \prec)$ with a subordination $\prec$, we have*

$$(B, \prec) \models \phi \;\Leftrightarrow\; (B, \prec)_+ \models \alpha(\phi).$$

Example 6.1.17. The following are the respective first-order correspondents of the examples provided in Example 6.1.2:

1. $\forall x, y \;:\; \Big(xRy \underline{\to} \exists w : \;(xRw \underline{\wedge}\, wRy)\Big);$
2. $\forall x, y \;:\; \Big(xRy \underline{\to} \exists w : \;(xRw \underline{\wedge}\, yRw)\Big);$
3. $\forall x, y \;:\; \Big(xRy \underline{\to} \exists w : \;(wRx \underline{\wedge}\, yRw)\Big);$
4. $\forall x, y \;:\; \Big(xRy \underline{\to} \exists w : \;(wRx \underline{\wedge}\, wRy)\Big).$

Here, we work out the algorithm resulting from the proof of Theorem 6.1.15 to compute the correspondent of the first of the above examples. Recall that the starting Sahlqvist formula of example 1. is $q \diamond r \to q \diamond p \vee \neg p \diamond r$.

Here, the antecedent $\theta = q \diamond r$ is a single disjunction of the single conjunction $q \diamond p$, and the minimal valuations satisfying this are those V_0 such that $V_0(q) = \{a\}, V_0(r) = \{b\}$ for some $(a, b) \in R$. As in our proof of correspondence, we use variables x, y to replace such elements a, b, and we use these variables to define the basic predicates for the Standard Translation:

$$\begin{aligned} ST(u, x, y, q) &= Q(u, x, y) \;:=\; (u \approx x) \\ ST(u, x, y, r) &= R(u, x, y) \;:=\; (u \approx y) \end{aligned}$$

resulting in the correspondent

$$\forall x, y : \Big(xRy \underline{\to} ST(x, y, q \diamond p \vee \neg p \diamond r)\Big).$$

By Definition 6.1.12, this is equal to

$$\forall x, y : \Big(xRy \underline{\to} \exists u, v, w : [uRw \underline{\wedge}\, wRv \underline{\wedge}\, ST(u, x, y, q) \underline{\wedge}\, ST(v, x, y, r)]\Big)$$

which, by our above definition of basic predicates, is equal to

$$\forall x, y : \Big(xRy \underline{\to} \exists u, v, w : [uRw \underline{\wedge}\, wRv \underline{\wedge}\, (u \approx x) \underline{\wedge}\, (v \approx y)]\Big)$$

which is finally equivalent to

$$\forall x, y : \Big(xRy \underline{\to} \exists w : (xRw \underline{\wedge}\, wRy)\Big).$$

Remark 6.1.18. As we noticed below Definition 2.1.3, property (Q6) is expressed in terms of the relation δ as "$a\delta b$ implies $b\delta a$". Satisfying this condition is thus equivalent to validating the formula $q \diamond r \to r \diamond q$, which is one of our Sahlqvist formulas. Working out the Sahlqvist correspondent of this formulas, one obtains that this expresses symmetry of the relation R. This is consistent with one of the facts showed in Lemma 2.1.12, that is that property (Q6) is satisfied by those algebras of which dual has a symmetric closed relation R.

Theorem 6.1.19. *Let $\phi = \theta \to \chi(\bar{p})$ be a Sahlqvist formula. Given any subordination space (X, R), we have*

$$(X, R) \Vdash \phi \quad \Leftrightarrow \quad (X, R, V) \Vdash \phi \textit{ for all valuations } V.$$

Proof. The direction $\Leftarrow$ is trivial. To prove $\Rightarrow$, assume $(X, R) \Vdash \phi$, and let V be any valuation. We need to show that $(X, R, V) \Vdash \phi$. By item 2. of Lemma 6.1.4, we can consider the following cases:

- Case $\theta = 1$:
  In this case, we always have $(X, R, V) \Vdash \theta$, and we need to show $(X, R, V) \Vdash \chi(\bar{p})$. By assumption, we have $(X, R, W) \Vdash \chi(\bar{p})$ for all clopen valuations W. Hence, if we let V_0 be the empty valuation, we have $(X, R, W) \Vdash \chi(\bar{p})$ for all clopen valuations such that $V_0 \leq W$. Hence, by Corollary 6.1.10, we obtain $(X, R, (V_0)^{\bar{A}}_{\bar{p}}) \Vdash \chi(\bar{p})$ for all subsets $\bar{A} \subseteq X$. So, by Lemma 6.1.6, we have $(X, R, W) \Vdash \chi(\bar{p})$ for all valuations such that $V_0 \leq W$. In particular, since $V \geq V_0$, we have $(X, R, V) \Vdash \chi(\bar{p})$.

  This shows that $(X, R, V) \Vdash \phi$.

- Case $\theta = \bigvee_{j=1}^{k} \theta_j$, where each θ_j is of the form $\bigwedge_{i=1}^{n} (\varphi_i \diamond \psi_i)$, where the φ_i's and the ψ_i's are either 1 or conjunctions of proposition letters:
  In this case, suppose we have $(X, R, V) \Vdash \theta$, and we need to show $(X, R, V) \Vdash \chi(\bar{p})$.

