Inquisitive Conditional-Doxastic Logic

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

This thesis develops inquisitive conditional-doxastic logic, obtained by enriching classical multi-agent plausibility models with issues for each agent. The aim of these models is to allow for a finer approach to modelling inquiry.

Issues capture informative and interrogative content, and so by associating an issue to an agent we are able to capture both their information state and their inquisitive state, while a plausibility map on worlds captures their doxastic state. Moreover, inquisitive plausibility models allow for conditionalisation on both informative and interrogative content.

Two conditional-doxastic modalities are introduced and axiomatised; a *considers* modality unique to inquisitive conditional-doxastic logic, which conditionalises on issues with respect to both an agent's doxastic and inquisitive state, and a generalisation of (conditional) belief, which conditionalises solely on an agent's doxastic state.

We show that inquisitive conditional-doxastic logic encodes the same assumptions concerning conditionalisation and conditional-doxastic logic. And, just as conditional-doxastic logic may be taken as the static counterpart to a dynamic logic of belief revision, inquisitive conditional-doxastic logic can be taken as the static counterpart to a dynamic logic of belief revision within an enriched setting that includes formal resources to model interrogatives.

Inquisitive conditional-doxastic logic is shown to be sound and complete with respect to inquisitive plausibility models, and it is shown that both (conditional) belief and knowledge modalities can be defined in terms of the considers modality. However, we also show that the entertains modality of inquisitive epistemic logic cannot be defined by the considers modality. Therefore, as the basic inquisitive conditional-doxastic logic is axiomatised solely by the considers modality over the base inquisitive semantics it cannot be used to reason about an agent's inquisitive state independently of their doxastic state.

For this reason, we also axiomatise inquisitive plausibility logic and show it is sound and complete with respect to the same class of inquisitive plausibility models. This logic introduces modalities which restrict the issue associated to each agent to the worlds considered at least as plausible as the current world or state of evaluation, and those strictly less plausible. The entertains modality of inquisitive epistemic logic is taken as basic in the axiomatisation of inquisitive plausibility logic, and we show the considers modality of basic inquisitive conditional-doxastic logic is definable. Therefore, inquisitive plausibility logic allows for a full study of the interaction between epistemic and conditional-doxastic modalities within the framework of inquisitive semantics.

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Contents

Introduction

Inquisitive semantics is a semantic framework based on a notion of meaning intended to capture both informative and inquisitive content, formally modelled via *issues*, in order to analyse information exchange—seen as process of raising and resolving issues.

We restrict ourselves to a propositional language in this thesis, which allows us to characterise issues as downward closed sets of sets of possible worlds. The logic InqD (Ciardelli, Groenendijk, and Roelofsen 2015) captures the relation of entailment between issues characterised this way. InqD is introduced and outlined in chapter 1.

Inquisitive epistemic logic (Ciardelli 2014b) extends InqD to an epistemic setting, by associating to each of a set of agents an issue, intended to capture the information available to an agent and the issues they entertain.

In this way the relationship between inquisitive epistemic logic and InqD parallels the relationship between epistemic logic and propositional logic (cf. Ciardelli 2015, §7.2) For, in epistemic logic an information state—a set of possible worlds—is associated to each agent in order to capture the information available to them. Using the information states associated to agent's in epistemic logic the propositional attitude of knowledge can be formally modelled. Following this parallel, inquisitive epistemic logic models the propositional attitudes of both knowledge and of *entertaining* an issue, with the attitude of entertaining standing in the same relation to issues as knowledge does to classical proposition. Chapter 2 introduces epistemic logic in section 1 and inquisitive epistemic logic in section 2.

However, epistemic logic does not have sufficient expressive power to capture the important propositional attitude of *belief*. This is captured by *doxastic* logic, or Moreover, an *epistemic-doxastic* logic may be used to capture to interaction between the propositional attitudes of knowledge and belief (cf. Stalnaker 2006). So, a natural progression of inquisitive epistemic logic may be to investigate a logic capturing the proposition attitude with respect to issues which parallels belief, and its interaction with the attitude of entertaining.

Alternatively, one may study the attitude of conditional belief—intending to capture what an agent believes conditional upon certain information being true, and how this relates to other conditional beliefs. From this point of view an analysis of belief is a special case of conditional belief, when no information is conditionalised on. Not only does this approach attempt to analyse a more general propositional attitude, but from a formal point of view the resulting logic allows for the reduction of a dynamic logic of conditional belief, where agent's may learn new information, to be reduced to a static logic. This establishes a strong connexion between conditional belief and belief revision (indeed, conditional beliefs can be thought of as a 'plan' for what will be believed in the information being conditionalised on is established to be true).

We take the latter approach in this thesis, constructing parallel to conditional-doxastic logic (Baltag and Smets 2006) within an inquisitive setting, and introduce the propositional attitude of (conditional) *considering* an issue in parallel to believing a classical proposition. This is the subject of chapter 3, which constitutes primary theoretical chapter of this thesis. Section 3.1 introduces conditional-doxastic logic, which is then generalised to an inquisitive setting in section 3.2, termed *inquisitive conditional-doxastic logic*.

The primary semantic structures for interpreting inquisitive conditional-doxastic logic are introduced in section 3.3. These are a generalisation of the plausibility models used to interpret conditional-doxastic logic, termed inquisitive plausibility models.

Given a base semantic structure section 3.3.1 contains a detailed study of (conditional) considering, while section 3.3.2 investigates the properties of conditional belief with respect to issues. Still, the theoretical point of view adopted is largely a formal one. We do not aim to connect the formally defined attitude of considering an issue with an informal concept, nor do we look to defend the assumptions made by Baltag and Smets in the base conditional-doxastic logic. These considerations are important, but due to the complexity of axiomatising inquisitive conditional-doxastic logic are beyond the scope of this thesis.

However, our axiomatisation of inquisitive conditional-doxastic logic allows us to observe that the interrogative content of issues can be 'factored out' of the process of conditionalisation. Importantly this establishes that the notion of conditionalisation captured by inquisitive conditional-doxastic logic is the same as that captured by conditional-doxastic logic. This is one of the key results of the thesis—belief revision as modelled by conditional-doxastic logic can be straightforwardly generalised to a notion of meaning that captures information and inquisitive content, and moreover the process of conditionalisation this relies upon can be isolated to the informative content of the propositions (i.e. issues) used to modelled this notion of meaning. Inquisitive conditional-doxastic logic, and this process of conditionalisation is the concern of the following three chapters.

The completeness of inquisitive conditional-doxastic logic with respect to the class of inquisitive plausibility models is established via showing that inquisitive conditional-doxastic logic is complete with respect to an alternative class of models, detailed in chapter 4 and the transformations between the two types of models is detailed in chapter 5. This strategy follows the approach of Baltag and Smets (2006) to the completeness of conditional-doxastic logic with respect to plausibility models. Still, we do pause in section 4.1 to note how the alternative semantic structures can offer a different perspective on inquisitive conditional-doxastic logic.

Chapter 6 then establishes the soundness and completeness of inquisitive conditional-doxastic logic with respect to both classes of models via the results of the preceding chapters.

However, while conditional-doxastic logic axiomatises both conditional belief and knowledge with respect to plausibility models, inquisitive conditional-doxastic logic fails to axiomatise entertaining with respect to inquisitive plausibility models. Moreover, we show in section 5.3 of chapter 5 that the modality corresponding to entertaining is not definable in terms of the modality correspond to considering (while in section 3.4.2 of chapter 3 we show that the modalities corresponding to conditional belief and knowledge on inquisitive plausibility models *are* definable in terms of conditional belief).

This leads us to explore and axiomatise inquisitive plausibility logic in chapter 7, in which the modalities corresponding to entertaining and considering can both be captured. Moreover, inquisitive plausibility logic is shown to be sound and complete with respect to inquisitive plausibility models. This allows the interaction between the two inquisitive attitudes—entertaining and considering—to be explored in full in future work.

Chapter 1

Inquisitive Semantics

1.1 Inquisitiveness, informally

At the core of inquisitive semantics is a notion of meaning, intended to provide an adequate foundation for the analysis of the exchange of information in linguistic discourse. A fundamental distinction is made between the *informative* and *inquisitive* content of a sentence. The former is understood as what must be the case if the sentence is true, and the latter as information sufficient to resolve the issue raised by the sentence. For example, in uttering '*has Sue contacted you yet?*,' I presuppose that Sue has intended to contact you, and I also raise the issue of whether or not she has done. Moreover, it seems I *request* an informative response to the issue raised, and thus the sentence contains inquisitive content. Contrast this to an utterance of '*Sue intends to contact you*,' corresponding to the presupposition on the previous utterance which contains only informative content, and invites no response. However, as Ciardelli, Groenendijk, and Roelofsen (2012, §6.2) note, when raising an issue a speaker may also merely *invite* a response to the issue raised. Not all interrogatives thus identified require a response.

Classically, the meaning of a sentence is identified with its *truth-conditions*, relative to a world of evaluation for intensional constructs. This gives an account of informative content; the restriction of logical space (conceived as a set of possible worlds) such that the sentence is true given any given world.

Worlds are maximally specific, in the sense that either φ or $\neg\varphi$ holds at a world for any given sentence φ, and therefore the truth conditions of a sentence partition logical space into two sets—those in which the sentence is true, and those in which the sentence is false. Given that worlds are maximally specific it follows that every interrogative is resolved in each worlds. For example, given a world w it will either be the case that Sue has contacted you, or that Sue has not contacted you, whence the issue raised by the interrogative '*has Sue contacted you yet?*' is settled.

However, given a set of worlds s it may be the case that in some of the worlds in s Sue has contacted you and in other Sue has not. Therefore, by evaluating sentences relative to a collection of worlds can be thought of corresponding to evaluating a sentence with respect to incomplete information. These information states can be characterised by the propositions true in all and only the worlds of the state. This perspective is familiar from epistemic logic, where the epistemic state of an agent is modelled by a collection of worlds, and an agent knows φ just in case φ is true at every world of an agent's information state. And so an agent will fail to know φ if there are two worlds in an agent's epistemic state which differ with respect to the truth of φ. Following this intuition we think of sets of worlds as *information states*. If an information state contains a single world, then that state has maximal information. However, if the information state contains multiple worlds, then it contains only partial information, crucially missing additional content that would allow one to distinguish between them.

Therefore, by evaluating sentences from arbitrary collections of worlds it is not guaranteed that there will be sufficient information to establish the information sufficient to resolve an issue, and so information states do not in general contain sufficient content to resolve any arbitrary interrogative.

This method of evaluation gives rise to a notion of support with respect to both informative and inquisitive content. In the case of the former an information state entails the informative content of a sentence just in case every world in the information state is one in which its informative content is true. While in the case of the latter, an information state entails the inquisitive content of a sentence just in case the information state entails at least one resolution of an interrogative, which in turn reduces to at least one resolution of the interrogative being true in every world of the information state.

The notion of support will be formalised in the following section by defining support conditions for formulas constructed from a dichotomous syntax of declaratives and interrogatives. Still, both types of formula will express a singular type of proposition, consisting of the information states in which the formula is supported. Coupled with a collection of rules of inference this system comprises the logic termed InqD, first detailed in Ciardelli, Groenendijk, and Roelofsen (2015). InqD will form the core of our investigations into inquisitive conditional-doxastic logic and inquisitive plausibility logic. However, the underlying conception of a proposition, how these ought to relate to one another, and an alternative (but equivalent) logic, termed InqB this gives rise to, was first presented in Ciardelli and Roelofsen (2009) and systematically investigated in Ciardelli (2009). Ciardelli and Roelofsen (2011) is a concise introduction to InqB and summarises many important results and properties of this logic.

1.2 InqD

Logical Language

The choice of InqD as the base language to formalise the propositional case of inquisitive semantics allows us to make a distinction between declaratives and interrogatives at a syntactic level. The language is constructed by simultaneous recursion as follows:

Definition 1.2.1 (Syntax of InqD)**.** Let At be a set of atomic formulas.

1. For any $p \in \mathsf{At}$, $p \in \mathcal{L}_!$
2. $\bot \in \mathcal{L}_!$
3. If $\alpha_1, \ldots, \alpha_n \in \mathcal{L}_!$ then $?\{\alpha_1, \ldots, \alpha_n\} \in \mathcal{L}_?$
4. If $\varphi \in \mathcal{L}_\circ$ and $\psi \in \mathcal{L}_\circ$ then $\varphi \wedge \psi \in \mathcal{L}_\circ$, where $\circ \in \{!, ?\}$
5. If $\varphi \in \mathcal{L}_! \cup \mathcal{L}_?$ and $\psi \in \mathcal{L}_\circ$ then $\varphi \to \psi \in \mathcal{L}_\circ$, where $\circ \in \{!, ?\}$
6. Nothing else belongs to either $\mathcal{L}_!$ or $\mathcal{L}_?$

We refer to the elements of $\mathcal{L}_!$ as declaratives and the elements of $\mathcal{L}_?$ as interrogatives. However, declaratives and interrogatives are tightly connected. For, the interrogative ? applies exclusively to a collections of declaratives, upon which *basic interrogatives*, of the form $?\{\alpha_1, \ldots, \alpha_n\}$, are constructed.

In addition to this we will adopt the convention of using lower case Latin letters $p, q, \ldots$ to denote elements of At, α, β, γ range over declaratives, μ, ν, λ range over interrogatives, and φ, ψ, χ range over the whole language. Furthermore, we define $\neg\varphi$ as $\varphi \to \bot$ (from this it is immediate that $\neg\varphi$ is always a declarative), and $\varphi \leftrightarrow \psi$ for $(\varphi \to \psi) \wedge (\psi \to \varphi)$. Finally a basic interrogative of the form $?\{\alpha, \neg\alpha\}$ will be abbreviated by $?\alpha$ and referred to as a *polar interrogative*.

Semantics

Models for InqD do not differ from possible world models for classical propositional logic, they consist of a set of possible worlds, and a valuation function specifying the atomic formulas true at each world.

Definition 1.2.2 (Models for InqD). An InqD model for a set At of atomic formulas is a pair $M = \langle W, V \rangle$, where:

- W is a set, whose elements we refer to as possible worlds
- $V : W \to \wp(\mathsf{At})$ is a valuation map, stating for each $w \in W$ the atomic formulas true at w

The inquisitive turn comes from evaluating formulas from *information states*, rather than worlds, and taking the fundamental semantic notion to be that of an *issue*—a collection of information states. As in Ciardelli and Roelofsen (2014a) the motivating idea is that an issue is identified with the information sufficient for its resolution, whence its characterisation.

Definition 1.2.3 (States and issues). Let $M = \langle W, V \rangle$ be an InqD model.

- An *information state* is a set $s \subseteq W$ of possible worlds.[1]
- An *issue* is a non-empty set I of information states which is downward closed: if $s \in I$ and $t \subseteq s$, then $t \in I$.
- Given a model M we denote by $\mathscr{I}_M$ the set of all issues over W. We will suppress the subscript M when the set of possible worlds is given by context.

As Ciardelli (2014b, 99, fn. 2) highlights, the notion of a resolution is a generalisation of the notion of a *basic answer* from the interrogative frameworks of Hintikka (1999, 2007) and Winiewski (1996). Still, the term *resolution* is used as a reminder that this is a technical notion, relative to the framework of inquisitive semantics.

Downward closure ensures that for any information state that resolves an issue, the refinement of that information state with further information will also resolve the issue. This corresponds to the property of persistence, defined below.

For any issue I there is a corresponding information state $|I| := \bigcup I$. Given $|I| = s$ for some information state s we may say that I is an issue *over* s, for any $t \in I$ will be such that $t \subseteq s$. The following fact is a consequence of downward closure.

Fact 1.2.4. *For an issue P, and world w, $\{w\} \in P$ iff $w \in |P|$.*

We now formulate the semantics of InqD via the notion of *support* between information states and formulas. Intuitively, an information state can be thought of as incomplete information about the actual world. For example, we can capture all and only the information contained in the formula p by taking the state consisting of exactly those worlds w such that $p \in V(w)$. For a declarative α to be supported in s amounts to α

[1] Given a state s we write $s^{\downarrow}$ for its downward closure. For example, $\{w, v\}^{\downarrow} = \{\{w, v\}, \{w\}, \{v\}, \emptyset\}$

being established by the information available in s, while for an interrogative μ to be supported amounts to μ being resolved by the information available in s.

Definition 1.2.5 (Support conditions for InqD).
Let $M = \langle W, V \rangle$ be an InqD model and s an information state:

1. $M, s \vDash p$ iff $p \in V(w)$ for all $w \in s$
2. $M, s \vDash \bot$ iff $s = \emptyset$
3. $M, s \vDash ?\{\alpha_1, \dots, \alpha_n\}$ iff $M, s \vDash \alpha_1$ or $\dots$ or $M, s \vDash \alpha_n$
4. $M, s \vDash \varphi \wedge \psi$ iff $M, s \vDash \varphi$ and $M, s \vDash \psi$
5. $M, s \vDash \varphi \to \psi$ iff for every $t \subseteq s$, if $M, t \vDash \varphi$ then $M, t \vDash \psi$

The characteristics of support for a propositional atom, and conjunction parallel the truth conditions for the same operators from classical propositional logic. The support condition for implication differs importantly, ensuring that φ implies ψ just in case any refinement of the information state chosen for evaluation which supports φ also supports ψ. Thus, that any possible resolution of φ given the information guaranteed by the state of evaluation, will also resolve ψ. Finally, support for basic interrogatives follows the idea that an interrogative is supported just in case one of its resolutions is supported.

A fundamental property of support conditions is that they are *persistent*, in the following sense.

Fact 1.2.6 (Persistence). *For all formulas φ, if $M, s \vDash \varphi$ and $t \subseteq s$, then $M, t \vDash \varphi$.*

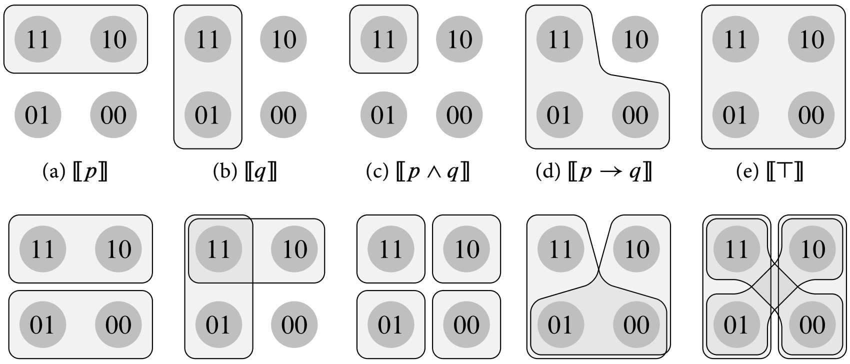
An intuitive consequence of persistence is that the refinement of any information state which resolves an interrogative will continue to do so.

We define the proposition expressed by φ, modulo a model M and denoted by $[\![\varphi]\!]_M$, as the set of all states in M which support φ. We will suppress the subscript if the relevant model is determined by context.

Definition 1.2.7 (Inquisitive propositions). For an InqD model M and formula φ, $[\![\varphi]\!]_M := \{s \mid M, s \vDash \varphi\}$ denotes the proposition expressed by φ.

By persistence every proposition is an issue, and we may often use the terms interchangeably, allowing context to disambiguate.

Figure 1.1 depicts the propositions expressed by some simple formulas. The figure also serves to highlight how the meaning of an interrogative is captured by InqD. For example, the polar interrogative $?p$ of 1.1f selects collections of possible worlds that support p and $\neg p$, corresponding to the components of the basic interrogative $?\{p, \neg p\}$. So, to resolve $?p$ it is sufficient to establish information that supports either p or $\neg p$ Similarly, $?\{p, q\}$ depicted in 1.1g selects collections of possible worlds into those that support p and those that support q.

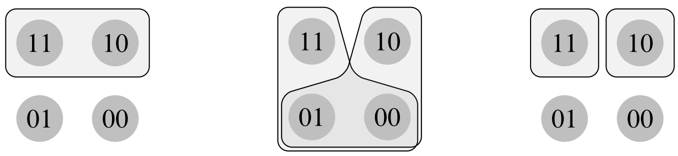


(a) $[\![p]\!]$ (b) $[\![q]\!]$ (c) $[\![p \wedge q]\!]$ (d) $[\![p \rightarrow q]\!]$ (e) $[\![\top]\!]$

(f) $[\![?p]\!]$ (g) $[\![?\{p, q\}]\!]$ (h) $[\![?p \wedge ?q]\!]$ (i) $[\![p \rightarrow ?q]\!]$ (j) $[\![?p \rightarrow ?q]\!]$

Figure 1.1: The inquisitive propositions expressed by some simple formulas. 11 represents a world where both p and q are true, 10 a world where only p is true, etc. For simplicity we draw only *maximal* supporting states. Thus, p has in total 4 supporting states, $?p \rightarrow ?q$ has 9, and $\top$ has 16.

Figure 1.2 depicts aspects of the complex interrogative $p \rightarrow ?q$ in greater detail. Here, for each information state in $[\![p \rightarrow ?q]\!]$, whenever p is supported, $?q$ is supported. So, as in the state the state $\{11, 10\}$ resolving p does not give rise to the issue of whether q it is not a constituent of the proposition. A still more perspicuous was to understand this is that intersecting $[\![p]\!]$ with $[\![p \rightarrow ?q]\!]$ constitutes establishing p and raising the issue of q or $\neg q$, as shown in figure 1.2.[2]



(a) $[\![p]\!]$ (b) $[\![p \rightarrow ?q]\!]$ (c) $[\![p]\!] \cap [\![p \rightarrow ?q]\!]$

Figure 1.2

Derivation System[3]

The natural deduction system of InqD is shown in figure 1.3. Here φ, ψ range over all propositions while α, β range only over declaratives.

Conjunction and implication both have their standard introduction and elimination rules, from falsum one can infer any proposition, while double negation is restricted to declaratives, an aspect of this category we shall explore below. Furthermore, the rules for the interrogative operator are disjunctive, following the correspondence between supports for an interrogative and n-ary disjunction. The introduction rule simply states

[2] Note that the language of InqD restricts the application of conjunction to formulas of the same type; either declarative or interrogative. Therefore, $p \wedge ?q$ is not a well-formed formula. However, as is clear from the support conditions of InqD, p is semantically equivalent to $?\{p\}$, and so $[\![?\{p\} \wedge ?q]\!]$ is the proposition corresponding to the intersection $[\![p]\!]$ with $[\![p \rightarrow ?q]\!]$ in this example.

[3] See Ciardelli (2014a)

Conjunction

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \frac{\varphi \wedge \psi}{\varphi} \qquad \frac{\varphi \wedge \psi}{\psi}$$

Implication

$$\frac{\begin{array}{c}[\varphi]\\ \vdots \\ \psi\end{array}}{(\varphi \to \psi)} \qquad \frac{(\varphi \to \psi) \quad \varphi}{\psi}$$

Interrogative

$$\frac{a_i}{?\{\alpha_1, \ldots, \alpha_n\}} \qquad \frac{\begin{array}{c}[\alpha_1]\\ \vdots \\ \varphi\end{array} \quad \ldots \quad \begin{array}{c}[\alpha_n]\\ \vdots \\ \varphi\end{array} \quad ?\{\alpha_1, \ldots, \alpha_n\}}{\varphi}$$

Kreisel-Putnam

$$\frac{\alpha \to ?\{\beta_1, \ldots, \beta_n\}}{?\{\alpha \to \beta_1, \ldots, \alpha \to \beta_n\}}$$

Falsum

$$\frac{\bot}{\varphi}$$

Declarative double negation

$$\frac{\neg\neg\alpha}{\alpha}$$

Figure 1.3: The natural deduction system of InqD.

that if some α_i is established, then any interrogative $?\{\alpha_1, \ldots, \alpha_n\}$ for which α_i is a resolution is resolved, while the elimination rule states that if an interrogative $?\{\alpha_1, \ldots, \alpha_n\}$ is resolved, and each of its resolutions entails φ, then we can infer φ, corresponding to the intuition that an interrogative can only be established if at least one its resolutions is. The unique rule of InqD is Kreisel-Putnam. Intuitively this rule states that if a declarative implies an interrogative, then it must imply some resolution of that interrogative. The formal requirement of Kreisel-Putnam within inquisitive semantics is observed in Ciardelli and Roelofsen (2011, §3.6), while the relation that holds when an interrogative implies another is explored in Ciardelli (2014a, §3).

As is standard we will write $\Phi \vdash \psi$ to mean that there exists a proof whose set of undischarged assumptions is included in Φ, and whose conclusion is ψ.

Theorem 1.2.8 (Soundness and completeness of InqD).
InqD is sound and (strongly) complete with respect to InqD models.

Proof. See Ciardelli (2014a). □

An important property of InqD is its status as an intermediate logic, a fact established by its equivalence to InqB (Ciardelli 2009), and the fact that the latter includes IPC, cf. Ciardelli (2009, p. 25). Therefore, inquisitive propositions inherit many characteristics of intuitionistic propositions, and in particular every intuitionistic tautology has an expression in InqD. Some are straightforward, as in the following fact.

Fact 1.2.9. *For any model M, state s, and formula φ, $M, s \vDash \varphi \to \neg\neg\varphi$.*

Another property of InqD inherited from its status as an intermediate logic are the *import-export* rules for the conditional. To demonstrate the proof system of InqD we show the derivations. We refer to the introduction rule for a connective $\circ$ as $(\circ e)$, and to the elimination rule as $(\circ i)$. We illustrate when an assumption has been discharged using implication elimination by superscripting the assumption with a number n, and noting n with the application of the rule.

Proposition 1.2.10. *For $\varphi, \psi, \chi \in \mathcal{L}^{\mathsf{InqD}}$ the following rules are derivable:*

$$\dfrac{\varphi \to (\psi \to \chi)}{(\varphi \wedge \psi) \to \chi}\ (\texttt{Imp.}) \qquad \dfrac{(\varphi \wedge \psi) \to \chi}{\varphi \to (\psi \to \chi)}\ (\texttt{Exp.})$$

Proof.

$$\dfrac{\dfrac{\varphi \to (\psi \to \chi) \qquad \dfrac{[\varphi \wedge \psi]^1}{\varphi}\,(\wedge e)}{\psi \to \chi}\,(\to e) \qquad \dfrac{[\varphi \wedge \psi]^1}{\psi}\,(\wedge e)}{(\varphi \wedge \psi) \to \chi}\,(\to i,1) \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^2 \quad [\psi]^1}{\varphi \wedge \psi}\,(\wedge i) \qquad (\varphi \wedge \psi) \to \chi}{\chi}\,(\to e)}{\psi \to \chi}\,(\to i,1)}{\varphi \to (\psi \to \chi)}\,(\to i,2)$$

□

As noted above, the semantics for InqD differs from classical propositional logic in its notion of support. However, classical truth conditional semantics can be recovered from support conditions by evaluating formulas relative to those states which contain exactly one world.

Definition 1.2.11 (Truth)**.** Given an InqD model M, φ is true at a world $w \in W$ just in case $M, \{w\} \vDash \varphi$, which we abbreviate to $M, w \vDash \varphi$.

From this definition, in conjunction with persistence, we can derive the following clauses.

Proposition 1.2.12 (Truth conditions for InqD)**.**
Let $M = \langle W, V \rangle$ be an InqD model and w an arbitrary element of W:

1. $M, w \vDash p$ *iff* $p \in V(w)$
2. $M, w \nvDash \bot$
3. $M, w \vDash ?\{\alpha_1, \ldots, \alpha_n\}$ *iff* $M, w \vDash \alpha_1$ *or* $\ldots$ *or* $M, w \vDash \alpha_n$
4. $M, w \vDash \varphi \wedge \psi$ *iff* $M, w \vDash \varphi$ *and* $M, w \vDash \psi$
5. $M, w \vDash \varphi \to \psi$ *iff* $M, w \nvDash \varphi$ *or* $M, w \vDash \psi$

Inspection shows that these are the familiar truth conditions from classical propositional logic, provided the interrogative operator is interpreted as n-ary disjunction.

Definition 1.2.13 (Truth set)**.** We define the *truth-set* of a formula φ in a model M as the set of all worlds in M in which φ is true: $|\varphi|_M := \{w \in W \mid M, w \vDash \varphi\}$.

Fact 1.2.14. *For any formula φ, $|\varphi|_M = \bigcup [\![\varphi]\!]_M$.*

We also observe that the support conditions for a declarative can be given in terms of its truth conditions.

Proposition 1.2.15. *$M, s \vDash \alpha$ iff $\forall w \in s, M, w \vDash \alpha$.*

And as a consequence of this, the proposition expressed by a declarative can be completely characterised by its truth conditions. Furthermore, for any proposition φ we can associate a declarative, denoted by $!\varphi$ such that $|\varphi| = |!\varphi|$, meaning φ and $!\varphi$ have the same truth conditions, by taking the double negation of φ, i.e. $!\varphi := \neg\neg\varphi$. Informally, this is because truth is local to a world of evaluation, and so as worlds behave classically, the truth of a proposition is equivalent to the truth of its double negation, fact 1.2.16 follows from proposition 1.2.12, above.

Fact 1.2.16. *For any model M, world w, and formula φ, $M, w \vDash \varphi$ iff $M, w \vDash \neg\neg\varphi$.*

Corollary 1.2.17. *For any declarative α, $M, s \vDash \alpha$ iff $M, s \vDash \neg\neg\alpha$.*

Corollary 1.2.18. *For any model M, state s, and formula φ, $|\varphi| = |\neg\neg\varphi| = |!\varphi|$.*

In particular, the proposition expressed by taking the declarative variant of a formula is uniquely characterised by its truth conditions, and conversely as a consequence of this, declaratives in general can be semantically characterised as those propositions wholly determined by their truth conditions.

Proposition 1.2.19. *For any formula φ, $[\![!\varphi]\!] = \wp(|!\varphi|) = \wp(|\varphi|)$.*

Proof. $!\varphi$ is a declarative, so by proposition 1.2.15 $M, s \vDash !\varphi$ iff $\forall w \in s, M, w \vDash \alpha$. Equivalently, $s \in [\![!\varphi]\!]$ iff $s \subseteq |!\varphi|$, iff $s \in \wp(|!\varphi|)$. Therefore $[\![!\varphi]\!] = \wp(|!\varphi|)$. That $\wp(!\varphi) = \wp(|\varphi|)$ follows immediately from corollary 1.2.18. □

We term $|\varphi|$ the *informative content* of φ. And, following proposition 1.2.19 we introduce the parallel notation for issues, allowing us to talk of the 'informative content' of an issue, independently of when an issue corresponds to a proposition.

Definition 1.2.20. For any issue P, $!P := \wp(|P|)$.

As the following corollary of proposition 1.2.19 in conjunction with corollary 1.2.17 shows, inquisitive propositions corresponding to declaratives are wholly determined by their informative content.

Corollary 1.2.21. *For any declarative α, $[\![\alpha]\!] = \wp(|\alpha|)$.*

One particular application of corollary 1.2.17 is the case of conditionals. The set of declaratives is inductively defined so that interrogatives can only occur in a declarative as the antecedent of an implication. As is shown below, only the informative content of such a formula is relevant to the evaluation of the conditional.

Proposition 1.2.22. $\mu \to \alpha \equiv !\mu \to \alpha$

Entailment of a formula by a set of formulas is defined as preservation of support.

Definition 1.2.23 (Entailment).
$\Phi \vDash \psi$ iff for any model M and state s, if $M, s \vDash \Phi$ then $M, s \vDash \psi$.

Ciardelli (2014a, p. 114) explores entailment in detail. For now, this definition allows us to observe that InqD is a conservative extension of classical proposition logic, in the following sense.

Fact 1.2.24 (Conservativity). *Let Γ be a set of classical formulas. Then, $\Gamma \vDash \alpha$ iff Γ entails α in classical propositional logic.*

Resolutions

The idea of information *resolving* an interrogative provided conceptual motivation for the support conditions of inquisitive semantics. We are now able to give a syntactic counterpart, for complex as well as basic interrogatives, by defining a set of resolutions for any given formula of InqD.

Definition 1.2.25 (Resolutions). The set $\mathcal{R}(\varphi)$ of resolutions for a given formula φ is defined inductively by:

- $\mathcal{R}(\alpha) = \{\alpha\}$
- $\mathcal{R}(?\{\alpha_1, \dots, \alpha_n\}) = \{\alpha_1, \dots, \alpha_n\}$
- $\mathcal{R}(\mu \wedge \nu) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\mu) \text{ and } \beta \in \mathcal{R}(\nu)\}$
- $\mathcal{R}(\varphi \to \mu) = \{\bigwedge_{\alpha \in \mathcal{R}(\varphi)}(\alpha \to f(\alpha)) \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\mu)\}$

The adequacy of this definition is shown by the following proposition.

Proposition 1.2.26. *For any M, s and φ; $M, s \vDash \varphi$ iff $M, s \vDash \alpha$, for some $\alpha \in \mathcal{R}(\varphi)$.*

This also gives rise to the following normal form result.

Proposition 1.2.27 (Normal form). *For any φ: $\varphi \equiv ?\mathcal{R}(\varphi)$.*[4]

This proposition states any given formula in the language of InqD is semantically equivalent to a 'basic' interrogative of the form $?\{\alpha_1, \dots, \alpha_n\}$. These results ground our basic perspective on InqD, and an understanding of the system that we will extend when we introduce doxastic modalities: a state supports an interrogative just in case it contains sufficient information to resolve the interrogative.

We introduce two final pieces of notation. First, proposition 1.2.15 and fact 1.2.16 allow us to define a 'pseudo negation' operator for declaratives, which is semantically equivalent to the negation of the declarative. More importantly, the double pseudo-negation of a declarative is syntactically equivalent to the declarative itself.

Definition 1.2.28 (Pseudo-negation).

$$\sim\alpha := \begin{cases} \beta & \text{if } \alpha \text{ is of the form } \neg\beta \\ \neg\alpha & \text{if } \alpha \text{ is of the form } \beta \end{cases}$$

Proposition 1.2.29. *For any declarative α:*

1. $\sim\sim\alpha = \alpha$
2. $\sim\alpha \dashv\vdash \neg\alpha$

Proof.

1. Immediate.

2. If α is of the form β then $\sim\alpha = \neg\alpha$, and so trivially $\sim\alpha \dashv\vdash \neg\alpha$. If α is of the form $\neg\beta$ then we need to show that $\beta \dashv\vdash \neg\neg\beta$. We observe that $\beta \vdash \neg\neg\beta$, as $\neg\neg\beta \vdash \beta$ follows immediately via the rule declarative double negation elimination. Recall $\neg\beta$ abbreviates $\beta \to \bot$.

$$\dfrac{\dfrac{\beta \quad [\neg\beta]^1}{\bot}\,(\to e)}{\neg\neg\beta}\,(\to i,1)$$

□

While we do not have current use for this operator, it will become useful when proving completeness of the conditional doxastic extension of InqD. The following fact is easily verifiable, given the soundness and completeness of InqD.

Fact 1.2.30. $\neg(\neg\alpha_1 \wedge \dots \wedge \neg\alpha_n) \dashv\vdash \neg(\sim\alpha_1 \wedge \dots \wedge \sim\alpha_n)$

[4] $\varphi \equiv \psi$ denotes that the two formulas φ and ψ are equivalent, in the sense that for all models M and states s, $M, s \vDash \varphi$ iff $M, s \vDash \psi$.

With this fact in hand we can define truth conditional disjunction with respect to declaratives.

Definition 1.2.31 (Truth conditional disjunction)**.** $\alpha_1 \vee \cdots \vee \alpha_n := \neg(\sim\alpha_1 \wedge \cdots \wedge \sim\alpha_n)$

By fact 1.2.30 support and truth conditions for disjunction can be derived.

S: $M, s \vDash \alpha \vee \beta$ iff $\forall w \in s, M, w \vDash \alpha \vee \beta$
T: $M, w \vDash \alpha \vee \beta$ iff $M, w \vDash \alpha$ or $M, w \vDash \beta$

Remark 1.2.32 (Alternative characterisation of informative content)**.** With disjunction defined we can associate an alternative formula to the informative content of a inquisitive proposition. For, given a formula φ, $!\varphi := \bigvee \mathcal{R}(\varphi)$.

There will be some useful formal applications of this alternative characterisation. Furthermore, it offers an alternative perspective on a familiar notion. It's what must be the case in a possible world for an interrogative to be solvable at that world.

However, we will generally avoid this terminology and notation.

We conclude this chapter by restating the resolution theorem from Ciardelli (2014b), with respect to InqD, a feature of the logic whose preservation will prove essential in its generalisations.

The concept of a resolution is generalised to sets of formulas in the following way.

Definition 1.2.33 (Resolutions of set)**.** The set $\mathcal{R}(\Phi)$ of resolutions of a set of formulas Φ contains those sets of declaratives Γ such that:

1. for all $\varphi \in \Phi$, there is an $\alpha \in \Gamma$ such that $\alpha \in \mathcal{R}(\varphi)$
2. for all $\alpha \in \Gamma$, there is a $\varphi \in \Phi$ such that $\alpha \in \mathcal{R}(\varphi)$

This allows a statement of the following theorem.

Theorem 1.2.34 (Resolution theorem)**.** *$\Phi \vdash \psi$ iff for all $\Gamma \in \mathcal{R}(\Phi)$ there exists some $\alpha \in \mathcal{R}(\psi)$ such that $\Gamma \vdash \alpha$.*

Which has a semantic counterpart in the following fact.

Fact 1.2.35. *$\Phi \vDash \psi$ iff for all $\Gamma \in \mathcal{R}(\Phi)$ there is an $\alpha \in \mathcal{R}(\psi)$ such that $\Gamma \vDash \alpha$.*

Chapter 2

Inquisitive Epistemic Logic

2.1 Epistemic Logic

In the previous chapter we saw how possible world semantics could be generalised via the notion of support to provide a framework for modelling interrogatives. An independent generalisation uses possible world semantics to describe and reason about what agent's know about the worlds and each other. We term the framework arising from this *epistemic logic*.

Inquisitive semantics and epistemic logic share a conceptual insight, viz. modelling information states as collections of possible worlds. While inquisitive semantics uses this to investigate a generalised notion of a proposition, epistemic logic uses information states to model the information an agent has. We have semantic structures of the following kind.

Definition 2.1.1 (Epistemic models). An epistemic model for a set At of atomic formulas and a set $\mathcal{A}$ of agents is a tuple: $M = \langle W, \{\sigma_a\}_{a\in\mathcal{A}}, V\rangle$, where:

- W is a set of possible worlds
- $V: W \to \wp(\mathsf{At})$ is a valuation map, stating for each world $w \in W$ the atomic formulas true at w
- σ_a is an epistemic map $W \to \wp(W)$ associating to each world an information state $\sigma_a(w)$, satisfying the following conditions:

  Factivity: for any $w \in W$, $w \in \sigma_a(w)$
  Introspection: for any $w, v \in W$, if $v \in \sigma_a(w)$, then $\sigma_a(v) = \sigma_a(w)$

As in InqD models, the valuation map specifies those atomic formulas true at a world, and the truth conditions of complex formulas are given on the basis of these. To interpret an agent's epistemic state we require additional vocabulary, achieved by augmenting the base language with modal operators K_a for each agent $a \in \mathcal{A}$, interpreted via the epistemic state of an agent at a world as follows:[1]

Definition 2.1.2 (Knowledge). $M, w \models K_a\varphi$ iff $\forall v \in \sigma_a(w), M, v \models \varphi$

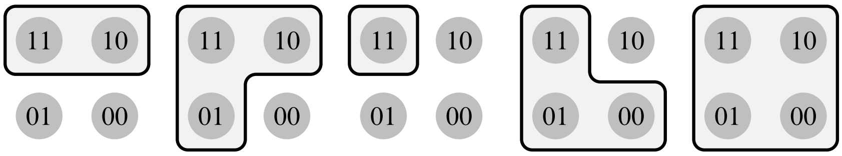
This clause states that an agent a at a world w in a model M knows that φ iff the truth of φ is established by the epistemic state of a at w. In other words, φ is true at any world compatible with the agent's current knowledge.

[1] The full syntax of epistemic logic is given by: $\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_a\varphi$.

While worlds in these models are described only by first-order information, by relativising an agent's epistemic state to a world we can think of a possible world as capturing not only first-order facts, but also higher-order facts about the epistemic state of agent's, just as first and higher-order facts are determinate in the actual world.

Factivity requires an agent's information state never rules out the actual world. As a consequence, an agent can know a piece of information only if it is true of the actual world. This ensures knowledge is factive, meaning that if a knows φ, then φ is true of the actual world. Formally, this means $K_a\varphi \to \varphi$ is a theorem of epistemic logic.

Introspection requires that if an agent has insufficient information to rule out a world, then their information state is the same in both worlds. This ensures an agent always knows their own information state, as their own information state must be constant over all the worlds compatible with the agent's information. In other worlds, it is impossible for an agent's to be uncertain about their own epistemic state. If an agent knows φ then she knows that she knows φ, and if she doesn't know φ, she knows this too. As with factivity, this means both $K_a\varphi \to K_aK_a\varphi$ and $\neg K_a\varphi \to K_a\neg K_a\varphi$ are theorems of epistemic logic.



(a) $K_a p$ (b) $K_a(p \vee q)$ (c) $K_a(p \wedge q)$ (d) $K_a(p \to q)$ (e) $K_a\top$

Figure 2.1: Epistemic states for an agent at world 11, paired with the strongest formula characterising that state. As in figure 1.1, 10 represents a world where p is true and q false, etc. In order to distinguish epistemic states, which are sets of worlds, from issues we use lines with a heavier weight.

2.2 Inquisitive Epistemic Logic

We have seen how information states play an important role in both inquisitive semantics and epistemic logic. Clearly, then, it is possible to construct a logic to reason about how the epistemic states of multiple agents relate to issues, by moving from truth to support conditions in order to interpret epistemic models. However, the framework of inquisitive semantics allows for a richer enhancement of epistemic logic by modelling not only the relation between an agent's epistemic state and issues, but also by providing sufficient semantic structure to model the inquisitive state of an agent in tandem with their epistemic state. To do so we use the framework of inquisitive semantics to associate to each agent an issue over their epistemic state.

Issues will be used to represent an agent's inquisitive state, with these capturing the *epistemic goals* of an agent.

We need not assume the issue used to model an agent's epistemic goals coincides with a unique issue held by the agent. For, given proposition 1.2.27 any interrogative can be represented as a basic interrogative, whence a complex formula comprised a number of distinct interrogatives may be taken to more naturally describe an agent's epistemic state, but can be modelled via a single issue.

We model an issue over an agent's epistemic state following the idea that a fundamental epistemic goal is to have true beliefs, and resolving an issue should be conductive to this. In other words, we assume an agent entertains an interrogative only if one or more of its resolutions could be true of the actual world from the agent's perspective.

Worlds outside an agent's epistemic state support information known to be false, and so cannot be possible resolutions. If the issue were over a subset of their epistemic state, then it would become possible for the agent to have, or learn, information which conflicts with any resolution of the issue held. Such an issue would have the potential to, and may in fact be, epistemically misleading.

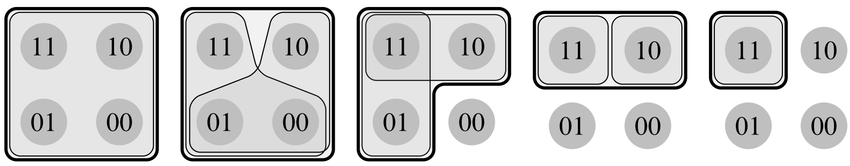
The requirements of factivity and introspection on epistemic states inherited from epistemic models, and the latter is extended to an agent's inquisitive state, following the idealisation away from agent's uncertainty about their own epistemic states.

Using issues to capture the inquisitive states of agents at a world we have *inquisitive epistemic models* (IEMS).

Definition 2.2.1 (Inquisitive epistemic models)**.** An inquisitive epistemic model for a set At of atomic formulas and a set $\mathcal{A}$ of agents is a tuple: $M = \langle W, \{\Sigma_a\}_{a\in\mathcal{A}}, V\rangle$, where:

- W is a set of possible worlds
- $V : W \to \wp(\mathsf{At})$ is a valuation map, stating for each $w \in W$ the atomic formulas true at w
- Σ_a is a state map $W \to \mathscr{I}$ associating to each world $w \in W$ an issue $\Sigma_a(w)$ satisfying the following conditions:

  Factivity: for any $w \in W$, $w \in \sigma_a(w)$, where $\sigma_a(w) := \bigcup\Sigma_a(w)$[2]
  Introspection: for any $w, v \in W$, if $v \in \sigma_a(w)$, then $\Sigma_a(v) = \Sigma_a(w)$



(a) $E_a\top$ (b) $E_a p \to ?q$ (c) $E_a?\{p,q\}$ (d) $E_a?q$ (e) $E_a(p\wedge q)$

Figure 2.2: Examples of issues over epistemic states.

The language used to interpret these models is obtained by enriching the language of InqD with operators for knowledge, as in EL, and a modality termed *entertains*.

Definition 2.2.2 (Syntax of IEL)**.** For At be a set of atomic formulas. We add to the syntax of InqD (definition 1.2.1) the following clause:

7. If $\varphi \in \mathcal{L}_! \cup \mathcal{L}_?$, then $K_a\varphi, E_a\varphi \in \mathcal{L}_!$, for $a \in \mathcal{A}$

In order to interpret the additional operators we introduce the following support clauses to those of InqD:

Definition 2.2.3 (Support conditions for IEL)**.** Let M be an IEL model, and s an information state.

[2]From this we observe that an agent's epistemic state can be recovered from their inquisitive state, whence an inquisitive epistemic model also determines a standard epistemic model.

6. $M, s \vDash K_a\varphi$ iff $\forall w \in s, M, \sigma_a(w) \vDash \varphi$
7. $M, s \vDash E_a\varphi$ iff $\forall w \in s$ and $\forall t \in \Sigma_a(w), M, t \vDash \varphi$

The truth condition for the entertains modality is derived from the above support condition, as the support condition straightforwardly quantifies over all the worlds in a given state of evaluation.

Definition 2.2.4 (Truth clause for entertains)**.** $M, w \vDash E_a\varphi$ iff $\forall t \in \Sigma_a(w), M, t \vDash \varphi$

Just as with knowledge, the semantic clause internalises the semantic consequence relation of the base logic; truth at a world for the former, and support by an information state for the latter.

Moreover, as inquisitive semantics defines support relative to information states we can simplify the semantic clause for knowledge, by directly assessing the agent's state.

Fact 2.2.5. $M, s \vDash K_a\varphi$ *iff* $\forall w \in s, \forall v \in \sigma_a(w), M, v \vDash \varphi$

That epistemic states are information states proves important, for any information state can be transformed into an issue by taking its downward closure, as issue are simply downward closed sets of states. So, just as $\sigma_a(w)$ describes an agent's epistemic state, $\wp(\sigma_a(w))$ can be interpreted to detail every possible refinement of an agent's epistemic state.[3] In this way, restrictions on $\wp(\sigma_a(w))$ correspond to selections of epistemic states an agent could, in principle, attain. This is analogous to how $\sigma_a(w) \cap |\varphi|$ will typically be a subset of $\sigma_a(w)$ specifying the refinement of the agent's epistemic state with the knowledge that φ is true. More generally, $\wp(\sigma_a(w)) = \{\sigma_a(w) \cap |P| \mid P \in \mathscr{I}\}$.

Our interest is in restrictions to $\wp(\sigma_a(w))$ that correspond to issues, capturing refinements to the agent's epistemic state sufficient to resolve certain refinements to the agent's epistemic state. For example, as in figure 2.2c an agent may know that $p \vee q$ is true of the actual world, and entertain whether p or q is true, but be disinterested in whether $p \wedge \neg q$, $p \wedge q$, or $\neg p \wedge q$ is true, though these will be sufficient to resolve the issue entertained by the agent. Therefore, the issue associated to the agent corresponds to the proposition $?\{p, q\}$.

Our interpretation of the entertains modality adds to this the notion that the agent intends, or desires, to attain any of the states selected, intuitively be establishing—or learning—sufficient information to resolve the issue entertained. This allows us to capture an agent's goals via interrogatives, specifically those whose resolutions are part of the goals of the agent. Therefore, via this interpretation of the entertains modality, an agent entertains an interrogative just in case they desire to have knowledge (equiv. be in an epistemic state) which resolves it.[4]

For example, figure 2.2 can be read left to right as the development of an agent's epistemic and inquisitive states. They begin in an ignorant state with no goals, and therefore entertain no proposition stronger than what is currently known, $\top$. And, subsequently, as in 2.2b they refine their epistemic goals, and the information states characterising the *weakest* states they consider desirable are $p \to q$ and $p \to \neg q$. These formulas are the weakest resolutions to $p \to ?q$, and entail all others. Therefore, we can say the agent entertains the issue of $p \to ?q$.

[3]This caveat is important, $\sigma_a(w)$ may contain epistemic states which cannot be achieved, in the same manner as conditionalisation, if interpreted as learning (discussed below) details the result of agents conditionalising on information that cannot be learnt. That $\{v\} \in \wp(\sigma_a(w))$ for every $v \in \sigma_a(w)$ expresses the fact that the agent doesn't know which world is actual, not that the actual world is undetermined.

[4]This gives an additional rationale for the requirement that $\Sigma_a(w)$ is an issue over $\sigma_a(w)$, for any desirable epistemic state must be factive, by virtue of its characterisation.

2.2c suggests a significant shift in the agent's goals upon learning that either p or q is the case, as she revises her goals to establishing which. Note that we are applying IEL purely descriptively, and make no assumptions about the agent's process of revision. In following chapters when conditionalisation, and eventually revision, are introduced, we will assume such leaps of reasoning are not made. Instead, we will suppose a process similar to the move from 2.2c to 2.2d, where the latter state is completely determined by the former and the information learnt, namely p.

In 2.2e the agent has reached a stage of complete knowledge, and consequently can desire no better epistemic state than that which they are currently in. However, and as with any epistemic state, by internalising the notion of support the agent (formally) entertains every proposition supported by their current state. This ambiguity between whether a given proposition is entertained on the basis of a resolution or not can be avoided by defining a modality to capture those issues the agent does not know a resolution to. Ciardelli and Roelofsen (2014a) term this *wondering*.

$$W_a\varphi := E_a\varphi \wedge \neg K_a\varphi$$

A simple inspection shows that in figures 2.2a and 2.2e the agent wonders about no proposition, while in the remaining figures that agent wonders about the propositions entertained.

For wondering to be defined with respect to states in addition to worlds the support condition for the knowledge modality is required. We first observe that, as the truth condition for knowledge universally quantifies over an information state, and a declarative is supported only if it is supported at every world in the information state, in the case of declaratives it can be simplified to the following definition, given the inquisitive notion of support.

Fact 2.2.6 (Truth condition for K modality). $M, w \models K_a\varphi$ *iff* $M, \sigma_a(w) \models \varphi$

The K modality describes an agent's epistemic state, and thus is naturally modelled as a declarative, allowing us to define the following support clause for it.

Fact 2.2.7 (Support condition for the K modality in terms of its truth conditions).

$$M, s \models K_a\varphi \text{ iff } \forall w \in s, M, w \models K_a\varphi$$

The following fact, used to axiomatise IEL (cf. Ciardelli (2014b)), shows knowledge is distributive with respect to interrogatives, meaning an agent knows an interrogative just in case they know (at least) one of its resolutions.

Fact 2.2.8. $K_a?\{\alpha_1, \dots, \alpha_n\} \rightarrow K_a\alpha_1 \vee \cdots \vee K_a\alpha_n$ *is valid with respect to* IEMS.

In full, the following axioms and rules of inference augment the rules of inference of InqD for a sound and complete system, IEL, with respect to IEMS.

