Action Models in Inquisitive Logic

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

In Dynamic Epistemic Logic, action models are used to encode situations in which agents are uncertain about information being exchanged. These models show how publicly or privately announced statements can bring about change in knowledge. In real communication, questions also play a big part. Asking a question can raise an issue, which typically triggers the recipient to resolve it. Therefore, adding questions and issues to action models makes them more suitable as models of communication.

In this thesis, we merge the ideas of Inquisitive Epistemic Logic (IEL) and Action Model Logic (AML) in three different ways. This results in three systems, which can encode the exchange of statements and questions and compute their effects on knowledge and issues. Each of these systems can be regarded as a conservative extension of both AML and IEL, and is provided with a sound and complete axiomatization.

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Contents

Chapter 1

Introduction

1.1 Motivation

Let us start by announcing a joyful occasion: Anna and Bob are getting married. The couple has invited all their friends and family to the ceremony. However, not everyone has responded to their invitation yet. Today, Anna receives a phone call from Peter, who tells her that either he will attend (p) or that he will not ($\neg p$). Meanwhile, Bob is watching television from the couch. Although he knows that Anna is on the phone with Peter, he cannot hear the conversation. When Peter announces to Anna that he is attending, Anna's knowledge about this fact changes, while Bob's knowledge about this fact does not: the only thing that Bob knows is that Anna now knows whether Peter is attending.

Situations like these can be represented in Dynamic Epistemic Logic (DEL). We encode the knowledge agents have in a model by specifying which possible worlds they can distinguish. An epistemic event takes place that transforms this epistemic model: namely, Anna learns that p. While Anna knows exactly which epistemic event takes place, Bob only knows what the possibilities are. As a consequence, Anna now considers only p-worlds to be possible. In contrast, Bill still considers both p- and $\neg p$-worlds.

Later, Anna gets another phone call. This time it is Quinn, who may also announce whether he is attending (q or $\neg q$). However, Anna and Bob know that he may also call for another reason: namely, to ask whether his ex-wife Rita will be attending ($?r$), so he can base his decision on this. In fact, this is what he asks Anna. She replies that she does not know this yet, but she promises him that she will find out.

This second situation is one that cannot be modelled in standard DEL, since the epistemic event that occurs involves a question rather than a statement. This question has an effect on Anna: namely, it becomes her goal to find out whether Rita is attending, so she can communicate this to Quinn. This question was not asked publicly, since Bob, who is still on the couch, has not heard. It seems that we cannot capture such a situation in a dynamic epistemic model in the familiar way: we need to rethink the notion of epistemic models and the ways in which they are updated.

1.2 Epistemic logic, dynamics and inquisitiveness

Traditionally, epistemic models are used to capture the knowledge of agents. These are standard Kripke models, which include an **S5** accessibility relation for each agent, specifying which possible worlds are indistinguishable to them. We express the knowledge of agents using the modality K: $K_a\varphi$ is true in a world w just in case φ is true in all worlds indistinguishable to agent a from w.

The idea behind dynamic epistemic logic is to add actions that have the ability to transform this epistemic model. In the most basic variant, Public Announcement Logic (PAL) (Plaza, 1989; Gerbrandy & Groeneveld, 1997), actions are public announcements of some formula φ, that have the effect that all worlds where φ is false are removed from the model. The language extends standard epistemic logic with new logical operators called *dynamic modalities*. The idea is that a formula with a dynamic modality, such as $[\varphi]\psi$, meaning 'after φ, ψ is the case', gets its truth value in world w of epistemic model M by evaluating ψ in w of the updated model M^φ, which is the model that results from announcing φ publicly in M.

But announcements do not necessarily have to be public: in more advanced frameworks such as Action Model Logic (AML) (Baltag et al., 1998), updates are performed using action models. These are Kripke models that encode possible epistemic events and accessibility relations for all agents, thereby encoding the knowledge agents have about which epistemic event is occurring. In this way, we can encode epistemic events that are not public, but only (fully) perceived by a subset of the agents.

The asking of a question is also an epistemic event, but a different one: intuitively, the primary effect of asking a question is not on the knowledge of agents, but on their *issues*. This idea is the basis of Inquisitive Dynamic Epistemic Logic (IDEL) (Ciardelli & Roelofsen, 2015). The models of this logic are not standard epistemic models but *inquisitive* epistemic models: they encode not only knowledge, but also issues of agents. The dynamic actions that IDEL models are public utterances: a generalization of public announcements that also includes the public asking of questions.

Because IDEL can encode only *public* utterances, all agents are always aware of the content of the utterance, just like in PAL. As a result, they will all learn that the announced information was true or, often, entertain the new question that was asked.[1] In our scenario, the utterance of Quinn is not a public one, since he utters it to Anna and not to Bob. That means the utterance will definitely have a different effect on Anna than on Bob. To model such scenarios, IDEL is not a suitable framework.

Both AML and IDEL can be seen as extensions of PAL: on the one hand, IDEL makes PAL more general by encoding the issues of agents, thereby making it possible to encode the effect of questions. On the other hand, AML makes PAL more general by allowing not only public announcements, but also more private announcements. However, to model situations like our example we need a mix of both: a model that can encode issues, questions and (semi-)private epistemic actions. The goal of this thesis is to provide a system that can encode the epistemic actions familiar from AML, but in which actions can change an epistemic scenario not only by providing new information, but also by raising new issues.

We start by discussing some background literature in the next chapter. At the end of that chapter, we will outline the structure of the rest of this thesis.

[1]Strictly speaking, it is not *always* the case that the question asked is entertained by all agents, because of the subtle fact that the presupposition of a question may have become false by the public utterance itself. This is analogous to self-defeating announcements such as 'φ and no one knows that φ', which, if true, becomes false once publicly announced.

Chapter 2

Background

2.1 Introduction

In this chapter we lay the foundation for this thesis, by discussing the literature on which we will build. We start the chapter with a brief introduction to questions in logic. The goal of this introduction is to illustrate the scientific context of this thesis. Additionally, it provides a motivation for the use of inquisitive epistemic logic, rather than some other logic, as a basis for a dynamic logic of questions.

After this overview, we describe inquisitive epistemic logic formally and we introduce some definitions and facts about this logic that we will use and build on in the coming chapters. Then, in Section 2.4, we briefly introduce standard action models and show how they are used to extend standard epistemic logic with dynamic modalities.

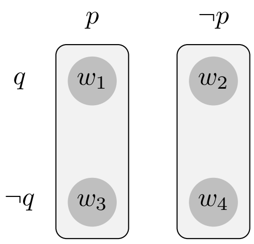
We end this chapter with a first step towards the original material presented in this thesis: we determine the strategy that we will follow and we explain how the rest of the thesis is set up.

2.2 Questions in logic

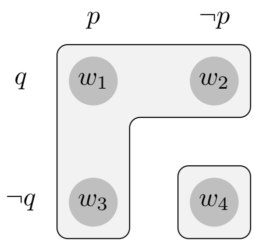
2.2.1 Partition semantics

The meaning of sentences is often associated with their truth conditions. While this is a suitable notion of meaning for statements, it is not for questions: we do not think of questions as being true or false. In most theories about the meaning of questions, predominantly originating in the philosophy of language and formal semantics, the meaning of a question is therefore not a basic proposition, but rather built up out of several propositions. Namely, the propositions denoted by the possible answers to the question in a certain world (Hamblin, 1973; Karttunen, 1977). Logic has traditionally been used to reason about the relations between propositions, which meant that questions did not play a role there (Ciardelli, 2016, chapter 1).

The idea that the meaning of a question is determined by its answers has been formalized by Groenendijk & Stokhof (1984). In their system, the meaning of a question is a function from worlds to propositions. Namely, questions denote in each world their true exhaustive answer. Because of this exhaustivity, different answers are always mutually exclusive. Moreover, in every world, one of the answers is true. Therefore, this conception of questions is known as *partition semantics*: we can think of the meaning of a question as dividing the logical space in cells, which form a partition (see Figure 2.1).



(a) The question whether p is the case



(b) The question whether $p \vee q$ is the case

Figure 2.1: Examples of question representations in partition semantics

This way of thinking about questions has been very influential, and has been used in various frameworks of formal semantics (see Aloni et al. (2007) for an overview). The idea behind these frameworks is that questions give rise to issues, which can be modelled by an indifference relation between possible worlds, reflecting the partition that the question gives rise to.

However, the partition conception of questions turned out to have some limitations: for instance, it cannot account for conditional questions such as 'If your knee hurts, will you play in the finals?'. Such a question, arguably, gives rise to two exhaustive answers that overlap (namely, 'if my knee hurts, I will play in the finals' and 'if my knee hurts, I will not play in the finals', see Figure 2.2). (Groenendijk & Roelofsen, 2009).

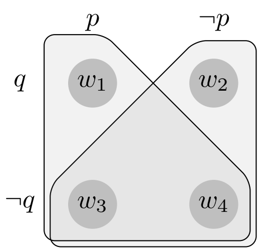


Figure 2.2: If your knee hurts (q), will you play in the finals (p)?

Although at first the formal properties of the indifference relation were adapted in order to solve these problems (Groenendijk, 2007; Mascarenhas, 2009), this eventually led to the development of a more general semantics of questions, in which the idea of an indifference relation was abandoned altogether (Ciardelli & Roelofsen, 2011).

2.2.2 Inquisitive semantics

An alternative way of representing the meaning of questions is by specifying what information is needed to resolve them. We say that an information state, which is a set of worlds, represents a body of information that someone may have. The meaning of a question is then a set of such information states, namely the ones in which the question is resolved. These information states are similar to the cells in a partition, familiar from partition semantics. However, these information states are not mutually exclusive, which means they can overlap.

This is the basic idea behind *inquisitive semantics* (Groenendijk & Roelofsen, 2009; Ciardelli & Roelofsen, 2011). The notion of question meaning in inquisitive semantics is inspired by partition semantics, but it is strictly more general than that.

In inquisitive semantics, the set of information states that represents the meaning of a question is called an *inquisitive proposition*. It is crucial to understand that unlike standard propositions, this is not a set of worlds, but a set of information states. Inquisitive propositions are always *downward closed*: they contain all subsets of all the sets they contain. This captures the fact that if being in some information state resolves a question, then so does being in a more specific information state.

To simplify notation of downward closed sets, we use the notation $S^{\downarrow}$ to indicate the downward closure of S. In this way, we can often write only the maximal elements of the set, instead of all subsets. For instance, the downward closure $\{\{w_1, w_3\}\}^{\downarrow}$ of $\{\{w_1, w_3\}\}$ is $\{\{w_1, w_3\}, \{w_1\}, \{w_3\}, \emptyset\}$.

Apart from the fact that inquisitive propositions are more general than partitions, there is also another advantage: statements can be modelled in the very same way. They also denote downward closed sets of information states, but a special case. Namely, the ones that have only one maximal element, and therefore do not give rise to multiple alternatives. This means that in inquisitive semantics, both statements and questions express inquisitive propositions and are thus both basic.

Although inquisitive semantics was primarily developed to model questions in natural language, it also provides a new perspective on questions in logic: since questions as well as statements can now denote (inquisitive) propositions, it becomes possible to investigate their role in logic as well. For instance, we now have a notion of entailment in which both statements and questions can participate (Ciardelli, 2016, chapter 1). It is worth noting that inquisitive semantics brought a new perspective on pragmatics as well, since it it allows for formalizations of pragmatic maxims that prescribe how to rationally exchange information (Groenendijk & Roelofsen, 2009; Westera, 2013). Although pragmatics are not within the scope of this thesis, we will return to this briefly in Section 6.2.4.

2.2.3 Dynamic epistemic logics

In this thesis we are interested in dynamic epistemic logic: in particular, we want to describe the exchange of information and questions and compute the results of this exchange. Although questions cannot be modelled in standard dynamic epistemic logic, some variants have been developed in which this is possible. Most importantly, Baltag (2001) and Van Benthem & Minică (2012) have both developed a dynamic logic in which they model the act of asking a question and the effect this has on the issues of agents. The common element in their approaches is to make questions special types of actions that, unlike regular actions, do not affect the epistemic relation, but a different relation: namely, the indifference relation which encodes the issue partition of each agent. However, in Baltag (2001), questions may also have an effect on the epistemic relation: namely, when some agent asks a question, under certain assumptions the other agents may come to know that this agent does not know the answer.

Although these logics are developed with similar intentions as the logics we develop in this thesis, they are based on the partition notion of questions and issues, which means the questions and issues that can be encoded in these systems is limited. We will come back to this in Section 4.10, when we compare one of the logics developed in this thesis to the logic of Van Benthem & Minică (2012).

As we mentioned before, inquisitive logic and dynamic epistemic logic are merged in the framework of Inquisitive Dynamic Epistemic Logic (IDEL) (Ciardelli & Roelofsen, 2015).

This logic can be used to express public utterances of statements and questions, and encode their effect on the knowledge and issues of agents. Since this system does not encode (semi-)private utterances, it is not yet enough to model the situation discussed in the introduction, but we will see that it is an important first step towards the systems that we develop in this thesis.

2.3 Inquisitive Epistemic Logic

2.3.1 Inquisitive Epistemic Models

Before we take a look at IDEL, we will first introduce its static fragment Inquisitive Epistemic Logic (IEL). Since this will be the static basis for the logics we develop in this thesis as well, we will discuss this in a more formal way. All definitions and facts given in this section are needed in later chapters.

We have already seen that in inquisitive logic, propositions are identified with downward closed sets of information states rather than with sets of worlds. As a result, inquisitive logics can express not just statements, but also questions. While a proper introduction would start with the most basic variant of inquisitive logic (InqB) (Ciardelli, 2016, chapter 2), in this context it makes more sense to look at its epistemic variant. This is the logic IEL, which extends InqB with epistemic modalities.

Models of inquisitive epistemic logic encode not only the knowledge agents have, but also their issues or epistemic goals: the things they want to find out. In IEL we can express propositions about the knowledge of agents, but also about the issues they entertain.

Definition 2.3.1. **Inquisitive Epistemic Model** (Ciardelli & Roelofsen, 2015, p. 1650)
An inquisitive epistemic model is a triple $M = \langle W, \{\Sigma_a \mid a \in \mathcal{A}\}, V \rangle$ where:

- W is the domain of worlds;
- $\mathcal{A}$ is the domain of agents;
- Σ_a is a state map for agent a. For each world w, $\Sigma_a(w)$ is the inquisitive proposition that encodes a's knowledge state and issues at w. $\Sigma_a(w)$ is a non-empty downward closed set of information states. The knowledge state of a is $\sigma_a(w) = \bigcup \Sigma_a(w)$: the set of worlds she considers possible at w. Her goal is to get her knowledge state to be one of the information states in $\Sigma_a(w)$;
- $V : \mathcal{P} \to \wp(W)$ is a valuation function that specifies for each atomic proposition in which worlds it is true.

In this thesis, we restrict ourselves to truly epistemic cases. That is, we think about knowledge as factive: agents always consider the actual world as one of the possible worlds. Additionally, we assume that the agents have knowledge about their knowledge and issues. Formally, this amounts to requiring that the state maps satisfy the following two conditions:

- *Factivity:* for all $w \in W$, $w \in \sigma_a(w)$
- *Introspection:* for all $w, v \in W$, if $v \in \sigma_a(w)$ then $\Sigma_a(v) = \Sigma_a(w)$

Example 2.3.1. **Inquisitive Epistemic Model**
As an example, take the following model M with $W = \{w_1, w_2, w_3, w_4\}$, $V(p) = \{w_1, w_3\}$, $V(q) = \{w_1, w_2\}$ and two agents a and b. Suppose a knows whether p and not whether q,

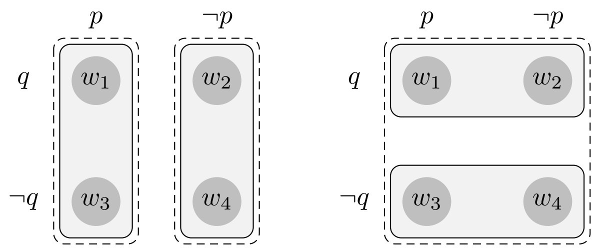
but does not care about it. b does not know whether p or whether q, but wants to know whether q. The model will look like this:

$\Sigma_a(w_1) = \Sigma_a(w_3) = \{\{w_1, w_3\}\}^{\downarrow}$

$\Sigma_a(w_2) = \Sigma_a(w_4) = \{\{w_2, w_4\}\}^{\downarrow}$

$\Sigma_b(w_1) = \Sigma_b(w_2) = \Sigma_b(w_3) = \Sigma_b(w_4) = \{\{w_1, w_2\}, \{w_3, w_4\}\}^{\downarrow}$

Throughout this thesis, we will represent inquisitive epistemic models by diagrams. We follow the conventions of Ciardelli (2016): for each world w, the worlds within the same dashed line are the worlds held possible by the agent in w. The solid lines represent the issues within each epistemic state: these states and their subsets are the ones that the agent strives to be in. Because of downward closure, we only need to draw the maximal states of the state map. The example model M is represented in Figure 2.3.



(a) State map for a (b) State map for b

Figure 2.3: Example of state maps in an inquisitive epistemic model

2.3.2 Syntax and semantics

As we will use the language $\mathcal{L}^{\mathsf{IEL}}$ to describe our models and action content, we now briefly introduce its syntax and semantics here.

Definition 2.3.2. **Syntax of $\mathcal{L}^{\mathsf{IEL}}$** (Ciardelli, 2016, p. 254)

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \rightarrow \varphi \mid \varphi \vee\!\!\!\vee \varphi \mid K_a\varphi \mid E_a\varphi$$

We use the following abbreviations:

Negation:	$\neg\varphi := \varphi \rightarrow \bot$
Classical disjunction:	$\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$
Wonder modality:	$W_a\varphi := \neg K_a\varphi \wedge E_a\varphi$
Question operator:	$?\varphi := \varphi \vee\!\!\!\vee \neg\varphi$
Tautology:	$\top := \bot \rightarrow \bot$

In order to be able to interpret both statements and questions, formulas in IEL are evaluated relative to information states rather than single worlds. Therefore, we cannot formulate the semantics of inquisitive epistemic logic in terms of truth conditions. Instead, we specify the conditions for a formula to be supported by an information state.

Definition 2.3.3. **Support conditions in IEL** (Ciardelli, 2016, p. 47, 50, 255)
Let s be an information state in inquisitive epistemic model M.

$M, s \models p$	iff	$s \subseteq V(p)$
$M, s \models \bot$	iff	$s = \emptyset$
$M, s \models \varphi \wedge \psi$	iff	$s \models \varphi$ and $s \models \psi$
$M, s \models \varphi \rightarrow \psi$	iff	for all $t \subseteq s$, $t \models \varphi$ implies $t \models \psi$
$M, s \models \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi$	iff	$s \models \varphi$ or $s \models \psi$
$M, s \models K_a\varphi$	iff	for all $w \in s : M, \sigma_a(w) \models \varphi$
$M, s \models E_a\varphi$	iff	for all $w \in s$, for all $t \in \Sigma_a(w) : M, t \models \varphi$

Let us also define the notion of support set, which is simply the set of all information states in a model that support φ.

Definition 2.3.4. **Support set** (Ciardelli, 2016, p. 12)
Let φ be a formula and M an inquisitive epistemic model. Then the support set $[\varphi]_M$ is the set of all information states in M where φ is supported:

$$[\varphi]_M = \{s \subseteq W_M \mid s \models \varphi\}$$

While not a primary notion but a derived one, the notion of truth still plays an important role in inquisitive logic. Truth in a world is defined as support in the corresponding singleton information state.

Definition 2.3.5. **Truth** (Ciardelli, 2016, p. 48)
Let M be an inquisitive epistemic model and w a world.

$$M, w \models \varphi \iff M, \{w\} \models \varphi$$

This lets us obtain the following truth conditions for our connectives:

Fact 2.3.1. **Truth conditions in IEL** (Ciardelli, 2016, p. 49, 50, 255)
Let w be a world in inquisitive epistemic model M.

$M, w \models p$	iff	$w \in V(p)$
$M, w \not\models \bot$		
$M, w \models \varphi \wedge \psi$	iff	$w \models \varphi$ and $w \models \psi$
$M, w \models \varphi \rightarrow \psi$	iff	$w \not\models \varphi$ or $w \models \psi$
$M, w \models \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi$	iff	$w \models \varphi$ or $w \models \psi$
$M, w \models K_a\varphi$	iff	$M, \sigma_a(w) \models \varphi$
$M, w \models E_a\varphi$	iff	for all $t \in \Sigma_a(w) : M, t \models \varphi$

Apart from a support set, we also define a truth set on any formula. This is the set of worlds in a model in which the formula is true.

Definition 2.3.6. **Truth set** (Ciardelli, 2016, p. 12)
Let φ be a formula and M an inquisitive epistemic model. Then the truth set $|\varphi|_M$ is the set of all worlds in M in which φ is true:

$$|\varphi|_M = \{w \in W_M \mid w \models \varphi\}$$

A fundamental notion in support-conditional semantics is that of truth-conditionality. A formula is truth-conditional just in case its meaning is completely determined by its truth conditions. This notion separates statements from questions on the semantic level.

DEFINITION 2.3.7. **Truth-conditionality** (Ciardelli, 2016, p. 26)
A formula φ is *truth-conditional* iff for all models M and states s:

$$s \models \varphi \iff w \models \varphi \text{ for all } w \in s$$

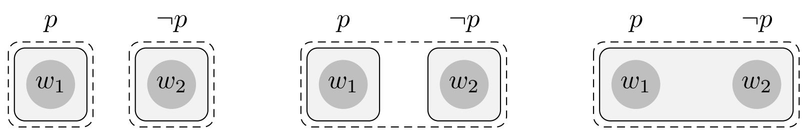
Following Ciardelli (2016), we will regard truth-conditional formulas as corresponding to statements and non truth-conditional formulas to questions. As a convention, statements are often denoted by α or β and questions by μ or ν.

The difference between questions and statements can be illustrated by looking at the difference between a classical disjunction ($p \vee q$) and an inquisitive disjunction ($p \mathbin{\vee\!\!\!\vee} q$): both formulas express the information that p or q is true, but the inquisitive disjunction also raises the issue whether it is p or q, which is resolved by finding out that p or finding out that q. In this way, the inquisitive disjunction expresses a question, while the classical disjunction expresses a statement.

It is important to note that if we look just at the fragment of formulas of IEL that do not contain any inquisitive disjunction or entertain modality, we have exactly the syntax of standard epistemic logic.

2.3.3 Modalities and questions

The modality K_a is familiar from standard epistemic logic and it has the same meaning whenever it is applied to a statement: namely, $K_a\alpha$ is true in a world w just in case α is true in any world indistinguishable from w to agent a. Thus, whenever α is a statement, it expresses that a knows this statement to be true. However, we can now also express knowledge about questions. For instance, $K_a?p$ is true just in case a knows whether p (she either knows that p or that $\neg p$).



(a) $K_a?p$ (b) $E_b?p$ (c) $\neg K_c?p$, $\neg E_c?p$

Figure 2.4: Examples of modalities ranging over questions

While we use the modality K_a to express the knowledge of agents, we use the entertain modality E_a to express their issues. For instance, $E_a?p$ means a *entertains* $?p$: she either knows or wants to know the answer to the question $?p$. It is easy to see that whenever α is a statement, $K_a\alpha$ and $E_a\alpha$ express the same thing, but whenever they range over a question, their meanings come apart.

The wonder modality W_a is an abbreviation of not knowing and entertaining. That is, $W_a\mu$ is true just in case a does not know the answer to the question μ and wants to know it. The wonder modality can only be meaningfully used to range over questions, since $W_a\alpha$ is a contradiction whenever α is a statement.

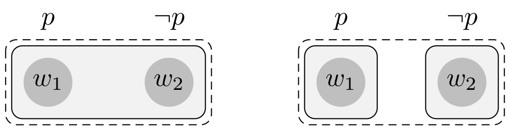
To understand better how these modalities work when they range over questions, let us have a look at Figure 2.4. In subfigure (a), we see from the dashed line that a can distinguish w_1 from w_2. This means that her information state is such that it resolves the question $?p$. Therefore, $K_a?p$ is true.

In subfigure (b), the dashed line indicates that w_1 and w_2 are not distinguishable to b. Therefore, her information state does not resolve the question $?p$. However, the solid lines indicate the information states she wants to be in: she wants to be either in $\{w_1\}$ or $\{w_2\}$. Both of these information states resolve the question $?p$. Therefore, we say that b entertains $?p$: $E_b?p$.

Subfigure (c) depicts the state map of agent c. Just like agent b, her information state does not resolve the question $?p$. The solid line, however, indicates that she is fine with being in this state: she does not strive to be in a more informative information state. Since this means that there is at least one information state in her state map that does not resolve $?p$ (namely $\{w_1, w_2\}$), she does not entertain $?p$: $\neg E_c?p$.

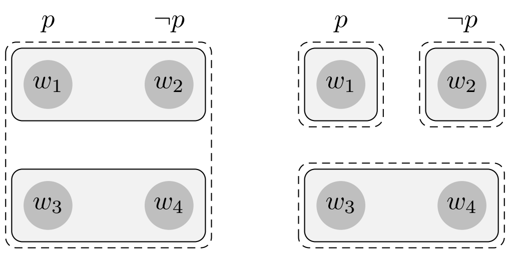
Agents do not just have knowledge and issues about general facts, but also about each others knowledge and issues. Consider the following examples of formulas, taken from Ciardelli (2016, p. 269):

- $K_a?K_b?p$ expresses that a knows whether b knows whether p (Figure 2.5);
- $K_aW_b?p$ expresses that a knows that b wonders whether p (idem);
- $W_a?K_bp$ expresses that a wonders whether b knows that p (Figure 2.6);
- $W_a?W_b?p$ expresses that a wonders whether b wonders whether p (Figure 2.7).



(a) State map for a (b) State map for b

Figure 2.5: a knows whether b knows whether p, a knows that b wonders whether p



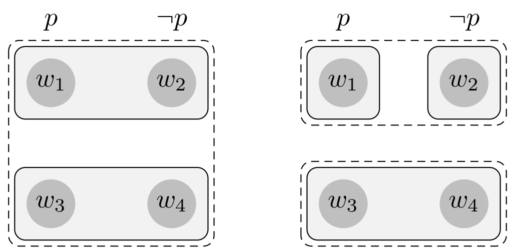
(a) State map for a (b) State map for b

Figure 2.6: a wonders whether b knows that p

2.3.4 Properties of IEL

We will now introduce a few interesting notions that are used in IEL. They will be useful to us when we extend this logic in the coming chapters.[1]

[1] In Ciardelli (2016), some of these definitions are presented in the context of inquisitive modal logic InqBM. Here we give the definitions for IEL, which is a multiagent epistemic interpretation of InqBM. Therefore, our notation differs from that of the original definitions.



(a) State map for a (b) State map for b

Figure 2.7: a wonders whether b wonders whether p

First, we take a closer look at the support relation of IEL. It is a principle of inquisitive logics that whenever some information supports a formula, every enhancement of this information still supports it. Furthermore, the empty set, which is the inconsistent information state, should support all formulas. It is easy to show from the support conditions that these principles indeed hold in IEL.

FACT 2.3.2. **Properties of the support relation** (Ciardelli, 2016, p. 50, 255)
For all models M and formulas $\varphi \in \mathcal{L}^{\mathsf{IEL}}$, we have the following properties:

- *Persistence property:* if $s \models \varphi$ and $t \subseteq s$, then $t \models \varphi$.
- *Empty state property:* $\emptyset \models \varphi$

As we will develop new inquisitive logics in this thesis, it is important to note that it is a desideratum of the support relation of these new logics to have these properties too.

We have seen in the previous section that truth-conditionality separates questions from statements on the semantic level. We now introduce the syntactic notion of declarativity. The set of declaratives of $\mathcal{L}^{\mathsf{IEL}}$ represents all truth-conditional meanings that can be expressed in IEL.

DEFINITION 2.3.8. **Declarative fragment of $\mathcal{L}^{\mathsf{IEL}}$** (Ciardelli, 2016, p. 260)
The set of declarative formulas $\mathcal{L}^{\mathsf{IEL}}_!$ is defined inductively as follows, where $\varphi \in \mathcal{L}^{\mathsf{IEL}}$:

$$\alpha ::= p \mid \bot \mid \alpha \wedge \alpha \mid \alpha \rightarrow \alpha \mid K_a\varphi \mid E_a\varphi$$

FACT 2.3.3. All declarative formulas of $\mathcal{L}^{\mathsf{IEL}}$ are truth-conditional (Ciardelli, 2016, p. 260)

This is a consequence of the fact that the only source of non truth-conditionality is the inquisitive disjunction. Furthermore, the support conditions of the modalities are such that any $K_a\varphi$ and $E_a\varphi$ are truth-conditional. Truth-conditionality is preserved by conjunction and implication.

With each $\varphi \in \mathcal{L}^{\mathsf{IEL}}$ we can associate a corresponding declarative, called the declarative variant of φ, which is defined as follows.

DEFINITION 2.3.9. **Declarative variant in $\mathcal{L}^{\mathsf{IEL}}$** (Ciardelli, 2016, p. 260)
The declarative variant $\varphi^!$ of a formula $\varphi \in \mathcal{L}^{\mathsf{IEL}}$ is defined inductively as follows:

- $\alpha^! = \alpha$ if α is an atom, $\bot$ or a modal formula $K_a\varphi$ or $E_a\varphi$
- $(\varphi \wedge \psi)^! = \varphi^! \wedge \psi^!$

- $(\varphi \to \psi)^! = \varphi^! \to \psi^!$
- $(\varphi \vee\!\!\!\vee \psi)^! = \varphi^! \vee \psi^!$

By induction it can be checked that although φ and $\varphi^!$ are not necessarily supported in the same information states, they are true in exactly the same worlds. We have the following fact:

FACT 2.3.4. **Declarative variants in $\mathcal{L}^{\mathsf{IEL}}$ have equal truth conditions** (Ciardelli, 2016, p. 261)
For all $\varphi \in \mathcal{L}^{\mathsf{IEL}}$, for every inquisitive epistemic model M and world w:

$$M, w \models \varphi \iff M, w \models \varphi^!$$

We have already seen that $\mathcal{L}^{\mathsf{EL}}$ is a fragment of $\mathcal{L}^{\mathsf{IEL}}$. We can now also see that by the definition of declaratives, all formulas of $\mathcal{L}^{\mathsf{EL}}$ are declaratives, and therefore truth-conditional. Since the truth conditions of $\mathcal{L}^{\mathsf{EL}}$-formulas are standard in IEL, IEL is a conservative extension of EL.

Note that while declaratives and declarative variants are truth-conditional, they are not necessarily formulas of standard epistemic logic, because they might still contain the entertain modality or an inquisitive disjunction (the latter only within the scope of a modality). The notion of declarative variants will be used to define the preconditions of epistemic actions that may be questions.

Notice that if φ is itself a truth-conditional formula, then $\varphi \equiv \varphi^!$. This means that declaratives are representative of truth-conditional formulas. We thus have the following fact.

FACT 2.3.5. **Any truth-conditional formula is equivalent to a declarative** (Ciardelli, 2016, p. 261)
For all $\varphi \in \mathcal{L}^{\mathsf{IEL}}$, if φ is truth-conditional, then there is some $\alpha \in \mathcal{L}_!^{\mathsf{IEL}}$ such that $\varphi \equiv \alpha$.

We now move on to the notion of resolutions. The set of resolutions of a formula in IEL is a set of declaratives that we associate with it.

DEFINITION 2.3.10. **Resolutions in $\mathcal{L}^{\mathsf{IEL}}$** (Ciardelli, 2016, p. 261)
For any formula $\varphi \in \mathcal{L}^{\mathsf{IEL}}$, its set of resolutions $\mathcal{R}(\varphi)$ is defined inductively as follows:

- $\mathcal{R}(\alpha) = \{\alpha\}$ if α is an atom, $\bot$ or a modal formula $K_a\varphi$ or $E_a\varphi$
- $\mathcal{R}(\varphi \wedge \psi) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\varphi) \text{ and } \beta \in \mathcal{R}(\psi)\}$
- $\mathcal{R}(\varphi \to \psi) = \{\bigwedge_{\alpha \in \mathcal{R}(\varphi)}(\alpha \to f(\alpha)) \mid f \text{ is a function from } \mathcal{R}(\varphi) \text{ to } \mathcal{R}(\psi)\}$
- $\mathcal{R}(\varphi \vee\!\!\!\vee \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$

These resolutions are relevant, because it can be shown by induction that for a state to support a formula φ is to support at least one of its resolutions. As these resolutions are declaratives, this fact allows us to connect support for declaratives with support for inquisitive formulas. This is often used in proofs, for instance in completeness proofs for inquisitive logics.

FACT 2.3.6. **A formula is supported iff some resolution of it is** (Ciardelli, 2016, p. 261)
For all $\varphi \in \mathcal{L}^{\mathsf{IEL}}$, for every inquisitive epistemic model M and state s:

$$M, s \models \varphi \iff M, s \models \alpha \text{ for some } \alpha \in \mathcal{R}(\varphi)$$

From this fact and the support condition of inquisitive disjunction it follows that any formula $\varphi \in \mathcal{L}^{\mathsf{IEL}}$ is equivalent to the inquisitive disjunction of its resolutions, which is called the *normal form* of the formula.

FACT 2.3.7. **Normal form**
For all $\varphi \in \mathcal{L}^{\mathsf{IEL}}$, $\varphi \equiv \bigvee\!\!\!\!\bigvee \mathcal{R}(\varphi)$.

The above notions all play an important part in IEL and inquisitive logics in general, and we will refer to them often in the coming chapters.

2.3.5 Dynamics

We end this section on IEL with a brief introduction to the dynamic variant IDEL (Ciardelli & Roelofsen, 2015; Ciardelli, 2016). This is especially relevant because the logics we develop in this thesis can be seen as generalizations of this logic.

A public utterance is the inquisitive counterpart of a public announcement: it is an epistemic action that transforms the model when it is executed. When a statement is announced, all worlds in which the statement is false are dropped, like in Public Announcement Logic (PAL). However, in IDEL a public utterance can also be a question. In that case, agents will typically come to entertain the question in the updated model: they will make it their goal to resolve it.

The following definition contains the procedure by which we can update a model with a public utterance, which makes use of the notions of truth set and support set.

DEFINITION 2.3.11. **Updated model** (Ciardelli, 2016, p. 311)
Let $M = \langle W, \{\Sigma_a \mid a \in \mathcal{A}\}, V\rangle$ be an inquisitive epistemic model and $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$. Then the model updated with φ is $M^\varphi = \langle W^\varphi, \{\Sigma_a^\varphi \mid a \in \mathcal{A}\}, V^\varphi\rangle$, defined as follows.[2]

- $W^\varphi = W \cap |\varphi|_M$
- $V^\varphi = V \restriction_{|\varphi|_M}$
- $\Sigma_a^\varphi(w) = \Sigma_a(w) \cap [\varphi]_M$

This definition says that updating an inquisitive epistemic model with a formula φ has two effects. First, the worlds in which φ is false are dropped from the model. Second, for each agent a and world w, the information states that count as resolving the issues of a in w become restricted to the ones that support φ. See Figure 2.8 for an illustration of the effect of a public utterance of a statement and a question.

The language $\mathcal{L}^{\mathsf{IDEL}}$ extends $\mathcal{L}^{\mathsf{IEL}}$ by adding a public utterance operator $[\varphi]$ to the inductive definition of the syntax. Then the following support condition is added:

DEFINITION 2.3.12. **Support condition for dynamic modalities** (Ciardelli, 2016, p. 313)
$M, s \models [\varphi]\psi \iff M^\varphi, s \cap |\varphi|_M \models \psi$

This means that $[\varphi]\psi$ is supported in information state s of model M just in case ψ is supported in the information state s, restricted to the worlds in which φ is true, in the

[2] The definition given here differs slightly from the original definition: we use formulas as the content of public utterances rather than inquisitive propositions, to make the update procedure of IDEL more easily comparable with the actions we introduce later.

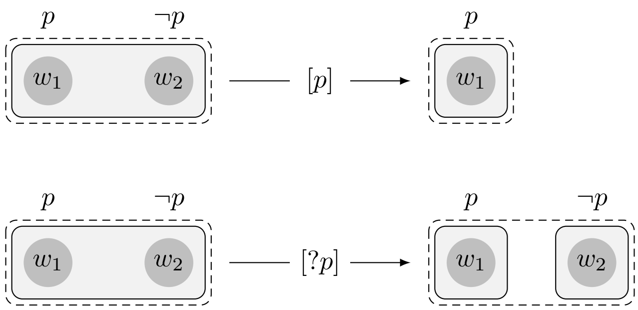


Figure 2.8: Public utterance in IDEL

updated model M^{φ}. In other words, our information state s is such that it would support ψ if it would be enhanced with a public utterance of φ.

As we mentioned before, utterances in IDEL are always public. In the next section, we will look at a way to encode epistemic events that are not public.

2.4 Action Models

In this section we will provide a more formal introduction to standard action models and their logic. Consider again the first scenario from the introduction: Peter tells Anna on the phone that he will be attending the wedding. While Bob is aware of the conversation, he also considers it possible that Peter calls to say he will not attend.

In dynamic epistemic logic, epistemic actions which are more complex than public announcements can be modelled using action models (Baltag et al., 1998). These models encode the knowledge that agents have about the action taking place, which can differ from agent to agent. The action model is a Kripke model containing all possible actions (in the example, the announcement that Peter either is or is not attending), an accessibility relation for each agent (which encodes their knowledge about the action taking place) and for each action a precondition: a formula that encodes what must be true for the action to be possible. Formally, action models are defined as follows.[3]

Definition 2.4.1. **Action Model** (Van Ditmarsch et al., 2007, p. 149)
An action model is a triple $\mathsf{M} = \langle \mathsf{S}, \{\sim_a \mid a \in \mathcal{A}\}, \mathsf{pre}\rangle$ where:

- S is a finite domain of action points;
- For each $a \in \mathcal{A}$, $\sim_a$ is an equivalence relation on S;
- $\mathsf{pre} : \mathsf{S} \to \mathcal{L}$ is a function that assigns a formula $\mathsf{pre}(\mathsf{x}) \in \mathcal{L}$ to each action point $\mathsf{x} \in \mathsf{S}$.[4]

The original epistemic model and the action model are combined to create a new epistemic model: the product update of the two. This new model encodes the knowledge of the agents after the action has taken place.

[3] Although the definitions have also been given elsewhere, we follow Van Ditmarsch et al. (2007, chapter 6).

[4] It is not very important at this point which language $\mathcal{L}$ we use. Throughout this thesis, we will assume that it is either $\mathcal{L}^{\mathsf{EL}}$ or $\mathcal{L}^{\mathsf{AML}}$. In the latter case, the definition of this language has to be given together with the definition of action models, in a recursive way.

Definition 2.4.2. **Updated model** (Van Ditmarsch et al., 2007, p. 151)
Let M be an epistemic model and M an action model. Then $M' = (M \otimes \mathsf{M})$ is the product update of M and M, defined as follows.

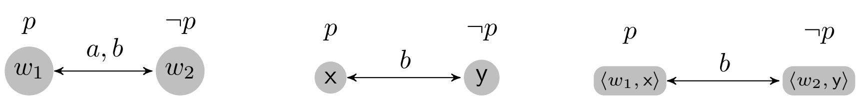
$M' = \langle W', \{\sim'_a \mid a \in \mathcal{A}\}, V' \rangle$, where:

- $W' = \{\langle w, \mathsf{x}\rangle \mid w \in W, \mathsf{x} \in \mathsf{S}$ and $M, w \models \mathsf{pre}(\mathsf{x})\}$
- $\langle w, \mathsf{x}\rangle \sim'_a \langle w', \mathsf{x}'\rangle$ iff $w \sim_a w'$ and $\mathsf{x} \sim_a \mathsf{x}'$
- $\langle w, \mathsf{x}\rangle \in V'(p)$ iff $w \in V(p)$

This update procedure says that the domain of the updated model is the Cartesian product of the domain of the original model and the domain of the updated model, restricted to the pairs that are compatible with each other. For instance, a world in which p is false is not compatible with an action that has precondition p. Hence, we associate with each world in the updated model a world from the original model and an action. Two worlds are indistinguishable in the updated model just in case the corresponding two worlds in the original model and the corresponding two actions are indistinguishable.

To illustrate this update procedure, we will look at two examples. The first one is the first scenario from the introduction.

Example 2.4.1. **Peter is attending**
Initially, neither Anna nor Bob knows whether Peter is attending. Then, action x occurs: Peter tells Anna that he will attend (p). Bob also considers it possible that Peter tells Anna that he will not attend ($\neg p$). Let us call this possible action y. To Bob, actions x and y are not distinguishable, but they are to Anna: by the update procedure, this goes for the corresponding worlds in the updated model as well. See Figure 2.9 for a visual representation of this (we omit all reflexive arrows).



(a) Original model (b) Action model (c) Updated model

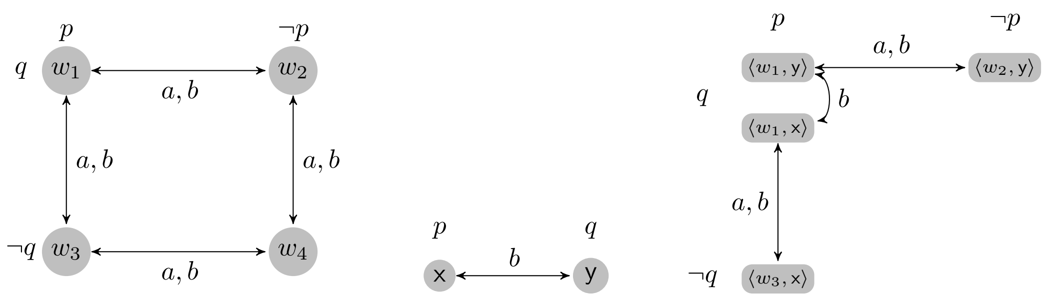
Figure 2.9: Example 2.4.1

Example 2.4.2. **Patrick or Qasim is attending**
This time, Anna and Bob get an RSVP message in the mail. They know that the sender is attending, but they are not sure yet about the identity of the sender: this may either be Patrick (p) or Qasim (q). Anna (a) opens the envelope, while Bob (b) watches from a distance and is not immediately aware of the contents.

Let the initial model and the action model be defined as in Figure 2.10 (this time, we omit both reflexive and transitive arrows). Let $\mathsf{pre}(\mathsf{x}) = p$ and $\mathsf{pre}(\mathsf{y}) = q$. Then the updated model is shown in subfigure (c).

A difference between this example and the previous one is that this time, both agents learn something: namely, since either p or q is true, w_4 is no longer a candidate for the actual world. However, a learns either that p or that q, while b only learns that $p \vee q$.

We end this section on action models by introducing the syntax and semantics of Action Model Logic (AML).



(a) Original model (b) Action model (c) Updated model

Figure 2.10: Example 2.4.2

DEFINITION 2.4.3. **Syntax of $\mathcal{L}^{\mathsf{AML}}$** (Van Ditmarsch et al., 2007, p. 149)
The language of Action Model Logic is defined as follows, where x is an action in action model M:[5]

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_a\varphi \mid [\lambda]\varphi$$
$$\lambda ::= \mathsf{M}, \mathsf{x} \mid \lambda \cup \lambda$$

DEFINITION 2.4.4. **Truth conditions in AML** (Van Ditmarsch et al., 2007, p. 151)
Let w be a world in epistemic model M.

$M, w \models p$	iff	$w \in V(p)$
$M, w \models \neg\varphi$	iff	$M, w \not\models \varphi$
$M, w \models \varphi \wedge \psi$	iff	$w \models \varphi$ and $w \models \psi$
$M, w \models K_a\varphi$	iff	$w \sim_a w'$ implies $M, w' \models \varphi$
$M, w \models [\lambda]\varphi$	iff	for all M', w' : $(M, w)[\![\lambda]\!](M', w')$ implies $M', w' \models \varphi$
$(M, w)[\![\mathsf{M}, \mathsf{x}]\!](M', w')$	iff	$M, w \models \mathsf{pre}(\mathsf{x})$ and $M', w' = (M \otimes \mathsf{M}), \langle w, \mathsf{x}\rangle$
$[\![\lambda \cup \lambda']\!]$	=	$[\![\lambda]\!] \cup [\![\lambda']\!]$

AML extends standard epistemic logic with dynamic modalities. These modalities can be used to express what would be true if some action were to take place: a formula of the form $[\mathsf{M}, \mathsf{x}]\varphi$ is true just in case φ is true after action x. For instance, $[\mathsf{M}, \mathsf{x}]K_a p$ expresses that after x, a knows that p. It is also possible to formulate non-deterministic actions: a formula of the form $[\mathsf{M}, \mathsf{x} \cup \mathsf{M}, \mathsf{y}]\varphi$ is true just in case φ is true after one of x and y takes place. This is a feature that will be of interest to us later.

2.5 Generalizing AML and IDEL

In the previous sections, we have discussed some extensions of basic epistemic logic (EL) in detail, in particular IEL and AML. Let us now zoom out a bit, to see how these systems relate to each other. The relation between them, and some other logics we have mentioned, is shown in Figure 2.11. We take the arrows down to represent generalization. Arrows to the left stand for an extension of the dynamic possibilities in the system: adding public utterances (EL → PAL, IEL → IDEL) or private utterances (PAL → AML). Arrows to the

[5] Modalities for common knowledge or other group knowledge are not included in this thesis. We will elaborate on this in Section 6.2.2.

right stand for making the logic inquisitive. Replacing the question mark in this diagram with an adequate new system is exactly the goal of this thesis. Notice that this logic can be seen as a generalization of AML (it is the inquisitive variant of it) and as a generalization of IDEL (it adds actions besides public announcements). Because of this, the rest of this thesis is intended to be read as a "possible next chapter" in Van Ditmarsch et al. (2007) (to fit after the introduction of EL, PAL and AML) as well as in Ciardelli (2016) (after the introduction of InqB, IEL and IDEL).

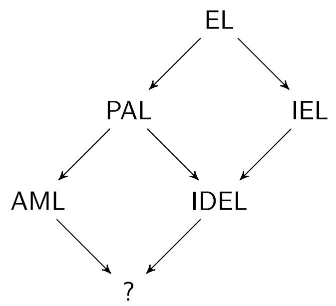


Figure 2.11: Relation between dynamic and inquisitive epistemic logics

Instead of developing one system to fill the gap in the diagram, we will look at more than one candidate. The reason for this is that there are two strategies we can take to make the logic of action models inquisitive. Obviously, we will replace the static language of AML with IEL. However, from this point on, there are still two directions possible. The first option is to make it possible for questions to be the content of an action in the action model. In this way, actions can raise issues as well as convey information. The second option is to change the structure of our action models: the issues that the actions raise can be encoded in the model itself, as issues agents have about which action is the actual one.

We will work out both options in detail. The resulting logic of the first option will be called Action Model Logic with Questions (AMLQ) and the second logic will be Inquisitive Action Model Logic (IAML). We will even take a step further and follow both strategies at once. The result will be a merge of the two previous systems, which we will call Inquisitive Action Model Logic with Questions (IAMLQ). The relation between these three logics and AML and IDEL is shown in Figure 2.12.