  Since $(X, R, V) \Vdash \theta$, there exists j such that $(X, R, V) \Vdash \theta_j$, where $\theta_j = \bigwedge_{i=1}^{n} (\varphi_i \diamond \psi_i)$. This means that there exist elements $\bar{a} = a_1, \ldots, a_n$ and $\bar{b} = b_1, \ldots, b_n$ in X such that $a_i R b_i$, $a_i \in V(\varphi_i)$ and $b_i \in V(\psi_i)$ for $i = 1 \ldots n$. If we define the finite valuation $V_0 := V_{\bar{a},\bar{b},\theta}$ by:

  $$V_{\bar{a},\bar{b},\theta} : q \mapsto \{a_i \mid q \trianglelefteq \varphi_i\} \cup \{b_i \mid q \trianglelefteq \psi_i\}$$

  then for any valuation $W \geq V_0$ we have $(X, R, W) \Vdash \theta_j$, and hence $(X, R, W) \Vdash \theta$. This holds in particular for all clopen valuations $W \geq V_0$, so by assumption we have $(X, R, W) \Vdash \chi(\bar{p})$ for all clopen valuations

$W \geq V_0$. Since $V_0(p_i) = \emptyset$ for $i = 1 \dots n$, by Corollary 6.1.10 we obtain $(X, R, (V_0)^{\bar{A}}_{\bar{p}}) \Vdash \chi(\bar{p})$ for all subsets $\bar{A} \subseteq X$. Hence, by Proposition 6.1.6, we have $(X, R, W') \Vdash \chi(\bar{p})$ for all valuations such that $V_0 \leq W'$. Since $V_0 \leq V$, in particular we have $(X, R, V) \Vdash \chi(\bar{p})$.

This shows $(X, R, V) \Vdash \phi$.

$\square$

We conclude this section with a completeness result which follows from our Sahlqvist correspondence theorem.

Corollary 6.1.20. *Let ϕ be a Sahlqvist formula. Then the system $\mathcal{S} + (\phi)$ is sound and complete with respect to the class of subordination spaces (X, R) such that R is reflexive and symmetric, and such that $(X, R) \models \alpha(\phi)$.*

Proof. By the completeness result of Chapter 3, we obtain that the system $\mathcal{S} + (\phi)$ is sound and complete with respect to contact algebras which validate ϕ. By Corollary 6.1.16, an algebra $(B, \prec)$ validates ϕ if and only if its dual subordination space $(X, R) = (B, \prec)_+$ satisfies $\alpha(\phi)$. Moreover, by Lemma 2.1.12, $(B, \prec)$ is a contact algebra if and only if the relation R in $(X, R) = (B, \prec)_+$ is reflexive and symmetric. Therefore, by the duality presented in Section 2.1.2, we have that $\mathcal{S} + (\phi)$ is sound and complete with respect to the class of subordination spaces which satisfy $\alpha(\phi)$ and in which the binary relation R is reflexive and symmetric. $\square$

6.2 Sahlqvist rules and correspondence

In Chapter 4 we showed that $\forall\exists$-statements are associated with Π_2-rules. In this section, we define a Sahlqvist fragment of $\forall\exists$-statements, and we prove a correspondence result which, in light of the aforementioned correspondence between statements and rules, can be regarded as a correspondence result for Π_2-rules. Unlike the results of the previous section, the results which we present here have no counterpart in the work of Balbiani and Kikot [2].

Recall that propositional formulas in the signature $(\wedge, \neg, 1, \diamond)$ can be seen as terms for first-order structures $(B, \prec)$ where B is a Boolean algebra, and $\prec$ a binary relation, where the binary function symbol $\diamond$ is interpreted as:

$$a \diamond b \ := \begin{cases} 1 & \text{if } a \not\prec \neg b \\ \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, if we consider a valuation $v : \mathit{Prop} \to B$, then we obtain that a first-order structure $(B, \prec, v)$ in the language $(\wedge, \neg, 1, \diamond, \mathit{Prop})$, where we regard the proposition letters in $Prop$ as constant symbols, and the valuation v as their interpretation in the structure.

In particular, given a subordination space (X, R) and a valuation $V : Prop \to \mathbf{Clop}(X)$, we obtain a first-order structure $(\mathbf{Clop}(X), \prec, V)$ as above.