1. $E_a(\varphi \rightarrow \psi) \rightarrow (E_a\varphi \rightarrow E_a\psi)$
2. $E_a\alpha \rightarrow \alpha$
3. i) $E_a\varphi \rightarrow E_aE_a\varphi$ and ii) $\neg E_a\varphi \rightarrow E_a\neg E_a\varphi$
4. $K_a(\varphi \rightarrow \psi) \rightarrow (K_a\varphi \rightarrow K_a\psi)$
5. $E_a\alpha \leftrightarrow K_a\alpha$
6. $K_a?\{\alpha_1, \dots, \alpha_n\} \rightarrow K_a\alpha_1 \vee \cdots \vee K_a\alpha_n$

$$\frac{\begin{array}{c}\emptyset\\ \vdots\\ \varphi\end{array}}{E_a\varphi} \qquad\qquad \frac{\begin{array}{c}\emptyset\\ \vdots\\ \varphi\end{array}}{K_a\varphi}$$

Finally we remark on an alternative approach to the definition of an agent's epistemic agenda. For, we can view the process as relativising the possible epistemic states to the resolutions of a collection of issues which the agent wishes to resolve. In this case an agent's epistemic agenda may be represented syntactically, as a set of interrogatives $\{\mu_1, \ldots, \mu_k\}$, with the natural assumption that the informative content of each μ_i is entailed by the agent's epistemic state. Semantically, this corresponds to taking $\Sigma_a(w)$ to be $\wp(\sigma_a(w)) \cap [\![\mu_1]\!] \cap \ldots \cap [\![\mu_j]\!]$.

Representing the epistemic agenda of an agent in this way is from a purely formal perspective unnecessary, as the catalogue of which interrogatives determine the agent's agenda is lost when condensed to the semantic object $\Sigma_a(w)$ when an IEM is defined. Therefore, beyond the intuitive appeal of this approach, we have little to say about it.

Fact 2.2.9 (Soundness and completeness of IEL)**.** *IEL is sound and complete with respect to IEMs.*

Proof. See Ciardelli (2014b). □

Chapter 3

Inquisitive Conditional-Doxastic Logic

The previous chapter brought together inquisitive semantics and epistemic logic. We now turn to synthesising inquisitive semantics with conditional-doxastic logic to form *inquisitive conditional-doxastic logic*.

Conditional-doxastic logic has two interrelated, but distinct interpretations. First, as a static logic, describing both an agent's doxastic and epistemic states (beliefs and knowledge, respectively) and how the states appear when conditionalising on arbitrary pieces of information. Second, as the static reduction of a dynamic logic which describes how an agent's doxastic and epistemic states are revised upon learning new information.

The latter interpretation allows conditional-doxastic logic to be treated as a formal tool, which need not have any value independent of its dynamic counterpart. The former, on the other hand, requires a clear interpretation of conditional-doxastic logic, from which its value can be understood. Furthermore, this perspective will be more general, precisely because it deals with conditionalising on arbitrary pieces of information—in particular, information that cannot (truthfully) be learnt. The following section takes this former perspective, and uses this perspective to motivate the way in which inquisitive semantics and conditional-doxastic logic are synthesised in the remainder of this chapter.

Before continuing let us briefly mention a third perspective that can be taken on conditional-doxastic logic, viz. as extending classical logic with non-classical and non-monotonic implications. This perspective can similarly be applied mutatis mutandis to inquisitive conditional-doxastic logic, we will only briefly return to this approach.

3.1 Conditional-Doxastic Logic

As with inquisitive semantics, epistemic logic, and inquisitive epistemic logic, conditional-doxastic logic builds on the basis of possible world semantics. Indeed, conditional-doxastic logic is a straightforward generalisation of epistemic logic, where instead of associating to each agent an single information state, an information state is associated to each agent for every classical proposition.

To our knowledge Board (2004) is the first to introduce operators for conditional belief, and axiomatise the logic we call CDL. However, our presentation and terminology follows Baltag and Smets (2006).

Both Board and Baltag and Smets highlight the connexion between the process of belief revision captured by CDL and the AGM account of belief revision, detailed in Alchourrón, Gärdenfors, and Makinson (1985). Furthermore, the process of conditionalisation that underlies the logic from a semantic point of view can be found in Stalnaker (1996, §3), who also connects this characterisation to Alchourrón, Gärdenfors, and Makinson (1985), and identifies this account as the first he is aware of.

In short, following Baltag and Smets (2006), conditional-doxastic logic can be interpreted using semantic structures termed *conditional-doxastic models* (CDMs).

We let $P, Q, \ldots$ denote arbitrary information states relative to logics with an interpretation based on classical propositional logic (i.e. in the remainder of this section).

Definition 3.1.1 (Conditional-doxastic models).

A conditional-doxastic model for a set At of atomic formulas and a set $\mathcal{A}$ of agents is a tuple $M = \langle W, \{s_a^P\}_{a \in \mathcal{A}, P \in \wp(W)}, V\rangle$, where:

- W is a set of possible worlds
- $V : W \to \wp(\mathsf{At})$ is a valuation map, stating for each $w \in W$ the atomic formulas true at w
- s_a^P is a doxastic map $W \to \wp(W)$ associating to each world $w \in W$ an information state $s_a^P(w)$ satisfying the following conditions:

Safety	if $w \in P$ then $s_a^P(w) \neq \emptyset$
Introspection	if $v \in s_a^P(w)$ then $s_a^Q(w) = s_a^Q(v)$
Adjustment	$s_a^P(w) \subseteq P$
Success	if $s_a^Q(w) \cap P \neq \emptyset$ then $s_a^P(w) \neq \emptyset$
Minimality	$s_a^{P \cap Q}(w) = s_a^P(w) \cap Q$, if $s_a^P(w) \cap Q \neq \emptyset$

Note the change in notation of an agent's state map from σ_a in epistemic models to s_a in conditional-doxastic models. This is intended to represent an underlying change in the *type* of information the state contains, from the current epistemic state of an agent, to their current (conditional) doxastic state.

Intuitively $s_a^P(w)$ captures those worlds in which an agent's *a*'s beliefs are true, given that they take P to be true.

The syntax of conditional-doxastic logic is given by adding a conditional belief operator to classical propositional logic, as in the following definition:

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid B_a^{\varphi}\varphi$$

With the enriched language of CDL and state maps for agents, we can interpret doxastic maps through the following semantic clause. As is standard we write $s_a^{\psi}(w)$ for $s_a^{|\psi|}(w)$.

Definition 3.1.2 (Conditional belief). $M, w \vDash B_a^{\psi}\varphi$ iff $\forall v \in s_a^{\psi}(w) \colon M, v \vDash \varphi$.

An agent taking P to be true and their resulting doxastic state can be interpreted in various ways. For example, an agent may take P to be true just in case they learn P, or just in case they are informed of P from a source they deem reliable.

More abstractly, we can interpret conditional belief as describing perceived entailment relations between propositions given the agent's current doxastic state, where $B_a^{\psi}\varphi$ reads 'the agent *a* believes φ conditionally on ψ.' For the moment we will not give a concrete interpretation of what conditionalising on P amounts to, and observe only that $s_a^P(w)$ encodes some function of an agent and a piece of information, satisfying certain conditions.

The first of these is *safety*, which ensures an agent's doxastic state will always be consistent with a proposition true of the actual world. In other words, an agent will never rule out the potential truth of, a proposition true of the actual world.

Introspection follows the same motivation as in epistemic models. We assume agents know their own conditional doxastic states, amounting to the assumption that agents have a fixed attitude to any given piece of information, and they cannot revise their own (conditional) beliefs about their conditional beliefs.

Adjustment ensures that an agent conditionalising on some information results in a state in which that information is true.

Success states that if an agent's knowledge is consistent with the truth of P, then they can consistently conditionalise on P.[1] Together with *adjustment*, success ensures that if P is consistent with an agent's epistemic state, then conditionalising on P will not lead to a trivial doxastic state supporting P.

In addition *minimality* ensures this non-trivial belief amounts to the agent minimally adjusting their doxastic state to exclude the possibility that P is false. More generally, minimality ensures that if the agent's beliefs will only be revised when they are contradicted by the information conditionalised on. As a consequence, when conditionalising on more specific information agents conditional doxastic states are the result of iterated conditionalisation, so long as each iterative step does not lead to an inconsistent state, as the following fact highlights. Therefore, *so long as* an agent can consistently iteratively conditionalise on $P_1, \ldots, P_k$, this amounts of conditionalising on $P_1 \cap \cdots \cap P_k$. It is only when for some i, j such that $\mathscr{s}_a^{P_1 \cap \cdots \cap P_i}(w) \cap P_j = \emptyset$ that an agent will revise their conditional doxastic state.[2]

These conditions characterise a certain kind of belief revision, given by a process of conditionalisation that, beyond some background assumptions about what can be conditionalised on, an the relationship between conditionalising at distinct worlds, is intersective when conditionalising on information that is consistent with an agent's given conditional doxastic state, but does not (directly) constrain the agent if they conditionalise on information inconsistent with that state. This latter case is left to be specified by the model, and thus the agent being modelled.

While conditional-doxastic models do not explicitly encode an agent's epistemic state, this can be recovered by taking the union of all their doxastic maps, following the intuition that an agent's epistemic map captures those worlds compatible with their knowledge, which in turn are those worlds, which, from the agent's point of view could be the actual world, and which the agent could given some additional information, revise their epistemic state in order to approximate. In other words, an agent knows P just in case P is believed under any conditions. Formally, $\sigma_a(w) := \bigcup_{P \in \mathscr{I}} \mathscr{s}_a^P(w)$, which in turn can be captured by the syntactic definition $K_a\varphi \leftrightarrow B_a^{\neg\varphi}\bot$,[3] stating an agent knows φ just in case conditionalising on its negation would lead them to a state of inconsistency. By this reduction it is immediate that the knowledge modality satisfies introspection, as conditional belief does. Furthermore, the safety condition ensures knowledge is factive.

[1]N.b. success can also be stated as: if $\sigma_a \cap P \neq \emptyset$, then $\mathscr{s}_a^P(w) \neq \emptyset$, given $\sigma_a(w) := \bigcup_{Q \in \mathscr{I}} \mathscr{s}_a^Q(w)$. As, for success to apply P need only be consistent with *some* conditional doxastic state $\mathscr{s}_a^Q(w)$.

[2]N.b. iterative conditionalisation is not associative. For example, given a plausibility model from the following section, consisting of three worlds w, v, u such that $w <_a^w v <_a^w u$, and propositions $P_1 = \{v, u\}$ and $P_2 = \{w, v, u\}$, then $\mathscr{s}_a^{P_1}(w) = \mathscr{s}_a^{P_1}(w) \cap P_2 = \{v\}$, which $\mathscr{s}_a^{P_2}(w) = \{w\} \neq \mathscr{s}_a^{P_2}(w) \cap P_1 = \emptyset$, whence $\mathscr{s}_a^{P_1}(w) \cap P_2 \neq \mathscr{s}_a^{P_2}(w) \cap P_1$.

[3]Note, we also have the following equivalence: $B_a^{\neg\varphi}\bot \leftrightarrow B_a^{\neg\varphi}\varphi$, allowing us to define knowledge as $K_a\varphi \leftrightarrow B_a^{\neg\varphi}\varphi$, as in CDL.

With epistemic states regained, we can observe that an agent's *current* doxastic state can be captured by conditionalising on any proposition they know to be true, as this will incur no revision of their doxastic map. Indeed, as it is trivially the case that every agent knows $\top$, be can define plan belief that φ as conditional belief that φ given $\top$. Thus, we write $B_a\varphi$ for $B_a^{\top}\varphi$.

Interpreting conditionalising on φ as *learning* φ interpretation of the state map conditions can be given a dynamic twist.

Safety, then, expresses that an agent can always consistently to learn a piece of information which is true of the actual world. Introspection states an agent continues to be introspective upon learning new information. Adjustment states that learning is effective—if an agent learns P, they come to believe P. Success states that if a piece of information is consistent with an agent's epistemic state it can be learnt. Finally, minimality states that an agent's beliefs are never given up when learning information consistent with their (conditional) doxastic state. Importantly, on this interpretation learning does not affect the agent's current epistemic state. Therefore, $\mathscr{s}_a^P(w)$ does not capture what the agent believes upon learning P, but how the agent's current doxastic state would appear upon learning P.

Following Baltag and Smets (2006, pp. 15–16), conditional-doxastic logic is axiomatised by augmenting any sound and complete axiomatisation of classical logic with the following collection of axioms, and an additional rule of inference.

1. $B_a^{\psi}(\varphi \to \chi) \to (B_a^{\psi}\varphi \to B_a^{\psi}\chi)$
2. $B_a^{\neg\varphi}\varphi \to \varphi$
3. i. $B_a^{\psi}\varphi \to B_a^{\chi}B_a^{\psi}\psi$ and ii. $\neg B_a^{\psi}\varphi \to B_a^{\chi}\neg B_a^{\psi}\varphi$
4. $B_a^{\varphi}\varphi$
5. $B_a^{\varphi}\neg\varphi \to B_a^{\psi}\neg\varphi$
6. $\neg B^{\psi}\neg\varphi \to (B_a^{\psi\wedge\varphi}\chi \leftrightarrow B_a^{\psi}(\varphi \to \chi))$
7. From $\vdash \varphi$ infer $\vdash B_a^{\psi}\varphi$

The first axiom ensures that conditional beliefs are closed under (conditionally) believed consequence, while the following axioms correspond in a one-one manner to the conditions imposed on doxastic maps. With this, Baltag and Smets (2006) show that CDL is sound and complete with respect to conditional-doxastic models. Furthermore, they establish ICDL is sound and complete with respect to *plausibility models* (PMS). These are semantic structures of the following kind.

Definition 3.1.3 (Plausibility models)**.** A plausibility model M for a set At of atomic formulas and a set $\mathcal{A}$ of agents, is a tuple: $\langle W, \{\leq_a^w\}_{a\in\mathcal{A}}^{w\in W}, V\rangle$, where:

1. W is a set of possible worlds
2. $V : W \to \wp(\mathsf{At})$ is a valuation map, stating for each $w \in W$ the atomic formulas true at w
3. $\leq_a^w$ associates to each world w and agent a a well-preorder[4] over a subset of W satisfying the following conditions:

Factivity $w \in \sigma_a(w)$, where $\sigma_a(w) := \{v \mid \exists u : v \leq_a^w u\}$

Introspection if $v \in \sigma_a(w)$ then $x \leq_a^v y$ if and only if $x \leq_a^w y$

[4] A binary relation $\leq_a^w$ which is reflexive, transitive, and such that for every set $s \subseteq \{v \mid \exists u : v \leq_a^w u\}$ there exists $v \in s$ such that $v \leq_a^w u$ for all $u \in s$.

Defining $\text{Min}_{\leq_a^w} P := \{v \in P \mid v \leq_a^w u \text{ for all } u \in P\}$ as the set of $\leq_a^w$ minimal elements of P, conditional belief is interpreted on plausibility models via the following clause:

Definition 3.1.4 (Belief). $M, w \models B_a^{\psi}\varphi$ iff $\forall v \in \text{Min}_{\leq_a^w}|\psi|, M, v \models \varphi$

Baltag and Smets (2006, p. 14) show that every PM can be transformed into a CDM, and conversely every *finite* CDM can be transformed into a PM, preserving the interpretation of CDL formulas.[5] This transformation is sufficient to establish CDL as sound and complete with respect to plausibility models. Therefore, plausibility models can be seen as quantitative counterparts to the essentially qualitative conditional-doxastic models for CDL. For, the interpretation of conditional belief now relies on an interpretation of an agent's ordering of worlds, often explicated in terms of 'degree of belief' or 'plausibility.' In the case of the latter $v \leq_a^w u$ is read to state that at w the agent a considers v at least as likely to be the actual world as u, following the mantra that belief aims at truth.

Reflexivity and transitivity follow immediately from the conception of $\leq_a^w$ representing a plausibility ordering, while the requirement of well-foundedness ensures that an agent will never have an infinite chain of worlds, each of which she considers strictly more plausible. Factivity and introspection follow from the same motivation as with respect to epistemic models, as they now pertain to the epistemic state of an agent. This assumes an agent has a plausibility ordering over all the worlds compatible with her knowledge, yet this assumption seems no more controversial as introspective conditionalisation with respect to an arbitrary proposition as assumed by conditional-doxastic models. Naturally, we can assume knowledge is not captured by plausibility models, in which case factivity amounts the assumption that an agent always assigns some plausibility to the actual world, and introspection ensures agents are certain of their own plausibility orderings.

The correspondence between CDMs and PMs, when interpreted via CDL, establishes both classes of models represent equivalent assumptions about how agents conditionalise on information, which is captured qualitatively by CDMs and quantitatively by PMs. In this sense axioms 1 – 6 capture general assumptions about conditionalisation, from which further principles can be derived. Indeed, axiom 2 is equivalent to the principle that knowledge entails belief, $K_\alpha\varphi \to B_a^{\psi}\varphi$, if knowledge is abbreviated through conditional belief, as above. And, using the same abbreviation, the principles of introspection can be rewritten as $B_a^{\psi}\varphi \to K_a B_a^{\psi}\varphi$ and $\neg B_a^{\psi}\varphi \to K_a \neg B_a^{\psi}\varphi$.

Two well-known examples of derived principles are cautious and rational monotonicity: $(B^{\psi}\varphi \wedge B^{\psi}\chi) \to B^{\psi\wedge\chi}\varphi$ and $(B_a^{\psi}\varphi \wedge \neg B^{\psi}\neg\chi) \to B_a^{\psi\wedge\chi}\varphi$, respectively. While the principles taken as axioms, with the exclusion of minimality, and cautious monotonicity, are largely uncontroversial rational monotonicity has generated some discussion. In effect, this principle states that when an agent conditionalises on information consistent with their (conditional) belief, this information will not undermine any of their current (conditional) beliefs, and derives from minimality.[6] See Stalnaker (1994) for a proposed counterexample, but note this would only show rational monotonicity/minimality should not be a *basic* principle of conditionalisation, requiring additional assumptions to be valid.

Finally, this abstract perspective on conditionalisation allows CDL to be related to

[5] A generalisation of this transformation is fully detailed in chapter 5.

[6] For an informal demonstration, given minimality we have the following as a theorem of CDL: $\neg B_a^{\psi}\neg\chi \to (B_a^{\psi}(\chi \to \varphi) \leftrightarrow B_a^{\psi\wedge\chi}\varphi)$. Therefore, under the assumption of $B_a^{\psi}\varphi \wedge \neg B_a^{\psi}\neg\chi$ we can infer $B_a^{\psi}(\chi \to \varphi) \leftrightarrow B_a^{\psi\wedge\chi}\varphi$, whence as $B_a^{\psi}\varphi$ entails $B_a^{\psi}(\chi \to \varphi)$, we infer $B_a^{\psi\wedge\chi}\varphi$.

the AGM theory of belief revision, e.g. Alchourrón, Gärdenfors, and Makinson (1985), as in Baltag and Smets (2006) and Baltag, Renne, and Smets (2015).

3.2 Inquisitive conditional-doxastic Logic

Inquisitive conditional-doxastic logic gives an inquisitive twist to conditional-doxastic logic by introducing interrogatives to the process of conditionalisation, mirroring the relationship between inquisitive epistemic logic and epistemic logic of chapter 2.

As with CDL the most abstract perspective on ICDL generalises away from semantic concrete interpretations of conditionalisation, and interprets theorems of ICDL as principles of conditionalisation, which can then be interpreted quantitatively or qualitatively. To introduce ICDL we will take the former approach, by enriching plausibility models for CDL with issues, familiar from IEL. The latter approach will be developed in chapter 4, and used to prove important properties of ICDL, such as soundness and completeness.

While the motivating semantics for ICDL with arise from synthesising IEMs with plausibility models the language of ICDL will omit the entertain modality (E_a) of IEL. This omission arises from technical issues axiomatising E_a on IPMs. In chapter 5 we show that entertains cannot be defined in terms of conditional modalities, and in chapter 6 we will briefly review the technical problems posed.

Still, in chapter 7 we will axiomatise an enriched language containing the entertains modality, of which ICDL is a fragment. With this in mind explore certain connexions between entertains and the conditional modality when setting out the interpretation of ICDL with respect to IPMs.

3.3 Inquisitive Plausibility Models

Inquisitive plausibility models (IPMs) are our principal structures for a semantic interpretation of ICDL. These arise from synthesising plausibility models for classical doxastic logic, as in section 3.1, and inquisitive epistemic models, as in chapter 2, to obtain models with structure to represent not only the doxastic state of an agent at a world, but also their inquisitive state.

Definition 3.3.1 (Inquisitive plausibility models).

An inquisitive plausibility model M for a set At of atomic formulas and a set $\mathcal{A}$ of agents,[7] is a tuple: $\langle W, \{\leq_a^w\}_{a\in\mathcal{A},w\in W}, \{\Sigma_a\}_{a\in\mathcal{A}}, V\rangle$, where:

1. W is a set of possible worlds
2. $\leq_a^w$ is a well-preorder over a subset of W
3. $\Sigma_a(w)$ is an issue over $\sigma_a(w)$, where $\sigma_a(w) := \{v \mid \exists u\colon v \leq_a^w u\}$
4. $V : W \to \wp(\mathsf{At})$ is a valuation map, stating for each $w \in W$ the atomic formulas true at w

And the following conditions are satisfied:

Factivity	$w \in \sigma_a(w)$, for all $w \in W$
Introspection 1	if $v \in \sigma_a(w)$, then $\Sigma_a(w) = \Sigma_a(v)$
Introspection 2	if $v \in \sigma_a(w)$, then $x \leq_a^v y$ if and only if $x \leq_a^w y$

The synthesis of PMs and IEMs is not irreversible, and any IPM can be segmented into these component parts, as figure 3.1 shows. Indeed, the assumptions made concerning

[7] We assume the set of agents is finite. However, modalities for common knowledge, belief and so on will not be explored in this thesis, and so there is no technical need for this assumption.

the preorder, and conditions of factivity and introspection stem from the same reasoning as in the previous chapter and section for EMs, IEMs, and PMs.

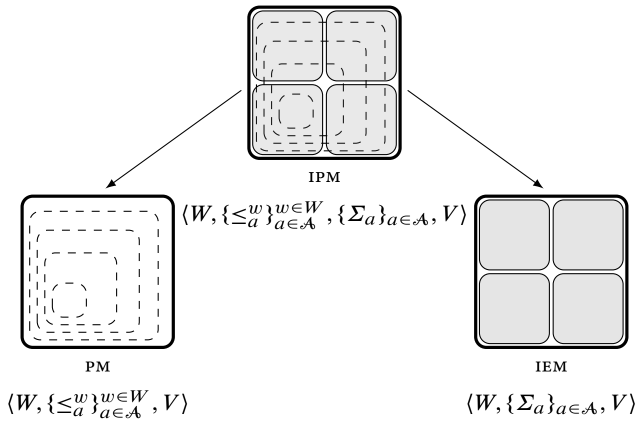


Figure 3.1: Factorization of an inquisitive plausibility model.

Remark 3.3.2. We may replace conditions 2 and 3 in definition inquisitive plausibility models (def. 3.3.1) as follows:

2′. $\Sigma_a(w)$ *is an issue*
3′. $\leq_a^w$ *is a well-preorder over* $\sigma_a(w) := \bigcup \Sigma_a(w)$

Proof. To see this alternative characterisation is equivalent we observe that if $\leq_a^w$ is a well-preorder over $\sigma_a(w)$ then $\sigma_a(w) = \{v \mid v \leq_a^w u \text{ for some u}\}$, for it is trivially the case that a pre-order is a pre-order over all the elements it orders. Therefore, we can rewrite 3′ as 2.

This allows us to observe that 2′ can be rewritten as 3, for we know that $\{v \mid v \leq_a^w u \text{ for some u}\}$ is some subset of W, and $w \leq_a^w w$, by factivity. □

This alternative characterisation allows us to take an 'issues first' approach to IPMs. It seems for many applications of ICDL this would be the preferred approach to take, given the ability to reason about issues is the primary distinguishing factor of ICDL from CDL. However, in reasoning about IPMs the alternative approach is often more transparent.

Our primary interest in inquisitive plausibility models lies in the interaction between an agent's (conditional) beliefs and the issues the hold. For example, in figure 3.2 the agent entertains whether $p \vee s$ or $q \vee t$, and moreover believes r to be the case, in conjunction with either p or q.

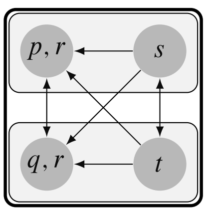


Figure 3.2

Through the modalities introduced for IEL and ICDL, we can use the formulas $E_a?\{p \vee s, q \vee t\}$ and $B_a((p \wedge r) \vee (q \wedge r))$ to express this. What we cannot capture with these two modalities alone is that if the agent's belief that r is correct, she would refine her epistemic goals to focus on establishing whether p or q, for s and t are sure to be false. Her epistemic goals are significantly simpler given the information she believes. We can also observe how the process of conditionalisation affects the considers of an agent. For example, conditioning on $\neg r$ the agent is able to make an corresponding simplification to her epistemic goals, with the alternative issue of whether s or t holds of the actual world being considered.

To formally represent this aspect of an agent's cognitive state we introduce the *considers* modality. First, we define the language of ICDL, which enriches the syntax of InqD by introducing an operator for the considers modality. As we will see below, further modalities can be defined in terms of the considers modality, but could be added as primitive operators if desired.

Definition 3.3.3 (Syntax of ICDL)**.** Let At be a set of atomic formulas:

1. For any $p \in \mathsf{At}$, $p \in \mathcal{L}_!$
2. $\bot \in \mathcal{L}_!$
3. If $\alpha_1, \ldots, \alpha_n \in \mathcal{L}_!$ then $?\{\alpha_1, \ldots, \alpha_n\} \in \mathcal{L}_?$
4. If $\varphi \in \mathcal{L}_\circ$ and $\psi \in \mathcal{L}_\circ$ then $\varphi \wedge \psi \in \mathcal{L}_\circ$, where $\circ \in \{!, ?\}$
5. If $\varphi \in \mathcal{L}_! \cup \mathcal{L}_?$ and $\psi \in \mathcal{L}_\circ$ then $\varphi \to \psi \in \mathcal{L}_\circ$, where $\circ \in \{!, ?\}$
6. If $\varphi, \psi \in \mathcal{L}_! \cup \mathcal{L}_?$ then $C_a^\psi \varphi \in \mathcal{L}_!$
7. Nothing else belongs to either $\mathcal{L}_!$ or $\mathcal{L}_?$

Just as with InqD we refer to elements of $\mathcal{L}_!$ as declaratives and elements of $\mathcal{L}_?$ as interrogatives. We continue to use the convention of using lower case Latin letters $p, q, \ldots$ to denote elements of At, while α, β, γ range over declaratives, μ, ν, λ range over interrogatives, and φ, ψ, χ range over the whole language. Similarly, $\neg\varphi$ is shorthand for $\varphi \to \bot$, and so on.

We also observe the definitions of information states and issues can be restated without adjustment for IPMs.

As with InqD *resolutions* of formulas have a key role in our theorising about ICDL, and thus we briefly pause to define these with respect to the enriched language. Furthermore, as ICDL differs from InqD only by the introduction of further declaratives no additional clause is required to define the resolutions of formulas containing modalities.

Definition 3.3.4 (Resolutions for ICDL)**.** The set $\mathcal{R}(\varphi)$ of resolutions for a given formula φ is defined inductively by:

- $\mathcal{R}(\alpha) = \{\alpha\}$
- $\mathcal{R}(?\{\alpha_1, \ldots, \alpha_n\}) = \{\alpha_1, \ldots, \alpha_n\}$
- $\mathcal{R}(\mu \wedge \nu) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\mu) \text{ and } \beta \in \mathcal{R}(\nu)\}$
- $\mathcal{R}(\varphi \to \mu) = \{\bigwedge_{\alpha \in \mathcal{R}(\varphi)} (\alpha \to f(\alpha)) \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\mu)\}$

To define the semantics for the considers modality we first define the restriction of an issue P to its $\leq_a^w$-minimal elements.

Definition 3.3.5 ($\mathrm{Min}_{\leq_a^w} P$)**.** For any issue P, $\mathrm{Min}_{\leq_a^w} P := \{s \in P \mid s \subseteq \mathrm{Min}_{\leq_a^w} |P|\}$, where $\mathrm{Min}_{\leq_a^w} |P| := \{v \in |P| \mid v \leq_a^w u \text{ for all } u \in |P|\}$.

Note that $\mathrm{Min}_{\leq_a^w} P$ is guaranteed to be an issue by construction, i.e. non-empty and downward closed. Recall that, on the other hand, $\mathrm{Min}_{\leq_a^w} |P|$ is an information state. The following proposition shows that $\mathrm{Min}_{\leq_a^w} |P|$ captures the agent's doxastic state when conditionalising on P, analogously to the way an agent's epistemic state can be obtained from $\Sigma_a(w)$.

Proposition 3.3.6. *For any issue P, $\mathrm{Min}_{\leq_a^w} |P| = |\mathrm{Min}_{\leq_a^w} P|$.*

Proof. By definition $\mathrm{Min}_{\leq_a^w} P := \{s \in P \mid s \subseteq \mathrm{Min}_{\leq_a^w} |P|\}$, so if $u \in \mathrm{Min}_{\leq_a^w} |P|$ then it is immediate that $\{u\} \in \mathrm{Min}_{\leq_a^w} P$, by the definition of $\mathrm{Min}_{\leq_a^w} P$, above. And, as $|\mathrm{Min}_{\leq_a^w} P| := \bigcup(\mathrm{Min}_{\leq_a^w} P)$, $u \in |\mathrm{Min}_{\leq_a^w} P|$.

Conversely, if $u \in |\mathrm{Min}_{\leq_a^w} P|$, then $\{u\} \in \mathrm{Min}_{\leq_a^w} P$, whence $\{u\} \subseteq \mathrm{Min}_{\leq_a^w} |P|$, so $u \in \mathrm{Min}_{\leq_a^w} |P|$, by the definition of $\mathrm{Min}_{\leq_a^w} P$, above. ☐

The ability to relativise an issue to a plausibility ordering allows us to state the support condition for a new modality we terms "considers" as follows.

Definition 3.3.7 (Support for considers)**.**

$$M, s \vDash C_a^{\psi}\varphi \text{ iff } \forall w \in s\colon \forall t \in \text{Min}_{\leq_a^w}(\llbracket\psi\rrbracket \cap \Sigma_a(w)), M, t \vDash \varphi$$

As with conditional belief, we will write C_a for $C_a^{\top}$. Moreover, we regard the epistemic, inquisitive, and doxastic state of an agent to be fixed at any given world of evaluation and the modal formula associated to these states, as with knowledge, entertains, and belief respectively, to express what facts hold of these worlds. Therefore, the support clause for considers straightforwardly quantifies over the worlds constitutive of the state of evaluation. From this we straightforwardly have the following truth condition.

Fact 3.3.8 (Truth for considers)**.**

$$M, w \vDash C_a^{\psi}\varphi \textit{ iff } \forall t \in \mathit{Min}_{\leq_a^w}(\llbracket\psi\rrbracket \cap \Sigma_a(w)), M, t \vDash \varphi$$

Given the support condition for considers we can state the support conditions for ICDL in full.

Definition 3.3.9 (Support conditions for ICDL)**.**
Let $M = \langle W, V\rangle$ be an InqD model and s an information state:

1. $M, s \vDash p$ iff $p \in V(w)$ for all $w \in s$
2. $M, s \vDash \bot$ iff $s = \emptyset$
3. $M, s \vDash ?\{\alpha_1, \ldots, \alpha_n\}$ iff $M, s \vDash \alpha_1$ or $\ldots$ or $M, s \vDash \alpha_n$
4. $M, s \vDash \varphi \wedge \psi$ iff $M, s \vDash \varphi$ and $M, s \vDash \psi$
5. $M, s \vDash \varphi \to \psi$ iff for every $t \subseteq s,$ if $M, t \vDash \varphi$ then $M, t \vDash \psi$
6. $M, s \vDash C_a^{\psi}\varphi$ iff $\forall w \in s\colon \forall t \in \text{Min}_{\leq_a^w}(\llbracket\psi\rrbracket \cap \Sigma_a(w)), M, t \vDash \varphi$

As with aspects relating to the syntax of ICDL, important definitions and results concerning InqD carry over to ICDL.

In particular the notion of an inquisitive proposition, the property of persistence, that the support conditions of declaratives can be characterised in terms of their truth conditions, and the semantic partner of the resolution theorem of proposition 1.2.26. In the following we will simply refer to these results as stated above to aid readability.

3.3.1 The Considers Modality

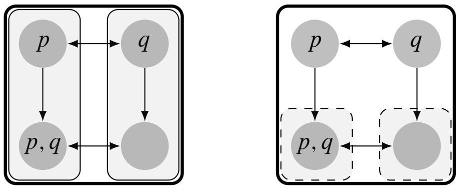
Figure 3.3 depicts a simple example of the difference between entertains and considers modalities. To understand exactly what considering amounts to we will examine four distinct cases of the modality, corresponding to the kind of proposition conditionalised on and considered.

First let us observe that the underlying assumptions about conditionalisation with respect to declaratives inherited from plausibility models can be factored out from the considers modality, as the following propositions' corollary shows.

Proposition 3.3.10. *For any issue P, $\mathit{Min}_{\leq_a^w} P = \wp(\mathit{Min}_{\leq_a^w}|P|) \cap P$.*

Proof. From left to right suppose $s \in \text{Min}_{\leq_a^w} P$. Then, $s \subseteq |\text{Min}_{\leq_a^w} P|$, and so by proposition 3.3.6, $s \subseteq \text{Min}_{\leq_a^w}|P|$, whence $s \in \wp(\text{Min}_{\leq_a^w}|P|)$. And, as $\text{Min}_{\leq_a^w} P \subseteq P$, this means $s \in \wp(\text{Min}_{\leq_a^w}|P|) \cap P$.

From right to left if $s \in \wp(\text{Min}_{\leq_a^w}|P|) \cap P$ then $s \in P$ and $s \subseteq \text{Min}_{\leq_a^w}|P|$, whence $s \in \text{Min}_{\leq_a^w} P$, by definition. □



(a) $E_a?p$ (b) $C_a?\{p \wedge q, \neg p \wedge \neg q\}$

Figure 3.3: The distinction between an agent's issues relative to their epistemic state and their doxastic state. The left hand figure shows the issue associated to an agent, and the right the result of conditionalising on $\top$, where the global issue is omitted and the agents doxastic state is represented using dashed lines. Each label gives a formula true of any world in the agent's depicted epistemic state.

Corollary 3.3.11. *For any P, $Min_{\leq_a^w}(P \cap \Sigma_a(w)) = \wp(Min_{\leq_a^w}|P|) \cap P \cap \Sigma_a(w)$.*

Proof. Clearly $P \cap \Sigma_a(w)$ is an issue. So, by proposition 3.3.10, $\mathrm{Min}_{\leq_a^w}(P \cap \Sigma_a(w)) = \wp(\mathrm{Min}_{\leq_a^w}|P \cap \Sigma_a(w)|) \cap P \cap \Sigma_a(w)$.

Therefore, we need only show that $\mathrm{Min}_{\leq_a^w}|P \cap \Sigma_a(w)| = \mathrm{Min}_{\leq_a^w}|P|$. But, as $\leq_a^w$ is defined to be over $\sigma_a(w) = \bigcup \Sigma_a(w)$, the equality is immediate. □

Corollary 3.3.12. *For any ψ, $Min_{\leq_a^w}(\llbracket\psi\rrbracket \cap \Sigma_a(w)) = \wp(Min_{\leq_a^w}|\psi|) \cap \llbracket\psi\rrbracket \cap \Sigma_a(w)$.*

Therefore, conditionalisation on the informative content of a proposition through the considers modality amounts to refining the doxastic state obtained through conditionalisation à la conditional belief to obtain a collection of doxastic states, a process familiar from interpreting the entertains modality.

However, doxastic states derived in this way should not be identified with the doxastic goals of the agent, understood as those information states desired on the basis of an agent's doxastic state, straightforwardly because the doxastic states are the result of refining the agent's *epistemic* goals.

We have no resources to define additional issues an agent may hold solely on the basis of believing a proposition, over and above those defined with respect to their epistemic state and the issue conditionalised upon.

Indeed, we could define a conditional variation of the entertains modality to capture an agent's epistemic goals conditional on the informative content of a proposition, with the familiar assumptions about conditionalisation by the following clause.

$$M, w \models E_a^{\psi}\varphi \text{ iff } \forall t \in \mathrm{Min}_{\leq_a^w}(\llbracket !\psi \rrbracket \cap \Sigma_a(w)), M, t \models \varphi$$

It is an easy exercise to check that $E_a^{\psi}\varphi$ is semantically equivalent to $C_a^{!\psi}\varphi$, and this is how we interpret the modality when conditionalising on the informative content of a proposition; the epistemic states desirable to the agent given their epistemic goals and the truth of the formula conditionalised on. As the following proposition suggests, systematic constraints hold between an agent's epistemic and conditional doxastic goals.

Proposition 3.3.13. $E_a(\psi \to \varphi) \to C_a^{\psi}\varphi$

Proof. Suppose $M, w \models E_a(\psi \to \varphi)$. So, $\forall t \in \Sigma_a(w)$, if $M, t \models \psi$ then $M, t \models \varphi$. Therefore, as it is the case for any $t' \in \mathrm{Min}_{\leq_a^w}(\llbracket\psi\rrbracket \cap \Sigma_a(w))$ that $t' \in \Sigma_a(w)$ and $M, t' \models \psi$, it is immediate that $\forall t \in \mathrm{Min}_{\leq_a^w}(\llbracket\psi\rrbracket \cap \Sigma_a(w)), M, t \models \varphi$, whence $M, w \models C_a^{\psi}\varphi$. □

Two intuitive consequences follow from this proposition. First, any proposition entertained is considered, for by taking ψ to be $\top$ the formula reduces to $E_a\varphi \to C_a\varphi$. Second, if φ is consistent with what the agent considers, then it is consistent with what the agent entertains. This can be seen by taking φ as $\neg\varphi$ in the previous reduction and contraposing the formula to obtain $\neg C_a^{\psi}\neg\varphi \to \neg E_a\neg\varphi$.

The effect of conditionalising on an inquisitive proposition ψ, then, captures i) the resulting doxastic state, given the information the agent is conditionalising on and ii) desired refinements of the agent's doxastic state, determined by the issues the agent wishes to resolve on the basis of their epistemic goals and the issue conditionalised on. These two components are captured by an issue analogously to the way an agent's epistemic state and goals are captured by $\Sigma_a(w)$. , whereby the agent's doxastic state is refined to those states the agent considers desirable, intuitively understood as information the agent wishes to establish, whose union is coextensive with the agent's (conditional) doxastic state.[8]

Considers, then, evaluates an agent's inquisitive doxastic state, derived from of both the agent's epistemic agent and some issue, and restricted to the worlds the agent considers most plausible given the informative content of the issue.

Therefore, we read $C_a^{\psi}\varphi$ as stating 'φ is supported in every information state the agent considers desirable and most plausible, given the truth of $!\psi$ and the intersection of the agent's epistemic issues with the inquisitive content of ψ,' where, following our interpretation of IEL, an agent considers an state desirable just in case it contains sufficient information to resolve their epistemic goals.

In particular, this means (just as with entertains) the strongest basic interrogative supported will identify the salient issues the agent desires to resolve, with weaker interrogatives specifying logical consequences of these.

A second parallel with entertains allows considers to be interpreted from an 'interrogative first' perspective, where the information states quantified over by the modality derive from issues the agent desires to resolve, over being given by information they wish to establish. The foremost distinction of this interpretation is the perspective taken on conditionalisation. For here the process can be interpreted as to include conditionalisation on interrogatives, whence corollary 3.3.11 shows not how conditionalisation can be factored out into truth-conditional conditionalisation, but instead how the process of conditionalisation can be broken down into a series of discrete steps, as in figure 3.4.

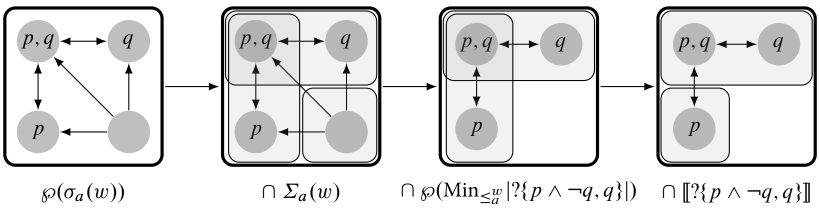


Figure 3.4: Breaking down the process of conditionalisation.

Indeed, while the considers modality can be conditionalised on in this way, the approach we have taken to IPMs means we have not made any assumptions about conditionalisation over those made with respect to declaratives from CDL.

[8]While the agent's epistemic goals are captured by an issue *over* $\sigma_a(w)$, it will not, in general, be the case that $[\![\psi]\!]$ is an issue over $\mathrm{Min}_{\leq_a^w}(\sigma_a(w) \cap |\psi|)$. Recall only part of this restriction carried interpretive value, while stipulating $|\Sigma_a(w)| \subseteq \sigma_a(w)$ made for formal simplicity.

We begin our case by case analysis of consider with conditionalisation applied declaratives. Here, only information content is conditionalised on and considered. This means the agent has no doxastic gaols, but moreover her epistemic goals are irrelevant, and the semantic clause reduces to that of conditional belief defined on PMs. So, when considering a declarative the considers modality boils down to truth in every world the agent believes most plausible.

Proposition 3.3.14. *$M, w \vDash C_a^{\psi}\beta$ iff $\forall v \in Min_{\leq_a^w}|\psi|, M, v \vDash \beta$, for any declarative β.*

Proof. $M, w \vDash C_a^{\psi}\beta$ iff $\forall t \in \mathrm{Min}_{\leq_a^w}(\llbracket\psi\rrbracket \cap \Sigma_a(w)), M, t \vDash \beta$. By proposition 1.2.15 this is the case iff $\forall v \in |\mathrm{Min}_{\leq_a^w}\llbracket\psi\rrbracket \cap \Sigma_a(w)|, M, v \vDash \beta$. And, by proposition 3.3.6 this is equivalent to $\forall v \in \mathrm{Min}_{\leq_a^w}|\llbracket\psi\rrbracket \cap \Sigma_a(w)|, M, v \vDash \beta$.

Now, suppose $v \in \mathrm{Min}_{\leq_a^w}|\llbracket\psi\rrbracket \cap \Sigma_a(w)|$. As $\Sigma_a(w)$ is an issue over $\sigma_a(w)$, for all $u \in |\psi|$, either $v \leq_a^w u$ or $u \notin \sigma_a(w)$. So, $v \in \mathrm{Min}_{\leq_a^w}|\psi|$. And, by analogous reasoning, if $u \in \mathrm{Min}_{\leq_a^w}|\psi|$, then $u \in \mathrm{Min}_{\leq_a^w}|\llbracket\psi\rrbracket \cap \Sigma_a(w)|$. Therefore, $\mathrm{Min}_{\leq_a^w}|\psi| = \mathrm{Min}_{\leq_a^w}|\llbracket\psi\rrbracket \cap \Sigma_a(w)|$. So, $M, w \vDash C_a^{\psi}\beta$ iff $\forall v \in \mathrm{Min}_{\leq_a^w}|\psi|, M, v \vDash \beta$. □

Considering an interrogative when conditionalising on a declarative is more interesting, as the agent's epistemic goals in the following clause are ineliminable.

$$M, w \vDash C_a^{\alpha}\nu \text{ iff } \forall t \in \mathrm{Min}_{\leq_a^w}(\llbracket\alpha\rrbracket \cap \Sigma_a(w)), M, t \vDash \nu$$

By introducing the resolutions[9] of ν, the following clause can be obtained.

$$M, w \vDash C_a^{\alpha}\nu \text{ iff } \forall t \in \mathrm{Min}_{\leq_a^w}(\llbracket\alpha\rrbracket \cap \Sigma_a(w)), M, t \vDash \beta_i \text{ for some } \beta_i \in \mathcal{R}(\nu)$$

This allows us to observe that considering an interrogative can be partly tied to—note the following fact is an *only if* statement—CDL-conditional belief, as follows.

$$M, w \vDash C_a^{\alpha}\nu \text{ only if } \forall v \in \mathrm{Min}_{\leq_a^w}|\alpha|, M, v \vDash \bigvee_{\beta_i \in \mathcal{R}(\nu)}\beta_i$$

This fact ties together the conceptual motivation behind both CDL-conditional belief and resolutions. For, such belief is captured by taking the most plausible worlds in which an agent's current beliefs are true, while resolutions specify the conditions under which an interrogative is settled. Thus, the fact reveals it is a necessary condition that the interrogative is resolved in every world the agent thinks most plausible for an her to consider an interrogative when conditionalising on a declarative.

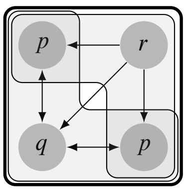


Figure 3.5

For many, and all polar, interrogatives $\bigvee_{\beta \in \mathcal{R}(\nu)}\beta \equiv \top$. This means, if the converse were to hold these would be trivially considered, conditional on any proposition, which is demonstrably not the case. Figure 3.5 depicts a model in which every world the agent believes most plausible the interrogative $?\{p, q\}$ is solvable, meaning that some resolution of $?\{p, q\}$ is true at each world, and hence a truthful resolution to $?\{p, q\}$ is always possible. Yet, the agent does not consider $?\{p, q\}$.

Still, if an agent does consider some resolution to an interrogative, as a consequence of the support condition they must also consider the interrogative. In this respect the considers modality internalises the right to left direction of proposition 1.2.27, which states $\varphi \equiv ?\mathcal{R}(\varphi)$, just as the forgoing entailment relied on the internalisation of the converse. For example, in figure 3.5 $\neg r$ holds in all the most plausible worlds, and so $C_a?r$ holds.

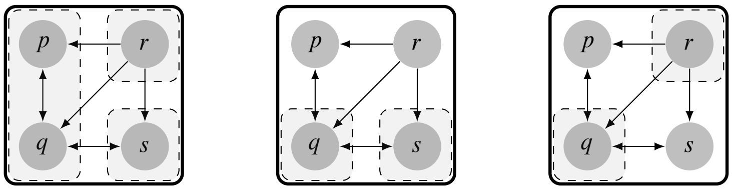
[9]Given ICDL only introduces declaratives to InqD the definition of resolutions for ICDL parallels that of InqD, as in section 1.2.

A simple variation of the considers modality akin to wonders for entertains can be defined to avoid this possibility, which is supported only if an agent considers an interrogative but not any of its resolutions, reading disjunction over resolutions as existential generalisation, as follows.

$$\mathcal{W}_a^{\psi}\varphi := C_a^{\psi}\varphi \wedge \neg\bigvee_{\alpha\in\mathcal{R}(\varphi)} C_a^{\psi}\alpha$$

This variation applies to conditionalising on both interrogatives and declaratives. Indeed, the latter case gives insight into an agent's 'short term' epistemic goals, as then the inquisitive state of an agent is the only source of (unresolved) interrogatives. Indeed, the entertains modality can be used capture which of an agent's goals arise uniquely in their doxastic state, as follows.

$$D_a^{\psi}\varphi := C_a^{\psi}\varphi \wedge \neg E_a\varphi$$



(a) $\Sigma_a(w)$ (b) $\Sigma_a(w) \cap [\![q \vee s]\!]$ (c) $\Sigma_a(w) \cap [\![q \vee r]\!]$

Figure 3.6

Figure 3.6 depicts the result of refining an inquisitive state with two different declaratives. Observe in 3.6a the agent (unconditionally) considers whether $p \vee q$ or s, formally $C_a?\{p \vee q, s\}$, as she believes p, q and s to be most plausible, while distinguishing p and q from s, but not p from q. Intersecting her inquisitive state with the proposition $q \vee r$, as in 3.6b, leaves the remaining world equiplausible, and moreover allows her to disregard p, thus $C_a^{q\vee s}?\{q, s\}$. Indeed, this holds under the conditional wonders modality also, and so we can write $\mathcal{W}_a^{q\vee s}?\{q, s\}$, expressing the fact that whether q or s is unresolved for the agent.

However, note that as q entails $p \vee q$, it remains in 3.6b the case that the agent considers whether $p \vee q$ or s under the standard considers modality, formally $C_a?\{p \vee q, s\}$. The considers modality does not allow us to rule out redundant interrogatives derived from truth conditional consequences of interrogatives considered, while the conditional wonders modality does.

Still, in figure 3.6c the agent considers the proposition q, conditional on $q \vee r$, and as a result considers whether q or r, despite her beliefs, formally $C_a^{q\vee r}?\{q, r\}$. Here we can write $C_a^{q\vee r} q \wedge \neg\mathcal{W}_a^{q\vee r}?\{q, r\}$ to express this aspect of the agent's cognitive state in greater detail.

3.3.2 Conditional Belief

We define conditional belief on IPMs as a straightforward generalisation of conditional belief with respect to PMs to interrogatives as well as declaratives.

Definition 3.3.15 (Truth for (conditional) belief)**.**

$$M, w \vDash B_a^{\psi}\varphi \text{ iff } \forall t \in \mathrm{Min}_{\leq_a^w} [\![\psi]\!], M, t \vDash \varphi$$

And, as with considers this gives rise to the following support condition.

Definition 3.3.16 (Support for (conditional) belief).

$$M, s \vDash B_a^{\psi} \varphi \text{ iff } \forall w \in s\colon \forall t \in \mathrm{Min}_{\leq_a^w} [\![\psi]\!], M, t \vDash \varphi$$

Indeed, as a corollary of following proposition shows, the semantic interpretation of belief and considers will coincide if an agent has no epistemic goals.[10]

Proposition 3.3.17. *$M, w \vDash B_a^{\psi} \beta$ iff $\forall v \in Min_{\leq_a^w} |\psi|, M, v \vDash \beta$.*

Proof. By definition, $M, w \vDash B_a^{\psi} \beta$ iff $\forall t \in \mathrm{Min}_{\leq_a^w} [\![\psi]\!], M, t \vDash \beta$. By proposition 1.2.15 this is the case iff $\forall v \in |\mathrm{Min}_{\leq_a^w} [\![\psi]\!]|, M, v \vDash \beta$, iff $\forall v \in \mathrm{Min}_{\leq_a^w} |\psi|, M, v \vDash \beta$, by proposition 3.3.6. □

Analogous to the reasoning with respect to considers, above, this means IPM belief is a conservative extension of PM belief, with respect to declaratives.

Still, the PM interpretation of (conditional) belief as truth in the most plausible worlds cannot be straightforwardly applied to interrogatives, as it relies on the fact that declaratives are truth-conditional. One way to interpret belief is as a special case of the considers modality, where only the conditional doxastic goals of the agent are taken into account, parallel to the conditional entertains modality. However, we choose to interpret conditional belief independently of an agent's inquisitive goals, and instead interpret belief to capture relations between propositions subject to an agent's doxastic state. Indeed, this perspective can was briefly raised with respect to CDL belief. Thus, $B_a^{\psi} \varphi$ reads 'on the basis of the agent's current beliefs, the agent holds ψ to entail φ.'

Interpreted with respect to declaratives on PMs, a declarative formula entails another just in case all the worlds the agent believes as good candidates for the actual world given the truth of the former proposition are worlds in which the latter is true. This is exactly what belief boils down to in the special case of declaratives.

$$M, w \vDash B_a^{\alpha} \beta \text{ iff } \forall v \in \mathrm{Min}_{\leq_a^w} |\alpha|, M, v \vDash \beta$$

Corollary 3.3.18. $C_a^{\alpha} \beta \equiv B_a^{\alpha} \beta$

It is when conditionalising on an interrogative that the generalised concept of entailment is apposite for interpreting conditional belief. For, by proposition 3.3.6 the states quantified over by conditional belief are the result of intersecting the agent's doxastic state, conditionalised on the informative content of ψ, with ψ. Therefore, the modality quantifies over the resolutions to ψ, given the agent's beliefs. More precisely, if $\forall t \in \mathrm{Min}_{\leq_a^w} [\![\psi]\!], M, t \vDash \varphi$, this means that, given the agent's beliefs given the informative content of ψ, for every resolution of ψ, some resolution of φ is supported. This is captured by the following reduction, proved as part of theorem 5.2.1, in section 4.3.[11]

Fact 3.3.19.

$$M, w \vDash B_a^{\psi} \varphi \text{ iff } \forall \alpha \in \mathcal{R}(\psi), \text{ if } M, w \nvDash B_a^{!\psi} \neg\alpha, \exists \beta \in \mathcal{R}(\varphi)\colon M, w \vDash B_a^{\alpha} \beta$$

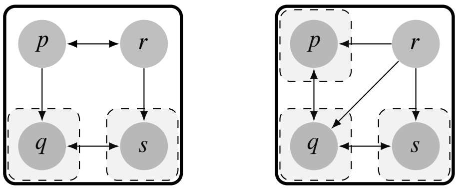
In other words, from the agent's perspective, given their current beliefs, resolving ψ would resolve φ. Alternatively, the issue of ψ implies the issue of φ.