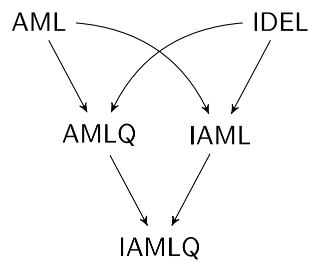


Figure 2.12: Relation between the three logics developed in this thesis and the two logics they are based on.

The rest of this thesis is set up as follows. We describe each of the three systems AMLQ, IAML and IAMLQ in a separate chapter. These chapters all have a similar structure: we give the necessary definitions, we test the results, and we provide a complete axiomatization. We will also discuss some interesting properties of the logics, and compare them to the other systems we have discussed so far. Then, in Chapter 6, we summarize our most important results and suggest some directions for further work.

Chapter 3

Action Model Logic with Questions

3.1 Introduction

In this chapter we will follow the first strategy mentioned in the previous section: we will define an Action Model Logic with Questions (AMLQ). In this system we use IEL as the language of our static models, rather than standard epistemic logic, and refine the definition of the content of actions. This means that our actions can be utterances of a statement as well as a question.

We will start by giving a new definition of action models. Then we define an update procedure that can deal with both statements and questions. In Section 3.3 we show some examples of situations our models can describe. In Section 3.4, we will show that some important properties of IEL are inherited by AMLQ.

We will then extend our language and action models in a recursive way, to allow for formulas of $\mathcal{L}^{\mathsf{AMLQ}}$ in AMLQ action models. After this, we will say a few words about how we interpret dynamic modalities of sets in Section 3.6. We define the notion of action model composition in Section 3.7. Then, Sections 3.8 and 3.9 are dedicated to developing a proof system for the logic AMLQ and showing that it is complete. In Section 3.10, we will show that our system is a conservative extension of both systems it is based on, after which we end the chapter with a short conclusion in which we summarize our findings.

3.2 Definitions

3.2.1 Action model

Making action models compatible with questions requires us to change the way we think about the content of actions. In AML, actions can only convey information. As a result, we only have to check in which worlds they can be executed, which is expressed by their precondition. In other words, the content of the action is equal to its precondition.

However, in the inquisitive setting there is more to actions than just information: the content of an action can also raise issues. For instance, a question like $?p$ has an effect on the agent that hears it: namely, if she does not know the answer, she will come to wonder about the question. This means that in the inquisitive setting, assigning a precondition to each action is not enough, as this assigns only informative content and not inquisitive content. We therefore replace pre with cont (for content) everywhere in the definition.

For now, the range of cont is restricted to formulas of IEL. The reason for this is that the language of AMLQ is not yet defined at this stage, and in fact depends on the definition of action models. To prevent our definitions from being circular, we give the full definition of AMLQ action models and their language at a later stage. What we define now are AMLQ_0 action models: AMLQ models of the lowest level. We will often omit the subscript to make our notation more concise.

Definition 3.2.1. **Action Model with Questions**
An AMLQ_0 action model is a triple $\mathsf{M} = \langle \mathsf{S}, \{\sim_a \mid a \in \mathcal{A}\}, \mathsf{cont}\rangle$, where:

- S is a finite domain of action points;
- For each $a \in \mathcal{A}$, $\sim_a$ is an equivalence relation on S;
- $\mathsf{cont} : \mathsf{S} \to \mathcal{L}^{\mathsf{IEL}}$ is a function that assigns a content $\mathsf{cont}(\mathsf{x}) \in \mathcal{L}^{\mathsf{IEL}}$ to each action point $\mathsf{x} \in \mathsf{S}$.

As the content of an action is a formula of IEL, it can be either a statement or a question. What remains unchanged is that we can still think of the precondition of an action as its informative content. In case the content is a statement, the precondition is the statement itself. In contrast, in case the content is a question, the precondition is a statement which we regard as capturing the question's presupposition that one of the proposed alternatives is true.

Instead of explicitly defining a precondition for each action, we can retrieve the precondition of an action from its content. If the content of an action is φ, the precondition for this action is its declarative variant $\varphi^!$, which is a statement with the same truth conditions as φ.

Definition 3.2.2. **Precondition of an action**
Given an action x, its precondition $\mathsf{pre}(\mathsf{x})$ is defined as:

$$\mathsf{pre}(\mathsf{x}) := \mathsf{cont}(\mathsf{x})^!$$

To illustrate this, consider the following examples: the precondition of an action with content p is p, because it can only truthfully be executed in worlds where p is true. In contrast, an action with content $?p$ can be truthfully executed in any world, as it contains no information. Therefore its precondition is $p \vee \neg p \equiv \top$.

As we identify the precondition of an action with the declarative variant of its content, we have the following proposition, which will play an important part in the rest of our proofs.

Proposition 3.2.1. **Action content and precondition have equal truth conditions**
For all actions x, for every inquisitive epistemic model M and world w:

$$M, w \models \mathsf{cont}(\mathsf{x}) \iff M, w \models \mathsf{pre}(\mathsf{x})$$

Proof: Immediate from Definition 3.2.2 and Fact 2.3.4. □

With this new definition we have generalized the notion of action models to make it compatible with contents that can be inquisitive as well as informative. In some parts of the definitions that follow, cont and pre may be interchangeable. To make things clearer, we use cont only in definitions if we expect the inquisitive content of the action (if any) to have an effect. In all other cases (e.g. when restricting a set of worlds to the ones compatible with an action) we use pre.

3.2.2 Update procedure

Now that we have generalized the notion of action models to make it compatible with questions, we also need to generalize the update procedure from Definition 2.4.2 to make it compatible with our new definition.

Like the update procedure for standard action models, our procedure should take care of preserving the knowledge from the original model and incorporating the knowledge obtained from the action. However, it should now also preserve the issues from the original model and incorporate the issues that arise from new action content.

To formulate our definition, we make use of two projection operators, defined as follows.

Definition 3.2.3. **Projection operators**
If $s \subseteq W \times \mathsf{S}$, then:

$$\pi_1(s) := \{w \mid \langle w, \mathsf{x}\rangle \in s \text{ for some } \mathsf{x}\}$$

$$\pi_2(s) := \{\mathsf{x} \mid \langle w, \mathsf{x}\rangle \in s \text{ for some } w\}$$

These projection operators are useful because they allow us to associate to each state s in the updated model a state $\pi_1(s)$ in the original model and a state $\pi_2(s)$ in the action model. For example: $\pi_1(\{\langle w_1, \mathsf{x}\rangle, \langle w_1, \mathsf{y}\rangle, \langle w_2, \mathsf{x}\rangle\}) = \{w_1, w_2\}$, while $\pi_2(\{\langle w_1, \mathsf{x}\rangle, \langle w_1, \mathsf{y}\rangle, \langle w_2, \mathsf{x}\rangle\}) = \{\mathsf{x}, \mathsf{y}\}$.

With these projection operators in place, we are now ready to define the update procedure. We will first give the formal definition and then provide a motivation.

Definition 3.2.4. **Updated IEL model**
Let M be an inquisitive epistemic model and M an AMLQ_0 action model. Then $M' = (M \otimes \mathsf{M})$ is the product update of M and M, defined as follows.

$M' = \langle W', \{\Sigma'_a \mid a \in \mathcal{A}\}, V'\rangle$, where:

- $W' = \{\langle w, \mathsf{x}\rangle \mid w \in W, \mathsf{x} \in \mathsf{S} \text{ and } M, w \models \mathsf{pre}(\mathsf{x})\}$
- $s \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$ iff
  - (i) $\pi_1(s) \in \Sigma_a(w)$
  - (ii) $\forall \mathsf{y} \in \pi_2(s) : \mathsf{x} \sim_a \mathsf{y}$
  - (iii) There is at most one $\mathsf{y} \in \pi_2(s)$
  - (iv) $\forall \mathsf{y} \in \pi_2(s) : M, \pi_1(s) \models \mathsf{cont}(\mathsf{y})$
- $\langle w, \mathsf{x}\rangle \in V'(p)$ iff $w \in V(p)$

Like in standard action models, the domain of the updated model consists of pairs containing a world from the original model and an action from the action model. We allow only those pairs $\langle w, \mathsf{x}\rangle$ of which the precondition of the action x is true in the world w. This restriction is necessary because only in worlds where the content of x is true, x can be truthfully uttered.

This definition of the domain restricts the set of worlds to the ones that we need our agents to consider. As this is a matter of information, not issues, there does not need to be any difference with respect to standard action models. Therefore, we can leave the definition of the domain in the updated model unchanged.

What also remains unchanged is the principle that epistemic actions do not change non-epistemic facts of the world: they only have an effect on the knowledge and issues of agents about the world and about each others knowledge and issues. This means that worlds in the new domain make the same atomic propositions true as the corresponding worlds in the old domain. Therefore, our definition of the valuation function can also be just like in standard action models.

This means that the crucial part of this definition is the definition of Σ'_a. The update procedure for standard action models formulates two conditions for two worlds to be indistinguishable in the updated model. Similarly, we formulate four conditions for an information state s to be in $\Sigma'_a(\langle w, \mathsf{x}\rangle)$. These conditions match the following four desiderata:

(i) **Preservation of knowledge and issues from the original model**

Epistemic actions can only enhance the knowledge of the agents, not decrease it. Formally, this means that if an agent can distinguish two worlds in the original model, she can also distinguish the corresponding worlds in the updated model. This is why in standard action models, one of the requirements for two worlds to be indistinguishable in the updated model ($\langle w, \mathsf{x}\rangle \sim'_a \langle w', \mathsf{x}'\rangle$) is that their corresponding worlds were indistinguishable in the original model ($w \sim_a w'$). In our current setting, this means that a requirement for $\langle w', \mathsf{x}'\rangle$ to be in $\sigma'_a(\langle w, \mathsf{x}\rangle)$ is that $w' \in \sigma_a(w)$. Consequently, for an information state s to be in $\Sigma'_a(\langle w, \mathsf{x}\rangle)$, we require that $\pi_1(s) \subseteq \sigma_a(w)$.

However, we should not just preserve knowledge in our updated model: we should also preserve issues. In the original model, the issues that the agent has in world w are encoded by the state map $\Sigma_a(w)$: this state map contains only information states that resolve her issues. As she does not lose interest in these issues after the update, we want all states in the updated state map to resolve these issues as well. We can achieve this by requiring that $\pi_1(s) \in \Sigma_a(w)$.

Because the requirement that $\pi_1(s) \in \Sigma_a(w)$ is stronger than $\pi_1(s) \subseteq \sigma_a(w)$, we need to require only the former.

(ii) **Preservation of knowledge from the action model**

In the same way that the original model encodes knowledge of the agents, so does the action model. As in standard action models, if two actions x and y are distinguishable for agent a ($\mathsf{x} \not\sim_a \mathsf{y}$), then so are any corresponding worlds in the updated model ($\langle w, \mathsf{x}\rangle \not\sim'_a \langle v, \mathsf{y}\rangle$). This means that a requirement for any world $\langle v, \mathsf{y}\rangle$ to be in $\sigma'_a(\langle w, \mathsf{x}\rangle)$ is that $\mathsf{x} \sim_a \mathsf{y}$. Hence, for a state s to be in $\Sigma'_a(\langle w, \mathsf{x}\rangle)$, we require that for all $\langle v, \mathsf{y}\rangle \in s$: $\mathsf{x} \sim_a \mathsf{y}$, which we can also write as $\forall \mathsf{y} \in \pi_2(s) : \mathsf{x} \sim_a \mathsf{y}$.

(iii) **Raising of issues about what the actual action is**

Standard action models encode only the knowledge agents have about which action is the actual one and not whether they consider this an issue. Action models with questions are no different in this respect. As our updated model is an inquisitive epistemic model, we have to decide in our update procedure how we interpret our action models: that is, whenever an agent does not know what the actual action is, does this become an issue or not? In other words, do we assume that agents are curious? This is a rather arbitrary decision, because we could interpret the action models either way.

In this system we choose the curious interpretation: we always consider it to be an issue to the agents what the actual action is. The reason for this choice is the fact that the non-curious interpretation has some complications that the curious interpretation does not have. We will deal with these in Chapter 5, when we develop action models that actually encode to what extent each agent is curious.

Hence, what we want is that agents who do not know whether x or y was the actual action, want to know this in the updated model. This means that a state only counts as resolving the agent's issues if it determines exactly what the actual action was. That is why we require for every state $s \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$ that $\pi_2(s)$ contains at most one element.

(iv) **Raising of issues by action content**

The novelty of action models with questions is that actions can have the effect of raising an issue. We can reflect this in the updated model by requiring information states in the state maps of the agents to support the action content. In that way, if the content is a question, the state maps of the agents indicate that they want to be in an information state that resolves the question.

However, there might be uncertainty about which action is the actual one. Suppose there are two actions x and y, with content $?p$ and $?q$ respectively. Intuitively, an agent who knows which of the two actions is the actual one (and had no issues in the original model) should entertain only the corresponding question after the update. In contrast, an agent who cannot distinguish between x and y has no reason to entertain $?p$ nor $?q$.

We can obtain this effect by raising issues locally in the state maps of our updated model. Recall that by condition (iii), all information states in the state maps of agents that are not empty, belong to exactly one action. This means that there are no mixed information states that contain both x-worlds and y-worlds. We require, for each action x, the states consisting of x-worlds to support the content of x. Formally, for a state s to be in $\Sigma'_a(\langle w, \mathsf{x}\rangle)$, we require that $\forall \mathsf{y} \in \pi_2(s) : M, \pi_1(s) \models \mathsf{cont}(\mathsf{y})$.

Now that we have formulated the update procedure, it is important to check that any updated model is indeed an inquisitive epistemic model. Most importantly, we need to check that the state map for every world and every agent is still an inquisitive proposition.

PROPOSITION 3.2.2. **Updates result in inquisitive epistemic models**
For any inquisitive epistemic model M and for any AMLQ action model M, $M' = (M \otimes \mathsf{M})$ is an inquisitive epistemic model.

Proof: We need to check that for every world $\langle w, \mathsf{x}\rangle$, $\Sigma'_a(\langle w, \mathsf{x}\rangle)$ is non-empty and downward closed. Furthermore, Σ'_a should satisfy factivity (for all $w \in W$, $w \in \sigma_a(w)$) and introspection (for all $w, v \in W$, if $v \in \sigma_a(w)$ then $\Sigma_a(v) = \Sigma_a(w)$).

Take an arbitrary world $\langle w, \mathsf{x}\rangle \in W'$.

We will first show that $\Sigma'_a(\langle w, \mathsf{x}\rangle)$ is downward closed. Take any state $s \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$ and any $t \subseteq s$.

(i) $\pi_1(t) \in \Sigma_a(w)$ since $\pi_1(t) \subseteq \pi_1(s)$ and $\Sigma_a(w)$ is downward closed.

(ii) $\forall \mathsf{y} \in \pi_2(t) : \mathsf{x} \sim_a \mathsf{y}$ since $\pi_2(t) \subseteq \pi_2(s)$ and $\forall \mathsf{y} \in \pi_2(s) : \mathsf{x} \sim_a \mathsf{y}$.

(iii) As $\pi_2(t) \subseteq \pi_2(s)$, it too contains at most one element.

(iv) As $\forall \mathsf{y} \in \pi_2(s) : M, \pi_1(s) \models \mathsf{cont}(\mathsf{y})$, we have $\forall \mathsf{y} \in \pi_2(t) : M, \pi_1(t) \models \mathsf{cont}(\mathsf{y})$ by persistence of support.

This concludes downward closure. We now show factivity and non-emptiness at the same time, by showing that the state map of $\langle w, \mathsf{x}\rangle$ contains its own singleton, $\{\langle w, \mathsf{x}\rangle\}$.

(i) $\{w\} \in \Sigma_a(w)$ by factivity and downward closure of $\Sigma_a(w)$.

(ii) $\mathsf{x} \sim_a \mathsf{x}$ by reflexivity of $\sim_a$.

(iii) $\pi_2(\{\langle w, \mathsf{x}\rangle\}) = \{\mathsf{x}\}$, which contains exactly one element.

(iv) By definition of W' we have $M, \{w\} \models \mathsf{pre}(\mathsf{x})$. Therefore, by Proposition 3.2.1, $M, \{w\} \models \mathsf{cont}(\mathsf{x})$.

We have $\{\langle w, \mathsf{x}\rangle\} \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$. It follows that $\langle w, \mathsf{x}\rangle \in \sigma'_a(w)$.

That leaves introspection. Take any two worlds $\langle w, \mathsf{x}\rangle$ and $\langle w', \mathsf{x}'\rangle$ from W' such that $\langle w', \mathsf{x}'\rangle \in \sigma'_a(\langle w, \mathsf{x}\rangle)$. By downward closure of $\Sigma'_a(\langle w, \mathsf{x}\rangle)$ we obtain $\{\langle w', \mathsf{x}'\rangle\} \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$. From condition (i) we learn that it must be the case that $w' \in \sigma_a(w)$. By introspection of Σ_a we obtain $\Sigma_a(w) = \Sigma_a(w')$. From condition (ii) we have $\mathsf{x} \sim_a \mathsf{x}'$, so for all actions y, $\mathsf{y} \sim_a \mathsf{x}$ iff $\mathsf{y} \sim_a \mathsf{x}'$. Then it is easy to check that for all states t, t satisfies conditions (i-iv) for $\Sigma'_a(\langle w, \mathsf{x}\rangle)$ iff it satisfies them for $\Sigma'_a(\langle w', \mathsf{x}'\rangle)$. Therefore $\Sigma'_a(\langle w, \mathsf{x}\rangle)$ and $\Sigma'_a(\langle w', \mathsf{x}'\rangle)$ are equal. □

3.2.3 Syntax and semantics

We have seen how IDEL extends IEL with a dynamic public utterance modality, and how AML extends epistemic logic with dynamic modalities containing actions in action models. In this section, we will show how these extensions can be generalized to an Action Model Logic with Questions (AMLQ), which is an extension of IEL with dynamic modalities for actions.

In what follows, we will use s and t to denote information states, which is common in inquisitive logic, and s and t for sets of action points x in an action model.

Like in AML, we add dynamic modalities for actions, to express that an action has been executed. However, we will see later that we need something more general. We need modalities for actions x, but also for sets of actions s. We interpret these set modalities as expressing information about which action is executed: it is one action $\mathsf{x} \in \mathsf{s}$, but it is not determined which one. We then recover the execution of a single action as a special case, namely the case where s is a singleton.

Definition 3.2.5. **Syntax of $\mathcal{L}^{\mathsf{AMLQ}_0}$**
Level 0 of the language of Action Model Logic with Questions is defined as follows, where s is a set of action points within the AMLQ_0 action model M:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \varphi \vee\!\!\!\vee \varphi \mid K_a\varphi \mid E_a\varphi \mid [\mathsf{M}, \mathsf{s}]\varphi$$

For now, we restrict the definition of our language to the fragment that has dynamic modalities of AMLQ_0 action models. The definition of the full language can only be given when action models of higher levels are defined, which we will do in Section 3.5. We denote this fragment by $\mathcal{L}^{\mathsf{AMLQ}_0}$. As this is the only part of the language defined so far, we will omit the subscript until we get to the full definition.

We take over the abbreviations from IEL. For actions we introduce two new notational conventions. Firstly, we allow ourselves to omit the action model M whenever no confusion can arise about the model that the action belongs to. Secondly, for single actions x we allow $[\mathsf{x}]$ as an abbreviation for $[\{\mathsf{x}\}]$.[1]

[1] In order to make $\mathcal{L}^{\mathsf{AMLQ}}$ a superset of $\mathcal{L}^{\mathsf{AML}}$, one might also allow the PDL-style notation of complex actions and let $[\mathsf{x}_1 \cup \ldots \cup \mathsf{x}_n]\varphi := [\{\mathsf{x}_1, \ldots, \mathsf{x}_n\}]\varphi$, although this notation is less natural in our setting. We will elaborate on this in Section 3.6.

The semantics of AMLQ consists of the support conditions of IEL, extended with a new support condition for the dynamic modalities. Before we can define this support condition, we need a way to connect states in the original model to the corresponding states in the updated model. Let M be an inquisitive epistemic model and s an information state in that model. Let M be an AMLQ action model and s a set of actions in that model. Furthermore, let $M' = M \otimes \mathsf{M}$.

Definition 3.2.6. **Updated state**
$s[\mathsf{M}, \mathsf{s}]$ is the information state in M' such that:

$$s[\mathsf{M}, \mathsf{s}] = \{\langle w, \mathsf{x}\rangle \in W' \mid w \in s \text{ and } \mathsf{x} \in \mathsf{s}\}$$

We allow omission of the action model M in the notation of updated states. Notice that by the above definition, the set $s[\mathsf{s}]$ consists of all the pairs $\langle w, \mathsf{x}\rangle \in s \times \mathsf{s}$ such that $M, w \models \mathsf{pre}(\mathsf{x})$. Using the notion of updated states, we can now give the support condition for dynamic modalities.

Definition 3.2.7. **Support condition for dynamic modalities**
The support condition for dynamic modalities is the following:

$$M, s \models [\mathsf{M}, \mathsf{s}]\varphi \iff M', s[\mathsf{M}, \mathsf{s}] \models \varphi$$

Like in IEL, truth in a world is defined as support in the corresponding singleton state. We therefore have the following truth condition for dynamic modalities.

Fact 3.2.1. **Truth condition for dynamic modalities**
Let $w[\mathsf{M}, \mathsf{s}] = \{\langle w, \mathsf{x}\rangle \in W' \mid \mathsf{x} \in \mathsf{s}\}$. Then the truth condition for dynamic modalities is the following:

$$M, w \models [\mathsf{M}, \mathsf{s}]\varphi \iff M', w[\mathsf{M}, \mathsf{s}] \models \varphi$$

Because s may contain multiple actions, s' may be an information state consisting of more than one world. This means that, for $[\mathsf{s}]\varphi$ to be *true* in w, φ must be *supported* in $w[\mathsf{s}]$. If we want to give a characterization of truth of dynamic formulas only in terms of truth, we can only do this for dynamic modalities of single actions:

$$M, w \models [\mathsf{M}, \mathsf{x}]\varphi \iff w \models \mathsf{pre}(\mathsf{x}) \text{ implies } M', \langle w, \mathsf{x}\rangle \models \varphi$$

This formulation makes it clear that $[\mathsf{x}]\varphi$ is true in w just in case either the action x is incompatible with w (which makes the formula vacuously true) or φ is true in the corresponding world $\langle w, \mathsf{x}\rangle$ in the product update. More importantly, this formulation corresponds exactly to the truth condition of dynamic modalities in AML.

The intuitive reading of a formula with a dynamic modality, like $[\mathsf{x}]\varphi$, should be: 'φ is supported after action x is executed'. A formula with a dynamic modality that contains a set s of actions, like $[\mathsf{s}]\varphi$, can be read as: 'after getting the information that *some action* of s is executed, φ is supported'. This means we think of sets of actions s in the same way as we think of information states s. Namely, as encoding the information that the actual world (action) is one of the worlds (actions) in the set.

3.2.4 Epistemic maps and state maps in updated models

Now that the definitions are in place, we will prove two lemmas that tell us more about how the knowledge and issues of original and updated models are related. These lemmas provide an alternative characterization of the update procedure, and will be useful when proving reduction equivalences in Section 3.8.

To simplify notation in what follows, we define $\delta_a(\mathsf{x}) := \{\mathsf{y} \mid \mathsf{y} \sim_a \mathsf{x}\}$. In this way, we can use $\delta_a(\mathsf{x})$ to refer to the set of actions indistinguishable from x to a. We can then prove the following lemma, which tells us that the information state of a in world $\langle w, \mathsf{x}\rangle$ of the updated model is simply the information state of a in world w of the original model updated with the set of actions $\delta_a(\mathsf{x})$.

Lemma 3.2.1. **Epistemic maps in updated models**
Let w be a world in an inquisitive epistemic model M, x an action in action model M and $M' = M \otimes \mathsf{M}$. Let $\langle w, \mathsf{x}\rangle$ be a world in M'. Then we have the following:

$$\sigma'_a(\langle w, \mathsf{x}\rangle) = \sigma_a(w)[\delta_a(\mathsf{x})]$$

Proof: ($\subseteq$) Take any $\langle v, \mathsf{y}\rangle \in \sigma'_a(\langle w, \mathsf{x}\rangle)$. This world is in this set because it belongs to a state that satisfies conditions (i)-(iv) of the update procedure in Definition 3.2.4. From condition (i) we obtain $v \in \sigma_a(w)$. From condition (ii) we obtain $\mathsf{y} \sim_a \mathsf{x}$. But these are, by Definition 3.2.6, exactly the conditions to be in $\sigma_a(w)[\delta_a(\mathsf{x})]$. So $\langle v, \mathsf{y}\rangle \in \sigma_a(w)[\delta_a(\mathsf{x})]$.

($\supseteq$) Take an arbitrary world $\langle v, \mathsf{y}\rangle \in \sigma_a(w)[\delta_a(\mathsf{x})]$. Then by Definition 3.2.6, we know that $v \in \sigma_a(w)$ and $\mathsf{y} \sim_a \mathsf{x}$. It follows that the state $\{\langle v, \mathsf{y}\rangle\}$ satisfies condition (i) and (ii) of Definition 3.2.4 to be in $\Sigma'_a(\langle w, \mathsf{x}\rangle)$. It also satisfies condition (iii) by being a singleton. As $\langle v, \mathsf{y}\rangle$ is in the domain of M', it must be the case that $M, v \models \mathsf{pre}(\mathsf{y})$. By Proposition 3.2.1 it follows that $M, v \models \mathsf{cont}(\mathsf{y})$. Therefore the state $\{\langle v, \mathsf{y}\rangle\}$ satisfies condition (iv) as well. This means that $\{\langle v, \mathsf{y}\rangle\} \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$ and hence that $\langle v, \mathsf{y}\rangle \in \sigma'_a(\langle w, \mathsf{x}\rangle)$. □

The above lemma shows us that epistemic maps in the updated model are obtained as in standard action model logic. The next lemma shows how information states in original and updated models are related.

Lemma 3.2.2. **State maps in updated models**
Let w be a world in an inquisitive epistemic model M, x an action in action model M and $M' = M \otimes \mathsf{M}$. Suppose $\langle w, \mathsf{x}\rangle \in W'$. Then we have the following:

$$\Sigma'_a(\langle w, \mathsf{x}\rangle) = \{s[\mathsf{y}] \mid s \in \Sigma_a(w),\ \mathsf{y} \sim_a \mathsf{x} \text{ and } M, s \models \mathsf{cont}(\mathsf{y})\}$$

Proof: ($\subseteq$) Assume $s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$.

Let s be $\pi_1(s')$. If s' is non-empty, by condition (iii) of Definition 3.2.4 there is exactly one y in $\pi_2(s')$. Then $s' = s[\mathsf{y}]$. For take any world $\langle v, \mathsf{y}\rangle \in s'$. It can only be in the domain of the updated model if $v \models \mathsf{pre}(\mathsf{y})$. Then by definition of $s[\mathsf{y}]$, $\langle v, \mathsf{y}\rangle \in s[\mathsf{y}]$. Now suppose $\langle v', \mathsf{y}\rangle \notin s'$. Then v' will not be in s, so $\langle v', \mathsf{y}\rangle \notin s[\mathsf{y}]$.

In case s' is empty, then so is s. In that case let $\mathsf{y} = \mathsf{x}$.

From conditions (i), (ii) and (iv) of Definition 3.2.4 it is immediate that $s \in \Sigma_a(w)$, $\mathsf{y} \sim_a \mathsf{x}$ and $M, s \models \mathsf{cont}(\mathsf{y})$.

($\supseteq$) Assume $s \in \Sigma_a(w), \mathsf{y} \sim_a \mathsf{x}$ and $M, s \models \mathsf{cont}(\mathsf{y})$.

We can show that $s[\mathsf{y}] \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$ by checking conditions (i)-(iv) of Definition 3.2.4 for a state to be in $\Sigma'_a(\langle w, \mathsf{x}\rangle)$.

(i) As $M, s \models \mathsf{cont}(\mathsf{y})$, all worlds in s satisfy the precondition of y. Therefore $\pi_1(s[\mathsf{y}]) = s$. By assumption we have $s \in \Sigma_a(w)$, so $\pi_1(s[\mathsf{y}]) \in \Sigma_a(w)$.

(ii) $\mathsf{y} \sim_a \mathsf{x}$ by assumption.

(iii) By definition, $\pi_2(s[\mathsf{y}])$ contains at most y.

(iv) $M, s \models \mathsf{cont}(\mathsf{y})$ by assumption. □

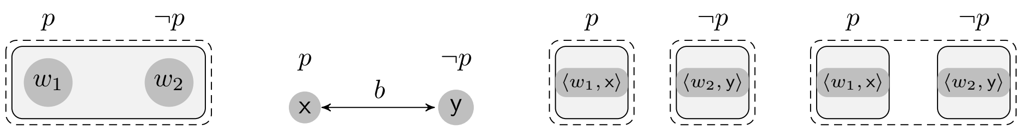
Notice that the above lemma may be viewed as an alternative formulation of the definition of Σ'_a.

3.3 Examples

Before we look at some more properties of the logical system, let us first get an idea of how our models and update mechanism work by looking at some examples. To make the examples easier to interpret, we skip the formal definitions of the models and give only the diagrams. We follow the conventions introduced in Chapter 2. Since we only consider reflexive relations in our action models, we skip the loops in their representation. All scenarios in these examples are variants of the scenarios about the wedding of Anna and Bob, introduced in Section 1.1.

Example 3.3.1. **Peter is attending**
Let us start with the phone call from Peter, in which he announces to Anna (a) that he will attend the wedding. Recall that Bob (b) is aware of the conversation, but unaware of what Peter announces. This means that there are two actions considered possible, namely Peter is attending (p) or he is not ($\neg p$).

Let the initial state maps for agents a and b and the equivalence relation of the action model be defined as in Figure 3.1. Let $\mathsf{cont}(\mathsf{x}) = p$ and $\mathsf{cont}(\mathsf{y}) = \neg p$. The state maps of the agents in the updated model are shown in subfigures (c) and (d).



(a) State map of a and b before action

(b) Action model

(c) State map of a after action

(d) State map of b after action

Figure 3.1: Example 3.3.1

The agents initially do not know whether Peter is attending and they do not entertain this question either. Then, the action model reflects that either x or y is occurring. The lack of arrows for a indicates that she can distinguish between these actions. As a result, she can also distinguish between the resulting worlds $\langle w_1, \mathsf{x}\rangle$ and $\langle w_2, \mathsf{y}\rangle$. This reflects that she now knows whether p is the case.

In contrast, b does not know which of the actions is taking place. He does not learn anything about whether p, which is reflected by the dashed line staying the same in his state map in the updated model. What changes, however, is that it becomes an issue for him whether p. This is an effect of condition (iii) of our update procedure, which makes agents curious about what action is taking place.

Notice also that, as the agents are only considering worlds $\langle w_1, \mathsf{x}\rangle$ and $\langle w_2, \mathsf{y}\rangle$, they are both aware of each others knowledge and issues. That is, a knows that b wonders whether p and b knows that a knows whether p.

Let us now look at a scenario we have also seen before, namely the scenario from Example 2.4.2.

Example 3.3.2. **Patrick or Qasim is attending**
Anna and Bob get an RSVP message in the mail. They know that the sender is attending, but they are not sure yet about the identity of the sender: this may either be Patrick (p) or Qasim (q). Anna (a) opens the envelope, while Bob (b) watches from a distance and is not immediately aware of the content.

Let the initial state maps for agents a and b and the equivalence relation of the action model be defined as in Figure 3.2. Let $\mathsf{cont}(\mathsf{x}) = p$ and $\mathsf{cont}(\mathsf{y}) = q$. The state maps of the agents in the updated model are shown in subfigures (c) and (d).

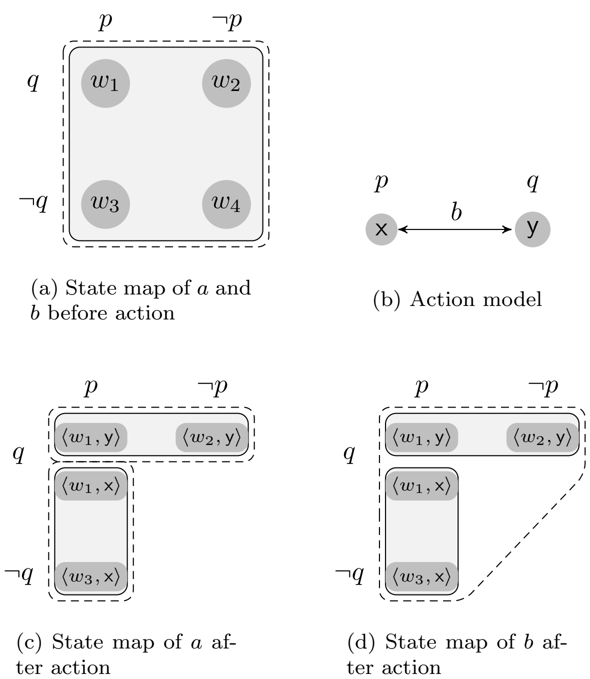


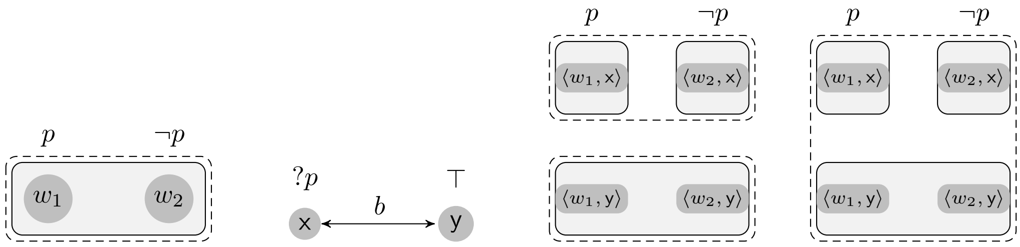
Figure 3.2: Example 3.3.2

In terms of knowledge, the outcome is exactly the same as we have seen in Example 2.4.2. The difference is that in this setting, Bob will entertain the issue whether it was Patrick or Qasim who just announced that he will be attending ($E_b(p \vee\!\!\!\vee q)$).

Now that we know how the mechanism works when the contents of actions are statements, it is time to look at some examples that involve questions. Let us start with a simple example in which either a question or nothing is communicated.

Example 3.3.3. **Is Penny attending?**
This time, the question whether Penny is attending ($?p$) is communicated semi-privately to Anna: that is, Bob does not know this, but since we are in the epistemic setting, he cannot be completely unaware, as he would then loose track of the actual action and consequently of the actual world, so he does consider it possible that this question was asked.

Let the initial state maps for agents a and b and the action model be defined as in Figure 3.3. This time, let $\mathsf{cont}(\mathsf{x}) = ?p$ and $\mathsf{cont}(\mathsf{y}) = \top$. The results are shown in the same figure.



(a) State map of a and b before action

(b) Action model

(c) State map of a after action

(d) State map of b after action

Figure 3.3: Example 3.3.3

The resulting state map for a reflects that she now entertains whether p: since the actual action was x, we know that the actual world is one of the x-worlds.

The state map for b looks similar, but there is an important difference: he cannot distinguish the x-worlds from the y-worlds. This means that whatever the actual world is, there is some information state he wants to be in that does not support $?p$, namely the state with the y-worlds. According to the support conditions of the entertain modality, b does not come to entertain $?p$.

Example 3.3.4. **One of two questions**
Let us now consider an action model with two questions: one of the guests calls, and Anna and Bob know that she will either ask whether Pablo is attending ($?p$) or whether Quentin is ($?q$). We let $\mathsf{cont}(\mathsf{x}) = ?p$ and $\mathsf{cont}(\mathsf{y}) = ?q$. The initial state map, action model and results are shown in Figure 3.4.

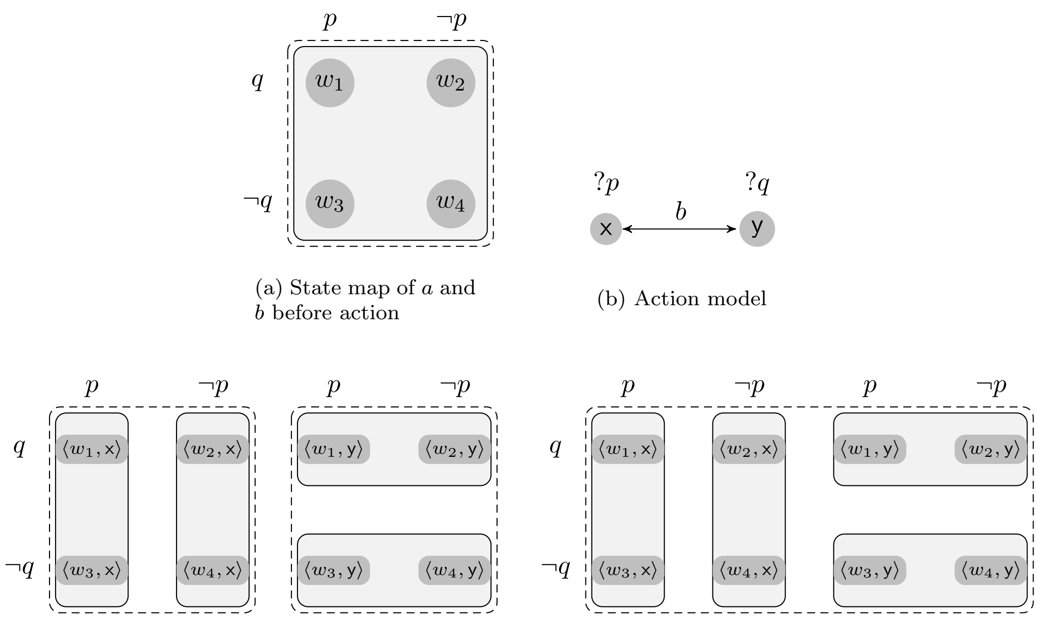
We obtain a result similar to the previous example. If the actual action was x, then in the updated model we have $E_a?p$: it becomes a's goal to be in an information state that settles whether p is true or false. But if the actual action was y, in the updated model we have $E_a?q$.

Since b cannot distinguish the x-worlds from the y-worlds, he considers both x-worlds and y-worlds, no matter what the actual action was. It is his goal to be in a state that settles what the actual action was and that also settles the corresponding question: $?p$ if x, and $?q$ if y. As there is at least one information state in his state map that does not support $?p$, he does not entertain $?p$. The same goes for $?q$. However, these issues are still encoded in his state map as conditional issues: if he found out what the actual action was, he would entertain the corresponding question.

Just like the contents of actions in AML can be about the knowledge of the agents, so can the contents of the actions in AMLQ. Additionally, they can also be about the issues they entertain. The following example illustrates this.

Example 3.3.5. **Bob wonders whether Paula is attending**
Initially, Anna does not know that Bob entertains the issue whether Paula is attending. Of course, Bob does know this himself (the actual world must be one of w_1 and w_2). After Bob tells her, Anna also knows this. Let $\mathsf{cont}(\mathsf{x}) = W_b?p$. The initial state maps of a and b, the action model (in this case consisting of just one action) and the updated state maps are shown in Figure 3.5.

Because the worlds w_3 and w_4 are incompatible with the action x, they do not appear in



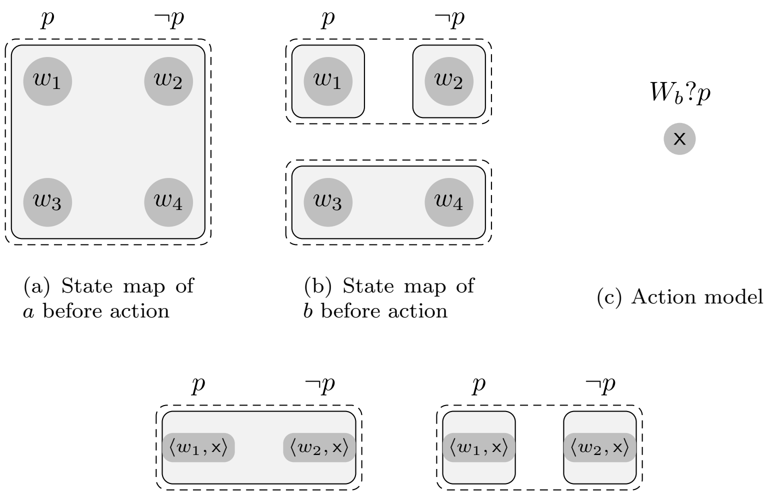
(a) State map of a and b before action

(b) Action model

(c) State map of a after action

(d) State map of b after action

Figure 3.4: Example 3.3.4



(a) State map of a before action

(b) State map of b before action

(c) Action model

(d) State map of a after action

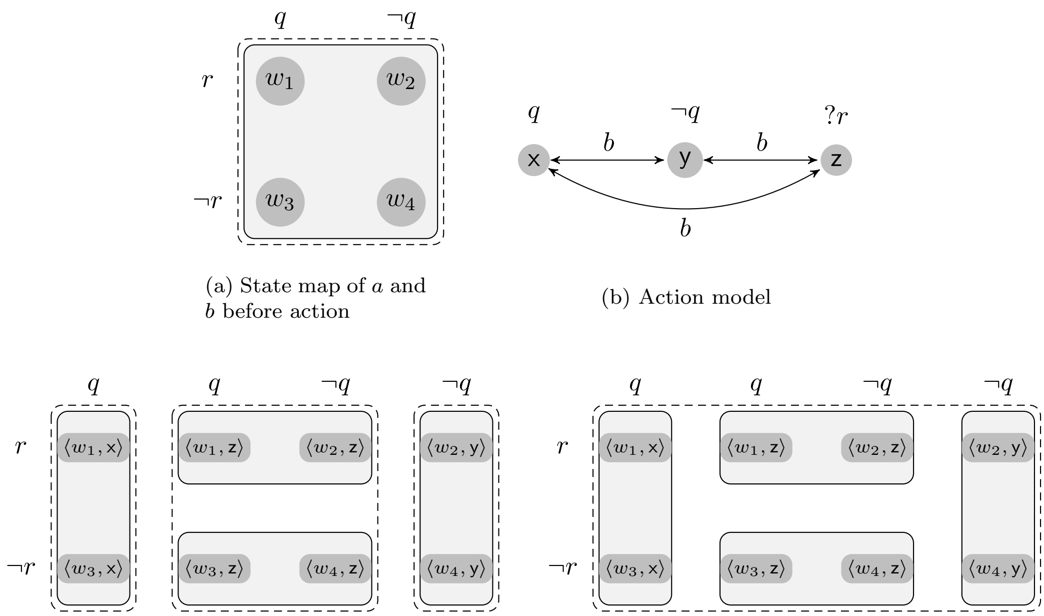
(e) State map of b after action

Figure 3.5: Example 3.3.5

the updated model. Therefore, what changes for a is that she no longer considers worlds in which $W_b?p$ is not true. Hence, after this utterance, Anna knows that Bob wonders whether Paula is coming: $[\mathsf{x}]K_aW_b?p$.

As a final example, and to demonstrate that AMLQ is in fact a solution to the example problem given in Section 1.1, let us consider Quinn's phone call again.

Example 3.3.6. **Quinn wants to know if Rita is attending**
Initially, both Anna (a) and Bob (b) have no knowledge about whether Quinn or Rita is attending, and let us assume they do not consider this an issue. We define an action model with three actions: x, y and z. Let $\mathsf{cont}(\mathsf{x}) = q$, $\mathsf{cont}(\mathsf{y}) = \neg q$ and $\mathsf{cont}(\mathsf{z}) = ?r$. Recall that according to the story, a knows which action is taking place and b does not. The original state maps, the action model and the results are shown in Figure 3.6.



(a) State map of a and b before action

(b) Action model

(c) State map of a after action

(d) State map of b after action

Figure 3.6: Example 3.3.6

In case the actual world is an x-world or a y-world, a now knows whether q, and therefore has no proper issues. If the actual world is a z-world, she will not learn anything about q, but instead entertain the issue whether $?r$.

In contrast, since b does not know the content of the message, his state cannot be affected by the actual content of the message. Instead, he considers the x-worlds and the y-worlds possible. This means he will have two information states in his state map that do not resolve $?r$, so he will not start entertaining this issue. He will only entertain a conditional issue: if the actual world is a z-world, he wants to know whether r.

What these examples illustrate is that the update procedure indeed gives us the desired results: looking only at knowledge, the results correspond with the results we get when using standard epistemic models and standard action models. We will prove in Section 3.10.2 that this is a general fact. The novelty of this update procedure, with respect to standard AML, is that there are now two sources for new issues in the updated model: firstly, an issue can be raised by uncertainty about what the actual action is. Secondly, the content of an action can raise issues.

3.4 Properties of AMLQ

Now that we have a feeling of how AMLQ works, we will show some properties that this logic shares with IEL. These will be important in the coming sections, for instance when we define reduction equivalences. In this section, whenever we write $\mathcal{L}^{\mathsf{AMLQ}}$, for now the reader can take this to denote $\mathcal{L}^{\mathsf{AMLQ}_0}$: the part of the language defined so far. As we will see later, all these definitions and proofs are independent of the language level and therefore generalize to the broader language that we will introduce in Section 3.5.

3.4.1 Persistence and empty state

We start by showing that the support relation of AMLQ has the persistence property and the empty state property.

PROPOSITION 3.4.1. **Properties of the support relation**
For all models M and formulas $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$, we have the following properties:

- *Persistence property:* if $s \models \varphi$ and $t \subseteq s$, then $t \models \varphi$.
- *Empty state property:* $\emptyset \models \varphi$

Proof: By induction on the complexity of φ. All steps of the proof proceed as in IEL (Fact 2.3.2). We only add the step for the dynamic modality.

Assume $s \models [\mathsf{s}]\psi$ and take any $t \subseteq s$. By the support condition of the dynamic modality, $s[\mathsf{s}] \models \psi$. As $t[\mathsf{s}] \subseteq s[\mathsf{s}]$, by the induction hypothesis we have $t[\mathsf{s}] \models \psi$. This means $t \models [\mathsf{s}]\psi$.

For the empty state property, by the induction hypothesis we have $\emptyset \models \psi$. Take any set of actions s. Then by Definition 3.2.6, $\emptyset = \emptyset[\mathsf{s}]$. From $\emptyset[\mathsf{s}] \models \psi$ we obtain by the support condition of the dynamic modality $\emptyset \models [\mathsf{s}]\psi$. □

It is noteworthy that the support relation also has analogous properties with respect to the information that is available about the action. This means that information about actions has the same effects on support as information about worlds: whenever the information s about the actual action would be enough to settle φ, and $\mathsf{t} \subseteq \mathsf{s}$ (that is, the information we have about which actions might be executed is enhanced), then t also gives us enough information to settle φ. Furthermore, we can regard the empty set as inconsistent information about actions, which makes it natural that $[\emptyset]\varphi$ is a validity.

PROPOSITION 3.4.2. **Properties of the support relation w.r.t. dynamic modalities**
For all models M formulas $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$ and sets of actions s, we have the following properties:

- *Modal persistence property:* If $s \models [\mathsf{s}]\varphi$ and $\mathsf{t} \subseteq \mathsf{s}$, then $s \models [\mathsf{t}]\varphi$.
- *Modal empty state property:* $\models [\emptyset]\varphi$

Proof: This follows from Proposition 3.4.1 and the following two facts about any information state s: firstly, $\mathsf{t} \subseteq \mathsf{s}$ implies $s[\mathsf{s}] \subseteq s[\mathsf{t}]$ and secondly, $s[\emptyset] = \emptyset$. □

3.4.2 Declaratives

As in IEL, we use the notion of declaratives. Since we have extended the language $\mathcal{L}^{\mathsf{IEL}}$ with dynamic modalities, we have to extend the definition of the declarative fragment with an extra clause.