Definition 6.2.1. *A* Sahlqvist statement *is a first-order statement* Ψ *in the signature* $(\wedge, \neg, 1, \diamond)$ *of the form*

$$\Psi \;=\; \forall \bar{q} : [\forall \bar{p} : (\theta \wedge (\bigwedge_{l=1}^{k} S_l(p_l)) \approx 1) \underline{\to} \forall \bar{r} : (\chi(\bar{r}) \approx 1)],$$

where:

- θ *is a Sahlqvist antecedent;*
- $\chi(\bar{r})$ *is a general positive formula;*
- *the* $S_l(p_l)$*'s are non-separating formulas;*
- $\bar{q}$ *are all proposition letters not among* $\bar{p} = p_1, \ldots, p_k$ *and* $\bar{r}$ *which occur in the formula;*
- *the proposition letters* $\bar{p}$ *and* $\bar{r}$ *do not occur anywhere but in their respective non-separating formulas.*

Example 6.2.2. The following are examples of Sahlqvist statements:

1. $\forall q, r : \;\Big[\forall p : \Big((q \diamond p \;\vee\; \neg p \diamond r) \;\approx 1\Big) \underline{\to} (q \diamond r \;\approx\; 1)\Big];$
2. $\forall q, r : \;\Big[\forall p : \Big((q \diamond p \;\vee\; r \diamond \neg p) \;\approx 1\Big) \underline{\to} (q \diamond r \;\approx\; 1)\Big];$
3. $\forall q, r : \;\Big[\forall p : \Big((p \diamond q \;\vee\; r \diamond \neg p) \;\approx 1\Big) \underline{\to} (q \diamond r \;\approx\; 1)\Big];$
4. $\forall q, r : \;\Big[\forall p : \Big((p \diamond q \;\vee\; \neg p \diamond r) \;\approx 1\Big) \underline{\to} (q \diamond r \;\approx\; 1)\Big];$

Note that Sahlqvist statements are statements in the language $(\wedge, \neg, 1, \diamond)$. Moreover, they are equivalent[11] to $\forall\exists$-statements. In fact, the generic Sahlqvist statement Ψ presented in Definition 6.2.1 can be rewritten as:

$$\Psi \;=\; \forall \bar{q}, \bar{r} : \exists \bar{p} : [(\theta \wedge (\bigwedge_{l=1}^{k} S_l(p_l)) \not\approx 1) \,\underline{\vee}\, (\chi(\bar{r}) \approx 1)].$$

By the aforementioned correspondence between Π_2-rules and $\forall\exists$-statements, this observation leads us to the defining Sahlqvist Π_2-rules as those which correspond to Sahlqvist statements:

[11] Here with *equivalent* we mean that they are either both true or both false under interpretation in any structure in the language $(\wedge, \neg, 1, \diamond)$.

Definition 6.2.3 (Sahlqvist rule)**.** *A Π_2-rule (ρ) is a* Sahlqvist rule *if its corresponding statement Φ_ρ* [12] *is equivalent to a Sahlqvist statement.*

In Theorem 6.2.5 we will show that, given a Sahlqvist statement Ψ, one can compute a first-order sentence $\beta(\Psi)$ in the language with one binary relation symbol R, and this is such that a Boolean algebra with a subordination $(B, \prec)$ satisfies Ψ if and only if its dual subordination space (X, R) satisfies $\beta(\Psi)$.

Then, using this result, we will derive a general completeness theorems for extensions of our system $\mathcal{S}$ [13] with Π_2-rules with respect to classes of subordination spaces (X, R) of which closed relation R satisfies some elementary conditions.

Observe that, by item 1. of Lemma 6.1.4, any non-separating formula $S(p) = (\varphi \diamond p) \vee (\psi \diamond \neg p)$ can be written in the form

$$\bigvee_i (\mu_i \diamond p) \ \vee \ \bigvee_j (\nu_j \diamond \neg p),$$

where each μ_i and ν_j is either 1 or a proper conjunction of proposition letters.

Writing non-separating formulas in this shape will be convenient for technical purposes.

In the proof of Theorem 6.2.5 we will make use of the following lemma:

Lemma 6.2.4. *Given a pair (X, R) and a valuation $V : Prop \to \mathbf{Clop}(X)$, we have:*

$$(\mathbf{Clop}(X), \prec, V) \models \forall p : \Big[\Big(\bigvee_i (\mu_i \diamond p) \ \vee \ \bigvee_j (\nu_j \diamond \neg p)\Big) \approx 1\Big] \quad \Leftrightarrow$$

there are i, j and there exist $x \in V(\mu_i), y \in V(\nu_j)$ and $w \in X$ s.t. xRw and yRw.

$$(\mathbf{Clop}(X), \prec, V) \models \forall p : \Big[\Big(\bigvee_i (\mu_i \diamond p) \ \vee \ \bigvee_j (\neg p \diamond \nu_j)\Big) \approx 1\Big] \quad \Leftrightarrow$$

there are i, j and there exist $x \in V(\mu_i), y \in V(\nu_j)$ and $w \in X$ s.t. xRw and wRy.

$$(\mathbf{Clop}(X), \prec, V) \models \forall p : \Big[\Big(\bigvee_i (p \diamond \mu_i) \ \vee \ \bigvee_j (\nu_j \diamond \neg p)\Big) \approx 1\Big] \quad \Leftrightarrow$$

there are i, j and there exist $x \in V(\mu_i), y \in V(\nu_j)$ and $w \in X$ s.t. wRx and yRw.

$$(\mathbf{Clop}(X), \prec, V) \models \forall p : \Big[\Big(\bigvee_i (p \diamond \mu_i) \ \vee \ \bigvee_j (\neg p \diamond \nu_j)\Big) \approx 1\Big] \quad \Leftrightarrow$$

there are i, j and there exist $x \in V(\mu_i), y \in V(\nu_j)$ and $w \in X$ s.t. xRw and yRw.