Similarly, if ψ is a declarative β and φ an interrogative μ, then the agent's learns a singular piece of information, and so if μ is resolved, then the agent's epistemic state must support a specific resolution of μ. We have the following reduction:

[10] As, in such a case $\Sigma_a(w) = \wp(\sigma_a(w))$.

[11] The fact is a natural language statement of one of the axiom which allows the reduction of conditional belief to the considers modality. Namely, $B_a^{\psi} \varphi \leftrightarrow \bigwedge_{\alpha \in \mathcal{R}(\psi)} (\neg B_a^{\psi} \neg\alpha \to \bigvee_{\beta \in \mathcal{R}(\varphi)} B_a^{\alpha} \beta)$.

Fact 3.3.20.
$M, w \vDash B_a^\beta \mu$ *iff* $\exists \alpha \in \mathcal{R}(\mu)\colon \forall v \in Min_{\leq_a^w} |\beta|, M, v \vDash \alpha,$ *iff* $\exists \alpha \in \mathcal{R}(\mu)\colon M, w \vDash B^\beta \alpha$



(a) $B_a^{?p} ?q$ (b) $B_a^{?\{p,q,s\}} ?\{\neg p, \neg q, \neg s\}$

Figure 3.7

Unlike knowledge, conditional belief generalised to interrogatives is not distributive in general, as can be seen from the examples in figure 3.7. Formally, the distributivity of knowledge is guaranteed by the fact that its semantic clause evaluates what is supported by the agent's epistemic state, which is in turn an information state. Indeed, an analogous clause for belief would evaluate what is supported by an agent's doxastic state.

Therefore, for such a clause would read $M, w \vDash B_a^\psi \varphi$ iff $M, \mathrm{Min}_{\leq_a^w} |\psi| \vDash \varphi$. But note that by persistence, if $M, \mathrm{Min}_{\leq_a^w} |\psi| \vDash \varphi$, then $\forall s \in \mathrm{Min}_{\leq_a^w} \wp(|\psi|), M, s \vDash \varphi$ and if for some $v \in \mathrm{Min}_{\leq_a^w} |\psi|, M, v \nvDash \varphi$, then there exists some $s \in \mathrm{Min}_{\leq_a^w} \wp(|\psi|)$ such that $M, s \nvDash \varphi$.

These two facts entail that $M, \mathrm{Min}_{\leq_a^w} |\psi| \vDash \varphi$ iff $\forall s \in \mathrm{Min}_{\leq_a^w} \wp(|\psi|), M, s \vDash \varphi$. Therefore, as $|\psi| = |!\psi|$, by corollary 1.2.18, and $[\![!\psi]\!] = \wp(|!\psi|)$, by proposition 1.2.19, this means $M, \mathrm{Min}_{\leq_a^w} |\psi| \vDash \varphi$ iff $\forall s \in \mathrm{Min}_{\leq_a^w} [\![!\psi]\!], M, s \vDash \varphi$. So, one can evaluate the propositions supported by the doxastic state of an agent as a special case of conditional belief, by conditionalising on the proposition expressed by the informative content of the proposition conditionalised on.

Furthermore, through complex formulas containing the believes modality we can distinguish certain aspects of an agent's epistemic state. For example, $B_a^\nu \mu \wedge \neg B_a^{!\nu} \mu$ expresses that the agent can resolve μ given a resolution of ν, but not given the presupposition of ν. Note, this is only possible for interrogatives, μ, ν.

Similarly, the added expressive power of the considers modality allows us to capture when an agent believes an issue on their epistemic agenda is resolved by conditionalising on a given proposition. Here, we are interested in capturing when conditionalising on ψ would lead to a resolution of μ, i.e. that some resolution of μ would be believed given ψ.[12] Formally:

$$M, w \vDash R_a^\psi \mu := W_a \mu \wedge \bigvee_{\alpha \in \mathcal{R}(\mu)} B_a^\psi \alpha$$

At this point we can observe an important distinction between support and truth conditions.

For, when evaluated at a world $\bigvee_{\alpha \in \mathcal{R}(\mu)} B_a^\psi \alpha$ reads, by interpreting disjunction as existential quantification, as there being some resolution α of μ which the agent believes, conditional on ψ. However, an agent may consider different resolutions of μ at different

[12] A weaker condition may define when an issue is resolved by the definition: $M, w \vDash R_a^\psi \mu := W_a \mu \wedge B_a^\psi \mu$. This stipulates that, conditional on ψ there is *some* resolution of μ true at the most plausible ψ worlds, but does not guarantee the agent is in a position to identify a single resolution. Here, then, conditionalising on ψ would not lead to a resolution of μ, but a guarantee that μ could in principle be resolved, given additional information.

worlds. For example, it may be the case that $M, \{w, v\} \vDash \bigvee_{\alpha \in \mathcal{R}(\mu)} B_a^{\psi} \alpha$, while $M, w \vDash B_a^{\psi} \alpha$ and $M, v \vDash B_a^{\psi} \beta$, for $\alpha \neq \beta \in \mathcal{R}(\mu)$. This coincides with an interpretation of the support condition as evaluating an agent's epistemic state from a position of incomplete information. More generally, every world in a state may agree on the agent's current beliefs, but differ with respect to how those beliefs will be revised if the agent were to learn new information.

Finally, we can generalise the reduction of knowledge to considers to knowledge of both declaratives and interrogatives by the following condition.

$$K_a \varphi := \bigvee_{\alpha \in \mathcal{R}(\varphi)} C_a^{\neg \alpha} \bot$$

This reduction mirrors the reduction of knowledge to conditional belief in CDL, with the proviso includes for interrogatives that at least one of the resolutions for φ is known. Note the reduction can be recast in terms of conditional belief, as in the case of CDL, given the connexion between considers and conditional belief observed in axiom 7, below.

3.4 Axioms & Rules

Axioms

To obtain a sound and complete axiomatisation of ICDL with respect to IPMs we enrich InqD with the following collections of axioms for each modality, and an additional rule of inference.

The axioms governing the considers modality run parallel to CDL belief. Indeed, as we have observed, CDL belief and considers share the same process of conditionalisation on IPMs, and stated axiomatically this means that under any semantic interpretation a common conception of conditionalisation is at the core of both CDL and ICDL, and the base framework of InqD generalises this from classical to inquisitive propositions. Recall α denotes an arbitrary declarative, while φ, ψ, χ denote arbitrary formulas which are either declaratives or interrogatives.

Considers

1. $C_a^{\psi}(\varphi \to \chi) \to (C_a^{\psi} \varphi \to C_a^{\psi} \chi)$
2. $C_a^{\varphi} \neg \varphi \to \neg \varphi$
3. $C_a^{\varphi} \varphi$
4. i. $C_a^{\psi} \varphi \to C_a^{\chi} C_a^{\psi} \varphi$ and ii. $\neg C_a^{\psi} \varphi \to C_a^{\chi} \neg C_a^{\psi} \varphi$
5. $\neg C^{\psi} \neg \varphi \to (C_a^{\psi \wedge \varphi} \chi \leftrightarrow C_a^{\psi}(\varphi \to \chi))$

3.4.1 Axioms for ICDL Enriched with Operators for Conditional Belief and Knowledge

Both conditional belief and knowledge can be defined in terms of the considers modality, and the following three axioms describe the reduction. These allow conditional belief and knowledge to be eliminated from the language of ICDL if added as primitive operators.

The first axiom with respect to conditional belief is a corollary of proposition 3.3.17, while the second is a syntactic expression of fact 3.3.19.

In short, axiom 7 states that the process of conditionalisation that belief and considers encode is equivalent with respect to informative content, with respect to evaluating declaratives.

Axiom 8, on the other hand, ensures that conditional belief that φ, given ψ, generalises over conditionalising on the resolutions of φ and ψ reading conjunction as universal and disjunction as existential quantification, so long as resolutions of the latter are consistent with what the agent believes given the informative content of ψ.

Belief

7. $B_a^\psi \alpha \leftrightarrow C_a^\psi \alpha$
8. $B_a^\psi \varphi \leftrightarrow \bigwedge_{\alpha\in\mathcal{R}(\psi)}(\neg B_a^\psi \neg\alpha \to \bigvee_{\beta\in\mathcal{R}(\varphi)} B_a^\alpha \beta)$[13]

Finally, axiom 9 states that knowledge can be defined in terms of the considers modality, by the fact that a proposition can be known if and only if the negation of one of its resolutions would lead to a contradiction. Via axiom 7 we know $\bigvee_{\alpha\in\mathcal{R}(\varphi)} B_a^{\neg\alpha}\bot \leftrightarrow \bigvee_{\alpha\in\mathcal{R}(\varphi)} C_a^{\neg\alpha}\bot$, and therefore axiom 2.2.8 can be rewritten as $K_a\varphi \leftrightarrow \bigvee_{\alpha\in\mathcal{R}(\varphi)} B_a^{\neg\alpha}\bot$, establishing the well established connexion between knowledge and conditional belief.

Knowledge

9. $K_a\varphi \leftrightarrow \bigvee_{\alpha\in\mathcal{R}(\varphi)} C_a^{\neg\alpha}\bot$

Rules

Necessitation and Replacement of Equivalents:

$$\dfrac{\begin{array}{c}\emptyset\\ \vdots\\ \varphi\end{array}}{C_a^\psi \varphi} \qquad\qquad \dfrac{\varphi \leftrightarrow \psi}{C_a^\varphi \chi \leftrightarrow C_a^\psi \chi}$$

3.4.2 Observations

Our axiomatisation of ICDL differs slightly from the axiomatisation of CDL in three respects. First, the axiom that corresponds to the safety condition on CDMs—axiom 2, $C_a^\varphi\neg\varphi \to \neg\varphi$—does not conditionalise on the negation of a formula, as it does in the case of CDL—axiom 2, $B^{\neg\varphi}\varphi \to \varphi$. Second, axiom 5 of CDL—$B^\varphi\neg\varphi \to B_a^\psi\neg\varphi$—is not reflected in the axiomatisation of ICDL.

These two observations are connected. For the revised axiom for the safety condition for ICDL allows for a straightforward derivation of a theorem corresponding to the CDL axiom 3.1 in ICDL.

Proposition 3.4.1. $\vdash_{ICDL} C_a^\varphi\neg\varphi \to C_a^\psi\neg\varphi$

Proof.

$$\dfrac{\dfrac{\dfrac{[C_a^\varphi\neg\varphi]^1 \qquad C_a^\varphi\neg\varphi \to C_a^\psi C_a^\varphi\neg\varphi}{C_a^\psi C_a^\varphi\neg\varphi}\,(\to e) \qquad \dfrac{\dfrac{C_a^\varphi\neg\varphi \to \neg\varphi}{C_a^\psi(C_a^\varphi\neg\varphi \to \neg\varphi)}\,(\texttt{Nec.})}{C_a^\psi C_a^\varphi\neg\varphi \to C_a^\psi\neg\varphi}\,(\texttt{Ax.1})}{C_a^\psi\neg\varphi}\,(\to e)}{C_a^\varphi\neg\varphi \to C_a^\psi\neg\varphi}\,(\to i,1)$$

The proof uses both implication introduction and elimination, necessitation (Nec.) for the considers modality, and axioms 1, 2, and 4ii of ICDL. The inference labelled (Ax. 1)

[13]Note the right hand side of this axiom can be rewritten as $\bigwedge_{\alpha\in\mathcal{R}(\psi)}\bigvee_{\beta\in\mathcal{R}(\varphi)}(\neg B_a^\psi\neg\alpha \to B_a^\alpha\beta)$.

abbreviates a straightforward inference using implication elimination on axiom 1 in conjunction from the formula obtained via necessitation. □

Third, our axiomatisation includes the rule of replacement of equivalents. We suspect this is also required for the axiomatisation of CDL, as it is included in Board's axiomatisation 2004, p. 55, and the axiomatisation of CDL in Baltag, Renne, and Smets (2015) establishes its derivability from a different axiomatisation of CDL from Baltag and Smets (2006).

The modalities for both conditional belief and knowledge can be eliminated from the language of ICDL when enriched with these operators.

Proposition 3.4.2 (Elimination of the B modality.).

$$B_a^{\psi}\varphi \dashv\vdash \bigwedge_{\alpha\in\mathcal{R}(\psi)}(\neg C_a^{\psi}\neg\alpha \to \bigvee_{\beta\in\mathcal{R}(\varphi)} C_a^{\alpha}\beta)$$

Proposition 3.4.3 (Elimination of the K modality.).

$$K_a\varphi \dashv\vdash \bigvee_{\alpha\in\mathcal{R}(\varphi)} C_a^{\neg\alpha}\bot$$

Remark 3.4.4. We can also define alternative reductions for the belief modality, using a two part reduction via the following axioms.

1. $B_a^{\psi}\varphi \leftrightarrow \bigwedge_{\alpha\in\mathcal{R}(\psi)}(\neg B^{\psi}\neg\alpha \to B_a^{\alpha}\varphi)$
2. $B_a^{\alpha}?\{\beta_1,\dots,\beta_n\} \leftrightarrow B_a^{\alpha}\beta_1 \vee \cdots \vee B_a^{\alpha}\beta_n$

However, this requires the additional rule of the *general* replacement of equivalents.

$$\frac{\varphi \leftrightarrow \psi}{\chi \leftrightarrow \chi[{}^{\psi}/_{\varphi}]}$$

3.5 The Road Ahead

Our primary goal now is to establish that ICDL is sound and (weakly) complete with respect to inquisitive plausibility models. However, the route to this theorem will be tortuous.

First, we establish ICDL can be interpreted via a different semantic structure, termed inquisitive conditional-doxastic models. We then establish a strong connexion between IPMs and ICDMs, namely that every IPM can be transformed into an ICDM, and conversely every *finite* ICDM can be transformed into an IPM, preserving the interpretation of ICDL.

While the main purpose of this connexion is transfer the soundness and weak completeness of ICDL with respect to ICDMs to soundness and completeness with respect to IPMs (and in this respect ICDMs can be considered a technical tool), it is also the case that ICDMs provide a qualitative counterpart to the quantitative nature of IPMs in an analogous fashion to CDMs and PMs with respect to CDL. In any case, we will observe some properties of ICDMs along the way, and illuminate some aspects of ICDL.

Chapter 4

Inquisitive Conditional-Doxastic Models

While inquisitive plausibility models are our primary semantic structures for interpreting ICDL, as in the case of classical conditional doxastic logic an alternative interpretation of ICDL can be given through an alternative semantic structure, which we term *inquisitive-conditional doxastic models*. These are a particular class of neighbourhood models, which allows us to interpret each basic modality of ICDL as a neighbourhood function.

This alternative semantic foundation will ease the proof of the soundness and completeness of ICDL with respect to IPMs, for we shall see in the following chapter that there is a tight connexion between IPMs and ICDMs and will also allow us to highlight further properties of the logic.

4.1 Inquisitive Conditional-Doxastic Models

We begin with the definition of inquisitive conditional-doxastic models (ICDMs).

Definition 4.1.1 (Inquisitive conditional-doxastic models).

An inquisitive conditional-doxastic model for a set At of atomic formulas and a set $\mathcal{A}$ of agents, is a tuple: $\langle W, \{\mathcal{S}_a^P\}_{a\in\mathcal{A}, P\in I}, V\rangle$, where:

- W is a set of possible worlds
- I is the set of all issues over W
- $V : W \to \wp(\mathsf{At})$ is a valuation map, stating for each $w \in W$ the atomic formulas true at w
- $\mathcal{S}_a^P$ is a map $W \to \mathscr{I}$ associating to each world an issue, $\mathcal{S}_a^P(w)$, satisfying the following conditions:

Safety	if $w \in \vert P\vert$ then $\mathcal{S}_a^P(w) \neq \{\emptyset\}$, where $\vert P\vert := \bigcup P$
Introspection	if $v \in s_a^P(w)$, then $\mathcal{S}_a^Q(w) = \mathcal{S}_a^Q(v)$
Adjustment	$\mathcal{S}_a^P(w) \subseteq P$
Success	$\mathcal{S}_a^P(w) \neq \{\emptyset\}$, if $\mathcal{S}_a^Q(w) \cap P \neq \{\emptyset\}$
Minimality	$\mathcal{S}_a^{P\cap Q}(w) = \mathcal{S}_a^P(w) \cap Q$, if $\mathcal{S}_a^P(w) \cap Q \neq \{\emptyset\}$

We write $s_a^P(w)$ for $|\mathcal{S}_a^P(w)| = \bigcup s_a^P(w)$, $\mathcal{S}_a(w)$ for $\bigcup_{P\in\mathscr{I}} \mathcal{S}_a^P(w)$, and $s_a(w)$ for $|\mathcal{S}_a(w)|$.

Recall that in the inquisitive setting $\{\emptyset\}$ is the inconsistent proposition, and so conditions such as $\mathcal{S}^P_a(w) \neq \{\emptyset\}$ amount to $\mathcal{S}^P_a(w)$ being consistent. For, as $\mathcal{S}^P_a(w)$ is an issue it is always the case that $\{\emptyset\} \subseteq \mathcal{S}^P_a(w)$.

Furthermore, we will write $\mathcal{S}^\varphi_a(w)$ for $\mathcal{S}^{[\![\varphi]\!]}_a(w)$ when a formula expressing the issue P is known, and similarly we write $s^\varphi_a(w)$ for $s^{[\![\varphi]\!]}_a(w)$.

Intuitively, each state map $\mathcal{S}^P_a(w)$ captures the agent's general doxastic state, given P, comprising of both the information they believe to be true of the actual world, captured by $s^P_a(w)$, and the issues they consider on the basis of this. Therefore, the following clause is used to interpret the considers modality on ICDMs.

Definition 4.1.2 (Support for considers).

$$M, s \vDash C^\psi_a \varphi \text{ iff } \forall w \in s \text{ and } \forall t \in \mathcal{S}^\psi_a(w), M, t \vDash \varphi$$

As with CDMs the conditions of safety through to minimality place constraints on the process of conditionalisation captured by ICDMs. We observed in chapter 3 that in terms of their axiomatisation, CDL and ICDL both encode the same process of conditionalisation, whether through the considers or the believes modality, and here we see the qualitative way in which this process was interpreted by CDMs straightforwardly generalises to issues.

As a consequence of this it is considers, rather than believes, that generalises CDL conditional belief from a semantic point of view,[1] by reformulating the interpretation of CDL-conditional belief with issues. Indeed, an epistemic issue paralleling $\Sigma_a(w)$ factors into the definition of considers, only indirectly via the union of all state maps, unlike the corresponding definition on IPMs. So, in order to interpret the considers modality with respect to ICDMs no explicit use of an agent's epistemic goals need be made.

This flexibility means that agents need not be modelled to have epistemic goals, or any inquisitive state over and above that relevant to the considers modality. While intriguing, this aspect of ICDMs and ICDL will not be explored further in this thesis. However, it will be shown in chapter 5 that epistemic states are able to capture properties of an agent which doxastic maps are unable to, by showing that the entertains modality cannot be defined in terms of the considers modality (cf. proposition 5.3.1). Intuitively this means that an agent's epistemic goals cannot be extracted from the combination of their epistemic and conditional doxastic goals.

Definition 4.1.3 (Support for conditional belief and knowledge).
Let $M = \langle W, \{\mathcal{S}^P_a(w)\}_{a\in\mathcal{A}, P\in\mathcal{I}}, V\rangle$ be an ICDM and s an arbitrary subset of W:

- $M, s \vDash B^\psi_a \varphi$ iff $\forall w \in s$ and $\forall t \in (\wp(s^\psi_a(w)) \cap [\![\psi]\!]), M, t \vDash \varphi$
- $M, s \vDash K_a \varphi$ iff $\forall w \in s$ and $M, s_a(w) \vDash \varphi$

The following lemma establishes a basic simplification of the support clause for state maps which we shall appeal to on numerous occasions.

Lemma 4.1.4 (Basic lemma).

- $M, s \vDash C^\psi_a \varphi$ *if and only if* $\forall w \in s, \mathcal{S}^\psi_a(w) \subseteq [\![\varphi]\!]$
- $M, s \vDash \neg C^\psi_a \varphi$ *if and only if* $\forall w \in s, \mathcal{S}^\psi_a(w) \not\subseteq [\![\varphi]\!]$

Proof. Immediate via the support conditions for the considers modality. □

[1] Though both considers and believes generalise CDL from a syntactic point of view, as these modalities are equivalent with respect to declaratives, and hence classical propositions, as evidenced by axiom 7 of ICDL. What we mean is that the semantic clause for the considers modality straightforwardly parallels that of CDL conditional belief.

4.2 Observations

In certain cases it is easier to work with a restatement of the support conditions for doxastic modalities, where the support for a modality at a given state is derived from the truth of the modality at each world in the state. These conditions are summarised in the following proposition.

Proposition 4.2.1 (Modal formulas are truth-conditional)**.** *Let M be an* ICDM *and $s \subseteq W$ arbitrary, then:*

- $M, s \vDash C_a^{\psi}\varphi$ *iff* $\forall w \in s, M, w \vDash C_a^{\psi}\varphi$ *iff* $\forall w \in s, \forall t \in \mathcal{S}_a^{\psi}(w), M, t \vDash \varphi$
- $M, s \vDash B_a^{\psi}\varphi$ *iff* $\forall w \in s, M, w \vDash B_a^{\psi}\varphi$ *iff* $\forall w \in s, \forall t \in \wp(\mathcal{s}_a^{\psi}(w)) \cap [\![\psi]\!], M, t \vDash \varphi$
- $M, s \vDash K_a\,\varphi$ *iff* $\forall w \in s, M, w \vDash K_a\,\varphi$ *iff* $\forall w \in s, M, \mathcal{s}_a(w) \vDash \varphi$

We now establish two properties of state maps that will be of use.

Proposition 4.2.2. $\mathcal{S}_a^P(w) = \{\emptyset\}$ *iff* $\mathcal{S}_a^{!P}(w) = \{\emptyset\}$*, for all* $w \in W$*,* $P \in \mathcal{I}$*.*[2]

Proof. From left to right suppose $\mathcal{S}_a^{!P}(w) \neq \{\emptyset\}$. Then, there is some $t \in \mathcal{S}_a^{!P}(w)$ such that $t \neq \emptyset$. By downward closure this means there exists some $v \in t$ such that $\{v\} \in \mathcal{S}_a^{!P}(w)$. By adjustment we know $\{v\} \in\, !P$. We know that $\{u\} \in P$ iff $\{u\} \in\, !P$, for by fact 1.2.4 $\{u\} \in P$ iff $u \in |P|$ and $!P := \wp(|P|)$ by definition 1.2.20. So, we can infer $\{v\} \in P$. Therefore, by minimality we know that $\{v\} \in \mathcal{S}_a^P(w) = \mathcal{S}_a^{P\cap !P}(w) = \mathcal{S}_a^{!P}(w) \cap P$, for $\mathcal{S}_a^{!P}(w) \cap P \neq \{\emptyset\}$. This means $\mathcal{S}_a^P(w) \neq \{\emptyset\}$.

From right to left suppose $\mathcal{S}_a^{!P}(w) = \{\emptyset\}$. Then, by success we know $\mathcal{S}_a^P(w) \cap !P = \{\emptyset\}$. Reasoning again by the fact that $|P| = |!P|$, this entails $\mathcal{S}_a^P(w) = \{\emptyset\}$. ☐

Proposition 4.2.3. $\mathcal{S}_a^P(w) = \mathcal{S}_a^{!P}(w) \cap P$*, for all* $w \in W$*,* $P \in \mathcal{I}$*.*

Proof. There are two cases. The first is if the outcome of either state map is inconsistent. If this is so, then the conclusion is immediate by lemma 4.2.2. The second is if both state maps are consistent. Suppose this is the case.

From left to right suppose $t \in \mathcal{S}_a^P(w)$ but $t \notin \mathcal{S}_a^{!P}(w) \cap P$. By adjustment we have that $t \in P$, as we have assumed $t \in \mathcal{S}_a^P(w)$, therefore it must be the case that $t \notin \mathcal{S}_a^{!P}(w)$.

We have assumed that $\mathcal{S}_a^{!P}(w)$ is consistent, which entails $\mathcal{S}_a^{!P}(w) \cap P$ is consistent. For, by our assumption of consistency we know that $\exists u \in \mathcal{S}_a^{!P}(w)$ such that $u \neq \emptyset$, whence there is some world $v \in t$. Furthermore, adjustment, $\mathcal{S}_a^{!P}(w) \subseteq\, !P$, and so $t \in\, !P$, and as $!P := \wp(|P|)$ this ensures $v \in |P|$ from which $\{v\} \in P$ by fact 1.2.4, whence $\{v\} \in \mathcal{S}_a^{!P}(w) \cap P$.

Therefore, as $\mathcal{S}_a^{!P}(w) \cap P \neq \{\emptyset\}$, $\mathcal{S}_a^{!P}(w) \cap P = \mathcal{S}_a^{!P\cap P}(w)$, by minimality Yet, as $!P \cap P = P$ this entails $t \notin \mathcal{S}_a^P(w)$, a contradiction.

From right to left suppose $t \in \mathcal{S}_a^{!P}(w) \cap P$. It is trivially the case that if $t = \emptyset$ then $t \in \mathcal{S}_a^P(w)$, therefore let us assume $t \neq \emptyset$. As, $\mathcal{S}_a^{!P}(w) \cap P \neq \{\emptyset\}$, we know by minimality that $\mathcal{S}_a^P(w) = \mathcal{S}_a^{!P\cap P}(w) = \mathcal{S}_a^{!P}(w) \cap P$. And so $t \in \mathcal{S}_a^P(w)$. ☐

Corollary 4.2.4. $\mathcal{s}_a^{!P}(w) = \mathcal{s}_a^P(w)$*, for all* $w \in W$*,* $P \in \mathcal{I}$*.*

Proof. By definition $\mathcal{s}_a^P(w) = \bigcup \mathcal{S}_a^P(w)$ and $\mathcal{s}_a^{!P}(w) = \bigcup \mathcal{S}_a^{!P}(w)$, so we need to show that $\bigcup \mathcal{S}_a^P(w) = \bigcup \mathcal{S}_a^{!P}(w)$.

[2] Recall that by definition 1.2.20 $!P := \wp(|P|)$.

Let $v \in \bigcup \mathcal{S}_a^P(w)$ be arbitrary. By proposition 4.2.3 we have the following equivalence $\{v\} \in \mathcal{S}_a^P(w)$ iff $\{v\} \in \mathcal{S}_a^{!P}(w) \cap P$. From which the left to right direction immediately follows, as $v \in \bigcup \mathcal{S}_a^P(w)$ iff $\{v\} \in \mathcal{S}_a^{!P}(w) \cap P$ by the equivalence and fact 1.2.4, whence $\{v\} \in \mathcal{S}_a^{!P}(w)$, and so $v \in \bigcup \mathcal{S}_a^{!P}(w)$.

For the right to left suppose $v \in \bigcup \mathcal{S}_a^{!P}(w)$, then $\{v\} \in \mathcal{S}_a^{!P}(w)$, whence by adjustment $\{v\} \in\, !P$. By definition 1.2.20 $!P = \wp(|P|)$, so $v \in |P|$ and so by fact 1.2.4 we know $\{v\} \in P$, ensuring $\{v\} \in \mathcal{S}_a^{!P}(w) \cap P$ from which we can reason that $v \in \bigcup \mathcal{S}_a^P(w)$ by the above equivalence from proposition 4.2.3. □

The following proposition will be of help in establishing the soundness of ICDL with respect to ICDMS.

Corollary 4.2.5. *For all formulas ψ and declaratives α, $C_a^{\psi} \alpha \equiv C_a^{!\psi} \alpha$*

Proof. We reason contrapositively.

From left to right suppose $M, w \not\vDash C_a^{!\psi} \alpha$. Then, $\exists t \in \mathcal{S}_a^{!\psi}(a), M, t \not\vDash \alpha$, whence by proposition 1.2.15 we know that for some $v \in s_a^{!\psi}(a), M, v \not\vDash \alpha$. By corollary 4.2.4 we know that $s_a^{!\psi}(a) = s_a^{\psi}(a)$, and so we know that for some $v \in s_a^{\psi}(w), M, v \not\vDash \alpha$. Therefore, $\exists t \in \mathcal{S}_a^{\psi}(w), M, t \not\vDash \alpha$, namely $t = \{v\}$. Therefore, $M, w \not\vDash C_a^{\psi} \alpha$.

The converse is established analogously. □

Remark 4.2.6. This is the conditional analogue of the fact that $\mu \to \alpha \equiv\, !\mu \to \alpha$ (cf. proposition 1.2.22).

Corollary 4.2.7. *A formula ψ is consistent with the beliefs of a given agent a, conditional on φ, if and only if it is consistent conditional on the informative content of φ. Formally:*

$$\neg B_a^{\varphi} \neg \psi \equiv \neg B_a^{!\varphi} \neg \psi$$

Proof. This follows from axiom 7 of ICDL, which states $B_a^{\psi} \alpha \leftrightarrow C_a^{\psi} \alpha$, that any formula of the form $\neg \psi$ is a declarative, and that declaratives behave classically. □

Theorem 5.2.1 of chapter 5 establishes that any IPM can be transformed into an equivalent ICDM, therefore to establish soundness of ICDL with respect to both IPMs and ICDMs it suffices to establish the soundness of ICDL with respect to ICDMs.

4.3 Soundness of ICDL with Respect to ICDMS

We show that axioms 1–9 are valid with respect to ICDMs, and that the two introduced rules of proof respect support, in the sense that if φ can be derived from $\varphi_1, \ldots, \varphi_n$ and $M, s \vDash \varphi_i$ for $i \leq n$ then $M, s \vDash \varphi$.

Theorem 4.3.1 (Soundness of ICDL wrt. ICDMs.)**.** *ICDL is sound with respect to ICDMs.*

Remark 4.3.2. We sketch the background required for a proof of soundness.

Each axiom of ICDL is a declarative, and so by proposition 1.2.15 soundness can be established via appeal to truth conditions. This observation allow us to sidestep many unnecessary repetitions and complications.

Proof.

1. $\boldsymbol{C_a^{\psi}(\varphi \to \chi) \to (C_a^{\psi}\varphi \to C_a^{\psi}\chi)}$

Suppose not. Then there exists some world w such that $M, w \vDash C_a^{\psi}(\varphi \to \chi)$ and $M, w \vDash C_a^{\psi}\varphi$ but $M, w \nvDash C_a^{\psi}\chi$. So, we know by the basic lemma that $\mathcal{S}_a^{\psi}(w) \subseteq [\![\varphi]\!]$ and $\mathcal{S}_a^{\psi}(w) \subseteq [\![\varphi \to \chi]\!]$. From the support clause from implication it is immediate that $\mathcal{S}_a^{\psi}(w) \subseteq [\![\chi]\!]$, yet this contradicts the fact that $M, w \nvDash C_a^{\psi}\chi$.

2. $\boldsymbol{C_a^{\varphi}\neg\varphi \to \neg\varphi}$

Suppose not. Then for some world w it is the case that $M, w \vDash C_a^{\varphi}\neg\varphi$ while $M, w \nvDash \neg\varphi$. However, it is then the case by the basic lemma that $\mathcal{S}_a^{\varphi}(w) \subseteq [\![\neg\varphi]\!]$. So, by adjustment that $\mathcal{S}_a^{\varphi}(w) \subseteq [\![\varphi]\!]$, whence it must be the case that $\mathcal{S}_a^{\varphi}(w) = \{\emptyset\}$.

However, as $M, w \nvDash \neg\varphi$ we know that $M, w \vDash \neg\neg\varphi$, and so by fact 1.2.16 we know $M, w \vDash \varphi$. Yet, this means that $\{w\} \in [\![\varphi]\!]$, whence $w \in |\varphi|$, and so by safety we know that $\mathcal{S}_a^{\varphi}(w) \neq \{\emptyset\}$, contradicting what we inferred above, and so our initial assumption must have been mistaken.

3. $\boldsymbol{C_a^{\varphi}\varphi}$

Suppose not. This means that there is a world w such that $M, w \nvDash C_a^{\varphi}\varphi$. By the support clause for C this means $\exists t \in \mathcal{S}_a^{\varphi}(w), M, t \nvDash \varphi$, which entails $\mathcal{S}_a^{\varphi}(w) \nsubseteq [\![\varphi]\!]$. This contradicts the condition adjustment on state maps.

4. $\boldsymbol{C_a^{\psi}\varphi \to C_a^{\chi}C_a^{\psi}\varphi}$ **and** $\boldsymbol{\neg C_a^{\psi}\varphi \to C_a^{\chi}\neg C_a^{\psi}\varphi}$

i $\boldsymbol{C_a^{\psi}\varphi \to C_a^{\chi}C_a^{\psi}\varphi}$

Suppose $M, w \vDash C_a^{\psi}\varphi$ but $M, w \nvDash C_a^{\chi}C_a^{\psi}\varphi$. So, for some $t \in \mathcal{S}_a^{\chi}(w), M, t \nvDash C_a^{\psi}\varphi$. Therefore, for some $v \in t, M, v \vDash \neg C_a^{\psi}\varphi$. As $v \in t$ and $t \in \mathcal{S}_a^{\chi}(w)$, $v \in s_a^{\chi}(w)$, whence by introspection we know $\mathcal{S}_a^{\psi}(w) = \mathcal{S}_a^{\psi}(v)$. Yet, this cannot be the case, for it would imply $M, w \vDash C_a^{\psi}\varphi$ iff $M, v \vDash C_a^{\psi}\varphi$.

ii $\boldsymbol{\neg C_a^{\psi}\varphi \to C_a^{\chi}\neg C_a^{\psi}\varphi}$

Suppose $M, w \vDash \neg C_a^{\psi}\varphi$, but $M, w \nvDash C_a^{\chi}\neg C_a^{\psi}\varphi$. So, for some $t \in \mathcal{S}_a^{\chi}(w), M, t \nvDash \neg C_a^{\psi}\varphi$. This means that for some $v \in t$, $M, v \vDash C_a^{\psi}\varphi$. But then, as $v \in t$ and $t \in \mathcal{S}_a^{\chi}(w)$ we have $v \in s_a^{\chi}(w)$, and so by introspection, $\mathcal{S}_a^{\psi}(w) = \mathcal{S}_a^{\psi}(v)$. This cannot be the case, for as before it would imply $M, w \vDash C_a^{\psi}\varphi$ iff $M, v \vDash C_a^{\psi}\varphi$.

5. $\boldsymbol{\neg C^{\psi}\neg\varphi \to (C_a^{\psi\wedge\varphi}\chi \leftrightarrow C_a^{\psi}(\varphi \to \chi))}$

Suppose $M, w \vDash \neg C_a^{\psi}\neg\varphi$. We now show $M, w \vDash C_a^{\psi\wedge\varphi}\chi \to C_a^{\psi}(\varphi \to \chi)$ and $M, w \vDash C_a^{\psi}(\varphi \to \chi) \to C_a^{\psi\wedge\varphi}\chi$

$\boldsymbol{M, w \vDash C_a^{\psi\wedge\varphi}\chi \to C_a^{\psi}(\varphi \to \chi)}$

Suppose $M, w \vDash C_a^{\psi\wedge\varphi}\chi$ while $M, w \nvDash C_a^{\psi}(\varphi \to \chi)$. The latter entails $\mathcal{S}_a^{\psi}(w) \nsubseteq [\![\varphi \to \chi]\!]$. From this we infer there exists some $t \in \mathcal{S}_a^{\psi}(w)$ such that $M, t \vDash \varphi$ but $M, t \nvDash \chi$.

We claim $\mathcal{S}_a^{\psi}(w) \cap [\![\varphi]\!] \neq \{\emptyset\}$. For, suppose not. Then, $\{\emptyset\} = \mathcal{S}_a^{\psi}(w) \cap [\![\varphi]\!]$. This means that the t we observed to exist above must be the empty set. However, $M, \emptyset \vDash \phi$ for all ϕ, and so $M, t \vDash \chi$, a contradiction. Therefore, by minimality we know $\mathcal{S}_a^{\psi\wedge\varphi}(w) = \mathcal{S}_a^{\psi}(w) \cap [\![\varphi]\!]$. Yet, by the basic lemma and the fact that $M, w \vDash C_a^{\psi\wedge\varphi}\chi$ we know $\mathcal{S}_a^{\psi\wedge\varphi}(w) \subseteq [\![\chi]\!]$ and so $\mathcal{S}_a^{\psi}(w) \cap [\![\varphi]\!] \subseteq [\![\chi]\!]$. This contradicts the existence of t.

$\boldsymbol{M, w \vDash C_a^{\psi}(\varphi \rightarrow \chi) \rightarrow C_a^{\psi \wedge \varphi}\chi}$

Suppose $M, w \vDash C_a^{\psi}(\varphi \rightarrow \chi)$. We show $M, w \vDash C_a^{\psi\wedge\varphi}\chi$.

First, we claim $\mathcal{S}_a^{\psi}(w) \cap [\![\varphi]\!] \neq \{\emptyset\}$. We know $M, w \vDash \neg C_a^{\psi}\neg\varphi$. Therefore, $\mathcal{S}_a^{\psi}(w) \nsubseteq [\![\neg\varphi]\!]$.

From this it follows that $\exists t \in \mathcal{S}_a^{\psi}(w), M, t \nvDash \neg\varphi$. Moreover, we know $t \neq \emptyset$, as if so it would be the case that $M, t \vDash \chi$ for all propositions χ. So, as $M, t \nvDash \neg\varphi$ we know that for some $v \in t$, $M, v \nvDash \neg\varphi$ by proposition 1.2.15, whence by fact 1.2.16, $M, v \vDash \varphi$. Moreover, as issues are downward closed we know $\{v\} \in \mathcal{S}_a^{\psi}(w)$. So, there exists some state $t' \in \mathcal{S}_a^{\psi}(w)$ such that $M, t' \vDash \varphi$. This establishes the claim and entails $\mathcal{S}_a^{\psi\wedge\varphi}(w) = \mathcal{S}_a^{\psi}(w) \cap [\![\varphi]\!]$, by minimality.

As $M, w \vDash C_a^{\psi}(\varphi \rightarrow \chi)$ this entails that if $u \in \mathcal{S}_a^{\psi}(w) \cap [\![\varphi]\!]$ then $M, u \vDash \chi$. Yet this means that for all $u \in \mathcal{S}_a^{\psi\wedge\varphi}(w)$ then $u \vDash \chi$. Therefore, $M, w \vDash C_a^{\psi\wedge\varphi}\chi$, by the basic lemma.

7. $\boldsymbol{B_a^{\psi}\alpha \leftrightarrow C_a^{\psi}\alpha}$

First we observe $\mathcal{S}_a^{\psi}(w) \subseteq \wp(s_a^{\psi}(w))$. For, if $t \in \mathcal{S}_a^{\psi}(w)$, then $t \subseteq s_a^{\psi}(w)$, and $s_a^{\psi}(w) := \bigcup \mathcal{S}_a^{\psi}(w)$, so $t \in \wp(s_a^{\psi}(w))$.

Suppose $M, w \vDash B_a^{\psi}\alpha$. Then, $\forall t \in (\wp(s_a^{\psi}(w)) \cap [\![\psi]\!]), M, t \vDash \alpha$. But then since $\mathcal{S}_a^{\psi}(w) \subseteq \wp(s_a^{\psi}(w))$ and $\mathcal{S}_a^{\psi}(w) \subseteq [\![\psi]\!]$ we can infer $\forall t \in \mathcal{S}_a^{\psi}(w), M, t \vDash \alpha$, whence $M, w \vDash C_a^{\psi}\alpha$. Therefore, $M, w \vDash B_a^{\psi}\alpha \rightarrow C_a^{\psi}\alpha$

Suppose $M, w \vDash \neg B_a^{\psi}\alpha$. So $M, w \nvDash B_a^{\psi}\alpha$, and for some $t \in (\wp(s_a^{\psi}(w)) \cap [\![\psi]\!])$, $M, t \nvDash \alpha$. From this we infer there is some $v \in t$ such that $M, v \nvDash \alpha$. It is simple to observe $v \in s_a^{\psi}(w)$, and so we know $\{v\} \in \mathcal{S}_a^{\psi}(w)$. And, as $\{v\} \in \mathcal{S}_a^{\psi}(w)$ and $M, v \nvDash \alpha$ we know $M, w \nvDash C_a^{\psi}\alpha$, whence $M, w \vDash \neg C_a^{\psi}\alpha$.

Therefore, $M, w \vDash \neg B_a^{\psi}\alpha \rightarrow \neg C_a^{\psi}\alpha$, which, as worlds behave classically, we contrapose to obtain $M, w \vDash C_a^{\psi}\alpha \rightarrow B_a^{\psi}\alpha$

8. $\boldsymbol{B_a^{\psi}\varphi \leftrightarrow \bigwedge_{\alpha\in\mathcal{R}(\psi)}(\neg B_a^{\psi}\neg\alpha \rightarrow \bigvee_{\beta\in\mathcal{R}(\varphi)} B_a^{\alpha}\beta)}$

We split the proof into two cases.

a) $\boldsymbol{B_a^{\psi}\varphi \rightarrow \bigwedge_{\alpha\in\mathcal{R}(\psi)}(\neg B_a^{\psi}\neg\alpha \rightarrow \bigvee_{\beta\in\mathcal{R}(\varphi)} B_a^{\alpha}\beta)}$

Suppose $M, w \vDash B_a^{\psi}\varphi$ while $M, w \nvDash \bigwedge_{\alpha\in\mathcal{R}(\psi)}(\neg B_a^{\psi}\neg\alpha \rightarrow \bigvee_{\beta\in\mathcal{R}(\varphi)} B_a^{\alpha}\beta)$. So, by the latter $M, w \vDash \neg\bigwedge_{\alpha\in\mathcal{R}(\psi)}(\neg B_a^{\psi}\neg\alpha \rightarrow \bigvee_{\beta\in\mathcal{R}(\varphi)} B_a^{\alpha}\beta)$. This, as worlds behave classically, is equivalent to $M, w \vDash \bigvee_{\alpha\in\mathcal{R}(\psi)}\neg(\neg B_a^{\psi}\neg\alpha \rightarrow \bigvee_{\beta\in\mathcal{R}(\varphi)} B_a^{\alpha}\beta)$. So, there must be some $\alpha \in \mathcal{R}(\psi)$ such that $M, w \vDash \neg(\neg B_a^{\psi}\neg\alpha \rightarrow \bigvee_{\beta\in\mathcal{R}(\varphi)} B_a^{\alpha}\beta)$. Therefore, for such an α, $M, w \vDash \neg B_a^{\psi}\neg\alpha$ while $M, w \nvDash \bigvee_{\beta\in\mathcal{R}(\varphi)} B_a^{\alpha}\beta$. The latter entails $M, w \vDash \bigwedge_{\beta\in\mathcal{R}(\varphi)}\neg B_a^{\alpha}\beta$.

Suppose $M, w \vDash B_a^{\alpha}\varphi$. Then, by proposition 1.2.27, $M, w \vDash B_a^{\alpha}?\mathcal{R}(\varphi)$. And, from this it follows by the semantics clause from conditional belief and the interrogative operator ? that $\forall t \in \wp(s_a^{\alpha}(w)) \cap [\![\alpha]\!], M, t \vDash \beta_1$ or $\ldots$ or $M, t \vDash \beta_n$, for $\beta_1, \ldots, \beta_n \in \mathcal{R}(\varphi)$. As α is a declarative, $[\![\alpha]\!]$ has a greatest state, as does $\wp(s_a^{\alpha}(w))$, whence $\wp(s_a^{\alpha}(w)) \cap [\![\alpha]\!]$ has a greatest state. Let g denote this state.

By the above reasoning we know that $M, g \vDash \beta_i$ for some $\beta_i \in \mathcal{R}(\varphi)$, and by persistence this holds for all $t \subseteq g$. Therefore, as g is the greatest state of $\wp(s_a^{\alpha}(w)) \cap [\![\alpha]\!]$ we know that $\forall t \in \wp(s_a^{\alpha}(w)) \cap [\![\alpha]\!], M, t \vDash \beta_i$, whence $M, w \vDash B_a^{\alpha}\beta_i$.

Yet, we know for all $\beta_j \in \mathcal{R}(\varphi)$ that $M, w \vDash \neg B_a^{\alpha}\beta_j$, which contradicts the fact that $M, w \vDash B_a^{\alpha}\beta_i$. Therefore, our assumption was mistaken and $M, w \nvDash B_a^{\alpha}\varphi$.

Taking stock, we have assumed $M, w \vDash B_a^{\psi}\varphi$ while $M, w \nvDash \bigwedge_{\alpha\in\mathcal{R}(\psi)}(\neg B_a^{\psi}\neg\alpha \to \bigvee_{\beta\in\mathcal{R}(\varphi)} B_a^{\alpha}\beta)$, and inferred that for some $\alpha \in \mathcal{R}(\psi)$, $M, w \vDash \neg B_a^{\psi}\neg\alpha$, while $M, w \nvDash B_a^{\alpha}\varphi$.

As $M, w \vDash \neg B_a^{\psi}\neg\alpha$ we know there exists some $t \in (\wp(s_a^{\psi}(w)) \cap [\![\psi]\!])$ such that $M, t \nvDash \neg\alpha$. So, by proposition 1.2.15 we know there exists some $v \in t$ such that $M, v \nvDash \neg\alpha$, which entails $M, v \vDash \alpha$. As $v \in t$ we know $v \in s_a^{\psi}(w)$, which entails $\{v\} \in \mathcal{S}_a^{\psi}(w)$. This means $\mathcal{S}_a^{\psi}(w) \cap [\![\alpha]\!] \neq \{\emptyset\}$. Therefore, by minimality we know $\mathcal{S}_a^{\psi}(w) \cap [\![\alpha]\!] = \mathcal{S}_a^{\psi\wedge\alpha}(w)$. Therefore, as $[\![\psi]\!] \cap [\![\alpha]\!] = [\![\alpha]\!]$, for $\alpha \in \mathcal{R}(\psi)$, this entails $\mathcal{S}_a^{\psi}(w) \cap [\![\alpha]\!] = \mathcal{S}_a^{\alpha}(w)$.

From the above it follows that $\mathcal{S}_a^{\alpha}(w) \subseteq \mathcal{S}_a^{\psi}(w)$, and so $s_a^{\alpha}(w) \subseteq s_a^{\psi}(w)$, whence $\wp(s_a^{\alpha}(w)) \subseteq \wp(s_a^{\psi}(w))$. Moreover, as α is a resolution of ψ, $[\![\alpha]\!] \subseteq [\![\psi]\!]$. From these facts we infer $\wp(s_a^{\alpha}(w)) \cap [\![\alpha]\!] \subseteq \wp(s_a^{\psi}(w)) \cap [\![\psi]\!]$.

We know $M, w \vDash B_a^{\chi}\xi$ iff $\wp(s_a^{\chi}(w)) \cap [\![\chi]\!] \subseteq [\![\xi]\!]$, for any formulas χ, ξ by the support clause for the conditional belief modality. Therefore, $\wp(s_a^{\psi}(w)) \cap [\![\psi]\!] \subseteq [\![\varphi]\!]$, by our assumption that $M, w \vDash B_a^{\psi}\varphi$, whence from the above reasoning $M, w \vDash B_a^{\alpha}\varphi$, contradicting our earlier sub-conclusion that $M, w \nvDash B_a^{\alpha}\varphi$.

b) $\boldsymbol{\bigwedge_{\alpha\in\mathcal{R}(\psi)}(\neg B_a^{\psi}\neg\alpha \to \bigvee_{\beta\in\mathcal{R}(\varphi)} B_a^{\alpha}\beta) \to B_a^{\psi}\varphi}$

Suppose $M, w \vDash \bigwedge_{\alpha\in\mathcal{R}(\psi)}(\neg B_a^{\psi}\neg\alpha \to \bigvee_{\beta\in\mathcal{R}(\varphi)} B_a^{\alpha}\beta)$ while $M, w \nvDash B_a^{\psi}\varphi$.

From these assumptions we begin by making five observations. i. As $M, w \nvDash B_a^{\psi}\varphi$ we know that $\exists t \in (\wp(s_a^{\psi}(w)) \cap [\![\psi]\!])$ such that $M, t \nvDash \varphi$. ii. As $t \in [\![\psi]\!]$ we know that $M, t \vDash \alpha$ for some $\alpha \in \mathcal{R}(\psi)$. iii. Yet, as $M, t \nvDash \varphi$ we know that $M, t \nvDash \beta$ for any $\beta \in \mathcal{R}(\varphi)$. iv. as $M, t \nvDash \varphi$ we know that $t \neq \emptyset$, for $M, \emptyset \vDash \varphi$. v. And, as $t \in (\wp(s_a^{\psi}(w))$ we know that $t \subseteq s_a^{\psi}(w)$, whence for all $v \in t$, $\{v\} \in \mathcal{S}_a^{\psi}(w)$.

By the first of the previous observations we know that for each $v \in t$, $\{v\} \in [\![\alpha]\!]$. From this and the last of the observations we know that $\mathcal{S}_a^{\psi}(w) \cap [\![\alpha]\!] \neq \{\emptyset\}$. So, by minimality we know $\mathcal{S}_a^{\psi\wedge\alpha}(w) = \mathcal{S}_a^{\alpha}(w) = \mathcal{S}_a^{\psi}(w) \cap [\![\alpha]\!]$.

Therefore, as $\forall v \in t$, $\{v\} \in [\![\alpha]\!]$ and $\{v\} \in \mathcal{S}_a^{\psi}(w)$ we infer that $\forall v \in t$, $\{v\} \in \mathcal{S}_a^{\alpha}(w)$. So, $t \subseteq s_a^{\alpha}(w)$, whence $t \in \wp(s_a^{\alpha}(w))$. And therefore, as $t \in [\![\alpha]\!]$ by the second observation, we know that $t \in \wp(s_a^{\alpha}(w)) \cap [\![\alpha]\!]$. This means that we have some $t \in \wp(s_a^{\alpha}(w)) \cap [\![\alpha]\!]$ such that $M, t \nvDash \beta$ for any $\beta \in \mathcal{R}(\varphi)$, whence $M, w \nvDash B_a^{\alpha}\varphi$.

By observation ii we know that $M, w \nvDash B_a^{\psi}\neg\alpha$, for $t \in \mathcal{S}_a^{\psi}(w)$ and $M, t \vDash \alpha$, whence $M, w \vDash \neg B_a^{\psi}\neg\alpha$ as worlds behave classically.

So, given that we know $M, w \vDash \bigwedge_{\alpha\in\mathcal{R}(\psi)}(\neg B_a^{\psi}\neg\alpha \to \bigvee_{\beta\in\mathcal{R}(\varphi)} B_a^{\alpha}\beta)$, we infer $M, w \vDash \bigvee_{\beta\in\mathcal{R}(\varphi)} B_a^{\alpha}\beta$. This means for some $\beta \in \mathcal{R}(\varphi)$ we have $M, w \vDash B_a^{\alpha}\beta$, and so $\forall s \in (\wp(s_a^{\alpha}(w)) \cap [\![\alpha]\!])$, $M, s \vDash \beta$. Yet, by the above reasoning we know $t \in (\wp(s_a^{\alpha}(w)) \cap [\![\alpha]\!])$, and by observation iii we know that $M, t \nvDash \beta$ for any $\beta \in \mathcal{R}(\varphi)$. Therefore we have derived a contradiction.

9. $\boldsymbol{K_a\varphi \leftrightarrow \bigvee_{\alpha\in\mathcal{R}(\varphi)} C_a^{\neg\alpha}\bot}$

From left to right suppose $M, w \vDash K_a\varphi$. Then $M, s_a(w) \vDash \varphi$, and from this we infer $M, s_a(w) \vDash \alpha$ for some $\alpha \in \mathcal{R}(\varphi)$ by proposition 1.2.26. As $s_a(w) := \bigcup_{P\in\mathscr{P}} s_a(w)$, we know $s_a^{\neg\alpha}(w) \subseteq s_a(w)$, and therefore by persistence we can infer $M, s_a^{\neg\alpha}(w) \vDash \alpha$.

Therefore, $\mathcal{S}_a^{\neg\alpha}(w) = \{\emptyset\}$ by the basic lemma and the condition of adjustment of ICDMs. So, $M, w \vDash C_a^{\neg\alpha}\bot$, whence $M, w \vDash \bigvee_{\alpha\in\mathcal{R}(\varphi)} B_a^{\neg\alpha}\bot$.