Definition 3.4.1. **Declarative fragment of $\mathcal{L}^{\mathsf{AMLQ}}$**
The set of declarative formulas $\mathcal{L}_!^{\mathsf{AMLQ}}$ is defined inductively as follows, where $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$:

$$\alpha ::= p \mid \bot \mid \alpha \wedge \alpha \mid \alpha \rightarrow \alpha \mid K_a\varphi \mid E_a\varphi \mid [\mathsf{s}]\alpha$$

Although $K_a\varphi$ and $E_a\varphi$ are truth-conditional even if φ is not, this is not the case for $[\mathsf{s}]\varphi$, as can be seen from the support condition of the dynamic modality. Therefore, to maintain that all declaratives are truth-conditional, we say that $[\mathsf{s}]\alpha$ is a declarative just in case α is. It is then easy to show that it is indeed still the case that all declaratives are truth-conditional.

Proposition 3.4.3. **Any $\alpha \in \mathcal{L}_!^{\mathsf{AMLQ}}$ is truth-conditional**

Proof: By induction of the complexity of α. All steps of the proof proceed as in IEL (Fact 2.3.3). We only add the step for the dynamic modality.

By the induction hypothesis, α is truth-conditional. Then we use the support condition of the dynamic modality and the definition of a state in an updated model to obtain:

$$\begin{aligned} M, s \models [\mathsf{s}]\alpha &\iff M', s[\mathsf{s}] \models \alpha \\ &\iff \text{for all } \langle w, \mathsf{x}\rangle \in s[\mathsf{s}] : M', \langle w, \mathsf{x}\rangle \models \alpha \\ &\iff \text{for all } \langle w, \mathsf{x}\rangle \in W' \text{ such that } w \in s \text{ and } \mathsf{x} \in \mathsf{s} : M', \langle w, \mathsf{x}\rangle \models \alpha \\ &\iff \text{for all } w \in s \text{ such that } \langle w, \mathsf{x}\rangle \in W' \text{ and } \mathsf{x} \in \mathsf{s} : M', \langle w, \mathsf{x}\rangle \models \alpha \\ &\iff \text{for all } w \in s : M', w[\mathsf{s}] \models \alpha \\ &\iff \text{for all } w \in s : M, w \models [\mathsf{s}]\alpha \end{aligned}$$

By Definition 2.3.7, $[\mathsf{s}]\alpha$ is truth-conditional. □

3.4.3 Resolutions and normal form

We can also obtain the normal form result familiar from IEL. That is, we can extend the definition of resolutions and show that every formula in $\mathcal{L}^{\mathsf{AMLQ}}$ is equivalent to the inquisitive disjunction of its resolutions.

Definition 3.4.2. **Resolutions in $\mathcal{L}^{\mathsf{AMLQ}}$**
For any formula $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$, its set of resolutions $\mathcal{R}(\varphi)$ is defined by extending Definition 2.3.10 with the following clause for the dynamic modality:

$$\mathcal{R}([\mathsf{s}]\varphi) = \{[\mathsf{s}]\alpha \mid \alpha \in \mathcal{R}(\varphi)\}$$

Proposition 3.4.4. **A formula is supported iff some resolution of it is**
For all $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$, for every inquisitive epistemic model M and state s:

$$M, s \models \varphi \iff M, s \models \alpha \text{ for some } \alpha \in \mathcal{R}(\varphi)$$

Proof: By induction of the complexity of α. All steps of the proof proceed as in IEL (Fact 2.3.6). We only add the step for the dynamic modality.

By the induction hypothesis, $M, s \models \varphi \iff M, s \models \alpha$ for some $\alpha \in \mathcal{R}(\varphi)$. Then we use the support condition of the dynamic modality and Definition 3.4.2 to obtain:

$$\begin{aligned} M, s \models [\mathsf{s}]\varphi &\iff M', s[\mathsf{s}] \models \varphi \\ &\iff M', s[\mathsf{s}] \models \alpha \text{ for some } \alpha \in \mathcal{R}(\varphi) \\ &\iff M, s \models [\mathsf{s}]\alpha \text{ for some } \alpha \in \mathcal{R}(\varphi) \\ &\iff M, s \models \alpha \text{ for some } \alpha \in \mathcal{R}([\mathsf{s}]\varphi) \end{aligned}$$ □

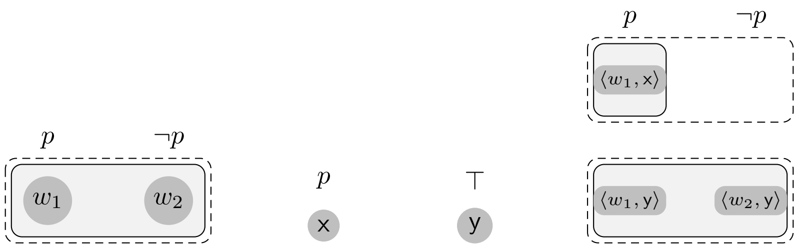
Proposition 3.4.5. **Normal form**
For all $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$, $\varphi \equiv \bigvee\!\!\!\!\bigvee \mathcal{R}(\varphi)$.

Proof: This follows immediately from Proposition 3.4.4 and the support condition of inquisitive disjunction. □

3.4.4 Declarative variant

We need the notion of declarative variant as well, especially because we want to be able to include dynamic modalities in the content of our action models. As the precondition of an action is defined as the declarative variant of its content, we need to extend the notion of declarative variant to formulas of AMLQ.

However, we cannot simply extend Definition 2.3.9 with the single clause $([\mathsf{s}]\varphi)^! = [\mathsf{s}]\varphi^!$. The reason for this is that $M, w \models [\mathsf{s}]\varphi \iff M, w \models [\mathsf{s}]\varphi^!$ holds if s is a singleton, but not in general. To see this, consider the counterexample depicted in Figure 3.7.



(a) State map of a before action

(b) Action model

(c) State map of a after action

Figure 3.7: Example to falsify $M, w \models [\mathsf{s}]\varphi \iff M, w \models [\mathsf{s}]\varphi^!$

Let $\mathsf{s} = \{\mathsf{x}, \mathsf{y}\}$. Then $w_1[\mathsf{s}] \not\models K_a p \vee\!\!\!\!\vee \neg K_a p$, but $w_1[\mathsf{s}] \models K_a p \vee \neg K_a p$, while the latter is the declarative variant of the former. The reason this can happen is because $w_1[\mathsf{s}]$ is not a single world, but an information state consisting of both $\langle w_1, \mathsf{x}\rangle$ and $\langle w_1, \mathsf{y}\rangle$. So although the two formulas do have the same truth conditions, they are now evaluated in a non-singleton information state, and they do not have the same support conditions. It follows that $w_1 \not\models [\mathsf{s}](K_a p \vee\!\!\!\!\vee \neg K_a p)$ but $w_1 \models [\mathsf{s}](K_a p \vee \neg K_a p)$.

This means we have to take a different approach to declarative variants in AMLQ. We can use the notion of resolutions, which is by definition a set of declaratives.

Definition 3.4.3. **Declarative variant in $\mathcal{L}^{\mathsf{AMLQ}}$**
The declarative variant $\varphi^!$ of a formula $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$ is defined by:

$$\varphi^! := \bigvee \mathcal{R}(\varphi)$$

With this definition of declarative variants, we now have a declarative variant for every formula in $\mathcal{L}^{\mathsf{AMLQ}}$. Furthermore, we can show that it is still the case that every formula has the same truth conditions as its declarative variant.

PROPOSITION 3.4.6. **Declarative variants in $\mathcal{L}^{\mathsf{AMLQ}}$ have equal truth conditions** For all $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$, for every inquisitive epistemic model M and world w:

$$M, w \models \varphi \iff M, w \models \varphi^{!}$$

Proof: By Proposition 3.4.5, φ has the same support conditions as $\bigvee\!\!\!\bigvee \mathcal{R}(\varphi)$, which means they have the same truth conditions as well. Furthermore, the truth conditions of classical disjunction and inquisitive disjunction are the same. □

We show that declaratives are representative of the truth-conditional formulas of our language, by proving the following proposition.

PROPOSITION 3.4.7. **Any truth-conditional formula is equivalent to a declarative** For all $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$, if φ is truth-conditional, then there is some $\alpha \in \mathcal{L}^{\mathsf{AMLQ}}_{!}$ such that $\varphi \equiv \alpha$.

Proof: Let α be $\varphi^{!}$. From the definition of declarative variants we can check that $\varphi^{!} \in \mathcal{L}^{\mathsf{AMLQ}}_{!}$. Furthermore, by truth-conditionality of φ, Definition 2.3.7 and Proposition 3.4.6 we have:

$$\begin{aligned} M, s \models \varphi &\iff \text{for all } w \in s : M, w \models \varphi \\ &\iff \text{for all } w \in s : M, w \models \varphi^{!} \\ &\iff M, s \models \varphi^{!} \end{aligned}$$ □

We have thus shown how the properties of IEL we discussed in Section 2.3.4 are inherited by AMLQ. This makes AMLQ fit perfectly in the family of inquisitive logics described in Ciardelli (2016): all these logics have in common that they have the persistence and empty state property, a notion of declaratives and declarative variants to describe the truth-conditional fragment of the logic, and a normal form result.[2]

3.5 Dynamic modalities in action content

In this section, we will expand our view to AMLQ action models of higher levels and the languages that they give rise to. We do this because we want to be able to use dynamic modalities in action content, as is also possible in AML. However, we cannot just adapt the definition of action models we gave earlier, as we need each formula to be well defined. For instance, if we have a formula $[\mathsf{x}]\varphi$, we do not want the content of the action x to be $[\mathsf{x}]\varphi$ again.

We start by defining higher level action models as well as the languages that can refer to them by dynamic modalities.

[2] An exception to this is first-order inquisitive logic, in which some formulas do not have a resolution set. See Ciardelli (2016, p. 109).

DEFINITION 3.5.1. **Higher level Action Model with Questions**
For every $i > 0$, an AMLQ_i action model is a triple $\mathsf{M} = \langle \mathsf{S}, \{\sim_a \mid a \in \mathcal{A}\}, \mathsf{cont}\rangle$, where:

- S is a finite domain of action points;
- For each $a \in \mathcal{A}$, $\sim_a$ is an equivalence relation on S;
- $\mathsf{cont} : \mathsf{S} \to \mathcal{L}^{\mathsf{AMLQ}_{(i-1)}}$ is a function that assigns a content $\mathsf{cont}(\mathsf{x}) \in \mathcal{L}^{\mathsf{AMLQ}_{(i-1)}}$ to each action point $\mathsf{x} \in \mathsf{S}$.

DEFINITION 3.5.2. **Syntax of $\mathcal{L}^{\mathsf{AMLQ}_i}$**
For every $i > 0$, the language of Action Model Logic with Questions of level i is defined as follows, where s is a set of action points within the AMLQ_j action model M, with the restriction that $j < i$:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \varphi \vee\!\!\!\vee \varphi \mid K_a\varphi \mid E_a\varphi \mid [\mathsf{M}, \mathsf{s}]\varphi$$

With these higher level languages, we can now define the full language of AMLQ in the following way.

DEFINITION 3.5.3. **Syntax of $\mathcal{L}^{\mathsf{AMLQ}}$**
The full language of Action Model Logic with Questions is defined as the union of all $\mathcal{L}^{\mathsf{AMLQ}_i}$ for all natural numbers i.

$$\mathcal{L}^{\mathsf{AMLQ}} := \bigcup_{i \geq 0} \mathcal{L}^{\mathsf{AMLQ}_i}$$

In a similar way, the set of all AMLQ action models is the union of all sets of AMLQ_i action models for $i \geq 0$. We then generalize the definition of updated models and thereby the semantics of our logic. Simultaneously, we generalize all the definitions and propositions from Section 3.4. It is easy to check that all proofs still hold.

The advantage of defining our action models and language in this way is that we can have dynamic modalities in our action content. Whenever we have a formula with a dynamic modality, the action of this dynamic modality can have a content with a dynamic modality as well, and so on. However, we can also be sure that this chain is finite: at some point there will be an action content that does not have a dynamic modality. We will see later that this fact is crucial to prove completeness.

3.6 Set modalities and non-deterministic actions

Before we continue, we draw a brief comparison between set modalities in AMLQ and non-deterministic action modalities in AML.[3]

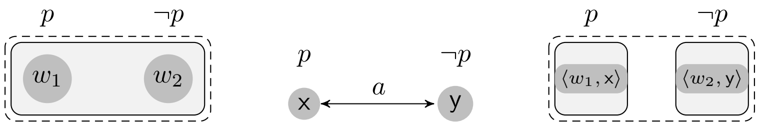
In AML we have dynamic modalities not just for simple actions, but also for complex actions, e.g. we have formulas of the form $[\mathsf{M}, \mathsf{x} \cup \mathsf{M}, \mathsf{y}]\alpha$. Like in PDL (Fischer & Ladner, 1979), the action $\mathsf{M}, \mathsf{x} \cup \mathsf{M}, \mathsf{y}$ is taken to be the single action of non-deterministically executing the action x or y. This seems very similar to $[\{\mathsf{x}, \mathsf{y}\}]\alpha$ in AMLQ, which we take to mean that α is true after getting the information that either of x and y is executed.

It is important to note that these two interpretations of complex modalities are the same as long as the language does not express questions: α is true after randomly executing either x or y just in case after getting the information that one of these has been executed we know

[3] We will sometimes refer to set modalities or non-deterministic modalities as *complex* modalities. By this, we mean that they are built up out of more than one action. This does not mean they have syntactic complexity: although they do have this in the notation used in AML, in our logic they are in fact basic.

that α is true. This means that the informational interpretation we give to these modalities is also available in AML.

However, the two interpretations come apart when adding questions to the language. For questions μ, $[\mathsf{s}]\mu$ is stronger than just stating that after any $\mathsf{x} \in \mathsf{s}$, μ is supported. To see this, consider the example depicted in Figure 3.8. The original information state of agent a supports both $[\mathsf{x}]p$ and $[\mathsf{y}]\neg p$, as can be seen from subfigure (c): given the information that x is occurring, it supports p, while given the information that y is occurring, it supports $\neg p$. So, after any action in the set $\{\mathsf{x}, \mathsf{y}\}$, $?p$ is supported. However, a's original information state does not support $[\{\mathsf{x}, \mathsf{y}\}]?p$: the information that either of x and y is executed is not enough to settle $?p$, since the information state in the updated model that corresponds with this information $(\{\langle w_1, \mathsf{x}\rangle, \langle w_2, \mathsf{y}\rangle\})$ does not support $?p$.



(a) State map of a before action

(b) Action model

(c) State map of a after action

Figure 3.8: Example to show that $[\{\mathsf{x}, \mathsf{y}\}]\varphi \not\equiv [\mathsf{x}]\varphi \wedge [\mathsf{y}]\varphi$

This shows the following:

Proposition 3.6.1. $[\{\mathsf{x}, \mathsf{y}\}]\varphi \not\equiv [\mathsf{x}]\varphi \wedge [\mathsf{y}]\varphi$

This is a fundamental difference between complex dynamic modalities in support-conditional semantics and in truth-conditional semantics. In logics based on the latter, like AML, this equivalence does hold (Van Ditmarsch et al., 2007, p. 152). This shows that in our setting, the interpretation of set modalities we have given is actually the only natural one.

As one would expect, this equivalence does hold in AMLQ whenever φ is truth-conditional. We can even be a bit more general and prove this for all sets s.[4]

Proposition 3.6.2. If α is truth-conditional, then $[\mathsf{s}]\alpha \equiv \bigwedge_{\mathsf{x}\in\mathsf{s}}[\mathsf{x}]\alpha$

Proof: As both formulas are declaratives, we only need to show that they have the same truth conditions. From the fact that α is truth-conditional, the definition of $s[\mathsf{s}]$ and the support conditions of the dynamic modality and conjunction we can obtain:

$$\begin{aligned}
M, w \models [\mathsf{s}]\alpha &\iff M', w[\mathsf{s}] \models \alpha \\
&\iff \text{for all } \langle w, \mathsf{x}\rangle \in w[\mathsf{s}] : M', \langle w, \mathsf{x}\rangle \models \alpha \\
&\iff \text{for all } \mathsf{x} \in \mathsf{s} : \text{if } \langle w, \mathsf{x}\rangle \in W', \text{ then } M', \langle w, \mathsf{x}\rangle \models \alpha \\
&\iff \text{for all } \mathsf{x} \in \mathsf{s} : M', w[\mathsf{x}] \models \alpha \\
&\iff \text{for all } \mathsf{x} \in \mathsf{s} : M, w \models [\mathsf{x}]\alpha \\
&\iff M, w \models \textstyle\bigwedge_{\mathsf{x}\in\mathsf{s}}[\mathsf{x}]\alpha
\end{aligned}$$

□

This proposition is important in what comes next, because we need to be able to reduce formulas with any dynamic modality to a formula containing only dynamic modalities of single actions.

[4]Notice that s can be empty. In that case, we have an empty conjunction, which we take to be $\top$.

3.7 Composition of action models

Like in AML, we want it to be possible in AMLQ to reduce a formula of the form $[\mathsf{s}][\mathsf{t}]\varphi$ to $[\mathsf{s};\mathsf{t}]\varphi$, where the set of actions $\mathsf{s};\mathsf{t}$ stands for the information that one action from the set s and one action from the set t are executed in sequence.

We can do this by taking the two action models from the two modalities and combining them into one action model that has, as actions, pairs of which the first element is an action from the first action model and the second element an action from the second action model. In this way, we construct an action model that encodes all sequences of epistemic actions from the first and the second action model.

Definition 3.7.1. **Composition of AMLQ action models**
Let $\mathsf{M} = \langle \mathsf{S}, \{\sim_a \mid a \in \mathcal{A}\}, \mathsf{cont}\rangle$ and $\mathsf{M}' = \langle \mathsf{S}', \{\sim'_a \mid a \in \mathcal{A}\}, \mathsf{cont}'\rangle$ be two AMLQ action models. Their composition $\mathsf{M};\mathsf{M}'$ is the action model $\langle \mathsf{S}'', \{\sim''_a \mid a \in \mathcal{A}\}, \mathsf{cont}''\rangle$ such that:

- $\mathsf{S}'' = \mathsf{S} \times \mathsf{S}'$
- $\langle \mathsf{x}, \mathsf{x}'\rangle \sim''_a \langle \mathsf{y}, \mathsf{y}'\rangle$ iff $\mathsf{x} \sim_a \mathsf{y}$ and $\mathsf{x}' \sim'_a \mathsf{y}'$
- $\mathsf{cont}''(\langle \mathsf{x}, \mathsf{x}'\rangle) = \mathsf{cont}(\mathsf{x}) \wedge [\mathsf{M}, \mathsf{x}]\mathsf{cont}'(\mathsf{x}')$

The composition of two pointed action models (M, s) and $(\mathsf{M}', \mathsf{s}')$ is the pointed action model $(\mathsf{M}'', \mathsf{s} \times \mathsf{s}')$ with $\mathsf{M}'' = \mathsf{M};\mathsf{M}'$ defined as above.

It is easy to show that $\sim''_a$ is an equivalence relation if $\sim_a$ and $\sim'_a$ are. This guarantees that if M and M' are action models, then so is $\mathsf{M};\mathsf{M}'$. We also need to show that updating with M first and then with M' gives the same result as updating with $\mathsf{M};\mathsf{M}'$ at once.

Strictly speaking, any $(M \otimes \mathsf{M}) \otimes \mathsf{M}'$ and $M \otimes (\mathsf{M};\mathsf{M}')$ will never be the same model, as their worlds are pairs with a different internal structure. However, this internal structure only shows the history of actions, which is not relevant for describing the knowledge and issues of agents. Therefore, for our purposes we can regard $\langle\langle w, \mathsf{x}\rangle, \mathsf{x}'\rangle$ and $\langle w, \langle \mathsf{x}, \mathsf{x}'\rangle\rangle$ as one world under two different names. We prove this in the following proposition.

Proposition 3.7.1. **Isomorphism between updated models**
For every inquisitive epistemic model M and every two action models M and M', $(M \otimes \mathsf{M}) \otimes \mathsf{M}'$ is isomorphic to $M \otimes (\mathsf{M};\mathsf{M}')$.

Proof: We define three updated models M', M'' and M''':

- $M' = M \otimes \mathsf{M} = \langle W', \{\Sigma'_a \mid a \in \mathcal{A}\}, V'\rangle$
- $M'' = M' \otimes \mathsf{M}' = \langle W'', \{\Sigma''_a \mid a \in \mathcal{A}\}, V''\rangle$
- $M''' = M \otimes (\mathsf{M};\mathsf{M}') = \langle W''', \{\Sigma'''_a \mid a \in \mathcal{A}\}, V'''\rangle$

We want to show that M'' and M''' are isomorphic. That is, there is a bijective function f that maps the worlds in M'' to the worlds in M''', which preserves the exact structure of the models. We show this by mapping each $\langle\langle w, \mathsf{x}\rangle, \mathsf{x}'\rangle \in W''$ to $\langle w, \langle \mathsf{x}, \mathsf{x}'\rangle\rangle \in W'''$.

We start by showing that $\langle\langle w, \mathsf{x}\rangle, \mathsf{x}'\rangle \in W''$ just in case $\langle w, \langle \mathsf{x}, \mathsf{x}'\rangle\rangle \in W'''$.

($\Rightarrow$) Assume $\langle\langle w, \mathsf{x}\rangle, \mathsf{x}'\rangle \in W''$

Then by definition of W'', we have $\langle w, \mathsf{x}\rangle \in W'$, $\mathsf{x}' \in \mathsf{S}'$ and $M', \langle w, \mathsf{x}\rangle \models \mathsf{pre}'(\mathsf{x}')$.

By definition of W', $\langle w, \mathsf{x}\rangle \in W'$ implies $w \in W$, $\mathsf{x} \in \mathsf{S}$ and $M, w \models \mathsf{pre}(\mathsf{x})$.

By Proposition 3.2.1 we obtain $M', \langle w, \mathsf{x}\rangle \models \mathsf{cont}'(\mathsf{x}')$ and $M, w \models \mathsf{cont}(\mathsf{x})$. Then as $\langle w, \mathsf{x}\rangle = w[\mathsf{x}]$, by the support condition of the dynamic modality, $M, w \models [\mathsf{M}, \mathsf{x}]\mathsf{cont}'(\mathsf{x}')$. This means that $M, w \models \mathsf{cont}(\mathsf{x}) \wedge [\mathsf{M}, \mathsf{x}]\mathsf{cont}'(\mathsf{x}')$. It follows that $M, w \models \mathsf{cont}''(\langle \mathsf{x}, \mathsf{x}'\rangle)$ and thus that $M, w \models \mathsf{pre}''(\langle \mathsf{x}, \mathsf{x}'\rangle)$. By $\mathsf{x} \in \mathsf{S}$ and $\mathsf{x}' \in \mathsf{S}'$ we also have $\langle \mathsf{x}, \mathsf{x}'\rangle \in \mathsf{S} \times \mathsf{S}'$. Then by definition of W''', $\langle w, \langle \mathsf{x}, \mathsf{x}'\rangle\rangle \in W'''$.

($\Leftarrow$) Assume $\langle w, \langle \mathsf{x}, \mathsf{x}'\rangle\rangle \in W'''$.

Then by definition of W''', we have $w \in W$, $\langle \mathsf{x}, \mathsf{x}'\rangle \in \mathsf{S}''$ and $M, w \models \mathsf{pre}''(\langle \mathsf{x}, \mathsf{x}'\rangle)$.

By Proposition 3.2.1, we obtain $M, w \models \mathsf{cont}''(\langle \mathsf{x}, \mathsf{x}'\rangle)$, which implies that $M, w \models \mathsf{cont}(\mathsf{x}) \wedge [\mathsf{M}, \mathsf{x}]\mathsf{cont}'(\mathsf{x}')$. As $\mathsf{S}'' = \mathsf{S} \times \mathsf{S}'$, $\mathsf{x} \in \mathsf{S}$ and $\mathsf{x}' \in \mathsf{S}'$.

We can use Proposition 3.2.1 and the support condition of the dynamic modality to obtain from $M, w \models \mathsf{cont}(\mathsf{x})$ and $M, w \models [\mathsf{M}, \mathsf{x}]\mathsf{cont}'(\mathsf{x}')$ that $M, w \models \mathsf{pre}(\mathsf{x})$, and $M, \langle w, \mathsf{x}\rangle \models \mathsf{pre}'(\mathsf{x}')$.

This also means it must be the case that $\langle w, \mathsf{x}\rangle \in W'$. This fact together with $M, \langle w, \mathsf{x}\rangle \models \mathsf{pre}'(\mathsf{x}')$ and $\mathsf{x}' \in \mathsf{S}'$, implies $\langle\langle w, \mathsf{x}\rangle, \mathsf{x}'\rangle \in W''$.

From this we can conclude that if we let $f(\langle\langle w, \mathsf{x}\rangle, \mathsf{x}'\rangle) = \langle w, \langle \mathsf{x}, \mathsf{x}'\rangle\rangle$, then f is a bijection between W'' and W'''. Now let us show that it is indeed an isomorphism. For this, we need to show two things:

(i) The mapping preserves the structure of the state maps. That is, if we let $f(s)$ be $\{f(w) \mid w \in s\}$, then we have:

$$s \in \Sigma''_a(\langle\langle w, \mathsf{x}\rangle, \mathsf{x}'\rangle) \iff f(s) \in \Sigma'''_a(f(\langle\langle w, \mathsf{x}\rangle, \mathsf{x}'\rangle)$$

(ii) The mapping preserves the valuation: $V''(\langle\langle w, \mathsf{x}\rangle, \mathsf{x}'\rangle) = V'''(f(\langle\langle w, \mathsf{x}\rangle, \mathsf{x}'\rangle))$

We start by showing (i). Take any world $\langle\langle w, \mathsf{x}\rangle, \mathsf{x}'\rangle \in W''$ and any state $s'' \subseteq W''$. Let $s''' = f(s'')$. Then we need to show that $s'' \in \Sigma''_a(\langle\langle w, \mathsf{x}\rangle, \mathsf{x}'\rangle) \iff s''' \in \Sigma'''_a(\langle w, \langle \mathsf{x}, \mathsf{x}'\rangle\rangle)$.

($\Rightarrow$) Assume $s'' \in \Sigma''_a(\langle\langle w, \mathsf{x}\rangle, \mathsf{x}'\rangle)$.

If s'' is empty, we are done, so assume it is not. We know that it satisfies conditions (i-iv) of Definition 3.2.4.

By condition (iii), $\pi_2(s'')$ contains one action y'. Let $s' = \pi_1(s'')$. Then by condition (iv), $M', s' \models \mathsf{cont}'(\mathsf{y}')$. By condition (ii), $\mathsf{x}' \sim_a \mathsf{y}'$ and by condition (i), $s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$. That means s' in turn satisfies conditions (i-iv) to be in $\Sigma'_a(\langle w, \mathsf{x}\rangle)$.

By condition (iii), $\pi_2(s')$ contains one action y. Then by condition (iv), we have $M, \pi_1(s') \models \mathsf{cont}(\mathsf{y})$. By condition (ii) we have $\mathsf{x} \sim_a \mathsf{y}$ and by condition (i), $\pi_1(s') \in \Sigma_a(w)$.

We can now show that s''' satisfies the four conditions to be in $\Sigma'''_a(\langle w, \langle \mathsf{x}, \mathsf{x}'\rangle\rangle)$. Because there is only one $\mathsf{y}' \in \pi_2(s'')$ and one $\mathsf{y} \in \pi_2(s')$, $\pi_2(s''') = \{\langle \mathsf{y}, \mathsf{y}'\rangle\}$. So condition (iii) is satisfied.

Notice that $\pi_1(s''') = \pi_1(s')$. As $\pi_2(s') = \{\mathsf{y}\}$, $s' = \pi_1(s')[\mathsf{y}] = \pi_1(s''')[\mathsf{y}]$. Therefore, we have $M, \pi_1(s''') \models \mathsf{cont}(\mathsf{y})$ by $M, \pi_1(s') \models \mathsf{cont}(\mathsf{y})$ and $M', \pi_1(s''')[\mathsf{y}] \models \mathsf{cont}'(\mathsf{y}')$ by $M', s' \models \mathsf{cont}'(\mathsf{y}')$. By the support condition of the dynamic modality, we have $M, \pi_1(s''') \models [\mathsf{y}]\mathsf{cont}'(\mathsf{y}')$. It follows that $M, \pi_1(s''') \models \mathsf{cont}''(\langle \mathsf{y}, \mathsf{y}'\rangle)$, which means condition (iv) is satisfied as well.

As for condition (ii), we have $\mathsf{x} \sim_a \mathsf{y}$ and $\mathsf{x}' \sim'_a \mathsf{y}'$, so by the definition of composition we have $\langle \mathsf{x}, \mathsf{x}'\rangle \sim''_a \langle \mathsf{y}, \mathsf{y}'\rangle$. Finally, we have condition (i) because $\pi_1(s''') = \pi_1(s')$ and $\pi_1(s') \in \Sigma_a(w)$.

($\Leftarrow$) Assume $s''' \in \Sigma'''_a(\langle w, \langle \mathsf{x}, \mathsf{x}' \rangle\rangle)$.

If s''' is empty, we are done, so assume it is not. We know that it satisfies conditions (i-iv) of Definition 3.2.4.

By condition (iii), $\pi_2(s''')$ contains one action $\langle \mathsf{y}, \mathsf{y}' \rangle$. By condition (iv), we have $M', \pi_1(s''') \models \mathsf{cont}''(\langle \mathsf{y}, \mathsf{y}' \rangle)$. By condition (ii), $\langle \mathsf{x}, \mathsf{x}' \rangle \sim''_a \langle \mathsf{y}, \mathsf{y}' \rangle$ and by condition (i), $\pi_1(s''') \in \Sigma_a(w)$.

By the definition of composition, by $M', \pi_1(s''') \models \mathsf{cont}''(\langle \mathsf{y}, \mathsf{y}' \rangle)$ we have that $M', \pi_1(s''') \models \mathsf{cont}(\mathsf{y}) \wedge [\mathsf{M}, \mathsf{y}]\mathsf{cont}'(\mathsf{y}')$ and $\langle \mathsf{x}, \mathsf{x}' \rangle \sim''_a \langle \mathsf{y}, \mathsf{y}' \rangle$ implies $\mathsf{x} \sim_a \mathsf{y}$ and $\mathsf{x}' \sim'_a \mathsf{y}'$.

Again, let $s' = \pi_1(s'')$. Then as $\pi_1(s') = \pi_1(s''')$ and $\pi_2(s') = \{\mathsf{y}\}$, it follows from the above that s' satisfies conditions (i-iv) to be in $\Sigma'_a(\langle w, \mathsf{x} \rangle)$. This, in turn, is condition (i) for s'' to be in $\Sigma''_a(\langle\langle w, \mathsf{x} \rangle, \mathsf{x}' \rangle)$. As $\pi_2(s'') = \{\mathsf{y}'\}$, and we already have $\mathsf{x}' \sim'_a \mathsf{y}'$, condition (iii) and (ii) are also satisfied. Then from $M', \pi_1(s') \models [\mathsf{M}, \mathsf{y}]\mathsf{cont}'(\mathsf{y}')$ we obtain $M, \pi_1(s')[\mathsf{y}] \models \mathsf{cont}'(\mathsf{y}')$. As $\pi_1(s')[\mathsf{y}] = \pi_1(s'')$, this establishes condition (iv).

Finally, by the definition of valuation in updated models, $\langle\langle w, \mathsf{x} \rangle, \mathsf{x}' \rangle \in V''(p)$ iff $\langle w, \mathsf{x} \rangle \in V'(p)$ iff $w \in V(p)$ iff $\langle w, \langle \mathsf{x}, \mathsf{x}' \rangle\rangle \in V'''(p)$, which settles condition (ii) for f to be an isomorphism. This concludes the proof that M'' and M''' are isomorphic. □

We continue by showing that updating a state in an original model twice, first with a set of actions s from one action model and then with a set of actions t from another action model, gives the same result as updating it once with the corresponding set of actions from the composed action model.

Proposition 3.7.2. **$s[\mathsf{s};\mathsf{t}]$ is the image of $s[\mathsf{s}][\mathsf{t}]$ under isomorphism f**
For every information state s and every two sets of actions s and t:

$$\langle\langle w, \mathsf{x} \rangle, \mathsf{x}' \rangle \in s[\mathsf{s}][\mathsf{t}] \iff \langle w, \langle \mathsf{x}, \mathsf{x}' \rangle\rangle \in s[\mathsf{s};\mathsf{t}]$$

Proof: ($\Rightarrow$) Assume $\langle\langle w, \mathsf{x} \rangle, \mathsf{x}' \rangle \in s[\mathsf{s}][\mathsf{t}]$.

Then by Definition 3.2.6, $M', \langle w, \mathsf{x} \rangle \models \mathsf{pre}'(\mathsf{x}')$, $\langle w, \mathsf{x} \rangle \in s[\mathsf{s}]$ and $\mathsf{x}' \in \mathsf{t}$. By Proposition 3.2.1 we obtain $M', \langle w, \mathsf{x} \rangle \models \mathsf{cont}'(\mathsf{x}')$. By the support condition of the dynamic modality, $M, w \models [\mathsf{x}]\mathsf{cont}'(\mathsf{x}')$.

As $\langle w, \mathsf{x} \rangle \in s[\mathsf{s}]$, by Definition 3.2.6 we also have $M, w \models \mathsf{pre}(\mathsf{x})$, $w \in s$ and $\mathsf{x} \in \mathsf{s}$. This means that $M, w \models \mathsf{cont}(\mathsf{x})$. As $\mathsf{x} \in \mathsf{s}$ and $\mathsf{x}' \in \mathsf{t}$, $\langle \mathsf{x}, \mathsf{x}' \rangle \in \mathsf{s} \times \mathsf{t}$. From $M, w \models \mathsf{cont}(\mathsf{x}) \wedge [\mathsf{M}, \mathsf{x}]\mathsf{cont}'(\mathsf{x}')$ it follows that $M, w \models \mathsf{cont}''(\langle \mathsf{x}, \mathsf{x}' \rangle)$ and thus that $M, w \models \mathsf{pre}''(\langle \mathsf{x}, \mathsf{x}' \rangle)$.

Then by Definition 3.2.6, $\langle w, \langle \mathsf{x}, \mathsf{x}' \rangle\rangle \in s[\mathsf{s};\mathsf{t}]$.

($\Leftarrow$) Assume $\langle w, \langle \mathsf{x}, \mathsf{x}' \rangle\rangle \in s[\mathsf{s};\mathsf{t}]$.

Then by Definition 3.2.6, $M, w \models \mathsf{pre}''(\langle \mathsf{x}, \mathsf{x}' \rangle)$, $w \in s$ and $\langle \mathsf{x}, \mathsf{x}' \rangle \in \mathsf{s} \times \mathsf{t}$, which means that $\mathsf{x} \in \mathsf{s}$ and $\mathsf{x}' \in \mathsf{t}$. We obtain $M, w \models \mathsf{cont}''(\langle \mathsf{x}, \mathsf{x}' \rangle)$ by Proposition 3.2.1, which means by the definition of composition that $M, w \models \mathsf{cont}(\mathsf{x}) \wedge [\mathsf{M}, \mathsf{x}]\mathsf{cont}'(\mathsf{x}')$. This in turn means that $M, w \models \mathsf{pre}(\mathsf{x})$ and $M', \langle w, \mathsf{x} \rangle \models \mathsf{pre}'(\mathsf{x}')$.

As $M, w \models \mathsf{pre}(\mathsf{x})$, $w \in s$ and $\mathsf{x} \in \mathsf{s}$, we have by Definition 3.2.6 that $\langle w, \mathsf{x} \rangle \in s[\mathsf{s}]$. Similarly, because $M', \langle w, \mathsf{x} \rangle \models \mathsf{pre}'(\mathsf{x}')$, $\langle w, \mathsf{x} \rangle \in s[\mathsf{s}]$ and $\mathsf{x}' \in \mathsf{t}$, we have $\langle\langle w, \mathsf{x} \rangle, \mathsf{x}' \rangle \in s[\mathsf{s}][\mathsf{t}]$. □

From the previous two propositions we can conclude that these states support the same formulas. A straightforward induction on the structure of formulas would suffice to show this. That means we can show that for any state s and formula φ, any two formulas of the form $[\mathsf{s}][\mathsf{t}]\varphi$ and $[\mathsf{s};\mathsf{t}]\varphi$ are indeed equivalent.

Proposition 3.7.3. $[\mathsf{s}][\mathsf{t}]\varphi \equiv [\mathsf{s};\mathsf{t}]\varphi$

Proof: From the support condition of the dynamic modality and the previous propositions, we obtain the following proof:

$$\begin{aligned} M, s \models [\mathsf{s}][\mathsf{t}]\varphi &\iff M', s[\mathsf{s}] \models [\mathsf{t}]\varphi \\ &\iff M'', s[\mathsf{s}][\mathsf{t}] \models \varphi \\ &\iff M''', s[\mathsf{s};\mathsf{t}] \models \varphi \\ &\iff M, s \models [\mathsf{s};\mathsf{t}]\varphi \end{aligned}$$ □

This proposition shows that any formula of the form $[\mathsf{s}][\mathsf{t}]\varphi$ can be reduced to $[\mathsf{s};\mathsf{t}]\varphi$.

3.8 Reduction

For the axiomatization of AMLQ, it is most natural to use the strategy that is used for the axiomatizations of both AML and IDEL. Both Van Ditmarsch et al. (2007) and Ciardelli (2016) use the fact that their logics are not more expressive than the static logics that they extend (epistemic logic and inquisitive epistemic logic, respectively). This is also the case for AMLQ. In this section, we show that for every formula $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$, we can find an equivalent formula φ^* that does not contain a dynamic modality and is therefore a formula of $\mathcal{L}^{\mathsf{IEL}}$.

3.8.1 Atom and $\bot$

We start by showing that, like in AML, the following equivalences hold for formulas in which a dynamic modality precedes an atom or $\bot$. As usual, x denotes a single action.

Proposition 3.8.1. $[\mathsf{x}]p \equiv \mathsf{pre}(\mathsf{x}) \to p$

Proof: As both formulas are declaratives, they are truth-conditional. This means that in order to prove the equivalence, we only need to show that they have the same truth conditions. We can show this using the truth condition of the dynamic modality, the definition of a state in an updated model and the truth condition of implication.

$$\begin{aligned} M, w \models [\mathsf{x}]p &\iff M', w[\mathsf{x}] \models p \\ &\iff w[\mathsf{x}] = \emptyset \text{ or } M', \langle w, \mathsf{x}\rangle \models p \\ &\iff w[\mathsf{x}] = \emptyset \text{ or } M, w \models p \\ &\iff M, w \not\models \mathsf{pre}(\mathsf{x}) \text{ or } M, w \models p \\ &\iff M, w \models \mathsf{pre}(\mathsf{x}) \to p \end{aligned}$$

As $[\mathsf{x}]p$ and $\mathsf{pre}(\mathsf{x}) \to p$ have the same truth conditions, by Definition 2.3.7 they have the same support conditions as well, which makes them equivalent. □

Because p is truth-conditional, the following equivalence immediately follows from the previous one and Proposition 3.6.2.

Corollary 3.8.1. $[\mathsf{s}]p \equiv \bigwedge_{\mathsf{x}\in\mathsf{s}}(\mathsf{pre}(\mathsf{x}) \to p)$

In a similar way as above, we can obtain a reduction equivalence for $\bot$.

PROPOSITION 3.8.2. $[\mathsf{x}]\bot \equiv \neg\mathsf{pre}(\mathsf{x})$

Proof: Again both formulas are declaratives, so we can show this with a proof similar to the previous one.

$$\begin{aligned} M, w \models [\mathsf{x}]\bot &\iff M', w[\mathsf{x}] \models \bot \\ &\iff w[\mathsf{x}] = \emptyset \\ &\iff M, w \not\models \mathsf{pre}(\mathsf{x}) \\ &\iff M, w \models \neg\mathsf{pre}(\mathsf{x}) \end{aligned}$$

□

COROLLARY 3.8.2. $[\mathsf{s}]\bot \equiv \bigwedge_{\mathsf{x}\in\mathsf{s}} \neg\mathsf{pre}(\mathsf{x})$

3.8.2 Conjunction and inquisitive disjunction

Whenever a dynamic modality precedes a conjunction, we can distribute it over the conjunction, just like in AML and IDEL. This goes for dynamic modalities of single actions as well as sets.

PROPOSITION 3.8.3. $[\mathsf{s}](\varphi \wedge \psi) \equiv [\mathsf{s}]\varphi \wedge [\mathsf{s}]\psi$

Proof: From the support conditions of the dynamic modality and conjunction we can obtain:

$$\begin{aligned} M, s \models [\mathsf{s}](\varphi \wedge \psi) &\iff M', s[\mathsf{s}] \models \varphi \wedge \psi \\ &\iff M', s[\mathsf{s}] \models \varphi \text{ and } M', s[\mathsf{s}] \models \psi \\ &\iff M, s \models [\mathsf{s}]\varphi \text{ and } M, s \models [\mathsf{s}]\psi \\ &\iff M, s \models [\mathsf{s}]\varphi \wedge [\mathsf{s}]\psi \end{aligned}$$

□

For dynamic modalities that precede an inquisitive disjunction, we can do the same as for conjunction.

PROPOSITION 3.8.4. $[\mathsf{s}](\varphi \vee\!\!\!\vee \psi) \equiv [\mathsf{s}]\varphi \vee\!\!\!\vee [\mathsf{s}]\psi$

Proof: From the support conditions of the dynamic modality and inquisitive disjunction we can obtain:

$$\begin{aligned} M, s \models [\mathsf{s}](\varphi \vee\!\!\!\vee \psi) &\iff M', s[\mathsf{s}] \models \varphi \vee\!\!\!\vee \psi \\ &\iff M', s[\mathsf{s}] \models \varphi \text{ or } M', s[\mathsf{s}] \models \psi \\ &\iff M, s \models [\mathsf{s}]\varphi \text{ or } M, s \models [\mathsf{s}]\psi \\ &\iff M, s \models [\mathsf{s}]\varphi \vee\!\!\!\vee [\mathsf{s}]\psi \end{aligned}$$

□

3.8.3 Implication

Although one might expect that we can have a similar reduction equivalence for implication, this is in fact not the case, as the following proposition shows.

PROPOSITION 3.8.5. $[\mathsf{s}](\varphi \to \psi) \not\equiv [\mathsf{s}]\varphi \to [\mathsf{s}]\psi$

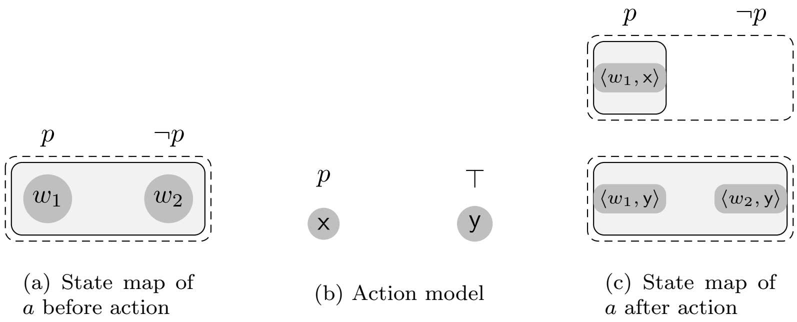


Figure 3.9: Example to show that $[\mathsf{s}](\varphi \to \psi) \not\equiv [\mathsf{s}]\varphi \to [\mathsf{s}]\psi$

Proof: Consider the original state map, action model and updated state map in Figure 3.9.

Let $s = \{w_1, w_2\}$ and $\mathsf{s} = \{\mathsf{x}, \mathsf{y}\}$. Then $s[\mathsf{s}]$ is the information state consisting of all the worlds in the updated model. This information state does not support $\neg K_a p$, because it contains a world $\langle w_1, \mathsf{x}\rangle$ in which a does know that p. So $s[\mathsf{s}] \not\models K_a p \to \bot$, which in turn means that $s \not\models [\mathsf{s}](K_a p \to \bot)$.

However, $s \models [\mathsf{s}]K_a p \to [\mathsf{s}]\bot$. For if we take a subset of s that supports $[\mathsf{s}]K_a p$, we only have the empty set to take, and the empty set indeed supports $[\mathsf{s}]\bot$. □

What this example shows is that $[\mathsf{s}](\varphi \to \psi)$ is actually stronger than $[\mathsf{s}]\varphi \to [\mathsf{s}]\psi$. In fact, if a state supports $[\mathsf{s}](\varphi \to \psi)$, it supports not only $[\mathsf{s}]\varphi \to [\mathsf{s}]\psi$, but $[\mathsf{t}]\varphi \to [\mathsf{t}]\psi$ for all $\mathsf{t} \subseteq \mathsf{s}$. But although $[\mathsf{s}](\varphi \to \psi) \models \bigwedge_{\mathsf{t}\subseteq\mathsf{s}}([\mathsf{t}]\varphi \to [\mathsf{t}]\psi)$, the converse is not the case.

The problem is that to prove from any assumption that a state s supports $[\mathsf{s}](\varphi \to \psi)$, we need to show that any $t' \subseteq s[\mathsf{s}]$ that supports φ, supports ψ as well.

However, this t' may be any subset: suppose s contains w and v and s contains x and y: then t' may be $\{\langle w, \mathsf{x}\rangle, \langle v, \mathsf{y}\rangle\}$. This means that t' is not equal to $t[\mathsf{t}]$ for any t and t. Therefore, there is no way we can use the support conditions of the dynamic modality and implication to find a reduction equivalence for $[\mathsf{s}](\varphi \to \psi)$.

This problem does not occur when s is a singleton: any subset of $s[\mathsf{x}]$ is in fact equal to some $t[\mathsf{x}]$ where $t \subseteq s$. We can use this to show the following equivalence.

Proposition 3.8.6. $[\mathsf{x}](\varphi \to \psi) \equiv [\mathsf{x}]\varphi \to [\mathsf{x}]\psi$

Proof: ($\Rightarrow$) Assume $M, s \models [\mathsf{x}](\varphi \to \psi)$.

Then by the support condition of the dynamic modality, $M', s[\mathsf{x}] \models \varphi \to \psi$. Take any $t \subseteq s$ such that $M, t \models [\mathsf{x}]\varphi$. Then $M', t[\mathsf{x}] \models \varphi$. As $t[\mathsf{x}]$ is a subset of $s[\mathsf{x}]$, $M', t[\mathsf{x}] \models \psi$ by the support condition of implication. This means that $M, t \models [\mathsf{x}]\psi$. By the support condition of implication we have $M, s \models [\mathsf{x}]\varphi \to [\mathsf{x}]\psi$.

($\Leftarrow$) Assume $M, s \models [\mathsf{x}]\varphi \to [\mathsf{x}]\psi$.