[12] Written in terms of $\diamond$ rather than $\rightsquigarrow$

[13] The system $\mathcal{S}$ is introduced in Chapter 3.

Proof. We prove only the first case, because the proof of the other three cases are analogous.

($\Rightarrow$) We prove the contrapositive. Suppose that for all i, j, and for all $x \in V(\mu_i)$ and $y \in V(\nu_j)$, there is no $w \in X$ such that xRw and yRw. This means that $(\bigcup_i R[V(\mu_i)]) \cap (\bigcup_j R[V(\nu_j)]) = \emptyset$. Since $\bigcup_i R[V(\mu_i)]$ and $\bigcup_j R[V(\nu_j)]$ are two disjoint closed sets, there exists a clopen subset U such that $\emptyset = (\bigcup_i R[V(\mu_i)]) \cap U = \bigcup_i (R[V(\mu_i)] \cap U)$ and $\bigcup_j R[V(\nu_j)] \subseteq U$, which means $\emptyset = (\bigcup_j R[V(\nu_j)]) \cap (X \setminus U) = \bigcup_j (R[V(\nu_j)] \cap (X \setminus U))$. So we have:

$$(\mathbf{Clop}(X), \prec, V) \models \Big(\Big(\bigvee_i (\mu_i \diamond p) \ \vee \ \bigvee_j (\nu_j \diamond \neg p)\Big) \not\approx 1\Big)[U/p]$$

hence

$$(\mathbf{Clop}(X), \prec, V) \not\models \forall p : \Big[\Big(\bigvee_i (\mu_i \diamond p) \ \vee \ \bigvee_j (\nu_j \diamond \neg p)\Big) \approx 1\Big].$$

($\Leftarrow$) Suppose there exist i, j and $x \in V(\mu_i), y \in V(\nu_j)$ and $w \in X$ such that xRw and yRw. Then we have $w \in R[V(\mu_i)] \cap R[V(\nu_j)]$. So, for any $U \in \mathbf{Clop}(X)$, we have either $w \in R[V(\mu_i)] \cap U \neq \emptyset$ or $w \in R[V(\nu_j)] \cap (X \setminus U) \neq \emptyset$, hence

$$(\mathbf{Clop}(X), \prec, V) \models \Big(\Big(\bigvee_i (\mu_i \diamond p) \ \vee \ \bigvee_j (\nu_j \diamond \neg p)\Big) \approx 1\Big)[U/p].$$

This shows that

$$(\mathbf{Clop}(X), \prec, V) \models \forall p : \Big[\Big(\bigvee_i (\mu_i \diamond p) \ \vee \ \bigvee_j (\nu_j \diamond \neg p)\Big) \approx 1\Big].$$

$\square$

Note that the proof of Lemma 6.2.4 relies on the properties stated in Remark 6.1.7.

Now, we are ready to prove the Sahlqvist correspondence result for our statements:

Theorem 6.2.5. *Let Ψ be a Sahlqvist statement. Then there exists a first-order formula $\beta(\Psi)$ in the language with one binary predicate R such that $\beta(\Psi)$ is effectively computable from Ψ and such that for any (X, R) we have*

$$(\mathbf{Clop}(X), \prec) \models \Psi \quad \Leftrightarrow \quad (X, R) \models \beta(\Psi).$$

Proof. Let $\Psi := \forall \bar{q} : [\forall \bar{p} : (\theta \wedge (\bigwedge_{l=1}^{k} S_l(p_l)) \approx 1) \underline{\rightarrow} \forall \bar{r} : (\chi(\bar{r}) \approx 1)]$.

We have $(\mathbf{Clop}(X), \prec) \models \Psi$ if and only if

$$\begin{array}{ll} & \text{for all } V : \{\bar{q}\} \to \mathbf{Clop}(X) \text{ it holds that} \\ \text{if} & (\mathbf{Clop}(X), \prec, V) \models \forall \bar{p} : [(\theta \wedge (\bigwedge_{l=1}^{k} S_l(p_l)) \approx 1] \\ \text{then} & (\mathbf{Clop}(X), \prec, V) \models \forall \bar{r} : (\chi(\bar{r}) \approx 1) \end{array} \tag{6.7}$$

We assume $\bar{p} = p_1, \ldots, p_k$, and $S_l(p_l) = \bigvee_i (\mu_{li} \diamond p_l) \ \vee \ \bigvee_j (\nu_{lj} \diamond \neg p_l)$ [14], and we consider the following cases:

- Case $\theta = 1$:
  Let V be a clopen valuation. We have

  $$(\mathbf{Clop}(X), \prec, V) \models \forall \bar{p} : [(\bigwedge_{l=1}^{k} S_l(p_l)) \approx 1]$$

  if and only if $(\mathbf{Clop}(X), \prec, V) \models \forall p_l : (S_l(p_l) \approx 1)$ for $l = 1 \ldots k$. By Lemma 6.2.4, the latter condition holds if and only if for all l there exists i_l, j_l and elements $c_l \in V(\mu_{li_l}), d_l \in V(\nu_{lj_l})$ and $e_l \in$X s.t. $c_l R e_l$ and $d_l R e_l$.