Conversely, suppose $M, w \vDash \bigvee_{\alpha\in\mathcal{R}(\varphi)} C_a^{\neg\alpha}\bot$. Then $M, w \vDash C_a^{\neg\alpha}\bot$ for some $\alpha \in \mathcal{R}(\varphi)$. So, we know that $\mathcal{S}_a^{\neg\alpha}(w) = \{\emptyset\}$ by the basic lemma. Therefore, by the success condition on ICDMs we know that $\mathcal{S}_a^{P}(w) \cap [\![\neg\alpha]\!] = \{\emptyset\}$. From this it follows that $\bigcup_{Q\in\mathscr{I}} \mathcal{S}_a^{Q}(w) \cap [\![\neg\alpha]\!] = \{\emptyset\}$, whence $\mathcal{S}_a(w) \cap [\![\neg\alpha]\!] = \{\emptyset\}$, and so $\mathcal{S}_a(w) \subseteq [\![\alpha]\!]$. So, we know that $\forall w \in s_a(w), \{w\} \in [\![\alpha]\!]$, whence $s_a(w) \in [\![\alpha]\!]$. Therefore, $M, s_a(w) \vDash \alpha$, whence $M, s_a(w) \vDash \varphi$ by proposition 1.2.26. So, $M, w \vDash K_a\varphi$.

10. Necessitation

Suppose φ is valid, and let M be an arbitrary model. Then for any $s \subseteq W$, $M, s \vDash \varphi$. This means that $[\![\varphi]\!]_M = \wp(W)$. Therefore, for an arbitrary ψ and world v, $\mathcal{S}_a^{\psi}(v) \subseteq [\![\varphi]\!]$. Therefore, $M, v \vDash C_a^{\psi}\varphi$ for any world v, whence $M, s \vDash C_a^{\psi}\varphi$, for any state s, by proposition 1.2.15.

11. Replacement of Equivalents

Given the proof system adopted for ICDL allows for undischarged assumptions want to show that from assumptions $\varphi_1, \ldots, \varphi_n$ that if $\varphi \leftrightarrow \psi$ is derivable then $C_a^{\varphi}\chi \leftrightarrow C_a^{\psi}\chi$ is derivable. This is shown via induction on the length of proof.

To use the rule of replacement of equivalents we must have a proof that $\{\varphi_1, \ldots, \varphi_n\} \vdash \varphi \leftrightarrow \psi$ of shorter length than the proof that $\{\varphi_1, \ldots, \varphi_n\} \vdash C_a^{\varphi}\chi \leftrightarrow C_a^{\psi}\chi$. Therefore, we can assume via the induction hypothesis that $\{\varphi_1, \ldots, \varphi_n\} \vDash \varphi \leftrightarrow \psi$.

Let M be an arbitrary ICDM and s as state such that, $M, s \vDash \varphi_i$ for $i \leq n$. Then we know that for all states s, $M, s \vDash \varphi \leftrightarrow \psi$, whence $[\![\varphi]\!]_M = [\![\psi]\!]_M$. Therefore, by definition $\mathcal{S}_a^{\varphi}(w) = \mathcal{S}_a^{\psi}(w)$ for any $w \in s$. Therefore by the basic lemma we infer $M, w \vDash C_a^{\varphi}\chi \leftrightarrow C_a^{\psi}\chi$, whence $M, s \vDash C_a^{\varphi}\chi \leftrightarrow C_a^{\psi}\chi$, for any state s, by proposition 1.2.15. □

4.4 Some Rules and Theorems of ICDL

Remark 4.4.1. The following rules and theorems of InqD will be used in order to establish the completeness of ICDL with respect to ICDMs. As these will be used before the completeness result we establish each result syntactically, and in full.

Lemma 4.4.2. *The following rule of inference is derivable in ICDL.*

$$\frac{C_a^{\varphi}(\psi \wedge \chi)}{C_a^{\varphi}\psi}\ (\texttt{Cw1.})$$

Proof. Note we only apply the rule of necessitation for the considers modality only after both the assumptions made are discharged.

$$\dfrac{C_a^{\varphi}(\psi \wedge \chi) \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\psi \wedge \chi]^1}{\psi}\,(\wedge e)}{(\psi \wedge \chi) \to \psi}\,(\to i,1)}{C_a^{\varphi}((\psi \wedge \chi) \to \psi)}\,(\texttt{Nec.})}{C_a^{\varphi}(\psi \wedge \chi) \to C_a^{\varphi}\psi}\,(\texttt{Ax.})}{C_a^{\varphi}\psi}\,(\to e)$$

□

Lemma 4.4.3. *Each instance of the following schemas is a theorem of* ICDL.

1. $C_a^{\varphi}\chi \to C_a^{\varphi}(\psi \to \chi)$
2. $\neg C_a^{\varphi}\neg\psi \to (C_a^{\varphi}\chi \to C_a^{\varphi\wedge\psi}\chi)$

Proof.

1 Note we only apply the rule of necessitation for the considers modality only after both the assumptions made are discharged.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\psi]^1 \quad [\chi]^2}{\chi}\,(\wedge i,\wedge e)}{\psi \to \chi}\,(\to i,1)}{\chi \to (\psi \to \chi)}\,(\to i,2)}{C_a^{\varphi}(\chi \to (\psi \to \chi))}\,(\texttt{Nec.})}{C_a^{\varphi}\chi \to C_a^{\varphi}(\psi \to \chi)}\,(\texttt{Ax.1})$$

2

$$\dfrac{\dfrac{\dfrac{\dfrac{[C_a^{\varphi}\chi]^1 \quad \dfrac{}{C_a^{\varphi}\chi \to C_a^{\varphi}(\psi \to \chi)}\,(\texttt{Thm.1})}{C_a^{\varphi}(\psi \to \chi)}\,(\to e) \qquad \dfrac{\dfrac{[\neg C_a^{\varphi}\neg\psi]^2 \quad \dfrac{}{\neg C_a^{\varphi}\neg\psi \to (C_a^{\varphi\wedge\psi}\chi \leftrightarrow C_a^{\varphi}(\psi \to \chi))}\,(\texttt{Ax.5})}{C_a^{\varphi\wedge\psi}\chi \leftrightarrow C_a^{\varphi}(\psi \to \chi)}\,(\to e)}{C_a^{\varphi}(\psi \to \chi) \to C_a^{\varphi\wedge\psi}\chi}\,(\wedge e)}{C_a^{\varphi\wedge\psi}\chi}\,(\to e)}{C_a^{\varphi}\chi \to C_a^{\varphi\wedge\psi}\chi}\,(\to i,1)}{\neg C_a^{\varphi}\neg\psi \to (C_a^{\varphi}\chi \to C_a^{\varphi\wedge\psi}\chi)}\,(\to i,2)$$

□

Finally we observe a 'meta-meta-rule' of ICDL in the following.

Lemma 4.4.4. *If* $\varphi \dashv\vdash \psi$ *then* $C_a^{\varphi}\chi \dashv\vdash C_a^{\psi}\chi$.

Proof. Suppose $\varphi \dashv\vdash \psi$. Then there exists a proof of φ on the (sole) assumption of ψ and conversely a proof of ψ on the assumption of φ. Therefore, by applying the rules of implication introduction to both proofs we can discharge the assumptions of each to obtain proofs of $\varphi \to \psi$ and $\psi \to \varphi$ from no assumptions. Therefore, by the rule of conjunction introduction we have $\vdash \varphi \leftrightarrow \psi$.

So, by the rule of replacements of equivalents we know $\vdash C_a^{\varphi}\chi \leftrightarrow C_a^{\psi}\chi$, for any χ. From this we can use the rule of conjunction elimination and modus ponens to obtain a proof of $C_a^{\varphi}\chi$ on the assumption of ψ and likewise of $C_a^{\psi}\chi$ on the assumption of $C_a^{\varphi}\chi$. Therefore, $C_a^{\varphi}\chi \dashv\vdash C_a^{\psi}\chi$. □

Chapter 5

Connexions

We now begin the road to establishing inquisitive conditional-doxastic logic is sound and complete with respect to plausibility models. This is established by showing each ICDM can be transformed into a inquisitive plausibility model, and each plausibility model can be transformed to an ICDM, each transformation preserving the interpretation of ICDL. Meaning, that for any given ICDM M and its corresponding IPM $M^\sharp$, $M, s \vDash \varphi$ iff $M^\sharp, s \vDash \varphi$, for any ICDL formula φ, and equivalently for the transformation of a given IPM M to an ICDM $M^\flat$.[1]

The two subcomponents of this goal, soundness and completeness respectively, require different aspects of the transformation.

For completeness we need to ensure we can transform the canonical model for ICDL, which will be an ICDM, into an IPM. And, as the canonical will be *finite* this means we can restrict our attention to transforming finite ICDMs into IPMs. Indeed, the assumption that our initial ICDM is finite will be necessary to ensure the ordering of the resulting IPM is a well-preorder

However, for soundness we require that any IPM can be transformed into an ICDM, to be sure that for any IPM countermodel to an given axiom a corresponding ICDM countermodel could be found.

5.1 From ICDMs to Plausibility Models

Theorem 5.1.1 (From ICDMs to IPMs). *Any finite ICDM can be transformed into an IPM preserving the interpretation of ICDL.*

To prove theorem 5.1.1 we first define a method to transform a given ICDM into an IPM, before proving that if the given ICDM is finite, then the structure it gives rise to is indeed an IPM. The assumption of finiteness turns out to be crucial in establishing the ordering on worlds generated by the method of transformation is indeed well-founded. However, this is not strictly a limitation on the method of transformation, as the axioms of ICDL are insufficient to establish its canonical structure has a well-founded ordering.

Definition 5.1.2 (Map from ICDMs to IPMs).

Given an arbitrary ICDM, $M = \langle W, \{\mathcal{S}_a^P\}_{a\in\mathcal{A}, P\in\mathcal{I}}, V\rangle$, we define a map $M \mapsto M^\sharp$, where $M^\sharp = \langle W^\sharp, \{\leq_a^w\}_{a\in\mathcal{A}, w\in W}, \{\Sigma_a\}_{a\in\mathcal{A}}, V^\sharp\rangle$ is constructed in the following way:

[1] Note our mappings will ensure the sets of possible worlds for both models is the same, and therefore we will not adjust the states from which a given formula is supported.

1. $W^\sharp := W$
2. $v \leq_a^w u$ if $v \in \mathfrak{s}_a^{\{\{v\},\{u\}\}^\downarrow}(w)$
3. $\Sigma_a := \mathcal{S}_a(w)$, where $\mathcal{S}_a(w) := \bigcup_{Q\in\mathcal{I}} \mathcal{S}_a^Q(w)$
4. $V^\sharp := V$

Lemma 5.1.3. *For every finite* ICDM M, $M^\sharp$ *is an* IPM.

Proof. Let $M = \langle W, \{\mathcal{S}_a^P\}_{a\in\mathcal{A},P\in\mathcal{I}}, V\rangle$ be an arbitrary (finite) ICDM, and take $M^\sharp = \langle W^\sharp, \{\leq_a^w\}_{a\in\mathcal{A},w\in W}, \{\Sigma_a\}_{a\in\mathcal{A}}, V^\sharp\rangle$ as defined above.

We first prove the model defined satisfies factivity and introspection, the conditions referred to can be found in chapter 4.

Factivity By conditions safety and adjustment on ICDMs, we know $w \in \mathfrak{s}_a^{\{\{w\}\}^\downarrow}(w)$. Therefore, by definition $w \leq_a^w w$, so $w \in \sigma_a(w)$.

Introspection 1 Suppose $v \in \sigma_a(w)$. As $\sigma_a(w) = \{v \mid v \leq_a^w u \text{ for some } u\}$, this means $v \in \mathfrak{s}_a^{\{\{v\},\}^\downarrow}(w)$. So, by introspection we infer $\mathcal{S}_a^Q(v) = \mathcal{S}_a^Q(w)$, for all $Q \in \mathcal{I}$. Therefore, $\Sigma_a(w) = \Sigma_a(v)$, by definition of $\Sigma_a(w)$.

Introspection 2 Suppose $x \in \sigma_a(w)$. Therefore, $x \leq_a^w x$. This means $x \in \mathfrak{s}_a^{\{\{x\}\}^\downarrow}(w)$, whence $\mathcal{S}_a^Q(w) = \mathcal{S}_a^Q(x)$ for all $Q \in \mathcal{I}$, by introspection. In particular, this means that $\mathfrak{s}_a^{\{\{y\},\{z\}\}^\downarrow}(w) = \mathfrak{s}_a^{\{\{y\},\{z\}\}^\downarrow}(x)$, for all $y, z \in W$. Therefore, $y \leq_a^x z$ iff $y \leq_a^w z$.

We now prove the ordering defined is a well-preorder, the conditions referred to can be found in chapter 4 definition 3.1.3.

Reflexivity Follows as a corollary of factivity.

Transitivity Suppose the ordering is not transitive. Then for some $x, y, z \in W$ we have $x \leq_a^w y$, $y \leq_a^w z$, but $x \not\leq_a^w z$. This means $x \in \mathfrak{s}_a^{\{\{x\},\{y\}\}^\downarrow}(w)$, $y \in \mathfrak{s}_a^{\{\{y\},\{z\}\}^\downarrow}(w)$ but $x \notin \mathfrak{s}_a^{\{\{x\},\{z\}\}^\downarrow}(w)$.

First, as, $x \in \mathfrak{s}_a^{\{\{x\},\{y\}\}^\downarrow}(w)$ we know $\mathcal{S}_a^Q(w) = \mathcal{S}_a^Q(x)$, by introspection. And, by the condition of safety we know that since $x \in \{x, y, z\}$ then $\mathcal{S}_a^{\{\{x\},\{y\},\{z\}\}^\downarrow}(x) \neq \{\emptyset\}$. So, it is the case that $\mathcal{S}_a^{\{\{x\},\{y\},\{z\}\}^\downarrow}(x) = \mathcal{S}_a^{\{\{x\},\{y\},\{z\}\}^\downarrow}(w) \neq \{\emptyset\}$.

Second, we observe that $\{x\} \notin \mathcal{S}_a^{\{\{x\},\{y\},\{z\}\}^\downarrow}(w)$. For suppose not. Then, it is the case that $\mathcal{S}_a^{\{\{x\},\{y\},\{z\}\}^\downarrow}(w) \cap \{\{x\},\{z\}\}^\downarrow \neq \{\emptyset\}$. So, by the condition of minimality we have $\mathcal{S}_a^{\{\{x\},\{z\}\}^\downarrow}(w) = \mathcal{S}_a^{\{\{x\},\{y\},\{z\}\}^\downarrow\cap\{\{x\},\{z\}\}^\downarrow}(w) = \mathcal{S}_a^{\{\{x\},\{y\},\{z\}\}^\downarrow}(w) \cap \{\{x\},\{z\}\}^\downarrow$. This would mean $\{x\} \in \mathcal{S}_a^{\{\{x\},\{z\}\}^\downarrow}(w)$, contradicting our original hypothesis, as then by definition $x \leq_a^w z$.

Third, we use the above to infer that $y \in \mathfrak{s}_a^{\{\{x\},\{y\},\{z\}\}^\downarrow}(w)$. For, by the previous observation $\{x\} \notin \mathcal{S}_a^{\{\{x\},\{y\},\{z\}\}^\downarrow}(w)$, but by the first $\mathcal{S}_a^{\{\{x\},\{y\},\{z\}\}^\downarrow}(w) \neq \{\emptyset\}$. So, as by adjustment we know $\mathcal{S}_a^{\{\{x\},\{y\},\{z\}\}^\downarrow}(w) \subseteq \{\{x\},\{y\},\{z\}\}^\downarrow$ we have $\mathcal{S}_a^{\{\{x\},\{y\},\{z\}\}^\downarrow}(w) \subseteq \{\{y\},\{z\}\}^\downarrow$. This ensures $\mathcal{S}_a^{\{\{x\},\{y\},\{z\}\}^\downarrow}(w) \cap \{\{y\},\{z\}\}^\downarrow \neq \{\emptyset\}$ from which it follows by minimality that $\mathcal{S}_a^{\{\{y\},\{z\}\}^\downarrow}(w) = \mathcal{S}_a^{\{\{x\},\{y\},\{z\}\}^\downarrow}(w) \cap \{\{y\},\{z\}\}^\downarrow$. As we know $y \in \mathfrak{s}_a^{\{\{x\},\{z\}\}^\downarrow}(w)$ this means that $y \in \mathfrak{s}_a^{\{\{x\},\{y\},\{z\}\}^\downarrow}(w)$.

Finally, $y \in \mathcal{s}_a^{\{\{x\},\{y\},\{z\}\}^{\downarrow}}(w)$ ensures $\mathcal{S}_a^{\{\{x\},\{y\},\{z\}\}^{\downarrow}}(w) \cap \{\{x\},\{y\}\}^{\downarrow} \neq \{\emptyset\}$. So $\mathcal{S}_a^{\{\{x\},\{y\}\}^{\downarrow}}(w) = \mathcal{S}_a^{\{\{x\},\{y\},\{z\}\}^{\downarrow}}(w) \cap \{\{x\},\{y\}\}^{\downarrow}$. As $\{x\} \notin \mathcal{S}_a^{\{\{x\},\{y\},\{z\}\}^{\downarrow}}(w)$ this means $\{x\} \notin \mathcal{S}_a^{\{\{x\},\{y\}\}^{\downarrow}}(w)$. This contradicts our assumption that $x \leq_a^w y$.

For every set $s \subseteq \{v \mid \exists u \colon v \leq_a^w u\}$ there exists $v \in s$ such that $v \leq_a^w u$ for all $u \in s$

Suppose for some set $s \subseteq \{v \mid \exists u \colon v \leq_a^w u\}$ that for every $v \in s$ there exists some $u \in s$ such that $v \nleq_a^w u$.

Let $v \in s$ be arbitrary and instantiate u such that $v \nleq_a^w u$. By definition of $\leq_a^w$, we know that $v \notin \mathcal{s}_a^{\{\{v\},\{u\}\}^{\downarrow}}(w)$. By adjustment we know $\mathcal{s}_a^{\{\{v\},\{u\}\}^{\downarrow}}(w) \subseteq \{v, u\}$, and as $v \in \{v, u\}$ by safety that $\mathcal{s}_a^{\{\{v\},\{u\}\}^{\downarrow}}(w) \neq \emptyset$. Therefore, it must be the case that $u \in \mathcal{s}_a^{\{\{v\},\{u\}\}^{\downarrow}}(w)$, whence $u <_a^w v$.

As $v \in s$ was arbitrary we have shown that for any $v \in s$ there exists some $u \neq v$ such that $u <_a^w v$. Yet, we know that s is finite as W is finite. Therefore, for some y it must be the case that there is no z such that $z <_a^w y$, whence we have derived a contradiction.

$\Sigma_a(w)$ is an issue over $\sigma_a(w)$

Suppose $v \in \sigma_a(w)$. Then $v \leq_a^w u$ for some u, by definition. But then $v \in \mathcal{s}_a^{\{\{v\},\{u\}\}^{\downarrow}}(w)$. In turn this means that $\{v\} \in \mathcal{S}_a^{\{\{v\},\{u\}\}^{\downarrow}}(w)$, and so $\{v\} \in \mathcal{S}_a(w)$, which entails $\{v\} \in \Sigma_a(w)$, by definition. This establishes $\sigma_a(w) \subseteq \bigcup \Sigma_a(w)$, as $\Sigma_a(w) := \mathcal{S}_a(w)$.

Suppose $v \notin \sigma_a(w)$. Then $\{v\} \notin \Sigma_a(w)$, whence $\{v\} \notin \mathcal{S}_a^Q(w)$ for any issue $Q \in \mathscr{I}$. In particular, then, $\{v\} \notin \mathcal{S}_a^{\{\{v\}\}^{\downarrow}}(w)$. However, by the conditions of safety and adjustment we know $\{v\} \in \mathcal{S}_a^{\{\{v\}\}^{\downarrow}}(v)$. Therefore, $\mathcal{S}_a^{\{\{v\}\}^{\downarrow}}(v) \neq \mathcal{S}_a^{\{\{v\}\}^{\downarrow}}(w)$. So, by introspection we know $v \notin \mathcal{s}_a^Q(w)$ for all issues Q, whence $\{v\} \notin \mathcal{S}_a^Q(w)$ for all $Q \in \mathscr{I}$. Therefore, $\{v\} \notin \mathcal{S}_a(w)$, whence $\{v\} \notin \Sigma_a(w)$, and so $v \notin \bigcup \Sigma_a(w)$. This establishes $\bigcup \Sigma_a(w) \subseteq \sigma_a(w)$. □

Before continuing to a proof of theorem 5.1.1 we establish a number of preliminary lemmas and corollaries to ease the proof. These lead to establishing a strong correspondence between the semantic structure used to interpret the considers modality on ICDMS and the structure used to interpret considers in its corresponding IPM.

Lemma 5.1.4. *For any ICDM;*

$$v \in \mathcal{s}_a^P(w) \text{ iff } v \in (\mathcal{s}_a(w) \cap |P|) \text{ and } \forall u \in |P|, v \in \mathcal{s}_a^{\{\{v\},\{u\}\}^{\downarrow}}(w).$$

Proof. **Left to right**

Assume $v \in \mathcal{s}_a^P(w)$. From this we know $\{v\} \in \mathcal{S}_a^P(w)$. So, $\{v\} \in \mathcal{S}_a(w)$, and by adjustment $\{v\} \in P$. So, $v \in (\mathcal{s}_a(w) \cap |P|)$.

Let $u \in |P|$ be arbitrary. So, $\{u\} \in P$, which entails $P \cap \{\{v\},\{u\}\}^{\downarrow} = \{\{v\},\{u\}\}^{\downarrow}$. We know $\{v\} \in \mathcal{S}_a^P(w)$, and so $\mathcal{S}_a^P(w) \cap \{\{v\},\{u\}\}^{\downarrow} \neq \{\emptyset\}$, therefore by minimality we have $\mathcal{S}_a^{P \cap \{\{v\},\{u\}\}^{\downarrow}}(w) = \mathcal{S}_a^{\{\{v\},\{u\}\}^{\downarrow}}(w) = \mathcal{S}_a^P(w) \cap \{\{v\},\{u\}\}^{\downarrow}$. This entails $\{v\} \in \mathcal{S}_a^{\{\{v\},\{u\}\}^{\downarrow}}(w)$. So, $v \in \mathcal{s}_a^{\{\{v\},\{u\}\}^{\downarrow}}(w)$.

Right to left

Assume $v \in (\mathcal{s}_a(w) \cap |P|)$ and $\forall u \in |P|, v \in \mathcal{s}_a^{\{\{v\},\{u\}\}^{\downarrow}}(w)$, while $v \notin \mathcal{s}_a^P(w)$.

We know $v \in (\mathcal{s}_a(w) \cap |P|)$, and so $v \in |P|$. From this we observe $\{v\} \in \mathcal{S}_a^{\{\{v\}\}^{\downarrow}}(w)$ and $\{v\} \in P$.

The latter follows by fact 1.2.4 that $\{v\} \in P$. The former follows from the fact that, as $v \in \mathcal{s}_a(w)$, $v \in \mathcal{s}_a^Q(w)$ for some $Q \in \mathcal{I}$. Therefore, $\{v\} \in \mathcal{S}_a^Q(w)$, whence $\mathcal{S}_a^Q(w) \cap \{\{v\}\}^{\downarrow} \neq \emptyset$, and so as by success $\{v\} \in Q$, we have $\mathcal{S}_a^Q(w) \cap \{\{v\}\}^{\downarrow} = \mathcal{S}_a^{Q \cap \{\{v\}\}^{\downarrow}}(w) = \mathcal{S}_a^{\{\{v\}\}}(w)$, by minimality.

Furthermore, as $v \in |P|$ we know $\{v\} \in P$, by fact 1.2.4, and this means $\mathcal{S}_a^{\{\{v\}\}^{\downarrow}}(w) \cap P \neq \{\emptyset\}$. Therefore, by success we know $\mathcal{S}_a^P(w) \neq \{\emptyset\}$.

But, as $v \notin \mathcal{s}_a^P(w)$, $\{v\} \notin \mathcal{S}_a^P(w)$, and therefore there must be some $u \neq v$ such that $\{u\} \in \mathcal{S}_a^P(w)$. By adjustment we know $\{u\} \in P$. Given $\{v\} \in P$ and $\{u\} \in P$ we know $P \cap \{\{v\},\{u\}\}^{\downarrow} = \{\{v\},\{u\}\}^{\downarrow}$. Therefore, as $\mathcal{S}_a^P(w) \cap \{\{v\},\{u\}\}^{\downarrow} = \{u\}^{\downarrow}$ we know $\mathcal{S}_a^{\{\{v\},\{u\}\}^{\downarrow}}(w) = \{u\}^{\downarrow}$, for $\mathcal{S}_a^P(w) \cap \{\{v\},\{u\}\}^{\downarrow} = \mathcal{S}_a^{P \cap \{\{v\},\{u\}\}^{\downarrow}}(w)$, by condition minimality This entails $\mathcal{s}_a^{\{\{v\},\{u\}\}^{\downarrow}}(w) = \{u\}$. However, as $u \in |P|$ we know $v \in \mathcal{s}_a^{\{\{v\},\{u\}\}^{\downarrow}}(w)$, which contradicts the fact that $\mathcal{s}_a^{\{\{v\},\{u\}\}^{\downarrow}}(w) = \{u\}$. □

Corollary 5.1.5.

$v \in \mathcal{s}_a^P(w)$ *iff* $v \in (\mathcal{s}_a(w) \cap |P|)$ *and* $\forall u \in (\mathcal{s}_a(w) \cap |P|), v \in \mathcal{s}_a^{\{\{v\},\{u\}\}^{\downarrow}}(w)$.

Proof. It is simple to repeat the steps of the previous argument to include the additional constraint on the choice of u. □

Lemma 5.1.6. *For an* ICDM M *and its corresponding plausibility model* $M^{\sharp}$*;*

$$\mathcal{s}_a(w) = \sigma_a(w).$$

Proof. From left to right suppose $v \in \mathcal{s}_a(w)$. By introspection whence $\mathcal{S}_a^Q(w) = \mathcal{S}_a^Q(v)$, for any Q. And, by safety and adjustment we know $\{v\} \in \mathcal{S}_a^{\{\{v\}\}^{\downarrow}}(v)$, we know $\{v\} \in \mathcal{S}_a^{\{\{v\}\}^{\downarrow}}(w)$, whence $v \in \mathcal{s}_a^{\{\{v\}\}^{\downarrow}}(w)$. By definition of $\leq_a^w$ this means $v \leq_a^w v$, whence by definition of $\sigma_a(w)$ we have $v \in \sigma_a(w)$.

From right to left suppose $v \in \sigma_a(w)$. Then we know $v \leq_a^w v$, for $\leq_a^w$ is reflexive. In turn this means $v \in \mathcal{s}_a^{\{\{v\}\}^{\downarrow}}(w)$, whence $v \in \mathcal{s}_a(w)$, by definition. □

Corollary 5.1.7. *Given an arbitrary* ICDM M *and a corresponding* IPM $M^{\sharp}$,

$$\mathcal{s}_a^P(w) = Min_{\leq_a^w}(\sigma_a(w) \cap |P|)$$

Proof. Expanding definitions we have $v \in \mathrm{Min}_{\leq_a^w}(\sigma_a(w) \cap |P|)$ iff $v \in (\sigma_a(w) \cap |P|)$ and $\forall u \in (\sigma_a(w) \cap |P|), v \in \mathcal{s}_a^{\{\{v\},\{u\}\}^{\downarrow}}(w)$. By lemma 5.1.6 we know $\sigma_a(w) = \mathcal{s}_a(w)$. Therefore, $v \in \mathrm{Min}_{\leq_a^w}(\sigma_a(w) \cap |P|)$ iff $v \in (\mathcal{s}_a(w) \cap |P|)$ and $\forall u \in (\mathcal{s}_a(w) \cap |P|), v \in \mathcal{s}_a^{\{\{v\},\{u\}\}^{\downarrow}}(w)$ iff $v \in \mathcal{s}_a^P(w)$, by corollary 5.1.5. □

Lemma 5.1.8. *Given an arbitrary* ICDM M *and a corresponding* IPM $M^{\sharp}$*;*

$$\mathcal{S}_a^P(w) = Min_{\leq_a^w}(\Sigma_a(w) \cap P).$$

Proof. **Left to right**

Let $t \in \mathcal{S}_a^P(w)$ be arbitrary. By definition we know $t \in P$, and $t \in \Sigma_a(w)$, for $\Sigma_a(w) = \bigcup_{Q \in \mathcal{I}} \mathcal{S}_a^Q(w)$, and $t \in \mathcal{S}_a^P(w)$. Furthermore, as $t \in \mathcal{S}_a^P(w)$ we know $t \subseteq \mathcal{s}_a^P(w)$, and therefore by corollary 5.1.7 we know $t \subseteq \mathrm{Min}_{\leq_a^w}(\sigma_a(w) \cap |P|)$.

Right to left

Let $t \in \mathrm{Min}_{\leq^w_a}(\Sigma_a(w) \cap P)$ be arbitrary. From this we infer $t \subseteq \mathrm{Min}_{\leq^w_a}(\sigma_a(w) \cap |P|)$, and so we know $t \subseteq s^P_a(w)$, by corollary 5.1.7. We know, then, that $t \in \Sigma_a(w)$, and $t \in P$. We now show that $t \in \mathcal{S}^P_a(w)$.

By the fact that $t \in \Sigma_a(w)$, $t \in \mathcal{S}_a(w) = \bigcup_{Q \in \mathscr{I}} \mathcal{S}^Q_a(w)$. Therefore $t \in \mathcal{S}^Q_a(w)$ for some $Q \in \mathscr{I}$. And, by success we know $\mathcal{S}^Q_a(w) \subseteq Q$, and so it is also the case that $t \in Q$ and we have established already that $t \in P$. If $t = \emptyset$ then it is trivially the case that $t \in \mathcal{S}^P_a(w)$, so let us assume $t \neq \emptyset$. Given this assumption we know $\mathcal{S}^Q_a(w) \cap P \neq \{\emptyset\}$, as $t \in \mathcal{S}^Q_a(w)$ and $t \in P$, and so by minimality we know $t \in \mathcal{S}^{Q \cap P}_a(w) = \mathcal{S}^Q_a(w) \cap P$.

Consider now $\mathcal{S}^P_a(w) \cap Q$. We know that $t \subseteq s^P_a(w)$, and it follows that for all $v \in t$, $\{v\} \in \mathcal{S}^P_a(w)$, given $s^P_a(w) = \bigcup \mathcal{S}^P_a(w)$. This means that $\mathcal{S}^P_a(w) \cap Q \neq \{\emptyset\}$, as each $v \in t$ is such that $\{v\} \in Q$, by the fact that issues are downward closed. Therefore, by minimality we know $\mathcal{S}^{P \cap Q}_a(w) = \mathcal{S}^P_a(w) \cap Q$. But then, as $t \in \mathcal{S}^{P \cap Q}_a(w)$, t it must also be the case that $t \in \mathcal{S}^P_a(w)$. □

We are now ready to prove theorem 5.1.1, by showing that for any finite ICDM, M, the mapping $M \mapsto M^\sharp$ preserves the interpretation of ICDL.

Proof of theorem 5.1.1. Let M be an arbitrary ICDM and $M^\sharp$ an arbitrary plausibility model constructed from M, given by the mapping defined above. We claim is that $M, s \vDash \phi$ iff $M^\sharp, s \vDash \phi$, for all $s \subseteq W$ and formulas ϕ.

Proof is via induction on the complexity of ϕ.

B1) $\boldsymbol{\phi := p}$ for some propositional letter $\boldsymbol{p}$. As each transformation between the models does not affect W nor the valuation function V, it is immediate that $M, s \vDash p$ iff $M^\sharp, s \vDash p$.

B2) $\boldsymbol{\phi := \bot}$. Trivial, using similar reasoning as above.

Induction Cases

Induction hypothesis:

I1) $\boldsymbol{\phi := \psi \wedge \chi}$. This follows from the support clauses for conjunction, which allow us to apply the induction hypothesis to simpler formulas.

I2) $\boldsymbol{\phi := ?\{\alpha_1, \ldots, \alpha_n\}}$. This follows in an analogous way to the case for conjunction.

I3) $\boldsymbol{\phi := \psi \to \chi}$. The process is similar to the previous two cases. The only aspect that changes is we must assume for some arbitrary $t \subseteq s$ that $M, t \vDash \psi$ implies $M, t \vDash \chi$. Yet, we know by the induction hypothesis that for all formulas of a lower complexity than $\psi \to \chi$ that for *all* $s \subseteq W$, $M, s \vDash \phi$ iff $M^\sharp, s \vDash \phi$. Therefore we can apply the induction hypothesis to t also.

I4) $\boldsymbol{\phi := C^\psi \varphi}$.

$M, w \vDash C^\psi_a \varphi$ iff $\forall t \in \mathcal{S}^\psi_a(w), M, t \vDash \varphi$. By the induction hypothesis $M, t \vDash \varphi$ iff $M^\sharp, t \vDash \varphi$. And, by lemma 5.1.8, $t \in \mathcal{S}^\psi_a(w)$ iff $t \in \mathrm{Min}_{\leq^w_a}(\Sigma_a(w) \cap [\![\psi]\!])$. Therefore, $t \in \mathcal{S}^\psi_a(w)$ and $M, t \vDash \varphi$ iff $t \in \mathrm{Min}_{\leq^w_a}(\Sigma_a(w) \cap [\![\psi]\!])$ and $M^\sharp, t \vDash \varphi$. So, $\forall t \in \mathcal{S}^\psi_a(w), M, t \vDash \varphi$ iff $\forall t \in \mathrm{Min}_{\leq^w_a}(\Sigma_a(w) \cap [\![\psi]\!]), M^\sharp, t \vDash \varphi$ This is the case iff $M^\sharp, w \vDash C_a^\psi \varphi$. By this we have established $M, w \vDash C_a^\psi \varphi$ iff $M^\sharp, w \vDash C_a^\psi \varphi$. So, we have $M, s \vDash C_a^\psi \varphi$ iff $M^\sharp, s \vDash C_a^\psi \varphi$, by the support clause for $C_a^\psi \varphi$ in both semantics. □

5.2 From Inquisitive Plausibility Models to ICDMS

Theorem 5.2.1 (From IPMS to ICDMS). *Any IMP can be transformed into an ICDM, preserving the interpretation of ICDL.*

As in the proof of theorem 5.1.1 we first define a method to transform a given IPM into an ICDM, before proving that the method of transformation does indeed give rise to an ICDM. However, unlike the transformation of ICDMS to IPMS we need not assume the IPM chosen is finite.

Definition 5.2.2 (Map from IPMS to ICDMS).

Given an arbitrary inquisitive plausibility model, $M = \langle W, \{\leq_a\}_{a\in\mathcal{A}}, \{\Sigma_a\}_{a\in\mathcal{A}}, V\rangle$, we define a map $M \mapsto M^\flat$, where $M^\flat = \langle W, \{\mathcal{S}_a^P\}_{a\in\mathcal{A}, P\in\mathscr{I}}, V\rangle$ is constructed in the following way:

1. $W^\flat := W$
2. $\mathcal{S}_a^P(w) := \mathrm{Min}_{\leq_a^w}(\Sigma_a(w) \cap P)$
3. $V^\flat := V$

Lemma 5.2.3. *For every IPM M, $M^\flat$ is an ICDM.*

Proof. Let, $M = \langle W, \{\leq_a\}_{a\in\mathcal{A}}, \{\Sigma_a\}_{a\in\mathcal{A}}, V\rangle$ be an arbitrary IPM, and take $M^\flat = \langle W, \{\mathcal{S}_a^P\}_{a\in\mathcal{A}, P\in\mathscr{I}}, V\rangle$ as defined above.

Safety

Suppose $w \in |P|$ for some $P \in \mathscr{I}$. Then, $\{w\} \in P$ by fact 1.2.4 and, by factivity, we know $\{w\} \in \Sigma_a(w)$. From this it follows that $w \in |P \cap \Sigma_a(w)|$. So, as $\mathrm{Min}_{\leq_a^w}(P \cap \Sigma_a(w)) = \wp(\mathrm{Min}_{\leq_a^w}|P \cap \Sigma_a(w)|) \cap P \cap \Sigma_a(w)$ by corollary 3.3.11 we know $\mathrm{Min}_{\leq_a^w}(P \cap \Sigma_a(w)) \neq \{\emptyset\}$, whence $\mathcal{S}_a^P(w) \neq \{\emptyset\}$, by definition.

Adjustment $\mathcal{S}_a^P(w) = \mathrm{Min}_{\leq_a^w}(\Sigma_a(w) \cap P)$, and so $\mathcal{S}_a^P(w) \subseteq P$.

Introspection

Let us assume $v \in s_a^P(w)$, to show is $\mathcal{S}_a^Q(w) = \mathcal{S}_a^Q(v)$. We can observe that $\{v\} \in s_a^P(w)$ and therefore we know $\{v\} \in \Sigma_a(w)$, whence $v \in \sigma_a(w)$.

Now, to show $\mathcal{S}_a^Q(w) = \mathcal{S}_a^Q(v)$, it is sufficient to show $\mathrm{Min}_{\leq_a^w}(\Sigma_a(w) \cap Q) = \mathrm{Min}_{\leq_a^v}(\Sigma_a(v) \cap Q)$. But, by introspection conditions 1. and 2. on plausibility model this is immediate.

Minimality

Assume $\mathcal{S}_a^P(w) \cap Q \neq \{\emptyset\}$. We show $\mathcal{S}_a^{P\cap Q}(w) = \mathcal{S}_a^P(w) \cap Q$.
First, we establish that $\mathrm{Min}_{\leq_a^w}(\sigma_a(w) \cap |P \cap Q|) = \mathrm{Min}_{\leq_a^w}(\sigma_a(w) \cap |P|) \cap |Q|$, on the basis of our assumption.

Let $u \in \text{Min}_{\leq_a^w}(\sigma_a(w) \cap |P \cap Q|)$ be arbitrary. Therefore, for all $v \in (\sigma_a(w) \cap |P \cap Q|)$, $u \leq_a^w v$. Let $v' \in \text{Min}_{\leq_a^w}(\sigma_a(w) \cap |P|) \cap |Q|$ be arbitrary. Observe we know such a v' exists, for we have assumed $\mathcal{S}_a^P(w) \cap Q \neq \{\emptyset\}$. Clearly $v' \in \sigma_a(w) \cap |P \cap Q|$, and therefore $u \leq_a^w v'$. As we know $v' \in \text{Min}_{\leq_a^w}(\sigma_a(w) \cap |P|)$ this entails $u \in \text{Min}_{\leq_a^w}(\sigma_a(w) \cap |P|)$, from which it follows that $u \in \text{Min}_{\leq_a^w}(\sigma_a(w) \cap |P|) \cap |Q|$.

Let $u \in \text{Min}_{\leq_a^w}(\sigma_a(w) \cap |P|) \cap |Q|$ be arbitrary. Therefore, $u \in |P|$ and $u \in |Q|$, whence either $u \in \text{Min}_{\leq_a^w}(\sigma_a(w) \cap |P \cap Q|)$, or for all $v \in \text{Min}_{\leq_a^w}(\sigma_a(w) \cap |P \cap Q|)$, $v <_a^w u$. However, the latter cannot be the case. For, $v \in |P|$, and therefore $v \in (\sigma_a(w) \cap |P|)$. So, as $v <_a^w u$ it must be the case that $u \notin \text{Min}_{\leq_a^w}(\sigma_a(w) \cap |P|)$. This means, $u \in \text{Min}_{\leq_a^w}(\sigma_a(w) \cap |P \cap Q|)$.

We are now ready to establish $\mathcal{S}_a^{P \cap Q}(w) = \mathcal{S}_a^P(w) \cap Q$.

1. $\mathcal{S}_a^{P \cap Q}(w) \subseteq \mathcal{S}_a^P(w) \cap Q$
   Suppose $t \in \mathcal{S}_a^{P \cap Q}(w)$. By definition of $\mathcal{S}_a^{P \cap Q}(w)$ we infer $t \in P, Q, \Sigma_a(w)$. Furthermore, $t \subseteq \text{Min}_{\leq_a^w}(\sigma_a(w) \cap |P \cap Q|)$. By the above this entails that $t \subseteq \text{Min}_{\leq_a^w}(\sigma_a(w) \cap |P|) \cap |Q|$. From this it follows that $t \in \wp(\text{Min}_{\leq_a^w}(\sigma_a(w) \cap |P|)) \cap \Sigma_a(w) \cap P \cap Q$, therefore $t \in \mathcal{S}_a^P(w) \cap Q$.
2. $\mathcal{S}_a^P(w) \cap Q \subseteq \mathcal{S}_a^{P \cap Q}(w)$
   Suppose $t \in (\mathcal{S}_a^P(w) \cap Q)$. By definition of $\mathcal{S}_a^P(w)$ we infer $t \in P$, $\Sigma_a(w)$ and $t \subseteq \text{Min}_{\leq_a^w}(\sigma_a(w) \cap |P|)$. Furthermore, $t \in Q$, and so $t \subseteq |Q|$. Therefore, $t \subseteq \text{Min}_{\leq_a^w}(\sigma_a(w) \cap |P|) \cap |Q|$, from which it follows that $t \subseteq \text{Min}_{\leq_a^w}(\sigma_a(w) \cap |P \cap Q|)$. Therefore, $t \in \text{Min}_{\leq_a^w}(\Sigma_a(w) \cap (P \cap Q))$, whence $t \in \mathcal{S}_a^{P \cap Q}(w)$.

Success

Assume $\mathcal{S}_a^Q(w) \cap P \neq \{\emptyset\}$. We want to show $\mathcal{S}_a^P(w) \neq \{\emptyset\}$. By definition $\mathcal{S}_a^Q(w) = \text{Min}_{\leq_a^w}(\Sigma_a(w) \cap Q)$. Therefore we know that there exists some $t \neq \emptyset$ in $\mathcal{S}_a^Q(w) \cap P$, whence by the former intersected set $t \in \Sigma_a(w)$, and so $t \subseteq \sigma_a(w)$. Likewise, $t \in P$, so $t \subseteq |P|$. Putting these two facts together we infer $t \subseteq (\sigma_a(w) \cap |P|)$, but as $t \neq \emptyset$ this means there must exist some $v \in \text{Min}_{\leq_a^w}(\sigma_a(w) \cap |P|)$, and in turn this means that $\text{Min}_{\leq_a^w}(\Sigma_a(w) \cap P) \neq \{\emptyset\}$, whence $\mathcal{S}_a^P(w) \neq \{\emptyset\}$. □

We now show the transformation $M \mapsto M^\flat$ preserves the interpretation of ICDL.

Proof of theorem 5.2.1. Let M be an arbitrary ICDM and take $M^\flat$, the ICDM constructed from M, given by the mapping defined above. We claim $M, s \vDash \phi$ iff $M^\flat, s \vDash \phi$, for all $s \subseteq W$ and formulas ϕ.

Proof is via induction on the complexity of ϕ. We prove only the considers modality, with the others established analogously to the proof of theorem 5.2.1.

14) $\boldsymbol{\phi := C_a^\psi \varphi}$.

It is immediate through the transformation that $\mathcal{S}_a^\psi(w) = \text{Min}_{\leq_a^w}(\Sigma_a(w) \cap [\![\psi]\!])$, And, through the induction hypothesis we know that $M, s \vDash \chi$ iff $M^\flat, s \vDash \chi$, for all formulas χ of a lower complexity than $C^\psi \varphi$. Therefore, for all $t \in \text{Min}_{\leq_a^w}(\Sigma_a(w) \cap [\![\psi]\!])$, $M^\flat, t \vDash \chi$ iff for all $t \in \mathcal{S}_a^\psi(w)$, $M, t \vDash \chi$ This holds for φ, in particular. Therefore, $M^\flat, s \vDash C_a^\psi \varphi$ iff $M, s \vDash C_a^\psi \varphi$. □

Theorem 5.2.4 (Soundness of ICDL wrt. IPMs).
ICDL is sound with respect to inquisitive plausibility models.

Proof. Immediate, given ICDL is sound with respect to ICDMs by theorem 4.3.1, and that we can transform any IPM into an ICDM, preserving the interpretation of ICDL by theorem 5.2.1.

For, any would-be inquisitive plausibility countermodel can be transformed into an inquisitive conditional-doxastic countermodel preserving the interpretation of ICDL by theorem 5.2.1. $\square$

5.3 The Entertains Modality Is Not Definable in ICDL

Proposition 5.3.1. *The entertains modality cannot be defined in ICDL.*

Proof. Consider the following IPMs for a single agent:

$$M^1 = \langle W, \{\leq_a^w\}_{w \in W}, \{\Sigma_a^1\}, V \rangle \text{ and } M^2 = \langle W, \{\leq_a^w\}_{w \in W}, \{\Sigma_a^2\}, V \rangle$$

where:

1. $W = \{w, v\}$, for both M^1 and M^2
2. $w <_a^w v$ and $w <_a^v v$
3. $\Sigma_a^1(x) = \{\{w, v\}\}^{\downarrow}$ and $\Sigma_a^2(x) = \{\{w\}, \{v\}\}^{\downarrow}$, for $x \in \{w, v\}$
4. $V = \{\langle w, \{p\}\rangle, \langle v, \emptyset\rangle\}$

These models have the following pictorial representation:

(a) M^1 (b) M^2

A simple inspection shows that both M^1 and M^2 are in fact IPMs. Furthermore, by the mapping defined in section 5.2, M^1 and M^2 give rise to the same ICDM.[2] Therefore, by theorem 5.2.1 $M^1, w \vDash \varphi$ iff $M^2, w \vDash \varphi$, for all $w \in W, \varphi \in \mathcal{L}^{\mathsf{ICDL}}$.

Yet, M^1 and M^2 disagree with respect to the entertains modality. For, we have $M^1, w \nvDash E_a?p$, yet $M^2, w \vDash E_a?p$. Therefore, $E_a?p$ is not equivalent to any formula in ICDL. $\square$

From an ICDM perspective, to obtain the correct interaction between an agent's epistemic state map and their conditional state maps we would require the following frame condition schema (presuming ICDMs were equipped with epistemic states):

$$\Sigma_a(w) \cap \wp(\mathfrak{s}_a^P(w)) = \mathcal{S}_a^{!P}(w)$$

corresponding to the following equivalence on IPMs.

$$\Sigma_a(w) \cap \wp(\mathrm{Min}_{\leq_a^w} |P|) = \mathrm{Min}_{\leq_a^w}(\Sigma_a(w) \cap !P)$$

The expression of this property is natural. However, we have not been able to find appropriate axioms to enforce this property on the canonical model.

[2] Note there are five distinct issues to consider when constructing state maps given the models defined; $\{\{w, v\}\}^{\downarrow}, \{\{w\}, \{v\}\}^{\downarrow}, \{\{w\}\}^{\downarrow}, \{\{v\}\}^{\downarrow}$, and $\{\emptyset\}$. So, the corresponding ICDM can easily be sketched.

After establishing the completeness of ICDL in chapter 6 we will turn to inquisitive plausibility logic in chapter 7, which will allow for an *indirect* axiomatisation of ICDL with the entertains modality. Inquisitive plausibility logic will be shown sound and complete with respect to inquisitive plausibility models, thus ensuring that inquisitive plausibility logic contains inquisitive conditional-doxastic logic.

Chapter 6

Completeness of ICDL

6.1 Introduction

Our approach to weak completeness is a blend of the proof of strong completeness of IEL with respect to IEMs from Ciardelli (2014b) and the proof of weak completeness of PDL with respect to PDLMs from Blackburn, Rijke, and Venema (2002, §4.8).

Indeed, the standard route to completeness, via constructing a canonical model with worlds corresponding to maximally consistent sets of declaratives cannot be taken. For, in order for the canonical model to be an ICDM it must be the case that at state map is associated to every downward closed set of states. However, as the size of At may be countably infinite this means the set of maximally consistent sets of declaratives will be uncountable. Letting W denote this set, it is immediate that the set $\wp(W)$ must be uncountably infinite, and therefore as this set corresponds one-one with the set of declaratives by taking the downward closure of each element, there are at least uncountably many issues. Yet, there are countably many formulas of InqD, and so there uncountably many potential inquisitive propositions for which no corresponding formula of InqD exists. This leaves us without syntactic resource to characterise the state maps which correspond to these propositions. Therefore, we will work with finite fragments of InqD to construct canonical models, leading to weak completeness.

A second limitation arises when transferring the completeness of ICDL with respect to ICDMs to IPMs, as we will use canonical ICDMs to create canonical IPMs. However, and as observed in chapter 5, only finite ICDMs can be transformed to IPMs.

These limitations combined put strong completeness beyond the scope of this thesis.

Instead, our approach to completeness will build models with respect to finite fragments of ICDL. For, given a finite fragment there will be finitely many maximally consistent sets of declaratives, whence there will only be finitely many possible issues with respect to the model.

We begin this chapter with a number of definitions, lemmas, and theorems in order to generate the finite fragment of interest given a finite set of formulas, and to construct maximally consistent sets of declaratives from this fragment which we term nuclei, which are in turn used to construct atoms which are used as worlds in our canonical model construction.

Section 6.2.1 then shows how every issue over a set of atoms can be characterised by using the fragment of ICDL used to generate the atoms. Section 6.3 then defines the canonical model construction with respect to an arbitrary finite fragment, and shows this model is an ICDM. Finally, section 6.4 outlines a number of results for ICDL, including completeness and the failure of compactness for ICDL.

6.2 Foundations for the Canonical Model

Definitions, lemmas, and theorems from Ciardelli (2014b)

In order to build maximally consistent sets of declaratives with respect to a fragment of ICDL to act as possible worlds we establish a connexion between declaratives and interrogatives. The following definition associates to each formula a set of 'resolutions' in the language of ICDL, repeated from section 3.2 while the following facts, lemmas, corollaries, and theorems outline the essentials of the relationship between a formula and its resolutions.

We begin by recalling the definition of resolutions for ICDL.

Definition 3.3.4 (Resolutions for ICDL)**.** The set $\mathcal{R}(\varphi)$ of resolutions for a given formula φ is defined inductively by:

- $\mathcal{R}(\alpha) = \{\alpha\}$
- $\mathcal{R}(?\{\alpha_1, \ldots, \alpha_n\}) = \{\alpha_1, \ldots, \alpha_n\}$
- $\mathcal{R}(\mu \wedge \nu) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\mu) \text{ and } \beta \in \mathcal{R}(\nu)\}$
- $\mathcal{R}(\varphi \to \mu) = \{\bigwedge_{\alpha \in \mathcal{R}(\varphi)} (\alpha \to f(\alpha)) \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\mu)\}$

Fact 6.2.1. *For any formula φ, $\mathcal{R}(\varphi)$ is finite.*

Definition 6.2.2 (Resolutions of a set)**.** Given a set of formulas Φ, we define the resolution of Φ as the set $\mathcal{R}(\Phi)$ containing sets of declaratives Γ satisfying:

1. $\forall \alpha \in \Gamma, \exists \varphi \in \Phi$ such that $\alpha \in \mathcal{R}(\varphi)$
2. $\forall \varphi \in \Phi, \exists \alpha \in \Gamma$ such that $\alpha \in \mathcal{R}(\varphi)$

Lemma 6.2.3. *For any φ, $\varphi \dashv\vdash ?\mathcal{R}(\varphi)$.*

Proof. Follows lemma 1 of Ciardelli, Groenendijk, and Roelofsen (2015). □

Corollary 6.2.4. *If $\alpha \in \mathcal{R}(\varphi)$, then $\alpha \vdash \varphi$.*

Corollary 6.2.5. *For any interrogative μ, $\mu \vdash !\mu$.*[1]

Lemma 6.2.6. *If $\psi_1, \ldots, \psi_n \vdash \varphi$ then $C_a^\chi \psi_1, \ldots, C_a^\chi \psi_n \vdash C_a^\chi \varphi$.*

Theorem 6.2.7 (Resolution Theorem for ICDL)**.**
$\Phi \vdash \psi$ iff every resolution of Φ derives some resolution of ψ. Formally;
$\Phi \vdash \psi$ if and only if $\forall \Gamma \in \mathcal{R}(\Phi)$, $\Gamma \vdash \alpha$ for some $\alpha \in \mathcal{R}(\psi)$.