Take any $t' \subseteq s[\mathsf{x}]$ such that $t' \models \varphi$. Let $t = \pi_1(t')$. Then by the definition of updated states, $t[\mathsf{x}] = t'$. As $M', t[\mathsf{x}] \models \varphi$, $M, t \models [\mathsf{x}]\varphi$. By the support condition of implication, $M, t \models [\mathsf{x}]\psi$. So $M', t[\mathsf{x}] \models \psi$. As t' was an arbitrary subset of $s[\mathsf{x}]$, by the support condition of implication, $M', s[\mathsf{x}] \models \varphi \to \psi$. Therefore $M, s \models [\mathsf{x}](\varphi \to \psi)$. □

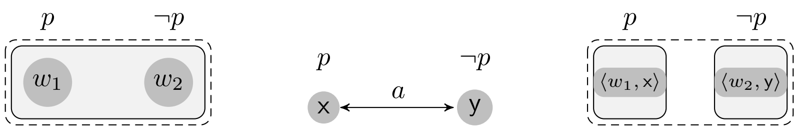
Since $\varphi \to \psi$ is not necessarily truth-conditional, we cannot use Proposition 3.6.2 to generalize this result to formulas with set modalities. We will see later that there is a way to work around this, although it complicates the completeness proof.

3.8.4 Knowledge modality

While in AML, $[\mathsf{x}]K_a\varphi$ is equivalent to $\mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} K_a[\mathsf{y}]\varphi$, this is not the case in AMLQ. The reason for this is that in AMLQ, φ can be a question. Indeed, for statements α, knowing α after x is (given that the precondition of x is met) the same as knowing that after any action indistinguishable from x, α will be the case. But this does not generalize to questions, as the following proposition shows.

PROPOSITION 3.8.7. $[\mathsf{x}]K_a\varphi \not\equiv \mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} K_a[\mathsf{y}]\varphi$

Proof: Consider the following counterexample, illustrated in Figure 3.10: agent a has no knowledge or issues in the original model and no knowledge about which action is taking place, which is in fact action x.



(a) State map of a before action

(b) Action model

(c) State map of a after action

Figure 3.10: Example to show that $[\mathsf{x}]K_a\varphi \not\equiv \mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} K_a[\mathsf{y}]\varphi$

Clearly, in the updated model, a still does not know whether p is true or false. So $[\mathsf{x}]K_a?p$ is false in w_1. However, $\mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} K_a[\mathsf{y}]?p$ is true in w_1, because a does know that $[\mathsf{x}]p$ and $[\mathsf{y}]\neg p$, which together make the consequent true. □

This example shows that, since φ can be a question in our setting, knowing φ after some action x is not the same as knowing that after y, φ, for each action y indistinguishable from x.

In fact, for $[\mathsf{x}]K_a?p$ to be true requires that either $K_a([\mathsf{x}]p \wedge [\mathsf{y}]p)$ or $K_a([\mathsf{x}]\neg p \wedge [\mathsf{y}]\neg p)$. This means we can come up with an equivalence for $[\mathsf{x}]K_a\varphi$ by quantifying over the resolutions of φ. However, there is a much shorter and more intuitive alternative: we can use a set modality to express the uncertainty about which action is the actual one.

We can show that after x, a knows φ just in case $\mathsf{pre}(\mathsf{x})$ is false or a's current information state is such that learning that one of $\delta_a(\mathsf{x})$ is executed is enough to settle φ.

PROPOSITION 3.8.8. $[\mathsf{x}]K_a\varphi \equiv \mathsf{pre}(\mathsf{x}) \to K_a[\delta_a(\mathsf{x})]\varphi$

Proof: As both formulas are declaratives, we only need to show that they have the same truth conditions.

($\Rightarrow$) Assume $M, w \models [\mathsf{x}]K_a\varphi$.

Then $M', w[\mathsf{x}] \models K_a\varphi$ by the truth condition of the dynamic modality. Assume $M, w \models \mathsf{pre}(\mathsf{x})$. By the truth condition of the knowledge modality, $M', \sigma'_a(\langle w, \mathsf{x}\rangle) \models \varphi$.

By Lemma 3.2.1, $\sigma'_a(\langle w, \mathsf{x}\rangle) = \sigma_a(w)[\delta_a(\mathsf{x})]$. So $M', \sigma_a(w)[\delta_a(\mathsf{x})] \models \varphi$. We use the truth condition of the dynamic modality to obtain $M, \sigma_a(w) \models [\delta_a(\mathsf{x})]\varphi$. By the truth condition of the knowledge modality, $M, w \models K_a[\delta_a(\mathsf{x})]\varphi$. We can then drop our assumption that $M, w \models \mathsf{pre}(\mathsf{x})$ to obtain $M, w \models \mathsf{pre}(\mathsf{x}) \to K_a[\delta_a(\mathsf{x})]\varphi$.

($\Leftarrow$) Assume $M, w \models \mathsf{pre}(\mathsf{x}) \to K_a[\delta_a(\mathsf{x})]\varphi$.

Either $M, w \models \mathsf{pre}(\mathsf{x})$ or $M, w \not\models \mathsf{pre}(\mathsf{x})$. In the latter case, we immediately have $M, w \models [\mathsf{x}]K_a\varphi$, since $w[\mathsf{x}] = \emptyset$. In the former case, we have $M, w \models K_a[\delta_a(\mathsf{x})]\varphi$. By the truth condition of the knowledge modality, $M, \sigma_a(w) \models [\delta_a(\mathsf{x})]\varphi$. Then by the truth condition the dynamic modality, $M', \sigma_a(w)[\delta_a(\mathsf{x})] \models \varphi$. Since by Lemma 3.2.1, $\sigma'_a(\langle w, \mathsf{x}\rangle) = \sigma_a(w)[\delta_a(\mathsf{x})]$, that means $M', \sigma'_a(\langle w, \mathsf{x}\rangle) \models \varphi$. Hence, $M', w[\mathsf{x}] \models K_a\varphi$. By the truth condition of the dynamic modality, $M, w \models [\mathsf{x}]K_a\varphi$. □

As both $K_a\varphi$ and $\mathsf{pre}(\mathsf{x}) \to K_a[\delta_a(\mathsf{x})]\varphi$ are truth-conditional, we can generalize the previous result to set modalities in the usual way.

Corollary 3.8.3. $[\mathsf{s}]K_a\varphi \equiv \bigwedge_{\mathsf{x}\in\mathsf{s}}(\mathsf{pre}(\mathsf{x}) \to K_a[\delta_a(\mathsf{x})]\varphi)$

3.8.5 Entertain modality

We now move on to the reduction equivalence for the entertain modality.

Proposition 3.8.9. $[\mathsf{x}]E_a\varphi \equiv \mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} E_a(\mathsf{cont}(\mathsf{y}) \to [\mathsf{y}]\varphi)$

Proof: As both formulas are declaratives, we only need to show that they have the same truth conditions.

($\Rightarrow$) Assume $M, w \models [\mathsf{x}]E_a\varphi$.

Then $M', w[\mathsf{x}] \models E_a\varphi$. Assume $M, w \models \mathsf{pre}(\mathsf{x})$. By the truth condition of the entertain modality, we have for all $s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle) : M', s' \models \varphi$.

Take any action z such that $\mathsf{z} \sim_a \mathsf{x}$ and some $s \in \Sigma_a(w)$. Then take any $t \subseteq s$ such that $M, t \models \mathsf{cont}(\mathsf{z})$. By Lemma 3.2.2, $t[\mathsf{z}] \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$. This means that $M', t[\mathsf{z}] \models \varphi$.

By the support condition of the dynamic modality, we have $M, t \models [\mathsf{z}]\varphi$. By the support condition of implication, we have $M, s \models \mathsf{cont}(\mathsf{z}) \to [\mathsf{z}]\varphi$. As s was an arbitrary state in $\Sigma_a(w)$, we have $M, w \models E_a(\mathsf{cont}(\mathsf{z}) \to [\mathsf{z}]\varphi)$. As z was an arbitrary action indistinguishable by a from x, we have $M, w \models \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} E_a(\mathsf{cont}(\mathsf{y}) \to [\mathsf{y}]\varphi)$. Then, finally, we drop our assumption that $M, w \models \mathsf{pre}(\mathsf{x})$ to obtain $M, w \models \mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} E_a(\mathsf{cont}(\mathsf{y}) \to [\mathsf{y}]\varphi)$.

($\Leftarrow$) Assume $M, w \models \mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} E_a(\mathsf{cont}(\mathsf{y}) \to [\mathsf{y}]\varphi)$.

Either $M, w \models \mathsf{pre}(\mathsf{x})$ or $M, w \not\models \mathsf{pre}(\mathsf{x})$. In the latter case, we immediately have $M, w \models [\mathsf{x}]E_a\varphi$, since $w[\mathsf{x}] = \emptyset$ and we are done, so assume the former. Then we have $M, w \models \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} E_a(\mathsf{cont}(\mathsf{y}) \to [\mathsf{y}]\varphi)$.

As $M, w \models \mathsf{pre}(\mathsf{x})$, we have a world $\langle w, \mathsf{x}\rangle$ in the updated model. Take any $s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$. Then by Lemma 3.2.2, $s' = s[\mathsf{z}]$ for some s, z such that $s \in \Sigma_a(w)$, $\mathsf{z} \sim_a \mathsf{x}$ and $M, s \models \mathsf{cont}(\mathsf{z})$. It follows from $\mathsf{z} \sim_a \mathsf{x}$ that $M, w \models E_a(\mathsf{cont}(\mathsf{z}) \to [\mathsf{z}]\varphi)$. Then by $s \in \Sigma_a(w)$ we have that $M, s \models \mathsf{cont}(\mathsf{z}) \to [\mathsf{z}]\varphi$. As $M, s \models \mathsf{cont}(\mathsf{z})$, we obtain $M, s \models [\mathsf{z}]\varphi$. By the support condition of the dynamic modality, $M', s' \models \varphi$.

As s' was an arbitrary state in $\Sigma'_a(\langle w, \mathsf{x}\rangle)$, we have $M', \langle w, \mathsf{x}\rangle \models E_a\varphi$, which means that $M', w[\mathsf{x}] \models E_a\varphi$. By the truth condition of the dynamic modality, we have $M, w \models [\mathsf{x}]E_a\varphi$. □

Finally, we also make this reduction equivalence more general in the same way as we did for the knowledge modality:

Corollary 3.8.4. $[\mathsf{s}]E_a\varphi \equiv \bigwedge_{\mathsf{x}\in\mathsf{s}}(\mathsf{pre}(\mathsf{x}) \rightarrow \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} E_a(\mathsf{cont}(\mathsf{y}) \rightarrow [\mathsf{y}]\varphi))$

3.8.6 Reduction procedure

Using the reduction equivalences from the previous sections, we will show that any formula of AMLQ can be reduced to a formula of the static language of IEL. The proof style we will use is different from that of AML and IDEL: while the reduction proofs for these logics proceed by a straightforward induction on the structure of formulas, we cannot do the same, due to the lack of a reduction equivalence for formulas of the form $[\mathsf{s}](\varphi \rightarrow \psi)$ (in particular, the case when s is a set of more than one action and $\varphi \rightarrow \psi$ is a question).

To show that we can reduce formulas even when they have this form, we need to use the fact that every formula is equivalent to the inquisitive disjunction of its resolutions (Proposition 3.4.5). We can then perform an induction on the structure of these resolutions. As these are declaratives by definition, we will have a way of reducing them.

However, another complication is that a subformula of a declarative is not necessarily a declarative itself (e.g. a declarative can be of the form $K_a\mu$ with μ a question). As a consequence, we cannot perform a straightforward induction on the structure of declaratives. Instead, we do our induction inside an induction on the modal depth of formulas, which we define as follows.

Definition 3.8.1. **Modal depth**
For any $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$, its modal depth $\mathsf{md}(\varphi)$ is the maximum number of nested epistemic modalities K_a and E_a occurring in φ, defined inductively as follows:

- $\mathsf{md}(p) = 0$
- $\mathsf{md}(\bot) = 0$
- $\mathsf{md}(\varphi \circ \psi) = \mathsf{max}(\mathsf{md}(\varphi), \mathsf{md}(\psi))$ for $\circ \in \{\wedge, \rightarrow, \vee\!\!\!\vee\}$
- $\mathsf{md}(\blacksquare\varphi) = \mathsf{md}(\varphi) + 1$ for $\blacksquare \in \{K_a, E_a\}$
- $\mathsf{md}([\mathsf{s}]\varphi) = \mathsf{md}(\varphi)$

In our reduction proof, we will use this notion together with the following lemma.

Lemma 3.8.1. **Resolutions never have a higher modal depth**
If $\alpha \in \mathcal{R}(\varphi)$, then $\mathsf{md}(\alpha) \leq \mathsf{md}(\varphi)$.

Proof: A straightforward induction on the complexity of φ, using the definitions of resolutions and modal depth, suffices to show this. □

We are now ready to prove that we can reduce every formula of $\mathcal{L}^{\mathsf{AMLQ}}$ to one of $\mathcal{L}^{\mathsf{IEL}}$.

Theorem 3.8.1. **Every formula of AMLQ is equivalent to some formula of IEL**
For any $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$, there is some $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$ such that $\varphi \equiv \varphi^*$.

Proof: We start by proving the claim for the most basic fragment of our logic, namely $\mathcal{L}^{\mathsf{AMLQ}_0}$. Take any $\varphi \in \mathcal{L}^{\mathsf{AMLQ}_0}$. We perform an induction on the complexity of

φ. All steps are immediate, except for the one where φ is $[\mathsf{s}]\psi$. By the induction hypothesis we have a $\psi^* \in \mathcal{L}^{\mathsf{IEL}}$ such that $\psi \equiv \psi^*$, so $\varphi \equiv [\mathsf{s}]\psi^*$. Therefore, what we need to show is that $[\mathsf{s}]\psi^* \equiv \varphi^*$ for some $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$.

We show this by induction on the modal depth of $[\mathsf{s}]\psi^*$.

- **Base case.** Let $\mathsf{md}([\mathsf{s}]\psi^*) = 0$. Take any resolution $\alpha \in \mathcal{R}([\mathsf{s}]\psi^*)$. Then by Definition 3.4.2, $\alpha = [\mathsf{s}]\beta^*$ with $\beta^* \in \mathcal{R}(\psi^*)$. This means that $\beta^* \in \mathcal{L}^{\mathsf{IEL}}$.

  We will show that we can find a formula $\alpha^* \in \mathcal{L}^{\mathsf{IEL}}$ that is equivalent to $[\mathsf{s}]\beta^*$. We do this by another induction, on the complexity of β^*.

  - **Base case.** Suppose β^* is an atom p or $\bot$. Take any set of actions t. Then by Corollary 3.8.1 or 3.8.2 we have:

    $[\mathsf{t}]\beta^* \equiv \bigwedge_{\mathsf{x}\in\mathsf{t}}(\mathsf{pre}(\mathsf{x}) \to \beta^*)$

    Which means that in case of $[\mathsf{s}]\beta^*$ we can let $\alpha^* := \bigwedge_{\mathsf{x}\in\mathsf{s}} \mathsf{pre}(\mathsf{x}) \to \beta^*$.

  - **Inductive step.** Induction hypothesis: for all γ less complex than β^*, for all sets of actions t, there is some $\gamma^* \in \mathcal{L}^{\mathsf{IEL}}$ such that $[\mathsf{t}]\gamma \equiv \gamma^*$.

    Notice that because β^* is a declarative with modal depth 0, the only connectives it can contain are conjunction and implication.

    ($\wedge$) Suppose β^* is $\gamma \wedge \gamma'$. Then by Proposition 3.8.3, $[\mathsf{s}]\beta^* \equiv [\mathsf{s}]\gamma \wedge [\mathsf{s}]\gamma'$. By the induction hypothesis, we have some $\gamma^*, \gamma'^* \in \mathcal{L}^{\mathsf{IEL}}$ which are equivalent to $[\mathsf{s}]\gamma$ and $[\mathsf{s}]\gamma'$ respectively. So we can let $\alpha^* := \gamma^* \wedge \gamma'^*$.

    ($\to$) Suppose β^* is $\gamma \to \gamma'$.

    Then by Proposition 3.6.2 and 3.8.6, $[\mathsf{s}]\beta^* \equiv \bigwedge_{\mathsf{x}\in\mathsf{s}}([\mathsf{x}]\gamma \to [\mathsf{x}]\gamma')$. Take any $\mathsf{x} \in \mathsf{s}$. By the induction hypothesis, we have some $\gamma^*, \gamma'^* \in \mathcal{L}^{\mathsf{IEL}}$ which are equivalent to $[\mathsf{x}]\gamma$ and $[\mathsf{x}]\gamma'$ respectively. Let $\theta_{\mathsf{x}} := \gamma^* \to \gamma'^*$. Then $\theta_{\mathsf{x}} \equiv [\mathsf{x}]\gamma \to [\mathsf{x}]\gamma'$. As we can define such a θ_{x} for all $\mathsf{x} \in \mathsf{s}$, we can let $\alpha^* := \bigwedge_{\mathsf{x}\in\mathsf{s}} \theta_{\mathsf{x}}$.

  This concludes the proof of the claim that we have some $\alpha^* \in \mathcal{L}^{\mathsf{IEL}}$ equivalent to every $\alpha \in \mathcal{R}([\mathsf{s}]\psi^*)$. Let Γ be the set of these formulas. Then $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}\Gamma$ is also a formula of $\mathcal{L}^{\mathsf{IEL}}$, which by the support condition of inquisitive disjunction is equivalent to $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}\mathcal{R}([\mathsf{s}]\psi^*)$. By Proposition 3.4.5, it is also equivalent to $[\mathsf{s}]\psi^*$. This means we can let $\varphi^* := \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}\Gamma$.

  What we have shown is that if $\mathsf{md}([\mathsf{s}]\psi^*) = 0$, then there is some $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$ such that $[\mathsf{s}]\psi^* \equiv \varphi^*$.

- **Inductive step.** By the induction hypothesis, we have for any $[\mathsf{s}]\chi^*$ of modal depth $< n$, some equivalent formula in $\mathcal{L}^{\mathsf{IEL}}$.

  Let $\mathsf{md}([\mathsf{s}]\psi^*) = n$.

  Like in the base case, we take an arbitrary resolution $\alpha \in \mathcal{R}([\mathsf{s}]\psi^*)$. Then $\alpha = [\mathsf{s}]\beta^*$ with $\beta^* \in \mathcal{R}(\psi^*)$. This means that $\beta^* \in \mathcal{L}^{\mathsf{IEL}}$.

  We will show that we can find a formula $\alpha^* \in \mathcal{L}^{\mathsf{IEL}}$ that is equivalent to $[\mathsf{s}]\beta^*$. Like in the base case, we perform an induction on the complexity of β^*.

  The base case and the inductive steps for conjunction and implication carry over. We only need to show the cases for the modalities K_a and E_a.

  Note that Lemma 3.8.1 guarantees that $\mathsf{md}(\beta^*) \leq n$.

(K) Suppose β^* is $K_a\chi$. Then by Corollary 3.8.3 we have $[\mathsf{s}]\beta^* \equiv \bigwedge_{\mathsf{x}\in\mathsf{s}}(\mathsf{pre}(\mathsf{x}) \to K_a[\delta_a(\mathsf{x})]\chi)$. As $\mathsf{md}(\chi) < n$, by the induction hypothesis we have some $\chi^*_{\mathsf{x}} \in \mathcal{L}^{\mathsf{IEL}}$ equivalent to each $[\delta_a(\mathsf{x})]\chi$. So we can let $\alpha^* := \bigwedge_{\mathsf{x}\in\mathsf{s}}(\mathsf{pre}(\mathsf{x}) \to K_a\chi^*_{\mathsf{x}})$.

(E) Suppose β^* is $E_a\chi$. Then by Corollary 3.8.4 we have $[\mathsf{s}]\beta^* \equiv \bigwedge_{\mathsf{x}\in\mathsf{s}}(\mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} E_a(\mathsf{cont}(\mathsf{y}) \to [\mathsf{y}]\chi))$. As $\mathsf{md}(\chi) < n$, by the induction hypothesis we have some $\chi^*_{\mathsf{y}} \in \mathcal{L}^{\mathsf{IEL}}$ equivalent to each $[\mathsf{y}]\chi$. So we can let α^* be defined as:

$$\bigwedge_{\mathsf{x}\in\mathsf{s}}(\mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} E_a(\mathsf{cont}(\mathsf{y}) \to \chi^*_{\mathsf{y}}))$$

Like in the base case, we have now shown that we have some $\alpha^* \in \mathcal{L}^{\mathsf{IEL}}$ equivalent to every $\alpha \in \mathcal{R}([\mathsf{s}]\psi^*)$. Again, we let Γ be the set of these formulas. Then $\bigvee\!\!\!\bigvee\Gamma$ is a formula of $\mathcal{L}^{\mathsf{IEL}}$ equivalent to $[\mathsf{s}]\psi^*$. This means we can let $\varphi^* := \bigvee\!\!\!\bigvee\Gamma$.

This concludes the inductive proof that no matter what the modal depth of $[\mathsf{s}]\psi^*$ is, we can find an equivalent formula $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$. As this was the only step we needed to prove, we have thereby shown that for every $\varphi \in \mathcal{L}^{\mathsf{AMLQ}_0}$, there is some $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$ such that $\varphi \equiv \varphi^*$.

We can now generalize this claim to all formulas of $\mathcal{L}^{\mathsf{AMLQ}}$. Recall that by definition, all formulas of $\mathcal{L}^{\mathsf{AMLQ}}$ are in $\mathcal{L}^{\mathsf{AMLQ}_i}$ for some $i \geq 0$. This means we can perform an induction on i.

- **Base case.** We have already shown that the claim holds for all $\varphi \in \mathcal{L}^{\mathsf{AMLQ}_0}$.
- **Inductive step.** By the induction hypothesis, for any $\varphi \in \mathcal{L}^{\mathsf{AMLQ}_i}$, there is some $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$ such that $\varphi \equiv \varphi^*$. We need to show that this is also the case for any $\varphi \in \mathcal{L}^{\mathsf{AMLQ}_{(i+1)}}$.

  The proof for the base case can be repeated. However, this time when we want to use $\mathsf{pre}(\mathsf{x})$ or $\mathsf{cont}(\mathsf{x})$ of some action x in our reduction, it might be the case that these are not formulas of IEL. But by Definition 3.5.1 they are formulas of $\mathcal{L}^{\mathsf{AMLQ}_i}$, which means that by the induction hypothesis we can obtain an equivalent formula from IEL that we can use instead. □

3.9 Axiomatizing AMLQ

In this section we use the results from the previous section to provide two complete axiomatizations for AMLQ. Both provide a different set of inference rules to reduce every formula of AMLQ to one of IEL. They are analogous to two existing axiomatizations for IDEL (Ciardelli, 2016, chapter 8) which in turn correspond to two axiomatizations for PAL (see Wang & Cao (2013)).

3.9.1 Completeness via replacement of equivalents

The first proof system we describe consists of all the inference rules for IEL (Ciardelli, 2014) and the rules in Figure 3.11. The rules in this figure are reduction rules that correspond with the equivalences we proved in Section 3.8, together with a rule of replacement of equivalents. The relation of derivability in this system is denoted by $\vdash_{\mathsf{AMLQ}^{\mathsf{RE}}}$ and the relation of inter-derivability by $\dashv\vdash_{\mathsf{AMLQ}^{\mathsf{RE}}}$.

To prove that this proof system is complete, we need to show that every formula of $\mathcal{L}^{\mathsf{AMLQ}}$ is inter-derivable with an equivalent formula in $\mathcal{L}^{\mathsf{IEL}}$. As our inference rules for dynamic modalities correspond with the reduction equivalences shown in Section 3.8, we can follow

!Atom	!∧	!K
$\dfrac{[\mathsf{x}]p}{\mathsf{pre}(\mathsf{x}) \to p}$	$\dfrac{[\mathsf{s}](\varphi \wedge \psi)}{[\mathsf{s}]\varphi \wedge [\mathsf{s}]\psi}$	$\dfrac{[\mathsf{x}]K_a\varphi}{\mathsf{pre}(\mathsf{x}) \to K_a[\delta_a(\mathsf{x})]\varphi}$
!⊥	!→	!E
$\dfrac{[\mathsf{x}]\bot}{\neg\mathsf{pre}(\mathsf{x})}$	$\dfrac{[\mathsf{x}](\varphi \to \psi)}{[\mathsf{x}]\varphi \to [\mathsf{x}]\psi}$	$\dfrac{[\mathsf{x}]E_a\varphi}{\mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{y}\sim_a\mathsf{x}} E_a(\mathsf{cont}(\mathsf{y}) \to [\mathsf{y}]\varphi)}$
AUD	!⩔	RE
$\dfrac{[\mathsf{s}]\alpha}{\bigwedge_{\mathsf{x}\in\mathsf{s}}[\mathsf{x}]\alpha}$	$\dfrac{[\mathsf{s}](\varphi \mathbin{\backslash\!\!/} \psi)}{[\mathsf{s}]\varphi \mathbin{\backslash\!\!/} [\mathsf{s}]\psi}$	$\dfrac{\varphi \leftrightarrow \psi}{\chi[\varphi/p] \leftrightarrow \chi[\psi/p]}$

Figure 3.11: The inference rules for dynamic modalities in AMLQ. The double lines indicate that the inference is allowed in both directions. The rule AUD (action uncertainty distribution) can only be applied to declaratives α.

the same strategy as in the proof for Theorem 3.8.1. Since this proof relies on the equivalence between formulas and their normal form, we first need to show that every formula is also inter-derivable with its normal form.

Lemma 3.9.1. **Provability of normal form in $\vdash_{\mathsf{AMLQ^{RE}}}$**
For any $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$, $\varphi \dashv\vdash_{\mathsf{AMLQ^{RE}}} \mathbin{\backslash\!\!/}\mathcal{R}(\varphi)$.

Proof: By induction on the complexity of φ. We can repeat the base case and the inductive step for ⩔, ∧ and → from Ciardelli (2016, p. 86). The steps for the modalities K_a and E_a are trivial: by definition, for all modal formulas α, $\mathcal{R}(\alpha) = \alpha$. We only need to add the inductive step for dynamic modalities.

Suppose φ is $[\mathsf{s}]\psi$. By the induction hypothesis, we have $\psi \dashv\vdash_{\mathsf{AMLQ^{RE}}} \mathbin{\backslash\!\!/}\mathcal{R}(\psi)$. We can use RE to obtain:

$$[\mathsf{s}]\psi \dashv\vdash_{\mathsf{AMLQ^{RE}}} [\mathsf{s}]\mathbin{\backslash\!\!/}\mathcal{R}(\psi)$$

Also, using !⩔ in both directions we can get:

$$[\mathsf{s}]\mathbin{\backslash\!\!/}\mathcal{R}(\psi) \dashv\vdash_{\mathsf{AMLQ^{RE}}} \mathbin{\backslash\!\!/}_{\alpha\in\mathcal{R}(\psi)}[\mathsf{s}]\alpha$$

By the definition of resolutions, we have the following equivalence:

$$\mathbin{\backslash\!\!/}_{\alpha\in\mathcal{R}(\psi)}[\mathsf{s}]\alpha \equiv \mathbin{\backslash\!\!/}\mathcal{R}([\mathsf{s}]\psi)$$

This means we can combine the two inter-derivabilities to:

$$[\mathsf{s}]\psi \dashv\vdash_{\mathsf{AMLQ^{RE}}} \mathbin{\backslash\!\!/}\mathcal{R}([\mathsf{s}]\psi)$$

This concludes the inductive step for the dynamic modality, which was all we needed to show that $\varphi \dashv\vdash_{\mathsf{AMLQ^{RE}}} \mathbin{\backslash\!\!/}\mathcal{R}(\varphi)$. □

Now that we have provability of normal form in this system, we can move on to the actual proof of the theorem.

THEOREM 3.9.1. **Every formula of AMLQ is inter-derivable with a formula of IEL**
For any $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$, there is some $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$ such that $\varphi \dashv\vdash_{\mathsf{AMLQ}^{\mathsf{RE}}} \varphi^*$.

Proof: The proof for this claim is analogous to the proof for Theorem 3.8.1, but using the relation of inter-derivability in $\mathsf{AMLQ}^{\mathsf{RE}}$ instead of logical equivalence. Instead of the reduction equivalences, we use the corresponding deduction rules from Figure 3.11. We use Lemma 3.9.1 instead of Proposition 3.4.5. □

First, notice that we have conservativity of entailment in AMLQ:

PROPOSITION 3.9.1. **Entailment in AMLQ is conservative over IEL**
For all $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{IEL}}$, $\Phi \models_{\mathsf{AMLQ}} \psi \iff \Phi \models_{\mathsf{IEL}} \psi$.

Proof: This is immediate from the fact that AMLQ has standard support conditions for all the connectives that are in the syntax of $\mathcal{L}^{\mathsf{IEL}}$. □

This means that we can use the entailment relation $\models$ for entailment in IEL and AMLQ interchangeably. The proof for the following theorem makes use of this fact.

THEOREM 3.9.2. **$\mathsf{AMLQ}^{\mathsf{RE}}$ is sound and complete**
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{AMLQ}}, \Phi \models \psi \iff \Phi \vdash_{\mathsf{AMLQ}^{\mathsf{RE}}} \psi$.

Proof: (⇒) Suppose $\Phi \models \psi$. Then by Theorem 3.8.1, for every $\varphi \in \Phi$ there is some $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$ such that $\varphi^* \equiv \varphi$ and some $\psi^* \in \mathcal{L}^{\mathsf{IEL}}$ such that $\psi^* \equiv \psi$. Let Φ^* be a set containing for each $\varphi \in \Phi$ a formula φ^* equivalent to φ. Then we have $\Phi^* \models \psi^*$. We obtain $\Phi^* \vdash_{\mathsf{AMLQ}^{\mathsf{RE}}} \psi^*$ by the completeness of the proof system for IEL. The previous theorem guarantees that $\Phi \vdash_{\mathsf{AMLQ}^{\mathsf{RE}}} \varphi^*$ for all $\varphi^* \in \Phi^*$ and $\psi^* \vdash_{\mathsf{AMLQ}^{\mathsf{RE}}} \psi$. This means that $\Phi \vdash_{\mathsf{AMLQ}^{\mathsf{RE}}} \psi$.

(⇐) Suppose $\Phi \vdash_{\mathsf{AMLQ}^{\mathsf{RE}}} \psi$. Then there is a proof from Φ to ψ using the inference rules of $\vdash_{\mathsf{AMLQ}^{\mathsf{RE}}}$, which consists of the inference rules in Figure 3.11 and the ones of IEL. The former are sound by Propositions 3.6.2 and 3.8.1 - 3.8.9 and the latter by Proposition 7.3.11 of Ciardelli (2016, p. 283). Therefore, since the proof is based on only sound inference rules, we have $\Phi \models \psi$. □

3.9.2 Completeness via monotonicity

An alternative way of providing a complete proof system for AMLQ is one that makes use of the monotonicity of dynamic modalities.

PROPOSITION 3.9.2. **Dynamic modalities are monotonic**
If $\varphi \models \psi$, then $[\mathsf{s}]\varphi \models [\mathsf{s}]\psi$.

Proof: Assume $\varphi \models \psi$. Take any information state s such that $s \models [\mathsf{s}]\varphi$. As $s[\mathsf{s}] \models \varphi$, $s[\mathsf{s}] \models \psi$. Therefore $s \models [\mathsf{s}]\psi$. □

To capture monotonicity in our proof system, we introduce the corresponding inference rule !Mon (Figure 3.12).

The proof system denoted by $\vdash_{\mathsf{AMLQ}^{!\mathsf{Mon}}}$ consists of the same inference rules as $\vdash_{\mathsf{AMLQ}^{\mathsf{RE}}}$, but we replace RE with !Mon. The soundness of this system is guaranteed by Proposition 3.9.2. As for the previous completeness proof, we need to show that a formula and its normal form are inter-derivable in this system. Since we do not have RE anymore, we have to replace one step of the proof.

!Mon

$$\dfrac{\begin{matrix}[\varphi] \\ \vdots \\ \psi\end{matrix} \qquad [\mathsf{s}]\varphi}{[\mathsf{s}]\psi}$$

Figure 3.12: The inference rule !Mon, where the proof of ψ has φ as its *only* undischarged assumption.

LEMMA 3.9.2. **Provability of normal form in $\vdash_{\mathsf{AMLQ}^{!\mathsf{Mon}}}$**
For any $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$, $\varphi \dashv\vdash_{\mathsf{AMLQ}^{!\mathsf{Mon}}} \bigvee\!\!\!\!\bigvee \mathcal{R}(\varphi)$.

Proof: Given the proof we already have for Lemma 3.9.1, we only need to show that we can do the inductive step for the dynamic modality we have performed there, using the inference rule !Mon instead of RE.

We can get from the induction hypothesis, $\psi \dashv\vdash_{\mathsf{AMLQ}^{!\mathsf{Mon}}} \bigvee\!\!\!\!\bigvee \mathcal{R}(\psi)$, to $[\mathsf{s}]\psi \dashv\vdash_{\mathsf{AMLQ}^{!\mathsf{Mon}}} [\mathsf{s}]\bigvee\!\!\!\!\bigvee \mathcal{R}(\psi)$ by applying !Mon in both directions. The rest of the proof carries over. □

We can now prove that the system is also complete.

THEOREM 3.9.3. **$\mathsf{AMLQ}^{!\mathsf{Mon}}$ is complete**
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{AMLQ}}, \Phi \models \psi$ implies $\Phi \vdash_{\mathsf{AMLQ}^{!\mathsf{Mon}}} \psi$.

Proof: We start by defining a non-standard complexity measure $c : \mathcal{L}^{\mathsf{AMLQ}} \to \mathbb{N}$ as follows:

- $c(p) = c(\bot) = 1$
- $c(\bigwedge \Gamma) = c(\bigvee\!\!\!\!\bigvee \Gamma) = |\Gamma| - 1 + \mathsf{max}(\{c(\varphi) \mid \varphi \in \Gamma\})$
- $c(\varphi \to \psi) = 1 + \mathsf{max}(c(\varphi), c(\psi))$
- $c(K_a\varphi) = c(E_a\varphi) = 1 + c(\varphi)$
- $c(\mathsf{M}, \mathsf{s}) = \mathsf{max}(\{\mathsf{max}(c(\mathsf{cont}(\mathsf{x})), c(\mathsf{pre}(\mathsf{x}))) \mid \mathsf{x} \in \mathsf{S}\})$ where S is the domain of M
- $c([\mathsf{M}, \mathsf{s}]\varphi) = (|\mathsf{s}| + |\mathsf{S}| + c(\mathsf{M}, \mathsf{s})) \cdot c(\varphi)$

This complexity measure satisfies the following conditions:

- $c(\varphi) > c(\psi)$ if ψ is a proper subformula of φ
- $c([\mathsf{s}](\varphi \circ \psi)) > c([\mathsf{s}]\varphi \circ [\mathsf{s}]\psi)$ for $\circ \in \{\wedge, \vee\!\!\!\vee\}$
- $c([\mathsf{s}](\varphi \to \psi)) > c(\bigwedge_{\mathsf{x} \in \mathsf{s}}([\mathsf{x}]\varphi \to [\mathsf{x}]\psi))$
- $c([\mathsf{s}]K_a\varphi) > c(\bigwedge_{\mathsf{x} \in \mathsf{s}}(\mathsf{pre}(\mathsf{x}) \to K_a[\delta_a(\mathsf{x})]\varphi))$
- $c([\mathsf{s}]E_a\varphi) > c(\bigwedge_{\mathsf{x} \in \mathsf{s}}(\mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{y} \sim_a \mathsf{x}} E_a(\mathsf{cont}(\mathsf{y}) \to [\mathsf{y}]\varphi)))$

Then we can define a translation $(\cdot)^* : \mathcal{L}^{\mathsf{AMLQ}} \to \mathcal{L}^{\mathsf{IEL}}$. We do this in several steps for fragments of $\mathcal{L}^{\mathsf{AMLQ}}$.

Let us first define a way of identifying the relevant fragments of the language. For every $i \geq 0$ and $n \geq 0$, let us denote by $\mathcal{L}^i_n$ the fragment of $\mathcal{L}^{\mathsf{AMLQ}_i}$ such that all formulas of this fragment have a modal depth of n. Furthermore, let us denote by $\mathcal{L}^i_{n!}$ the declarative fragment of $\mathcal{L}^i_n$, defined as usual.

We first define a translation to IEL for a very small fragment of our language, namely the level 0 formulas that are declarative and that do not contain any epistemic modalities. The translation $f^0_{0!} : \mathcal{L}^0_{0!} \to \mathcal{L}^{\mathsf{IEL}}$ is defined as follows, by recursion on c:

- $f^0_{0!}(p) = p$
- $f^0_{0!}(\bot) = \bot$
- $f^0_{0!}(\alpha \circ \beta) = f^0_{0!}(\alpha) \circ f^0_{0!}(\beta)$ for $\circ \in \{\wedge, \to\}$
- $f^0_{0!}([\mathsf{s}]p) = \bigwedge_{\mathsf{x} \in \mathsf{s}}(\mathsf{pre}(\mathsf{x}) \to p)$
- $f^0_{0!}([\mathsf{s}]\bot) = \bigwedge_{\mathsf{x} \in \mathsf{s}} \neg\mathsf{pre}(\mathsf{x})$
- $f^0_{0!}([\mathsf{s}](\alpha \wedge \beta)) = f^0_{0!}([\mathsf{s}]\alpha) \wedge f^0_{0!}([\mathsf{s}]\beta)$
- $f^0_{0!}([\mathsf{s}](\alpha \to \beta)) = \bigwedge_{\mathsf{x} \in \mathsf{s}}(f^0_{0!}([\mathsf{x}]\alpha) \to f^0_{0!}([\mathsf{x}]\beta))$
- $f^0_{0!}([\mathsf{s}][\mathsf{t}]\alpha) = f^0_{0!}([\mathsf{s}]f^0_{0!}([\mathsf{t}]\alpha))$

Based on this translation of the declarative fragment of $\mathcal{L}^0_0$, we can define a translation $f^0_0 : \mathcal{L}^0_0 \to \mathcal{L}^{\mathsf{IEL}}$ for all level 0 formulas that have no epistemic modalities. We do this as follows:

$$f^0_0(\varphi) = \bigvee_{\alpha \in \mathcal{R}(\varphi)} f^0_{0!}(\alpha)$$

Next, we move on to declaratives that do contain epistemic modalities. We define for all $n > 0$, the following translation $f^0_{n!} : \mathcal{L}^0_{n!} \to \mathcal{L}^{\mathsf{IEL}}$, by recursion on c:

- $f^0_{n!}(p) = p$
- $f^0_{n!}(\bot) = \bot$
- $f^0_{n!}(\alpha \circ \beta) = f^0_{n!}(\alpha) \circ f^0_{n!}(\beta)$ for $\circ \in \{\wedge, \to\}$
- $f^0_{n!}(\blacksquare\varphi) = \blacksquare f^0_{(n-1)}(\varphi)$ for $\blacksquare \in \{K_a, E_a\}$
- $f^0_{n!}([\mathsf{s}]p) = \bigwedge_{\mathsf{x} \in \mathsf{s}}(\mathsf{pre}(\mathsf{x}) \to p)$
- $f^0_{n!}([\mathsf{s}]\bot) = \bigwedge_{\mathsf{x} \in \mathsf{s}} \neg\mathsf{pre}(\mathsf{x})$
- $f^0_{n!}([\mathsf{s}](\alpha \wedge \beta)) = f^0_{n!}([\mathsf{s}]\alpha) \wedge f^0_{n!}([\mathsf{s}]\beta)$
- $f^0_{n!}([\mathsf{s}](\alpha \to \beta)) = \bigwedge_{\mathsf{x} \in \mathsf{s}}(f^0_{n!}([\mathsf{x}]\alpha) \to f^0_{n!}([\mathsf{x}]\beta))$
- $f^0_{n!}([\mathsf{s}]K_a\varphi) = \bigwedge_{\mathsf{x} \in \mathsf{s}}(\mathsf{pre}(\mathsf{x}) \to K_a f^0_{(n-1)}([\delta_a(\mathsf{x})]\varphi))$
- $f^0_{n!}([\mathsf{s}]E_a\varphi) = \bigwedge_{\mathsf{x} \in \mathsf{s}}(\mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{y} \sim_a \mathsf{x}} E_a(\mathsf{cont}(\mathsf{y}) \to f^0_{(n-1)}([\mathsf{y}]\varphi)))$
- $f^0_{n!}([\mathsf{s}][\mathsf{t}]\alpha) = f^0_{n!}([\mathsf{s}]f^0_{n!}([\mathsf{t}]\alpha))$

Now that we have a translation for the declarative fragment of $\mathcal{L}^0_n$ for every n, we can define a translation $f^0_n : \mathcal{L}^0_n \to \mathcal{L}^{\mathsf{IEL}}$ as follows:

$$f^0_n(\varphi) = \bigvee_{\alpha \in \mathcal{R}(\varphi)} f^0_{n!}(\alpha)$$

We now have several translation functions f^0_n that together translate all level 0 formulas with any modal depth n. We can combine these translations to one single translation by defining $f^0 : \mathcal{L}^{\mathsf{AMLQ}_0} \to \mathcal{L}^{\mathsf{IEL}}$ in the following way:

$$f^0(\varphi) = f^0_{\mathsf{md}(\varphi)}$$

Thereby, we have a translation for all level 0 formulas of $\mathcal{L}^{\mathsf{AMLQ}}$. We generalize this to the full language by defining for all $i > 0$, the translations $f^i_{0!}$, f^i_0, $f^i_{n!}$ and f^i_n in the same way as the first four translations, with the only difference that we replace all occurrences of $\mathsf{pre}(\mathsf{x})$ and $\mathsf{cont}(\mathsf{y})$ with $f^{(i-1)}(\mathsf{pre}(\mathsf{x}))$ and $f^{(i-1)}(\mathsf{cont}(\mathsf{y}))$ respectively. Then we define the translation $f^i : \mathcal{L}^{\mathsf{AMLQ}_i} \to \mathcal{L}^{\mathsf{IEL}}$ as $f^i(\varphi) = f^i_{\mathsf{md}(\varphi)}$

Finally, we can define the translation for the full language, $(\cdot)^* : \mathcal{L}^{\mathsf{AMLQ}} \to \mathcal{L}^{\mathsf{IEL}}$. We simply let $\varphi^* = f^i(\varphi)$ where i is the smallest number such that $\varphi \in \mathcal{L}^{\mathsf{AMLQ}_i}$.

Since our proof system includes a complete proof system for IEL, we only need to show that $\varphi \dashv\vdash_{\mathsf{AMLQ}^{!\mathsf{Mon}}} \varphi^*$, which we can do by several inductions, following the inductive structure by which we have defined the translation. Notice that we need provability of normal form and the inference rule !Mon along the way. □

We have thus provided two complete axiomatizations of $\mathcal{L}^{\mathsf{AMLQ}}$: one using replacement of equivalents and one using monotonicity of dynamic modalities. Interestingly, for both PAL (Van Ditmarsch et al., 2007, chapter 7) and IDEL (Ciardelli, 2016, chapter 8) there exists a third axiomatization strategy in which a composition rule ($[\mathsf{s}][\mathsf{t}]\varphi \dashv\vdash [\mathsf{s};\mathsf{t}]\varphi$) is added instead of RE or !Mon. However, while we do have composition of dynamic modalities in our system, completeness via composition is blocked. The reason for this is that the workaround we use because of the lack of a reduction equivalence for formulas of the form $[\mathsf{s}](\varphi \to \psi)$ depends on the provability of normal form, which we do not get when we add a composition rule instead of RE and !Mon. An investigation of alternative completeness proofs could be an interesting direction for further research.

3.10 Comparison

In this section we will compare AMLQ to the systems we discussed in Chapter 2. We will show that AMLQ can be seen as a conservative extension of both IDEL and AML.

3.10.1 Inquisitive Dynamic Epistemic Logic

As we have already seen in Chapter 2, IDEL can encode public utterances of questions as well as statements, and in the current chapter it has been our goal to extend this to more private utterances. In this section, we will show that AMLQ is conservative over IDEL: that is, the notion of public utterance in AMLQ is no different than it is in IDEL. Let us first see how we would encode a public utterance in an AMLQ action model.

Definition 3.10.1. **Public utterance in AMLQ**
For any formula $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$, the public utterance of φ is modelled as an update with the action model M^φ, defined as follows:

$\mathsf{M}^\varphi = \langle \{\mathsf{pub}^\varphi\}, \{\sim_a \mid a \in \mathcal{A}\}, \mathsf{cont}\rangle$, where $\mathsf{cont}(\mathsf{pub}^\varphi) = \varphi$ and for all $a \in \mathcal{A}$: $\mathsf{pub}^\varphi \sim_a \mathsf{pub}^\varphi$.

We can show that updating with this action model is essentially the same as performing an update in IDEL. The resulting updated models are not the same, but this is only because the individual worlds of the models do not have the same name. In other words, we can show that they are isomorphic.

Proposition 3.10.1. **Isomorphism between updated models**
For any inquisitive epistemic model M and for any $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$ and $\psi \in \mathcal{L}^{\mathsf{AMLQ}}$: if $\varphi \equiv \psi$, then $M' = M \otimes \mathsf{M}^\psi$ is isomorphic to M^φ.

Proof: Take any $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$ and $\psi \in \mathcal{L}^{\mathsf{AMLQ}}$ such that $\varphi \equiv \psi$. We define the following models:

- $M = \langle W, \{\Sigma_a \mid a \in \mathcal{A}\}, V\rangle$ (our original model)
- $M' = M \otimes \mathsf{M}^{\psi} = \langle W', \{\Sigma'_a \mid a \in \mathcal{A}\}, V'\rangle$
- $M^{\varphi} = \langle W^{\varphi}, \{\Sigma^{\varphi}_a \mid a \in \mathcal{A}\}, V^{\varphi}\rangle$

We start by showing that $w \in W^{\varphi}$ just in case $\langle w, \mathsf{pub}^{\psi}\rangle \in W'$, by unpacking the definitions of W^{φ} and W':

$$\begin{aligned} w \in W^{\varphi} &\iff w \in W \cap |\varphi|_M \\ &\iff w \in W \text{ and } w \in |\varphi|_M \\ &\iff w \in W \text{ and } M, w \models \varphi \\ &\iff w \in W \text{ and } M, w \models \mathsf{pre}(\mathsf{pub}^{\psi}) \\ &\iff \langle w, \mathsf{pub}^{\psi}\rangle \in W' \end{aligned}$$

Let $f(w) = \langle w, \mathsf{pub}^{\psi}\rangle$. Then f is a bijection between W^{φ} and W'. Now let us show that it is indeed an isomorphism. For this, we need to show two things:

(i) The mapping preserves the structure of the state maps. That is, if we let $f(s)$ be $\{f(w) \mid w \in s\}$, then we have:

$$s \in \Sigma^{\varphi}_a(w) \iff f(s) \in \Sigma'_a(f(w))$$

(ii) The mapping preserves the valuation: $V^{\varphi}(w) = V'(f(w))$.