  Hence, if given $\{i_l, j_l\}_l$ and $\{c_l, d_l, e_l\}_l$ such that $c_l R e_l$ and $d_l R e_l$ we define the finite valuation $V_0 = V(\{i_l, j_l, c_l, d_l, e_l\}_l)$ by:

  $$V_0 : q \mapsto \{c_l \mid q \trianglelefteq \mu_{li_l}\} \cup \{d_l \mid q \trianglelefteq \nu_{lj_l}\}$$

  we have that $(\mathbf{Clop}(X), \prec, V) \models \forall \bar{p} : [(\bigwedge_{l=1}^{k} S_l(p_l)) \approx 1]$ if and only if there exist $\{i_l, j_l\}_l$ and $\{c_l, d_l, e_l\}_l$ such that $c_l R e_l$ and $d_l R e_l$ and $V \geq V_0$, where $V_0 = V(\{i_l, j_l, c_l, d_l, e_l\}_l)$.

  So (6.7) holds if and only if for all $\{i_l, j_l\}_l$ we have

  $$\begin{array}{l} \text{for all } \{c_l, d_l, e_l\}_l \text{ s.t. } c_l R e_l \text{ and } d_l R e_l \ : \\ \quad \Big[(\mathbf{Clop}(X), \prec, V) \models \forall \bar{r} : (\chi(\bar{r}) \approx 1) \quad \text{for all } V \geq V_0\Big] \\ \quad \text{where } V_0 = V(\{i_l, j_l, c_l, d_l, e_l\}_l) \quad . \end{array} \tag{6.8}$$

  Now, (6.8) can be rewritten as

  $$\begin{array}{l} \text{for all } \{c_l, d_l, e_l\}_l \text{ s.t. } c_l R e_l \text{ and } d_l R e_l \ : \\ \qquad \Big[(X, R, V) \Vdash \chi(\bar{r}) \quad \text{for all } V \geq V_0\Big] \\ \quad \text{where } V_0 = V(\{i_l, j_l, c_l, d_l, e_l\}_l), \end{array} \tag{6.9}$$

[14] If we let some $S_l(p_l)$ be of one of the other three forms in which non-separating formulas can be, the proof would be similar to the one we give below.

which by Corollary 6.1.10 is equivalent to

$$\begin{array}{l}\text{for all } \{c_l, d_l, e_l\}_l \text{ s.t. } c_l R e_l \text{ and } d_l R e_l \ : \\ \qquad\qquad \Big[(X, R, (V_0)^{\bar{A}}_{\bar{r}}) \Vdash \chi(\bar{r}) \quad \text{for all subsets } \bar{A} \subseteq X\Big] \\ \qquad \text{where } V_0 = V(\{i_l, j_l, c_l, d_l, e_l\}_l).\end{array} \tag{6.10}$$

Since for each proposition letter q we have that $V_0(q)$ is definable with parameters $\{c_l, d_l\}_l$ by the formula

$$Q(u, \{c_l, d_l\}_l) := \underline{\bigvee}\{u \approx c_l \mid q \trianglelefteq \mu_{l i_l}\} \veebar \underline{\bigvee}\{u \approx d_l \mid q \trianglelefteq \nu_{l j_l}\},$$

by Proposition 6.1.13 we obtain that (6.10) is equivalent to

$$\begin{array}{l}\text{for all } \{c_l, d_l, e_l\}_l \text{ s.t. } c_l R e_l \text{ and } d_l R e_l \ : \\ \qquad\qquad (X, R) \models ST(\bar{x}, \bar{y}, \chi(\bar{r}))[c_l/x_l, d_l/y_l]\end{array} \tag{6.11}$$

that is

$$\Big[(X, R) \models \forall \bar{x}, \bar{v}, w\Big((\underline{\bigwedge_{l=1}^{k}}(x_l R w \mathbin{\underline{\wedge}} y_l R w)) \mathbin{\underline{\to}} ST(\bar{x}, \bar{y}, \chi(\bar{r}))\Big)\Big] \tag{6.12}$$

So we can conclude that $(\mathbf{Clop}(X), \prec) \models \Psi$ if and only if $(X, R) \models \underline{\bigwedge}_{\{i_l, j_l\}_l} \beta_{\{i_l, j_l\}_l}$, where [15]

$$\beta_{\{i_l, j_l\}_l} \ := \ \forall \bar{x}, \bar{y}, w\Big((\underline{\bigwedge_{l=1}^{k}}(x_l R w \mathbin{\underline{\wedge}} y_l R w)) \mathbin{\underline{\to}} ST(\bar{x}, \bar{y}, \chi(\bar{r}))\Big).$$

Therefore, we have obtained the correspondent $\beta(\Psi) := \underline{\bigwedge}_{\{i_l, j_l\}_l} \beta_{\{i_l, j_l\}_l}$.