Proof. See Ciardelli (2014b, p. 109). While a number of cases need to be added to the proof, the axioms introduced are declaratives, and so these are straightforward. □

Corollary 6.2.8 (Split)**.**
For a set of declaratives Γ, $\Gamma \vdash \varphi$ implies $\Gamma \vdash \alpha$ for some $\alpha \in \mathcal{R}(\varphi)$.

[1] Recall $!\mu := \neg\neg\mu$.

Finite fragments of ICDL

Definition 6.2.9 (Subformulas)**.** Let $\mathcal{F}$ be a set of formulas, we define $sub(\mathcal{F})$ to be the smallest set satisfying the following conditions:

1. If $\varphi \circ \psi \in \mathcal{F}$ then $\varphi, \psi \in sub(\mathcal{F})$ for $\circ \in \{\wedge, \to\}$.
2. If $?\{\alpha_1, \ldots, \alpha_n\} \in \mathcal{F}$, then $\alpha_1, \ldots, \alpha_n \in sub(\mathcal{F})$.
3. If $C_a^{\psi}\varphi \in \mathcal{F}$, then $\varphi, \psi \in sub(\mathcal{F})$.

Definition 6.2.10 ($\mathfrak{F}$)**.** We define five successive sets based on a set of formulas $\mathcal{F}$.

1. We define $\mathcal{F}^{\mathsf{I}}$ to be the smallest set satisfying the following:
   (a) $\mathcal{F} \subseteq \mathcal{F}^{\mathsf{I}}$,
   (b) if $\varphi \in \mathcal{F}^{\mathsf{I}}$ and $\psi \in sub(\varphi)$, then $\psi \in \mathcal{F}^{\mathsf{I}}$,
   (c) if $\varphi \in \mathcal{F}^{\mathsf{I}}$, and $\alpha \in \mathcal{R}(\varphi)$, then $\alpha \in \mathcal{F}^{\mathsf{I}}$,
   (d) if $\alpha \in \mathcal{F}^{\mathsf{I}}$, then $\sim\alpha \in \mathcal{F}^{\mathsf{I}}$.[2]
2. We define $\mathcal{F}^{\mathsf{II}}$ to be the smallest set satisfying the following:
   (e) $\mathcal{F}^{\mathsf{I}} \subseteq \mathcal{F}^{\mathsf{II}}$,
   (f) if $\alpha_1, \ldots, \alpha_n \in \mathcal{F}^{\mathsf{I}}$ and are distinct then $\alpha_1 \wedge \cdots \wedge \alpha_n \in \mathcal{F}^{\mathsf{II}}$.
3. We define $\mathcal{F}^{\mathsf{III}}$ to be the smallest set satisfying the following:
   (g) $\mathcal{F}^{\mathsf{II}} \subseteq \mathcal{F}^{\mathsf{III}}$,
   (h) if $\alpha_1, \ldots, \alpha_n \in \mathcal{F}^{\mathsf{II}}$ and are distinct then $\alpha_1 \vee \cdots \vee \alpha_n \in \mathcal{F}^{\mathsf{III}}$.
4. We define $\mathcal{F}^{\mathsf{IV}}$ to be the smallest set satisfying the following:
   (i) $\mathcal{F}^{\mathsf{III}} \subseteq \mathcal{F}^{\mathsf{IV}}$,
   (j) if $\alpha_1, \ldots, \alpha_n \in \mathcal{F}^{\mathsf{III}}$ and are distinct then: $?\{\alpha_1, \ldots, \alpha_n\} \in \mathcal{F}^{\mathsf{IV}}$,
   (k) if $\alpha_1, \ldots, \alpha_n \in \mathcal{F}^{\mathsf{III}}$ and are distinct then $\neg(?\{\alpha_1, \ldots, \alpha_n\}) \in \mathcal{F}^{\mathsf{IV}}$.
5. We define $\mathfrak{F}$ to be the smallest set satisfying the following:
   (l) $\mathcal{F}^{\mathsf{IV}} \subseteq \mathfrak{F}$,
   (m) if $\alpha \in \mathcal{F}^{\mathsf{IV}}$, then $\sim\alpha \in \mathfrak{F}$.

The construction of $\mathfrak{F}$ is tortuous but motivated as follows.

$\mathcal{F}^{\mathsf{I}}$ takes the declarative counterpart to $\mathcal{F}$, and closes it under subformulas, resolutions and their quasi-negations. $\mathcal{F}^{\mathsf{II}}$ introduces conjunctions of these resolutions. Note in particular that for each maximally consistent subset of $\mathcal{F}^{\mathsf{I}}$, there will be a conjunction of all the elements in that set, which will in turn characterise the set. $\mathcal{F}^{\mathsf{III}}$ then allows us to take disjunctions of those characteristic conjunctions, with these we will be able to characterise sets of maximally consistent sets. $\mathcal{F}^{\mathsf{IV}}$ introduces basic interrogatives constructed from arbitrary declaratives in $\mathcal{F}^{\mathsf{III}}$. This allows us to characterise issues, as for any collection of states $S_1, \ldots, S_n$ there will be characteristic formulas $\gamma_{S_1}, \ldots, \gamma_{S_n}$, whence $?\{\gamma_{S_1}, \ldots, \gamma_{S_n}\}$ will collect together those states, and their downward closure. Furthermore, $\neg(?\{\gamma_{S_1}, \ldots, \gamma_{S_n}\})$ will give a syntactic account of relative-pseudo complement in terms of negation, as opposed to the characteristic formula for the relative-pseudo complement of the issue. Finally, $\mathfrak{F}$ then ensures that $\mathfrak{F}$ will be closed under pseudo-negations with respect to declaratives, which will prove useful in establishing a number of results.

[2] Recall $\sim\alpha := \beta$ if α is of the form $\neg\beta$, and $\neg\alpha$ otherwise.

The ability to characterise issues means that, given an agent, we will be able to define state maps for each issue the agent can conditionalise on via the characteristic formula for that state map. Moreover, we will use such characteristic formulas and their negations in conjunction with the axiomatisation of ICDL to show that the canonical models we construct are in fact ICDMs.

Proposition 6.2.11. *$\mathfrak{F}$ has the following properties:*

1. *$\mathfrak{F}$ is complete with respect to declaratives, in the sense that $\alpha \in \mathfrak{F}$ iff $\sim\alpha \in \mathfrak{F}$.*
2. *if $\mathcal{F}$ is finite then $\mathfrak{F}$ is finite.*
3. *if $\varphi \in \mathfrak{F}$ and $\psi \in sub(\varphi)$, then $\psi \in \mathfrak{F}$.*
4. *if $\varphi \in \mathfrak{F}$, then $\alpha \in \mathfrak{F}$ for all $\alpha \in \mathcal{R}(\varphi)$.*

Proof.

1 Immediate by clause 5m and the fact that $\sim\sim\alpha = \alpha$.

2 As $\mathcal{F}$ is finite it contains only a finite number of formulas.

Moreover, the set resolutions to a formula φ are inductively constructed from resolutions to subformulas of φ, which must terminate in resolutions to basic interrogatives. Therefore, as every resolution is a declarative, the only interrogatives present in $\mathcal{F}^{\mathsf{I}}$ will be those found in the subformulas of the elements of $\mathcal{F}$. So, closing a formula under resolutions will introduce no interrogatives as the subformulas to a resolution not found in the original formula. Therefore, as there are only finitely many subformulas of any given formula, there can be only finitely many interrogatives in $\mathcal{F}^{\mathsf{I}}$.

Furthermore, this means that the closure of $\mathcal{F}$ under subformulas, resolutions, and then subformulas again is also closed under resolutions, as the second closure under subformulas will introduce at most only new declarative which are their own resolutions. Clearly each operation introduces only a finite number of formulas, and therefore closing $\mathcal{F}$ under both resolutions and subformulas results in a finite set. As this set is finite, there must be only finitely many declaratives, and therefore only finitely many pseudo-negations need to be added to achieve the conditions of $\mathcal{F}^{\mathsf{I}}$, from which we can conclude that $\mathcal{F}^{\mathsf{I}}$ is finite if $\mathcal{F}$ is.

Clearly, then, if $\mathcal{F}^{\mathsf{I}}$ is finite then there are only finitely many distinct α_i, and thus finitely many possible conjunctions of those α_i. Therefore, clause 2f introduces only finitely many formulas. The same reasoning holds for clause 3h, and so $\mathcal{F}^{\mathsf{III}}$ is finite. Therefore, so too must it be the case that $\mathcal{F}^{\mathsf{IV}}$ is finite, for there are only finitely many $\alpha_i \in \mathcal{F}^{\mathsf{III}}$. Finally, the same reasoning applies to $\mathfrak{F}$, as there will be only finitely many declaratives in $\mathcal{F}^{\mathsf{IV}}$.

3 Proof is by cases, and at each case we look solely at the new formulas introduced by the closure procedure for $\mathfrak{F}$.

a. $\varphi \in \mathcal{F}^{\mathsf{I}}$

Suppose $\psi \in sub(\varphi)$. Then, by clause 1b we know $\psi \in \mathcal{F}^{\mathsf{I}}$. As $\mathcal{F}^{\mathsf{I}} \subseteq \mathcal{F}^{\mathsf{II}} \subseteq \mathcal{F}^{\mathsf{III}} \subseteq \mathcal{F}^{\mathsf{IV}} \subseteq \mathfrak{F}$, $\psi \in \mathfrak{F}$.

b. $\varphi \in \mathcal{F}^{\mathsf{II}}$

Either $\varphi \in \mathcal{F}^{\mathsf{I}}$ or φ is of the form $\alpha_1 \wedge \cdots \wedge \alpha_n$ for distinct $\alpha_1, \ldots, \alpha_n \in \mathcal{F}^{\mathsf{I}}$. Any subformula of the latter is either an element of $\mathcal{F}^{\mathsf{I}}$, or a conjunction of distinct $\alpha_i, \ldots, \alpha_k \in \mathcal{F}^{\mathsf{I}}$, where $1 \leq i \leq k \leq n$. As $\alpha_i, \ldots, \alpha_k \in \mathcal{F}^{\mathsf{I}}$ are distinct clause 2f ensures $\alpha_i \wedge \cdots \wedge \alpha_k \in \mathcal{F}^{\mathsf{II}}$, and by the previous case we know that $\mathcal{F}^{\mathsf{I}}$ is closed under subformulas, whence $\mathcal{F}^{\mathsf{II}}$ must be too.

c. $\varphi \in \mathcal{F}^{\mathsf{III}}$

Either $\varphi \in \mathcal{F}^{\mathsf{II}}$ or φ is of the form $\alpha_1 \vee \cdots \vee \alpha_n$ for distinct $\alpha_1, \ldots, \alpha_n \in \mathcal{F}^{\mathsf{II}}$. Given $\alpha_1 \vee \cdots \vee \alpha_n := \neg(\sim\alpha_1 \wedge \cdots \wedge \sim\alpha_n) = (\sim\alpha_1 \wedge \cdots \wedge \sim\alpha_n) \to \bot$, we need to show that $\bot \in \mathfrak{F}$ and any formula of the form $\sim\alpha_i \wedge \cdots \wedge \sim\alpha_k \in \mathfrak{F}$ for $1 \leq i \leq k \leq n$.

By clauses 1c and 1d we know there exists some $\alpha \in \mathcal{F}^{\mathsf{I}}$, and that $\neg\alpha$ in $\mathcal{F}^{\mathsf{I}}$. As $\neg\alpha := \alpha \to \bot$, by clause 1b we know, as $\bot \in sub(\alpha \to \bot)$, so $\bot \in \mathcal{F}^{\mathsf{I}}$.

As $\alpha_i, \ldots, \alpha_k \in \mathcal{F}^{\mathsf{I}}$ we know $\sim\alpha_i, \ldots, \sim\alpha_k \in \mathcal{F}^{\mathsf{I}}$ by clause 1d. Therefore, by clause 2f $\sim\alpha_i \wedge \cdots \wedge \sim\alpha_n \in \mathcal{F}^{\mathsf{II}} \subseteq \mathfrak{F}$.

d. $\varphi \in \mathcal{F}^{\mathsf{IV}}$

Either $\varphi \in \mathcal{F}^{\mathsf{III}}$ or φ has the form $?\{\alpha_1, \ldots, \alpha_n\}$, or $\neg(?\{\alpha_1, \ldots, \alpha_n\})$, with distinct $\alpha_1, \ldots, \alpha_n \in \mathcal{F}^{\mathsf{III}}$.

If $\varphi := ?\{\alpha_1, \ldots, \alpha_n\}$, then $\alpha_1, \ldots, \alpha_n \in \mathcal{F}^{\mathsf{III}}$, and so any $\psi \in sub(\varphi) \in \mathcal{F}^{\mathsf{III}}$ which we have established is closed under subformulas.

If $\varphi := \neg(?\{\alpha_1, \ldots, \alpha_n\})$ then we know $?\{\alpha_1, \ldots, \alpha_n\} \in \mathfrak{F}$, by clause 4j, and thus by the previous reasoning that $sub(?\{\alpha_1, \ldots, \alpha_n\}) \subseteq \mathfrak{F}$, and by prior reasoning we know $\bot \in \mathcal{F}^{\mathsf{IV}}$.

e. $\varphi \in \mathfrak{F}$

Either $\varphi \in \mathcal{F}^{\mathsf{IV}}$ or φ is of the form $\sim\alpha$ for some $\alpha \in \mathcal{F}^{\mathsf{IV}}$. But as $\sim\alpha$ is either of the form $\neg\beta$ or β, with $\beta \in \mathcal{F}^{\mathsf{IV}}$. So, as $\alpha \in \mathcal{F}^{\mathsf{IV}}$ we know that $\psi \in sub(\alpha)$, $\psi \in \mathcal{F}^{\mathsf{IV}}$, and as $\bot \in \mathcal{F}^{\mathsf{IV}}$ by prior reasoning, for any $\psi \in sub(\sim\alpha)$, $\psi \in \mathcal{F}^{\mathsf{IV}} \subseteq \mathfrak{F}$.

4 By clause 1c we know that if $\varphi \in \mathcal{F}^{\mathsf{I}}$, then $\alpha \in \mathcal{F}^{\mathsf{I}}$ for all $\alpha \in \mathcal{R}(\varphi)$. Furthermore, for any clause that introduce declaratives, as $\mathcal{R}(\alpha) = \{\alpha\}$, we know the property is preserved by these.

Therefore, we only need to concern ourselves with clause 5. However, this clause only introduces basic interrogatives, and as the resolutions of a basic interrogative are its declarative components, for any $?\{\alpha_1, \ldots, \alpha_n\} \in \mathcal{F}^{\mathsf{IV}} - \mathcal{F}^{\mathsf{III}}$, $\alpha_i \in \mathcal{F}^{\mathsf{III}}$, whence $\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathfrak{F}$. $\square$

In order to construct a finite canonical model we will isolate the declarative part of $\mathfrak{F}$. To do so we introduce the following notation, with '$\mathfrak{D}$' relating to 'declaratives' just as $\mathfrak{F}$ relates to 'formulas.'

Definition 6.2.12 ($\mathfrak{D}$). $\mathfrak{D} := \{\alpha \mid \alpha \in \mathfrak{F}\}$.

From $\mathfrak{D}$ we will build complete theories of declaratives relative to $\mathcal{F}$, and these will function as the set of worlds. However, not all properties of proposition 6.2.11 carry over to $\mathfrak{D}$. For example, the formula $C_a^p ?\{q, r\}$ is a declarative, but contains an interrogative as a subformula. Still, property 1b follows in a modified form, taking only *declarative* subformulas.

Proposition 6.2.13. *$\mathfrak{D}$ has the following properties:*

1. *$\mathfrak{D}$ is complete with respect to declaratives, in the sense that $\alpha \in \mathfrak{D}$ iff $\sim\alpha \in \mathfrak{D}$.*
2. *if $\mathcal{F}$ is finite then $\mathfrak{D}$ is finite.*
3. *if $\alpha \in \mathfrak{D}$ and $\beta \in sub(\alpha)$, then $\beta \in \mathfrak{D}$.*
4. *if $\varphi \in \mathfrak{F}$, then $\alpha \in \mathfrak{D}$ for all $\alpha \in \mathcal{R}(\varphi)$.*

Proof.

1. $\alpha \in \mathfrak{D}$ iff $\alpha \in \mathfrak{F}$ iff $\sim\alpha \in \mathfrak{F}$ iff $\sim\alpha \in \mathfrak{D}$, by proposition 6.2.11 and the definition of $\mathfrak{D}$.
2. Immediate, as $\mathfrak{D} \subseteq \mathfrak{F}$, and we have by proposition 6.2.11 that $\mathfrak{F}$ is finite.
3. If $\alpha \in \mathfrak{D}$ then $\alpha \in \mathfrak{F}$, whence if $\beta \in sub(\alpha)$ then $\beta \in \mathfrak{F}$ by proposition 6.2.11, and thus as β is a declarative, $\beta \in \mathfrak{D}$ by definition of $\mathfrak{D}$.
4 If $\varphi \in \mathfrak{F}$ then by proposition 6.2.11 we know that $\alpha \in \mathfrak{F}$ for all $\alpha \in \mathcal{R}(\varphi)$, whence $\alpha \in \mathfrak{D}$ for all $\alpha \in \mathcal{R}(\varphi)$ by definition of $\mathfrak{D}$. □

These observations will be important in constructing our complete theories of declaratives relative to $\mathcal{F}$. First, we define complete theories of declaratives simpliciter.

Definition 6.2.14 (Complete theory of declaratives)**.** A set of declaratives Γ is a complete theory of declaratives (CTD) if it is the smallest set satisfying the following conditions:

1. Γ is consistent. I.e. $\Gamma \nvdash \bot$
2. Γ is complete, in the sense that for every declarative α, either α or $\sim\alpha \in \Gamma$.

We observe Lindenbaum's lemma, which we will appeal to later.

Fact 6.2.15 (Lindenbaum's lemma)**.** *If Δ is a consistent theory of declaratives, then $\Delta \subseteq \Gamma$ for some complete theory of declaratives Γ.*

We now turn to constructing a canonical model relative to a set of formulas $\mathcal{F}$. Our approach is loosely modelled on the proof of weak completeness for PDL as found in Blackburn, Rijke, and Venema (2002).

Definition 6.2.16 (Nuclei)**.** For a set of formulas $\mathcal{F}$ we define a set of declaratives N to be an nucleus over $\mathcal{F}$ if it is a maximally consistent theory of declaratives in $\mathfrak{D}$. So, N is an nucleus over $\mathcal{F}$ if *a*) A is a set of declaratives, *b*) A is consistent, *c*) $A \subseteq \mathfrak{D}$, and *d*) if $A \subset B \subseteq \mathfrak{D}$, then B is inconsistent. Let $\mathrm{Nu}(\mathcal{F})$ be the set of all nuclei over $\mathcal{F}$.

Lemma 6.2.17. *$Nu(\mathcal{F}) = \{\Gamma \cap \mathfrak{D} \mid \Gamma$ is a complete theory of declaratives.$\}$*

Proof. We split the proof into two cases, establishing each set is a subset of the other.

From left to right suppose $N \in \mathrm{Nu}(\mathcal{F})$. Then, N is a consistent theory of declaratives, by definition, and so can be extended to some complete theory of declaratives Γ by Lindenbaum's lemma. We now want to show $N = \Gamma \cap \mathfrak{D}$, but this is immediate given that N is a complete theory of declaratives with respect to $\mathfrak{D}$, as for any $\alpha \in (\Gamma - N)$, $\alpha \notin \mathfrak{D}$.

From right to left, let $\Gamma \cap \mathfrak{D}$ be arbitrary. By definition Γ is a maximally consistent set, and trivially $\Gamma \cap \mathfrak{D} \subseteq \mathfrak{D}$. Therefore, $\Gamma \cap \mathfrak{D}$ is a consistent subset of $\mathfrak{D}$. To see that it is maximal with respect to the other consistent subsets of $\mathfrak{D}$ suppose there is some $\alpha \in \mathfrak{D}$ such that $\Gamma \cap \mathfrak{D} \cup \{\alpha\}$ is consistent yet $\alpha \notin \Gamma \cap \mathfrak{D}$.

However, by proposition 6.2.13 if $\alpha \in \mathfrak{D}$ then $\sim\alpha \in \mathfrak{D}$, whence $\alpha \notin \Gamma$. And so, by definition 6.2.14 we know $\sim\alpha \in \Gamma$. So, $\sim\alpha \in \Gamma \cap \mathfrak{D}$, contradicting the assumption that $\Gamma \cap \mathfrak{D} \cup \{\alpha\}$ is consistent.

Therefore, as $\Gamma \cap \mathfrak{D}$ is a consistent subset of $\mathfrak{D}$ and is maximal with respect to the other consistent subsets of $\mathfrak{D}$, $\Gamma \cap \mathfrak{D} \in \mathrm{Nu}(\mathcal{F})$. □

Remark 6.2.18. This an adaptation of the 'heavy handed' approach from Blackburn, Rijke, and Venema (2002, p. 242), the finitary proof given for the case of PDL does not straightforwardly carry over to ICDL. For our purposes, this approach is sufficient.

We have defined our nuclei to be maximally consistent subsets of $\mathfrak{D}$. However, in order to show that the canonical model over $\mathscr{F}$ is an ICDM it seems we need further syntactic structure.[3] In order to do this we extend each nucleus to an atom in a manner analogous to the construction of a maximally consistent set of formulas in Lindenbaum's lemma (Cf. Blackburn, Rijke, and Venema (2002, p. 199)).

Definition 6.2.19 (Atoms). Let $\alpha_1, \ldots, \alpha_i, \ldots$ be an enumeration of the declaratives of $\mathscr{L}^{\mathsf{ICDL}}$. We define an atom A relative to a nucleus N as the union of a chain of $\mathscr{L}^{\mathsf{ICDL}}$-consistent sets as follows:

$$A_0 = N$$
$$A_{n+1} = \begin{cases} A_n \cup \{\alpha_n\}, & \text{if } A_n \vdash \alpha_n \\ A_n \cup \{\neg\alpha_n\}, & \text{otherwise} \end{cases}$$
$$A = \textstyle\bigcup_{n \geq 0} A_n.$$

While nuclei would seem to give us insufficient syntactic structure, atoms give us an abundance. And, while we will not require the full structure of atoms, their definition is simpler than carefully crafting sets of formulas that we do need for nuclei.

The complication remains in taking a finite set of formulas relative to a fragment of ICDL and expanding these to any consistent set of declaratives relative to ICDL (giving up maximally consistent sets of declaratives), before then restricting these maximally consistent sets relative to the finite fragment (giving us nuclei), and then enlarging the sets again to a greater fragment of ICDL (giving us atoms). However, the construction of atoms differs from the construction of maximally consistent sets by constraining *which* declaratives are added to the nuclei.

For, given a nucleus N and declarative β such that $N \nvdash \beta$ and $N \nvdash \neg\beta$ then there would be two maximally consistent sets, corresponding to extensions of $N \cup \{\beta\}$ and $N \cup \{\neg\beta\}$. However, there will only be one atom; either the extension of $N \cup \{\beta\}$ or of $N \cup \{\neg\beta\}$.[4] Moreover, as there are a finite number of nuclei over $\mathfrak{F}$, this ensures there are a finite number of atoms over $\mathfrak{F}$.

We denote by $\mathrm{At}(\mathscr{F})$ the set of atoms over $\mathscr{F}$.

Proposition 6.2.20. *Let A be an atom, then:*

1. *A is consistent.*
2. *If $A \cap \mathfrak{D} = B \cap \mathfrak{D}$, then $A = B$.*
3. *For any declaratives $\alpha \in \mathscr{L}^{\mathsf{ICDL}}$, $A \vdash \alpha$ or $A \vdash \neg\alpha$.*

Proof.

1 Straightforward.

2 Suppose $A \cap \mathfrak{D} = B \cap \mathfrak{D}$. Then $A_0 = B_0$, which establishes the base case. For the induction case suppose $A_n = B_n$. Then $A_n \vdash \alpha$ iff $B_n \vdash \alpha$ for any $\alpha \in \mathscr{L}^{\mathsf{ICDL}}$, and so $A_{n+1} = B_{n+1}$, by the construction of A_{n+1} and B_{n+1}.

[3] The issue comes in particular for formulas of the form $\neg C_a^{\psi} \neg\varphi$, for $\varphi, \psi \in \mathfrak{F}$. For, axioms 4.ii and 5 both have antecedents of this form, yet we are not guaranteed these are elements of $\mathfrak{F}$, nor is it obvious that formulas of this kind would follow from the elements of each nucleus …

[4] If β is the first formula in the enumeration we can be sure that the atom is an extension of $N \cup \{\beta\}$. However, if β occurs after so formula it may be the case that the partial atom entails β, whence the complete atom will entail β even if N does not.

3 Let α_i in the enumeration of the declaratives of $\mathscr{L}^{\mathsf{ICDL}}$ be arbitrary. As atoms are consistent then it is the case that either $A \vdash \alpha_i$ or $A \nvdash \alpha_i$. By construction of A, if $A \nvdash \alpha_i$ then we know $\neg\alpha_i \in A$, whence $A \vdash \neg\alpha_i$. $\square$

Lemma 6.2.21. *If* $\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathfrak{F}$ *and* $\{\alpha_1, \ldots, \alpha_n\}$ *is consistent, there is an atom* $A \in At(\mathcal{F})$ *such that* $\{\alpha_1, \ldots, \alpha_n\} \subseteq A$.

Proof. $\{\alpha_1, \ldots, \alpha_n\}$ is a consistent set of declaratives, so long as it is consistent. Therefore by fact 6.2.15 there is some CTD Γ such that $\{\alpha_1, \ldots, \alpha_n\} \subseteq \Gamma$. We now apply lemma 6.2.17 to infer $N = \Gamma \cap \mathfrak{D}$ is an nucleus containing α. Therefore, by definition 6.2.19 there is an atom extending N. $\square$

Proposition 6.2.22 (Deduction of declaratives).
For a set of formulas $\mathcal{F}$ *and every* A *in* $At(\mathcal{F})$, *if* $A \vdash \beta$, *then* $\beta \in A$.

Proof. Let $A \in \mathrm{At}(\mathcal{F})$ be arbitrary and suppose for some β that $A \vdash \beta$ but $\beta \notin A$. By the former fact in conjunction with the fact that atoms are consistent by proposition 6.2.20 we know that $A \nvdash \neg\beta$ and therefore we know that $\neg\beta \notin A$. However, as β is a declarative it must occur in the enumeration of the declaratives of ICDL, which means that every atom contains either β or $\neg\beta$ by construction. So, we have contradicted our assumption that A is an atom. $\square$

Proposition 6.2.23 (Disjunction property for Atoms). *If* $\alpha_1 \vee \cdots \vee \alpha_n \in A$, *for some atom* A, *then* $\alpha_i \in A$ *for some* $i \leq n$.

Proof. As $\alpha_1 \vee \cdots \vee \alpha_n \in A$, $A \vdash \alpha_1 \vee \cdots \vee \alpha_n$. Therefore, $A \vdash \neg({\sim}\alpha_1 \wedge \cdots \wedge {\sim}\alpha_n)$. So, suppose toward a contradiction that $A \nvdash \alpha_i$ for $i \leq n$. We know by proposition 6.2.20.3 that $A \vdash \alpha_i$ or $A \vdash \neg\alpha_i$, and so it must be the case that $A \vdash \neg\alpha_i$ for all $i \leq n$. So, $A \vdash {\sim}\alpha_i$ for all $i \leq n$, given that we know ${\sim}\alpha \dashv\vdash \neg\alpha$ by proposition 1.2.29. Therefore, by the introduction rule for conjunction we know that $A \vdash {\sim}\alpha_1 \wedge \cdots \wedge {\sim}\alpha_n$, but this contradicts the consistency of A, whence $A \vdash \alpha_i$ for some $i \leq n$ $\square$

6.2.1 Syntactic Characterisation of Atoms, States, and Propositions.

Before giving the recipe to construct a canonical model relative to $\mathcal{F}$ we establish some properties about the set of atoms. Let us assume a fixed $\mathcal{F}$. By $A, B, \ldots$ we denote atoms, $S, T, \ldots$ sets of atoms, and by $P, Q \ldots$ non-empty downward closed subsets of $\wp(\mathrm{At}(\mathcal{F}))$. Given the recipe to construct the canonical model, below, atoms correspond to possible words, $S, T, \ldots$ to states, and $P, Q, \ldots$ to issues.

Lemma 6.2.24. *If* $A\restriction_{\mathcal{F}'} = B\restriction_{\mathcal{F}'}$, *then* $A = B$.

Proof. Suppose $A\restriction_{\mathcal{F}'} = B\restriction_{\mathcal{F}'}$. We establish a series of sublemmas, corresponding to restrictions of A and B.

1. If $\alpha \in \mathcal{F}^{\mathsf{I}}$, **then** $\alpha \in A$ **iff** $\alpha \in B$

Immediate, given our assumption that $A\restriction_{\mathcal{F}'} = B\restriction_{\mathcal{F}'}$.

2. If $\alpha \in \mathcal{F}^{\mathsf{II}}$, then $\alpha \in A$ iff $\alpha \in B$

Suppose $\alpha \in \mathcal{F}^{\mathsf{II}}$. Then, either $\alpha \in \mathcal{F}^{\mathsf{I}}$ or $\alpha \in \mathcal{F}^{\mathsf{II}} - \mathcal{F}^{\mathsf{I}}$. The previous sublemma establishes the claim holds if the former is the case. Suppose, then, that $\alpha \in \mathcal{F}^{\mathsf{II}} - \mathcal{F}^{\mathsf{I}}$. Therefore, α is of the form $\alpha_1 \wedge \cdots \wedge \alpha_n$ where each $\alpha_i \in \mathcal{F}^{\mathsf{I}}$.

Suppose $\alpha \in A$. Then, by closure under deduction of declaratives in $\mathfrak{F}$ (proposition 6.2.22), it must be the case that $\alpha_i \in A$, for each i. Then, by the previous sublemma we can infer $\alpha_i \in B$ for each i. So, again by the fact that B is complete and consistent with respect to $\mathfrak{F}$ it must be the case that $\alpha_1 \wedge \cdots \wedge \alpha_n \in B$. The converse is established analogously.

3. If $\alpha \in \mathcal{F}^{\mathsf{III}}$, then $\alpha \in A$ iff $\alpha \in B$

Suppose $\alpha \in \mathcal{F}^{\mathsf{III}}$ and $\alpha \in A$. As before, the case where $\alpha \in \mathcal{F}^{\mathsf{II}}$ follows from the previous sublemma. Therefore, let us assume $\alpha \in \mathcal{F}^{\mathsf{III}} - \mathcal{F}^{\mathsf{II}}$. So, α is of the form $\alpha_1 \vee \cdots \vee \alpha_n$ where each $\alpha_i \in \mathcal{F}^{\mathsf{II}}$. However, atoms have the disjunction property (proposition 6.2.23), and therefore $\alpha_i \in A$ for some i. So, as $\alpha_i \in \mathcal{F}^{\mathsf{II}}$ and $\alpha_i \in A$ we have by the previous sublemma that $\alpha_i \in B$. Finally, the fact that B is complete and consistent guarantees that $\alpha_1 \vee \cdots \vee \alpha_n \in B$. The converse is established analogously.

4. If $\alpha \in \mathcal{F}^{\mathsf{IV}}$, then $\alpha \in A$ iff $\alpha \in B$

Note the operation for $\mathcal{F}^{\mathsf{IV}}$ only introduces declaratives of the form $\neg(?\{\alpha_1, \ldots, \alpha_n\})$.

So, from left to right, if $\neg(?\{\alpha_1, \ldots, \alpha_n\}) \in A$ then $A \vdash \neg(?\{\alpha_1, \ldots, \alpha_n\})$, whence as A is consistent this means $A \nvdash ?\{\alpha_1, \ldots, \alpha_n\}$. So, by theorem 6.2.7 we know that $A \nvdash \alpha_i$ for all $i \leq n$, as $\mathcal{R}(?\{\alpha_1, \ldots, \alpha_n\}) = \{\alpha_1, \ldots, \alpha_n\}$. Therefore, $\alpha_i \notin A$, but we know each $\alpha_i \in \mathcal{F}^{\mathsf{III}}$, whence by the previous case we know that $\alpha_i \notin B$, and therefore $B \nvdash ?\{\alpha_1, \ldots, \alpha_n\}$ by theorem 6.2.7 again. So, $B \vdash \neg(?\{\alpha_1, \ldots, \alpha_n\})$ by proposition 6.2.20, whence $\neg(?\{\alpha_1, \ldots, \alpha_n\}) \in B$.

The right to left direction is established analogously.

5. If $\alpha \in \mathfrak{D}$, then $\alpha \in A$ iff $\alpha \in B$

Suppose $\alpha \in \mathfrak{F} - \mathcal{F}^{\mathsf{IV}}$ and $\alpha \in A$. This means α is of the form $\sim\beta$, for some $\beta \in \mathcal{F}^{\mathsf{IV}}$. As atoms are consistent, this means $\beta \notin A$. So, from the previous sublemma $\beta \notin B$. Therefore, by the fact that atoms are complete by proposition 6.2.20, it must be the case that $\sim\beta \in B$. The converse is established analogously.

6. As we have shown that for all $\alpha \in \mathfrak{F}$, $\alpha \in A$ iff $\alpha \in B$ it follows immediately that for all $\alpha \in \mathfrak{D}$, $\alpha \in A$ iff $\alpha \in B$. Therefore, we now know that $A \cap \mathfrak{D} = B \cap \mathfrak{D}$, and therefore by proposition 6.2.202 we know that $A = B$. □

By the previous lemma we can associate to every atom a canonical formula, defined by taking the conjunction of the restriction of A to $\mathcal{F}^{\mathsf{I}}$, and by the following definition this is used to build canonical formulas for states.

Definition 6.2.25. $\gamma_A := \bigwedge_{\alpha \in A\restriction_{\mathcal{F}^{\prime}}} \alpha$

The following lemma establishes that γ_A is indeed canonical.

Lemma 6.2.26. $\gamma_A \in B \iff B = A$

Proof. Suppose $\gamma_A \in B$. Therefore, as B is closed under deduction of declaratives by proposition 6.2.22 it must be the case that $\alpha_i \in B$ for all $\alpha_i \in A\restriction_{\mathcal{F}'}$, so $A\restriction_{\mathcal{F}'} \subseteq B\restriction_{\mathcal{F}'}$. Conversely, if $\alpha \notin A\restriction_{\mathcal{F}'}$, then $\sim\alpha \in A\restriction_{\mathcal{F}'}$, whence $\sim\alpha \in B\restriction_{\mathcal{F}'}$ by the previous inference, and so $\alpha \notin B\restriction_{\mathcal{F}'}$ Therefore, as $A\restriction_{\mathcal{F}'} = B\restriction_{\mathcal{F}'}$ by lemma 6.2.24 we know $A = B$.

From right to left we begin by observing that $A = B$ implies $A\restriction_{\mathcal{F}'} = B\restriction_{\mathcal{F}'}$. Second, we observe $\gamma_A \in \mathcal{F}^{\text{II}}$, by the fact that each conjunct of γ_A is an element of $\mathcal{F}^{\text{I}}$. Therefore, by closure of atoms under deduction of declaratives in $\mathfrak{F}$, we know $\gamma_A \in B$. □

Lemma 6.2.27. *For any atom A, $\gamma_A \in \mathcal{F}^{\text{II}}$.*

Proof. By definition 6.2.25 $\gamma_A := \bigwedge_{\alpha \in A\restriction_{\mathcal{F}'}} \alpha$. And, by definition 6.2.10 clause 2f if $\alpha_1, \ldots, \alpha_n \in \mathcal{F}^{\text{I}}$ and are distinct then $\alpha_1 \wedge \cdots \wedge \alpha_n \in \mathcal{F}^{\text{II}}$, whence $\bigwedge_{\alpha \in A\restriction_{\mathcal{F}'}} \alpha \in \mathcal{F}^{\text{II}}$. □

From atoms we move to states. These are canonically defined by taking the disjunction of the canonical formula for every atom in the state.

Definition 6.2.28. $\gamma_S := \bigvee_{A \in S} \gamma_A$, where $\gamma_\emptyset := \bot$.

For canonicity we require S and its substates derive the associated formula. This weakening follows from the fact that states in ICDL are persistent, and so states cannot be uniquely characterised. To show that γ_S is indeed canonical for S we begin by showing that an atom derives γ_S just in case it is an element of S.

Lemma 6.2.29. $A \vdash \gamma_S$ *iff* $A \in S$.

Proof. From left to right suppose $A \vdash \gamma_S$. Then, $A \vdash \bigvee_{B \in S} \gamma_B$, by definition of γ_S. So, we know that $\bigvee_{B \in S} \gamma_B \in A$ by proposition 6.2.22. From this we know that $\gamma_B \in A$ for some $B \in S$ by proposition 6.2.23. And, by lemma 6.2.26 we know $A \vdash \gamma_B$ iff $B = A$, whence $A \in S$.

From right to left suppose $A \in S$. This means we have γ_A as a disjunct of $\bigvee_{B \in S} \gamma_B$. However, we know $A \vdash \gamma_A$, and therefore it follows that $A \vdash \bigvee_{B \in S} \gamma_B$, which is equivalent to $A \vdash \gamma_S$. □

Lemma 6.2.30. *For any state S, $\gamma_S \in \mathcal{F}^{\text{III}}$.*

Proof. First note that as there are only finitely many atoms, each state S is of a finite size.

By definition 6.2.28 $\gamma_S := \bigvee_{A \in S} \gamma_A$, and by lemma 6.2.27 we know $\gamma_A \in \mathcal{F}^{\text{II}}$. By definition 6.2.10 clause 3h if $\alpha_1, \ldots, \alpha_n \in \mathcal{F}^{\text{II}}$ and are distinct then $\alpha_1 \vee \cdots \vee \alpha_n \in \mathcal{F}^{\text{III}}$, therefore $\gamma_{A_1} \vee \cdots \vee \gamma_{A_n} \in \mathcal{F}^{\text{III}}$ for $A_1, \ldots, A_n \in S$, and so $\gamma_S \in \mathcal{F}^{\text{III}}$. □

From this we observe the intersection of all the atoms in a state derive its canonical formula.

Corollary 6.2.31. $\bigcap S \vdash \gamma_S$, *where* $\bigcap \emptyset := \mathfrak{D}$.[5]

Proof. If $S = \emptyset$ then $\bigcap S = \mathfrak{D}$. Clearly, $\mathfrak{D} \vdash \bot$, as $\mathfrak{D}$ contains α and $\sim\alpha$ for all $\alpha \in \mathfrak{F}$.

So, assume $S \neq \{\emptyset\}$. Then, we know by lemma 6.2.29 that $A \vdash \gamma_S$ for all $A \in S$. And, $\gamma_S \in \mathfrak{F}$, for by lemma 6.2.30 $\gamma_S \in \mathcal{F}^{\text{III}} \subseteq \mathfrak{F}$. Therefore, by proposition 6.2.22 we have $\gamma_S \in A$ Therefore, as $\bigcap S = \{\alpha \mid \alpha \in A \text{ for all } A \in S\}$ we know $\gamma_S \in \bigcap S$, whence $\bigcap S \vdash \gamma_S$. □

[5] We have defined $\bigcap S = \{\alpha \in \mathfrak{D} \mid \forall A \in S, \alpha \in A\}$. So, on this basis we define $\bigcap \emptyset := \{\alpha \in \mathfrak{D} \mid \forall A \in \emptyset, \alpha \in A\} = \mathfrak{D}$.

By the previous lemma and its corollary we are ready to show that γ_S is canonical in the required sense.

Lemma 6.2.32. $\bigcap T \vdash \gamma_S \iff T \subseteq S$

Proof. From left to right suppose $\bigcap T \vdash \gamma_S$ and let $A \in T$ be arbitrary. As $\bigcap T \subseteq A$, we know $A \vdash \gamma_S$. Therefore, $A \in S$, by lemma 6.2.29.

From right to left suppose $T \subseteq S$. From this assumption we infer, $\bigcap S \subseteq \bigcap T$. By corollary 6.2.31 we know $\bigcap S \vdash \gamma_S$, and so $\bigcap T \vdash \gamma_S$. $\square$

Using canonical formulas for states, we can syntactically characterise issues. Like atoms and states, each issue can be identified with a unique characteristic formula, and thus an inquisitive proposition, defined as follows.

Definition 6.2.33. $\chi_P := ?\{\gamma_S \mid S \in P\}$

Lemma 6.2.34. $\bigcap S \vdash \chi_P \iff S \in P$

Proof. From left to right suppose $\bigcap S \vdash \chi_P$. As $\bigcap S$ is a set of declaratives, then by corollary 6.2.8 we know $\bigcap S \vdash \alpha$ for some $\alpha \in \mathcal{R}(\chi_P)$. But then by definition of χ_P, $\bigcap S \vdash \gamma_T$ for some $T \in P$. Therefore, $S \subseteq T$, by 6.2.32. But then $S \in P$ as $T \in P$ and P is downward closed by definition.

From right to left suppose $S \in P$. We know by corollary 6.2.31 that $\bigcap S \vdash \gamma_S$. Therefore, $\bigcap S \vdash ?\{\gamma_S \mid S \in P\}$. So, $\bigcap S \vdash \chi_P$, by definition. $\square$

Proposition 6.2.35. *Given a set of formulas $\mathcal{F}$, then for every issue P over At($\mathcal{F}$) there is a formula ϕ in $\mathfrak{F}$ such that $S \in P$ iff $\bigcap S \vdash \phi$.*

Proof. By definition 6.2.33, $\chi_P := ?\{\gamma_S \mid S \in P\}$. By lemma 6.2.30, $\gamma_S \in \mathcal{F}^{\text{III}}$, as so by 6.2.10 clause 4j, $\chi_P := ?\{\gamma_S \mid S \in P\} \in \mathfrak{F}$. $\square$

Remark 6.2.36. Note also that as $\chi_P = ?\{\gamma_S \mid S \in P\}$ and each $\gamma_S \in \mathcal{F}^{\text{III}}$ by lemma 6.2.30 we also know by clause 4k, that $\neg\chi_P = \neg(?\{\gamma_S \mid S \in P\}) \in \mathfrak{F}$.

6.3 The Canonical Model

Definition 6.3.1 (Canonical model over $\mathcal{F}$). Let $\mathcal{F}$ be a finite set of formulas. The canonical model over $\mathcal{F}$ is the tuple: $M^{\mathcal{F}} = \langle \text{At}(\mathcal{F}), \{\mathcal{S}_a^P\}_{a \in \mathcal{A}, P \in \mathcal{I}}, V\rangle$, defined as follows:

- At($\mathcal{F}$) is the set of atoms over $\mathcal{F}$
- $V(A) = \{p \in \mathsf{At} \mid p \in A\}$
- For every issue $P \in \mathcal{I}$, $\mathcal{S}_a^P(w)$ is the set of states $S \subseteq \text{At}(\mathcal{F})$ defined by:

$$S \in \mathcal{S}_a^P(A) \iff \bigcap S \vdash \varphi \text{ whenever } A \vdash C_a^{\chi_P} \varphi$$

Lemma 6.3.2. $\forall S \subseteq At(\mathcal{F})$ *and* $\forall \alpha \in \mathfrak{F}, \bigcap S \vdash \alpha \iff \alpha \in \bigcap S$.

Proof. Our reasoning closely follows Ciardelli (2014b, p. 111). The right to left direction is immediate. For the left to right direction suppose $\bigcap S \vdash \alpha$. Then as for any $A \in S$ we have $\bigcap S \subseteq A$ we know that $A \vdash \alpha$. We have assumed $\alpha \in \mathfrak{F}$, and therefore by proposition 6.2.22 we must have $\alpha \in A$, and so $\alpha \in \bigcap S$. $\square$

Lemma 6.3.3. *For $A \in At(\mathcal{F})$, $\varphi \in \mathfrak{F}$, if $A \nvdash C_a^{\chi_P} \varphi$ then $\exists S \in \mathcal{S}_a^P(A)$: $\bigcap S \nvdash \varphi$.*

Proof. Our reasoning closely follows Ciardelli (2014b, pp. 111–112).

Suppose $A \nvdash C_a^{XP}\varphi$. Define $A^{C_a^{XP}} := \{\chi \mid A \vdash C_a^{XP}\chi$ and $\chi \in \mathfrak{F}\}$. We now claim $A^{C_a^{XP}} \nvdash \varphi$. Suppose toward a contradiction that $A^{C_a^{XP}} \vdash \varphi$. This means we have $\chi_i \in A^{C_a^{XP}}$ such that $\chi_1, \dots, \chi_n \vdash \varphi$. Therefore, we can apply lemma 6.2.6 to infer $C_a^{XP}\chi_1, \dots, C_a^{XP}\chi_n \vdash C_a^{XP}\varphi$. However, we know that $A \vdash C_a^{XP}\chi_1, \dots, C_a^{XP}\chi_n$, and therefore $A \vdash C_a^{XP}\varphi$, contrary to our initial assumption.

As $A^{C_a^{XP}} \nvdash \varphi$ we know by lemma 6.2.7 that there must be a resolution Θ of $A^{C_a^{XP}}$ which does not entail any resolution α of φ. Therefore, for any $\alpha \in \mathcal{R}(\varphi)$ we have a consistent set of declaratives $\Theta \cup \{\sim\alpha\}$. Furthermore, we know $\Theta \cup \{\sim\alpha\} \subseteq \mathfrak{F}$, for we have assumed $\varphi \in \mathfrak{F}$, and so each of its resolutions and their pseudo-negations is in $\mathfrak{F}$, and $A^{C_a^{XP}} \subseteq \mathfrak{F}$, by construction, guaranteeing $\Theta \in \mathfrak{F}$. Therefore there exists some atom A_α such that $\Theta \cup \{\sim\alpha\} \subseteq A$ by lemma 6.2.21.

Consider now the state $S := \{A_\alpha \mid \alpha \in \mathcal{R}(\varphi)\}$. As $\Theta \subseteq A_\alpha$ for all $A_\alpha \in T$ we know $\Theta \subseteq \bigcap S$. Suppose $A \vdash C_a^{XP}\rho$ for some $\rho \in \mathfrak{F}$. Then $\rho \in A^{C_a^{XP}}$, by definition, and since Θ is a resolution of $A^{C_a^{XP}}$ it must contain some resolution β of ρ. As $\beta \in \Theta \subseteq \bigcap T$ it follows that $\bigcap S \vdash \rho$, by corollary 6.2.4. This means that $\bigcap S \vdash \rho$ whenever $A \vdash C_a^{XP}\rho$, which entails $S \in \mathcal{S}_a^P(A)$.

Suppose toward a contradiction that $\bigcap S \vdash \varphi$. Then by corollary 6.2.8 we have $\bigcap S \vdash \alpha$ for some $\alpha \in \mathcal{R}(\varphi)$. As $\bigcap S \subseteq A_\alpha$ we know that $\bigcap S$ contains no such resolution of φ, given A_a contains $\sim\alpha$ and is consistent. Therefore $\bigcap S \nvdash \varphi$. □

Lemma 6.3.4 (Support lemma). *For a set of formulas $\mathcal{F}$ and the canonical model over $\mathcal{F}$, $M^{\mathcal{F}}$, for any $S \subseteq At(\mathcal{F})$ and any $\varphi \in \mathfrak{F}$,*

$$M^{\mathcal{F}}, S \vDash \varphi \iff \bigcap S \vdash \varphi$$

Proof. Our proof follows Ciardelli (2014b, pp. 112–1123), and proceeds via induction on the complexity of φ. Complications arise, and therefore we prove the lemma in full.

B1) $\phi := p$, for some proposition letter $p \in \mathfrak{F}$

$M^{\mathcal{F}} S \vDash p$ iff $p \in V(A)$ for all $A \in S$ iff $p \in \bigcap S$ iff $\bigcap S \vdash p$.

B2) $\phi := \bot$

From left to right, $M^{\mathcal{F}}, S \vDash \bot$ iff $S = \emptyset$ iff $\bigcap S = \mathfrak{D}$. Therefore, $\bigcap S \vdash \bot$.

From right to left, if $\bigcap S \vdash \bot$ then it must be the case that $S = \emptyset$, as if there is some $A \in S$ then $\bigcap S \subseteq A$. So, were it the case that $\bigcap S \vdash \bot$, it would follow that $A \vdash \bot$. Yet, we know that atoms are consistent, whence $S = \emptyset$, and so $M^{\mathcal{F}}, S \vDash \bot$.

We now suppose the claim holds for all formulas ψ of a lower complexity than φ.

I1) $\phi := \psi \wedge \chi$

$M^{\mathcal{F}}, S \vDash \psi \wedge \chi$ iff $M^{\mathcal{F}}, S \vDash \psi$ and $M^{\mathcal{F}}, S \vDash \chi$. As $\psi \wedge \chi \in \mathfrak{F}$, and $\mathfrak{F}$ is closed under subformulas we know $\psi, \chi \in \mathfrak{F}$. Therefore, by the induction hypothesis we have $M^{\mathcal{F}}, S \vDash \psi \wedge \chi$ iff $M^{\mathcal{F}}, S \vDash \psi$ and $M^{\mathcal{F}}, S \vDash \chi$ iff $\bigcap S \vdash \psi$ and $\bigcap S \vdash \chi$.

Moreover, it is the case that $\bigcap S \vdash \psi$ and $\bigcap S \vdash \chi$ iff $\bigcap S \vdash \psi \wedge \chi$, by the rules governing conjunction, whence $M^{\mathcal{F}}, S \vDash \psi \wedge \chi$ iff $\bigcap S \vdash \psi \wedge \chi$.

I2) $\phi := \psi \to \chi$

Suppose $\bigcap S \nvdash \psi \to \chi$. So, by the introduction rule for implication $\bigcap S, \psi \nvdash \chi$. Therefore, by theorem 6.2.7 there is some resolution of $\bigcap S \cup \{\psi\}$ which doesn't derive χ. As $\bigcap S$ is a set of declaratives this resolution will be of the form $\bigcap S \cup \{\alpha\}$, where $\alpha \in \mathcal{R}(\psi)$. Therefore, there must exist a resolution α of ψ such that $\bigcap S, \alpha \nvdash \chi$.

Let $T = \{A \in S \mid \alpha \in A\}$. As $T \neq \emptyset$, then by definition we have $\alpha \in T$, whence by corollary 6.2.4 we have $\bigcap T \vdash \psi$. Therefore, $\bigcap T \vdash ?\mathcal{R}(\psi)$, and so by the induction hypothesis we have $M^{\mathcal{F}}, \bigcap T \vDash \psi$.

Toward a contradiction suppose $\bigcap T \vdash \chi$. As $\bigcap T$ is a set of declaratives it follows by corollary 6.2.8 that $\bigcap T \vdash \beta$ for some $\beta \in \mathcal{R}(\chi)$. By lemma 6.3.2 this means $\beta \in \bigcap T$, and so $\alpha \to \beta \in \bigcap T$. Consider now any $A \in S - T$. By definition of T this means $\alpha \notin A'$, so $\sim\alpha \in A'$. Therefore, $\sim\alpha \in \bigcap(S - T)$, so $\alpha \to \beta \in \bigcap(S - T)$. As $\alpha \to \beta \in \bigcap(S - T)$ and $\alpha \to \beta \in \bigcap T$ we know $\alpha \to \beta \in \bigcap S$, therefore, $\bigcap S \vdash \alpha \to \beta$, thus $\bigcap S, \alpha \vdash \beta$. Since β is a resolution of χ this means $\bigcap S, \alpha \vdash \chi$. This contradicts our assumption that $\bigcap S, \alpha \nvdash \chi$. Therefore, $\bigcap T \nvdash \chi$ and so by the induction hypothesis we know $M^{\mathcal{F}}, T \nvDash \chi$. But then S has a substate of T which supports ψ but not χ, which shows that $M^{\mathcal{F}}, S \nvDash \psi \to \chi$.

Conversely, suppose $\bigcap S \vdash \psi \to \chi$. Let $T \subseteq S$ be arbitrary. If $M^{\mathcal{F}}, T \vDash \psi$ then by the induction hypothesis $\bigcap T \vdash \psi$. As $T \subseteq S$, $\bigcap S \subseteq \bigcap T$. And since $\bigcap S \vdash \psi \to \chi$ this means $\bigcap T \vdash \psi \to \chi$. So, it follows that $\bigcap T \vdash \chi$. This implies $M^{\mathcal{F}}, T \vDash \chi$, by the induction hypothesis. Therefore, as $T \subseteq S$ was arbitrary we have shown $M^{\mathcal{F}}, S \vDash \psi \to \chi$.