Given the definition of valuation, the latter is trivial, so we only show (i). Take any world $w \in W^{\varphi}$ and any state $s \subseteq W^{\varphi}$. Let $s' = f(s)$. Then we need to show that $s \in \Sigma^{\varphi}_a(w) \iff s' \in \Sigma'_a(\langle w, \mathsf{pub}^{\psi}\rangle)$. Then the update procedure of IDEL gives us conditions (i) and (iv) of the update procedure of AMLQ, while conditions (ii) and (iii) are trivial given the definition of the action model M^{ψ}. We have:

$$\begin{aligned} s \in \Sigma^{\varphi}_a(w) &\iff s \in \Sigma_a(w) \cap [\varphi]_M \\ &\iff s \in \Sigma_a(w) \text{ and } s \in [\varphi]_M \\ &\iff \pi_1(f(s)) \in \Sigma_a(w) \text{ and } \pi_1(f(s)) \in [\varphi]_M \\ &\iff \pi_1(s') \in \Sigma_a(w) \text{ and } \pi_1(s') \models \mathsf{cont}(\mathsf{pub}^{\psi}) \\ &\iff s' \in \Sigma'_a(\langle w, \mathsf{pub}^{\psi}\rangle) \end{aligned}$$

□

PROPOSITION 3.10.2. **$s[\mathsf{pub}^{\psi}]$ is the image of $s \cap |\varphi|_M$ under isomorphism f**
For every information state s: $w \in s \cap |\varphi|_M \iff \langle w, \mathsf{pub}^{\psi}\rangle \in s[\mathsf{pub}^{\psi}]$

Proof: We only need to unpack the definition of updated states:

$$\begin{aligned} w \in s \cap |\varphi|_M &\iff w \in s \text{ and } w \in |\varphi|_M \\ &\iff w \in s \text{ and } M, w \models \mathsf{pre}(\mathsf{pub}^{\psi}) \\ &\iff \langle w, \mathsf{pub}^{\psi}\rangle \in s[\mathsf{pub}^{\psi}] \end{aligned}$$

□

DEFINITION 3.10.2. **Translation of IDEL formulas to AMLQ formulas**
For every $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$, its translation $\varphi^* \in \mathcal{L}^{\mathsf{AMLQ}}$ is defined recursively as follows:

- $p^* = p$

- $\bot^* = \bot$
- $(\varphi \circ \psi)^* = \varphi^* \circ \psi^*$ for $\circ \in \{\wedge, \rightarrow, \mathbin{\rotatebox[origin=c]{90}{$\geqslant$}}\}$
- $(\blacksquare\varphi)^* = \blacksquare\varphi^*$ for $\blacksquare \in \{K_a, E_a\}$
- $([\varphi]\psi)^* = [\mathsf{M}^{\varphi^*}, \mathsf{pub}^{\varphi^*}]\psi^*$

Proposition 3.10.3. **Every IDEL formula is equivalent to its AMLQ translation**
For every $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$ and its translation $\varphi^* \in \mathcal{L}^{\mathsf{AMLQ}}$, $\varphi \equiv \varphi^*$.

Proof: By induction on the complexity of φ. The only step which is not immediate is the step for the dynamic modality.

Suppose φ is $[\psi]\chi$. By the induction hypothesis, we have some $\psi^*, \chi^* \in \mathcal{L}^{\mathsf{AMLQ}}$ equivalent to ψ and χ respectively. Take any state s in any inquisitive epistemic model M. Let $M' = M \otimes \mathsf{M}^{\psi^*}$. Then by Proposition 3.10.1, M' and M^{ψ} are isomorphic. Together with Proposition 3.10.2 and the support conditions of both dynamic modalities, we obtain:

$$\begin{aligned} M, s \models [\psi]\chi &\iff M^{\psi}, s \cap |\psi|_M \models \chi \\ &\iff M', s[\mathsf{M}^{\psi^*}, \mathsf{pub}^{\psi^*}] \models \chi \\ &\iff M', s[\mathsf{M}^{\psi^*}, \mathsf{pub}^{\psi^*}] \models \chi^* \\ &\iff M, s \models [\mathsf{M}^{\psi^*}, \mathsf{pub}^{\psi^*}]\chi^* \\ &\iff M, s \models ([\psi]\chi)^* \end{aligned}$$

□

We have thereby shown that AMLQ can express everything that IDEL can and that the interpretation of dynamic modalities of public utterance coincides. We may even define $[\varphi]\psi$ in AMLQ as an abbreviation for $[\mathsf{M}^{\varphi}, \mathsf{pub}^{\varphi}]\psi$. In that case, we can view AMLQ as a conservative extension of IDEL. It follows from Proposition 8.2.6 of Ciardelli (2016) that the result of a public utterance of a statement in AMLQ is the same as the result of a public announcement in PAL.

3.10.2 Action Model Logic

In this section we will show that AMLQ is a conservative extension of AML. We start by showing that, in terms of updating knowledge, our procedure coincides with the standard one. First, we define a way in which we associate a standard Kripke model with any inquisitive epistemic model.

Definition 3.10.3. **Kripke model determined by an inquisitive epistemic model**
For any inquisitive epistemic model $M = \langle W, \{\Sigma_a \mid a \in \mathcal{A}\}, V\rangle$, M^K is the Kripke model determined by M, defined by:

$M^K = \langle W, \{\sim_a \mid a \in \mathcal{A}\}, V\rangle$ where $\sim_a = \{\langle w, w'\rangle \mid w' \in \sigma_a(w)\}$

Any Kripke model M^K is a standard epistemic model that encodes the same knowledge as M.

Definition 3.10.4. **AMLQ variant of standard action model**
Let $\mathsf{M} = \langle \mathsf{S}, \{\sim_a \mid a \in \mathcal{A}\}, \mathsf{pre}\rangle$ be a standard action model. Then its AMLQ variant $\widehat{\mathsf{M}} = \langle \mathsf{S}, \{\sim_a \mid a \in \mathcal{A}\}, \mathsf{cont}\rangle$ is the AMLQ action model such that $\mathsf{cont} = \mathsf{pre}$.

As all preconditions in standard action models are formulas of AML, they are also formulas of AMLQ. Furthermore, as they cannot contain an inquisitive disjunction, they are declaratives and therefore truth-conditional. This makes cont and pre interchangeable.

Proposition 3.10.4. **The AMLQ update procedure is standard in terms of knowledge**
For any inquisitive epistemic model M and any standard action model M: $(M \otimes \widehat{\mathsf{M}})^K = M^K \otimes \mathsf{M}$.

Proof: Take any inquisitive epistemic model $M = \langle W, \{\Sigma_a \mid a \in \mathcal{A}\}, V\rangle$ and standard action model M. We define the following two standard updated models:

- $M' = (M \otimes \widehat{\mathsf{M}})^K = \langle W', \{\sim'_a \mid a \in \mathcal{A}\}, V'\rangle$
- $M'' = M^K \otimes \mathsf{M} = \langle W'', \{\sim''_a \mid a \in \mathcal{A}\}, V''\rangle$

From the definitions of both update procedures, it is immediate that $W' = W''$ and $V' = V''$. It remains to show that $\sim'_a = \sim''_a$.

($\Rightarrow$) Assume $\langle w, \mathsf{x}\rangle \sim'_a \langle w', \mathsf{x}'\rangle$.

Then by Definition 3.10.3, $\langle w', \mathsf{x}'\rangle \in \sigma'_a(\langle w, \mathsf{x}\rangle)$. This means that $\{\langle w', \mathsf{x}'\rangle\} \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$. By condition (i) of Definition 3.2.4, $\{w'\} \in \Sigma_a(w)$ of M. This means by the definition of M^K that $w \sim_a w'$. By condition (ii), $\mathsf{x} \sim_a \mathsf{x}'$. Then by the update procedure of standard action models, $\langle w, \mathsf{x}\rangle \sim''_a \langle w', \mathsf{x}'\rangle$.

($\Leftarrow$) Assume $\langle w, \mathsf{x}\rangle \sim''_a \langle w', \mathsf{x}'\rangle$.

Then by the update procedure of standard action models, $w \sim_a w'$ and $\mathsf{x} \sim_a \mathsf{x}'$.

We want to show that $\{\langle w', \mathsf{x}'\rangle\} \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$, which means we need to check that this state satisfies conditions (i)-(iv) of Definition 3.2.4.

As $w \sim_a w'$, by the definition of M^K we have $\{w'\} \in \Sigma_a(w)$, which means condition (i) is satisfied. By $\mathsf{x} \sim_a \mathsf{x}'$ we already have condition (ii). Condition (iii) is trivial as $\{\langle w', \mathsf{x}'\rangle\}$ is a singleton. We have $M, w' \models \mathsf{pre}(\mathsf{x}')$ because otherwise the pair $\langle w', \mathsf{x}'\rangle$ would not have been in the domain of the updated model. By Proposition 3.2.1, $M, w' \models \mathsf{cont}(\mathsf{x}')$, which means condition (iv) is satisfied too.

Because $\{\langle w', \mathsf{x}'\rangle\} \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$, by Definition 3.10.3 we have $\langle w, \mathsf{x}\rangle \sim'_a \langle w', \mathsf{x}'\rangle$. □

We have shown that the update procedure of AMLQ action models is standard with respect to knowledge. Furthermore, we can check that all connectives of the language of AML have the same truth conditions as those of AMLQ and that all $\alpha \in$ AML are truth-conditional. It follows that the semantics of these formulas is standard in our setting. Therefore, for every $\Gamma \cup \{\alpha\} \in \mathcal{L}^{\mathsf{AML}}$: $\Gamma \models_{\mathsf{AML}} \alpha \iff \Gamma \models_{\mathsf{AMLQ}} \alpha$. This makes AMLQ a conservative extension of AML.

3.11 Conclusion

In this chapter, we have defined AMLQ action models and a corresponding logic, which is a conservative extension of both IEL and AML. Before we continue, we briefly summarize the most important findings of this chapter.

We have generalized the notion of action preconditions, familiar from AML, to incorporate questions. From our perspective, a precondition function is no longer the primary source for the contents of actions. Instead, this role is passed on to the content function cont. While pre still works in the same way as it does in AML, cont only has an effect if the content of one of the actions is a question.

The examples in Section 3.3 show that AMLQ is a natural generalization of IDEL: it encodes semi-private utterances of statements and questions, which in turn give us updated models that intuitively seem to be correct.

By allowing for modalities of sets of actions as well as single actions, we have found a new way of interpreting complex dynamic modalities in a support semantics. Namely, the interpretation we already gave to sets of worlds as information states is also available for sets of actions. This means that we understand $[\mathsf{s}]\varphi$ as supported just in case our information, enhanced with the information that one of the actions in s is executed, supports φ. This perspective on complex modalities differs from the perspective given in AML as non-deterministic actions.

We have shown that any formula of AMLQ can be translated to an equivalent formula of the static logic IEL. Since we have a sound and complete axiomatization of IEL, this allows for a sound and complete axiomatization of AMLQ, using reduction axioms and either a rule for replacement of equivalents or a monotonicity rule.

Chapter 4

Inquisitive Action Model Logic

4.1 Introduction

In the previous chapter, we have seen that we can add action models to inquisitive epistemic logic by making it possible to have questions as the content of actions. We did not change the structure of action models in any way. That is, action models in AMLQ encode only knowledge the agents have about which action is the actual one, not their issues about what action is taking place.

In this chapter we will investigate a different approach. Instead of adding the possibility to make the content of an action a question, we now change the structure of action models. This will allow us to encode issues the agents have about which action is the actual one.

This chapter follows the structure of the previous one. That means we start with the relevant definitions and give some examples. We will discuss the same properties of the logic in Section 4.4. Then, we will extend the language to higher levels and define the notion of composition. In Sections 4.7 and 4.8, we provide a reduction and axiomatization of the logic. We end with two comparisons: first, we compare IAML to AMLQ. Then, we compare it to a similar system, namely ELQm, as proposed by Van Benthem & Minică (2012).

4.2 Definitions

4.2.1 Action model

First, let us consider the difference between a standard epistemic model, used in AML, and an inquisitive epistemic model used in IEL. As we have already seen, a standard epistemic model encodes just the knowledge agents have about the world, by having an equivalence relation for every agent, that encodes which worlds are indistinguishable to this agent. An inquisitive epistemic model encodes this knowledge too, but it also encodes the issues agents have, by having a state map $\Sigma_a(w)$ for each world w, which defines what information states an agent wants to reach.

We have seen in the previous chapter that action models are very similar to standard epistemic models: they use equivalence relations to encode which actions are indistinguishable to agents. We have also seen that this leaves open the question whether an agent considers it an issue which action is actually occurring.

In the previous chapter, we chose to always interpret uncertainty about epistemic actions as being an issue to the agents. In this chapter we will encode these issues explicitly, by changing the definition of action models to make them more like inquisitive epistemic models. We replace the equivalence relation with a state map for each action, which encodes not only which actions are indistinguishable from it, but also what information the agent desires to have about which action is the actual one.

This change also allows us to encode the action of asking a question in a different way. We have seen in the previous chapters that in inquisitive epistemic logic, a question has the effect of raising an issue: after the question, the agents want to know the answer. Take for example the question $?p$. Asking this question means raising the issue whether p or $\neg p$ is the case. Now instead of raising the issue by making the question the content of an action, we can also encode the issue directly in the action model itself, by adding two actions with contents p and $\neg p$ instead of one action with content $?p$.

Because this new structure allows us to encode questions in such a way, we do not have to allow questions as action content. This means we can restrict the content to declaratives for now. Like in the previous chapter, we start by defining level 0 action models, which have declarative formulas of IEL as their content. The general definition of action models will follow in Section 4.5.

Definition 4.2.1. **Inquisitive Action Models**
An IAML_0 action model is a triple $\mathsf{M} = \langle \mathsf{S}, \{\Delta_a \mid a \in \mathcal{A}\}, \mathsf{cont}\rangle$, where:

- S is a finite domain of action points;
- For each $a \in \mathcal{A}$, Δ_a is a function that maps an action point to a non-empty downward closed set of sets of action points;
- $\mathsf{cont} : \mathsf{S} \to \mathcal{L}_!^{\mathsf{IEL}}$ is a function that assigns a content $\mathsf{cont}(\mathsf{x}) \in \mathcal{L}_!^{\mathsf{IEL}}$ to each action point $\mathsf{x} \in \mathsf{S}$.

Recall that in our epistemic model, the information state $\sigma_a(w)$ is defined as the union of the inquisitive proposition $\Sigma_a(w)$. This is simply the set of worlds that are indistinguishable from w to agent a, which represents the information at w. Similarly, in inquisitive action models, we define $\delta_a(\mathsf{x}) := \bigcup \Delta_a(\mathsf{x})$ for each action x. This means that $\delta_a(\mathsf{x})$ is the set of actions indistinguishable from x to agent a.

As we are still dealing with epistemic logic, it is natural to require that Δ_a satisfies the same factivity and introspection conditions as Σ_a. This means that the actual action is always considered possible and that agents have full knowledge of their own knowledge and issues. Formally, this means that Δ_a satisfies the following conditions:

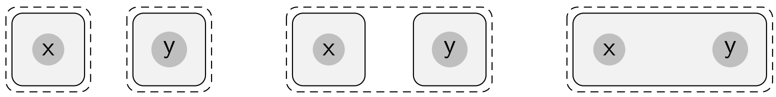
- *Factivity:* for all $\mathsf{x} \in \mathsf{S}$, $\mathsf{x} \in \delta_a(\mathsf{x})$
- *Introspection:* for all $\mathsf{x}, \mathsf{y} \in \mathsf{S}$, if $\mathsf{y} \in \delta_a(\mathsf{x})$ then $\Delta_a(\mathsf{y}) = \Delta_a(\mathsf{x})$

Note that the accessibility relation $\sim_a$ can still be retrieved, because it is completely determined by Δ_a:

$$\sim_a := \{\langle \mathsf{x}, \mathsf{x}' \rangle \mid \mathsf{x}' \in \delta_a(\mathsf{x})\}$$

Because we require Δ_a to satisfy factivity and introspection, it follows that $\sim_a$ is still an equivalence relation.

Example 4.2.1. **Inquisitive Action Model**
Consider the two actions x and y. If agent a knows which action is the actual one, $\Delta_a(\mathsf{x}) = \{\{\mathsf{x}\}\}^{\downarrow}$ and $\Delta_a(\mathsf{y}) = \{\{\mathsf{y}\}\}^{\downarrow}$. If agent b does not know which action is the actual one, but *wants* to know, $\Delta_b(\mathsf{x}) = \Delta_b(\mathsf{y}) = \{\{\mathsf{x}\}, \{\mathsf{y}\}\}^{\downarrow}$. If agent c does not know which action is the actual one, and does not care either, then $\Delta_c(\mathsf{x}) = \Delta_c(\mathsf{y}) = \{\{\mathsf{x}, \mathsf{y}\}\}^{\downarrow}$. The state maps are represented visually in Figure 4.1.



(a) State map of a in action model
(b) State map of b in action model
(c) State map of c in action model

Figure 4.1: Example 4.2.1

A consequence of the restriction of action contents to declaratives is that the notions of action content and precondition coincide in the current setting.

Proposition 4.2.1. **The precondition of an action in IAML is equivalent to its content**
For all actions x, $\mathsf{pre}(\mathsf{x}) \equiv \mathsf{cont}(\mathsf{x})$.

Proof: By Fact 2.3.4, every formula $\varphi \in \mathcal{L}^{\mathsf{IEL}}$ has the same truth conditions as its declarative variant $\varphi^!$. As $\mathsf{cont}(\mathsf{x}) \in \mathcal{L}^{\mathsf{IEL}}_!$, $\mathsf{cont}(\mathsf{x})$ is truth-conditional. As $\mathsf{cont}(\mathsf{x})$ and $\mathsf{cont}(\mathsf{x})^!$ are both truth-conditional and share the same truth conditions, by Definition 2.3.7 they also share the same support conditions. □

Although cont and pre are equivalent in IAML, we will, like in the rest of this thesis, still use cont when appropriate. The difference between the two concepts remains relevant, especially because at a later stage they will come apart again.

4.2.2 Update procedure

Now that we have refined the definition of action models, we need to do the same with the update procedure.

Just as in the previous system, the basic idea behind the update procedure stays the same, so there is no need to make any changes to the definitions of the domain and valuation of our updated model. What we do need to change is the definition of the state map in the updated model. Recall that in AMLQ, we formulated four conditions for an information state to be in $\Sigma'_a(\langle w, \mathsf{x}\rangle)$.

Condition (i) was in place to preserve the knowledge and issues from the original model. Because this has nothing to do with the structure of the action model, we can take over this condition in the current update procedure.

We added condition (ii) to preserve knowledge from the action model and condition (iii) to make sure that it was always an issue for the agents what the actual action was. The latter is not the case in the current setting: whether agents consider it an issue what the actual action is, is now encoded by their state maps in the action model.

Since both knowledge and issues about agents now depend on their state maps, it is easy to preserve this in the updated model using one single condition that takes care of both.

Recall that knowledge and issues about the actions are encoded in Δ in exactly the same way knowledge and issues about the worlds are encoded in Σ. Therefore, if we make the new condition (ii) completely analogous to condition (i), our update procedure preserves the knowledge and issues about the actions in the same way as well.

This means that a state s can be in $\Sigma'_a(\langle w, \mathsf{x}\rangle)$ only if $\pi_2(s) \in \Delta_a(\mathsf{x})$. Firstly, this guarantees that there is no $\langle v, \mathsf{y}\rangle \in \sigma'(\langle w, \mathsf{x}\rangle)$ such that $\mathsf{y} \notin \delta(\mathsf{x})$, so we preserve distinguishability of actions. Secondly, a state can only count as resolving the issues of the agent in the updated model if it resolves the issues the agent has about the action.

Finally, the reason we added condition (iv) in the update procedure of AMLQ was to make sure that action content could raise issues. Since we do not allow questions as action content in the current setting, we do not need such a condition now. This means we only need to impose two conditions for a state to be in a state map.

Definition 4.2.2. **IEL model updated with an Inquisitive Action Model**
Let M be an inquisitive epistemic model and M an IAML_0 action model. Then $M' = (M \otimes \mathsf{M})$ is the product update of M and M, defined as follows.

$M' = \langle W', \{\Sigma'_a \mid a \in \mathcal{A}\}, V'\rangle$, where:

- $W' = \{\langle w, \mathsf{x}\rangle \mid w \in W, \mathsf{x} \in \mathsf{S} \text{ and } M, w \models \mathsf{pre}(\mathsf{x})\}$
- $s \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$ iff
  - (i) $\pi_1(s) \in \Sigma_a(w)$
  - (ii) $\pi_2(s) \in \Delta_a(\mathsf{x})$
- $\langle w, \mathsf{x}\rangle \in V'(p)$ iff $w \in V(p)$

Notice that the definition of state maps in updated models is now very symmetric, which is a consequence of giving the action models the same structure as the epistemic models. Like in the previous chapter, we show again that our definitions guarantee that updates do not break our inquisitive epistemic models.

Proposition 4.2.2. **Updates result in inquisitive epistemic models**
For any inquisitive epistemic model M and for any IAML action model M, $M' = (M \otimes \mathsf{M})$ is an inquisitive epistemic model.

Proof: We need to check that for every world $\langle w, \mathsf{x}\rangle$, $\Sigma'_a(\langle w, \mathsf{x}\rangle)$ is non-empty and downward closed. Furthermore, Σ'_a should satisfy factivity (for all $w \in W$, $w \in \sigma_a(w)$) and introspection (for all $w, v \in W$, if $v \in \sigma_a(w)$ then $\Sigma_a(v) = \Sigma_a(w)$).

Take an arbitrary world $\langle w, \mathsf{x}\rangle \in W'$.

We will first show that $\Sigma'_a(\langle w, \mathsf{x}\rangle)$ is downward closed. Take any state $s \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$ and any $t \subseteq s$.

(i) $\pi_1(t) \in \Sigma_a(w)$ since $\pi_1(t) \subseteq \pi_1(s)$ and $\Sigma_a(w)$ is downward closed.

(ii) $\pi_2(t) \in \Delta_a(\mathsf{x})$ since $\pi_2(t) \subseteq \pi_2(s)$ and $\Delta_a(\mathsf{x})$ is downward closed.

This concludes downward closure. We now show factivity and non-emptiness at the same time, by showing that the state map of $\langle w, \mathsf{x}\rangle$ contains its own singleton, $\{\langle w, \mathsf{x}\rangle\}$.

(i) $\{w\} \in \Sigma_a(w)$ by factivity and downward closure of $\Sigma_a(w)$.

(ii) $\{\mathsf{x}\} \in \Delta_a(\mathsf{x})$ by factivity and downward closure of $\Delta_a(\mathsf{x})$.

Since $\{w\} = \pi_1(\langle w, \mathsf{x}\rangle)$ and $\{\mathsf{x}\} = \pi_2(\langle w, \mathsf{x}\rangle)$, this means that we have $\{\langle w, \mathsf{x}\rangle\} \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$. It follows that $\langle w, \mathsf{x}\rangle \in \sigma'_a(w)$.

That leaves introspection. Take any two worlds $\langle w, \mathsf{x}\rangle$ and $\langle w', \mathsf{x}'\rangle$ from W' such that $\langle w', \mathsf{x}'\rangle \in \sigma'_a(\langle w, \mathsf{x}\rangle)$. By downward closure of $\Sigma'_a(\langle w, \mathsf{x}\rangle)$ we obtain $\{\langle w', \mathsf{x}'\rangle\} \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$. From condition (i) we learn that it must be the case that $w' \in \sigma_a(w)$. By introspection of Σ_a we obtain $\Sigma_a(w) = \Sigma_a(w')$. Via condition (ii) we can obtain in a similar way that $\Delta_a(\mathsf{x}) = \Delta_a(\mathsf{x}')$. Then it is easy to check that for all states t, t satisfies conditions (i) and (ii) for $\Sigma'_a(\langle w, \mathsf{x}\rangle)$ iff it satisfies them for $\Sigma'_a(\langle w', \mathsf{x}'\rangle)$. Therefore $\Sigma'_a(\langle w, \mathsf{x}\rangle)$ and $\Sigma'_a(\langle w', \mathsf{x}'\rangle)$ are equal. □

4.2.3 Syntax and semantics

We can now define the syntax and semantics of IAML, which is much like that of AMLQ. The language, like that of AMLQ, extends IEL with dynamic modalities for sets of actions, but this time these are actions in IAML action models instead.

Definition 4.2.3. **Syntax of $\mathcal{L}^{\mathsf{IAML}_0}$**
Level 0 of the language of Inquisitive Action Model Logic is defined as follows, where s is a set of action points within the IAML_0 action model M:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \varphi \mid K_a\varphi \mid E_a\varphi \mid [\mathsf{M}, \mathsf{s}]\varphi$$

We restrict the definition of our language to the fragment that has dynamic modalities of IAML_0 action models for now. The definition of the full language will be given in Section 4.5, and until then we will often omit the subscripts.

For the language of IAML, we take over the abbreviations and notational conventions from AMLQ. Also, we take over Definition 3.2.6 as the definition of updated states and the support condition of the dynamic modality from Definition 3.2.7, restated below:

- $s[\mathsf{M}, \mathsf{s}] = \{\langle w, \mathsf{x}\rangle \in W' \mid w \in s \text{ and } \mathsf{x} \in \mathsf{s}\}$
- $M, s \models [\mathsf{M}, \mathsf{s}]\varphi \iff (M \otimes \mathsf{M}), s[\mathsf{M}, \mathsf{s}] \models \varphi$

It is important to keep in mind that, while both the syntax and semantics of $\mathcal{L}^{\mathsf{IAML}}$ look the same as in the previous chapter, there is a big difference between the two: they both have their own set of action models and these sets are disjoint. This means that a formula of $\mathcal{L}^{\mathsf{AMLQ}}$ is not necessarily a formula of $\mathcal{L}^{\mathsf{IAML}}$ and vice versa.

4.2.4 Epistemic maps and state maps in updated models

For AMLQ, we gave two lemmas about epistemic maps and state maps to provide an alternative characterization of the update procedure. We will do the same for IAML_0. We start by showing that Lemma 3.2.1 holds for IAML_0 as well.

Lemma 4.2.1. **Knowledge in updated models**
Let w be a world in an inquisitive epistemic model M, x an action in inquisitive action model M and $M' = M \otimes \mathsf{M}$. Let $\langle w, \mathsf{x}\rangle$ be a world in M'. Then we have the following:

$$\sigma'_a(\langle w, \mathsf{x}\rangle) = \sigma_a(w)[\delta_a(\mathsf{x})]$$

Proof: ($\subseteq$) Take any $\langle v, \mathsf{y}\rangle \in \sigma'_a(\langle w, \mathsf{x}\rangle)$. This world is in this set because it belongs to a state that satisfies conditions (i) and (ii) of the update procedure in Definition 4.2.2. From condition (i) we obtain $v \in \sigma_a(w)$. From condition (ii) we obtain $\mathsf{y} \in \delta_a(\mathsf{x})$.

These are, by Definition 3.2.6, exactly the conditions to be in $\sigma_a(w)[\delta_a(\mathsf{x})]$. So $\langle v, \mathsf{y}\rangle \in \sigma_a(w)[\delta_a(\mathsf{x})]$.

($\supseteq$) Take an arbitrary world $\langle v, \mathsf{y}\rangle \in \sigma_a(w)[\delta_a(\mathsf{x})]$. Then by Definition 3.2.6, we know that $v \in \sigma_a(w)$ and $\mathsf{y} \in \delta_a(\mathsf{x})$. It follows that the state $\{\langle v, \mathsf{y}\rangle\}$ satisfies condition (i) and (ii) of Definition 4.2.2 to be in $\Sigma'_a(\langle w, \mathsf{x}\rangle)$. This means that $\{\langle v, \mathsf{y}\rangle\} \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$. Hence, $\langle v, \mathsf{y}\rangle \in \sigma'_a(\langle w, \mathsf{x}\rangle)$. □

As for the state maps of agents, there is an important difference between AMLQ and IAML. In IAML, an information state does not necessarily specify what action has taken place, to count as resolving the agent's issues. As a consequence, we do not have Lemma 3.2.2 in IAML. However, we can refine this lemma in the following way.

Lemma 4.2.2. **State maps in updated models**
Let w be a world in an inquisitive epistemic model M, x an action in inquisitive action model M and $M' = M \otimes \mathsf{M}$. Suppose $\langle w, \mathsf{x}\rangle \in W'$. Then we have the following:

$$s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle) \iff s' \subseteq s[\mathsf{s}] \text{ for some } s \subseteq W \text{ and } \mathsf{s} \subseteq \mathsf{S} \text{ such that:}$$

- $s \in \Sigma_a(w)$
- $\mathsf{s} \in \Delta_a(\mathsf{x})$

Proof: ($\Rightarrow$) Assume $s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$.

Let s be $\pi_1(s')$ and s be $\pi_2(s')$. Then clearly $s' \subseteq s[\mathsf{s}]$. By Definition 4.2.2, since $s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$, it must be the case that $s \in \Sigma_a(w)$ and $\mathsf{s} \in \Delta_a(\mathsf{x})$.

($\Leftarrow$) Assume $s' \subseteq s[\mathsf{s}]$ such that $s \in \Sigma_a(w)$ and $\mathsf{s} \in \Delta_a(\mathsf{x})$.

We can show that $s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$ by checking conditions (i) and (ii) of Definition 4.2.2:

(i) As $s' \subseteq s[\mathsf{s}]$, $\pi_1(s') \subseteq s$. Since $s \in \Sigma_a(w)$, by downward closure we have $\pi_1(s') \in \Sigma_a(w)$.

(ii) As $s' \subseteq s[\mathsf{s}]$, $\pi_2(s') \subseteq \mathsf{s}$. Since $\mathsf{s} \in \Delta_a(\mathsf{x})$, by downward closure we have $\pi_2(s') \in \Delta_a(\mathsf{x})$. □

Just like Lemma 3.2.2 can be read as an alternative definition of Σ'_a in AMLQ, the same goes for the above lemma in IAML.

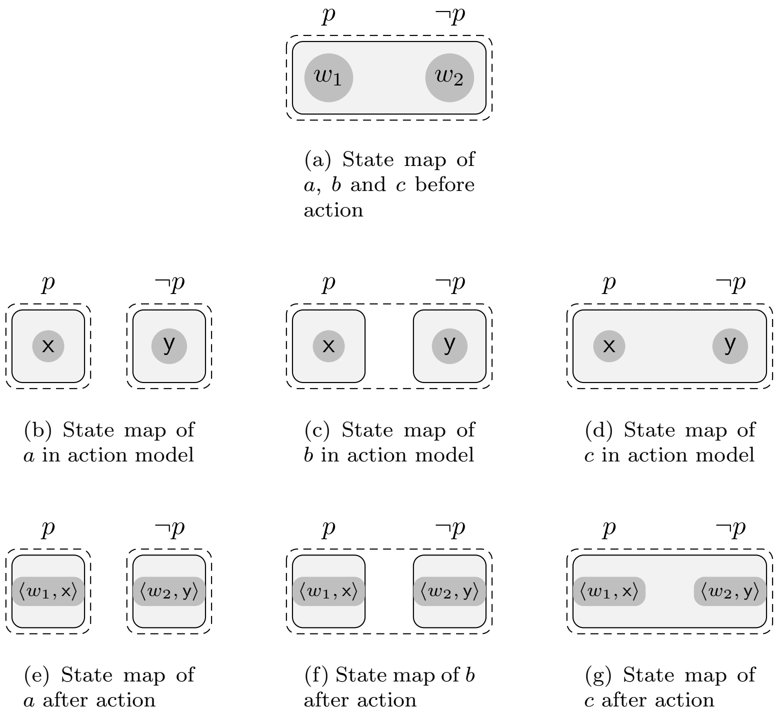
4.3 Examples

We will now look at some examples. First, let us look at two scenarios with statements, which we also discussed in the previous chapter.

Example 4.3.1. **Peter is attending**
We repeat Example 3.3.1: Peter calls to tell Anna that he will be attending the wedding. Like before, Bob is unaware what the actual action is, but we now explicitly encode that he wants to know. Let us add a third character, Calvin (c), a friend of Bob who is present by coincidence, but does not care which action is taking place.

The relevant diagrams are in Figure 4.2. Let $\mathsf{cont}(\mathsf{x}) = p$ and $\mathsf{cont}(\mathsf{y}) = \neg p$. The state maps of the agents in the updated model are shown in subfigures (e), (f) and (g).

The resulting state maps for a and b are the same as last time. However, agent c has a resulting state map that we could not obtain in the previous system: the fact that he does



(a) State map of a, b and c before action

(b) State map of a in action model

(c) State map of b in action model

(d) State map of c in action model

(e) State map of a after action

(f) State map of b after action

(g) State map of c after action

Figure 4.2: Example 4.3.1

not consider it an issue whether x or y was the actual action is reflected in the updated model.

Now let us also add our uninterested agent Calvin to the scenario from Example 4.3.2.

Example 4.3.2. **Patrick or Qasim is attending**
The RSVP message is either from Patrick (p) or Qasim (q). Anna (a) reads it, Bob (b) is only interested and Calvin (c) does not care.

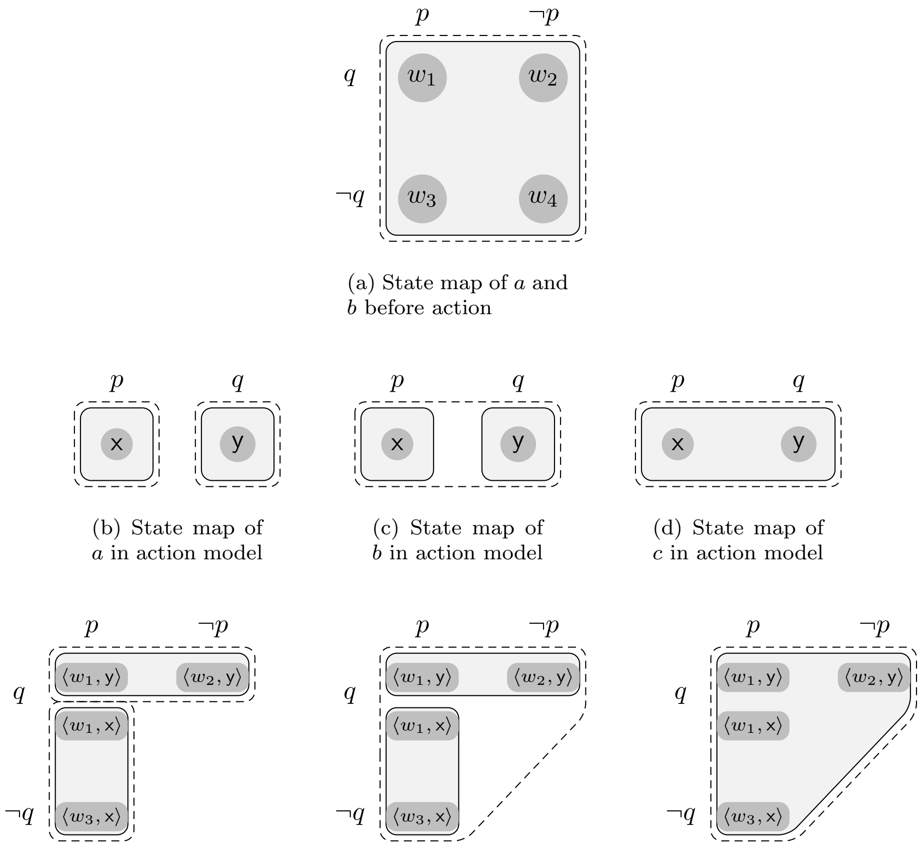
All diagrams are in Figure 4.3. Let $\mathsf{cont}(\mathsf{x}) = p$ and $\mathsf{cont}(\mathsf{y}) = q$. The state maps of the agents in the updated model are shown in subfigures (e), (f) and (g).

Again, the outcome for agents a and b is the same as last time. But while Bob entertains the issue whether Patrick or Qasim is attending ($E_b(p \mathbin{\vee\!\!\!\vee} q)$), this is not the case for Calvin: although he does learn something, namely that w_4 is no longer a candidate for the actual world, he does not entertain $p \mathbin{\vee\!\!\!\vee} q$ in the updated model.

The previous two examples illustrate the advantage of explicitly encoding issues in the action model: it allows us to add agents that are not interested in the epistemic action that is taking place. In the next and final example, we will show that we can still encode the action of asking a question in our current setting, by repeating Example 3.3.3.

Example 4.3.3. **Is Penny attending?**
Like last time, the question whether Penny is attending ($?p$) is communicated to Anna without Bob knowing (although he does consider it possible that this question was asked).

This time, we encode the question $?p$ by adding not one but two actions x_1 and x_2 to the model, one for each resolution of $?p$. We let $\mathsf{cont}(\mathsf{x}_1) = p, \mathsf{cont}(\mathsf{x}_2) = \neg p$ and $\mathsf{cont}(\mathsf{y}) = \top$.



(a) State map of a and b before action

(b) State map of a in action model

(c) State map of b in action model

(d) State map of c in action model

(e) State map of a after action

(f) State map of b after action

(g) State map of c after action

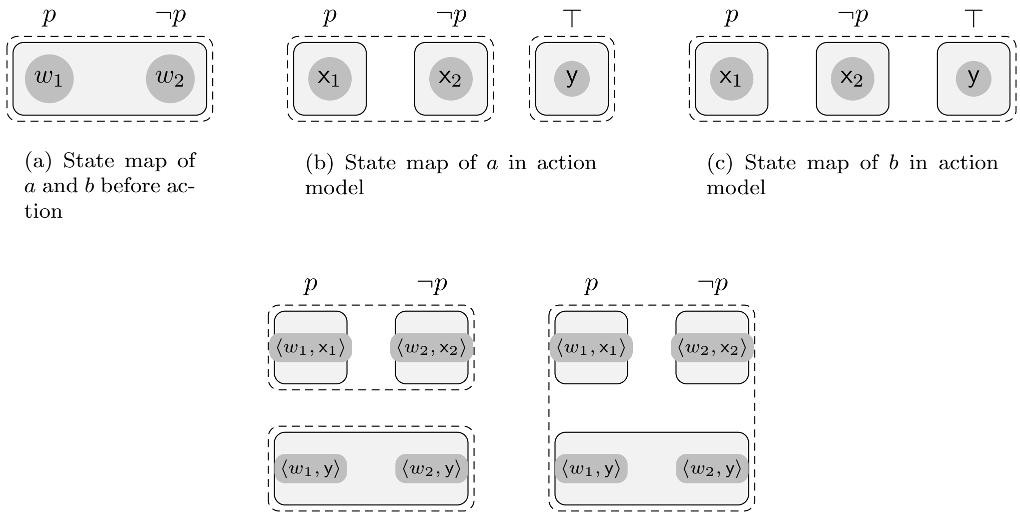
Figure 4.3: Example 4.3.2

We represent all models in Figure 4.4.

We see that the updated model we get using this strategy is very similar to the updated model in Example 3.3.3. In fact, apart from the names that are given to the worlds, they are the same. We will show in Section 4.9.1 that we can in fact define a translation from AMLQ action models to IAML action models. Using this translation, we can simulate any epistemic action from AMLQ in IAML.

4.4 Properties of IAML

We now briefly discuss the same properties we discussed in Section 3.4. In short, all definitions and propositions we give here are the same as for AMLQ. As the proofs only rely on definitions that were not changed when we moved from AMLQ to IAML, we omit all proofs in this section. As all definitions and propositions are independent of the level of the language, we write $\mathcal{L}^{\mathsf{IAML}}$ instead of $\mathcal{L}^{\mathsf{IAML}_0}$, even though the full language $\mathcal{L}^{\mathsf{IAML}}$ has not been defined yet.



(a) State map of a and b before action

(b) State map of a in action model

(c) State map of b in action model

(d) State map of a after action

(e) State map of b after action

Figure 4.4: Example 4.3.3

4.4.1 Persistence and empty state

PROPOSITION 4.4.1. **Properties of the support relation**
For all models M and formulas $\varphi \in \mathcal{L}^{\mathsf{IAML}}$, we have the following properties:

- *Persistence property:* if $s \models \varphi$ and $t \subseteq s$, then $t \models \varphi$.
- *Empty state property:* $\emptyset \models \varphi$
- *Modal persistence property:* If $s \models [\mathsf{s}]\varphi$ and $\mathsf{t} \subseteq \mathsf{s}$, then $s \models [\mathsf{t}]\varphi$.
- *Modal empty state property:* $\models [\emptyset]\varphi$

4.4.2 Declaratives

The declarative fragment of $\mathcal{L}^{\mathsf{IAML}}$ is defined in the same way as that of $\mathcal{L}^{\mathsf{AMLQ}}$.

DEFINITION 4.4.1. **Declarative fragment of $\mathcal{L}^{\mathsf{IAML}}$**
The set of declarative formulas $\mathcal{L}_!^{\mathsf{IAML}}$ is defined inductively as follows, where $\varphi \in \mathcal{L}^{\mathsf{IAML}}$:

$$\alpha ::= p \mid \bot \mid \alpha \wedge \alpha \mid \alpha \to \alpha \mid K_a\varphi \mid E_a\varphi \mid [\mathsf{s}]\alpha$$

PROPOSITION 4.4.2. **Any $\alpha \in \mathcal{L}_!^{\mathsf{IAML}}$ is truth-conditional**

4.4.3 Resolutions and normal form

We can also obtain the normal form result familiar from IEL. That is, we can extend the definition of resolutions and show that every formula in $\mathcal{L}^{\mathsf{AMLQ}}$ is equivalent to the inquisitive disjunction of its resolutions.

Definition 4.4.2. **Resolutions in $\mathcal{L}^{\mathsf{IAML}}$**
For any formula $\varphi \in \mathcal{L}^{\mathsf{IAML}}$, its set of resolutions $\mathcal{R}(\varphi)$ is defined by extending Definition 2.3.10 with the following clause for the dynamic modality:

$$\mathcal{R}([\mathsf{s}]\varphi) = \{[\mathsf{s}]\alpha \mid \alpha \in \mathcal{R}(\varphi)\}$$

Proposition 4.4.3. **A formula is supported iff some resolution of it is**
For all $\varphi \in \mathcal{L}^{\mathsf{IAML}}$, for every inquisitive epistemic model M and state s:

$$M, s \models \varphi \iff M, s \models \alpha \text{ for some } \alpha \in \mathcal{R}(\varphi)$$

Proposition 4.4.4. **Normal form**
For all $\varphi \in \mathcal{L}^{\mathsf{IAML}}$, $\varphi \equiv \bigvee\!\!\!\!\bigvee \mathcal{R}(\varphi)$.

4.4.4 Declarative variant

Definition 4.4.3. **Declarative variant in $\mathcal{L}^{\mathsf{IAML}}$**
The declarative variant $\varphi^!$ of a formula $\varphi \in \mathcal{L}^{\mathsf{IAML}}$ is defined by:

$$\varphi^! := \bigvee \mathcal{R}(\varphi)$$

Proposition 4.4.5. **Declarative variants in $\mathcal{L}^{\mathsf{IAML}}$ have equal truth conditions**
For all $\varphi \in \mathcal{L}^{\mathsf{IAML}}$, for every inquisitive epistemic model M and world w:

$$M, w \models \varphi \iff M, w \models \varphi^!$$

We show that declaratives are representative of the truth-conditional formulas of our language, by proving the following proposition.

Proposition 4.4.6. **Any truth-conditional formula is equivalent to a declarative**
For all $\varphi \in \mathcal{L}^{\mathsf{IAML}}$, if φ is truth-conditional, then there is some $\alpha \in \mathcal{L}_!^{\mathsf{IAML}}$ such that $\varphi \equiv \alpha$.

It is important that we have checked that these properties all carry over, because we will need them again for later proofs, in the same way as we used the properties of AMLQ in the previous chapter. For instance, without resolutions we cannot prove completeness.

4.5 Dynamic modalities in action content

Like in the previous chapter, we now have all definitions in place to expand the definition of action models, so that we can start using dynamic modalities in action content. We give the definitions in the exact same way we did for AMLQ in Section 3.5.

Definition 4.5.1. **Higher level Inquisitive Action Models**
For every $i > 0$, an IAML_i action model is a triple $\mathsf{M} = \langle \mathsf{S}, \{\Delta_a \mid a \in \mathcal{A}\}, \mathsf{cont}\rangle$, where:

- S and Δ_a are defined as before;
- $\mathsf{cont} : \mathsf{S} \to \mathcal{L}^{\mathsf{IAML}_{(i-1)}}$ is a function that assigns a content $\mathsf{cont}(\mathsf{x}) \in \mathcal{L}^{\mathsf{IAML}_{(i-1)}}$ to each action point $\mathsf{x} \in \mathsf{S}$.

Definition 4.5.2. **Syntax of $\mathcal{L}^{\mathsf{IAML}_i}$**
For every $i > 0$, the language of Inquisitive Action Model Logic of level i is defined as follows, where s is a set of action points within the IAML_j action model M, with the restriction that $j < i$:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \rightarrow \varphi \mid \varphi \vee\!\!\!\vee \varphi \mid K_a\varphi \mid E_a\varphi \mid [\mathsf{M}, \mathsf{s}]\varphi$$

Definition 4.5.3. **Syntax of $\mathcal{L}^{\mathsf{IAML}}$**
The full language of Inquisitive Action Model Logic is defined as the union of all $\mathcal{L}^{\mathsf{IAML}_i}$ for all natural numbers i.

$$\mathcal{L}^{\mathsf{IAML}} := \bigcup_{i \geq 0} \mathcal{L}^{\mathsf{IAML}_i}$$

The set of all IAML action models is the union of all sets of IAML_i action models for $i \geq 0$. We now simultaneously generalize the definition of updated models and all the definitions and propositions from Section 4.4, like we also did for AMLQ.

4.6 Composition of inquisitive action models

We also define the notion of composition for inquisitive action models. We do this in the following straightforward way.

Definition 4.6.1. **Composition of IAML action models**
Let $\mathsf{M} = \langle \mathsf{S}, \{\Delta_a \mid a \in \mathcal{A}\}, \mathsf{cont}\rangle$ and $\mathsf{M}' = \langle \mathsf{S}', \{\Delta'_a \mid a \in \mathcal{A}\}, \mathsf{cont}'\rangle$ be two IAML action models. Their composition $\mathsf{M}; \mathsf{M}'$ is the action model $\langle \mathsf{S}'', \{\Delta''_a \mid a \in \mathcal{A}\}, \mathsf{cont}''\rangle$ such that:

- $\mathsf{S}'' = \mathsf{S} \times \mathsf{S}'$
- $\mathsf{s} \in \Delta''_a(\langle \mathsf{x}, \mathsf{x}'\rangle)$ iff $\pi_1(\mathsf{s}) \in \Delta_a(\mathsf{x})$ and $\pi_2(\mathsf{s}) \in \Delta'_a(\mathsf{x}')$
- $\mathsf{cont}''(\langle \mathsf{x}, \mathsf{x}'\rangle) = \mathsf{cont}(\mathsf{x}) \wedge [\mathsf{M}, \mathsf{x}]\mathsf{cont}'(\mathsf{x}')$

The composition of two pointed action models (M, s) and $(\mathsf{M}', \mathsf{s}')$ is the pointed action model $(\mathsf{M}'', \mathsf{s} \times \mathsf{s}')$ with $\mathsf{M}'' = \mathsf{M}; \mathsf{M}'$ defined as above.

It is easy to show that any composition of inquisitive action models is an inquisitive action model (in fact, as the composition of two inquisitive action models is very similar to the product update of an inquisitive epistemic model and an inquisitive action model, a proof for this claim would follow the same structure as the proof for Proposition 4.2.2).

Like in the previous chapter, we can show that it is indeed the case that updating with the composed action model gives the same results as updating with the separate action models in sequence.

Proposition 4.6.1. **Isomorphism between updated models**
For every inquisitive epistemic model M and every two action models M and M', $(M \otimes \mathsf{M}) \otimes \mathsf{M}$ is isomorphic to $M \otimes (\mathsf{M}; \mathsf{M}')$.