- Case $\theta \neq 1$:
  In this case, by item 2. of Lemma 6.1.4, θ can be written as $\bigvee_h \theta_h$ where the θ_h's are conjunctions of formulas $\varphi \diamond \psi$ where each φ, ψ is either 1 or a conjunction of proposition letters.

  Then we have $(\mathbf{Clop}(X), \prec, V) \models \forall \bar{p} : [(\theta \wedge \bigwedge_{l=1}^{k} S_l(p_l)) \approx 1]$ if and only if $(\mathbf{Clop}(X), \prec, V) \models \theta \approx 1$ and $(\mathbf{Clop}(X), \prec, V) \models \forall \bar{p} : [(\bigwedge_{l=1}^{k} S_l(p_l)) \approx 1]$, that is if and only if:

  - there is $\theta_h = \wedge_s(\varphi_{hs} \diamond \psi_{hs})$ and elements $\bar{a}, \bar{b}$ such that $\bar{a} R \bar{b}$ and $a_s \in V(\varphi_{hs}), b_s \in V(\psi_{hs})$;
    (note that this is similar to what happened in Theorem 6.1.15)

[15] Note that the definition of $ST(\cdot)$ depends on the choice of $\{i_l, j_l\}_l$.

– for all l there exists i_l, j_l and elements $c_l \in V(\mu_{li_l}), d_l \in V(\nu_{lj_l})$ and $e_l \in$X s.t. $c_l R e_l$ and $d_l R e_l$, as in the previous case.

Hence, if we define the finite valuation $V_0 = V(\{h, i_l, j_l, c_l, d_l, e_l\}_l)$ by:

$V_0 : q \mapsto \{a_s \mid q \trianglelefteq \varphi_{hs}\} \cup \{b_s \mid q \trianglelefteq \psi_{hs}\} \cup \{c_l \mid q \trianglelefteq \mu_{li_l}\} \cup \{d_l \mid q \trianglelefteq \nu_{lj_l}\}$

arguing as in the previous case and as in Theorem 6.1.15 we have that (6.7) holds if and only if for all $h, \{i_l, j_l\}_l$ we have

$$(X, R) \models \forall \bar{x}, \bar{y}, \bar{u}, \bar{v}, w\Big(\bar{x}R\bar{y} \mathbin{\underline{\wedge}} (\underline{\bigwedge_{l=1}^{k}}(u_l R w \mathbin{\underline{\wedge}} v_l R w)) \mathbin{\underline{\to}} ST(\bar{x}, \bar{y}, \bar{u}, \bar{v}, \chi(\bar{r}))\Big). \tag{6.13}$$

So we can conclude that $(\mathbf{Clop}(X), \prec) \models \Psi$ if and only if $(X, R) \models \underline{\bigwedge}_{h,\{i_l,j_l\}_l} \beta_{h,\{i_l,j_l\}_l}$, where

$$\beta_{h,\{i_l,j_l\}_l} := \forall \bar{x}, \bar{y}, \bar{u}, \bar{v}, w\Big(\bar{x}R\bar{y} \mathbin{\underline{\wedge}} (\underline{\bigwedge_{l=1}^{k}}(u_l R w \mathbin{\underline{\wedge}} v_l R w)) \mathbin{\underline{\to}} ST(\bar{x}, \bar{y}, \bar{u}, \bar{v}, \chi(\bar{r}))\Big).$$

Therefore, we have obtained the correspondent $\beta(\Psi) := \underline{\bigwedge}_{h,\{i_l,j_l\}_l} \beta_{h,\{i_l,j_l\}_l}$.

□

Example 6.2.6. The following are the correspondents of the Sahlqvist statements provided in Example 6.2.2:

1. $\forall x, y, w \;:\; (xRw \mathbin{\underline{\wedge}} wRy \mathbin{\underline{\to}} xRy)$;
2. $\forall x, y, w \;:\; (xRw \mathbin{\underline{\wedge}} yRw \mathbin{\underline{\to}} xRy)$;
3. $\forall x, y, w \;:\; (wRx \mathbin{\underline{\wedge}} yRw \mathbin{\underline{\to}} xRy)$;
4. $\forall x, y, w \;:\; (wRx \mathbin{\underline{\wedge}} wRy \mathbin{\underline{\to}} xRy)$.