I3) $\phi := ?\{\alpha_1, \ldots, \alpha_n\}$

If $M^{\mathcal{F}}, S \vDash ?\{\alpha_1, \ldots, \alpha_n\}$, then $M^{\mathcal{F}}, S \vDash \alpha_i$ for some i, and $?\{\alpha_1, \ldots, \alpha_n\} \in \mathfrak{F}$ implies $\alpha_i \in \mathfrak{F}$ for all i. Therefore, by the induction hypothesis we have $\bigcap S \vdash \alpha_i$, whence $\bigcap S \vdash ?\{\alpha_1, \ldots, \alpha_n\}$.

Conversely, suppose $\bigcap S \vdash ?\{\alpha_1, \ldots, \alpha_n\}$. As $\bigcap S$ is a set of declaratives it follows from corollary 6.2.8 that $\bigcap S \vdash \alpha_i$ for some i. Therefore, by the induction hypothesis we have $M^{\mathcal{F}}, \bigcap S \vDash \alpha_i$, and so $M^{\mathcal{F}}, \bigcap S \vDash ?\{\alpha_1, \ldots, \alpha_n\}$.

I4) $\phi := C_a^{\psi} \chi$

We begin by showing that $\psi \dashv\vdash \chi_Q$ for some issue Q in $M^{\mathcal{F}}$. From this it will follow that $C_a^{\psi} \varphi \dashv\vdash C_a^{\chi_Q} \varphi$ by the rule of replacement of equivalents, from which this induction step will follow by definition of $\mathcal{S}_a^{Q}(A)$ and lemma 6.3.3.

By the induction hypothesis we know that for any state S, $\bigcap S \vdash \psi$ iff $M^{\mathcal{F}}, S \vDash \psi$, given ψ is a subformula of $C_a^{\psi} \chi$. Let $[\![\psi]\!]_{M^{\mathcal{F}}} = Q$ for an issue Q over $\mathrm{At}(\mathcal{F})$. Recall by lemma 6.2.34 that for any issue P in $M^{\mathcal{F}}$ there is a corresponding formula χ_P such that $\bigcap S \vdash \chi_P$ iff $S \in P$. Therefore, we can infer that $\bigcap S \vdash \chi_Q$ iff $S \in [\![\psi]\!]$ iff $M^{\mathcal{F}}, S \vDash \psi$. For ease of comprehension let us write χ_ψ for χ_Q, whence we have $\bigcap S \vdash \chi_\psi$ iff $M^{\mathcal{F}}, S \vDash \psi$, etc.

We now show that $\psi \dashv\vdash \chi_\psi$.

Let $\mathcal{R}(\psi) = \{\alpha_1, \ldots, \alpha_n\}$ and let $\mathcal{R}(\chi_\psi) = \{\beta_1, \ldots, \beta_m\}$. As $C_a^{\psi} \chi \in \mathfrak{F}$, we know $\psi \in \mathfrak{F}$, and $\chi_\psi \in \mathfrak{F}$ by construction, so by proposition 6.2.11 we know that α_i and $\beta_j \in \mathfrak{F}$, as are $\sim\alpha_i$ and $\sim\beta_j$.

Suppose $\psi \nvdash \chi_\psi$. Then, by theorem 6.2.7 we know that for some $i \leq n$, we have $\alpha_i \nvdash \beta_1, \ldots, \alpha_i \nvdash \beta_m$. This implies that for $j \leq m$ the sets $\{\alpha_i, \sim\beta_j\}$ are consistent, whence each $\{\alpha_i, \sim\beta_j\}$ is a consistent theory of declaratives and can be extended to an atom A_j by lemma 6.2.21.

Consider now the state $T = \{A_1, \ldots, A_m\}$. As $\alpha_i \in A_j$ for all j we know that $\alpha_i \in \bigcap T$, whence $\bigcap T \vdash \alpha_i$, and by corollary 6.2.4 $\alpha_i \vdash \psi$, whence $\bigcap T \vdash \psi$. By the induction hypothesis this means that $M^{\mathcal{F}}, T \vDash \psi$.

However, we also have that $\bigcap T \nvdash \chi_\psi$. For , suppose otherwise. Then $\bigcap T \vdash \chi_\psi$ and so by corollary 6.2.8 we have that $\bigcap T \vdash \beta_j$ for some j, but then it must be the case that $A_j \vdash \beta_j$, which we know to be impossible as A_j contains $\sim\beta_j$ and is consistent. But then this implies that $M^{\mathcal{F}}, T \nvDash \psi$, by the fact that $\bigcap S \vdash \chi_\psi$ iff $M^{\mathcal{F}}, S \vDash \psi$, established above. But this contradicts our prior observation that $M^{\mathcal{F}}, T \vDash \psi$. Thus, we have established $\psi \vdash \chi_\psi$. By an analogous argument we establish $\chi_\psi \nvdash \psi$, whence $\psi \dashv\vdash \chi_\psi$. By lemma 4.4.4 it then follows that $C_a^\psi \chi \dashv\vdash C_a^{\chi_\psi} \chi$.

Now, suppose $\bigcap S \vdash C_a^\psi \chi$. Since $C_a^\psi \chi \dashv\vdash C_a^{\chi_\psi} \chi$, we also have $\bigcap S \vdash C_a^{\chi_\psi} \chi$.

For an arbitrary $A \in S$, we know $\bigcap S \subseteq A$, and so $A \vdash C_a^{\chi_\psi} \chi$.

As we have assumed $C_a^\psi \chi \in \mathfrak{F}$, we know $\chi \in \mathfrak{F}$. By definition of $\mathcal{S}_a^\psi(A)$ we know that, as $A \vdash C_a^{\chi_\psi} \chi$, for all $T \in \mathcal{S}_a^\psi(A)$, $\bigcap T \vdash \chi$. Still, by the induction hypothesis we know $\bigcap T \vdash \chi$ iff $M^{\mathcal{F}}, T \vDash \chi$. Therefore, for any $T \in \mathcal{S}_a^\psi(A)$, $M^{\mathcal{F}}, T \vDash \chi$. But then $M^{\mathcal{F}}, A \vDash C_a^{\chi_\psi} \chi$, and as A was arbitrary this holds for all $A \in S$, so $M^{\mathcal{F}}, S \vDash C_a^{\chi_\psi} \chi$. And, since $\psi \dashv\vdash \chi_\psi$ and our system is sound we know $[\![\psi]\!] = [\![\chi_\psi]\!]$, whence $M^{\mathcal{F}}, S \vDash C_a^\psi \chi$.

Conversely, suppose $\bigcap S \nvdash C_a^\psi \chi$. Reasoning by the fact that $C_a^\psi \chi \dashv\vdash C_a^{\chi_\psi} \chi$ we infer $\bigcap S \nvdash C_a^{\chi_\psi} \chi$. Since $\bigcap S \nvdash C_a^{\chi_\psi} \chi$ and $C_a^{\chi_\psi} \chi \in \mathfrak{D}$ there must be some $A \in S$ such that $C_a^{\chi_\psi} \chi \notin A$.

So, by lemma 6.3.3 there exists some state $T \in \mathcal{S}_a^\psi(A)$ such that $\bigcap T \nvdash \chi$. By the induction hypothesis we infer $M^{\mathcal{F}}, T \nvDash \chi$ and so we know it is not the case that $M^{\mathcal{F}}, T \vDash \chi$ for every $T \in \mathcal{S}_a^\psi(A)$. Therefore, for any atom $A \in S$ we can infer that $M^{\mathcal{F}}, A \nvDash C_a^\psi \chi$, which entails $M^{\mathcal{F}}, S \nvDash C_a^{\chi_\psi} \chi$, whence $M^{\mathcal{F}}, S \nvDash C_a^\psi \chi$. □

A number of important results follow from the support lemma. We begin by giving the semantic parallel to proposition 6.2.35, establishing that there is a corresponding inquisitive proposition to every issue over At($\mathcal{F}$).

Corollary 6.3.5. *For any $P \in \mathscr{I}_{M^{\mathcal{F}}}$, $P = [\![\chi_P]\!]$.*

Proof. $S \in P$ iff $\bigcap S \vdash \chi_P$ by lemma 6.2.34. By corollary 6.3.4 this is case iff $M^{\mathcal{F}}, S \vDash \chi_P$ iff $S \in [\![\chi_P]\!]$. □

Corollary 6.3.6. *For $P, Q \in \mathscr{I}_{M^{\mathcal{F}}}$: $M^{\mathcal{F}}, S \vDash \chi_P \wedge \chi_Q$ iff $M^{\mathcal{F}}, S \vDash \chi_{(P \cap Q)}$.*

Proof. For suppose $S \in [\![\chi_{(P\cap Q)}]\!]$. Then $S \in (P \cap Q)$, by corollary 6.3.5, whence $S \in P$ and $S \in Q$. By the same lemma we then know that $S \in [\![\chi_P]\!]$ and $S \in [\![\chi_Q]\!]$. Therefore, $M^{\mathcal{F}}, S \vDash \chi_P$ and $M^{\mathcal{F}}, S \vDash \chi_P$, whence $M^{\mathcal{F}}, S \vDash \chi_P \wedge \chi_Q$, thus $S \in [\![\chi_P \wedge \chi_Q]\!]$. As each step appealed to is an equivalence, the converse is also established. □

We further observe that, by generalising reasoning of induction steps for the considers modality in the support lemma, for every inquisitive formula $\psi \in \mathfrak{F}$, ψ is interderivable with the formula that characterises the corresponding inquisitive proposition over At($\mathcal{F}$), and a slight generalisation.

Corollary 6.3.7. *For $\psi \in \mathfrak{F}$:*

i. $\psi \dashv\vdash \chi_\psi$
ii. $C_a^\varphi \psi \dashv\vdash C_a^\varphi \chi_\psi$

Proof. i follows by same reasoning as the induction step for the considers modality in the support lemma and ii follows from lemma 6.2.6. □

Corresponding to, and as a consequence of, corollary 6.3.7.i we have the following semantic property of our canonical models.

Corollary 6.3.8. *For ψ in $\mathfrak{F}$, $[\![\psi]\!]_{M^{\mathcal{F}}} = [\![\chi_\psi]\!]_{M^{\mathcal{F}}}$.*

Proof. $S \in [\![\psi]\!]_{M^{\mathcal{F}}}$ iff $M^{\mathcal{F}}, S \vDash \psi$ iff $\bigcap S \vdash \psi$, by the support lemma, which is the case iff $\bigcap S \vdash \chi_\psi$, by 6.3.7.i This is the case iff $M^{\mathcal{F}}, S \vDash \chi_\psi$, by the support lemma, iff $S \in [\![\chi_\psi]\!]_{M^{\mathcal{F}}}$. □

The last corollary we state generalises the support lemma slightly, to instances of the considers modality not necessarily part of $\mathfrak{F}$.

Corollary 6.3.9. *For a set of formulas $\mathcal{F}$ and the canonical model over $\mathcal{F}$, $M^{\mathcal{F}}$, for any atom $A \in At(\mathcal{F})$, and issues $P, Q \in \mathscr{I}_{M^{\mathcal{F}}}$:*

1. $A \vdash C_a^{\chi_Q} \chi_P$ *iff* $M^{\mathcal{F}}, A \vDash C_a^{\chi_Q} \chi_P$
2. $A \vdash C_a^{\chi_Q} \neg\chi_P$ *iff* $M^{\mathcal{F}}, A \vDash C_a^{\chi_Q} \neg\chi_P$

Proof. We prove the former case, with the latter being established analogously.

From left to right suppose $A \vdash C_a^{\chi_Q} \chi_P$, then by definition $S \in \mathcal{S}_a^Q(A)$ implies $\bigcap S \vdash \chi_P$. As $\chi_P \in \mathfrak{F}$ we know by the support lemma that $M^{\mathcal{F}}, S \vDash \chi_P$ implies $\bigcap S \vdash \chi_P$, and so $S \in \mathcal{S}_a^Q(A)$ iff $M^{\mathcal{F}}, S \vDash \chi_P$. Therefore, $M^{\mathcal{F}}, A \vDash C_a^{\chi_Q} \chi_P$.

From right to left suppose $A \nvdash C_a^{\chi_Q} \chi_P$. We know that $\chi_P \in \mathfrak{F}$, so by our assumption and lemma 6.3.3 we know $\exists S \in \mathcal{S}_a^Q(A)$ such that $\bigcap S \nvdash \chi_P$, whence by the support lemma $M^{\mathcal{F}}, S \nvDash \chi_P$. So, for some $S \in \mathcal{S}_a^Q(A)$, $M^{\mathcal{F}}, S \nvDash \chi_P$. Therefore, it follows that $M^{\mathcal{F}}, A \nvDash C_a^{\chi_Q} \chi_P$. □

Finally, the three following lemmas establish important properties of $M^{\mathcal{F}}$ that we will appeal to in order to show that it satisfies the condition of minimality.

Lemma 6.3.10. *For $P, Q \in \mathscr{I}_{M^{\mathcal{F}}}$: $C_a^{\chi_P \wedge \chi_Q} \varphi \dashv\vdash C_a^{\chi_{(P \cap Q)}} \varphi$.*

Proof. Our method of proof generalises the induction step for the considers modality in the support lemma.

We first show that $\chi_{(P\cap Q)} \dashv\vdash \chi_P \wedge \chi_Q$.

Let $\mathcal{R}(\chi_P) = \{\alpha_1, \ldots, \alpha_n\}$, $\mathcal{R}(\chi_Q) = \{\beta_1, \ldots, \beta_m\}$, and $\mathcal{R}(\chi_{(P\cap Q)}) = \{\gamma_1, \ldots, \gamma_l\}$. As $\chi_P, \chi_Q, \chi_{(P\cap Q)} \in \mathfrak{F}$ we know that $\alpha_i, \beta_j, \gamma_k \in \mathfrak{D}$, as are $\sim\alpha_i, \sim\beta_j, \sim\gamma_k$.

Suppose $\chi_P \wedge \chi_Q \nvdash \chi_{(P\cap Q)}$. By definition 1.2.25 we have $\mathcal{R}(\chi_P \wedge \chi_Q) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\chi_P) \text{ and } \beta \in \mathcal{R}(\chi_Q)\}$. So, by theorem 6.2.7 we know that for some $i \leq n$, $j \leq m$, we have $\alpha_i \wedge \beta_j \nvdash \gamma_1, \ldots, \alpha_i \wedge \beta_j \nvdash \gamma_l$. This implies that for $k \leq l$ the sets $\{\alpha_i \wedge \beta_j, \sim\gamma_k\}$ are consistent. While we cannot be sure that $\alpha_i \wedge \beta_j \in \mathfrak{D}$, by the fact that $\{\alpha_i \wedge \beta_j, \sim\gamma_k\}$ is consistent we know $\{\alpha_i, \beta_j, \sim\gamma_k\}$ is consistent. Moreover, $\{\alpha_i, \beta_j, \sim\gamma_k\} \subseteq \mathfrak{D}$. Therefore, as $\{\alpha_i, \beta_j, \sim\gamma_k\}$ is a consistent theory of declaratives it can be extended to an atom A_k by lemma 6.2.21.

Consider now the state $T = \{A_1, \ldots, A_k\}$. As $\alpha_i, \beta_j \in A_k$ for all $k \leq l$ we know that $\alpha_i, \beta_j \in \bigcap T$, whence $\bigcap T \vdash \alpha_i$ and $\bigcap T \vdash \beta_j$. By corollary 6.2.4 $\bigcap T \vdash \chi_P$ and

$\bigcap T \vdash \chi_Q$, whence by the support lemma $M^{\mathcal{F}}, T \vDash \chi_P$ and $M^{\mathcal{F}}, T \vDash \chi_Q$. Therefore, $M^{\mathcal{F}}, T \vDash \chi_P \wedge \chi_Q$.

However, we also have $\bigcap T \nvdash \chi_{(P \cap Q)}$. For, suppose otherwise. Then $\bigcap T \vdash \chi_{(P \cap Q)}$ and therefore by corollary 6.2.4 we have that $\bigcap T \vdash \gamma_k$ for some k, yet then it must be the case that $A_k \vdash \gamma_k$ for all $A_k \in T$ which we know to be impossible as each A_k contains $\sim\gamma_k$ and is consistent. So, $\bigcap T \nvdash \chi_{(P \cap Q)}$ By the support lemma this entails $M^{\mathcal{F}}, T \nvDash \chi_{(P \cap Q)}$. Yet, we know that $M^{\mathcal{F}}, S \vDash \chi_P \wedge \chi_Q$ iff $M^{\mathcal{F}}, S \vDash \chi_{(P \cap Q)}$ by corollary 6.3.6, and therefore we have derived a contradiction.

Conversely, suppose $\chi_{(P \cap Q)} \nvdash \chi_P \wedge \chi_Q$. So, reasoning as before we have, by definition 1.2.25 and theorem 6.2.7, we know that for some $k \leq l$, we have $\gamma_k \nvdash \alpha_i \wedge \beta_j$ for all $i \leq n$, $j \leq m$. This implies that for $i \leq n$, $j \leq m$ the set $\{\gamma_k, \sim(\alpha_i \wedge \beta_j)\}$ is consistent. Therefore, for each pairing of i, j either $\{\gamma_k, \sim\alpha_i\}$ or $\{\gamma_k, \sim\beta_j\}$ is consistent.[6]

Therefore, as either $\{\gamma_k, \sim\alpha_i\}$ or $\{\gamma_k, \sim\beta_j\}$ is consistent set of declaratives either $\{\gamma_k, \sim\alpha_i\}$ or $\{\gamma_k, \sim\beta_j\}$ can be extended to an atom by lemma 6.2.21.

Clearly there exists an injective mapping $f : (i, j) \mapsto \mathbb{N}$, and therefore by the reasoning above we can associate to each $f(i, j)$ an atom $A_{f(i,j)}$ such that either $\sim\alpha_i$ or $\sim\beta_j \in A_{f(i,j)}$, whence as atoms are consistent either $\alpha_i \notin A_{f(i,j)}$ or $\beta_j \notin A_{f(i,j)}$.

Consider now the state $T = \{A_{f(1,1)}, \ldots, A_{f(n,m)}\}$. As $\gamma_k \in A_{f(i,j)}$ for all i, j we know that $\gamma_k \in \bigcap T$, whence $\bigcap T \vdash \gamma_k$, and so by corollary 6.2.4 we know $\bigcap T \vdash \chi_{(P \cap Q)}$, whence by the support lemma we know $M^{\mathcal{F}}, T \vDash \chi_{(P \cap Q)}$.

We also have $\bigcap T \nvdash \chi_P \wedge \chi_Q$. For, suppose otherwise. Then $\bigcap T \vdash \chi_P \wedge \chi_Q$, whence $\bigcap T \vdash \chi_P$ and $\bigcap T \vdash \chi_Q$, by the elimination rule for conjunction. But then by corollary 6.2.4 we have $\bigcap T \vdash \alpha_i$ for some $i \leq n$ and $\bigcap T \vdash \beta_j$ for some $j \leq m$. However, it must then be the case that $A_{f(i,j)} \vdash \alpha_i$ and $A_{f(i,j)} \vdash \beta_j$ for every $A_{f(i,j)} \in T$. Yet, it then follows that $A_{f(i,j)}$ (taking the indices i, j to be fixed by α_i and β_j) that $\alpha_i \notin A_{f(i,j)}$ or $\beta_j \notin A_{f(i,j)}$, as $A_{f(i,j)}$ is consistent. We know the former to be impossible, and therefore $\bigcap T \nvdash \chi_P \wedge \chi_Q$. Therefore, by the support lemma we know $M^{\mathcal{F}}, T \nvDash \chi_P \wedge \chi_Q$. However, we know that $M^{\mathcal{F}}, S \vDash \chi_P \wedge \chi_Q$ iff $M^{\mathcal{F}}, S \vDash \chi_{(P \cap Q)}$ by corollary 6.3.6, and therefore we have derived a contradiction.

Given we have established $\chi_{(P \cap Q)} \dashv\vdash \chi_P \wedge \chi_Q$ it then follows by the rule of replacement of equivalents that $C_a^{\chi_P \wedge \chi_Q} \varphi \dashv\vdash C_a^{\chi_{(P \cap Q)}} \varphi$. $\square$

Finally, we establish an important lemma for showing the canonical model satisfies the condition of minimality.

Lemma 6.3.11. *For $P, Q, R \in \mathscr{I}_{M^{\mathcal{F}}}$, $(P \cap Q) \subseteq R$ iff $P \subseteq [\![\chi_Q \to \chi_R]\!]$.*

Proof. From left to right suppose $(P \cap Q) \subseteq R$.

Let $S \in P$ be arbitrary, and let $T \subseteq S$ be arbitrary such that $M^{\mathcal{F}}, T \vDash \chi_Q$. Given that $S \in P$ we know via corollary 6.3.5 that $M^{\mathcal{F}}, S \vDash \chi_P$. From this, and as $T \subseteq S$, we know via persistence that $M^{\mathcal{F}}, T \vDash \chi_P$, whence $M^{\mathcal{F}}, T \vDash \chi_P \wedge \chi_Q$. So, observing that $[\![\chi_{(P \cap Q)}]\!]_{M^{\mathcal{F}}} = [\![\chi_P \wedge \chi_Q]\!]_{M^{\mathcal{F}}}$ by corollary 6.3.6, we infer $M^{\mathcal{F}}, T \vDash \chi_{(P \cap Q)}$, and therefore by corollary 6.3.5, $T \in (P \cap Q)$. It then follows via our initial assumption that $T \in R$. So, via corollary 6.3.5 we know $M^{\mathcal{F}}, T \vDash \chi_R$.

[6] For suppose the sets $\{\gamma_k, \sim\alpha_i\}$ or $\{\gamma_k, \sim\beta_j\}$ are inconsistent. Then $\gamma_k \vdash \neg\sim\alpha_i$ and $\gamma_k \vdash \neg\sim\beta_j$. So, $\gamma_k \vdash \neg\sim\alpha_i \wedge \neg\sim\beta_j$.

Now, $\neg\sim\alpha \dashv\vdash \alpha$ for any declarative α. For, suppose α is of the form β. Then $\sim\alpha = \neg\beta$, whence $\neg\sim\alpha = \neg\neg\beta$. So, as $\beta \dashv\vdash \neg\neg\beta$, we know that $\alpha \dashv\vdash \neg\sim\alpha$. And, if α is of the form $\neg\beta$ then $\sim\alpha$ is of the form β. So, $\neg\sim\alpha = \neg\beta$. As α is either of the two preceding forms we then know that $\alpha \dashv\vdash \neg\sim\alpha$. Therefore we know that $\gamma_k \vdash \alpha_i \wedge \beta_j$. But then we know $\{\gamma_k, \alpha_i \wedge \beta_j\}$ is a consistent set, which contradicts the fact that $\{\gamma_k, \sim(\alpha_i \wedge \beta_j)\}$ is a consistent set.

Now, as $T \subseteq S$ was arbitrary such that $M^{\mathcal{F}}, T \vDash \chi_Q$ and from this we inferred that $M^{\mathcal{F}}, T \vDash \chi_R$, we have shown that $M^{\mathcal{F}}, S \vDash \chi_Q \to \chi_R$ and so $S \in [\![\chi_Q \to \chi_R]\!]$. Therefore, given $S \in P$ was arbitrary we have established $P \subseteq [\![\chi_Q \to \chi_R]\!]$.

Conversely, suppose $P \subseteq [\![\chi_Q \to \chi_R]\!]$, and let $S \in (P \cap Q)$ be arbitrary. As $S \in (P \cap Q)$, $S \in P$ and $S \in Q$. Therefore, using the latter observation $S \in [\![\chi_Q]\!]$ by corollary 6.3.5 and so $M^{\mathcal{F}}, S \vDash \chi_Q$. Furthermore, as $S \in P$ we know by our initial assumption that $S \in [\![\chi_Q \to \chi_R]\!]$, whence $M^{\mathcal{F}}, S \vDash \chi_Q \to \chi_R$. Therefore, as $M^{\mathcal{F}}, S \vDash \chi_Q$ we know $M^{\mathcal{F}}, S \vDash \chi_R$. So, we know that $S \in R$ by corollary 6.3.5. Therefore, we have shown that $(P \cap Q) \subseteq R$. □

Lemma 6.3.12.
For $S \subseteq At(\mathcal{F})$, and $P, Q \in \mathscr{I}_{M^{\mathcal{F}}}$: if $\bigcap S \vdash \chi_P \to \chi_Q$ then $M^{\mathcal{F}}, S \vDash \chi_P \to \chi_Q$.

Proof. Suppose $\bigcap S \vdash \chi_P \to \chi_Q$. Let $T \subseteq S$ be arbitrary, and suppose $M^{\mathcal{F}}, T \vDash \chi_P$. Then, by the support lemma we know $\bigcap T \vdash \gamma_T$. As $T \subseteq S$ we know $\bigcap S \subseteq \bigcap T$, whence $\bigcap T \vdash \chi_P \to \chi_Q$. So, $\bigcap T \vdash \chi_Q$ and therefore $M^{\mathcal{F}}, T \vDash \chi_Q$ by the support lemma. By this we have shown $M^{\mathcal{F}}, S \vDash \chi_P \to \chi_Q$. □

With the support lemma and the above properties, lemmas, and corollaries established we can show that each canonical model over $\mathcal{F}$, for some $\mathcal{F}$, is an ICDM.

Lemma 6.3.13. *For every $P \in \mathscr{I}_{M^{\mathcal{F}}}$, $\mathcal{S}_a^P(A)$ is an issue.*

Proof. Let $A \in \mathrm{At}(\mathcal{F})$, $P \in \mathscr{I}_{M^{\mathcal{F}}}$ be arbitrary.

First, we observe $\mathcal{S}_a^P(A)$ is non-empty. For, by definition of $\mathcal{S}_a^P(A)$, $S \in \mathcal{S}_a^P(A)$ iff $\bigcap S \vdash \varphi$ whenever $A \vdash C_a^{\chi_P}\varphi$, for $\varphi \in \mathfrak{F}$. Furthermore, we have defined $\bigcap \emptyset$ to be $\mathfrak{F}$, whence $\bigcap \emptyset \vdash \varphi$ for any φ. Therefore, $\emptyset \in \mathcal{S}_a^P(A)$.

Second, we observe that $\mathcal{S}_a^P(A)$ is downward closed. For, suppose $S \in \mathcal{S}_a^P(A)$ and $T \subseteq S$. So, $\bigcap S \vdash \varphi$ whenever $A \vdash C_a^{\chi_P}\varphi$. But then, as $T \subseteq S$, $\bigcap S \subseteq \bigcap T$. Consequently, $\bigcap T \vdash \varphi$ whenever $\bigcap S \vdash \varphi$, which means $\bigcap T \vdash \varphi$ whenever $A \vdash C_a^{\chi_P}\varphi$, and so $T \in \mathcal{S}_a^P(A)$. □

Lemma 6.3.14. *The state maps of any canonical model over a set of formulas $\mathcal{F}$ satisfy conditions safety through to minimality.*

Proof.

Safety: if $A \in |P|_{M^{\mathcal{F}}}$ then $\mathcal{S}_a^P(A) \neq \{\emptyset\}$.

Let A be an arbitrary atom such that $A \in |P|$. From this we know $\{A\} \in P$, and so by corollary 6.3.5, $A \vdash \chi_P$. Therefore, by proposition 6.2.20 $A \nvdash \neg\chi_P$. By axiom 2, $\vdash C_a^{\chi_P}\neg\chi_P \to \neg\chi_P$, and so we know that $A \nvdash C_a^{\chi_P}\neg\chi_P$. So, by corollary 6.3.9 we know that $M^{\mathcal{F}}, A \nvDash C_a^{\chi_P}\neg\chi_P$, whence it must be the case that $\mathcal{S}_a^P(A) \neq \{\emptyset\}$.

Introspection: if $B \in \mathcal{s}_a^P(A)$, then $\mathcal{S}_a^Q(A) = \mathcal{S}_a^Q(B)$.

Suppose $B \in \mathcal{s}_a^P(A)$. First we establish that $M^{\mathcal{F}}, A \vDash C_a^{\chi_Q}\varphi$ iff $M^{\mathcal{F}}, B \vDash C_a^{\chi_Q}\varphi$, for any issue Q over $\mathrm{At}(\mathcal{F})$.

As $B \in \mathcal{s}_a^P(A)$ we know that $\{B\} \in \mathcal{S}_a^P(A)$.

Now, suppose $M, A \vDash C_a^{\chi_Q}\chi_R$. Then, by corollary 6.3.9 we know $A \vdash C_a^{\chi_Q}\chi_R$ By axiom 4.i we have have $\vdash C_a^{\chi_Q}\chi_R \to C_a^{\chi_P}C_a^{\chi_Q}\chi_R$, whence $A \vdash C_a^{\chi_P}C_a^{\chi_Q}\chi_R$.

As $A \vdash C_a^{\chi_P} C_a^{\chi_Q} \chi_R$ then $S \in \mathcal{S}_a^P(A)$ implies $\bigcap S \vdash C_a^{\chi_Q} \chi_R$. Therefore, as $\{B\} \in \mathcal{S}_a^P(A)$ we know that $B \vdash C_a^{\chi_Q} \chi_R$. Now, by corollary 6.3.9 it follows that $M^{\mathcal{F}}, B \vDash C_a^{\chi_Q} \chi_R$.

Conversely, suppose $M, A \nvDash C_a^{\chi_Q} \chi_R$. By corollary 6.3.9 it follows that $A \nvdash C_a^{\chi_Q} \chi_R$. Therefore, we know that $A \vdash \neg C_a^{\chi_Q} \chi_R$ by proposition 6.2.20, from which it follows that $A \vdash C_a^{\chi_P} \neg C_a^{\chi_Q} \chi_R$ by axiom 4.ii

Reasoning as before, given $A \vdash C_a^{\chi_P} \neg C_a^{\chi_Q} \chi_R$ and $\{B\} \in \mathcal{S}_a^P(A)$ we know that $B \vdash \neg C_a^{\chi_Q} \chi_R$, whence $B \nvdash C_a^{\chi_Q} \chi_R$, and so $M^{\mathcal{F}}, B \nvDash C_a^{\chi_Q} \chi_R$ by corollary 6.3.9.

Therefore, $M, A \vDash C_a^{\chi_Q} \chi_R$ iff $M, B \vDash C_a^{\chi_Q} \chi_R$.

By the basic lemma we have $M, A \vDash C_a^{\chi_Q} \chi_R$ iff $\mathcal{S}_a^Q(A) \subseteq [\![\chi_R]\!]$, and by corollary 6.3.5 this means $M, A \vDash C_a^{\chi_Q} \chi_R$ iff $\mathcal{S}_a^Q(A) \subseteq R$. As this for any arbitrary issue R over $\mathrm{At}(\mathcal{F})$ we know that $\mathcal{S}_a^Q(A) \subseteq P$ iff $\mathcal{S}_a^Q(B) \subseteq P$, for any $P \in \mathscr{I}$.

Therefore, as $\mathcal{S}_a^Q(A), \mathcal{S}_a^Q(B)$ are issues over $\mathrm{At}(\mathcal{F})$ we know $\mathcal{S}_a^Q(B) \subseteq \mathcal{S}_a^Q(A)$, and by analogous reasoning $\mathcal{S}_a^Q(A) \subseteq \mathcal{S}_a^Q(B)$, whence $\mathcal{S}_a^Q(A) = \mathcal{S}_a^Q(B)$.

Adjustment: $\mathcal{S}_a^P(A) \subseteq P$.

First, given corollary 6.3.5 we know that for any $P \in \mathscr{I}$, $P = [\![\chi_P]\!]_{M^{\mathcal{F}}}$. Let $P \in \mathscr{I}$ be arbitrary.

By definition $S \in \mathcal{S}_a^P(A) \iff \bigcap S \vdash \varphi$ whenever $A \vdash C_a^{\chi_P} \varphi$ and $\varphi \in \mathfrak{F}$. As $\vdash C_a^{\chi_P} \chi_P$ by axiom 3, it must be the case that for all $S \in \mathcal{S}_a^P(A)$ that $\bigcap S \vdash \chi_P$. So, by the support lemma $M^{\mathcal{F}}, S \vDash \chi_P$, for all $S \in \mathcal{S}_a^P(A)$. Therefore, for any arbitrary $S \in \mathcal{S}_a^P(A)$, we have $S \in [\![\chi_P]\!]$, and so $\mathcal{S}_a^P(A) \subseteq [\![\chi_P]\!]$, whence $\mathcal{S}_a^P(A) \subseteq P$ by our initial observation.

Success: $\mathcal{S}_a^P(A) \neq \{\emptyset\}$, if $\mathcal{S}_a^Q(A) \cap P \neq \{\emptyset\}$.

Assume $\mathcal{S}_a^Q(A) \cap P \neq \{\emptyset\}$. So, $\mathcal{S}_a^Q(A) \cap [\![\chi_P]\!] \neq \{\emptyset\}$, by corollary 6.3.5. Therefore, $M^{\mathcal{F}}, A \nvDash C_a^{\chi_Q} \neg\chi_P$. For otherwise $M^{\mathcal{F}}, A \vDash C^{\chi_Q} \neg\chi_P$, whence by the basic lemma, $\mathcal{S}_a^Q(A) \subseteq [\![\neg\chi_P]\!]$, contradicting our assumption.

As $M^{\mathcal{F}}, A \nvDash C_a^{\chi_Q} \neg\chi_P$ we know by corollary 6.3.9 that , $A \nvdash C_a^{\chi_Q} \neg\chi_P$.

By proposition 3.4.1 we know that $\vdash C^{\chi_P} \neg\chi_P \to C^{\chi_Q} \neg\chi_P$, and therefore it must be the case that $A \nvdash C^{\chi_P} \neg\chi_P$, which means $M^{\mathcal{F}}, A \nvDash C^{\chi_P} \neg\chi_P$ by corollary 6.3.9. However, by the basic lemma we know that if $M^{\mathcal{F}}, A \nvDash C_a^{\chi_P} \neg\chi_P$ then $\mathcal{S}_a^P(A) \nsubseteq [\![\neg\chi_P]\!]$. Therefore, it cannot be the case that $\mathcal{S}_a^P(A) = \{\emptyset\}$, for $\{\emptyset\} \subseteq R$ for all issues R over $\mathrm{At}(\mathcal{F})$ and $[\![\chi_P]\!]$ is such an issue.

Minimality: $\mathcal{S}_a^{P \cap Q}(A) = \mathcal{S}_a^P(A) \cap Q$, if $\mathcal{S}_a^P(A) \cap Q \neq \{\emptyset\}$.

Suppose $\mathcal{S}_a^P(A) \cap Q \neq \{\emptyset\}$. By corollary 6.3.5 this entails $M^{\mathcal{F}}, A \vDash \neg C_a^{\chi_P} \neg\chi_Q$.

As $M^{\mathcal{F}}, A \vDash \neg C_a^{\chi_P} \neg\chi_Q$ we know that $M^{\mathcal{F}}, A \nvDash C_a^{\chi_P} \neg\chi_Q$. So, by corollary 6.3.9 we know that $A \nvdash C_a^{\chi_P} \neg\chi_Q$, whence $A \vdash \neg C_a^{\chi_P} \neg\chi_Q$ by proposition 6.2.20.

As we know $A \vdash \neg C_a^{\chi_P} \neg\chi_Q$ it follows that $A \vdash C^{\chi_P} \varphi \to C_a^{\chi_P \wedge \chi_Q} \varphi$ by lemma 4.4.3, theorem 2.

It is trivially the case that $\mathcal{S}_a^P(A) \subseteq \mathcal{S}_a^P(A)$, we know $\mathcal{S}_a^P(A) \subseteq [\![\chi_{\mathcal{S}_a^P(A)}]\!]$, by corollary 6.3.5. Therefore, by the basic lemma we know $M^{\mathcal{F}}, A \vDash C_a^{\chi_P} \chi_{\mathcal{S}_a^P(A)}$. So, by corollary 6.3.9 we know $A \vdash C_a^{\chi_P} \chi_{\mathcal{S}_a^P(A)}$, whence $A \vdash C_a^{\chi_P \wedge \chi_Q} \chi_{\mathcal{S}_a^P(A)}$ by lemma 4.4.2, using rule (Cw1.). From lemma 6.3.10 it then follows that $A \vdash C_a^{\chi(P \cap Q)} \chi_{\mathcal{S}_a^P(A)}$. Therefore, by corollary 6.3.9 we know $M^{\mathcal{F}}, A \models C_a^{\chi(P \cap Q)} \chi_{\mathcal{S}_a^P(A)}$, whence $\mathcal{S}_a^{P \cap Q}(A) \subseteq \mathcal{S}_a^P(A)$.

We know by axiom 3 that $A \vdash C_a^{\chi_P \wedge \chi_Q}(\chi_P \wedge \chi_Q)$. From reasoning analogous to the above this allows us to infer that $M^{\mathcal{F}}, A \models C^{\chi(P \cap Q)}(\chi_P \wedge \chi_Q)$, whence $M^{\mathcal{F}}, A \models C^{\chi(P \cap Q)} \chi_Q$. Therefore, by the basic lemma we infer that $\mathcal{S}_a^{P \cap Q}(A) \subseteq Q$.

So, as $\mathcal{S}_a^{P \cap Q}(A) \subseteq \mathcal{S}_a^P(A)$ and $\mathcal{S}_a^{P \cap Q}(A) \subseteq Q$, we know $\mathcal{S}_a^{P \cap Q}(A) \subseteq \mathcal{S}_a^P(A) \cap Q$.

For the converse direction we use axiom 5.

We know $\mathcal{S}_a^{(P \cap Q)}(A) \subseteq [\![\chi_{\mathcal{S}_a^{(P \cap Q)}(A)}]\!]$, using reasoning analogous to that found in the previous direction. So, $M^{\mathcal{F}}, A \models C_a^{\chi(P \cap Q)} \chi_{\mathcal{S}_a^{(P \cap Q)}(A)}$. By corollary 6.3.9 this entails $A \vdash C_a^{\chi(P \cap Q)} \chi_{\mathcal{S}_a^{(P \cap Q)}(A)}$, and so via lemma 6.3.10 we have $A \vdash C_a^{\chi_P \wedge \chi_Q} \chi_{\mathcal{S}_a^{(P \cap Q)}(A)}$. Therefore, as we have assumed that $A \vdash \neg C_a^{\chi_P} \neg \chi_Q$ it follows by axiom 5 that $A \vdash C_a^{\chi_P \wedge \chi_Q} \chi_{\mathcal{S}_a^{(P \cap Q)}(A)} \rightarrow C^{\chi_P}(\chi_Q \rightarrow \chi_{\mathcal{S}_a^{(P \cap Q)}(A)})$, by the rules of modus ponens and conjunction elimination. So, by the prior observation we have $A \vdash C^{\chi_P}(\chi_Q \rightarrow \chi_{\mathcal{S}_a^{(P \cap Q)}(A)})$, by modus ponens again. Therefore, $S \in \mathcal{S}_a^P(A)$ implies $\bigcap S \vdash \chi_Q \rightarrow \chi_{\mathcal{S}_a^{(P \cap Q)}(A)}$. From this and lemma 6.3.12 we know that if $S \in \mathcal{S}_a^P(A)$ then $M^{\mathcal{F}}, S \models \chi_Q \rightarrow \chi_{\mathcal{S}_a^{(P \cap Q)}(A)}$. Therefore, $\mathcal{S}_a^P(A) \subseteq [\![\chi_Q \rightarrow \chi_{\mathcal{S}_a^{(P \cap Q)}(A)}]\!]_{M^{\mathcal{F}}}$. From lemma 6.3.11 it follows that $\mathcal{S}_a^P(A) \cap Q \subseteq \mathcal{S}_a^{P \cap Q}(A)$.

So, we have established that if $\mathcal{S}_a^P(A) \cap Q \neq \{\emptyset\}$, then $\mathcal{S}_a^P(A) \cap Q = \mathcal{S}_a^{P \cap Q}(A)$. □

6.4 Results

Theorem 6.4.1 (Completeness of ICDL with respect to ICDMs).
If for all ICDMs $M = \langle W, \{\mathcal{S}_a^P\}_{a \in \mathcal{A}, P \in \mathcal{I}}, V \rangle$, $M \models \psi$, then $\vdash \psi$.

Proof. Suppose $\nvdash \psi$. By theorem 6.2.7 we know that $\nvdash \alpha_i$ for all $\alpha_i \in \mathcal{R}(\psi)$. We claim $\sim\alpha_i$ is consistent. For suppose not, then $\vdash \neg\sim\alpha_i$. Since α_i is a declarative $\neg\sim\alpha_i \dashv\vdash \alpha_i$, which entails $\vdash \alpha_i$. Let $\mathcal{F} = \{\psi\}$, and construct the canonical model over $\mathcal{F}$. As $\psi \in \mathcal{F}$ we know $\sim\alpha_i \in \mathfrak{F}$, and as each $\sim\alpha_i$ is consistent it can be extended to an atom A_i by lemma 6.2.21. Take $S = \{A_1, \ldots, A_n\}$. We claim that $M^{\mathcal{F}}, S \nvDash \psi$. For suppose not. Then, by the support lemma $\bigcap S \vdash \psi$. So, by theorem 6.2.7 $\bigcap S \vdash \alpha_i$ for some $\alpha_i \in \mathcal{R}(\psi)$. As $\bigcap S \subseteq A_i$ this entails $A_i \vdash \alpha_i$. But, A_i is consistent and contains $\sim\alpha_i$ by construction, and so this is not possible. Therefore $M^{\mathcal{F}}, S \nvDash \psi$. So, there exists some ICDM model M and state $s \subseteq W$ such that $M, s \nvDash \psi$. □

Theorem 6.4.2 (Weak completeness for plausibility models). *ICDL is weakly complete with respect to IPMs.*

Proof. Suppose $\nvdash \varphi$, then by theorem 6.4.1 we know there is a finite ICDM M such that $M \nvDash \varphi$. By theorem 5.1.1 this can be transformed into an IPM $M^\sharp$ preserving the interpretation of ICDL, and therefore $M^\sharp \nvDash \varphi$. So, $M^\sharp$ witnesses that there is some plausibility model M (i.e. $M^\sharp$) such that $M \nvDash \varphi$. □

Theorem 6.4.3 (Decidability of ICDL). *ICDL is decidable*

Proof. Theorem 6.4.2 establishes that ICDL has the finite model property, therefore we can test for the validity of any given formula by constructing two Turing machines.

One to enumerate the ICDL-validities using the proof system of ICDL and the other to generate all the finite models. Either a proof of the formula can be derived, or a countermodel constructed. Therefore, one of the machines must halt in finite time. □

Theorem 6.4.4 (ICDL is not compact). *There exists a set of formulas Θ in the language of ICDL such that every finite subset of Θ is satisfiable, but Θ is not.*

Proof. Our method of proof follows Veltman (1985, pp. 104–105).

Let $p_1, \ldots, p_n, \ldots$ be countably many distinct atomic formulas and define φ_k and ψ_k for $k \in \mathbb{N}$ in the following way:

$$\begin{aligned} \varphi_k &:= C_a^{p_1 \vee \cdots \vee p_{k+1}} \neg(p_1 \vee \cdots \vee p_k) \\ \psi_k &:= \neg(C^{p_1 \vee \cdots \vee p_{k+1}}(p_1 \vee \cdots \vee p_k)) \end{aligned}$$

Observe for an arbitrary $k \in \mathbb{N}$, $C_a^{p_1 \vee \cdots \vee p_{k+1}} \neg(p_1 \vee \cdots \vee p_k)$ ensures it is the case that $\mathrm{Min}_{\leq_a^w} [\![(p_1 \vee \cdots \vee p_{k+1})]\!] \subseteq (\wp(W) - [\![p_1 \vee \cdots \vee p_k]\!])$, and therefore it is the case that $\mathrm{Min}_{\leq_a^w} [\![(p_1 \vee \cdots \vee p_{k+1})]\!] \cap [\![p_1 \vee \cdots \vee p_k]\!] = \{\emptyset\}$.

However, it is also the case that $\mathrm{Min}_{\leq_a^w} [\![(p_1 \vee \cdots \vee p_{k+1})]\!] \subseteq [\![p_1 \vee \cdots \vee p_{k+1}]\!]$, whence it must be the case that $\mathrm{Min}_{\leq_a^w} [\![(p_1 \vee \cdots \vee p_{k+1})]\!] \subseteq [\![p_{k+1}]\!]$.

Moreover $\neg(C^{p_1 \vee \cdots \vee p_{k+1}}(p_1 \vee \cdots \vee p_k))$ ensures $\mathrm{Min}_{\leq_a^w} [\![(p_1 \vee \cdots \vee p_{k+1})]\!] \neq \{\emptyset\}$.

Therefore, given $\mathrm{Min}_{\leq_a^w} [\![(p_1 \vee \cdots \vee p_{k+1})]\!] \subseteq ([\![p_{k+1}]\!] - [\![p_1 \vee \cdots \vee p_k]\!])$ and $\mathrm{Min}_{\leq_a^w} [\![(p_1 \vee \cdots \vee p_{k+1})]\!] \neq \{\emptyset\}$, we obtain the following constraints:

$$\begin{aligned} \mathrm{Min}_{\leq_a^w} [\![(p_1 \vee p_2)]\!] &\subseteq ([\![p_2]\!] - [\![p_1]\!]) \\ \mathrm{Min}_{\leq_a^w} [\![(p_1 \vee p_2 \vee p_3)]\!] &\subseteq ([\![p_3]\!] - [\![p_1 \vee p_2]\!]) \\ \mathrm{Min}_{\leq_a^w} [\![(p_1 \vee p_2 \vee p_3 \vee p_4)]\!] &\subseteq ([\![p_4]\!] - [\![p_1 \vee p_2 \vee p_3]\!]) \\ &\vdots \end{aligned}$$

Thus closer and closer worlds are generated by the conjunction of φ_k and ψ_k, as $k \in \mathbb{N}$ increases. For, if w satisfies $\varphi_k \wedge \psi_k$ there would need to exist a world u such that $u <_a^w v$ for $v \in \mathrm{Min}_{\leq_a^w} [\![(p_1 \vee \cdots \vee p_{k+1})]\!]$ in order for w to be able to satisfy $\varphi_{k+1} \wedge \psi_{k+1}$.

Therefore, the set $\{\varphi_k \wedge \psi_k \mid k \in \mathbb{N}\}$ is unsatisfiable by the fact that $\leq_a^w$ is required to be well-founded.

It remains to be shown that each finite subset can be satisfied. To do so we take the greatest k in the subset and construct an IPM $M = \langle W, \{\leq_a^w\}_{a \in \mathcal{A}, w \in W}, \{\Sigma_a\}_{a \in \mathcal{A}}, V \rangle$ such that $W = \{w_1, \ldots, w_k\}$, $w_{j+1} <_a^w w_j$, and $V(p_j) = \{w_j\}$, for all $j \leq k$. Clearly this will satisfy the finite subset. □

Corollary 6.4.5. *CDL is not compact.*[7]

Proof. As we made no appeal to issues in the proof of theorem 6.4.4 the same reasoning can be applied to CDL, using conditional belief in place of considers. □

[7] We assume this is a known result, but we have been unable to find work in which it is explicitly stated.

Chapter 7

Inquisitive Plausibility Logic

In order to axiomatise belief revision with both local and global issues we enrich the base language of IEL with two modal operators, familiar from logics of preference, $E_a^{\leq}$ and $E_a^{\not\leq}$. The logic defined is based on the work of Boutilier, and is closely connected to his logic CO (1994, pp. 100–101), with a slight variation in axiomatisation in order to enrich the logic to a multi-agent setting with issues for each agent and in order to adapt it to our method of proving completeness.

We term this logic *inquisitive plausibility logic* on the basis of a natural interpretation. From standpoint of the previous chapters IPL is primarily of interest for the fact that it is an extension of ICDL with sufficient expressive power to ensure the correct relation between entertains and considers modalities, thus affording us an indirect axiomatisation of ICDL with the entertains modality. Still, the logic also has a broader interest given the motivating ideas of IEL and ICDL, by having sufficient power to express additional properties of an agent's epistemic and conditional doxastic goals. Aspects of this perspective will be sketched after the basic semantic notions have been established, and the connexion with ICDL shown.

7.1 Inquisitive Plausibility Logic

Logic Language

We begin by enriching the language of IEL sans the K modality, to obtain the language of inquisitive plausibility logic, IPL.

Definition 7.1.1 (Syntax of IPL). Let At be a set of atomic formulas:

1. For any $p \in \mathsf{At}$, $p \in \mathcal{L}_!$
2. $\bot \in \mathcal{L}_!$
3. If $\alpha_1, \ldots, \alpha_n \in \mathcal{L}_!$ then $?\{\alpha_1, \ldots, \alpha_n\} \in \mathcal{L}_?$
4. If $\varphi \in \mathcal{L}_\circ$ and $\psi \in \mathcal{L}_\circ$ then $\varphi \wedge \psi \in \mathcal{L}_\circ$, where $\circ \in \{!, ?\}$
5. If $\varphi \in \mathcal{L}_! \cup \mathcal{L}_?$ and $\psi \in \mathcal{L}_\circ$ then $\varphi \to \psi \in \mathcal{L}_\circ$, where $\circ \in \{!, ?\}$
6. If $\varphi \in \mathcal{L}_! \cup \mathcal{L}_?$, then $E_a\varphi \in \mathcal{L}_!$, for $a \in \mathcal{A}$
7. If $\varphi \in \mathcal{L}_! \cup \mathcal{L}_?$, then $E_a^{\Box}\varphi \in \mathcal{L}_!$, where $\Box \in \{\leq, \not\leq\}$
8. Nothing else belongs to either $\mathcal{L}_!$ or $\mathcal{L}_?$

As with InqD and ICDL resolutions of formulas play a key role in our theorising about IPL. Just as with ICDL, IPL differs from InqD only by the introduction of further declaratives, as so no additional clause is required to define the resolutions of formulas containing IPL modalities.

Definition 3.3.4 (Resolutions for ICDL)**.** The set $\mathcal{R}(\varphi)$ of resolutions for a given formula φ is defined inductively by:

- $\mathcal{R}(\alpha) = \{\alpha\}$
- $\mathcal{R}(?\{\alpha_1, \ldots, \alpha_n\}) = \{\alpha_1, \ldots, \alpha_n\}$
- $\mathcal{R}(\mu \wedge \nu) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\mu) \text{ and } \beta \in \mathcal{R}(\nu)\}$
- $\mathcal{R}(\varphi \to \mu) = \{\bigwedge_{\alpha \in \mathcal{R}(\varphi)}(\alpha \to f(\alpha)) \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\mu)\}$

Semantics

As with inquisitive conditional-doxastic logic, inquisitive plausibility models are the primary semantic structures used to interpret IPL. We restate the definition of these.

Definition 3.3.1 (Inquisitive plausibility models)**.**
An inquisitive plausibility model M for a set At of atomic formulas and a set $\mathcal{A}$ of agents,[1] is a tuple: $\langle W, \{\leq_a^w\}_{a \in \mathcal{A}, w \in W}, \{\Sigma_a\}_{a \in \mathcal{A}}, V\rangle$, where:

1. W is a set of possible worlds
2. $\leq_a^w$ is a well-preorder over a subset of W
3. $\Sigma_a(w)$ is an issue over $\sigma_a(w)$, where $\sigma_a(w) := \{v \mid \exists u\colon v \leq_a^w u\}$
4. $V : W \to \wp(\mathsf{At})$ is a valuation map, stating for each $w \in W$ the atomic formulas true at w

And the following conditions are satisfied:

Factivity	$w \in \sigma_a(w)$, for all $w \in W$
Introspection 1	if $v \in \sigma_a(w)$, then $\Sigma_a(w) = \Sigma_a(v)$
Introspection 2	if $v \in \sigma_a(w)$, then $x \leq_a^v y$ if and only if $x \leq_a^w y$

Just as with our original account of plausibility models it is the interpretation of binary relation on worlds that distinguishes our understanding of these, where $v \leq_a^w u$ reads 'at world w agent a considers v at least as plausible as u.'

We define the semantics clauses for the introduced modalities as follows.