Proof: We define three updated models M', M'' and M''':

- $M' = M \otimes \mathsf{M} = \langle W', \{\Sigma'_a \mid a \in \mathcal{A}\}, V'\rangle$
- $M'' = M' \otimes \mathsf{M}' = \langle W'', \{\Sigma''_a \mid a \in \mathcal{A}\}, V''\rangle$
- $M''' = M \otimes (\mathsf{M}; \mathsf{M}') = \langle W''', \{\Sigma'''_a \mid a \in \mathcal{A}\}, V'''\rangle$

We want to show that M'' and M''' are isomorphic.

Because the domain and valuation of the composed action model are defined in the same way as in Definition 3.7.1, we can repeat the proof for the facts that $\langle\langle w, \mathsf{x}\rangle, \mathsf{x}'\rangle \in W''$ iff $\langle w, \langle \mathsf{x}, \mathsf{x}'\rangle\rangle \in W'''$ and $\langle\langle w, \mathsf{x}\rangle, \mathsf{x}'\rangle \in V''(p)$ iff $\langle w, \langle \mathsf{x}, \mathsf{x}'\rangle\rangle \in V'''(p)$ from the proof for Proposition 3.7.1. This means that we only need to show that the state maps have the same structure.

Let $s''' = \{\langle w, \langle \mathsf{x}, \mathsf{x}'\rangle\rangle \mid \langle\langle w, \mathsf{x}\rangle, \mathsf{x}'\rangle \in s''\}$.

We need to show that $s'' \in \Sigma''_a(\langle\langle w, \mathsf{x}\rangle, \mathsf{x}'\rangle) \iff s''' \in \Sigma'''_a(\langle w, \langle \mathsf{x}, \mathsf{x}'\rangle\rangle)$.

(⇒) Assume $s'' \in \Sigma''_a(\langle\langle w, \mathsf{x}\rangle, \mathsf{x}'\rangle)$.

If s'' is empty, we are done, so assume that it is not. We know that it satisfies conditions (i) and (ii) of Definition 4.2.2.

Let $s' = \pi_1(s'')$ and $\mathsf{s}' = \pi_2(s'')$. By condition (i), $s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$ and by condition (ii), $\mathsf{s}' \in \Delta'_a(\mathsf{x}')$. From the former we know that s' in turn satisfies conditions (i) and (ii) to be in $\Sigma'_a(\langle w, \mathsf{x}\rangle)$.

Let $s = \pi_1(s')$ and $\mathsf{s} = \pi_2(s')$. By condition (i), $s \in \Sigma_a(w)$ and by condition (ii) we have $\mathsf{s} \in \Delta_a(\mathsf{x})$.

We can now show that s''' satisfies the two conditions to be in $\Sigma'''_a(\langle w, \langle \mathsf{x}, \mathsf{x}'\rangle\rangle)$.

For condition (i), notice that $\pi_1(s''') = s$. This means that from $s \in \Sigma_a(w)$, we have $\pi_1(s''') \in \Sigma_a(w)$.

For condition (ii), notice that $\pi_1(\pi_2(s''')) = \mathsf{s}$ and $\pi_2(\pi_2(s''')) = \mathsf{s}'$. This means that from $\mathsf{s} \in \Delta_a(\mathsf{x})$ and $\mathsf{s}' \in \Delta'_a(\mathsf{x}')$ we have by the definition of composition that $\pi_2(s''') \in \Delta''_a(\langle \mathsf{x}, \mathsf{x}'\rangle)$.

(⇐) Assume $s''' \in \Sigma'''_a(\langle w, \langle \mathsf{x}, \mathsf{x}'\rangle\rangle)$.

If s''' is empty, we are done, so assume it is not. We know that it satisfies conditions (i) and (ii) of Definition 4.2.2.

By condition (i), $\pi_1(s''') \in \Sigma_a(w)$ and by condition (ii), $\pi_2(s''') \in \Delta''_a(\langle \mathsf{x}, \mathsf{x}'\rangle)$.

By the definition of composition, $\pi_2(s''') \in \Delta''_a(\langle \mathsf{x}, \mathsf{x}'\rangle)$ implies $\pi_1(\pi_2(s''')) \in \Delta_a(\mathsf{x})$ and $\pi_2(\pi_2(s''')) \in \Delta'_a(\mathsf{x}')$.

Like in the proof for the other direction, let $s' = \pi_1(s'')$ and $\mathsf{s}' = \pi_2(s'')$. Then, let $s = \pi_1(s')$ and $\mathsf{s} = \pi_2(s')$. Because $s = \pi_1(s''')$ and $\mathsf{s} = \pi_1(\pi_2(s'''))$, we have $s \in \Sigma_a(w)$ and $\mathsf{s} \in \Delta_a(w)$. That means $s' \in \Sigma_a(\langle w, \mathsf{x}\rangle)$. This, in turn, is condition (i) for s'' to be in $\Sigma''_a(\langle\langle w, \mathsf{x}\rangle, \mathsf{x}'\rangle)$. Condition (ii) requires that $\mathsf{s}' \in \Delta'(\mathsf{x}')$. We already have this, since $\mathsf{s}' = \pi_2(\pi_2(s'''))$. □

Now that we have this result, the proofs for the other propositions in Section 3.7 carry over. It is then easy to show that we have the same equivalence in IAML as the one we already had in AMLQ.

Proposition 4.6.2. $[\mathsf{s}][\mathsf{t}]\varphi \equiv [\mathsf{s};\mathsf{t}]\varphi$

Proof: We can show that for any information state s, the updated state $s[\mathsf{s};\mathsf{t}]$ is the image of $s[\mathsf{s}][\mathsf{t}]$ under the isomorphism just defined, by repeating the proof for Proposition 3.7.2. With this result, we can simply repeat the proof for Proposition 3.7.3. □

4.7 Reduction

For our axiomatization, we follow the same strategy as for AMLQ. This means we have to give some equivalences to show that IAML is not more expressive than IEL. Most of these equivalences can be immediately taken over from AMLQ.

PROPOSITION 4.7.1. The following equivalences hold in IAML:

- If α is truth-conditional, then $[\mathsf{s}]\alpha \equiv \bigwedge_{\mathsf{x}\in\mathsf{s}}[\mathsf{x}]\alpha$
- $[\mathsf{x}]p \equiv \mathsf{pre}(\mathsf{x}) \to p$
- $[\mathsf{x}]\bot \equiv \neg\mathsf{pre}(\mathsf{x})$
- $[\mathsf{s}](\varphi \wedge \psi) \equiv [\mathsf{s}]\varphi \wedge [\mathsf{s}]\psi$
- $[\mathsf{s}](\varphi \vee\!\!\!\vee \psi) \equiv [\mathsf{s}]\varphi \vee\!\!\!\vee [\mathsf{s}]\psi$
- $[\mathsf{x}](\varphi \to \psi) \equiv [\mathsf{x}]\varphi \to [\mathsf{x}]\psi$

Proof: The proofs of Propositions 3.8.1 - 3.8.6 and 3.6.2 establish this, as they only depend on definitions that remain unchanged in IAML with respect to AMLQ. □

For the knowledge modality, we can also prove the same equivalence we already had in AMLQ.

PROPOSITION 4.7.2. $[\mathsf{x}]K_a\varphi \equiv \mathsf{pre}(\mathsf{x}) \to K_a[\delta_a(\mathsf{x})]\varphi$

Proof: Repeat the proof for Proposition 3.8.8, using Lemma 4.2.1 instead of Lemma 3.2.1 □

In contrast with all the other equivalences, the equivalence we obtained for the entertain modality in the previous chapter no longer holds. The reason for this is that the issues that the agents entertain in the updated model now depend on the issues that are encoded in the state maps of the inquisitive action model.

For instance, let there be an agent who does not entertain $?p$ in the original model. Let the updated model contain two actions x and y, with content p and $\neg p$ respectively. Then whether the agent entertains $?p$ after the update depends on whether it is an issue for her whether x or y is the actual action. This difference between AMLQ and IAML is also reflected in the difference between Lemma 3.2.2 and Lemma 4.2.2. We use the latter to show that in IAML we following equivalence holds.

PROPOSITION 4.7.3. $[\mathsf{x}]E_a\varphi \equiv \mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{s}\in\Delta_a(\mathsf{x})} E_a[\mathsf{s}]\varphi$

Proof: As both formulas are declaratives, we only need to show that they have the same truth conditions.

($\Rightarrow$) Assume $M, w \models [\mathsf{x}]E_a\varphi$.

Then $M', w[\mathsf{x}] \models E_a\varphi$. Assume $M, w \models \mathsf{pre}(\mathsf{x})$. Then by the truth condition of the entertain modality, we have for all $s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle) : M', s' \models \varphi$.

Take any state s in the action model such that $\mathsf{s} \in \Delta_a(\mathsf{x})$ and any state s in the original model such that $s \in \Sigma_a(w)$. By Lemma 4.2.2, $s[\mathsf{s}] \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$. This means that $M', s[\mathsf{s}] \models \varphi$.

By the support condition of the dynamic modality, we have $M, s \models [\mathsf{s}]\varphi$. As s was an arbitrary state in $\Sigma_a(w)$, we have $M, w \models E_a[\mathsf{s}]\varphi$. As s was chosen arbitrarily, we have $M, w \models \bigwedge_{\mathsf{s}\in\Delta_a(\mathsf{x})} E_a[\mathsf{s}]\varphi$. Then finally, we drop our assumption that $M, w \models \mathsf{pre}(\mathsf{x})$ to obtain $M, w \models \mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{s}\in\Delta_a(\mathsf{x})} E_a[\mathsf{s}]\varphi$.

($\Leftarrow$) Assume $M, w \models \mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{s} \in \Delta_a(\mathsf{x})} E_a[\mathsf{s}]\varphi$.

Either $M, w \models \mathsf{pre}(\mathsf{x})$ or $M, w \not\models \mathsf{pre}(\mathsf{x})$. In the latter case, we immediately have $M, w \models [\mathsf{x}]E_a\varphi$ and we are done, so assume the former. Then we have $M, w \models \bigwedge_{\mathsf{s} \in \Delta_a(\mathsf{x})} E_a[\mathsf{s}]\varphi$.

As $M, w \models \mathsf{pre}(\mathsf{x})$, we have a world $\langle w, \mathsf{x} \rangle$ in the updated model. Take any $s' \in \Sigma'_a(\langle w, \mathsf{x} \rangle)$. Then by Lemma 4.2.2, $s' \subseteq t[\mathsf{t}]$ for some $t \in \Sigma_a(w)$ and $\mathsf{t} \in \Delta_a(\mathsf{x})$. Since we have $M, w \models \bigwedge_{\mathsf{s} \in \Delta_a(\mathsf{x})} E_a[\mathsf{s}]\varphi$ by assumption, from $\mathsf{t} \in \Delta_a(\mathsf{x})$ we obtain $M, w \models E_a[\mathsf{t}]\varphi$, so by $t \in \Sigma_a(w)$ we have that $M, t \models [\mathsf{t}]\varphi$. By the support condition of the dynamic modality, $M', t[\mathsf{t}] \models \varphi$. As $s' \subseteq t[\mathsf{t}]$, $M', s' \models \varphi$.

As s' was an arbitrary state in $\Sigma'_a(\langle w, \mathsf{x} \rangle)$, we have $M', \langle w, \mathsf{x} \rangle \models E_a\varphi$, which means that $M', w[\mathsf{x}] \models E_a\varphi$. By the truth condition of the dynamic modality, we have $M, w \models [\mathsf{x}]E_a\varphi$. □

Like in the previous chapter, these equivalences can be used to show that any formula of IAML can be reduced to a formula of our static language. We take over the definition of modal depth that we defined for AMLQ in the previous chapter. It is then easy to check that everything we have used in the proof for Theorem 3.8.1 can be reused in the same way, except that in IAML we have a different reduction equivalence for the entertain modality.

Theorem 4.7.1. **Every formula of IAML is equivalent to some formula of IEL**
For any $\varphi \in \mathcal{L}^{\mathsf{IAML}}$, there is some $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$ such that $\varphi \equiv \varphi^*$.

Proof: The proof proceeds exactly like the proof of Theorem 3.8.1, except for the case of the entertain modality.

(E) Suppose β^* is $E_a\chi$. Then by Proposition 4.7.1 and 4.7.3 we have $[\mathsf{s}]\beta^* \equiv \bigwedge_{\mathsf{x} \in \mathsf{s}}(\mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{t} \in \Delta_a(\mathsf{x})} E_a[\mathsf{t}]\chi)$. As $\mathsf{md}(\chi) < n$, by the induction hypothesis we have some $\chi^*_{\mathsf{t}} \in \mathcal{L}^{\mathsf{IEL}}$ equivalent to each $[\mathsf{t}]\chi$. So we can let $\alpha^* := \bigwedge_{\mathsf{x} \in \mathsf{s}}(\mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{t} \in \Delta_a(\mathsf{x})} E_a\chi^*_{\mathsf{t}})$ □

4.8 Axiomatizing IAML

4.8.1 Completeness via replacement of equivalents

We start by defining the proof system that consists of all the inference rules for IEL (Ciardelli, 2014) and the rules in Figure 4.5. The rules in this figure correspond with the equivalences from Section 4.7. We also add a rule that allows us to replace equivalent formulas again. The relation of derivability in this system is denoted by $\vdash_{\mathsf{IAML}^{\mathsf{RE}}}$ and the relation of inter-derivability by $\dashv\vdash_{\mathsf{IAML}^{\mathsf{RE}}}$.

The completeness proof for this system proceeds in the same way as the proof for completeness of $\vdash_{\mathsf{AMLQ}^{\mathsf{RE}}}$. This means that it is essential to show that we have provability of normal form in this system.

Lemma 4.8.1. **Provability of normal form in $\vdash_{\mathsf{IAML}^{\mathsf{RE}}}$**
For any $\varphi \in \mathcal{L}^{\mathsf{IAML}}$, $\varphi \dashv\vdash_{\mathsf{IAML}^{\mathsf{RE}}} \bigvee \mathcal{R}(\varphi)$.

Proof: Repeat the proof for Lemma 3.9.1. □

We can then show that all formulas are inter-derivable with their reduced equivalents in the same way that we did in the previous chapter.

!Atom	!∧	!K
$\dfrac{[\mathsf{x}]p}{\mathsf{pre}(\mathsf{x}) \to p}$	$\dfrac{[\mathsf{s}](\varphi \wedge \psi)}{[\mathsf{s}]\varphi \wedge [\mathsf{s}]\psi}$	$\dfrac{[\mathsf{x}]K_a\varphi}{\mathsf{pre}(\mathsf{x}) \to K_a[\delta_a(\mathsf{x})]\varphi}$
!⊥	!→	!E
$\dfrac{[\mathsf{x}]\bot}{\neg\mathsf{pre}(\mathsf{x})}$	$\dfrac{[\mathsf{x}](\varphi \to \psi)}{[\mathsf{x}]\varphi \to [\mathsf{x}]\psi}$	$\dfrac{[\mathsf{x}]E_a\varphi}{\mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{s}\in\Delta_a(\mathsf{x})} E_a[\mathsf{s}]\varphi}$
AUD	!⩛	RE
$\dfrac{[\mathsf{s}]\alpha}{\bigwedge_{\mathsf{x}\in\mathsf{s}}[\mathsf{x}]\alpha}$	$\dfrac{[\mathsf{s}](\varphi \vee\!\!\!\vee \psi)}{[\mathsf{s}]\varphi \vee\!\!\!\vee [\mathsf{s}]\psi}$	$\dfrac{\varphi \leftrightarrow \psi}{\chi[\varphi/p] \leftrightarrow \chi[\psi/p]}$

Figure 4.5: The inference rules for dynamic modalities in IAML. The double lines indicate that the inference is allowed in both directions. The rule AUD can only be applied to declaratives α.

Theorem 4.8.1. **Every formula of IAML is inter-derivable with a formula of IEL**
For any $\varphi \in \mathcal{L}^{\mathsf{IAML}}$, there is some $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$ such that $\varphi \dashv\vdash_{\mathsf{IAML}^{\mathsf{RE}}} \varphi^*$.

Proof: The proof is the same as the proof for Theorem 3.9.1, but using the inference rules from Figure 4.5. □

With Theorem 4.7.1 and 4.8.1 in place, it is easy to show that the proof system denoted by $\vdash_{\mathsf{IAML}^{\mathsf{RE}}}$ is sound and complete.

Theorem 4.8.2. **IAML$^{\mathsf{RE}}$ is sound and complete**
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{IAML}}, \Phi \models \psi \iff \Phi \vdash_{\mathsf{IAML}^{\mathsf{RE}}} \psi$.

Proof: Repeat the proof for Theorem 3.9.2, using Theorem 4.7.1 and 4.8.1. □

4.8.2 Completeness via monotonicity

We also provide an alternative complete proof system for IAML using monotonicity of dynamic modalities, like we also did for AMLQ. As the support condition for dynamic modalities has not changed, the proof for the claim that dynamic modalities are monotonic carries over. We capture monotonicity in our proof system by introducing the inference rule !Mon (Figure 4.6).

The proof system denoted by $\vdash_{\mathsf{IAML}^{\mathsf{!Mon}}}$ consists of the inference rules of $\vdash_{\mathsf{IAML}^{\mathsf{RE}}}$, with !Mon instead of RE. The soundness of this system is guaranteed by checking that an analogue of Proposition 3.9.2 can be proved for IAML modalities. Like $\vdash_{\mathsf{AMLQ}^{\mathsf{!Mon}}}$, we have provability of normal form in this system too:

Lemma 4.8.2. **Provability of normal form in $\vdash_{\mathsf{IAML}^{\mathsf{!Mon}}}$**
For any $\varphi \in \mathcal{L}^{\mathsf{IAML}}$, $\varphi \dashv\vdash_{\mathsf{IAML}^{\mathsf{!Mon}}} \bigvee\!\!\!\bigvee \mathcal{R}(\varphi)$.

Proof: Repeat the proof for Lemma 3.9.2. □

!Mon

$$\dfrac{\begin{array}{c}[\varphi] \\ \vdots \\ \psi\end{array} \qquad [\mathsf{s}]\varphi}{[\mathsf{s}]\psi}$$

Figure 4.6: The inference rule !Mon, where the proof of ψ has φ as its *only* undischarged assumption.

We can now prove that the system is also complete.

THEOREM 4.8.3. **$\mathsf{IAML}^{!\mathsf{Mon}}$ is complete**
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{IAML}}$, $\Phi \models \psi$ implies $\Phi \vdash_{\mathsf{IAML}^{!\mathsf{Mon}}} \psi$.

Proof: We repeat the proof for Theorem 3.9.3, but with two slight changes: first, the complexity measure c has to be redefined, since the complexity of the reduction equivalence of the entertain modality now depends on the amount of subsets $\Delta_a(\mathsf{x})$ has. This amount is maximally the number of subsets of the domain of the action model.

We therefore redefine the complexity of formulas of the form $[\mathsf{s}]\varphi$ as follows, where S is the domain of M:

$$c([\mathsf{M}, \mathsf{s}]\varphi) = (|\mathsf{s}| + |\wp(\mathsf{S})| + c(\mathsf{M}, \mathsf{s})) \cdot c(\varphi)$$

It is then easy to check that the complexity of $[\mathsf{s}]E_a\varphi$ is guaranteed to be higher than that of its translation.

Second, we replace the clause in which we translate formulas of the form $[\mathsf{s}]E_a\varphi$:

$$f^0_{n!}([\mathsf{s}]E_a\varphi) = \bigwedge_{\mathsf{x}\in\mathsf{s}}(\mathsf{pre}(\mathsf{x}) \rightarrow \bigwedge_{\mathsf{t}\in\Delta_a(\mathsf{x})} E_a f^0_{(n-1)!}([\mathsf{t}]\varphi))$$

Since this translation corresponds to the reduction equivalence for the entertain modality in IAML, this change is enough to make the proof carry over. □

We have thereby provided two complete axiomatizations of $\mathcal{L}^{\mathsf{IAML}}$, in the same way we did for $\mathcal{L}^{\mathsf{AMLQ}}$.

4.9 Comparison to AMLQ, IDEL and AML

4.9.1 AMLQ

We have seen in the examples that, although in IAML actions cannot have questions as content, we can still model the action of asking a question by having separate actions for each resolution of the question. In this section, we will show that in fact everything we can encode in AMLQ action models can also be encoded by an IAML action model.

We will first define a method of translating any AMLQ action model to an IAML action model in such a way that updating any inquisitive epistemic model with either of the two action models has the same updated model as a result.

Definition 4.9.1. **Translation of AMLQ action model to IAML action model**
Let $\mathsf{M} = \langle \mathsf{S}, \{\sim_a \mid a \in \mathcal{A}\}, \mathsf{cont}\rangle$ be an AMLQ action model. We will define a corresponding IAML action model M^*.

First, we define a function C that maps each action to the set of all combinations of resolutions of its content. For every action $\mathsf{x} \in \mathsf{S}$, let $\mathsf{cont}(\mathsf{x})^* \in \mathcal{L}^{\mathsf{IAML}}$ be a formula that is equivalent to $\mathsf{cont}(\mathsf{x})$.[1]

$$\text{For every } \mathsf{x} \in \mathsf{S},\ \text{let } C(\mathsf{x}) := \wp(\mathcal{R}(\mathsf{cont}(\mathsf{x})^*)) - \emptyset$$

Note that every member of $C(\mathsf{x})$ is a set of statements. For each set $C(\mathsf{x})$, let $C(\mathsf{x}) = \{\Gamma^{\mathsf{x}}_1, \ldots, \Gamma^{\mathsf{x}}_n\}$. Then let $\overline{\Gamma^{\mathsf{x}}_i} = \mathcal{R}(\mathsf{cont}(\mathsf{x})^*) - \Gamma^{\mathsf{x}}_i$ be the set of resolutions of the content of x that are not in Γ^{x}_i for this particular i.

Let the IAML action model $\mathsf{M}^* = \langle \mathsf{S}^*, \{\Delta^*_a \mid a \in \mathcal{A}\}, \mathsf{cont}^*\rangle$ be defined as follows:

- $\mathsf{S}^* := \{\mathsf{x}_i \mid \mathsf{x} \in \mathsf{S} \text{ and } \Gamma^{\mathsf{x}}_i \in C(\mathsf{x})\}$
- $\mathsf{s} \in \Delta^*_a(\mathsf{x}_i)$ iff
  - For all $\mathsf{y}_j \in \mathsf{s} : \mathsf{x} \sim_a \mathsf{y}$
  - For all $\mathsf{y}_j, \mathsf{y}'_k \in \mathsf{s} : \mathsf{y} = \mathsf{y}'$
  - There is some α such that for all $\mathsf{y}_j \in \mathsf{s} : \alpha \in \Gamma^{\mathsf{y}}_j$
- For every $\mathsf{x}_i \in \mathsf{S}^*$, $\mathsf{cont}^*(\mathsf{x}_i) := \bigwedge \Gamma^{\mathsf{x}}_i \wedge \bigwedge_{\alpha \in \overline{\Gamma^{\mathsf{x}}_i}} \neg\alpha$

Before we prove that the updated models produced by M and M^* are isomorphic, we will first show that the way we defined the translation method makes sure that any M^* is actually a proper IAML action model according to Definition 4.2.1.

Proposition 4.9.1. **M^* is an IAML action model**
For every AMLQ action model M, M^* is an IAML action model.

Proof: Take any AMLQ action model $\mathsf{M} = \langle \mathsf{S}, \{\sim_a \mid a \in \mathcal{A}\}, \mathsf{cont}\rangle$ and the corresponding $\mathsf{M}^* = \langle \mathsf{S}^*, \{\Delta^*_a \mid a \in \mathcal{A}\}, \mathsf{cont}^*\rangle$.

As the content of the actions in M^* is built up from declaratives using conjunction and negation, it is clear that the contents of its actions are itself always declaratives. We only need to show that for each action x_i, $\Delta^*_a(\mathsf{x}_i)$ is non-empty and downward closed and that it satisfies factivity and introspection.

Take any $\mathsf{s} \in \Delta^*_a(\mathsf{x}_i)$ and any $\mathsf{t} \subseteq \mathsf{s}$. It is easy to check that if s satisfies the conditions to be in $\Delta^*_a(\mathsf{x}_i)$, then so does t. So $\Delta^*_a(\mathsf{x}_i)$ is downward closed.

Next, to show factivity and non-emptiness, consider the singleton set $\{\mathsf{x}_i\}$. It satisfies the first condition by reflexivity of $\sim_a$ and the second two conditions by being a singleton. So $\{\mathsf{x}_i\} \in \Delta^*_a(\mathsf{x}_i)$, which means that $\mathsf{x}_i \in \delta^*_a(\mathsf{x}_i)$.

For introspection, take any two actions $\mathsf{x}_i, \mathsf{z}_j \in \mathsf{S}^*$. Suppose $\mathsf{z}_j \in \delta^*_a(\mathsf{x}_i)$. Then by the first condition, $\mathsf{x} \sim_a \mathsf{z}$. Now take any set of action s. As $\sim_a$ is an equivalence relation and $\mathsf{x} \sim_a \mathsf{z}$, s satisfies the first condition to be in $\Delta^*_a(\mathsf{x}_i)$ just in case it satisfies the first condition to be in $\Delta^*_a(\mathsf{z}_j)$. As the other two conditions are independent of x_i and z_j, $\Delta^*_a(\mathsf{x}_i) = \Delta^*_a(\mathsf{z}_j)$. This concludes the proof that M^* is an IAML action model. □

[1] Recall that $\mathsf{cont}(\mathsf{x})$ may contain dynamic modalities of AMLQ action models. As we define an IAML action model, they cannot be part of the content of our actions here. We can always find an equivalent formula $\mathsf{cont}(\mathsf{x})^* \in \mathcal{L}^{\mathsf{IAML}}$, either by reducing $\mathsf{cont}(\mathsf{x})$ to a formula of $\mathcal{L}^{\mathsf{IEL}}$ using Theorem 3.8.1, or using the translation procedure from $\mathcal{L}^{\mathsf{AMLQ}}$ to $\mathcal{L}^{\mathsf{IAML}}$ that we define below.

We now show that updating any inquisitive epistemic model with either M or M^* gives us the same updated model.

PROPOSITION 4.9.2. **Isomorphism between updated models**
For every AMLQ action model M and its IAML counterpart M^*, for every inquisitive epistemic model M : $M \otimes \mathsf{M}$ is isomorphic to $M \otimes \mathsf{M}^*$.

Proof: Take any AMLQ action model $\mathsf{M} = \langle \mathsf{S}, \{\sim_a \ \mid a \in \mathcal{A}\}, \mathsf{cont}\rangle$.

Now take any inquisitive epistemic model $M = \langle W, \{\Sigma_a \mid a \in \mathcal{A}\}, V\rangle$. We claim that $M \otimes \mathsf{M} = \langle W', \{\Sigma'_a \mid a \in \mathcal{A}\}, V'\rangle$ is isomorphic to $M \otimes \mathsf{M}^* = \langle W'', \{\Sigma''_a \mid a \in \mathcal{A}\}, V''\rangle$.

We start by showing that $\langle w, \mathsf{x}\rangle \in W'$ just in case $\langle w, \mathsf{x}_i\rangle \in W''$ for some i.

($\Rightarrow$) Take any $\langle w, \mathsf{x}\rangle \in W'$. Then by the definition of W', $M, w \models \mathsf{pre}(\mathsf{x})$. As w is a single world, we also have $M, w \models \mathsf{cont}(\mathsf{x})$. This means that $M, w \models \alpha$ for some $\alpha \in \mathcal{R}(\mathsf{cont}(\mathsf{x}))$. Let $\Gamma = \{\alpha \in \mathcal{R}(\mathsf{cont}(\mathsf{x})) \mid M, w \models \alpha\}$ and $\overline{\Gamma} = \{\alpha \in \mathcal{R}(\mathsf{cont}(\mathsf{x})) \mid M, w \models \neg\alpha\}$.

By construction, Γ is some $\Gamma^{\mathsf{x}}_i \in C(\mathsf{x})$ and $\overline{\Gamma}$ is the corresponding $\overline{\Gamma^{\mathsf{x}}_i}$. Then by the definition of S', there is a corresponding action $\mathsf{x}_i \in \mathsf{S}^*$. We have constructed Γ and $\overline{\Gamma}$ in such a way that $M, w \models \bigwedge \Gamma^{\mathsf{x}}_i \wedge \bigwedge_{\alpha \in \overline{\Gamma^{\mathsf{x}}_i}} \neg\alpha$, so $M, w \models \mathsf{cont}^*(\mathsf{x}_i)$. Therefore $M, w \models \mathsf{pre}^*(\mathsf{x}_i)$. It follows by the definition of W'' that $\langle w, \mathsf{x}_i\rangle \in W''$.

($\Leftarrow$) Take any $\langle w, \mathsf{x}_i\rangle \in W''$. Then by the definition of W'', $M, w \models \mathsf{pre}^*(\mathsf{x}_i)$. As w is a single world, we also have $M, w \models \mathsf{cont}^*(\mathsf{x}_i)$. This means that $M, w \models \bigwedge \Gamma^{\mathsf{x}}_i$. As Γ^{x}_i is non-empty, there is some $\alpha \in \Gamma^{\mathsf{x}}_i$ such that $M, w \models \alpha$. As this α is a resolution of $\mathsf{cont}(\mathsf{x})$, $M, w \models \mathsf{cont}(\mathsf{x})$. Therefore $M, w \models \mathsf{pre}(\mathsf{x})$. It follows by the definition of W' that $\langle w, \mathsf{x}\rangle \in W'$.

To guarantee that we can make a bijection between any world $\langle w, \mathsf{x}\rangle \in W'$ and $\langle w, \mathsf{x}_i\rangle \in W''$, we also need to show that for any $\langle w, \mathsf{x}\rangle \in W'$ there is exactly one $\langle w, \mathsf{x}_i\rangle \in W''$. That is, for every $\langle w, \mathsf{x}_i\rangle, \langle w, \mathsf{x}_j\rangle \in W''$, it must be that $i = j$. To see that this is the case, suppose that $i \neq j$. Then by the definition of W'', $M, w \models \mathsf{pre}'(\mathsf{x}_i)$ and $M, w \models \mathsf{pre}^*(\mathsf{x}_j)$. It follows that $M, w \models \mathsf{cont}^*(\mathsf{x}_i)$ and $M, w \models \mathsf{cont}^*(\mathsf{x}_j)$. However, as $\Gamma^{\mathsf{x}}_i \neq \Gamma^{\mathsf{x}}_j$, there must be some α that is in one of these sets and not in the other. But then from the definition of cont^* we can see that w supports both this α and its negation. We have obtained a contradiction, so $i = j$.

From this we can conclude that if we let $f(\langle w, \mathsf{x}_i\rangle) = \langle w, \mathsf{x}\rangle$, then f is a bijection between W'' and W'. Now let us show that it is indeed an isomorphism. For this, we need to show two things:

(i) The mapping preserves the structure of the state maps. That is, if we let $f(s)$ be $\{f(w) \mid w \in s\}$, then we have:

$$s \in \Sigma''_a(\langle w, \mathsf{x}_i\rangle) \iff f(s) \in \Sigma'_a(f(\langle w, \mathsf{x}_i\rangle))$$

(ii) The mapping preserves the valuation: $V''(\langle w, \mathsf{x}_i\rangle) = V'(f(\langle w, \mathsf{x}_i\rangle))$

Given the definition of valuation, the latter is trivial, so we only show (i). Take any world $\langle w, \mathsf{x}_i\rangle \in W''$ and any state $s'' \subseteq W''$. Let $s' = f(s'')$. Then we need to show that $s'' \in \Sigma''_a(\langle w, \mathsf{x}_i\rangle) \iff s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$.

($\Leftarrow$) Assume $s'' \in \Sigma''_a(\langle w, \mathsf{x}_i\rangle)$

Then s'' satisfies conditions (i) and (ii) of Definition 4.2.2 to be in $\Sigma''_a(\langle w, \mathsf{x}_i\rangle)$. We need to show that s' satisfies conditions (i)-(iv) of Definition 3.2.4 to be in $\Sigma'_a(\langle w, \mathsf{x}\rangle)$. Since $\pi_1(s') = \pi_1(s'')$ and the first condition is the same, we only need to check conditions (ii), (iii) and (iv). If s' is empty we are already done, so assume it is not. From the fact that s'' satisfies condition (ii) of Definition 4.2.2 we know that $\pi_2(s'')$ satisfies the three conditions to be in $\Delta^*_a(\mathsf{x})$ formulated in the definition of M^*. From the first condition of that definition we have that $\forall \mathsf{y} \in \pi_2(s') : \mathsf{x} \sim_a \mathsf{y}$, which means condition (ii) is satisfied.

We know that there is exactly one $\mathsf{y} \in \pi_2(s')$ because for all $\mathsf{y}_j, \mathsf{y}'_k \in \pi_2(s'')$: $\mathsf{y} = \mathsf{y}'$, so condition (iii) is satisfied. By definition there is some α such that for all $\mathsf{y}_j \in \pi_2(s'')$, $\alpha \in \Gamma^{\mathsf{y}}_j$. Now take any $\langle v, \mathsf{y}_j\rangle \in s''$. Then $M, v \models \mathsf{cont}^*(\mathsf{y}_j)$, so $M, v \models \alpha$. As $\langle v, \mathsf{y}_j\rangle$ was chosen arbitrarily, this goes for all $v \in \pi_1(s'')$, so $M, \pi_1(s'') \models \alpha$ by the truth-conditionality of α. As $\alpha \in \mathcal{R}(\mathsf{cont}(\mathsf{y}))$, $M, \pi_1(s'') \models \mathsf{cont}(\mathsf{y})$. So condition (iv) is satisfied too.

As s satisfies conditions (i)-(iv) of Definition 3.2.4, $s \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$.

($\Rightarrow$) Assume $s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$

Then s' satisfies conditions (i)-(iv) of Definition 3.2.4 to be in $\Sigma'_a(\langle w, \mathsf{x}\rangle)$. We need to show that s'' satisfies conditions (i) and (ii) of Definition 4.2.2 as well. As the first condition is the same, we only need to check that $\pi_2(s'') \in \Delta^*_a(\mathsf{x})$.

In our definition of M^*, we formulated three conditions for a set of actions to be in $\Delta^*_a(\mathsf{x})$. These are all satisfied if s'' is empty, so suppose it is not. We have for all $\mathsf{y}_j \in \pi_2(s'') : \mathsf{x} \sim_a \mathsf{y}$ by condition (ii) of Definition 3.2.4. By condition (iii) there is just one $\mathsf{y} \in \pi_2(s')$. This guarantees us that for all $\mathsf{y}_j, \mathsf{y}'_k \in \pi_2(s'')$: $\mathsf{y} = \mathsf{y}'$.

That leaves only the third condition. By condition (iv), $M, \pi_1(s') \models \mathsf{cont}(\mathsf{y})$. This means that $M, \pi_1(s'') \models \alpha$ for some $\alpha \in \mathcal{R}(\mathsf{cont}(\mathsf{y}))$. Now take any world $\langle v, \mathsf{y}_j\rangle$ in s''. Suppose for reductio that $\alpha \notin \Gamma^{\mathsf{y}}_j$. By definition, $\alpha \in \overline{\Gamma^{\mathsf{y}}_j}$. Then from $M, v \models \mathsf{cont}^*(\mathsf{y}_j)$ we obtain $M, v \models \neg\alpha$. But as v is in $\pi_1(s'')$ and $M, \pi_1(s'') \models \alpha$, we have $M, v \models \alpha$, so we have a contradiction. This means $\alpha \in \Gamma^{\mathsf{y}}_j$. Because the world was chosen arbitrarily, this goes for all worlds in s''. Hence, $\pi_2(s'')$ satisfies the third condition.

As $\pi_2(s'')$ satisfies all three conditions to be in $\Delta^*_a(\mathsf{x})$, s'' satisfies condition (ii) of Definition 4.2.2, which means $s'' \in \Sigma''_a(\langle w, \mathsf{x}_i\rangle)$.

We have thereby shown that f is indeed an isomorphism between $M \otimes \mathsf{M}$ and $M \otimes \mathsf{M}^*$. □

PROPOSITION 4.9.3. **$s[\mathsf{M}, \mathsf{s}]$ is the image of $s[\mathsf{M}^*, \{\mathsf{x}_i \mid \mathsf{x} \in \mathsf{s}\}]$ under isomorphism f**
For every information state s: $\langle w, \mathsf{x}\rangle \in s[\mathsf{M}, \mathsf{s}] \iff \langle w, \mathsf{x}_i\rangle \in s[\mathsf{M}^*, \{\mathsf{x}_i \mid \mathsf{x} \in \mathsf{s}\}]$

Proof: We use the definition of updated states, together with Definition 4.9.1, to obtain:

$$\begin{aligned}
\langle w, \mathsf{x}\rangle \in s[\mathsf{M}, \mathsf{s}] &\iff w \in s, \mathsf{x} \in \mathsf{s} \text{ and } M, w \models \mathsf{pre}(\mathsf{x}) \\
&\iff w \in s, \mathsf{x} \in \mathsf{s} \text{ and } M, w \models \mathsf{cont}(\mathsf{x}) \\
&\iff w \in s, \mathsf{x} \in \mathsf{s} \text{ and } M, w \models \alpha \text{ for some } \alpha \in \mathcal{R}(\mathsf{cont}(\mathsf{x})) \\
&\iff w \in s, \mathsf{x} \in \mathsf{s} \text{ and } M, w \models \mathsf{cont}^*(\mathsf{x}_i) \text{ for exactly one } \mathsf{x}_i \in \mathsf{S}^* \\
&\iff w \in s, \mathsf{x} \in \mathsf{s} \text{ and } M, w \models \mathsf{pre}^*(\mathsf{x}_i) \text{ for exactly one } \mathsf{x}_i \in \mathsf{S}^* \\
&\iff \langle w, \mathsf{x}_i\rangle \in s[\mathsf{M}^*, \{\mathsf{x}_i \mid \mathsf{x} \in \mathsf{s}\}]
\end{aligned}$$ □

DEFINITION 4.9.2. **Translation of AMLQ formulas to IAML formulas**
For every $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$, its translation $\varphi^* \in \mathcal{L}^{\mathsf{IAML}}$ is defined recursively as follows, where M is an AMLQ action model and M^* its IAML counterpart.

- $p^* = p$
- $\bot^* = \bot$
- $(\varphi \circ \psi)^* = \varphi^* \circ \psi^*$ for $\circ \in \{\wedge, \rightarrow, \vee\!\!\!\vee\}$
- $(\blacksquare\varphi)^* = \blacksquare\varphi^*$ for $\blacksquare \in \{K_a, E_a\}$
- $([\mathsf{M}, \mathsf{s}]\varphi)^* = [\mathsf{M}^*, \{\mathsf{x}_i \mid \mathsf{x} \in \mathsf{s}\}]\varphi^*$

PROPOSITION 4.9.4. **Every AMLQ formula is equivalent to its IAML translation** For every $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$ and its translation $\varphi^* \in \mathcal{L}^{\mathsf{IAML}}$, $\varphi \equiv \varphi^*$.

Proof: By induction on the complexity of φ. The only step which is not immediate is the step for the dynamic modality.

Suppose φ is $[\mathsf{M}, \mathsf{s}]\psi$. By the induction hypothesis, $\psi \equiv \psi^*$. Take any state s in any inquisitive epistemic model M. Let $M' = M \otimes \mathsf{M}$ and $M'' = M \otimes \mathsf{M}^*$. Then by Proposition 4.9.2, M' and M'' are isomorphic. Furthermore, by Proposition 4.9.3, the information state $s[\mathsf{M}, \mathsf{s}]$ is the image of $s[\mathsf{M}^*, \{\mathsf{x}_i \mid \mathsf{x} \in \mathsf{s}\}]$. We use theses facts, together with the support condition of the dynamic modality, to obtain:

$$\begin{aligned} M, s \models [\mathsf{M}, \mathsf{s}]\psi &\iff M', s[\mathsf{M}, \mathsf{s}] \models \psi \\ &\iff M', s[\mathsf{M}, \mathsf{s}] \models \psi^* \\ &\iff M'', s[\mathsf{M}^*, \{\mathsf{x}_i \mid \mathsf{x} \in \mathsf{s}\}] \models \psi^* \\ &\iff M, s \models [\mathsf{M}^*, \{\mathsf{x}_i \mid \mathsf{x} \in \mathsf{s}\}]\psi^* \end{aligned}$$ □

We have thereby shown that there is a translation of all AMLQ action models to IAML action models, and of all $\varphi \in \mathcal{L}^{\mathsf{AMLQ}}$ to an equivalent $\varphi^* \in \mathcal{L}^{\mathsf{IAML}}$. Examples such as Example 4.3.1 show that the other way around is not possible, at least not for the action models, since we cannot encode the uninterested agent in an AMLQ action model. It is however possible to translate any $\varphi \in \mathcal{L}^{\mathsf{IAML}}$ to an equivalent $\varphi^* \in \mathcal{L}^{\mathsf{AMLQ}}$, but in case it involves uninterested agents, this can only be done by reducing it to a formula without dynamic modalities. We conclude that IAML is in fact more general than AMLQ.

4.9.2 IDEL, AML

In Section 3.10.1, we have defined an AMLQ action model for every public utterance in IDEL. Using these models, we provided a translation method from $\mathcal{L}^{\mathsf{IDEL}}$ formulas to $\mathcal{L}^{\mathsf{AMLQ}}$ formulas. In the previous section, we have shown how to convert AMLQ action models to IAML action models, and we have shown that all formulas of $\mathcal{L}^{\mathsf{AMLQ}}$ can be translated to an equivalent formula of $\mathcal{L}^{\mathsf{IAML}}$. This means we also have a way of encoding any public utterance in IAML, and combining the two translations yields a translation from $\mathcal{L}^{\mathsf{IDEL}}$ to $\mathcal{L}^{\mathsf{IAML}}$. However, we do not regard IAML as a conservative extension of IDEL, since a translation of IDEL-formulas like $[\mu]\varphi$ with μ a question is not as straightforward as in AMLQ.

With respect to AML, observe that we can easily rewrite Section 3.10.2 for IAML. We can simply regard a standard action model as an inquisitive action model. Then, regardless of how we encode the issues of the agents in this inquisitive action model, the update procedure will be standard with respect to knowledge. Since $\mathcal{L}^{\mathsf{AML}}$ is a subset of $\mathcal{L}^{\mathsf{IAML}}$, which consists only of truth-conditional formulas, and all connectives in AML have the same truth conditions in IAML, we get for every $\Gamma \cup \{\alpha\} \in \mathcal{L}^{\mathsf{AML}}$: $\Gamma \models_{\mathsf{AML}} \alpha \iff \Gamma \models_{\mathsf{IAML}} \alpha$. Hence, IAML is a conservative extension of AML.

4.10 Comparison to ELQm

4.10.1 Introduction

In this section, we will compare IAML to a similar system called ELQm, developed in Van Benthem & Minică (2012). The setup of this system is very similar to ours: action models are used to extend the static issue logic ELQ to a logic with dynamic modalities. It therefore makes sense to compare our results with theirs and to compare the scope of the scenarios that the two systems can encode.

The static systems IEL and ELQ, which form the basis of IAML and ELQm respectively, have already been compared extensively in Ciardelli & Roelofsen (2015). We will only briefly repeat the three most important differences, since these differences also apply to the extensions IAML and ELQm. Let us look at the definition of the static models of ELQ, called *epistemic issue models*.

Definition 4.10.1. **Epistemic Issue Model** (Van Benthem & Minică, 2012, p. 635)
An epistemic issue model is a structure $M = \langle W, \sim_a, \approx_a, V \rangle$ where:

- W is the domain of worlds;
- $\sim_a$ is an equivalence relation on W (epistemic indistinguishability) for agent a;
- $\approx_a$ is an equivalence relation on W (issue relation) for agent a;
- V is a valuation function.

Firstly, issues in ELQ are modelled as an equivalence relation on the set of worlds. This relation induces a partition: each cell in the partition is an answer to the issue. This means that alternative answers to the issue cannot overlap, in contrast to alternative answers in IEL (e.g. the issue whether p or q is the case). That makes the notion of issues in IEL strictly more general than that in ELQ.

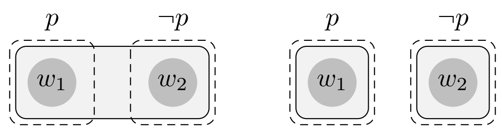
Secondly, the equivalence relation $\approx_a$ encodes only one issue for agent a, which is the same in all worlds. In IEL, agents can have different issues in different worlds. It seems more natural to assume that an agent has different issues in different possible worlds, in the same way that she might have different information there.

The third difference is in the expressiveness of the language: in ELQ, all expressions are statements, while the asking of questions can only be encoded as a special type of action. As we have seen, in IEL questions are formulas of the language in the same way statements are. This has important advantages. For instance, it allows us to express embedded questions such as $W_a?p$, and conditional questions such as $p \to ?q$.

4.10.2 Refinement

Another important difference between IEL and ELQ concerns the refinement of issues. As a consequence of the way issues are encoded in ELQ, cells of issue partitions may extend beyond the boundaries of cells of epistemic partitions, as is the case in Figure 4.7a.

In this model, the issue relation encodes that the agent is indifferent about whether p is the case. However, according to the epistemic indistinguishability relation, she *knows* whether p is the case. Therefore, what the model seems to encode is that although she happens to know whether p, she does not care whether she knows this or not.



(a) Unrefined issue relation

(b) Refined issue relation

Figure 4.7: Example of unrefined and refined issue relations in ELQ

In contrast, in IEL we do not have alternatives that include epistemically distinguishable worlds, since we ware interested in the issues that the agent entertains in view of her knowledge. From the perspective of IEL, it therefore seems to make sense to refine the issue relation in cases like these. This amounts to restricting issue cells to the boundaries of epistemic cells. In the dynamic extension of ELQ, Van Benthem & Minică (2012) provide an operation that does exactly this: they introduce an *issue refinement* action (denoted by '?') which transforms the issue relation in such a way that $\approx_? = \approx \cap \sim$. This means that the issue relation depicted in Figure 4.7a would be like Figure 4.7b after refinement.

When we interpret ELQ models as models of information flow, it is difficult to see what real world communicative action would trigger such an issue refinement action. This makes the action of issue refinement a somewhat artificial addition to the system. Notice that in IEL models, issues are refined automatically. Hence, a communicative action of stating or asking can be modelled in a straightforward way: updated models do not need any refinement afterwards.

As a consequence of the above, the only ELQ models we can translate to IEL models without loss of information are ones with refined issue relations. Whenever we translate an unrefined ELQ model to an IEL model, it is inevitable that we perform a refinement action along the way. However, given the approach to issues taken in IEL, this is quite natural. With this in mind, we can give the translation method of ELQ models to IEL models.

Definition 4.10.2. **Translation of epistemic issue model to IEL model**
Let $M = \langle W, \sim_a, \approx_a, V \rangle$ be an epistemic issue model. Then the corresponding IEL model is $M^* = \langle W, \{\Sigma_a \mid a \in \mathcal{A}\}, V \rangle$, with the state maps defined as follows:[2]

$$\Sigma_a(w) := \{s \mid s \subseteq W \text{ and for all } v, v' \in s : w \sim_a v,\ w \sim_a v' \text{ and } v \approx_a v'\}$$

What this definition says is that information states count as resolving a's issues in world w just in case: firstly, their worlds are epistemically indistinguishable from w (that is, a considers these to be possible candidates for the actual world) and secondly, all worlds in s are in the same cell in the issue partition (that is, it is not important to a *which* world in s is the actual one).