As we did in Example 6.1.17, here we work out the algorithm resulting from the proof of Theorem 6.1.15 to compute the first correspondent, which is relative to the Sahlqvist statement:

$$\forall q, r : \Big[\forall p : \Big((q \diamond p \;\vee\; \neg p \diamond r) \;\approx 1\Big) \mathbin{\underline{\to}} (q \diamond r \;\approx\; 1)\Big].$$

Here we are in the case $\theta = 1$. Given a subordination space (X, R), minimal valuations making $(\mathbf{Clop}(X), \prec, V_0)$ into a model of $\forall p : \Big((q \diamond p \vee \neg p \diamond r) \;\approx 1\Big)$ are those such that there are elements $c \in V(q), d \in V(r), e \in X$ such that cRe and eRd. As in our proof of correspondence, we use variables x, y, w to replace such elements c, d, e, and we use these variables to define the basic predicates for the Standard Translation:

$$\begin{array}{rclcl} ST(u,x,y,q) & = & Q(u,x,y) & := & (u \approx x) \\ ST(u,x,y,r) & = & R(u,x,y) & := & (u \approx y) \end{array}$$

and we obtain the correspondent

$$\forall x, y, w\Big((xRw \mathrel{\underline{\wedge}} wRy) \mathrel{\underline{\to}} ST(x, y, q \diamond r)\Big).$$

By Definition 6.1.12, this is equal to

$$\forall x, y, w\Big((xRw \mathrel{\underline{\wedge}} wRy) \mathrel{\underline{\to}} \exists u, v : [uRv \mathrel{\underline{\wedge}} ST(u, x, y, q) \mathrel{\underline{\wedge}} ST(v, x, y, r)]\Big)$$

which, by our above definition of basic predicates, is equal to

$$\forall x, y, w\Big((xRw \mathrel{\underline{\wedge}} wRy) \mathrel{\underline{\to}} \exists u, v : [uRv \mathrel{\underline{\wedge}} (u \approx x) \mathrel{\underline{\wedge}} (v \approx y)]\Big)$$

which is finally equvalent to

$$\forall x, y, w\Big((xRw \mathrel{\underline{\wedge}} wRy) \mathrel{\underline{\to}} xRy\Big).$$

Remark 6.2.7. Note that the correspondent which we computed in the previous example, which expresses transitivity of R, is the same which corresponds to property (Q7), as proved in Lemma 2.1.12. Indeed, it is easy to check that an algebra $(B, \prec)$ satisfies (Q7) if and only if it satisfies statement 1. of Example 6.2.2.

We use Theorem 6.2.5 to conclude this section with the following completeness result:

Corollary 6.2.8. *If (ρ) is a Sahlqvist rule, then the system $\mathcal{S} + (\rho)$ is sound and complete with respect to the class of subordination spaces (X, R) where R is a reflexive and symmetric binary relation, and $(X, R) \models \beta(\Phi_\rho)$.*

Proof. By Theorem 4.1.5, the system $\mathcal{S} + (\rho)$ is sound and complete with respect to the class of algebras $K_\rho := \{(B, \prec) \text{ contact algebra} \mid (B, \prec) \models \Phi_\rho\}$. As we discussed earlier in this chapter, by the duality between subordination spaces and Boolean algebras with subordinations, interpreting our formulas in an algebra $(B, \prec)$ is equivalent to interpreting them in subordination spaces (X, R) under clopen valuations. Therefore, the system $\mathcal{S} + (\rho)$ is sound and complete with respect to the class of duals of algebras in K_ρ, namely

$$\{(X, R) \text{ subordination space } \mid (X, R)^+ \text{ contact algebra and } (X, R)^+ \models \Phi_\rho\}.$$

As it follows by Lemma 2.1.12, we have that $(X, R)^+$ is a contact algebra if and only if the relation R is reflexive and symmetric. Moreover, since ρ is a Sahlqvist rule, and hence by definition Φ_ρ is a Sahlqvist statement, by Theorem

6.2.5 we obtain that $(X,R)^+ \models \Phi_\rho$ if and only if $(X,R) \models \beta(\Phi_\rho)$. Thus, we can conclude that $\mathcal{S}+(\rho)$ is sound and complete with respect to the following class:

$$\{(X,R) \text{ subordination space } \mid R \text{ reflexive and symmetric, and } (X,R) \models \beta(\Phi_\rho)\}$$

and this proves the statement of this corollary. $\square$

Conclusion

In this chapter, we work with semantics with respect to subordination spaces. Following the work of Balbiani and Kikot [2], we define a fragment of Sahlqvist formulas and we prove a Sahlqvist correspondence theorem. Moreover, as in this thesis we introduced Π_2-rules associated to $\forall\exists$-statements, we define Sahlqvist statements and we prove a correspondence theorem also for them. This results in a completeness theorem for logics with our Π_2-rules with respect to subordination spaces satisfying some elementary conditions. The correspondence results of this chapter extend Lemma 2.1.12, in finding conditions on algebras which correspond to elementary condition on the closed relation R of their dual subordination spaces (X,R).

Chapter 7

Conclusion and Future Work

In this chapter, we summarize the content of this thesis, and point directions for future work.

7.1 Conclusion

In this thesis we presented a finitary system which we showed to be sound and complete with respect to compact Hausdorff spaces. Before obtaining this completeness result we took several steps.

First, we introduced a finitary system $\mathcal{S}$ which we showed to be complete with respect to contact algebras (see Definition 2.1.3). We also introduced Π_2-rules, and we considered extensions of $\mathcal{S}$ with such rules. Thus we contributed to developing the theory of Π_2-rules, by giving a model-theoretic criterion for establishing admissibility of Π_2-rules in $\mathcal{S}$, and by showing that there is a correspondence between such extensions of $\mathcal{S}$ and inductive classes of contact algebras. More precisely, we showed that given an extension of $\mathcal{S}$ with a set of Π_2-rules, there exists an inductive class of contact algebras with respect to which the system is strongly complete, and vice versa starting from an inductive class K of contact algebras we can find a set of Π_2-rules which, if added to $\mathcal{S}$, result in a system complete with respect to K.