Definition 7.1.2 (Support conditions for IPL modalities)**.**

1. $M, w \vDash E_a^{\leq}\varphi$ iff $\forall t \in \Sigma_a^{\leq}(w), M, t \vDash \varphi$
2. $M, w \vDash E_a^{\not\leq}\varphi$ iff $\forall t \in \Sigma_a^{\not\leq}(w), M, t \vDash \varphi$

where:

a. $\Sigma_a^{\leq}(w) := \wp(\sigma_a^{\leq}(w)) \cap \Sigma_a(w)$, and $\sigma_a^{\leq}(w) := \{v \mid v \leq_a^w w\}$
b. $\Sigma_a^{\not\leq}(w) := \wp(\sigma_a^{\not\leq}(w)) \cap \Sigma_a(w)$, and $\sigma_a^{\not\leq}(w) := \{v \in \sigma_a(w) \mid v \not\leq_a^w w\}$

We make four initial observations.

Proposition 7.1.3.

1. *For all $v, u \in \sigma_a(w)$, $v \leq_a^w u$ or $u \leq_a^w v$.*
2. *$v \in \sigma_a^{\not\leq}(w)$ iff $w <_a^w v$, where $x <_a^w y := x \leq_a^w y$ and $y \not\leq_a^w x$.*
3. *$\sigma_a^{\leq}(w) \cap \sigma_a^{\not\leq}(w) = \emptyset$.*

[1] We assume the set of agents is finite. However, modalities for common knowledge, belief and so on will not be explored in this thesis, and so there is no technical need for this assumption.

4. $\sigma_a^{\leq}(w) \cup \sigma_a^{\nleq}(w) = \sigma_a(w)$

Proof.

1. Let v, u in $\sigma_a(w)$ be arbitrary. Now, as $<_a^w$ is a well-preorder we know that for every set $s \subseteq \{v \mid \exists u\colon v \leq_a^w u\} = \sigma_a(w)$ there exists $v \in s$ such that $v \leq_a^w u$ for all $u \in s$. So, as $\{v, u\} \subseteq \sigma_a(w)$ by assumption, it must be the case that either $v \leq_a^w u$ or $u \leq_a^w v$.

2. From left to right suppose $v \in \sigma_a^{\nleq}(w)$. Then, $v \in \sigma_a(w)$ and $v \nleq_a^w w$. So, by the previous observation we know that $w \leq_a^w v$, whence $w <_a^w v$.

   From right to left if $w <_a^w v$ then $w \leq_a^w v$, and as $\leq_a^w$ is a well-preorder we know the relation is reflexive, whence $v \leq_a^w v$, which ensures that $v \in \sigma_a(w)$. Furthermore, as $w <_a^w v$ we know $v \nleq_a^w w$, and so from this and the previous observation we know that $v \in \sigma_a^{\nleq}(w)$.

3. Suppose $v \in \sigma_a^{\leq}(w)$ and $v \in \sigma_a^{\nleq}(w)$. Then, $v \leq_a^w w$ and $v \nleq_a^w w$, and immediate contradiction.

4. The left to right direction is immediate by the definitions of $\sigma_a^{\leq}(w)$ and $\sigma_a^{\nleq}(w)$.

   From right to left let $v \in \sigma_a(w)$ be arbitrary. We know that $w \in \sigma_a(w)$ by the condition of factivity on IPMS. Therefore, we know that $v, w \in \sigma_a(w)$. So, by the first item of this proposition we know $v \leq_a^w w$ or $w \leq_a^w v$. If $v \leq_a^w w$, then $v \in \sigma_a^{\leq}(w)$ by definition. So, the only other case that remain is if $w \leq_a^w v$ and $v \nleq_a^w w$, and then $v \in \sigma_a^{\nleq}(w)$ by definition. □

So, $E_a^{\leq}$ quantifies over the intersection of an agent's inquisitive state at w with the powerset of the worlds that are at least as plausible as w—and thus all possible issues over the worlds that are at least as plausible as w. So, we obtain the restriction of an agent's inquisitive state to the worlds they consider at least as plausible as the current world of evaluation. Therefore, $E_a^{\leq}$ evaluated at w reads 'when the issues that the agent a entertains at w is restricted to the worlds the agent considers at least as plausible as w, φ is supported.'

Just as the language of IPL is bisected into declaratives and interrogatives we may choose to split our interpretation of the entertains modality according to whether it scopes over a declarative or an interrogative. For a declarative under the scope of the $E_a^{\leq}$ modality allows us to directly describe the worlds at least as plausible as the world of evaluation, as the following proposition shows.

Proposition 7.1.4. *$M, w \vDash E_a^{\Box}\alpha$ iff $\forall v \in \sigma_a^{\Box}(w), M, v \vDash \alpha$, for $\Box \in \{\leq, \nleq\}$.*

Proof. We prove the case for $E_a^{\leq}$. The case of $E_a^{\nleq}$ is established analogously.

From left to right suppose $M, w \vDash E_a^{\leq}\alpha$, and let $v \in \sigma_a^{\leq}(w)$ be arbitrary. By the latter assumption we know that $v \in \sigma_a(w)$, whence $\{v\} \in \Sigma_a(w)$, and clearly $\{v\} \in \wp(\sigma_a^{\leq}(w))$, whence $\{v\} \in E_a^{\leq}w$. By the former assumption we have that $\forall t \in \Sigma_a^{\leq}(w), M, t \vDash \alpha$. Therefore, $M, v \vDash \alpha$, whence $\forall v \in \sigma_a^{\leq}(w), M, v \vDash \alpha$.

From right to left suppose that $\forall v \in \sigma_a^{\leq}(w), M, v \vDash \alpha$, and let $t \in \Sigma_a^{\leq}(w)$ be arbitrary. As $t \in \Sigma_a^{\leq}(w)$ we know that $t \subseteq \sigma_a^{\leq}(w)$ and therefore that $\forall u \in t, u \in \sigma_a^{\leq}(w)$. So, $\forall u \in t, M, u \vDash \alpha$, by the former assumption made. Therefore, by proposition 1.2.15 we know that $M, t \vDash \alpha$, whence $\forall t \in \Sigma_a^{\leq}(w), M, t \vDash \alpha$. □

So, we can read $E_a^{\leq}\alpha$ evaluated at a world w straightforwardly as 'at all the worlds the agent a considers at least as plausible as w, α is the case.' This allows us to restrict the clumsy reading of the $E_a^{\leq}$ modality above to interrogatives.

For interrogatives a more elegant reading seems unavailable. For example, reading $E_a^{\leq}\mu$ evaluated at w as 'agent a entertains μ over the worlds a considers at least as plausible as w' relies on assuming that the attitude of entertaining is always (implicitly) related to an issue over an agent's epistemic state. The information states contained in $E_a^{\leq}w$, for example, are merely restrictions of $\Sigma_a(w)$ to worlds at least as plausible as w. These worlds do not, as in the case of the considers modality, have any clear doxastic interpretation, connecting the issue over them to the agent's beliefs or otherwise.

However, and importantly, we show below that the considers modality of ICDL can be defined in the language of IPL. Thus ICDL can be considered as a fragment of IPL, or IPL as a conservative extension of ICDL with respect to conditional-doxastic formulas, as we one will be able to translate any formula of ICDL into a semantically equivalent formula of IPL.

Similarly, $E_a^{\not\geq}$ quantifies over intersection of an agent's inquisitive state at w with the powerset of the worlds that are considered strictly less plausible than w—but note those still considered possible, as these are derived from the agent's epistemic state—as is shown in the following proposition.

As noted above, an important aspect to IPL (and fundamental to its motivation with regards to modelling doxastic states) is the definability of the considers modality of ICDL in terms of $E_a^{\leq}$, which we now show.

Definition 7.1.5 (IPL conditional belief). $C_a^{\psi}\varphi := E_a(\psi \to \langle E_a^{\leq}\rangle(\psi \wedge E_a^{\leq}(\psi \to \varphi)))$[2]

This definition parallels the definition for conditional preference given by Liu (2011, p. 39). In order to establish the semantic adequacy of the definition we begin with the following lemma.

Lemma 7.1.6. *If* $w \in |\varphi|$ *then* $Min_{\leq_a^w}(\llbracket\varphi\rrbracket \cap \Sigma_a(w)) \subseteq \Sigma_a^{\leq}(w)$.

Proof. Suppose $w \in |\varphi|$ and let $t \in \mathrm{Min}_{\leq_a^w}(\llbracket\varphi\rrbracket \cap \Sigma_a(w))$ be arbitrary. We know $w \in \sigma_a(w)$ by the condition of factivity on IPMs. Furthermore, as $w \in |\varphi|$ and $w \in \sigma_a(w)$ then $\mathrm{Min}_{\leq_a^w}|\varphi| = \{v_1, \dots, v_n\}$ such that $v_i \in |\varphi|$ and $v_i \leq_a^w w$ for $i \leq n$. From this it follows that $\mathrm{Min}_{\leq_a^w}|\varphi| \subseteq \sigma_a^{\leq}(w)$, by definition of $\sigma_a^{\leq}(w)$. Furthermore, as $t \in \mathrm{Min}_{\leq_a^w}(\llbracket\varphi\rrbracket \cap \Sigma_a(w))$ this means that $t \subseteq \mathrm{Min}_{\leq_a^w}|\varphi|$ by corollary 3.3.11, and so $t \subseteq \{v_1, \dots, v_n\}$, whence $t \in \wp(\sigma_a^{\leq}(w))$. And, as $t \in \mathrm{Min}_{\leq_a^w}(\llbracket\varphi\rrbracket \cap \Sigma_a(w))$ it follows immediately that $t \in \Sigma_a(w)$. So, $t \in (\wp(\sigma_a^{\leq}(w)) \cap \Sigma_w) = \Sigma_a^{\leq}(w)$. □

We now show that the defined considers modality satisfies the support clause for the considers modality on inquisitive plausibility models.

Proposition 7.1.7. $M, s \vDash C_a^{\psi}\varphi$ *iff* $\forall w \in s, \forall t \in Min_{\leq_a^w}(\llbracket\psi\rrbracket \cap \Sigma_a(w)), M, t \vDash \varphi$

Proof. We prove $M, w \vDash C_a^{\psi}\varphi$ iff $\forall t \in \mathrm{Min}_{\leq_a^w}(\llbracket\psi\rrbracket \cap \Sigma_a(w)), M, t \vDash \varphi$, with the desired result following immediately from this by proposition 1.2.15, as the formula $C_a^{\psi}\varphi$ abbreviates—and which we work with—is a declarative.

From left to right suppose $M, w \vDash E_a(\psi \to \langle E_a^{\leq}\rangle(\psi \wedge E_a^{\leq}(\psi \to \varphi)))$. So, we know that $\forall t \in \Sigma_a(w), M, t \vDash \psi \to \langle E_a^{\leq}\rangle(\psi \wedge E_a^{\leq}(\psi \to \varphi))$.

[2] We write $\langle\Box\rangle\varphi$ for $\neg\Box\neg\varphi$ for $\Box \in \{E_a^{\leq}, E_a^{\not\geq}, E_a\}$ to aid legibility.

We want to show that $\forall t \in \mathrm{Min}_{\leq_a^w}(\llbracket\psi\rrbracket \cap \Sigma_a(w)), M, t \vDash \varphi$. Therefore, let $t \in \mathrm{Min}_{\leq_a^w}(\llbracket\psi\rrbracket \cap \Sigma_a(w))$ be arbitrary. By this assumption it follows that $t \in \Sigma_a(w)$ and $t \in \llbracket\psi\rrbracket$. So, $M, t \vDash \langle E_a^{\leq}\rangle(\psi \wedge E_a^{\leq}(\psi \rightarrow \varphi))$, by our previous observation concerning $\Sigma_a(w)$. Restated, this reads $M, t \vDash \neg E_a^{\leq}\neg(\psi \wedge E_a^{\leq}(\psi \rightarrow \varphi))$.

By the above it follows that for all $v \in t, \exists t' \in \Sigma_a^{\leq}(v)$ such that $M, t' \nvDash \neg(\psi \wedge E_a^{\leq}(\psi \rightarrow \varphi))$, whence via proposition 1.2.15 and fact 1.2.16 we have some $u \in t'$ such that $M, u \vDash \psi \wedge E_a^{\leq}(\psi \rightarrow \varphi)$. By the latter conjunct we know that $\forall t'' \in \Sigma_a^{\leq}(u), M, t'' \vDash \psi \rightarrow \varphi$, and by the former conjunct we know that $u \in |\psi|$.

Given that $u \in |\psi|$ we know it must be the case that $\mathrm{Min}_{\leq_a^u}(\llbracket\psi\rrbracket \cap \Sigma_a(u)) \subseteq \Sigma_a^{\leq}(u)$ by lemma 7.1.6. So, then $\forall t''' \in \mathrm{Min}_{\leq_a^u}(\llbracket\psi\rrbracket \cap \Sigma_a(u))$ we have $M, t''' \vDash \varphi$, as we know that $\forall t'' \in \Sigma_a^{\leq}(u), M, t'' \vDash \psi \rightarrow \varphi$.

Now, we know $u \in t'$ and $t' \Sigma_a^{\leq}(v)$, whence $u \in \sigma_a^{\leq}(v)$. From this it follows that $u \in \sigma_a(v)$ and so $\Sigma_a(u) = \Sigma_a(v)$ by the condition of introspection 1. on IPMs. Furthermore, $v \in t$ and $t \in \mathrm{Min}_{\leq_a^w}(\llbracket\psi\rrbracket \cap \Sigma_a(w))$, whence $t \in \Sigma_a(w)$, from which it follows that $v \in \sigma_a(w)$. So, by the condition of introspection 1. on IPMs we know $\Sigma_a(v) = \Sigma_a(w)$. Therefore, $\Sigma_a(u) = \Sigma_a(w)$, and $\sigma_a(u) = \sigma_a(w)$. As $u \in \sigma_a(u)$ by the condition of factivity on IPMs it follows from the previous observations that $u \in \sigma_a(w)$.

As $u \in \sigma_a(w)$ it follows that $x \leq_a^u y$ iff $x \leq_a^w y$ by the condition of introspection 2. on IPMs, and we have observed that $\Sigma_a(w) = \Sigma_a(u)$. Therefore, $\mathrm{Min}_{\leq_a^u}(\llbracket\psi\rrbracket \cap \Sigma_a(u)) = \mathrm{Min}_{\leq_a^w}(\llbracket\psi\rrbracket \cap \Sigma_a(w))$. So, $\forall t \in \mathrm{Min}_{\leq_a^w}(\llbracket\psi\rrbracket \cap \Sigma_a(w)), M, t \vDash \varphi$.

Conversely, suppose $M, w \nvDash E_a(\psi \rightarrow \langle E_a^{\leq}\rangle(\psi \wedge E_a^{\leq}(\psi \rightarrow \varphi)))$, then $\exists t \in \Sigma_a(w), M, t \nvDash \psi \rightarrow \langle E_a^{\leq}\rangle(\psi \wedge E_a^{\leq}(\psi \rightarrow \varphi))$. From this we infer that for some $t' \subseteq t$ it is the case that $M, t' \vDash \psi$ while $M, t' \nvDash \langle E_a^{\leq}\rangle(\psi \wedge E_a^{\leq}(\psi \rightarrow \varphi))$. The latter abbreviates $M, t' \nvDash \neg E_a^{\leq}\neg(\psi \wedge E_a^{\leq}(\psi \rightarrow \varphi))$, and from this we infer via proposition 1.2.15 and fact 1.2.16 that for some world $v \in t', M, v \vDash E_a^{\leq}\neg(\psi \wedge E_a^{\leq}(\psi \rightarrow \varphi))$, whence $\forall t'' \in \Sigma_a^{\leq}(v), M, t'' \vDash \neg(\psi \wedge E_a^{\leq}(\psi \rightarrow \varphi))$.

We know by lemma 7.1.6 that $\mathrm{Min}_{\leq_a^v}(\llbracket\psi\rrbracket \cap \Sigma_a(v)) \subseteq \Sigma_a^{\leq}(v)$. Therefore, by persistence we know that $\forall t'' \in \mathrm{Min}_{\leq_a^v}(\llbracket\psi\rrbracket \cap \Sigma_a(v)), M, t'' \vDash \neg(\psi \wedge E_a^{\leq}(\psi \rightarrow \varphi))$. As $\mathrm{Min}_{\leq_a^v}(\llbracket\psi\rrbracket \cap \Sigma_a(v)) \subseteq \llbracket\psi\rrbracket$ by definition, it then follows by the previous observation that $\forall t'' \in \mathrm{Min}_{\leq_a^v}(\llbracket\psi\rrbracket \cap \Sigma_a(v)), M, t'' \vDash \neg E_a^{\leq}(\psi \rightarrow \varphi)$, by persistence and proposition 1.2.15.

Now, as $v \in t'$ was arbitrary and we have inferred that $M, t' \vDash \psi$, we know via persistence that $\{v\} \in \llbracket\psi\rrbracket$. Furthermore, we know $\{v\} \in \Sigma_a(v)$ by the condition of factivity on IPMs, and that $v \leq_a^v v$ by the fact that $\leq_a^v$ is a well-preorder. From these facts it follows that $\mathrm{Min}_{\leq_a^v}(\llbracket\psi\rrbracket \cap \Sigma_a(v)) \neq \{\emptyset\}$, for $\llbracket\psi\rrbracket \cap \Sigma_a(v) \neq \{\emptyset\}$ and so this can be restricted to its minimal elements.

So, let $t''' \in \mathrm{Min}_{\leq_a^v}(\llbracket\psi\rrbracket \cap \Sigma_a(v))$ such that $t \neq \emptyset$ be arbitrary. We know $M, t''' \vDash \neg E_a^{\leq}(\psi \rightarrow \varphi)$. So, by proposition 1.2.15, we know that $\forall u \in t''', M, u \vDash \neg E_a^{\leq}(\psi \rightarrow \varphi)$. As we have observed that $t''' \neq \emptyset$, let $u \in t'''$ be arbitrary.

From the above, we know that $M, u \vDash \neg E_a^{\leq}(\psi \rightarrow \varphi)$ and so there exists some $t^{\mathrm{IV}} \in \Sigma_a^{\leq}(u)$ such that $M, t^{\mathrm{IV}} \nvDash \psi \rightarrow \varphi$. Therefore, for some $t^{\mathrm{V}} \subseteq t^{\mathrm{IV}}$ we have $M, t^{\mathrm{V}} \vDash \psi$ while $M, t^{\mathrm{V}} \nvDash \varphi$. From this and proposition 1.2.15 we know that for some $x \in t^{\mathrm{V}}, M, x \nvDash \varphi$. Whence, as worlds behave classically, $M, x \vDash \neg\varphi$ and by persistence $M, x \vDash \psi$, as $M, t^{\mathrm{V}} \vDash \psi$.

As $x \in t^{\mathrm{V}}$ and $t^{\mathrm{V}} \in \Sigma_a^{\leq}(u)$ we infer that $x \in \sigma_a^{\leq}(u)$. So, $x \leq_a^u u$. Furthermore, $u \in t''', t''' \in \mathrm{Min}_{\leq_a^v}(\llbracket\psi\rrbracket \cap \Sigma_a(v))$, and $M, v \vDash \psi$. So, it follows immediately that $u \leq_a^v v$. And, by the latter observation we know $u \in \sigma_a(v)$, whence $y \leq_a^u z$ iff $y \leq_a^v z$, so by the former observation $x \leq_a^v u$.

We know $u \in \mathrm{Min}_{\leq_a^v}(\llbracket\psi\rrbracket \cap \Sigma_a(v))$ and so by corollary 3.3.12 we know $u \in$

$\text{Min}_{\leq_a^v}|\psi|$. So, as $x \leq_a^v u$ and $M, x \vDash \psi$, we know that $x \in \text{Min}_{\leq_a^v}|\psi|$. Furthermore, as $x \leq_a^v u$ we know $x \in \sigma_a(v)$, whence $\{x\} \in \Sigma_a(v)$ and as $M, x \vDash \psi$ we know $\{x\} \in [\![\psi]\!]$. Therefore, by corollary 3.3.12 we know that $\{x\} \in \text{Min}_{\leq_a^v}([\![\psi]\!] \cap \Sigma_a(v))$. So, given that $M, x \vDash \neg\varphi$ we know that $\exists t \in \text{Min}_{\leq_a^v}([\![\psi]\!] \cap \Sigma_a(v)), M, t \nvDash \varphi$.

Finally, as $v \in \sigma_a(w)$ we can observe via the conditions of introspection 1. and 2. on IPMs that $\text{Min}_{\leq_a^v}([\![\psi]\!] \cap \Sigma_a(v)) = \text{Min}_{\leq_a^w}([\![\psi]\!] \cap \Sigma_a(w))$, whence it follows that $\exists t \in \text{Min}_{\leq_a^w}([\![\psi]\!] \cap \Sigma_a(w)), M, t \nvDash \varphi$. □

Given we have chosen to continue to interpret the binary relation on worlds as that of plausibility, the reading of the considers modality outlined in 3 carries over to IPL, and we obtain corresponding formulas to define belief and knowledge, following the results established in chapter 3 section 3.4.2.

Naturally the interpretation of the considers modality differs when preference is used to interpret the binary relation between worlds, however much of the investigation into the considers modality from chapter 3 carries over *mutatis mutandis*. Indeed, $C_a^\psi \varphi$ when interpreted on IPMs can be read as 'agent a considers φ conditional on ψ,' so long as the conditionalisation process is now understood to rely on the preferences and not the doxastic state of the agent. This highlights the extent to which considers is, for better or worse, a technical notion in this thesis, for while we can derive a corresponding definition for conditional belief, the term is intuitively at odds with preference relations. Still, the distinction between the considers and conditional belief modalities may be used to distinguish conditionalising on ψ given certain background assumption and conditionalising on ψ only, corresponding to the former and latter modalities respectively.

Furthermore, we observe that not only can one define what issues remain open when an agent entertains only the worlds taken as most plausible or preferable, but given the expressive power of IPL one can identify the *unique* issues open at the most preferable worlds, e.g. one can define considering φ *only* given ψ.

$$C_a^\psi \varphi \wedge C_a^\psi E_a^{\not\leq} \neg\varphi$$

In turn this shows the considerable expressive power of IPL, for one can now has the possibility of a syntactic representation of the most plausible worlds for an agent, by letting $\psi = \top$. From this a formula for the 'second most' plausible worlds can be constructed, by letting $\psi = \neg\varphi$, where φ is consider only given $\top$, and so on. In this way a full syntactic characterisation of an agent's plausibility ordering may be constructed.

Indeed, if one is aiming to model certain pragmatic phenomena the ability to express the fact that an agent has full or partial knowledge of another agent's plausibility ordering, or at least that the agent has some understanding of another agent's doxastic state seems desirable. One may, for example, hope to capture a notion of relevance by pairing together the structure of a speakers own epistemic state with what that agent takes to be the broader doxastic state of their interlocutor.

We leave the potential applications of IPL aside in this thesis, pausing only to note that the plausibility ordering for each agent can be lifted from worlds to propositions, following the work of Girard (2008) and Liu (2011).

7.1.1 Binary Operators

One may hope to induce a plausibility relation on worlds from a plausibility relation on propositions, in an analogous way to how the plausibility ordering used to describe (part of) an agent's doxastic state on IPMs could be given qualitatively via ICDMs. To this end we briefly explore the definability of binary plausibility operators between propositions in IPL.

This allows us to look for representation of such terms as 'agent a holds φ to be at least as plausible as ψ.' As the plausibility relation on IPMs is between worlds the binary operators lifted from this relation holds between declaratives, for these are the formulas which characterise what does and does not hold at any given world. These binary operators defined with respect to declaratives can then be lifted to interrogatives via the ability to characterise interrogatives in terms of their resolutions. We make suggest some avenues to pursue at the end of this section, but leave a complete development to later work.

Our presentation follows Girard (2008, Ch. 4).

Definition 7.1.8 (Binary plausibility operators). The following seven binary plausibility operators given below are defined following Girard (2008, p. 60).[3]

$$\begin{aligned}
M, w &\vDash \alpha \leq_a^{\exists\exists} \beta \text{ iff } \exists v, \exists u : M, v \vDash \alpha \;\&\; M, u \vDash \beta \;\&\; v \leq_a^w u \\
M, w &\vDash \alpha \leq_a^{\forall\exists} \beta \text{ iff } \forall v, \exists u : M, v \vDash \alpha \Rightarrow M, u \vDash \beta \;\&\; v \leq_a^w u \\
M, w &\vDash \alpha <_a^{\exists\exists} \beta \text{ iff } \exists v, \exists u : M, v \vDash \alpha \;\&\; M, u \vDash \beta \;\&\; v <_a^w u \\
M, w &\vDash \alpha <_a^{\forall\exists} \beta \text{ iff } \forall v, \exists u : M, v \vDash \alpha \Rightarrow M, u \vDash \beta \;\&\; v <_a^w u \\
M, w &\vDash \alpha <_a^{\forall\forall} \beta \text{ iff } \forall v, \forall u : M, v \vDash \alpha \;\&\; M, u \vDash \beta \Rightarrow v <_a^w u \\
M, w &\vDash \alpha \leq_a^{\forall\forall} \beta \text{ iff } \forall v, \forall u : M, v \vDash \alpha \;\&\; M, u \vDash \beta \Rightarrow v \leq_a^w u \\
M, w &\vDash \alpha \geq_a^{\exists\forall} \beta \text{ iff } \exists v, \forall u : M, v \vDash \alpha \;\&\; M, u \vDash \beta \Rightarrow u \leq_a^w v
\end{aligned}$$

To aid in establishing the definability of these operators in IPL we establish a number of lemmas.

Lemma 7.1.9. *$M, w \vDash \langle E_a \rangle \varphi$ iff $\exists v \in \sigma_a(w), M, v \vDash \varphi$*

Proof. From left to right suppose $M, w \vDash \langle E_a \rangle \varphi$. Expanding, this reads: $M, w \vDash \neg E_a \neg \varphi$. From this it follows that $M, w \nvDash E_a \neg \varphi$, and so, $\exists t \in \Sigma_a(w), M, t \nvDash \neg \varphi$. By proposition 1.2.15 this entails that for some $v \in t$, $M, v \nvDash \neg \varphi$, whence $M, v \vDash \varphi$. And, as $v \in t$ and $t \in \Sigma_a(w)$, it is immediate that $v \in \sigma_a(w)$.

From right to left the reasoning is analogous in the converse direction. ☐

Lemma 7.1.10. *$M, w \vDash \langle E_a^{\leq} \rangle \varphi$ iff $\exists v : v \leq_a^w w$ and $M, v \vDash \varphi$*

Proof. The proof is analogous to lemma 7.1.9. ☐

Lemma 7.1.11. *$M, w \vDash \langle E_a^{\nleq} \rangle \varphi$ iff $\exists v : w <_a^w v$ and $M, v \vDash \varphi$*

Proof. The proof is analogous to lemma 7.1.9, with the aid of proposition 7.1.3 to interchange $v \nleq_a^w w$ and $w <_a^w v$. ☐

Proposition 7.1.12. *The plausibility operators of definition 7.1.8 can be defined in IPL.*

$$\begin{aligned}
M, w &\vDash \alpha \leq_a^{\exists\exists} \beta := \langle E_a \rangle (\beta \wedge \langle E_a^{\leq} \rangle \alpha) \\
M, w &\vDash \alpha \leq_a^{\forall\exists} \beta := E_a(\alpha \to \langle E_a^{\leq} \rangle \beta) \\
M, w &\vDash \alpha <_a^{\exists\exists} \beta := \langle E_a \rangle (\langle E_a^{\leq} \rangle \alpha \wedge \langle E_a^{\nleq} \rangle \beta) \\
M, w &\vDash \alpha <_a^{\forall\exists} \beta := E_a(\alpha \to \langle E_a^{\nleq} \rangle \beta) \\
M, w &\vDash \alpha <_a^{\forall\forall} \beta := E_a(\alpha \to E_a^{\leq} \neg \beta) \\
M, w &\vDash \alpha \leq_a^{\forall\forall} \beta := E_a(\beta \to E_a^{\nleq} \neg \alpha) \\
M, w &\vDash \alpha \geq_a^{\exists\forall} \beta := \langle E_a \rangle (\alpha \wedge E_a^{\nleq} \neg \beta)
\end{aligned}$$

[3]Note, Girard defines eight binary operators. Here, we omit the operator $\alpha >_a^{\exists\forall} \beta$, defined by $\exists v, \forall u : M, v \vDash \alpha \;\&\; M, u \vDash \beta \Rightarrow u <_a^w v$. Seven, naturally, a more perfect number than eight.

Proof.

$M, w \vDash \langle E_a \rangle(\beta \wedge \langle E_a^{\leq} \rangle \alpha)$ **iff** $\exists v, \exists u \colon M, v \vDash \alpha \;\&\; M, u \vDash \beta \;\&\; v \leq_a^w u$

From left to right suppose $M, w \vDash \langle E_a \rangle(\beta \wedge \langle E_a^{\leq} \rangle \alpha)$. Therefore, $\exists u \in \sigma_a(w), M, u \vDash \beta \wedge \langle E_a^{\leq} \rangle \alpha$, by lemma 7.1.9. Moreover, $M, u \vDash \beta \wedge \langle E_a^{\leq} \rangle \alpha$ iff $M, u \vDash \beta$ and $M, u \vDash \langle E_a^{\leq} \rangle \alpha$. So, by lemma 7.1.10 and the latter conjunct, $\exists v \leq_a^u u, M, v \vDash \alpha$. As $u \in \sigma_a(w)$ we infer by the condition of introspection 2. on IPMs that $\exists v \leq_a^w u, M, v \vDash \alpha$. So, we have shown $\exists v, \exists u \colon M, v \vDash \alpha \;\&\; M, u \vDash \beta \;\&\; v \leq_a^w u$.

From right to left the reasoning is analogous in the converse direction.

$M, w \vDash E_a(\alpha \to \langle E_a^{\nleq} \rangle \beta)$ **iff** $\forall v \in \sigma_a(w), \exists u \colon M, u \vDash \alpha \Rightarrow M, u \vDash \beta \;\&\; v \leq_a^w u$

From left to right, suppose $M, w \vDash E_a(\alpha \to \langle E_a^{\leq} \rangle \beta)$. So, $\forall t \in \Sigma_a(w), M, t \vDash \alpha \to \langle E_a^{\leq} \rangle \beta$. Therefore, by persistence we know that $\forall v \in \sigma_a(w), M, v \vDash \alpha \to \langle E_a^{\leq} \rangle \beta$. So, as worlds behave classically, if $M, v \vDash \alpha$ then $M, v \vDash \langle E_a^{\nleq} \rangle \beta$, whence by lemma 7.1.11 we have that $\exists u \colon M, u \vDash \beta$ and $v \leq_a^v u$. As before, we know $v \in \sigma_a(w)$ and so by introspection 2. on IPMs this entails that $\exists u \colon M, u \vDash \beta$ and $v \leq_a^w u$. Therefore, if $M, v \vDash \alpha$ then $\exists u \colon M, u \vDash \beta$ and $v \leq_a^w u$.

From right to left the reasoning is analogous in the converse direction.

$M, w \vDash \langle E_a \rangle(\langle E_a^{\leq} \rangle \alpha \wedge \langle E_a^{\nleq} \rangle \beta)$ **iff** $\exists v, \exists u \colon M, v \vDash \alpha \;\&\; M, u \vDash \beta \;\&\; v <_a^w u$

From left to right suppose $M, w \vDash \langle E_a \rangle(\langle E_a^{\leq} \rangle \alpha \wedge \langle E_a^{\nleq} \rangle \beta)$. So, by lemma 7.1.9 we know that $\forall x \in \sigma_a(w)$ that $M, x \vDash \langle E_a^{\leq} \rangle \alpha \wedge \langle E_a^{\nleq} \rangle \beta$. Let $x \in \sigma_a(w)$ be arbitrary. As $M, x \vDash \langle E_a^{\leq} \rangle \alpha \wedge \langle E_a^{\nleq} \rangle \beta$ we know $M, x \vDash \langle E_a^{\leq} \rangle \alpha$ and $M, x \vDash \langle E_a^{\nleq} \rangle \beta$.

By the former conjunct and lemma 7.1.10 we know that $\exists v \colon v \leq_a^x x$ and by the latter conjunct and lemma 7.1.11 we know that $\exists u \colon x <_a^x u$ and $M, u \vDash \beta$. So, we can infer that $v <_a^x u$. And, as $x \in \sigma_a(w)$ this implies that $v <_a^x u$ by the condition of introspection 2. on IPMs.

From right to left suppose $\exists v, \exists u \colon M, v \vDash \alpha \;\&\; M, u \vDash \beta \;\&\; v <_a^w u$. Let v, u instantiate their respective quantifiers. Then, $v <_a^w u$. From this we know that $v \in \sigma_a(w)$ and therefore by the condition of introspection 2. on IPMs we know $v <_a^v u$.

So, as $v \leq_a^v v$ and $M, v \vDash \alpha$ we know by lemma 7.1.10 that $M, v \vDash \langle E_a^{\leq} \rangle \alpha$. Similarly by the fact that $v <_a^v u$ and $M, u \vDash \beta$ we know by lemma 7.1.11 that $M, v \vDash \langle E_a^{\nleq} \rangle \beta$. So, $M, v \vDash \langle E_a^{\leq} \rangle \alpha \wedge \langle E_a^{\nleq} \rangle \beta$. And, as $v \in \sigma_a(w)$ we know by lemma 7.1.9 that $M, w \vDash \langle E_a \rangle(\langle E_a^{\leq} \rangle \alpha \wedge \langle E_a^{\nleq} \rangle \beta)$.

$M, w \vDash E_a(\alpha \to \langle E_a^{\nleq} \rangle \beta)$ **iff** $\forall v, \exists u \colon M, v \vDash \alpha \Rightarrow M, u \vDash \beta \;\&\; v <_a^w u$

From left to right suppose $M, w \vDash E_a(\alpha \to \langle E_a^{\nleq} \rangle \beta)$. Let $v \in \sigma_a(w)$ be arbitrary, and suppose $M, v \vDash \alpha$. So, as $M, w \vDash E_a(\alpha \to \langle E_a^{\nleq} \rangle \beta)$ we know that $\forall t \in \Sigma_a(w), M, t \vDash \alpha \to \langle E_a^{\nleq} \rangle \beta$, and by persistence this entails $M, v \vDash \alpha \to \langle E_a^{\nleq} \rangle \beta$, whence we infer that $M, v \vDash \langle E_a^{\nleq} \rangle \beta$. So, by lemma 7.1.11 we know that $\exists u, v <_a^v u$ and $M, u \vDash \beta$. As before, we use the fact that $v \in \sigma_a(w)$ to observe via the condition of introspection 2. on IPMs and the previous fact that $v <_a^w u$, whence the result follows.

From right to left the reasoning is analogous in the converse direction.

$M, w \models E_a(\alpha \to E_a^{\leq}\neg\beta)$ **iff** $\forall v, \forall u\colon M, v \models \alpha \;\&\; M, u \models \beta \Rightarrow v <_a^w u$

From left to right suppose $M, w \models E_a(\alpha \to E_a^{\leq}\neg\beta)$. So, $\forall t \in \Sigma_a(w), M, t \models \alpha \to E_a^{\leq}\neg\beta$.

Now, suppose for an arbitrary $v, u \in \sigma_a(w), M, v \models \alpha$ and $M, u \models \beta$. From the above observation and our assumption that $M, v \models \alpha$ we infer $M, v \models E_a^{\leq}\neg\beta$. So, $\forall t' \in \Sigma_a^{\leq}(v), M, t' \models \neg\beta$.

Now, suppose $u \leq_a^w v$. Using the condition of introspection 2. on IPMs and the fact that $v \in \sigma_a(w)$ we infer $u \leq_a^v v$. Therefore, $\{u\} \in \Sigma_a^{\leq}(v)$, whence $M, u \models \neg\beta$. This contradicts our inference that $M, u \models \beta$, and therefore we infer that $u \not\leq_a^w v$. However, as $v, u \in \sigma_a(w)$ we know by this and proposition 7.1.3 that $v \leq_a^w u$, whence $v <_a^w u$.

From right to left suppose $M, w \not\models E_a(\alpha \to E_a^{\leq}\neg\beta)$. Then $\exists t \in \Sigma_a(w), M, t \models \alpha$ while $M, t \not\models E_a^{\leq}\neg\beta$. So, as worlds behave classically it must be the case that for some $v \in t, M, v \models \langle E_a^{\leq}\rangle\beta$ So, by lemma 7.1.10 we have some $u \leq_a^v v$ such that $M, u \models \beta$. By the fact that $v \in \sigma_a(w)$ and the condition of introspection 2. on IPMs it follows that $u \leq_a^w v$, whence $v \not<_a^w u$.

$M, w \models E_a(\beta \to E_a^{\not\leq}\neg\alpha)$ **iff** $\forall v, \forall u\colon M, v \models \alpha \;\&\; M, u \models \beta \Rightarrow v \leq_a^w u$

From left to right suppose $M, w \models E_a(\beta \to E_a^{\not\leq}\neg\alpha)$, so $\forall t \in \Sigma_a(w), M, t \models \beta \to E_a^{\not\leq}\neg\alpha$. So, suppose for arbitrary $v, u \in \sigma_a(w), M, v \models \alpha$ and $M, u \models \beta$. Then $M, u \models E_a^{\not\leq}\neg\alpha$, whence $v \notin \sigma_a^{\not\leq}(u)$, and so $v \leq_a^u u$. As $u \in \sigma_a(w)$ this entails, via the condition of introspection 2. on IPMs, that $v \leq_a^w u$.

From right to left suppose $\forall v, \forall u\colon (M, v \models \alpha \;\&\; M, u \models \beta) \Rightarrow v \leq_a^w u$ but $M, w \not\models E_a(\beta \to E_a^{\not\leq}\neg\alpha)$. Then $M, w \models \langle E_a\rangle(\beta \wedge \langle E_a^{\not\leq}\rangle\alpha)$, and so, by lemma 7.1.9, for some $u \in \sigma_a(w), M, u \models \beta \wedge \langle E_a^{\not\leq}\rangle\alpha$, whence by lemma 7.1.11, for some $v\colon u <_a^u v, M, v \models \alpha$, but then $u \leq_a^w v$, via the fact that $u \in \sigma_a(w)$ and the condition of introspection 2. on IPMs, and we have a contradiction.

$M, w \models \langle E_a\rangle(\varphi \wedge E_a^{\not\leq}\neg\psi)$ **iff** $\exists v, \forall u\colon M, v \models \alpha \;\&\; M, u \models \beta \Rightarrow u \leq_a^w v$

From left to right suppose $M, w \models \langle E_a\rangle(\alpha \wedge E_a^{\not\leq}\neg\beta)$. Then by lemma 7.1.9, for some $v \in \sigma_a(w), M, v \models \alpha \wedge E_a^{\not\leq}\neg\beta$. For an arbitrary $u \in \sigma_a(w)$ suppose $u \models \beta$, then $u \notin \sigma_a^{\not\leq}(v)$ on pain of contradiction.

As $u \notin \sigma_a^{\not\leq}(v)$ we know that $u \leq_a^v v$, and by the fact that $v \in \sigma_a(w)$ and the condition of introspection 2. this entails $u \leq_a^w v$.

From right to left suppose $M, w \not\models \langle E_a\rangle(\alpha \wedge E_a^{\not\leq}\neg\beta)$. Therefore, as worlds behave classically we know $M, w \models E_a\neg(\alpha \wedge E_a^{\not\leq}\neg\beta)$ So, for all $t \in \Sigma_a(w), M, t \models \neg(\alpha \wedge E_a^{\not\leq}\neg\beta)$.

Now, suppose $\exists v, \forall u\colon M, v \models \alpha \;\&\; (M, u \models \beta \Rightarrow u \leq_a^w v)$. Let v instantiate the existential quantifier. Then, as $v \in \sigma_a(w)$ we know via persistence that $M, v \models \neg(\alpha \wedge E_a^{\not\leq}\neg\beta)$. So, as $M, v \models \alpha$ we know $M, v \not\models E_a^{\not\leq}\neg\beta$. From this we infer $M, v \models \neg E_a^{\not\leq}\neg\beta$, which can be rewritten as $M, v \models \langle E_a^{\not\leq}\rangle\beta$. From the latter observation and lemma 7.1.11 we know that for some $u\colon v <_a^v u, M, u \models \beta$. So, by the fact that $v \in \sigma_a(w)$ and the condition of introspection 2. on IPMs we infer that $v <_a^w u$. This contradicts our assumption that $\exists v, \forall u\colon M, v \models \alpha \;\&\; (M, u \models \beta \Rightarrow u \leq_a^w v)$. □

Using the preceding definitions binary plausibility operators can be defined with respect to interrogatives in a number of ways. For example, we may identify that μ is a more plausible interrogative to ν by observing that for every resolution to ν there is a more plausible resolution to μ. To capture this we may say $\mu <_a^{\exists\forall} \nu$ iff $\forall\beta \in \mathcal{R}(\nu)\exists\alpha \in \mathcal{R}(\mu) \colon \alpha <_a^{\forall\forall} \beta$, which is captured by the formula $\bigwedge_{\beta\in\mathcal{R}(\nu)} \bigvee_{\alpha\in\mathcal{R}(\mu)}(\alpha <_a^{\forall\forall} \beta)$, etc.

7.2 Inquisitive Plausibility Neighbourhood Models

As with EL, IEL, CDL, and ICDL we define a class of neighbourhood models termed *inquisitive plausibility neighbourhood models* (IPNMs) to which IPL corresponds, and as with the inquisitive variations of these logics it is to this class of models that we prove soundness and completeness, before showing that every IPNM can be transformed into an IPM. And, while in the case of ICDL this proved to be of more interest than a mere technical tool, here we regard IPNMs and strictly inferior as an interpretation of IPL than IPMs, and so we will not explore an interpretation.

Definition 7.2.1 (Inquisitive plausibility neighbourhood models).
An Inquisitive plausibility neighbourhood model is a tuple: $\langle W, \{\Sigma_a^{\leq}\}, \{\Sigma_a^{\not\leq}\}, \{\Sigma_a\}, V\rangle$, where:

- W is a set of possible worlds
- $V \colon W \to \wp(\mathsf{At})$ is a valuation map, stating for each $w \in W$ the atomic formulas true at w
- Each $\Sigma_a^{\square}$ for $\square \in \{\leq, \not\leq, \cdot\}$ is a map $W \to \Pi$ associating to each world an issue in accordance with the following conditions:
  1. if $v \in \sigma_a^{\leq}(w)$, then $\Sigma_a^{\leq}(v) \subseteq \Sigma_a^{\leq}(w)$, where $\sigma_a^{\leq}(w) := \bigcup \Sigma_a^{\leq}(w)$
  2. if $u \in \sigma_a^{\not\leq}(w)$, then $\Sigma_a^{\not\leq}(u) \subseteq \Sigma_a^{\not\leq}(w)$, where $\sigma_a^{\not\leq}(w) := \bigcup \Sigma_a^{\not\leq}(w)$
  3. $w \in \sigma_a^{\leq}(w)$
  4. $\sigma_a^{\leq}(w) \cap \sigma_a^{\not\leq}(w) = \emptyset$
  5. $\sigma_a^{\leq}(w) \cup \sigma_a^{\not\leq}(w) = \sigma_a(w)$
  6. if $v \in \sigma_a(w)$, then $\Sigma_a(w) = \Sigma_a(v)$, where $\sigma_a(w) := \bigcup \Sigma_a(w)$
  7. $\Sigma_a^{\leq}(w) = (\wp(\sigma_a^{\leq}(w)) \cap \Sigma_a(w))$
  8. $\Sigma_a^{\not\leq}(w) = (\wp(\sigma_a^{\not\leq}(w)) \cap \Sigma_a(w))$

Conditions 1–8 are intended to ensure that any IPM can be translated into an IPNM, and any finite IPNM can be transformed into an IPM, preserving the interpretation of IPL, following the methods of chapter 5. This will be established in section 7.5.

The conditions have the following readings.

1. If v is at least as plausible as w, then any issue over the worlds at least as plausible v can be obtained from the issue over the worlds at least as plausible as w.
2. Similar to 1, if u is less plausible than w, then any issue over the plausible worlds at w can be obtained from the issue over the plausible worlds at u.
3. The plausibility relation is reflexive.
4. No world is both plausible and implausible.
5. The plausibility relation is over an agent's epistemic state.
6. Agents have both positive and negative introspection.
7. An issue over the plausible worlds at some world w is obtained from restricting the agent's epistemic issue to those worlds at least as plausible as w.

8. An issue over the implausible worlds at some world w is obtained from restricting the agent's epistemic issue to those worlds at least as plausible as w.

Lemma 7.2.2.

1. $v \in \sigma_a^{\leq}(w)$ *iff* $\Sigma_a^{\leq}(v) \subseteq \Sigma_a^{\leq}(w)$.
2. If $v, u \in \sigma_a(w)$*, then either* $u \in \sigma_a^{\leq}(v)$ *or* $u \in \sigma_a^{\not\leq}(v)$.

Proof.

1. From left to right it precisely condition 1. From right to left, by condition 3 we know $v \in \sigma_a^{\leq}(v)$, and so if $\Sigma_a^{\leq}(v) \subseteq \Sigma_a^{\leq}(w)$ it is immediate that $v \in \sigma_a^{\leq}(w)$.

2. Let $v, u \in \sigma_a(w)$ be arbitrary. So, by condition 6 on IPNMs we know that $\Sigma_a(w) = \Sigma_a(v) = \Sigma_a(u)$. By condition 3 we know that $u \in \sigma_a^{\leq}(u)$, whence $\{u\} \in \Sigma_a^{\leq}(u) = \Sigma_a^{\leq}(v)$, and so $u \in \sigma_a(v)$. So, by condition 5 we know it must be the case that $u \in \sigma_a^{\leq}(v)$ or $u \in \sigma_a^{\not\leq}(v)$. □

The support conditions for IPL on IPNMs mirror those of IPL on IPMs, given our duplication of notation.

Definition 7.2.3 (Support conditions for IPL modalities on IPNMs)**.**

1. $M, w \vDash E_a^{\leq}\varphi$ iff $\forall t \in \Sigma_a^{\leq}(w), M, t \vDash \varphi$
2. $M, w \vDash E_a^{\not\leq}\varphi$ iff $\forall t \in \Sigma_a^{\not\leq}(w), M, t \vDash \varphi$

where:

a. $\Sigma_a^{\leq}(w) := \wp(\sigma_a^{\leq}(w)) \cap \Sigma_a(w)$, and $\sigma_a^{\leq}(w) := \{v \mid v \leq_a^w w\}$
b. $\Sigma_a^{\not\leq}(w) := \wp(\sigma_a^{\not\leq}(w)) \cap \Sigma_a(w)$, and $\sigma_a^{\not\leq}(w) := \{v \in \sigma_a(w) \mid v \not\leq_a^w w\}$

7.3 Axioms and Rules of IPL

The following 11 axiom schemas and rule of inference augment the base system of InqD to provide a sound and complete logic with respect to IPMs.

Definition 7.3.1. Axioms and rules of IPL

1. $E_a(\varphi \to \psi) \to (E_a\varphi \to E_a\psi)$
2. $E_a^{\leq}(\varphi \to \psi) \to (E_a^{\leq}\varphi \to E_a^{\leq}\psi)$
3. $E_a^{\not\leq}(\varphi \to \psi) \to (E_a^{\not\leq}\varphi \to E_a^{\not\leq}\psi)$
4. $E_a^{\leq}\varphi \to E_a^{\leq}E_a^{\leq}\varphi$
5. $E_a^{\not\leq}\varphi \to E_a^{\not\leq}E_a^{\not\leq}\varphi$
6. i. $E_a\varphi \to E_aE_a\varphi$ and ii. $\neg E_a\varphi \to E_a\neg E_a\varphi$
7. $E_a\varphi \to (E_a^{\leq}\varphi \wedge E_a^{\not\leq}\varphi)$
8. $(E_a^{\leq}\varphi \wedge E_a^{\not\leq}\psi) \to E_a(\bigvee_{\alpha\in\mathcal{R}(\varphi)}\bigvee_{\beta\in\mathcal{R}(\psi)}(\alpha \vee \beta))$
9. $E_a^{\leq}\alpha \to \alpha$
10. $(E_a^{\leq}\varphi \wedge E_a^{\not\leq}\neg\alpha) \to E_a(\alpha \to \varphi)$
11. $(E_a^{\not\leq}\varphi \wedge E_a^{\leq}\neg\alpha) \to E_a(\alpha \to \varphi)$

In conjunction with the rules of inference of InqD we add necessitation for entertains:[4]

$$\frac{\begin{array}{c}\emptyset\\ \vdots\\ \varphi\end{array}}{E_a\varphi}$$

Unlike ICDL, the axioms of IPL do not have a straightforward epistemic interpretation, and indeed our choice of axioms are made on the basis of their corresponding semantic property on IPNMs, and not IPMs, leading to additional obscurity. In what follows we briefly overview some basic properties of the axioms, however their role is best understood by their function in constructing the canonical model for IPL and showing the regular model for IPL is an IPNM as in lemma 7.4.8.

Axioms 1, 6.i, and 6.ii axiomatise the basic properties of the entrains modality familiar from IEL, with factivity following from the conjunction with axioms 7 and 3 (We give an explicit proof in proposition 7.3.2, below.) Axioms 2 and 3 mirror axiom 1 in ensuring that the plausibility modalities, like the entertains modality, are monotonic operators.

Axioms 7 and 8 constrain the plausibility modalities to range over propositions entertained by an agent, while axioms 9, and 4 ensure the plausibility relation is reflexive and transitive.

Finally axioms 10 and 11 describe the interaction between the entertains modality and the plausibility modalities in specific contexts—in effect ensuring that the inquisitive states of an agent relativised to aspects of the plausibility relation are determined solely by the agent's general inquisitive state.

We also pause to note factivity of the entertains modality.

Proposition 7.3.2. *All instances of the schema $E_a\alpha \to \alpha$ are axioms of IPL.*

Proof. We give the proof in a condensed form, abbreviating applications of implication elimination on a conditional to immediate derivations of the consequent from the antecedent of the relevant conditional.

$$\cfrac{\cfrac{\cfrac{\cfrac{[E_a\alpha]^1}{E_a^{\leq}\alpha \wedge E_a^{\not\leq}\alpha}\ (\mathsf{Ax.7})}{E_a^{\leq}\alpha}\ (\wedge e)}{\alpha}\ (\mathsf{Ax.9})}{E_a\alpha \to \alpha}\ (\to i, 1)$$

□

Theorem 7.3.3 (Soundness of IPL wrt. IPNMs). *IPL is sound with respect to IPNMs.*

Proof. As in theorem 4.3.1 we observe that each axiom of IPL is a declarative, and so by proposition 1.2.15 soundness can be established via appeal to truth conditions. This observation allow us to sidestep many unnecessary repetitions and complications.

1 $E_a(\varphi \to \psi) \to (E_a\varphi \to E_a\psi)$

Suppose $M, w \vDash E_a(\varphi \to \psi)$ and $M, w \vDash E_a\varphi$. Then, $\forall t \in \Sigma_a(w), M, t \vDash \varphi \to \psi$ and $\forall t \in \Sigma_a(w), M, t \vDash \varphi$, whence $\forall t \in \Sigma_a(w), M, t \vDash \psi$, and so $M, w \vDash E_a\psi$.

[4]Observe that necessitation for the two plausibility modalities follows from necessitation for the entertains modality and axiom 7

2. $E_a^{\leq}(\varphi \to \psi) \to (E_a^{\leq}\varphi \to E_a^{\leq}\psi)$

Analogous to the previous axiom.

3. $E_a^{\nleq}(\varphi \to \psi) \to (E_a^{\nleq}\varphi \to E_a^{\nleq}\psi)$

Analogous to the previous axiom.

4. $E_a^{\leq}\varphi \to E_a^{\leq}E_a^{\leq}\varphi$

Suppose $M, w \vDash E_a^{\leq}\varphi$ while $M, w \nvDash E_a^{\leq}E_a^{\leq}\varphi$. By the former we infer that $\forall t \in \Sigma_a^{\leq}(w), M, t \vDash \varphi$. By the latter, as worlds behave classically, we infer that $M, w \vDash \neg E_a^{\leq}E_a^{\leq}\varphi$. So, $\exists t' \in \Sigma_a^{\leq}(w)$ such that $M, t' \nvDash E_a^{\leq}\varphi$, whence $M, w \vDash \neg E_a^{\leq}\varphi$. By persistence this means that for some world $v \in t', M, v \vDash \neg E_a^{\leq}\varphi$, whence $\exists t'' \in \Sigma_a^{\leq}(v)$ such that $M, t'' \nvDash \varphi$. So, by proposition 1.2.15 this means that there's some world $u \in t''$ such that $M, u \nvDash \varphi$. As worlds behave classically we can infer $M, w \vDash \neg\varphi$. However, as $t' \in \Sigma_a^{\leq}(w)$ and $v \in t'$ we know that $\{v\} \in \Sigma_a^{\leq}(w)$ as $\Sigma_a^{\leq}(w)$ is an issue and hence is downward closed. And, as $u \in t''$ and $t'' \in \Sigma_a^{\leq}(u)$ $\{u\} \in \Sigma_a^{\leq}(v)$. So, we can infer by the previous observation and condition 1 on IPNMs that $\{u\} \in \Sigma_a^{\leq}(w)$. But then it must be the case that $M, u \vDash \varphi$, a contradiction.