4.10.3 Dynamics

We now move on to the dynamics of ELQ. The definition of action models in ELQm is analogous to that of the static models. Knowledge and issues about the actual action are encoded in the same way, by two separate equivalence relations. As all formulas in ELQm

[2] Note that this cannot be simplified to $\{s \mid s \subseteq W \text{ and for all } v \in s : w \sim_a v \text{ and } w \approx_a v\}$, since the requirement that for all $v \in s : w \approx_a v$ would be to strong. We only need to require that the worlds in s are $\approx_a$-related to *each other*, not to w itself.

are statements, the standard notion of preconditions is enough to define action contents, just like in AML and IAML.

Definition 4.10.3. **ELQm action model** (Van Benthem & Minică, 2012, p. 654)
An ELQm action model is a structure $\mathsf{M} = \langle \mathsf{S}, \sim_a, \approx_a, \mathsf{pre} \rangle$ where:

- S is a set of events (possible future answer events);
- $\sim_a$ is an equivalence relation on S (prediction uncertainty) for agent a;
- $\approx_a$ is an equivalence relation on S (issue highlight relation) for agent a;
- $\mathsf{pre} : \mathsf{S} \to \mathcal{L}^{\mathsf{ELQm}}$ is a function that assigns a formula $\mathsf{pre}(\mathsf{x}) \in \mathcal{L}^{\mathsf{ELQm}}$ to each action point $\mathsf{x} \in \mathsf{S}$.

Notice that, since issues are encoded in action models in the same way they are encoded in epistemic models, the differences between the static models of IEL and ELQ (issues are world-independent and alternatives cannot overlap) also apply to the action models of IAML and ELQm.

The update procedure is a straightforward extension of the standard AML update procedure, with an extra clause for the issue relation.

Definition 4.10.4. **Updated ELQ model** (Van Benthem & Minică, 2012, p. 655)
Let M be an epistemic issue model and M an ELQm action model. Then $M' = (M \otimes \mathsf{M})$ is the product update of M and M, defined as follows.

$M' = \langle W', \sim'_a, \approx'_a, V' \rangle$, where:

- $W' = \{\langle w, \mathsf{x} \rangle \mid w \in W, \mathsf{x} \in \mathsf{S} \text{ and } M, w \models \mathsf{pre}(\mathsf{x})\}$
- $\langle w, \mathsf{x} \rangle \sim'_a \langle w', \mathsf{x}' \rangle$ iff $w \sim_a w'$ and $\mathsf{x} \sim_a \mathsf{x}'$
- $\langle w, \mathsf{x} \rangle \approx'_a \langle w', \mathsf{x}' \rangle$ iff $w \approx_a w'$ and $\mathsf{x} \approx_a \mathsf{x}'$
- $\langle w, \mathsf{x} \rangle \in V'(p)$ iff $w \in V(p)$

Let us also look at the syntax and semantics of the language ELQm.

Definition 4.10.5. **Syntax of $\mathcal{L}^{\mathsf{ELQm}}$** (Van Benthem & Minică, 2012, p. 657)
The language of $\mathcal{L}^{\mathsf{ELQm}}$ is defined as follows, where x is an action point within the ELQm action model M:

$$\varphi ::= i \mid p \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid U\varphi \mid K_a\varphi \mid Q_a\varphi \mid R_a\varphi \mid [\mathsf{M}, \mathsf{x}]\varphi$$

Definition 4.10.6. **Truth conditions in ELQm** (Van Benthem & Minică, 2012, p. 637, p. 657)[3]
Let w be a world in epistemic issue model M.

[3]We adapted the clause for the dynamic modality slightly to make things more clear. Here, we state explicitly that $[\mathsf{M}, \mathsf{x}]\varphi$ is true in w if $w \not\models \mathsf{pre}(\mathsf{x})$. This is implicit in the original definition, since in that case $\langle w, \mathsf{x} \rangle$ would not be in the domain of the updated model.

$$\begin{array}{lll}
M, w \models p & \text{iff} & w \in V(p) \\
M, w \models i & \text{iff} & w \in V(i) \\
M, w \models \neg\varphi & \text{iff} & M, w \not\models \varphi \\
M, w \models \varphi \wedge \psi & \text{iff} & M, w \models \varphi \text{ and } M, w \models \psi \\
M, w \models U\varphi & \text{iff} & \text{for all } w \in W : M, w \models \varphi \\
M, w \models K_a\varphi & \text{iff} & w \sim_a v \text{ implies } M, v \models \varphi \\
M, w \models Q_a\varphi & \text{iff} & w \approx_a v \text{ implies } M, v \models \varphi \\
M, w \models R_a\varphi & \text{iff} & w(\sim_a \cap \approx_a)v \text{ implies } M, v \models \varphi \\
M, w \models [\mathsf{M}, \mathsf{x}]\varphi & \text{iff} & M, w \not\models \mathsf{pre}(\mathsf{x}) \text{ or } M \otimes \mathsf{M}, \langle w, \mathsf{x}\rangle \models \varphi
\end{array}$$

The language of ELQm is just the static language of ELQ extended with a dynamic modality, in the same way that IAML extends IEL. However, the static language of ELQ includes three modalities that are not available in IEL: a universal modality U that quantifies over all worlds, a modality Q_a that quantifies over all $\approx_a$-equivalent worlds and, finally, a modality R_a that quantifies over all $(\sim_a \cap \approx_a)$-equivalent worlds. These modalities are needed to describe the issues entertained by agents in ELQ models. For instance, $U(Q_a\varphi \vee Q_a\neg\varphi)$ expresses that agent a entertains whether φ is the case (see Van Benthem & Minică (2012, p. 637)).

4.10.4 Translation

In this section, we provide a translation from ELQm to IAML, for a fragment of the set of action models and formulas.

The language of IAML is not equipped with a universal modality, Q-modality or R-modality in the syntax. This is mostly because we do not need them: we already have embedded questions and the E-modality to express the issues of agents in a way that is arguably more elegant and intuitive.[4] We will therefore only translate the fragment of ELQm that is free of the modalities U, Q and R, defined as follows.

Definition 4.10.7. **UQR-free fragment of $\mathcal{L}^{\mathsf{ELQm}}$**
For any $\varphi \in \mathcal{L}^{\mathsf{ELQm}}$, φ is UQR-free just in case it satisfies the following conditions:

(i) φ does not contain any modality U, Q or R;

(ii) For all dynamic modalities $[\mathsf{M}, \mathsf{x}]$ occurring in φ, the preconditions of the actions of M are UQR-free.

We denote the UQR-free fragment of $\mathcal{L}^{\mathsf{ELQm}}$ by $\mathcal{L}^{\mathsf{ELQm}}_{-}$.

With our restricted language in place, we can now also give the translation method for action models, which is similar to the translation we gave for the epistemic issue models.

[4] See Ciardelli & Roelofsen (2015) for a discussion. Notice that, since all IEL models have a refined issue relation, the modality Q cannot be defined in IEL, because it would collapse into R. In IEL, we cannot express whether something is true of worlds that are *just* $\approx_a$-equivalent. However, the modalities U and R could be added to IEL, and generalized to questions, with the following support conditions:

$$M, s \models U\varphi \iff M, W \models \varphi$$
$$M, s \models R_a\varphi \iff \text{for all } w \in s : \text{for all } t \in \mathsf{alt}(\Sigma_a(w)) : w \in t \text{ implies } M, t \models \varphi$$

Definition 4.10.8. **Translation of ELQm action model to IAML action model**
Let $\mathsf{M} = \langle \mathsf{S}, \sim_a, \approx_a, \mathsf{pre}\rangle$ be an ELQm action model. Then the corresponding IAML action model is $\mathsf{M}^* = \langle \mathsf{S}, \{\Delta_a \mid a \in \mathcal{A}\}, \mathsf{cont}\rangle$ is defined as follows:

- $\Delta_a(\mathsf{x}) := \{\mathsf{s} \mid \mathsf{s} \subseteq \mathsf{S}$ and for all $\mathsf{y}, \mathsf{y}' \in \mathsf{s} : \mathsf{x} \sim_a \mathsf{y}$, $\mathsf{x} \sim_a \mathsf{y}'$ and $\mathsf{y} \approx_a \mathsf{y}'\}$
- For every $\mathsf{x} \in \mathsf{S}$, $\mathsf{cont}(\mathsf{x}) := \varphi \in \mathcal{L}^{\mathsf{IAML}}$ such that $\varphi \equiv \mathsf{pre}(\mathsf{x})$[5]

We can now show that epistemic actions in ELQm can be modelled in IAML as well, and that this gives us the same updated models.

Proposition 4.10.1. **Actions in ELQm can be simulated in IAML**
For every epistemic issue model M and every ELQm action model M, $(M \otimes \mathsf{M})^* = M^* \otimes \mathsf{M}^*$.

Proof: We define the following models:

- $M = \langle W, \sim_a, \approx_a, V\rangle$ is our original epistemic issue model
- $\mathsf{M} = \langle \mathsf{S}, \sim_a, \approx_a, \mathsf{pre}\rangle$ is the ELQm action model
- $M' = M \otimes \mathsf{M} = \langle W', \sim'_a, \approx'_a, V'\rangle$ is the result of the update performed using the ELQm update procedure
- $M'' = M^* \otimes \mathsf{M}^* = \langle W'', \{\Sigma''_a \mid a \in \mathcal{A}\}, V''\rangle$ is the result we get from performing the update using the IAML update procedure, after first translating the models M and M to the inquisitive setting.

We claim that $M'^* = M''$. As it is immediate from the definitions that their domains and valuations are the same, we only need to check that for every agent $a \in \mathcal{A}$ and for every world $\langle w, \mathsf{x}\rangle$, $\Sigma'^*(\langle w, \mathsf{x}\rangle) = \Sigma''(\langle w, \mathsf{x}\rangle)$.

($\subseteq$) Assume $s \in \Sigma'^*(\langle w, \mathsf{x}\rangle)$

Then by Definition 4.10.2, the model M' is such that for all $\langle v, \mathsf{y}\rangle, \langle v', \mathsf{y}'\rangle \in s$, we have:

- $\langle w, \mathsf{x}\rangle \sim_a \langle v, \mathsf{y}\rangle$
- $\langle w, \mathsf{x}\rangle \sim_a \langle v', \mathsf{y}'\rangle$
- $\langle v, \mathsf{y}\rangle \approx_a \langle v, \mathsf{y}\rangle$

By Definition 4.10.4, for all $\langle v, \mathsf{y}\rangle, \langle v', \mathsf{y}'\rangle \in s$:

- $w \sim_a v$ and $\mathsf{x} \sim_a \mathsf{y}$
- $w \sim_a v'$ and $\mathsf{x} \sim_a \mathsf{y}'$
- $v \approx_a v'$ and $\mathsf{y} \approx_a \mathsf{y}'$

This means that for any $v, v' \in \pi_1(s) : w \sim_a v$, $w \sim_a v'$ and $v \approx_a v'$ (in M), so by Definition 4.10.2, $\pi_1(s) \in \Sigma_a(w)$ (in M^*). Similarly, for any $\mathsf{y}, \mathsf{y}' \in \pi_2(s)$: $\mathsf{x} \sim_a \mathsf{y}$, $\mathsf{x} \sim_a \mathsf{y}'$ and $\mathsf{y} \approx_a \mathsf{y}'$ (in M). Therefore, by Definition 4.10.8, $\pi_2(s) \in \Delta_a(\mathsf{x})$ (in M^*).

Since $\pi_1(s) \in \Sigma_a(w)$ and $\pi_2(s) \in \Delta_a(\mathsf{x})$, by Definition 4.2.2 we have $s \in \Sigma''_a(\langle w, \mathsf{x}\rangle)$.

($\supseteq$) Assume $s \in \Sigma''(\langle w, \mathsf{x}\rangle)$.

Then by Definition 4.2.2, $\pi_1(s) \in \Sigma_a(w)$ (in M^*) and $\pi_2(s) \in \Delta_a(\mathsf{x})$ (in M^*).

[5] Note that as $\mathsf{pre}(\mathsf{x})$ is a formula of $\mathcal{L}_{-}^{\mathsf{ELQm}}$, it might contain a dynamic modality that refers to an ELQm action model M. In that case we translate this to M^*.

By Definition 4.10.2 this means that for any $v, v' \in \pi_1(s)$, $w \sim_a v$, $w \sim_a v'$ and $v \approx_a v'$ (in M). Similarly, by Definition 4.10.8 this means that for any $\mathsf{y}, \mathsf{y}' \in \pi_2(s)$, $\mathsf{x} \sim_a \mathsf{y}$, $\mathsf{x} \sim_a \mathsf{y}'$ and $\mathsf{y} \approx_a \mathsf{y}'$ (in M).

Take an arbitrary $\langle v, \mathsf{y}\rangle, \langle v', \mathsf{y}'\rangle \in s$. Then as $w \sim_a v$ and $\mathsf{x} \sim_a \mathsf{y}$, by Definition 4.10.4 we have $\langle w, \mathsf{x}\rangle \sim'_a \langle v, \mathsf{y}\rangle$. Similarly, as $v \approx_a v'$ and $\mathsf{y} \approx_a \mathsf{y}'$, we have $\langle v, \mathsf{y}\rangle \approx'_a \langle v', \mathsf{y}'\rangle$. As $\langle v, \mathsf{y}\rangle$ and $\langle v', \mathsf{y}'\rangle$ were chosen arbitrarily, this goes for all worlds in s, so by Definition 4.10.2, we have $s \in \Sigma'^*(\langle w, \mathsf{x}\rangle)$ □

Using the proof above, we can now show that all formulas of $\mathcal{L}_{-}^{\mathsf{ELQm}}$ can indeed be translated to IAML, while keeping the same truth conditions.

Definition 4.10.9. **Translation of ELQm formulas to IAML formulas**
For every $\varphi \in \mathcal{L}_{-}^{\mathsf{ELQm}}$, its translation $\varphi^* \in \mathcal{L}^{\mathsf{IAML}}$ is defined recursively as follows:

- $p^* = p$
- $\bot^* = \bot$
- $(\neg\varphi)^* = \neg\varphi^*$
- $(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$
- $(K_a\varphi)^* = K_a\varphi^*$
- $([\mathsf{M}, \mathsf{x}]\varphi)^* = [\mathsf{M}^*, \mathsf{x}]\varphi^*$

Proposition 4.10.2. **Every ELQm formula is equivalent to its $\mathcal{L}^{\mathsf{IAML}}$ translation**
Let w be a world in epistemic issue model M. If $\varphi \in \mathcal{L}_{-}^{\mathsf{ELQm}}$ then:

$$M, w \models \varphi \iff M^*, w \models \varphi^*$$

Proof: By induction on the complexity of φ. As all $\varphi \in \mathcal{L}_{-}^{\mathsf{ELQm}}$ are truth-conditional, we can compare the truth conditions of ELQm and IAML, which is trivial for the classical connectives. Furthermore, by unpacking the relevant definitions, it is easy to see that the truth conditions of the modality K_a are the same in both semantics. Therefore, the only step we need to show is the step for the dynamic modality.

Suppose φ is $[\mathsf{M}, \mathsf{x}]\psi$. By the induction hypothesis, $\psi \equiv \psi^*$. Using Proposition 4.10.1 and the semantics of dynamic modalities in ELQm and IAML, we obtain the following:

$$\begin{aligned} M, w \models [\mathsf{M}, \mathsf{x}]\psi &\iff M, w \not\models \mathsf{pre}(\mathsf{x}) \text{ or } M \otimes \mathsf{M}, \langle w, \mathsf{x}\rangle \models \psi \\ &\iff M^*, w \not\models \mathsf{pre}(\mathsf{x})^* \text{ or } (M \otimes \mathsf{M})^*, \langle w, \mathsf{x}\rangle \models \psi^* \\ &\iff M^*, w \not\models \mathsf{pre}(\mathsf{x})^* \text{ or } M^* \otimes \mathsf{M}^*, \langle w, \mathsf{x}\rangle \models \psi^* \\ &\iff M^{*\prime}, w[\mathsf{M}^*, \mathsf{x}] \models \psi^* \\ &\iff M^*, w \models [\mathsf{M}^*, \mathsf{x}]\psi^* \\ &\iff M^*, w \models ([\mathsf{M}, \mathsf{x}]\psi)^* \end{aligned}$$ □

4.10.5 Global issues in updated models

We have already seen that issues in ELQ are global, since $\approx_a$ encodes just one issue for agent a, which is the same in every world. We want to point out a subtle consequence this has to the dynamic system ELQm. Consider the following example from Van Benthem & Minică (2012, p. 656), in which either the question whether p or the question whether q is asked, and agent a knows which. Then by the update procedure of ELQm, the updated model will be as in Figure 4.8. This looks exactly the same as it would look in IAML.

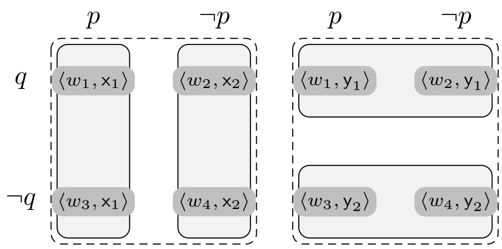


Figure 4.8: Updated model in ELQm

Now let us assume that the actual world is $\langle w_1, \mathsf{x}_1\rangle$. Then in IAML, it is true that a wants to know $?p$ and not $?q$. However, since issues are not relativized to worlds in ELQ models, we interpret this picture differently there: although a knows that the question was about p, the cells on the right, that encode the issue about q, still count as resolving her issue. This happens because there is no principle that tells us only to look at the cells within the dashed lines (the $\sim_a$-equivalent worlds).

This is a rather subtle difference between ELQm and IAML, which is easily overlooked but quite important: because issues are defined globally, they do not depend in the same way on which question was actually asked. If we interpret this model in the way that is suggested by Van Benthem & Minică (2012), this seems to be an unexpected outcome.

4.10.6 Conclusion

We conclude this section with a brief summary of what we have seen when comparing IAML to ELQm.

On the static level, we have seen that the notion of issues in inquisitive logic is more general than the notion of issues on which ELQ is based. We have observed that this difference extends to the dynamic level, since the action models of ELQm and IAML are based on the same notions of issues as the respective epistemic models.

Because of this, it is impossible to translate inquisitive epistemic models and inquisitive action models to ELQ models and ELQm action models. Furthermore, the inquisitive disjunction and E-modality from IEL cannot be defined in ELQ.

However, if we restrict the set of ELQ models and ELQm action models slightly, we can simulate the dynamics of ELQm in IAML. Also, we can translate a significant fragment of the language of ELQm to IAML. This shows that, if we set aside the differences in the static languages that these logics build on, the effect of updating with an ELQm-action model can be simulated in IAML, while the converse is not always the case.

4.11 Conclusion

In this chapter, we investigated an alternative strategy of merging IEL with AML. Namely, instead of allowing questions as the contents of actions in our action models, we encoded issues about the actions in our action models. We did this by making action models inquisitive. Like in inquisitive epistemic models, agents were assigned a state map instead of just an equivalence relation.

As for the logic, we were able to build on the foundations that we have already developed for AMLQ in Chapter 3. Much of what we have already seen, like the properties of the logic in Section 4.4, could immediately be taken over.

The novelty of IAML is that we can now encode issues explicitly in our action models. This, in contrast to AMLQ, allows us to encode agents who are not necessarily curious about which action has occurred. Although questions cannot be the contents of actions in IAML, we can still encode them, by taking a different perspective on the action of asking a question. We have even shown that everything we can encode in AMLQ can be encoded in IAML. Furthermore, IAML is, just like AMLQ, a conservative extension of IEL and AML.

In our comparison with ELQm we have argued that IAML, since it inherits the more general notion of issues from IEL and the possibility to embed questions under modal operators, is arguably a more natural framework to model the dynamics of information and questions in a communicative setting. Additionally, we have shown that we can simulate the dynamics of ELQm in IAML.

Obviously, not being able to model the action of asking a question in such a direct way as in AMLQ is a limitation of IAML. This means that questions are treated differently than statements, which does not match the basic ideas behind inquisitive semantics. However, even with this limitation we have shown that IAML can encode more epistemic scenarios than AMLQ can. Moreover, what we get in return is a very elegant update procedure and a logic that is relatively easily axiomatizable.

Chapter 5

Inquisitive Action Model Logic with Questions

5.1 Introduction

In the last two chapters we have seen two natural ways of extending inquisitive epistemic logic with action models: first, we kept the structure of action models standard, and generalized the notion of action content to make it possible to model the asking of a question as a normal action. The resulting logic was Action Model Logic with Questions (AMLQ).

Then, we changed the structure of action models and modelled the asking of a question as an update with an action model that contained separate actions for the alternative answers. This approach led to Inquisitive Action Model Logic (IAML).

We argued that both systems are natural extensions of inquisitive epistemic logic, since both are conservative over IEL as well as AML. Furthermore, both logics behave well, their update procedures give intuitively correct results and they have a complete axiomatization.

However, from the perspective of the inquisitive tradition, both systems are still limited in a way: in AMLQ, we cannot encode the issues agents have about which action is actual. In IAML, the asking of a question is not modelled directly as a single action with the question as its content. Instead, it is modelled as the execution of one of a group of actions, about which the agents are uncertain, and that have statements as their contents, not questions.

Therefore, on the one hand, AMLQ does not make full use of the advantages of the structure in which knowledge and issues can be encoded in the inquisitive setting. On the other hand, IAML does not make full use of the fact that in the inquisitive setting, questions are formulas in the same way statements are. Clearly, both systems have some component that is unnatural in the inquisitive approach. With respect to this, they are each others inverse: the limitation of the one is the advantage of the other.

Because of this, it makes sense to combine the insights from both systems into an even richer system, which is what we will do in this chapter. We are not after more expressive power: we already have two systems in which we can accurately model the situation we described in our motivation for this thesis. What we want to see is if the merge of these systems gives us perhaps an even more natural result from the perspective of the inquisitive approach.

This chapter follows the structure familiar from the previous two chapters. However, we will go through many of the definitions and proofs much quicker, since they will just be repetitions of what we have already seen for the previous systems.

5.2 Definitions

5.2.1 Action model

We start again with defining our level 0 action models. We take over the structure of the action models from IAML: for each agent, we define a state map $\Delta_a(w)$ for each world. The content of the actions is taken from the definition of AMLQ: we assign a formula of $\mathcal{L}^{\mathsf{IEL}}$, which means we do not restrict this to declaratives.

Definition 5.2.1. **Inquisitive Action Model with Questions**
An IAMLQ_0 action model is a triple $\mathsf{M} = \langle \mathsf{S}, \{\Delta_a \mid a \in \mathcal{A}\}, \mathsf{cont}\rangle$, where:

- S is a finite domain of action points;
- For each $a \in \mathcal{A}$, Δ_a is a function that maps an action point to a non-empty downward closed set of sets of action points;
- $\mathsf{cont} : \mathsf{S} \to \mathcal{L}^{\mathsf{IEL}}$ is a function that assigns a content $\mathsf{cont}(\mathsf{x}) \in \mathcal{L}^{\mathsf{IEL}}$ to each action point $\mathsf{x} \in \mathsf{S}$.

5.2.2 Update procedure

We now move on to the crucial part of the definition of IAMLQ: the definition of the update procedure. Here we have to make a decision on how to work with some new situations that did not occur in the previous two systems. Consider Figure 5.1. We know how to deal with situations like in subfigure (a) and (b) (from AMLQ) and (c) (from IAML), but situations like in subfigures (d) and (e) were not possible in these systems.

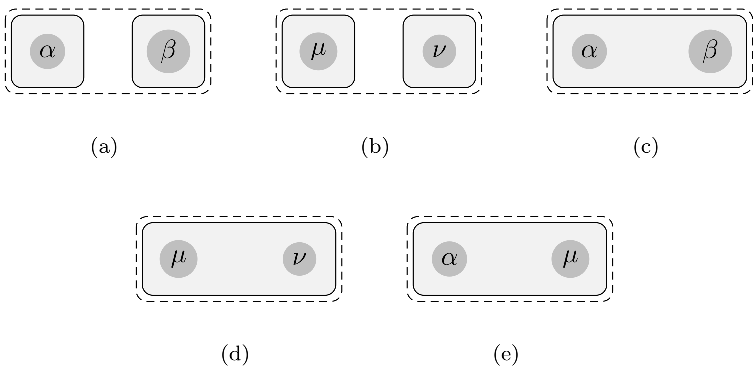


Figure 5.1: Combinations of statements, questions and issues in action models

It seems that to come up with a system that accurately computes the result from a situation like in subfigure (d) is quite difficult: for instance, suppose we have an agent who is not curious about whether the question asked was $?p$ or $?q$. We do not really have clear intuitions to tell us what her issues should be like in the updated model. Perhaps it would even seem realistic that she would have no issues at all, since she is unsure and uninterested about the question. But then, what would we do with a situation in which she is uncertain and uninterested about $?p$ and $?p \wedge q$? Arguably, it seems unintuitive that this would not raise any issue for her, since she can be sure at least about some part of the question.

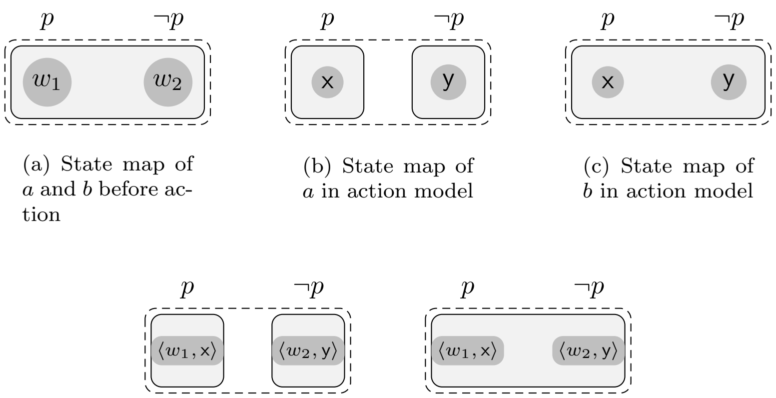
Since such cases do not seem to have just one solution, perhaps the most important result for this combined system should be that whenever we have a situation that we could also encode in one of AMLQ and IAML, the outcome should be the same. In other words, we want IAMLQ to be backwards compatible: it should not suddenly give us different results for the cases that worked fine in the previous systems.

This desideratum has two consequences: first, action models of which the contents of actions are statements should be handled just like in IAML. Second, whenever actions have questions as their contents, and the agent considers it an issue which one of the questions is being asked, the procedure should give the same results as AMLQ.

In AMLQ, information states that contain x-worlds as well as y-worlds are ruled out from the start. That made it easy to define the issues raised by the possible questions: we just said that the x-states have to resolve the question associated with x, and the y-states the question associated with y. This time, we can have information states containing both x-worlds and y-worlds, like we had in IAML.

Since we want IAMLQ to deal with statements in the way IAML does, we can take over condition (i) and (ii) from the update procedure of IAML. This update procedure provides a natural way of reflecting the issues from the original model and the action model in the updated model. The only thing it does not yet do, is make the contents of actions raise issues. This is taken care of by condition (iv) of the update procedure of AMLQ.

However, although this condition works well in AMLQ, we cannot just take over the exact formulation in IAMLQ, since it would have unwanted effects for actions with statements as their contents. To see this, consider the example in Figure 5.2, which is similar to Example 4.3.1.



(a) State map of a and b before action
(b) State map of a in action model
(c) State map of b in action model
(d) State map of a after action
(e) State map of b after action

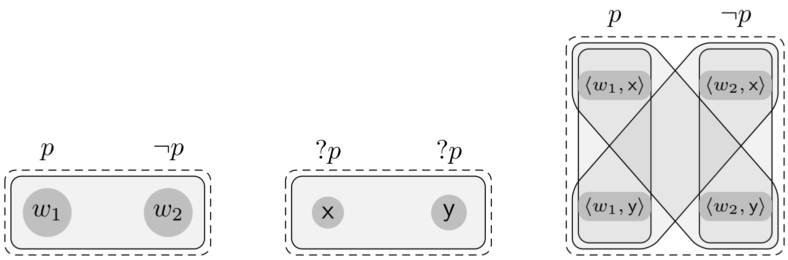
Figure 5.2: Example to show that we cannot take over condition (iv) from AMLQ in IAMLQ

Clearly, given the original state maps and the state map of b in the action model, we want the state map of b in the updated model to be as in subfigure (e). However, if we would add condition (iv), the information state $\{\langle w_1, \mathsf{x}\rangle, \langle w_2, \mathsf{y}\rangle\}$ would not satisfy this condition, because $M, \{w_1, w_2\} \not\models p$ and $M, \{w_1, w_2\} \not\models \neg p$. Therefore, the state maps of both agents would be as in subfigure (d). Apart from this being unintuitive, this would also break backwards compatibility with IAML (cf. Example 4.3.1).

It seems that we need something weaker than condition (iv). Of course, we still want an information state to support the content(s) of the associated action(s), to make sure that it resolves the question(s) asked in the associated action(s). We have seen in Chapter 3 that information states that consist of only x-worlds should support the content of x.

To keep this intact, a natural way to weaken condition (iv) is to evaluate not the information state itself, but to check whether, if we enhance the information state until it determines an action x, this enhancement supports the content of x. That is, if an information state s contains x-worlds as well as y-worlds, we check that the restriction to only the x-worlds supports the content of x, and the restriction to y-worlds supports the content of y.

This approach solves the problem of backwards compatibility. However, it has at least one very unintuitive consequence, illustrated in Figure 5.3.



(a) State map before action

(b) State map in action model

(c) State map after action

Figure 5.3: Unintuitive result of weakening condition (iv)

In this example, we have two actions with the same content. Since the agent can be sure that the content of the action is $?p$, the presence of the information states $\{\langle w_1, \mathsf{x}\rangle, \langle w_2, \mathsf{y}\rangle\}$ and $\{\langle w_2, \mathsf{x}\rangle, \langle w_1, \mathsf{y}\rangle\}$ in the updated model is unexpected, because they do not resolve the issue whether p is the case. However, when we enhance them until they determine the action, we get singleton states which do support $?p$, which is why they are allowed in the state map. This result is especially unintuitive since, had there been only one action with content $?p$, then we would have had $E_a?p$ in the updated model.[1]

This problem illustrates that the restriction we make when we determine whether an information state is allowed in a state map should perhaps not be made with respect to the *action*, but with respect to the *content* of this action.

To avoid results such as Figure 5.3, we will work out this idea. Whenever we want to check if an information state s is allowed in a state map, and it contains both x-worlds and y-worlds, we will restrict the state to the worlds in which the content of x is true to check whether this state supports the content of x, and similarly for y.

Formally, we require that for all y in $\pi_2(s)$: $M, \pi_1(s) \cap |\mathsf{cont}(\mathsf{y})|_M \models \mathsf{cont}(\mathsf{y})$. Since the truth set of cont(y) is exactly the truth set of pre(y), we can write this more elegantly as $M, \pi_1(s) \models \mathsf{pre}(\mathsf{y}) \rightarrow \mathsf{cont}(\mathsf{y})$. In this notation, it is clear that this condition does not change anything with respect to IAML, since $\mathsf{pre}(\mathsf{y}) \rightarrow \mathsf{cont}(\mathsf{y})$ is a tautology whenever cont(y) is truth-conditional.

With this new condition as our replacement of condition (iv) of AMLQ, we are now ready to define the update procedure in IAMLQ.

Definition 5.2.2. **IEL model updated with an IAMLQ action model**
Let M be an inquisitive epistemic model and M an IAMLQ_0 action model. Then $M' = (M \otimes \mathsf{M})$ is the product update of M and M, defined as follows.

[1] Similarly unintuitive results occur with two different, but logically equivalent, formulas, or with formulas of which one entails the other.

$M' = \langle W', \{\Sigma'_a \mid a \in \mathcal{A}\}, V' \rangle$, where:

- $W' = \{\langle w, \mathsf{x}\rangle \mid w \in W, \mathsf{x} \in \mathsf{S}$ and $M, w \models \mathsf{pre}(\mathsf{x})\}$
- $s \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$ iff
  - (i) $\pi_1(s) \in \Sigma_a(w)$
  - (ii) $\pi_2(s) \in \Delta_a(\mathsf{x})$
  - (iii) For all $\mathsf{y} \in \pi_2(s)$: $M, \pi_1(s) \models \mathsf{pre}(\mathsf{y}) \rightarrow \mathsf{cont}(\mathsf{y})$
- $\langle w, \mathsf{x}\rangle \in V'(p)$ iff $w \in V(p)$

Although we do not claim that this update procedure is the only possible solution, it can be shown that it satisfies the desiderata formulated above. Furthermore, we will see in Section 5.8 that the system can still be axiomatized in this way.

Proposition 5.2.1. **Updates result in inquisitive epistemic models**
For any inquisitive epistemic model M and for any IAMLQ action model M, $M' = (M \otimes \mathsf{M})$ is an inquisitive epistemic model.

Proof: We need to check that for every world $\langle w, \mathsf{x}\rangle$, $\Sigma'_a(\langle w, \mathsf{x}\rangle)$ is non-empty and downward closed. Furthermore, Σ'_a should satisfy factivity (for all $w \in W$, $w \in \sigma_a(w)$) and introspection (for all $w, v \in W$, if $v \in \sigma_a(w)$ then $\Sigma_a(v) = \Sigma_a(w)$).

Take an arbitrary world $\langle w, \mathsf{x}\rangle \in W'$.

We will first show that $\Sigma'_a(\langle w, \mathsf{x}\rangle)$ is downward closed. Take any state $s \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$ and any $t \subseteq s$.

(i) $\pi_1(t) \in \Sigma_a(w)$ since $\pi_1(t) \subseteq \pi_1(s)$ and $\Sigma_a(w)$ is downward closed.

(ii) $\pi_2(t) \in \Delta_a(\mathsf{x})$ since $\pi_2(t) \subseteq \pi_2(s)$ and $\Delta_a(\mathsf{x})$ is downward closed.

(iii) As $\forall \mathsf{y} \in \pi_2(s) : M, \pi_1(s) \models \mathsf{pre}(\mathsf{y}) \rightarrow \mathsf{cont}(\mathsf{y})$, we have $\forall \mathsf{y} \in \pi_2(t) : M, \pi_1(t) \models \mathsf{pre}(\mathsf{y}) \rightarrow \mathsf{cont}(\mathsf{y})$ by persistence of support.

This concludes downward closure. We now show factivity and non-emptiness at the same time, by showing that the state map of $\langle w, \mathsf{x}\rangle$ contains its own singleton, $\{\langle w, \mathsf{x}\rangle\}$.

(i) $\{w\} \in \Sigma_a(w)$ by factivity and downward closure of $\Sigma_a(w)$.

(ii) $\{\mathsf{x}\} \in \Delta_a(\mathsf{x})$ by factivity and downward closure of $\Delta_a(\mathsf{x})$.

(iii) By Proposition 3.2.1, $\mathsf{pre}(\mathsf{x})$ and $\mathsf{cont}(\mathsf{x})$ have the same truth conditions. This makes $\mathsf{pre}(\mathsf{x}) \rightarrow \mathsf{cont}(\mathsf{x})$ true in every singleton information state. Therefore, $M, \{w\} \models \mathsf{pre}(\mathsf{x}) \rightarrow \mathsf{cont}(\mathsf{x})$.

Since $\{w\} = \pi_1(\langle w, \mathsf{x}\rangle)$ and $\{\mathsf{x}\} = \pi_2(\langle w, \mathsf{x}\rangle)$, this means that we have $\{\langle w, \mathsf{x}\rangle\} \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$. It follows that $\langle w, \mathsf{x}\rangle \in \sigma'_a(w)$.

That leaves introspection. Take any two worlds $\langle w, \mathsf{x}\rangle$ and $\langle w', \mathsf{x}'\rangle$ from W' such that $\langle w', \mathsf{x}'\rangle \in \sigma'_a(\langle w, \mathsf{x}\rangle)$. By downward closure of $\Sigma'_a(\langle w, \mathsf{x}\rangle)$ we obtain $\{\langle w', \mathsf{x}'\rangle\} \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$. From condition (i) we learn that it must be the case that $w' \in \sigma_a(w)$. By introspection of Σ_a we obtain $\Sigma_a(w) = \Sigma_a(w')$. Via condition (ii) we can obtain in a similar way that $\Delta_a(\mathsf{x}) = \Delta_a(\mathsf{x}')$. Then it is easy to check that for all states t, t satisfies conditions (i)-(iii) for $\Sigma'_a(\langle w, \mathsf{x}\rangle)$ iff it satisfies them for $\Sigma'_a(\langle w', \mathsf{x}'\rangle)$. Therefore $\Sigma'_a(\langle w, \mathsf{x}\rangle)$ and $\Sigma'_a(\langle w', \mathsf{x}'\rangle)$ are equal. □

5.2.3 Syntax and semantics

For the definition of the syntax, we follow the same strategy as in the previous two chapters.

Definition 5.2.3. **Syntax of $\mathcal{L}^{\mathsf{IAMLQ}_0}$**
Level 0 of the language of Inquisitive Action Model Logic with Questions is defined as follows, where s is a set of action points within the IAMLQ_0 action model M:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \varphi \vee\!\!\!\vee\, \varphi \mid K_a\varphi \mid E_a\varphi \mid [\mathsf{M}, \mathsf{s}]\varphi$$

We restrict the definition of our language to the fragment that has dynamic modalities of IAMLQ_0 action models, until Section 5.5. We take over the abbreviations and notational conventions from AMLQ and IAML. The definition of updated states (Definition 3.2.6) and the support condition for dynamic modalities (Definition 3.2.7) are identical. We restate them here:

- $s[\mathsf{M}, \mathsf{s}] = \{\langle w, \mathsf{x}\rangle \in W' \mid w \in s \text{ and } \mathsf{x} \in \mathsf{s}\}$
- $M, s \models [\mathsf{M}, \mathsf{s}]\varphi \iff (M \otimes \mathsf{M}), s[\mathsf{M}, \mathsf{s}] \models \varphi$

5.2.4 Epistemic maps and state maps in updated models

As usual, we give two lemmas about epistemic maps and state maps to provide an alternative characterization of the update procedure. We start by showing that Lemma 4.2.1 carries over. That is, the addition of condition (iii) does not change anything to the way we update epistemic maps in IAMLQ.

Lemma 5.2.1. **Knowledge in updated models**
Let w be a world in an inquisitive epistemic model M, x an action in action model M and $M' = M \otimes \mathsf{M}$. Let $\langle w, \mathsf{x}\rangle$ be a world in M'. Then we have the following:

$$\sigma'_a(\langle w, \mathsf{x}\rangle) = \sigma_a(w)[\delta_a(\mathsf{x})]$$

Proof: ($\subseteq$) Take any $\langle v, \mathsf{y}\rangle \in \sigma'_a(\langle w, \mathsf{x}\rangle)$. This world is in this set because it belongs to a state that satisfies conditions (i)-(iii) of the update procedure in Definition 5.2.2. From condition (i) we obtain $v \in \sigma_a(w)$. From condition (ii) we obtain $\mathsf{y} \in \delta_a(\mathsf{x})$. These are, by Definition 3.2.6, exactly the conditions to be in $\sigma_a(w)[\delta_a(\mathsf{x})]$. So $\langle v, \mathsf{y}\rangle \in \sigma_a(w)[\delta_a(\mathsf{x})]$.

($\supseteq$) Take an arbitrary world $\langle v, \mathsf{y}\rangle \in \sigma_a(w)[\delta_a(\mathsf{x})]$. Then by Definition 3.2.6, we know that $v \in \sigma_a(w)$ and $\mathsf{y} \in \delta_a(\mathsf{x})$. It follows that the state $\{\langle v, \mathsf{y}\rangle\}$ satisfies condition (i) and (ii) of Definition 5.2.2 to be in $\Sigma'_a(\langle w, \mathsf{x}\rangle)$.

By Proposition 3.2.1, $\mathsf{pre}(\mathsf{y})$ and $\mathsf{cont}(\mathsf{y})$ have the same truth conditions. Therefore, $M, \{v\} \models \mathsf{pre}(\mathsf{y}) \to \mathsf{cont}(\mathsf{y})$, which is condition (iii).

This means that $\{\langle v, \mathsf{y}\rangle\} \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$. Hence, $\langle v, \mathsf{y}\rangle \in \sigma'_a(\langle w, \mathsf{x}\rangle)$. □

The lemma about state maps has to be slightly adapted, to reflect the addition of condition (iii).

Lemma 5.2.2. **State maps in updated models**
Let w be a world in an inquisitive epistemic model M, x an action in action model M and $M' = M \otimes \mathsf{M}$. Suppose $\langle w, \mathsf{x}\rangle \in W'$. Then we have the following:

$$s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle) \iff s' \subseteq s[\mathsf{s}] \text{ for some } s \subseteq W \text{ and } \mathsf{s} \subseteq \mathsf{S} \text{ such that:}$$

- $s \in \Sigma_a(w)$
- $\mathsf{s} \in \Delta_a(\mathsf{x})$
- For all $\mathsf{y} \in \mathsf{s} : M, s \models \mathsf{pre}(\mathsf{y}) \to \mathsf{cont}(\mathsf{y})$

Proof: ($\Rightarrow$) Assume $s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$.

Let s be $\pi_1(s')$ and s be $\pi_2(s')$. Then clearly $s' \subseteq s[\mathsf{s}]$. By Definition 5.2.2, since $s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$, it must be the case that $s \in \Sigma_a(w)$, $\mathsf{s} \in \Delta_a(\mathsf{x})$ and for all $\mathsf{y} \in \mathsf{s} : M, s \models \mathsf{pre}(\mathsf{y}) \to \mathsf{cont}(\mathsf{y})$.

($\Leftarrow$) Assume $s' \subseteq s[\mathsf{s}]$ such that $s \in \Sigma_a(w)$, $\mathsf{s} \in \Delta_a(\mathsf{x})$ and for all $\mathsf{y} \in \mathsf{s} : M, s \models \mathsf{pre}(\mathsf{y}) \to \mathsf{cont}(\mathsf{y})$.

We can show that $s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$ by checking conditions (i)-(iii) of Definition 5.2.2:

(i) As $s' \subseteq s[\mathsf{s}]$, $\pi_1(s') \subseteq s$. Since $s \in \Sigma_a(w)$, by downward closure we have $\pi_1(s') \in \Sigma_a(w)$.

(ii) As $s' \subseteq s[\mathsf{s}]$, $\pi_2(s') \subseteq \mathsf{s}$. Since $\mathsf{s} \in \Delta_a(\mathsf{x})$, by downward closure we have $\pi_2(s') \in \Delta_a(\mathsf{x})$.

(iii) We know that for all $\mathsf{y} \in \mathsf{s} : M, s \models \mathsf{pre}(\mathsf{y}) \to \mathsf{cont}(\mathsf{y})$. Since $\pi_1(s') \subseteq s$ and $\pi_2(s') \subseteq \mathsf{s}$, it follows that for all $\mathsf{y} \in \pi_2(s') : M, \pi_1(s') \models \mathsf{pre}(\mathsf{y}) \to \mathsf{cont}(\mathsf{y})$. □

5.3 Examples

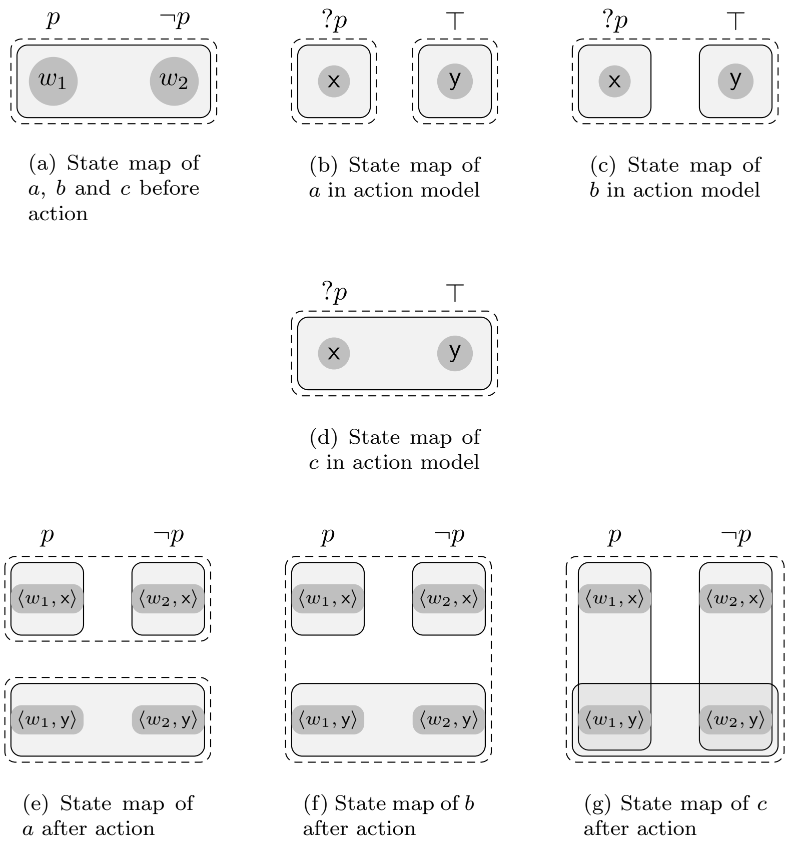
We do not repeat examples that only have statements as action contents, since in these cases the examples of IAML already count as examples of IAMLQ (see Example 4.3.1 and 4.3.2). Let us instead look at some examples with questions, to see to what extent IAMLQ can encode things that IAML or AMLQ cannot.[2]

Example 5.3.1. **Is Penny attending?**
We consider again the question whether Penny is attending ($?p$), which is communicated to Anna without Bob knowing (although he does consider it possible that this question was asked). This time, we also add the uninterested Calvin (c), who does not know what was communicated, does not consider it an issue, but knows about the possibilities. We represent all models in Figure 5.4.

In particular, we are interested in the difference between the resulting state maps of b and c. We see that, like b, c wants to know whether p, conditional on the actual action being x. However, while for b it is also a goal to find out whether x or y was the actual action, this is not the case for c: he is also fine with just knowing that the action was one of x and y.

Clearly, the outcome for a and b is what we would expect, given what we have seen in AMLQ. The outcome for c seems correct in the sense that the issue $?p$ is still encoded in the system, conditional on the action being x, and in the sense that it is not an issue for him what the actual action was. However, one may argue that it would be natural to

[2] We provide a computational tool that can be used to generate and test more examples of updates in IAMLQ. See Appendix A.