Since the class of compingent algebras (see Definition 2.1.4) is an inductive class of contact algebras, by the aforementioned correspondence we could identify two specific Π_2-rules (ρ7) and (ρ8) which make the system $\mathcal{S} + (\rho7) + (\rho8)$ sound and complete with respect to compingent algebras. Moreover, using our criterion of admissibility, we proved that rules (ρ7) and (ρ8) are admissibile in $\mathcal{S}$.

As a last step towards a completeness result with respect to compact Hausdorff spaces, we used MacNeille completions to show that $\mathcal{S} + (\rho7) + (\rho8)$, and then by de Vries duality we could conclude that this system is also complete with respect to compact Hausdorff spaces.

Finally, we proved Sahlqvist correspondence theorems with respect to semantics of subordination spaces (see Definition 2.1.5). After defining Sahlqvist formulas for our language, we showed that a subordination space (X, R) validates a Sahlqvist formula φ if and only if (X, R) satisfies a first-order sentence which is effectively computable from φ. Then, we identified particular $\forall\exists$-sentences in the language of the algebras which we are considering in this thesis, and we showed that an algebra satisfies[1] the $\forall\exists$-sentence Φ if and only if its dual subordination space satisfies a first-order correspondent of Φ, again effectively computable from the starting $\forall\exists$-sentence. Since we showed that $\forall\exists$-sentences correspond to Π_2-rules, the latter result can be regarded as a Sahlqvist correspondence for Π_2-rules.

7.2 Future work

In this section we list some questions and ideas for future research. We discuss some of them in detail:

- Topo-bisimulations:

  In Definition 5.2.1, we defined topological models for our formulas. The next natural question is how to define a notion of *topo-bisimulation* between such models. An interesting direction for the future could be an investigation of the properties and behaviour of topo-bisimulations in this context. For example, can we prove an analogue of van Benthem's bisimulation characterization theorem in this framework? We could also investigate how this notion of bisimulation fits into the categorical setting presented in Chapter 2, and study its coalgebraic aspects.

  We have some initial results in this direction. We can define a topo-bisimulation in the following way:

  Definition 7.2.1. *Given two topological models (X, V) and (X', V'), a* topo-bisimulation *between the two is a binary relation $Z \subseteq X \times X'$ which satisfies the following properties:*

  1. $\forall p \in Prop, x \in X, x' \in X'$, if xZx' then $(x \in V(p) \Leftrightarrow x' \in V'(p))$;
  2. $\forall U \subseteq X, U' \subseteq X'$ open subsets, both $Z[U]$ and $Z^{-1}[U']$ are open subsets;
  3. Z is total, that is for all $x \in X$ there exists $x' \in X'$ such that xZx', and for all $x' \in X$ there exists $x \in X$ such that xZx'.

  As a starting point, we can prove that this notion of topo-bisimulation satisfies the following desirable property:

  Proposition 7.2.2. *If Z is a topo-bisimulation between topological models (X, V) and (X', V'), and if $x \in X, x' \in X'$ are such that xZx', then for all formulas φ we have*

  $$x \in V(\varphi) \quad \Leftrightarrow \quad x' \in V'(\varphi).$$

[1] Here with satisfies we mean as a first-order structure.

- Further non-standard rules:
  Standard inference rules correspond to quasi-equations, multi-conclusion rules[2] correspond to universal sentences, and as we showed in Chapter 4 Π_2-rules correspond to $\forall\exists$ sentences. In light of this, it is natural to ask ourselves whether we can find other forms of non-standard rules, corresponding to other kinds of first-order formulas.

- Enriching our language:
  By enriching our language, we might be able to express more topological properties, and hence obtain further logics for classes of topological spaces. For example, we can consider studying the μ-version of our language. This language can be interpreted in de Vries algebras extended with the least and greatest fixed point operators. These operators are well defined, because de Vries algebras are complete.

  Another idea is to extend our language with the standard modal operator $\Box$. As semantics for this extension we can use modal de Vries algebras, which were introduced in [6].

- Given a compact Hausdorff space X, we consider the relation $\prec$ on $RO(X)$ defined as $U \prec V$ if $\mathbf{Cl}(U) \subseteq V$. Our logical investigation stems from the properties which characterize algebras of the form $(RO(X), \prec)$. One could investigate the properties of algebras $(RO(X), \prec)$ in which $\prec$ is defined, instead, as $U \prec V$ if $d(U) \subseteq V$, where d is the derivative operator[3].

[2] An introduction to multi-conclusion rules can be found, e.g., in [7]

[3] The derivative of U is defined as the set of limit points of U. See [57, Section 3.1] for the survey of the semantics of modal logic via the d-operator.

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