(a) State map of a, b and c before action

(b) State map of a in action model

(c) State map of b in action model

(d) State map of c in action model

(e) State map of a after action

(f) State map of b after action

(g) State map of c after action

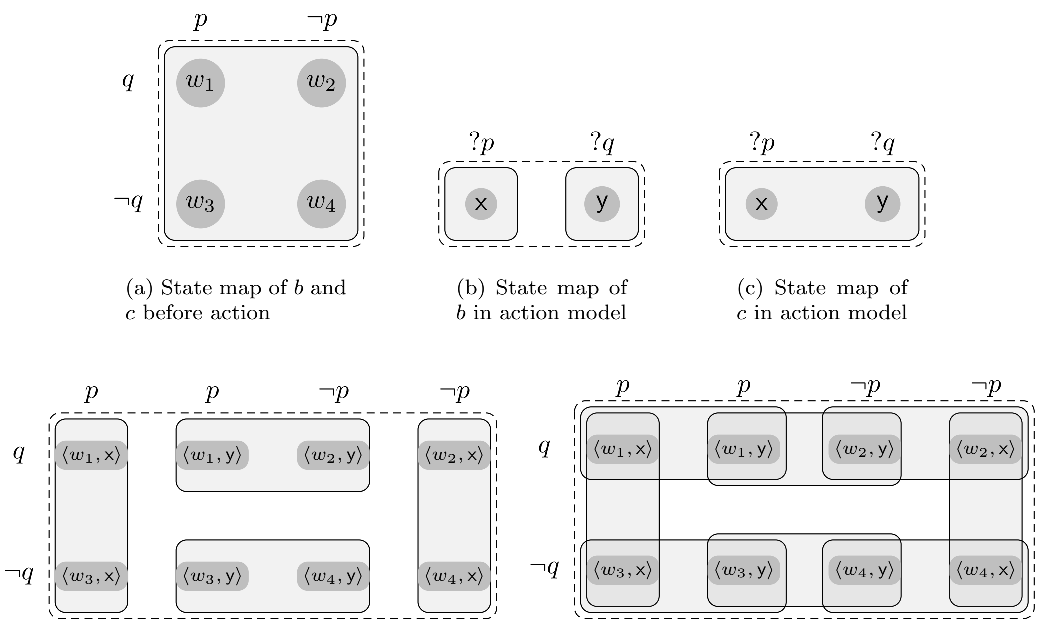
Figure 5.4: Example 5.3.1

also have $\{\langle w_1, \mathsf{x}\rangle, \langle w_1, \mathsf{y}\rangle, \langle w_2, \mathsf{y}\rangle\}$ and $\{\langle w_2, \mathsf{x}\rangle, \langle w_1, \mathsf{y}\rangle, \langle w_2, \mathsf{y}\rangle\}$ as information states that c would want to be in.[3]

Example 5.3.2. **One of two questions**
Let us have another look at an example with two questions: one of the guests calls, and Anna, Bob and Calvin know that she will either ask whether Pablo is attending ($?p$) or whether Quentin is ($?q$). We let $\mathsf{cont}(\mathsf{x}) = ?p$ and $\mathsf{cont}(\mathsf{y}) = ?q$. The diagrams can be found in Figure 5.5. For space reasons, we omit those of agent a, as we are primarily interested in the difference between b and c.

When we look at the state maps of b and c in the updated model, we see something similar to the previous example. While both want to know $?p$ conditional on the action being x and $?q$ conditional on the action being y, none of them entertain any of these issues now. While b wants to be in a state that specifies what the action was, c is fine without knowing this, which is reflected by some extra information states that c wants to be in but b does not. Notice that the states that do not specify the action still resolve both $?p$ and $?q$. This means that, if c found out that the actual action was x (for instance, an announcement of $W_a?p$ would cause him to stop considering all y-worlds), he would still start wondering whether p.

[3] There is an alternative update procedure available in which this result is indeed obtained. The idea in this update procedure is that states associated with more than one action support the content of *some* action. To maintain downward closure, we need to check that this goes for all enhancements as well. We can replace our condition (iii) with the following: for all non-empty $t \subseteq s$, for some $\mathsf{y} \in \pi_2(t) : \pi_1(t) \models \mathsf{pre}(\mathsf{y}) \rightarrow \mathsf{cont}(\mathsf{y})$. Although this is a serious alternative candidate for the update procedure we chose in this thesis, we have not yet axiomatized IAMLQ using this alternative update procedure.



(a) State map of b and c before action

(b) State map of b in action model

(c) State map of c in action model

(d) State map of b after action

(e) State map of c after action

Figure 5.5: Example 5.3.2

We have seen in these examples that IAMLQ is indeed backwards compatible with AMLQ and IAML, which was the most important desideratum we formulated. Arguably, the outcome of updates in which some agent is uninterested and considers multiple questions, is difficult to match with intuitions. However, we do see that the uninterested agent still keeps the conditional issues and that he is less interested in finding out what the actual action was.

5.4 Properties of IAMLQ

In the previous two chapters we have seen that AMLQ and IAML inherit some interesting properties from IEL. It is easy to check that the definitions and proofs from Section 3.4 and 4.4 carry over to IAMLQ. For convenience, we repeat the definitions and propositions here again, but we omit the proofs.

5.4.1 Persistence and empty state

PROPOSITION 5.4.1. **Properties of the support relation**
For all models M and formulas $\varphi \in \mathcal{L}^{\mathsf{IAMLQ}}$, we have the following properties:

- *Persistence property:* if $s \models \varphi$ and $t \subseteq s$, then $t \models \varphi$.
- *Empty state property:* $\emptyset \models \varphi$
- *Modal persistence property:* If $s \models [\mathsf{s}]\varphi$ and $\mathsf{t} \subseteq \mathsf{s}$, then $s \models [\mathsf{t}]\varphi$.
- *Modal empty state property:* $\models [\emptyset]\varphi$

5.4.2 Declaratives

Definition 5.4.1. **Declarative fragment of $\mathcal{L}^{\mathsf{IAMLQ}}$**
The set of declarative formulas $\mathcal{L}_!^{\mathsf{IAMLQ}}$ is defined inductively as follows, where $\varphi \in \mathcal{L}^{\mathsf{IAMLQ}}$:

$$\alpha ::= p \mid \bot \mid \alpha \wedge \alpha \mid \alpha \rightarrow \alpha \mid K_a\varphi \mid E_a\varphi \mid [\mathsf{s}]\alpha$$

Proposition 5.4.2. **Any $\alpha \in \mathcal{L}_!^{\mathsf{IAMLQ}}$ is truth-conditional**

5.4.3 Resolutions and normal form

Definition 5.4.2. **Resolutions in $\mathcal{L}^{\mathsf{IAMLQ}}$**
For any formula $\varphi \in \mathcal{L}^{\mathsf{IAMLQ}}$, its set of resolutions $\mathcal{R}(\varphi)$ is defined by extending Definition 2.3.10 with the following clause for the dynamic modality:

$$\mathcal{R}([\mathsf{s}]\varphi) = \{[\mathsf{s}]\alpha \mid \alpha \in \mathcal{R}(\varphi)\}$$

Proposition 5.4.3. **A formula is supported iff some resolution of it is**
For all $\varphi \in \mathcal{L}^{\mathsf{IAMLQ}}$, for every inquisitive epistemic model M and state s:

$$M, s \models \varphi \iff M, s \models \alpha \text{ for some } \alpha \in \mathcal{R}(\varphi)$$

Proposition 5.4.4. **Normal form**
For all $\varphi \in \mathcal{L}^{\mathsf{IAMLQ}}$, $\varphi \equiv \bigvee\mathcal{R}(\varphi)$.

5.4.4 Declarative variant

Definition 5.4.3. **Declarative variant in $\mathcal{L}^{\mathsf{IAMLQ}}$**
The declarative variant $\varphi^!$ of a formula $\varphi \in \mathcal{L}^{\mathsf{IAMLQ}}$ is defined by:

$$\varphi^! := \bigvee \mathcal{R}(\varphi)$$

Proposition 5.4.5. **Declarative variants in $\mathcal{L}^{\mathsf{IAMLQ}}$ have equal truth conditions**
For all $\varphi \in \mathcal{L}^{\mathsf{IAMLQ}}$, for every inquisitive epistemic model M and world w:

$$M, w \models \varphi \iff M, w \models \varphi^!$$

Proposition 5.4.6. **Any truth-conditional formula is equivalent to a declarative**
For all $\varphi \in \mathcal{L}^{\mathsf{IAMLQ}}$, if φ is truth-conditional, then there is some $\alpha \in \mathcal{L}_!^{\mathsf{IAMLQ}}$ such that $\varphi \equiv \alpha$.

5.5 Dynamic modalities in action content

We can now give the definitions for the full set of action models and the full language in the same way we did for AMLQ and IAML.

Definition 5.5.1. **Higher level Inquisitive Action Model**
For every $i > 0$, an IAMLQ_i action model is a triple $\mathsf{M} = \langle \mathsf{S}, \{\Delta_a \mid a \in \mathcal{A}\}, \mathsf{cont}\rangle$, where:

- S and Δ_a are defined as before;
- $\mathsf{cont} : \mathsf{S} \rightarrow \mathcal{L}^{\mathsf{IAMLQ}_{(i-1)}}$ is a function that assigns a content $\mathsf{cont}(\mathsf{x}) \in \mathcal{L}^{\mathsf{IAMLQ}_{(i-1)}}$ to each action point $\mathsf{x} \in \mathsf{S}$.

DEFINITION 5.5.2. **Syntax of $\mathcal{L}^{\mathsf{IAMLQ}_i}$**
For every $i > 0$, the language of Inquisitive Action Model Logic with Questions of level i is defined as follows, where s is a set of action points within the IAMLQ_j action model M, with the restriction that $j < i$:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \rightarrow \varphi \mid \varphi \vee\!\!\!\vee \varphi \mid K_a \varphi \mid E_a \varphi \mid [\mathsf{M}, \mathsf{s}]\varphi$$

DEFINITION 5.5.3. **Syntax of $\mathcal{L}^{\mathsf{IAMLQ}}$**
The full language of Inquisitive Action Model Logic with Questions is defined as the union of all $\mathcal{L}^{\mathsf{IAMLQ}_i}$ for all natural numbers i.

$$\mathcal{L}^{\mathsf{IAMLQ}} := \bigcup_{i \geq 0} \mathcal{L}^{\mathsf{IAMLQ}_i}$$

The set of all IAMLQ action models is the union of all sets of IAMLQ_i action models for $i \geq 0$. We generalize the definition of updated models and all the definitions and propositions from Section 5.4, like we did for the previous logics.

5.6 Composition of action models

As IAMLQ action models have the same structure as IAML action models, we can define the composition of action models in exactly the same way. We restate the definition below.

DEFINITION 5.6.1. **Composition of IAMLQ action models**
Let $\mathsf{M} = \langle \mathsf{S}, \{\Delta_a \mid a \in \mathcal{A}\}, \mathsf{cont}\rangle$ and $\mathsf{M}' = \langle \mathsf{S}', \{\Delta'_a \mid a \in \mathcal{A}\}, \mathsf{cont}'\rangle$ be two IAMLQ action models. Their composition $\mathsf{M};\mathsf{M}'$ is the action model $\langle \mathsf{S}'', \{\Delta''_a \mid a \in \mathcal{A}\}, \mathsf{cont}''\rangle$ such that:

- $\mathsf{S}'' = \mathsf{S} \times \mathsf{S}'$
- $\mathsf{s} \in \Delta''_a(\langle \mathsf{x}, \mathsf{x}'\rangle)$ iff $\pi_1(\mathsf{s}) \in \Delta_a(\mathsf{x})$ and $\pi_2(\mathsf{s}) \in \Delta'_a(\mathsf{x}')$
- $\mathsf{cont}''(\langle \mathsf{x}, \mathsf{x}'\rangle) = \mathsf{cont}(\mathsf{x}) \wedge [\mathsf{M}, \mathsf{x}]\mathsf{cont}'(\mathsf{x}')$

The composition of two pointed action models (M, s) and $(\mathsf{M}', \mathsf{s}')$ is the pointed action model $(\mathsf{M}'', \mathsf{s} \times \mathsf{s}')$ with $\mathsf{M}'' = \mathsf{M};\mathsf{M}'$ defined as above.

It is easy to check that any composition of IAMLQ action models is indeed an IAMLQ action model itself. Furthermore, we can check that any $(M \otimes \mathsf{M}) \otimes \mathsf{M}'$ is isomorphic to $M \otimes (\mathsf{M};\mathsf{M}')$ We omit the proofs this time, as they are very similar to the proofs in Sections 3.7 and 4.6. We end this section by stating the following equivalence for any $\varphi \in \mathcal{L}^{\mathsf{IAMLQ}}$.

PROPOSITION 5.6.1. $[\mathsf{s}][\mathsf{t}]\varphi \equiv [\mathsf{s};\mathsf{t}]\varphi$

5.7 Reduction

To make sure that we can reduce all formulas of IAMLQ to a formula of IEL, we start by checking that we can take over the equivalences from AMLQ and IAML that we formulated in the previous chapters. With the exception of the equivalence for the entertain modality, this is indeed the case. We have the following proposition.

PROPOSITION 5.7.1. The following equivalences hold in IAMLQ:

- If α is truth-conditional, then $[\mathsf{s}]\alpha \equiv \bigwedge_{\mathsf{x} \in \mathsf{s}} [\mathsf{x}]\alpha$

- $[\mathsf{x}]p \equiv \mathsf{pre}(\mathsf{x}) \to p$
- $[\mathsf{x}]\bot \equiv \neg\mathsf{pre}(\mathsf{x})$
- $[\mathsf{s}](\varphi \wedge \psi) \equiv [\mathsf{s}]\varphi \wedge [\mathsf{s}]\psi$
- $[\mathsf{s}](\varphi \vee\!\!\!\vee \psi) \equiv [\mathsf{s}]\varphi \vee\!\!\!\vee [\mathsf{s}]\psi$
- $[\mathsf{x}](\varphi \to \psi) \equiv [\mathsf{x}]\varphi \to [\mathsf{x}]\psi$
- $[\mathsf{x}]K_a\varphi \equiv \mathsf{pre}(\mathsf{x}) \to K_a[\delta_a(\mathsf{x})]\varphi$

Proof: All equivalences except the last one are established by the proofs of Propositions 3.8.1 - 3.8.6 and 3.6.2. For the knowledge modality, observe that Lemma 5.2.1 says the same as Lemma 3.2.1, so for this equivalence we can repeat the proof for Proposition 3.8.8. □

Then, like was the case for IAML, all we need to provide is a new equivalence for the entertain modality. It is no surprise that the conditions under which $[\mathsf{x}]E_a\varphi$ is true are different in IAMLQ compared to IAML, since we have strictly more actions $[\mathsf{x}]$ to consider. We have already seen that Lemma 5.2.2, which gives a characterization of the state maps in updated models, says something slightly different then Lemma 4.2.2. We adapt our equivalence by accommodating this difference.

Proposition 5.7.2. $[\mathsf{x}]E_a\varphi \equiv \mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{s}\in\Delta_a(\mathsf{x})} E_a(\bigwedge_{\mathsf{y}\in\mathsf{s}}(\mathsf{pre}(\mathsf{y}) \to \mathsf{cont}(\mathsf{y})) \to [\mathsf{s}]\varphi)$

Proof: As both formulas are declaratives, we only need to show that they have the same truth conditions.

($\Rightarrow$) Assume $M, w \models [\mathsf{x}]E_a\varphi$.

Then $M', w[\mathsf{x}] \models E_a\varphi$. Assume $M, w \models \mathsf{pre}(\mathsf{x})$. Then by the truth condition of the entertain modality, we have for all $s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle) : M', s' \models \varphi$.

Take any state s in the action model such that $\mathsf{s} \in \Delta_a(\mathsf{x})$ and any state s in the original model such that $s \in \Sigma_a(w)$.

Now take an arbitrary $t \subseteq s$ such that $M, t \models \bigwedge_{\mathsf{y}\in\mathsf{s}}(\mathsf{pre}(\mathsf{y}) \to \mathsf{cont}(\mathsf{y}))$. Then by downward closure of $\Sigma_a(w)$, $t \in \Sigma_a(w)$. So by Lemma 5.2.2, $t[\mathsf{s}] \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$. This means that $M', t[\mathsf{s}] \models \varphi$. By the support condition of the dynamic modality, we have $M, t \models [\mathsf{s}]\varphi$. Since t was an arbitrary subset of s, by the support condition of implication we have $M, s \models \bigwedge_{\mathsf{y}\in\mathsf{s}}(\mathsf{pre}(\mathsf{y}) \to \mathsf{cont}(\mathsf{y})) \to [\mathsf{s}]\varphi$.

As s was an arbitrary state in $\Sigma_a(w)$, we have:

$$M, w \models E_a(\bigwedge_{\mathsf{y}\in\mathsf{s}}(\mathsf{pre}(\mathsf{y}) \to \mathsf{cont}(\mathsf{y})) \to [\mathsf{s}]\varphi)$$

As s was chosen arbitrarily, we have:

$$M, w \models \bigwedge_{\mathsf{s}\in\Delta_a(\mathsf{x})} E_a(\bigwedge_{\mathsf{y}\in\mathsf{s}}(\mathsf{pre}(\mathsf{y}) \to \mathsf{cont}(\mathsf{y})) \to [\mathsf{s}]\varphi)$$

Then finally, we drop our assumption that $M, w \models \mathsf{pre}(\mathsf{x})$ to obtain:

$$M, w \models \mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{s}\in\Delta_a(\mathsf{x})} E_a(\bigwedge_{\mathsf{y}\in\mathsf{s}}(\mathsf{pre}(\mathsf{y}) \to \mathsf{cont}(\mathsf{y})) \to [\mathsf{s}]\varphi)$$

($\Leftarrow$) Assume $M, w \models \mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{s}\in\Delta_a(\mathsf{x})} E_a(\bigwedge_{\mathsf{y}\in\mathsf{s}}(\mathsf{pre}(\mathsf{y}) \to \mathsf{cont}(\mathsf{y})) \to [\mathsf{s}]\varphi)$.

Either $M, w \models \mathsf{pre}(\mathsf{x})$ or $M, w \not\models \mathsf{pre}(\mathsf{x})$. In the latter case, we immediately have $M, w \models [\mathsf{x}]E_a\varphi$ and we are done, so assume the former. Then we have $M, w \models \bigwedge_{\mathsf{s}\in\Delta_a(\mathsf{x})} E_a(\bigwedge_{\mathsf{y}\in\mathsf{s}}(\mathsf{pre}(\mathsf{y}) \to \mathsf{cont}(\mathsf{y})) \to [\mathsf{s}]\varphi)$.

As $M, w \models \mathsf{pre}(\mathsf{x})$, we have a world $\langle w, \mathsf{x}\rangle$ in the updated model. Take any $s' \in \Sigma'_a(\langle w, \mathsf{x}\rangle)$. Then by Lemma 5.2.2, $s' \subseteq t[\mathsf{t}]$ for some $t \in \Sigma_a(w)$ and $\mathsf{t} \in \Delta_a(\mathsf{x})$ such that $M, t \models \mathsf{pre}(\mathsf{y}) \to \mathsf{cont}(\mathsf{y})$ for all $\mathsf{y} \in \mathsf{t}$.

Since we assumed that $M, w \models \bigwedge_{\mathsf{s}\in\Delta_a(\mathsf{x})} E_a(\bigwedge_{\mathsf{y}\in\mathsf{s}}(\mathsf{pre}(\mathsf{y}) \to \mathsf{cont}(\mathsf{y})) \to [\mathsf{s}]\varphi)$, from $\mathsf{t} \in \Delta_a(\mathsf{x})$ we get $M, w \models E_a(\bigwedge_{\mathsf{y}\in\mathsf{t}}(\mathsf{pre}(\mathsf{y}) \to \mathsf{cont}(\mathsf{y})) \to [\mathsf{t}]\varphi)$. As $t \in \Sigma_a(w)$ we have that $M, t \models \bigwedge_{\mathsf{y}\in\mathsf{t}}(\mathsf{pre}(\mathsf{y}) \to \mathsf{cont}(\mathsf{y})) \to [\mathsf{t}]\varphi$. Then, since $M, t \models \mathsf{pre}(\mathsf{y}) \to \mathsf{cont}(\mathsf{y})$ for all $\mathsf{y} \in \mathsf{t}$, we have $M, t \models [\mathsf{t}]\varphi$. By the support condition of the dynamic modality, $M', t[\mathsf{t}] \models \varphi$. As $s' \subseteq t[\mathsf{t}]$, $M', s' \models \varphi$.

As s' was an arbitrary state in $\Sigma'_a(\langle w, \mathsf{x}\rangle)$, we have $M', \langle w, \mathsf{x}\rangle \models E_a\varphi$, which means that $M', w[\mathsf{x}] \models E_a\varphi$. By the truth condition of the dynamic modality, we have $M, w \models [\mathsf{x}]E_a\varphi$. □

Although the resulting formula becomes quite complex, from Lemma 5.2.2 it is in fact easy to see why this is valid. Now that we have the equivalences above, we go forward using the same procedure as in Chapter 3. We only need to adapt the proof for Theorem 3.8.1 slightly to obtain a proof for the next proposition.

THEOREM 5.7.1. **Every formula of IAMLQ is equivalent to some formula of IEL**
For any $\varphi \in \mathcal{L}^{\mathsf{IAMLQ}}$, there is some $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$ such that $\varphi \equiv \varphi^*$.

Proof: The proof proceeds exactly like the proof of Theorem 3.8.1, except for the case of the entertain modality.

(E) Suppose β^* is $E_a\chi$. Then by Proposition 5.7.1 and 5.7.2 we have:

$$[\mathsf{s}]\beta^* \equiv \bigwedge_{\mathsf{x}\in\mathsf{s}}(\mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{t}\in\Delta_a(\mathsf{x})} E_a(\bigwedge_{\mathsf{y}\in\mathsf{t}}(\mathsf{pre}(\mathsf{y}) \to \mathsf{cont}(\mathsf{y})) \to [\mathsf{t}]\chi))$$

As $\mathsf{md}(\chi) < n$, by the induction hypothesis we have some $\chi^*_{\mathsf{t}} \in \mathcal{L}^{\mathsf{IEL}}$ equivalent to each $[\mathsf{t}]\chi$. So we can let α^* be defined as:

$$\bigwedge_{\mathsf{x}\in\mathsf{s}}(\mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{t}\in\Delta_a(\mathsf{x})} E_a(\bigwedge_{\mathsf{y}\in\mathsf{t}}(\mathsf{pre}(\mathsf{y}) \to \mathsf{cont}(\mathsf{y})) \to \chi^*_{\mathsf{t}}))$$

Then α^* is a formula of a level lower than the level of α. □

5.8 Axiomatizing IAMLQ

In Chapter 4, we have already shown how to adapt the proof systems and completeness proofs given in Chapter 3 to turn them into complete proof systems for IAML. We only needed to adapt the clauses for the entertain modality, since this is the only case in which the reduction equivalences of AMLQ and IAML differ. The same is the case when we compare IAMLQ with AML or IAML. Therefore, we can follow the same procedure we followed in Chapter 4 to obtain two complete axiomatizations for IAMLQ. We skip the proofs in this section.

5.8.1 Completeness via replacement of equivalents

The proof system denoted by $\vdash_{\mathsf{IAMLQ^{RE}}}$ consists of all the inference rules for IEL (Ciardelli, 2014) and the rules in Figure 5.6. We denote the relation of inter-derivability by $\dashv\vdash_{\mathsf{IAMLQ^{RE}}}$. Using the same proof strategy as in the previous chapter, we can obtain the following theorem.

THEOREM 5.8.1. **$\mathsf{IAMLQ^{RE}}$ is sound and complete**
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{IAMLQ}}$, $\Phi \models \psi \iff \Phi \vdash_{\mathsf{IAMLQ^{RE}}} \psi$.

!Atom	!∧	!K
$\dfrac{[\mathsf{x}]p}{\mathsf{pre}(\mathsf{x}) \to p}$	$\dfrac{[\mathsf{s}](\varphi \wedge \psi)}{[\mathsf{s}]\varphi \wedge [\mathsf{s}]\psi}$	$\dfrac{[\mathsf{x}]K_a\varphi}{\mathsf{pre}(\mathsf{x}) \to K_a[\delta_a(\mathsf{x})]\varphi}$
!⊥	!→	!E
$\dfrac{[\mathsf{x}]\bot}{\neg\mathsf{pre}(\mathsf{x})}$	$\dfrac{[\mathsf{x}](\varphi \to \psi)}{[\mathsf{x}]\varphi \to [\mathsf{x}]\psi}$	$\dfrac{[\mathsf{x}]E_a\varphi}{\mathsf{pre}(\mathsf{x}) \to \bigwedge_{\mathsf{s}\in\Delta_a(\mathsf{x})} E_a(\bigwedge_{\mathsf{y}\in\mathsf{s}}(\mathsf{pre}(\mathsf{y}) \to \mathsf{cont}(\mathsf{y})) \to [\mathsf{s}]\varphi)}$
AUD	!⩛	RE
$\dfrac{[\mathsf{s}]\alpha}{\bigwedge_{\mathsf{x}\in\mathsf{s}}[\mathsf{x}]\alpha}$	$\dfrac{[\mathsf{s}](\varphi \mathbin{\vee\!\!\!\vee} \psi)}{[\mathsf{s}]\varphi \mathbin{\vee\!\!\!\vee} [\mathsf{s}]\psi}$	$\dfrac{\varphi \leftrightarrow \psi}{\chi[\varphi/p] \leftrightarrow \chi[\psi/p]}$

Figure 5.6: The inference rules for dynamic modalities in IAMLQ. The double lines indicate that the inference is allowed in both directions. The rule AUD can only be applied to declaratives α.

5.8.2 Completeness via monotonicity

Once again, we provide an alternative complete proof system for IAMLQ using monotonicity of dynamic modalities. The proof system denoted by $\vdash_{\mathsf{IAMLQ}^{!\mathsf{Mon}}}$ consists of the inference rules of $\vdash_{\mathsf{IAMLQ}^{\mathsf{RE}}}$, with !Mon (Figure 4.6) instead of RE. By adapting the proofs in the previous chapter slightly, we obtain:

THEOREM 5.8.2. **$\mathsf{IAMLQ}^{!\mathsf{Mon}}$ is sound and complete**
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{IAMLQ}}, \Phi \models \psi \iff \Phi \vdash_{\mathsf{IAMLQ}^{!\mathsf{Mon}}} \psi$.

Like for AMLQ and IAML, we have thereby provided two complete axiomatizations of IAMLQ.

5.9 Comparison

We now briefly compare IAMLQ to the other logics discussed in this thesis. Let us start with AMLQ. It is easy to transform any AMLQ action model into an IAMLQ action model. We can take over the same actions, and whenever there is uncertainty for some agent which action is the actual one, we always let this be an issue (formally: $\Delta_a(\mathsf{x}) = \{\{\mathsf{y}\} \mid \mathsf{x} \sim_a \mathsf{y}\} \cup \emptyset$). Then we have a subset of IAMLQ action models for which we can show that the two update procedures coincide. We can then give a very straightforward translation in which we only have to translate the dynamic modalities, and we can show that IAMLQ is conservative over AMLQ.

With respect to IAML, it is even easier to see that IAMLQ is a conservative extension: every IAML action model is already an IAMLQ action model, namely one without questions as the contents of actions. This makes $\mathcal{L}^{\mathsf{IAML}}$ a strict subset of $\mathcal{L}^{\mathsf{IAMLQ}}$. Since condition (iii) of the update procedure of IAMLQ is void whenever y is a statement (because this makes pre(y) $\rightarrow$ cont(y) a tautology), the update procedure of IAMLQ is conservative over that of IAML. Therefore, the logic is also a conservative extension of IAML.

Since IAMLQ is conservative over AMLQ and IAML, it is also conservative over AML and IDEL. Furthermore, everything we said in Section 4.10, when we compared IAML to ELQm, carries over to IAMLQ. We may even add, as an extra argument for IAMLQ over ELQm, that the action of asking a question can be represented in IAMLQ as a regular action with a question as its content, rather than as a special action consisting of multiple statements, like in ELQm and IAML. As we argued earlier in this chapter, this representation is more natural.

5.10 Conclusion

We have argued that a lot of relevant situations can already be accurately encoded in AMLQ or IAML. The step to IAMLQ is motivated mainly by conceptual considerations. We can profit from both the advantage of AMLQ (asking a question is a regular action, just like uttering a statement) and the advantage of IAML (we can encode issues explicitly in action models).

It may be seen as an extra advantage of IAMLQ that we can now also encode situations in which we have uncertain and uninterested agents and one of the possible actions concerns a question. However, our update procedure gives us issues in the updated model that are, arguably, difficult to match with any intuitions.

The most important contribution of this chapter is that we have formulated an update procedure that is conservative over both AMLQ and IAML. Furthermore, we have checked that all the properties we discussed for AMLQ and IAML carry over. For the axiomatization, we followed the same strategies as before, but with an altered and more complex reduction axiom for the entertain modality. The length and complexity of this reduction axiom is a nice illustration of how useful the addition of dynamic modalities in fact is.

Chapter 6

Conclusions and further work

6.1 Conclusions

Since Chapters 3, 4 and 5 end with their own conclusions with respect to the systems developed in these chapters, in this section we only repeat what we consider to be the most important contributions of this thesis in general.

We started with the observation that inquisitive dynamic epistemic logic, in which we can encode public utterances of statements and questions and compute their results, lacked the means to encode (semi-)private utterances. At the same time, (semi-)private announcements are encoded in dynamic epistemic logic using action models. We identified two different strategies towards a combination of these two insights: adapting the contents of actions and adapting the structure of action models. We developed the former in AMLQ, the latter in IAML, and both strategies together in IAMLQ.

We have seen that if we want to have questions as normal actions in our system, action contents cannot be reduced to preconditions, since preconditions are truth-conditional. Instead, we generalized this notion of action content, and retrieve preconditions as a derived notion: the presupposition that we associate with the action content.

The combination of inquisitive logic and dynamic modalities gives a new perspective on complex dynamic modalities, built up out of more than one action. While they are commonly interpreted in terms of non-deterministic actions, we interpret them in terms of partial information about the action taking place. In this way, we can make sense of complex dynamic modalities in combination with questions.

We have provided two complete axiomatizations for each of the systems. Both strategies rely on a reduction to IEL. Interestingly, we have found that formulas of the form $[\mathsf{s}](\varphi \to \psi)$ cannot be elegantly reduced. However, we have found a way around this using the normal form result familiar from inquisitive logics.

The update procedures of AMLQ, IAML and IAMLQ are all standard with respect to knowledge. This means they differ only in the issues that result from updates. These differences are reflected in their axiomatizations: they differ only in the reduction rule for formulas of the form $[\mathsf{s}]E_a\varphi$, which express which issues an agent entertains after an update.

IAMLQ broadens the scope of epistemic situations that can be modelled. The possibility to encode semi-private utterances of statements carries over from AML. However, the range of statements is extended with statements about issues (a entertains φ) and statements that embed questions (a knows whether μ), which are not available in AML.

Furthermore, while we can encode only the knowledge agents have about epistemic actions in AML, in IAMLQ we can also encode their issues: this means that we can allow uncertainty about epistemic actions to raise issues for agents, which reflect the extent to which they are interested in which action is occurring.

An even more important addition is the possibility to encode semi-private questions in IAMLQ. While we already had public questions in IDEL, we could not yet model the act of asking a question in a situation where agents are not necessarily certain about its content. At least, not the range of questions that we know from inquisitive logics, which include conditional questions and questions with presuppositions.

IAMLQ can be regarded as conservative over both AML and IDEL. We can view AML action models as IAMLQ action models and there is a straightforward way of encoding a public utterance in IAMLQ. Formulas of AML are formulas of IAML, and their semantics remain standard. Formulas of IDEL can be defined as abbreviations in IAMLQ.

This thesis is accompanied by a computational tool using which product updates in IAMLQ can be calculated. Its source code has been made available to the community (see Appendix A).

6.2 Further work

In this final section, we point out four directions for possible future research: axiomatization strategies, common knowledge and issues, doxastic logic and pragmatics. We describe each of these directions in more detail below.

6.2.1 Axiomatization strategies

We have developed two complete axiomatizations for AMLQ, IAML and IAMLQ. As we pointed out in Section 3.9, a third axiomatization strategy using the composition rule, that is applied to IDEL (Ciardelli, 2016, chapter 8) is blocked for the logics in this thesis. The reason for that is that we use a workaround using normal form, to avoid having to use a reduction axiom for formulas of the form $[\mathsf{s}](\varphi \to \psi)$. It would be interesting to investigate other workarounds for this.

Moreover, as is also suggested for IDEL by Ciardelli (2016, chapter 8), it would be interesting to see if it is possible to have a direct axiomatization like Wang & Cao (2013) develop for PAL, without resorting to a reduction to IEL, using some non-standard semantics for dynamic modalities.

6.2.2 Common knowledge and common issues

In epistemic logics, we are often interested in notions of group knowledge, of which common knowledge is a particularly important one. The information that is available to the group often explains certain epistemic puzzles. Ciardelli (2016, p. 272) defines not only a common knowledge modality K_* for IEL, but also a common entertain modality E_* to express common issues: the issues that are open for the group.

Common issues are a very interesting addition to a dynamic model of communication, since they can give us an explanation for the actions of the agents, perhaps even more so than private issues. For instance, we may assume that agents exchange information with the goal of resolving a common issue. It is then also natural to assume that if it is a private goal for some agent to find out whether p, she may want to make this a public goal, to trigger the other agents to help resolve it. This would then explain her utterance of the question $?p$.

Therefore, these operators would be an interesting addition to IAMLQ. This amounts to defining a public state map Σ_* on the initial model, the definition of which is given in Ciardelli (2016, p. 272, 274). On our action models we may then define a public state map Δ_*, which encodes common knowledge and issues about the actions, in a similar way.

However, it can be shown that Lemma 5.2.1 and Lemma 5.2.2 do not generalize to public epistemic maps and state maps: that is, we cannot characterize σ'_* and Σ'_* in this way. Therefore, the reduction axioms we have for $[\mathsf{s}]K_a\varphi$ and $[\mathsf{s}]E_a\varphi$ do not hold for $[\mathsf{s}]K_*\varphi$ and $[\mathsf{s}]E_*\varphi$. This means we do not have a complete axiomatization for this logic. We know that action model logic with common knowledge (AMC, see Van Ditmarsch et al. (2007, chapter 7)) is not reducible to epistemic logic with common knowledge, so we can safely assume that IAMLQ with common knowledge is also not reducible to IEL. Providing a completeness proof for IAMLQ with common knowledge and issues is therefore left to further research.

6.2.3 Cheating and deceiving

In this thesis we have only looked at the relatively safe realm of epistemic situations: our agents never lost track of the actual world or action. Although this has been a good starting point, it is not a very realistic assumption in a model of communication. We also want to be able to look at situations in which cheating and deceiving play a role, since these are often even more interesting.

In Ciardelli (2016, chapter 7) it is shown that when we relax the constraints on Σ_a, we can broaden our view. In particular, if we keep introspection and replace factivity with consistency (for all $w \in W, \sigma_a(w) \neq \emptyset$), we can describe models in which agents may not consider the actual world as one of the possible ones. The modality K_a can then be interpreted as expressing belief instead of knowledge. We can take over this replacement of factivity with consistency for Δ_a.

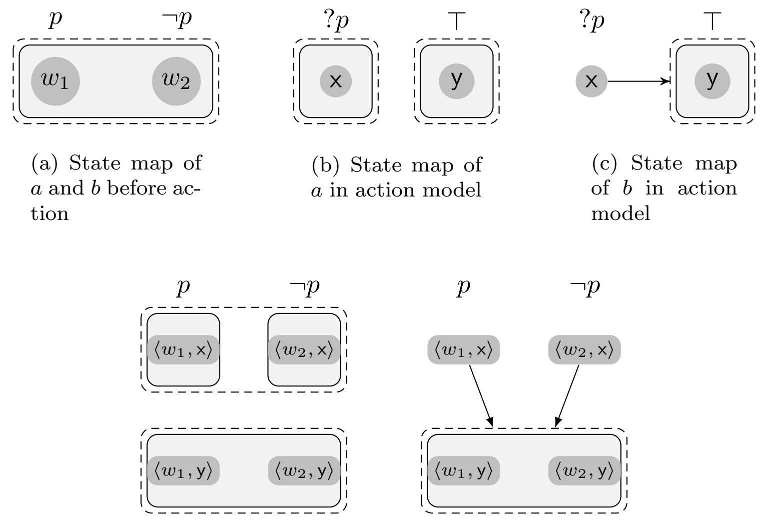
Let us look at one example of such a doxastic case, which is a completely privately asked question to agent a. Let $\mathsf{cont}(\mathsf{x}) = ?p$ and $\mathsf{cont}(\mathsf{y}) = \top$. We then simply let $\Delta_b(\mathsf{x}) = \Delta_b(\mathsf{y}) = \{\{\mathsf{y}\}\}^{\downarrow}$, while $\Delta_a(\mathsf{x}) = \{\{\mathsf{x}\}\}^{\downarrow}$ and $\Delta_a(\mathsf{y}) = \{\{\mathsf{y}\}\}^{\downarrow}$. See Figure 6.1 for a visual representation. Whenever a world does not belong to its own epistemic map, we use an arrow to indicate the epistemic map in this world.

Let us assume that the actual action was indeed x. Then, we can see in this example that since agent b does not consider the actual action possible, the updated model is a doxastic one: she does not consider the actual world anymore. This very brief illustration serves just to show that relaxing the constraints on the models gives us many more situations to encode and investigate. Further research on inquisitive doxastic logic, and in particular dynamic variants, should show what else there is to discover.

6.2.4 Pragmatics

As a fourth and final direction for further work, we suggest that IAMLQ may be used as the basis for a framework in which to investigate the pragmatics of communicative situations. For instance, in the system of Baltag (2001), action points x are assigned a sender $\mathsf{sender}(\mathsf{x})$. The advantage of this is that we can then add extra preconditions to actions.

If we assume that our model describes a communicative setting in which the agents are co-operative, we can assign an action of asking a question μ an extra precondition $\neg K_{\mathsf{sender}(\mathsf{x})}\mu$, indicating that the sender does not know the answer to the question. In this way, when some



(a) State map of a and b before action

(b) State map of a in action model

(c) State map of b in action model

(d) State map of a after action

(e) State map of b after action

Figure 6.1: Example of a doxastic case: a question asked privately to a

agent asks a question, the other agents learn that the sender does not know the answer to the question, which seems natural in a cooperative setting.

We may also assume that when someone asks a question, she knows or believes that one of the other agents knows the answer ($K_{\mathsf{sender(x)}}(K_a\mu \vee K_b\mu)$). There are much more assumptions we can make, depending on the sort of communication that we want to describe. For now, we only want to suggest that IAMLQ may be an elegant basis for more complex models in which agents are aware of such pragmatic principles of communication.

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Appendix A

Computing updates in IAMLQ

A.1 Introduction

Throughout the research for this thesis, we have considered several variants for the definitions of update procedures, especially for IAMLQ. In order to quickly test these definitions on several models, we developed a script in Python. This script includes classes in which we can encode inquisitive epistemic models and inquisitive action models, so that the product update can be calculated. The source code has been made available as an open source project, so that anyone working with dynamics in inquisitive epistemic logic can use it and contribute to it.[1]

The purpose of this appendix is to explain how the framework can be used. We do this by means of an example: we show how to define an inquisitive epistemic model and an action model, and we calculate their product update.

A.2 Inquisitive epistemic model

We start by defining the basic objects we will use in this example: we need three agents, two atomic formulas and four worlds:

```
from iamlq import *

#Define agents
a = Agent('a')
b = Agent('b')
c = Agent('c')

#Define atoms
p = Atom('p')
q = Atom('q')

#Define worlds
w1 = World('w1')
w2 = World('w2')
w3 = World('w3')
```

[1] The source code is available at github.com/thomvangessel/inquisitive-action-models

```
w4 = World('w4')
```

In order to define our initial epistemic model, we first need to define the domain, the state map and the valuation. We use a `Set` for the domain, which we initiate by giving it a `list` of worlds.

```
W = Set([w1,w2,w3,w4])
```

The `StateMap` takes a `dict` that specifies a state map for each agent, for each world. In this example we want that every agent is completely ignorant and has no issues in the initial model. Recall that this means that for each $a \in \mathcal{A}$, for each $w \in W$, $\Sigma_a(w) = \wp(W)$. We denote the powerset of W by `W.powerset()`.

```
Sigma = StateMap({
  a: {
    w1: W.powerset(),
    w2: W.powerset(),
    w3: W.powerset(),
    w4: W.powerset()
  },
  b: {
    w1: W.powerset(),
    w2: W.powerset(),
    w3: W.powerset(),
    w4: W.powerset()
  },
  c: {
    w1: W.powerset(),
    w2: W.powerset(),
    w3: W.powerset(),
    w4: W.powerset()
  }
})
```

Finally, the valuation of the initial model is a `Valuation` that takes a `dict` to specify, for each atomic formula, the `Set` of worlds in which it is true.

```
V = Valuation({
  p: Set([w1,w3]),
  q: Set([w1,w2])
})
```

Using the domain `W`, the state map `Sigma` and the valuation `V`, we can now define the initial inquisitive epistemic model `M`.

```
M = Model(W,Sigma,V)
```

We can verify our definition of the model by outputting it.

```
print M
```

In this case, this should give us the following output.

```
Domain:
{w1, w2, w3, w4}
```

```
Statemap:
a:
w1: {{w1, w2, w3, w4}}↓
w2: {{w1, w2, w3, w4}}↓
w3: {{w1, w2, w3, w4}}↓
w4: {{w1, w2, w3, w4}}↓
b:
w1: {{w1, w2, w3, w4}}↓
w2: {{w1, w2, w3, w4}}↓
w3: {{w1, w2, w3, w4}}↓
w4: {{w1, w2, w3, w4}}↓
c:
w1: {{w1, w2, w3, w4}}↓
w2: {{w1, w2, w3, w4}}↓
w3: {{w1, w2, w3, w4}}↓
w4: {{w1, w2, w3, w4}}↓

Valuation:
p: {w1, w3}
q: {w1, w2}
```

We can see above that in this framework, the output of a `Set` is similar to the notation we use in this thesis. Namely, whenever a `Set` is downward closed, only its maximal elements are printed. This makes the output much more comprehensible.

A.3 Formulas

Formulas are defined recursively. `Atom('p')`, `Top()` and `Bottom()` are formulas. If `phi` and `psi` are formulas and `a` is an agent, then the following are formulas:

- `Conjunction(phi,psi)`
- `InqDisjunction(phi,psi)`
- `Implication(phi,psi)`
- `Knows(a,phi)`
- `Entertains(a,phi)`
- `Negation(phi)`
- `Question(phi)`
- `ClassicDisjunction(phi,psi)`
- `Wonders(a,phi)`

This means that all formulas of $\mathcal{L}^{\mathsf{IEL}}$ can be defined in this framework. Consider the following examples:

```
p = Atom('p')
q = Atom('q')
a = Agent('a')

phi = Conjunction(p,Negation(q))
```

```
print phi #Outputs: (p & ¬q)
print Knows(a,Question(p)) #Outputs: Ka?p
```

A.4 Inquisitive action model

We can now specify an inquisitive action model, which we specify in a similar way as the inquisitive epistemic model.

First, we define the elements that will be in the domain of the action model. These are elements of the type `ActionPoint`. We specify a name and a content for them, the latter being a `Formula`. In this example, we let cont(x) = $?p$ and cont(y) = q.

```
x = ActionPoint('x',Question(p))
y = ActionPoint('y',q)
S = Set([x,y])
```

We now define `Delta` in the same way we defined `Sigma`. Like in most of the examples in this thesis, we want agent `a` to know which action is the actual one, while `b` and `c` do not know. We let this be an issue to `b` and not to `c`. We get the following definition.

```
Delta = StateMap({
  a: {
    x: Set([[x]]).downset(),
    y: Set([[y]]).downset()
  },
  b: {
    x: Set([[x],[y]]).downset(),
    y: Set([[x],[y]]).downset()
  },
  c: {
    x: S.powerset(),
    y: S.powerset()
  }
})
```

We then define the action model `N` and verify the definition by printing it.

```
N = ActionModel(S,Delta)
print N
```

This should give us the following output.

```
Domain:
{x, y}

Statemap:
a:
x: {{x}}↓
y: {{y}}↓
b:
x: {{x}, {y}}↓
y: {{x}, {y}}↓
c:
```

```
x: {{x, y}}↓
y: {{x, y}}↓

Contents and Preconditions:
cont(x) = ?p
cont(y) = q
pre(x) = (p v ¬p)
pre(y) = q
```

Notice that the different state maps of the agents correspond to the examples in Section 5.3. Also, notice that we do not need to specify the preconditions of the actions, since these are automatically derived from their content.

A.5 Product update

After defining the inquisitive epistemic model and action model, we can calculate their product update. We use the following notation.

```
O = M * N
```

Thereby we have defined `O`, which is of the type `Model` (an inquisitive epistemic model). This is the product update of `M` and `N` according to Definition 5.2.2. We can verify the update product by printing it.

```
print O
```

The output will be a full specification of the inquisitive epistemic model as we have seen for the initial model.

```
Domain:
{(w1,x), (w1,y), (w2,x), (w2,y), (w3,x), (w4,x)}

Statemap:
a:
(w1,x): {{(w1,x), (w3,x)}, {(w2,x), (w4,x)}}↓
(w1,y): {{(w1,y), (w2,y)}}↓
(w2,x): {{(w1,x), (w3,x)}, {(w2,x), (w4,x)}}↓
(w2,y): {{(w1,y), (w2,y)}}↓
(w3,x): {{(w1,x), (w3,x)}, {(w2,x), (w4,x)}}↓
(w4,x): {{(w1,x), (w3,x)}, {(w2,x), (w4,x)}}↓
b:
(w1,x): {{(w1,y), (w2,y)}, {(w1,x), (w3,x)}, {(w2,x), (w4,x)}}↓
(w1,y): {{(w1,y), (w2,y)}, {(w1,x), (w3,x)}, {(w2,x), (w4,x)}}↓
(w2,x): {{(w1,y), (w2,y)}, {(w1,x), (w3,x)}, {(w2,x), (w4,x)}}↓
(w2,y): {{(w1,y), (w2,y)}, {(w1,x), (w3,x)}, {(w2,x), (w4,x)}}↓
(w3,x): {{(w1,y), (w2,y)}, {(w1,x), (w3,x)}, {(w2,x), (w4,x)}}↓
(w4,x): {{(w1,y), (w2,y)}, {(w1,x), (w3,x)}, {(w2,x), (w4,x)}}↓
c:
(w1,x): {{(w1,y), (w2,y)}, {(w1,x), (w1,y), (w3,x)}, {(w2,x),
        (w2,y), (w4,x)}}↓
(w1,y): {{(w1,y), (w2,y)}, {(w1,x), (w1,y), (w3,x)}, {(w2,x),
        (w2,y), (w4,x)}}↓
(w2,x): {{(w1,y), (w2,y)}, {(w1,x), (w1,y), (w3,x)}, {(w2,x),
```

```
         (w2,y), (w4,x)}}↓
(w2,y): {{(w1,y), (w2,y)}, {(w1,x), (w1,y), (w3,x)}, {(w2,x),
         (w2,y), (w4,x)}}↓
(w3,x): {{(w1,y), (w2,y)}, {(w1,x), (w1,y), (w3,x)}, {(w2,x),
         (w2,y), (w4,x)}}↓
(w4,x): {{(w1,y), (w2,y)}, {(w1,x), (w1,y), (w3,x)}, {(w2,x),
         (w2,y), (w4,x)}}↓

Valuation:
p: {(w1,x), (w1,y), (w3,x)}
q: {(w1,x), (w1,y), (w2,x), (w2,y)}
```

If we want, we may then define another action model with which we can update `O`. Of course, the usefulness of the framework becomes much more clear when we think about more complicated examples. Since our goal here is only to explain how to use the framework, the reader is referred to the repository, which contains some more examples.