5. $E_a^{\nleq}\varphi \to E_a^{\nleq}E_a^{\nleq}\varphi$

Analogous to the previous axiom.

6.i $E_a\varphi \to E_aE_a\varphi$ and 6.ii $\neg E_a\varphi \to E_a\neg E_a\varphi$

Each follows the proof of positive/negative introspection for IEL.

7. $E_a\varphi \to (E_a^{\leq}\varphi \wedge E_a^{\nleq}\varphi)$

Suppose $M, w \vDash E_a\varphi$. Then, $\forall t \in \Sigma_a(w), M, t \vDash \varphi$. Therefore, as $(\wp(\sigma_a^{\leq}(w)) \cap \Sigma_a(w)) \subseteq \Sigma_a(w))$, we know that $\forall t \in (\wp(\sigma_a^{\leq}(w)) \cap \Sigma_a(w)), M, t \vDash \varphi$. Therefore, for all $t \in \Sigma_a^{\leq}(w), M, t \vDash \varphi$, whence $M, w \vDash E_a^{\leq}\varphi$. The reasoning is analogous for $E_a^{\nleq}$, and therefore $M, w \vDash E_a^{\leq}\varphi \wedge E_a^{\nleq}\varphi$.

8. $(E_a^{\leq}\varphi \wedge E_a^{\nleq}\psi) \to E_a(\bigvee_{\alpha\in\mathcal{R}(\varphi)}\bigvee_{\beta\in\mathcal{R}(\psi)}(\alpha \vee \beta))$

Suppose $M, w \vDash E_a^{\leq}\varphi \wedge E_a^{\nleq}\psi$. Therefore, $\forall t \in \Sigma_a^{\leq}(w), M, t \vDash \varphi$ and $\forall t' \in \Sigma_a^{\nleq}(w), M, t' \vDash \psi$. So, by theorem 6.2.7 we know that $\forall t \in \Sigma_a^{\leq}(w), M, t \vDash \alpha$ for some $\alpha \in \mathcal{R}(\varphi)$ and $\forall t' \in \Sigma_a^{\nleq}(w), M, t' \vDash \beta$ for some $\beta \in \mathcal{R}(\psi)$. So, for an arbitrary $v \in \sigma_a^{\leq}(w)$ we know $\{v\} \in \Sigma_a^{\leq}(w)$, whence $M, v \vDash \alpha$ for some $\alpha \in \mathcal{R}(\varphi)$. So, $M, v \vDash (\bigvee_{\alpha\in\mathcal{R}(\varphi)}\bigvee_{\beta\in\mathcal{R}(\psi)}(\alpha \vee \beta))$. Similarly, for an arbitrary $u \in \sigma_a^{\nleq}(w)$ we know $\{u\} \in \Sigma_a^{\nleq}(w)$, whence $M, u \vDash \beta$ for some $\beta \in \mathcal{R}(\psi)$. So, $M, u \vDash (\bigvee_{\alpha\in\mathcal{R}(\varphi)}\bigvee_{\beta\in\mathcal{R}(\psi)}(\alpha \vee \beta))$.

Therefore, as $\sigma_a(w) = \sigma_a^{\leq}(w) \cup \sigma_a^{\nleq}(w)$ by condition 5 on IPNMs we know that for all $v \in \sigma_a(w), M, v \vDash (\bigvee_{\alpha\in\mathcal{R}(\varphi)}\bigvee_{\beta\in\mathcal{R}(\psi)}(\alpha \vee \beta))$. By proposition 1.2.15 this means that for all $t \in \Sigma_a(w), M, t \vDash (\bigvee_{\alpha\in\mathcal{R}(\varphi)}\bigvee_{\beta\in\mathcal{R}(\psi)}(\alpha \vee \beta))$, as $\forall t \in \Sigma_a(w), t \subseteq \sigma_a(w)$. So, $M, w \vDash E_a(\bigvee_{\alpha\in\mathcal{R}(\varphi)}\bigvee_{\beta\in\mathcal{R}(\psi)}(\alpha \vee \beta))$.

9. $E_a^{\leq}\alpha \to \alpha$

Suppose $M, w \vDash E_a^{\leq}\alpha$. Then, it is the case that $\forall t \in \Sigma_a^{\leq}(w), M, t \vDash \alpha$. We know by condition 3 that $w \in \sigma_a^{\leq}(w)$, whence $\{w\} \in \Sigma_a^{\leq}(w)$, and so it is immediate that $M, w \vDash \alpha$.

10. $(E_a^{\leq}\varphi \wedge E_a^{\not\leq}\neg\alpha) \to E_a(\alpha \to \varphi)$

Suppose $M, w \vDash E_a^{\leq}\varphi \wedge E_a^{\not\leq}\neg\alpha$, while $M, w \nvDash E_a(\alpha \to \varphi)$. From this it follows that $\forall t \in \Sigma_a^{\leq}(w), M, t \vDash \varphi$, $\forall t' \in \Sigma_a^{\not\leq}(w), M, t' \vDash \neg\alpha$ and $\exists t'' \in \Sigma_a(w), M, t'' \vDash \alpha$ while $M, t'' \nvDash \varphi$.

By the fact that $M, t'' \nvDash \varphi$ we know that $t'' \neq \emptyset$. So, there is some $v \in t''$ such that $M, v \nvDash \varphi$ by proposition 1.2.15, while by persistence $M, v \vDash \alpha$. Furthermore, as $\Sigma_a^{\not\leq}(w) \subseteq [\![\neg\alpha]\!]$ by the support condition for $E_a^{\not\leq}$, we know $\sigma_a^{\not\leq}(w) \subseteq |\neg\alpha|$, whence $t'' \cap \sigma_a^{\not\leq}(w) = \emptyset$. However, as $t'' \in \Sigma_a(w)$ we know that $t \subseteq \sigma_a(w)$. So, as $\sigma_a = \sigma_a^{\leq}(w) \cup \sigma_a^{\not\leq}(w)$ by condition 5 on IPNMS, we know it must be the case that $t'' \subseteq \sigma_a^{\leq}(w)$. Therefore, as $\Sigma_a^{\leq}(w) = \wp(\sigma_a^{\leq}(w)) \cap \Sigma_a(w)$ we can infer $t'' \in \Sigma_a^{\leq}(w)$. So, as $\forall t \in \Sigma_a^{\leq}(w), M, t \vDash \varphi$ we know $M, t'' \vDash \varphi$ via persistence, a contradiction.

11. $(E_a^{\not\leq}\varphi \wedge E_a^{\leq}\neg\alpha) \to E_a(\alpha \to \varphi)$

Analogous to the previous axiom. □

7.4 Completeness

We are in a position to establish completeness for IPNMS. The method taken to establish completeness of IPL with respect to IPNMS follows that of ICDL with respect to ICDMS, from chapter 6.

7.4.1 Preliminaries

We will not restate the preliminaries for completeness for ICDL from chapter 6, as these straightforwardly carry over to IPL. We note only the adjustment required to adapt the definition of the finite fragment of ICDL to that of IPL, arising from a difference in the modal operators between the two languages, in the definition of a subformula.

Definition 7.4.1 (Subformulas). Let $\mathcal{F}$ be a set of formulas, we define $sub(\mathcal{F})$ to be the smallest set satisfying the following conditions:

1. If $\varphi \circ \psi \in \mathcal{F}$ then $\varphi, \psi \in sub(\mathcal{F})$ for $\circ \in \{\wedge, \to\}$.
2. If $?\{\alpha_1, \ldots, \alpha_n\} \in \mathcal{F}$, then $\alpha_1, \ldots, \alpha_n \in sub(\mathcal{F})$.
3. If $E_a^{\Box}\varphi \in \mathcal{F}$, then $\varphi \in sub(\mathcal{F})$, for $\Box \in \{\leq, \not\leq, \cdot\}$.

As in ICDL to establish completeness for IPL we begin by taking a finite set of formulas $\mathcal{F}$ from IPL which then close under successive operations.

As definition 6.2.10 makes no reference to ICDL we apply the same method to define the finite fragment of IPL, $\mathfrak{F}$, via successive closures on a finite set of formulas $\mathcal{F}$.

The fragment of IPL, $\mathfrak{F}$, is then used as a basis for the construction of nuclei and then atoms of the canonical model, as in ICDL.

As the syntactic characterisation of atoms, states, and propositions for fragments of ICDL makes no specific reference to the language of ICDL, nor to $\mathfrak{F}$, the definitions and lemmas established in section 6.2.1 carry over to the fragment of IPL, as do the relevant lemmas of section 6.3.

7.4.2 Constructing the Canonical Model

Definition 7.4.2 (Canonical model for IPL over $\mathcal{F}$). The canonical model for IPL is the tuple: $M^{\mathcal{F}} = \langle W, \{\Sigma_a^{\leq}\}_{a\in\mathcal{A}}, \{\Sigma_a^{\not\leq}\}_{a\in\mathcal{A}}\{\Sigma_a\}_{a\in\mathcal{A}}, V\rangle$, defined as follows:

- W is the set of complete theories of declaratives of IPL.
- $V(A) = \{p \in \mathsf{At} \mid p \in A\}$
- For $\Box \in \{\leq, \not\leq, \cdot\}$, $\Sigma_a^{\Box}(A)$ is the set of states $S \subseteq W$ defined by:

$$S \in \Sigma_a^{\Box}(A) \iff \bigcap S \vdash \varphi \text{ whenever } A \vdash E_a^{\Box}\varphi$$

7.4.3 The Support Lemma

Lemma 7.4.3. *For $A \in At(\mathcal{F})$, $\varphi \in \mathfrak{F}$, if $A \nvdash E_a^{\Box}\varphi$ then $\exists S \in \Sigma_a^{\Box}(A): \bigcap S \nvdash \varphi$, for $\Box \in \{\leq, \not\leq, \cdot\}$.*

Proof. Analogous to lemma 6.3.3. □

Lemma 7.4.4 (Support lemma for the canonical model over $\mathcal{F}$). *For a set of formulas $\mathcal{F}$ and the canonical model over $\mathcal{F}$, $M^{\mathcal{F}}$, for any $S \subseteq At(\mathcal{F})$ and any $\varphi \in \mathfrak{F}$;*

$$M^{\mathcal{F}}, S \vDash \varphi \iff \bigcap S \vdash \varphi$$

Proof. Proof of the support lemma for IPL differs from the the support lemma for ICDL only with respect to modal formulas, therefore we consider only these steps.

I4) $\phi := E_a\varphi$

From left to right suppose $M^{\mathcal{F}}, S \vDash E_a\varphi$. Then $\forall A \in S, \forall T \in \Sigma_a(A), M^{\mathcal{F}}, T \vDash \varphi$, whence by the induction hypothesis we have $\forall A \in S, \forall T \in \Sigma_a(A), \bigcap T \vdash \varphi$. So, by lemma 7.4.3 we have $\forall A \in S, A \vdash E_a\varphi$, whence $\bigcap S \vdash E_a\varphi$.

From right to left suppose $\bigcap S \vdash E_a\psi$, so $\forall A \in S, A \vdash E_a\psi$. Therefore, as $T \in \Sigma_a(A) \iff \bigcap T \vdash \chi$ whenever $E_a\chi \in A$ we can infer by the induction hypothesis that $\forall T \in \Sigma_a(A), M, T \vDash \psi$, whence $M, A \vDash E_a\psi$ for all $A \in S$, whence $M, S \vDash E_a\psi$.

I5) $\phi := E_a^{\leq}\varphi$

Analogous to the case for the entertains modality.

I6) $\phi := E_a^{\not\leq}\varphi$

Analogous to the case for the entertains modality. □

It is certainly not clear that the canonical model for IPL is an IPNM, as it appears plausible that for some $A, B \in \mathrm{At}(\mathcal{F})$, $B \in \sigma_a^{\leq}(A) \cap \sigma_a^{\not\leq}(A)$—it is certainly possible for some formula φ that $A \vdash \langle E_a^{\leq}\rangle\varphi \wedge \langle\not\leq_a\rangle\varphi$, whence it seems that $A \vdash \langle E_a^{\leq}\rangle\gamma_B \wedge \langle\not\leq_a\rangle\gamma_B$ cannot be excluded. We therefore define a transformation of any given canonical IPNM into a regular IPNM.

Definition 7.4.5 (Regular IPL model over $\mathcal{F}$). Given the canonical IPL model over $\mathcal{F}$

$$M = \langle W, \{\Sigma_a^{\leq}\}_{a\in\mathcal{A}}, \{\Sigma_a^{\not\leq}\}_{a\in\mathcal{A}}\{\Sigma_a\}_{a\in\mathcal{A}}, V\rangle$$

we define the regular model for IPL as the tuple:

$$M' = \langle W', \{\Sigma_a'^{\leq}\}_{a\in\mathcal{A}}, \{\Sigma_a'^{\not\leq}\}_{a\in\mathcal{A}}\{\Sigma_a'\}_{a\in\mathcal{A}}, V'\rangle$$

such that for every $B \in \sigma_a^{\leq}(A) \cap \sigma_a^{\nleq}(A)$ we create two copies, $B^{\leq}$ and $B^{\nleq}$ distinguished by adding some set-theoretic element outside of IPL to B to obtain $B^{\leq}$ and omitting it from $B^{\nleq}$.

We omit B from W' in place of $B^{\leq}$, $B^{\nleq}$, which are then defined to belong to $\sigma_a^{\leq}(A)$ and $\sigma_a^{\nleq}(A)$ respectively, for any A such that $B \in \sigma_a^{\leq}(A) \cap \sigma_a^{\nleq}(A)$ and with $\Sigma_a(A)$, $\Sigma_a^{\leq}(A)$, and $\Sigma_a^{\nleq}(A)$ modified accordingly. We then define $\Sigma_a^{\leq}(B^{\leq})$ as $\Sigma_a^{\leq}(B)$ with $B^{\leq}$ in place of B, and $\Sigma_a^{\nleq}(B^{\nleq})$ as $\Sigma_a^{\nleq}(B)$. With both $B^{\leq}$ and $B^{\nleq}$ in place of B we then revise the valuation function V' accordingly, following the definition of V from the canonical model.

The rest of the canonical model remains unchanged.

Lemma 7.4.6 (Support lemma for the regular model over $\mathcal{F}$)**.** *For a set of formulas $\mathcal{F}$ and the regular model over $\mathcal{F}$, $M^{\mathcal{F}}$, for any $S \subseteq At(\mathcal{F})$ and any $\varphi \in \mathfrak{F}$;*

$$M^{\mathcal{F}}, S \vDash \varphi \Longleftrightarrow \bigcap S \vdash \varphi$$

Proof. Inherited from the canonical model. For, it is straightforward to see that the changes made to the canonical modal are purely cosmetic—duplicating certain atoms, which the fragment of IPL cannot distinguish between, to ensure the intersection of certain sets are indeed empty. □

Corollary 7.4.7. *For all $\varphi \in \mathfrak{F}$, any $S \subseteq At(\mathcal{F})$, and $\Box \in \{\leq, \nleq, \cdot\}$:*

$$\bigcap S \vdash E_a^{\Box}\varphi \textit{ iff } M^{\mathcal{F}}, S \vDash E_a^{\Box}\varphi$$

Proof. If $S = \emptyset$ the result is trivial, so we assume $S = \{A_1, \ldots, A_n\}$ for $A_i \in \mathrm{At}(\mathcal{F})$.

From left to right suppose $\bigcap S \vdash E_a^{\Box}\varphi$. Then, as $S = \{A_1, \ldots, A_n\}$ we know that $\bigcap S \subseteq A_i$ for $i \leq n$, whence $A_i \vdash E_a^{\Box}\varphi$ for all $i \leq n$.

Therefore, by definition of $E_a^{\Box}\varphi$ we know that $T \in \Sigma_a^{\Box}(A_i)$ iff $\bigcap T \vdash \varphi$. Therefore, as $\varphi \in \mathfrak{F}$ we know by the support lemma that $\bigcap T \vdash \varphi$ iff $M^{\mathcal{F}}, T \vDash \varphi$, whence we infer $M^{\mathcal{F}}, A_i \vDash E_a^{\Box}\varphi$. As A_i was arbitrary this holds for all $i \leq n$, and so via proposition 1.2.15 we can infer that $M^{\mathcal{F}}, S \vDash E_a^{\Box}\varphi$.

As each inference appealed to used one direction of an equivalence the left to right direction follows easily from the same reasoning in reverse. □

Lemma 7.4.8. *The regular model is an* IPNM.

Proof.

1 if $B \in \sigma_a^{\leq}(A)$ and $T \in \Sigma_a^{\leq}(B)$, then $T \in \Sigma_a^{\leq}(A)$

Suppose $B \in \sigma_a^{\leq}(A)$ and $T \in \Sigma_a^{\leq}(B)$. As $T \in \Sigma_a^{\leq}(B)$ we know that $\bigcap T \vdash \psi$ whenever $B \vdash E_a^{\leq}\psi$. And, as $B \in \sigma_a^{\leq}(A)$, whence $\{B\} \in \Sigma_a^{\leq}(A)$, which means that $B \vdash \psi$ whenever $A \vdash E_a^{\leq}\psi$. Our task is to show that $\bigcap T \vdash \chi$ whenever $A \vdash E_a^{\leq}\chi$, so suppose $A \vdash E_a^{\leq}\chi$. By axiom 4 it follows that $A \vdash E_a^{\leq}E_a^{\leq}\chi$, whence $B \vdash E_a^{\leq}\chi$, and so $\bigcap T \vdash \chi$.

2 if $B \in \sigma_a^{\nleq}(A)$ and $T \in \Sigma_a^{\nleq}(B)$, then $T \in \Sigma_a^{\nleq}(A)$

Analogous to the previous case, using axiom axiom 5.

3 $A \in \sigma_a^{\leq}(A)$

Suppose $A \notin \sigma_a^{\leq}(A)$. Then $M^{\mathcal{F}}, A \vDash E_a^{\leq}\neg\gamma_A$.

For, if $\Sigma_a^{\leq}(A) = \{\emptyset\}$ then $M^{\mathcal{F}}, A \vDash E_a^{\leq}\varphi$ for any φ. And, if $\Sigma_a^{\leq}(A) \neq \{\emptyset\}$ then, as we know that $A \notin \sigma_a^{\leq}(A)$, and $\gamma_A \in B$ iff $B = A$ by lemma 6.2.26, it must be the case that $M^{\mathcal{F}}, B \vDash \neg\gamma_A$ for all $B \in \sigma_a^{\leq}(A)$. So, by proposition 1.2.15 $M^{\mathcal{F}}, S \vDash \neg\gamma_A$ for all $S \in \Sigma_a^{\leq}(A)$, whence $M^{\mathcal{F}}, A \vDash E_a^{\leq}\neg\gamma_A$. Therefore, by corollary 7.4.7 we know that $A \vdash E_a^{\leq}\neg\gamma_A$.

So, by axiom 9 we have $\vdash E_a^{\leq}\neg\gamma_A \to \neg\gamma_A$. Therefore, as $A \vdash E_a^{\leq}\neg\gamma_A$ we have that $A \vdash \neg\gamma_A$ by modus ponens. However, we know by IPL counterpart to lemma 6.2.26 and proposition 6.2.22 that $A \vdash \gamma_A$. A contradiction.

4 $\sigma_a^{\leq}(A) \cap \sigma_a^{\nleq}(A) = \emptyset$

By construction.

5 $\sigma_a^{\leq}(A) \cup \sigma_a^{\nleq}(A) = \sigma_a(A)$

From left to right assume $B \in \sigma_a^{\leq}(A) \cup \sigma_a^{\nleq}(A)$. Then it is either the case that $B \vdash \psi$ whenever $A \vdash E_a^{\leq}\psi$ or $B \vdash \psi$ whenever $A \vdash E_a^{\nleq}\psi$. Without loss of generality let us suppose the latter is the case. To show $B \in \sigma_a(A)$ it is sufficient to show that $B \vdash \chi$ whenever $A \vdash E_a\chi$.

By axiom 7, $\vdash E_a\chi \to (E_a^{\leq}\chi \wedge E_a^{\nleq}\chi)$, whence $A \vdash E_a\chi \to (E_a^{\leq}\chi \wedge E_a^{\nleq}\chi)$. So, if $A \vdash E_a\chi$ then $A \vdash E_a^{\nleq}\chi$, whence $B \vdash \chi$. Therefore, it is the case that $B \vdash \chi$ whenever $A \vdash E_a\chi$.

From right to left suppose $B \in \sigma_a(A)$, while $B \notin \sigma_a^{\leq}(A) \cup \sigma_a^{\nleq}(A)$. So, $B \notin \sigma_a^{\leq}(A)$ and $B \notin \sigma_a^{\nleq}(A)$, whence $\{B\} \notin \Sigma_a^{\leq}(A)$ and $\{B\} \notin \Sigma_a^{\nleq}(A)$.

So, $B \vdash \psi$ whenever $A \vdash E_a\psi$, but for some χ_1, χ_2, $A \vdash E_a^{\leq}\chi_1$ and $A \vdash E_a^{\nleq}\chi_2$, yet $B \nvdash \chi_1$ and $B \nvdash \chi_2$.

However, by axiom 8 it follows that $A \vdash E_a(\bigvee_{\alpha\in\mathcal{R}(\chi_1)}\bigvee_{\beta\in\mathcal{R}(\chi_2)}(\alpha \vee \beta))$, whence $B \vdash (\bigvee_{\alpha\in\mathcal{R}(\chi_1)}\bigvee_{\beta\in\mathcal{R}(\chi_2)}(\alpha \vee \beta))$. As B is an atom it has the disjunction property by proposition 6.2.23. So, $B \vdash \alpha \vee \beta$, for some $\alpha \in \mathcal{R}(\chi_1)$ and $\beta \in \mathcal{R}(\chi_2)$. Appealing again to the fact that B has the disjunction property we know that $B \vdash \alpha$ for $\alpha \in \mathcal{R}(\chi_1)$ or $B \vdash \beta$ for $\beta \in \mathcal{R}(\chi_2)$. Therefore, by theorem 6.2.7 we know that either $B \vdash \chi_1$ or $B \vdash \chi_2$, a contradiction.

6 if $B \in \sigma_a(A)$, then $\Sigma_a(A) = \Sigma_a(B)$

This comes via the introspection axioms for the entertains modality, paralleling the proof for introspection with respect to considers on ICDMs.

7 $\Sigma_a^{\leq}(A) = (\wp(\sigma_a^{\leq}(A)) \cap \Sigma_a(A))$

From left to right, if $T \in \Sigma_a^{\leq}(A)$ then $T \subseteq \sigma_a^{\leq}(A)$, and so $T \in \wp(\sigma_a^{\leq}(A))$. Furthermore, $\bigcap T \vdash \psi$ whenever $A \vdash E_a^{\leq}$. Suppose $A \vdash E_a\psi$ then by axiom 7 and the elimination rule for conjunction, $A \vdash E_a^{\leq}\psi$, whence $\bigcap T$. So, $\bigcap T \vdash \psi$ whenever $A \vdash E_a\psi$, and so $T \in \Sigma_a(A)$. Therefore, $T \in \wp(\sigma_a^{\leq}(A)) \cap \Sigma_a(A)$.

From right to left, let $T \in \wp(\sigma_a^{\leq}(A)) \cap \Sigma_a(A)$ be arbitrary.

We begin by establishing that $A \vdash E_a^{\nleq}\neg\gamma_T$. For, as $T \in \Sigma_a(A)$ we know $\bigcap T \vdash \psi$ whenever $A \vdash E_a\psi$, and as $T \in \wp(\sigma_a^{\leq}(A))$ we know $T \subseteq \sigma_a^{\leq}(A)$. Therefore, by condition 4 on IPNMs, proved to be satisfied by $M^{\mathcal{F}}$ above, we know $T \cap \sigma_a^{\not\leq}(A) = \emptyset$, whence $T \notin \wp(\sigma_a^{\not\leq}(A))$. So, as $\Sigma_a^{\not\leq}(A) \subseteq \wp(\sigma_a^{\not\leq}(A))$ it follows that $T \notin \Sigma_a^{\not\leq}(A)$.

As $B \vdash \gamma_T$ iff $B \in T$ by lemma 6.2.29, this means that $\forall B \in \sigma_a^{\not\leq}(A), B \nvdash \gamma_T$, whence $B \vdash \neg\gamma_T$, as B is a maximally consistent set of declaratives. By the support lemma, then, as $\neg\gamma_T \in \mathfrak{D}$, we know that for all $B \in \sigma_a^{\not\leq}(A)$, $M^{\mathcal{F}}, B \vDash \neg\gamma_T$. So, by lemma 1.2.15 we know that for all $T \in \Sigma_a^{\not\leq}(A)$, $M^{\mathcal{F}}, T \vDash \neg\gamma_T$, and so $M^{\mathcal{F}}, A \vDash E_a^{\not\leq}\neg\gamma_T$. Therefore, by the support lemma we know $A \vdash E_a^{\not\leq}\neg\gamma_T$.

We now show that if $A \vdash E_a^{\leq}\varphi$ then $\bigcap T \vdash \varphi$. For, suppose $A \vdash E_a^{\leq}\varphi$ for an arbitrary $\varphi \in \mathcal{F}$. By the previously established fact that $A \vdash E_a^{\not\leq}\neg\gamma_T$ and the introduction rule for conjunction we have $A \vdash E_a^{\leq}\varphi \wedge E_a^{\not\leq}\neg\gamma_T$ So, as an instantiation of axiom 10 we have $A \vdash (E_a^{\leq}\varphi \wedge E_a^{\not\leq}\neg\gamma_T) \to E_a(\gamma_T \to \varphi)$, from which we infer $A \vdash E_a(\gamma_T \to \varphi)$. Therefore, if $S \in \Sigma_a(A)$ then $\bigcap S \vdash \gamma_T \to \varphi$.

As we know $\bigcap T \in \Sigma_a(A)$ this means that $\bigcap T \vdash \gamma_T \to \varphi$, whence as $\bigcap T \vdash \gamma_T$ by lemma 6.2.32 we know that $\bigcap T \vdash \varphi$. Therefore, we have show that if $A \vdash E_a^{\leq}\varphi$ then $T \in \Sigma_a^{\leq}(A)$. And, as T was an arbitrary element of $\wp(\sigma_a^{\leq}(A)) \cap \Sigma_a(A)$ this establishes $\wp(\sigma_a^{\leq}(A)) \cap \Sigma_a(A) \subseteq \Sigma_a^{\leq}(A)$.

8 $\boldsymbol{\Sigma_a^{\not\leq}(A) = (\wp(\sigma_a^{\not\leq}(A)) \cap \Sigma_a(A))}$

Proof is analogous to the previous condition using axiom 11. □

Theorem 7.4.9 (Completeness of IPL wrt. IPNMs.). *IPL is weakly complete with respect to IPNMs.*

Proof. As with completeness of ICDL with respect to ICDMs, theorem 6.4.1. □

7.5 Connexions Between IPMs and IPNMs

We now turn to establishing soundness and completeness of IPL with respect to IPMs. This follows the same strategy as chapter 5, by defining transformations between the class of IPNMs and the class of IPMs that preserve the interpretation of IPL.

7.5.1 From IPNMs to IPMs

Theorem 7.5.1 (From IPNMs to IPMs). *Any finite IPNM can be transformed into an IPM preserving the interpretation of IPL.*

Definition 7.5.2 (Map from IPNMs to IPMs).

Given an arbitrary IPNM, $M = \langle W, \{\Sigma_a^{\leq}\}, \{\Sigma_a^{\not\leq}\}, \{\Sigma_a\}, V\rangle$ we define a map $M \mapsto M^\sharp$, where $M^\sharp = \langle W^\sharp, \{\leq_a^w\}_{a\in\mathcal{A}, w\in W^\sharp}, \{\Sigma_a\}_{a\in\mathcal{A}}, V^\sharp\rangle$ is constructed in the following way:

1. $W^\sharp := W$
2. $v \leq_a^w u$ if $v, u \in \sigma_a(w)$ and $v \in \sigma_a^{\leq}(u)$
3. $\Sigma_a^\sharp := \Sigma_a$
4. $V^\sharp := V$

Lemma 7.5.3. *For every finite IPNM M, $M^\sharp$ is an IPM.*

Proof. Note in the following $\sigma_a^\sharp(w) := \bigcup \Sigma_a^\sharp(w)$.

Factivity: $w \in \sigma_a^\sharp(w)$, for all $w \in W$

We know that $w \in \sigma_a^\leq(w)$, by condition 3 on IPNMS. Furthermore, from this it follows that $w \in \sigma_a(w)$, by condition 5. So, as $w \in \sigma_a(w)$ and $w \in \sigma_a^\leq(w)$ we have $w \leq_a^w w$, by definition of $\leq_a^w$, whence $w \in \sigma_a^\sharp(w)$.

Introspection 1: if $v \in \sigma_a^\sharp(w)$, then $\Sigma_a^\sharp(w) = \Sigma_a^\sharp(v)$

Suppose $v \in \sigma_a^\sharp(w)$. Then $v \leq_a^w u$ for some u. So, it must be the case that $v \in \sigma_a(w)$ and $v \in \sigma_a^\leq(u)$. And, as $v \in \sigma_a(w)$ we know that $\Sigma_a(w) = \Sigma_a(v)$ by condition 6 on IPNMS, but then $\Sigma_a^\sharp(w) = \Sigma_a^\sharp(v)$ by definition.

Introspection 2: if $v \in \sigma_a^\sharp(w)$, then $x \leq_a^v y$ if and only if $x \leq_a^w y$

Suppose $v \in \sigma_a^\sharp(w)$, then $v \leq_a^w u$ for some u, and so $v \in \sigma_a(w)$ and $v \leq u$. By the former observation we note that by condition 6 on IPNMS, $\sigma_a(w) = \sigma_a(v)$.

Now from right to left, if $x \leq_a^w y$ this means that $x, y \in \sigma_a(w)$ and $x \in \sigma_a^\leq(y)$. So, as $x, y \in \sigma_a(w)$ and $\sigma_a(w) = \sigma_a(v)$ we know that $x, y \in \sigma_a(v)$. Therefore, as $x, y \in \sigma_a(v)$ and $x \in \sigma_a^\leq(y)$ we have $x \leq_a^v y$ by definition of $\leq_a^v$.

For the left to right direction the argument proceeds analogously.

We begin by showing the ordering defined is a well-preorder.

Transitivity

Suppose $x \leq_a^w y$ and $y \leq_a^w z$. This means that $x, y, z \in \sigma_a(w)$ and $x \in \sigma_a^\leq(y)$ and $y \in \sigma_a^\leq(z)$.

Now, as $x \in \sigma_a^\leq(y)$ we know $\Sigma_a^\leq(x) \subseteq \Sigma_a^\leq(y)$, and as $y \in \sigma_a^\leq(z)$ we know $\Sigma_a^\leq(y) \subseteq \Sigma_a^\leq(z)$ by condition 1 on IPNMS. Therefore, by transitivity of the subset relation we know that $\Sigma_a^\leq(x) \subseteq \Sigma_a^\leq(z)$. Furthermore, by condition 3 we know that $x \in \sigma_a^\leq(x)$, whence $\{x\} \in \Sigma_a^\leq(x)$, from which it follows by the previous observation that $\{x\} \in \Sigma_a^\leq(z)$, and so $x \in \sigma_a^\leq(z)$. Therefore, $x \leq_a^w z$ by definition of $\leq_a^w$.

Reflexivity Follows as a corollary of factivity established above.

For every set $s \subseteq \{v \mid \exists u \colon v \leq_a^w u\}$ there exists $v \in s$ such that $v \leq_a^w u$ for all $u \in s$

Suppose for some set $s \subseteq \{v \mid \exists u \colon v \leq_a^w u\}$ that for every $v \in s$ there exists some $u \in s$ such that $v \not\leq_a^w u$.

Let $v \in s$ be arbitrary, and instantiate u such that $v \not\leq_a^w u$. As $v \not\leq_a^w u$ it must be the case that either $v, u \notin \sigma_a(w)$ or $v \notin \sigma_a^\leq(u)$.

So, as $v \in s$ we know that $\exists x \colon v \leq_a^w x$, and so by definition of $\leq_a^w$ this means that $v, x \in \sigma_a(w)$ and $v \in \sigma_a^\leq(x)$. Analogous reasoning applies to u, and therefore we know that $v, u \in \sigma_a(w)$. From this it follows that $v \notin \sigma_a^\leq(u)$. And, as $v, u \in \sigma_a(w)$ it follows by condition 6 on IPNMS that $\sigma_a(v) = \sigma_a(w) = \sigma_a(u)$, and therefore we know that $v \in \sigma_a(u)$ and $u \in \sigma_a(v)$.

Now, as $v \in \sigma_a^{\not\leq}(u)$ we know that $\Sigma_a^\leq(v) \subseteq \Sigma_a^\leq(u)$ by condition 2. From this it follows that $u \notin \sigma_a^{\not\leq}(v)$. For, suppose otherwise. Then, as $u \in \sigma_a^{\not\leq}(v)$ we know $\{u\} \in \Sigma_a^{\not\leq}(v)$, and therefore $\{u\} \in \Sigma_a^{\not\leq}(u)$, which entails $u \in \sigma_a^{\not\leq}(u)$. However, we know by condition 3 on IPNMs that $u \in \sigma_a^{\leq}(u)$ and by condition 4 that $\sigma_a^{\leq}(u) \cap \sigma_a^{\not\leq}(u) = \emptyset$. Therefore, we have derived a contradiction.

So, as $u \notin \sigma_a^{\not\leq}(v)$ while $u \in \sigma_a(v)$ we know by condition 5 that $u \in \sigma_a^{\leq}(v)$. Therefore, by definition of $\leq_a^w$ we have that $u <_a^w v$, and furthermore we know $u \neq v$ as $v \leq_a^w v$ by the fact that $\leq_a^w$ is reflexive, as established above.

As $v \in s$ was arbitrary we have shown that for any $v \in s$ there exists some $u \neq v$ such that $u <_a^w v$. Yet, we know that s is finite as W is finite. Therefore, for some y it must be the case that there is no z such that $z <_a^w y$, whence we have derived a contradiction.

We now show the remaining properties required for $M^\sharp$ to be an IPM are satisfied.

$\Sigma_a^\sharp(w)$ is an issue over $\sigma_a^\sharp(w)$, where $\sigma_a^\sharp(w) := \{v \mid v \leq_a^w u \text{ for some u}\}$

We have defined $\Sigma_a^\sharp(w)$ as $\Sigma_a(w)$, and as $\Sigma_a(w)$ is an issue over $\sigma_a(w)$, it will suffice to show that $\sigma_a^\sharp(w) = \sigma_a(w)$.

So, from left to right suppose $v \in \sigma_a^\sharp(w)$. Then, $v \leq_a^w u$ for some u. So, given the definition of $\leq_a^w$ it must be the case that $v, u \in \sigma_a(w)$ and $v \in \sigma_a^{\leq}(u)$ for some u, whence by the former conjunct $v \in \sigma_a(w)$. Therefore, $\sigma_a^\sharp(w) \subseteq \sigma_a(w)$.

Conversely, if $v \in \sigma_a(w)$ then by condition 3 that $v \in \sigma_a^{\leq}(v)$, and therefore as $v \in \sigma_a(w)$ and $v \in \sigma_a^{\leq}(v)$ it follows by definition of $\leq_a^w$ that $v \leq_a^w v$, whence $v \in \sigma_a^\sharp(w)$. Therefore, $\sigma_a(w) \subseteq \sigma_a^\sharp(w)$. □

Lemma 7.5.4. *Given an arbitrary IPNM M and a corresponding IPM $M^\sharp$:*

1. $\Sigma_a^{\leq}(w) = \Sigma_a^{\sharp,\leq}(w)$
2. $\Sigma_a^{\not\leq}(w) = \Sigma_a^{\sharp,\not\leq}(w)$

Proof. Recall $\Sigma_a^{\leq}(w) = (\wp(\sigma_a^{\leq}(w)) \cap \Sigma_a(w))$ and $\Sigma_a^{\not\leq}(w) = (\wp(\sigma_a^{\not\leq}(w)) \cap \Sigma_a(w))$, while $\Sigma_a^{\sharp,\leq}(w) := \wp(\sigma_a^{\sharp,\leq}(w)) \cap \Sigma_a^\sharp(w)$ and $\Sigma_a^{\sharp,\not\leq}(w) := \wp(\sigma_a^{\sharp,\not\leq}(w)) \cap \Sigma_a^\sharp(w)$. So, as $\Sigma_a^\sharp(w) = \Sigma_a(w)$, showing both $\sigma_a^{\leq}(w) = \sigma_a^{\sharp,\leq}(w)$ and $\sigma_a^{\leq}(w) = \sigma_a^{\sharp,\leq}(w)$ will be sufficient to establish the lemma. Furthermore, recall $\sigma_a^{\sharp,\leq}(w) := \{v \in \sigma_a^\sharp(w) \mid v \leq_a^w w\}$ and $\sigma_a^{\sharp,\not\leq}(w) := \{v \in \sigma_a^\sharp(w) \mid v \not\leq_a^w w\}$.

1 From left to right suppose $v \in \sigma_a^{\leq}(w)$. Then by condition 5 on IPNMs we know that $v \in \sigma_a(w)$, and be analogous reasoning using condition 3 we know $w \in \sigma_a(w)$. So, as $v, w \in \sigma_a(w)$ and $v \in \sigma_a^{\leq}(w)$, then $v \leq_a^w w$ by definition of $\leq_a^w$, whence $v \in \sigma_a^{\sharp,\leq}(w)$.

Conversely, if $v \in \sigma_a^{\sharp,\leq}(w)$ then it must be the case $v \leq_a^w w$. So, $v \in \sigma_a(w)$ and $v \in \sigma_a^{\leq}(w)$.

2 We know by lemma 7.5.3 that $\sigma_a(w) = \sigma_a^\sharp(w)$, and by the previous case we know $\sigma_a^{\leq}(w) = \sigma_a^{\sharp,\leq}(w)$, so it must be the case that $(\sigma_a(w) - \sigma_a^{\leq}(w)) = (\sigma_a^\sharp(w) - \sigma_a^{\sharp,\leq}(w))$. Therefore, as $(\sigma_a(w) - \sigma_a^{\leq}(w)) = \sigma_a^{\not\leq}(w)$ by condition 5 on IPNMs, and $(\sigma_a^\sharp(w) - \sigma_a^{\sharp,\leq}(w)) = \{v \mid v \not\leq_a^w w\} = \sigma_a^{\sharp,\not\leq}(w)$ we have that $\sigma_a^{\not\leq}(w) = \sigma_a^{\sharp,\not\leq}(w)$. □

Proof of theorem 7.5.1. Let M be an arbitrary IPNM and take $M^{\sharp}$, the IPM constructed from M, given by the mapping defined above. We claim $M, s \vDash \phi$ iff $M^{\sharp}, s \vDash \phi$, for all $s \subseteq W$ and formulas ϕ.

Proof is via induction on the complexity of ϕ, and here we consider only the cases unique to IPL, with the others established as in the proof of theorem 5.2.1.

15) $\phi := E_a^{\leq}\varphi$

By lemma 7.5.4 we know that $\Sigma_a^{\leq}(w) = \Sigma_a^{\sharp,\leq}(w)$. So, $M, s \vDash E_a^{\leq}\varphi$ iff $\forall w \in s, \forall t \in \Sigma_a^{\leq}(w), M, t \vDash \varphi$ iff (by the induction hypothesis and noted lemma) $\forall w \in s, \forall t \in \Sigma_a^{\sharp,\leq}(w), M^{\sharp}, t \vDash \varphi$ iff $M^{\sharp}, s \vDash E_a^{\leq}\varphi$.

16) $\phi := E_a^{\nleq}\varphi$

Analogous to the previous case.

17) $\phi := E_a\varphi$

Immediate. $\square$

7.6 From IPMs to IPNMs

Theorem 7.6.1 (From IPMs to IPNMs). *Any finite IPM can be transformed into an IPNM preserving the interpretation of IPL.*

Definition 7.6.2 (Map from IPMs to IPNMs).

Given an arbitrary IPM, $M = \langle W, \{\leq_a^w\}_{a \in \mathcal{A}, w \in W}, \{\Sigma_a\}_{a \in \mathcal{A}}, V\rangle$ we define a map $M \mapsto M^{\flat}$, where $M^{\flat} = \langle W^{\flat}, \{\Sigma_a^{\leq,\flat}\}, \{\Sigma_a^{\nleq,\flat}\}, \{\Sigma_a^{\flat}\}, V^{\flat}\rangle$ is constructed in the following way:

1. $W^{\flat} := W$
2. $\Sigma_a^{\leq,\flat}(w) = \wp(\sigma_a^{\leq}(w)) \cap \Sigma_a(w)$
3. $\Sigma_a^{\nleq,\flat}(w) = \wp(\sigma_a^{\nleq}(w)) \cap \Sigma_a(w)$
4. $\Sigma_a^{\flat} := \Sigma_a$
5. $V^{\flat} := V$

Lemma 7.6.3. *Any model $M^{\flat}$, defined as in definition 7.6.2, is an* IPNM.

Proof. Note $\sigma_a^{\leq,\flat}(w) := \bigcup\Sigma_a^{\leq,\flat}(w) = \sigma_a^{\leq}(w) \cap \sigma_a(w) = \sigma_a^{\leq}(w)$, $\sigma_a^{\nleq,\flat}(w) := \bigcup\Sigma_a^{\nleq,\flat}(w) = \sigma_a^{\nleq}(w) \cap \sigma_a(w) = \sigma_a^{\nleq}(w)$, and $\sigma_a^{\flat}(w) := \bigcup\Sigma_a^{\flat}(w) = \sigma_a(w) \cap \sigma_a(w) = \sigma_a(w)$.

Therefore, $\sigma_a^{\leq,\flat}(w) = \sigma_a^{\leq}(w)$, $\sigma_a^{\nleq,\flat}(w) = \sigma_a^{\nleq}(w)$, and $\sigma_a^{\flat}(w) = \sigma_a(w)$.

1. If $v \in \sigma_a^{\leq,\flat}(w)$, then $\Sigma_a^{\leq,\flat}(v) \subseteq \Sigma_a^{\leq,\flat}(w)$

Suppose $v \in \sigma_a^{\leq,\flat}(w)$. So, by the observations above we know $v \in \sigma_a^{\leq}(w)$. Therefore, $v \leq_a^w w$. By expanding definitions, to show that $\Sigma_a^{\leq,\flat}(v) \subseteq \Sigma_a^{\leq,\flat}(w)$ is to show that $(\wp(\sigma_a^{\leq}(v)) \cap \Sigma_a(v)) \subseteq (\wp(\sigma_a^{\leq}(w)) \cap \Sigma_a(w))$. To do this we show 1. $\Sigma_a(v) = \Sigma_a(w)$ and 2. $\sigma_a^{\leq}(v) \subseteq \sigma_a^{\leq}(w)$. From these two facts the result is immediate.

1. As $v \leq_a^w w$ we know that $v \in \sigma_a(w)$, and therefore by introspection 1 on IPMs we know that $\Sigma_a(v) = \Sigma_a(w)$.

2. By definition, $\sigma_a^{\leq}(v) = \{x \mid x \leq_a^v v\}$ and $\sigma_a^{\leq}(w) = \{y \mid y \leq_a^w w\}$. Furthermore, we know that $v \leq_a^w w$, so $v \in \sigma_a(w)$, whence by the condition of introspection 2 on IPMs we know that $x \leq_a^v y$ iff $x \leq_a^w y$.
   So, let $u \in \sigma_a^{\leq}(v)$ be arbitrary. Then $u \leq_a^v v$, whence $u \leq_a^w v$, by the previous observation. And, as $v \leq_a^w w$ it follows by the transitivity of the ordering that $u \leq_a^w w$, whence $u \in \sigma_a^{\leq}(w)$.

2. If $u \in \sigma^{\not\leq,\flat}(w)$, then $\Sigma_a^{\not\leq,\flat}(u) \subseteq \Sigma_a^{\not\leq,\flat}(w)$

Suppose $u \in \sigma^{\not\leq,\flat}(w)$. Therefore, we know that $u \in \sigma_a(w)$ but $u \not\leq_a^w w$, whence $w <_a^w u$.

Following the strategy of the previous condition we observe that by expanding definitions it suffices to show that $\wp(\sigma_a^{\not\leq}(u)) \cap \Sigma_a(u) \subseteq \wp(\sigma_a^{\not\leq}(w)) \cap \Sigma_a(w)$. And, analogously to before we show 1. $\Sigma_a(u) = \Sigma_a(w)$ and 2. $\sigma_a^{\not\leq}(u) \subseteq \sigma_a^{\not\leq}(w)$. For from these two facts the result is immediate.

1. As $u \in \sigma_a(w)$ we know by introspection 1 on IPMs we know that $\Sigma_a(u) = \Sigma_a(w)$.
2. We have established that $\Sigma_a(u) = \Sigma_a(w)$, and so by the condition of introspection 2 on IPMs we know that $x \leq_a^u y$ iff $x \leq_a^w y$. Now, let $v \in \sigma_a^{\not\leq}(u)$ be arbitrary. From this we know $v \not\leq_a^u u$, whence $v \not\leq_a^w u$. So, as $v \not\leq_a^w u$ we know by proposition 7.1.3 that $u \leq_a^w v$, whence $u <_a^w v$. We now have $w <_a^w u$ and $u <_a^w v$. And, as $\leq_a^w$ is a well-preorder it is transitive, so the preceding implies $w <_a^w v$, whence $v \not\leq_a^w w$, and so $v \in \sigma_a^{\not\leq}(w)$.

3. $w \in \sigma^{\leq,\flat}(w)$

We observed above that $\sigma_a^{\leq,\flat}(w) = \sigma_a^{\leq}(w)$. Therefore, as $w \leq_a^w w$ by the fact that $\leq_a^w$ is a well-preorder and hence reflexive, we know $w \in \sigma_a^{\leq}(w)$, and so $w \in \sigma_a^{\leq,\flat}(w)$.

4. $\sigma^{\leq,\flat}(w) \cap \sigma^{\not\leq,\flat}(w) = \emptyset$

As before, we observed above that $\sigma_a^{\leq,\flat}(w) = \sigma_a^{\leq}(w)$ and $\sigma_a^{\not\leq,\flat}(w) = \sigma_a^{\not\leq}(w)$, therefore we need only observe that $\sigma_a^{\leq}(w) \cap \sigma_a^{\not\leq}(w) = \emptyset$. This is established in proposition 7.1.3.

5. $\sigma^{\leq,\flat}(w) \cup \sigma^{\not\leq,\flat}(w) = \sigma_a^{\flat}(w)$

Follows the same reasoning as the previous condition.

6. If $v \in \sigma_a^{\flat}(w)$, then $\Sigma_a^{\flat}(w) = \Sigma_a^{\flat}(v)$

Suppose $v \in \sigma_a^{\flat}(w)$. Then, following the above observations, we know $v \in \sigma_a(w)$, whence $\Sigma_a(v) = \Sigma_a(w)$, by introspection 1. Therefore, $\Sigma_a^{\flat}(v) = \Sigma_a^{\flat}(w)$, by definition.

7. $\Sigma_a^{\leq,\flat}(w) = (\wp(\sigma^{\leq,\flat}(w)) \cap \Sigma_a^{\flat}(w))$

By definition of the transformation $\Sigma_a^{\leq,\flat}(w) = (\wp(\sigma^{\leq}(w)) \cap \Sigma_a(w))$. However, we also know $\sigma_a^{\leq,\flat}(w) = \sigma_a^{\leq}(w)$ by the above observation, and $\Sigma_a^{\flat}(w) = \Sigma_a(w)$ by definition of the transformation. Therefore, the condition is immediate.

8. $\Sigma_a^{\not\leq,\flat}(w) = (\wp(\sigma^{\not\leq,\flat}(w)) \cap \Sigma_a^{\flat}(w))$

Analogous to the previous case. □

Proof of theorem 7.6.3. As in lemma 7.6.3 the proof, proceeding as in theorems 5.1.1, 5.2.1, and 7.5.1, is trivial given the transformation defined in 7.6.2 and therefore is omitted. □

7.7 Results

Theorem 7.7.1 (Soundness of IPL wrt. IPMS.). *IPL is sound with respect to IPMS.*

Proof. We know by theorem 7.3.3 that IPL is sound with respect to IPNMS, and by theorem 7.6.1 that any IPM can be transformed into an IPNM. Therefore, using the reasoning of theorem 5.2.4 we can infer that IPL is sound with respect to IPMS. □

Theorem 7.7.2 (Completeness of IPL wrt. IPMS). *IPL is weakly complete with respect to IPMS.*

Proof. As with completeness of ICDL with respect to IPMS, theorem 6.4.2, using theorem 7.5.1. □

Theorem 7.7.3 (Compactness of IPL). *IPL is not compact.*

Proof. Consider the set of formulas $\{\langle E_a^{\not\leq}\rangle\bot, \langle E_a^{\not\leq}\rangle p_0, \langle E_a^{\not\leq}\rangle(\neg p_0 \wedge p_1), \dots\}$. Clearly any finite subset is satisfiable, yet the whole set cannot be satisfied on any IPNM. □

Our final important result is the decidability of IPL, and by implication ICDL enriched with the entertains modality.

Theorem 7.7.4 (Decidability of IPL). *IPL is decidable.*

Proof. Follows the proof of theorem 6.4.3. □

Corollary 7.7.5. *ICDL enriched with the entertains modality is axiomatisable.*

Proof. We are able to translate any formula in the enriched language to a formula of IPL, whence we can determine whether or not it is a theorem of IPL and hence of the extension of ICDL. □

7.8 Additional Observations

Given we have established completeness of IPL with respect to IPNMS we can use semantic reasoning to prove some additional theorems of IPL.

Proposition 7.8.1 (Further theorems of IPL). *The initial three theorems of IPL are notable as being adaptations of axioms of CO. (Cf. Boutilier (1994, p. 101) axioms S. and H. and K′. respectively.) The final theorem partially characterises the interaction between the two plausibility modalities.*

1. $\varphi \to E_a^{\not\leq}\langle E_a^{\leq}\rangle\varphi$
2. $(\langle E_a^{\leq}\rangle(E_a^{\leq}\varphi \wedge E_a^{\not\leq}\psi) \vee \langle\not\leq_a\rangle(E_a^{\leq}\varphi \wedge E_a^{\not\leq}\psi)) \to$
   $(E_a^{\leq}(\bigvee_{\alpha\in\mathcal{R}(\varphi)}\bigvee_{\beta\in\mathcal{R}(\psi)}(\alpha \vee \beta)) \wedge E_a^{\not\leq}(\bigvee_{\alpha\in\mathcal{R}(\varphi)}\bigvee_{\beta\in\mathcal{R}(\psi)}(\alpha \vee \beta)))$
3. $E_a^{\not\leq}\bot \to (E_a^{\leq}\varphi \to E_a\varphi)$
4. $\langle E_a^{\not\leq}\rangle\varphi \to E_a^{\leq}\langle E_a^{\not\leq}\rangle\varphi$

Proof. We show each theorem is valid on an arbitrary state in an arbitrary IPNM. All but the first theorems are declaratives, and therefore we use proposition 1.2.15 to simply the proofs.

1. $\varphi \to E_a^{\not\leq}\langle E_a^{\leq}\rangle\varphi$

Suppose $M, s \vDash \varphi$ but $M, s \nvDash E_a^{\not\leq}\langle E_a^{\leq}\rangle\varphi$. The latter is shorthand for $M, s \nvDash E_a^{\not\leq}\neg E_a^{\leq}\neg\varphi$, which by proposition 1.2.15 entails that for some $v \in s$ it is the case that $M, v \nvDash E_a^{\not\leq}\neg E_a^{\leq}\neg\varphi$. So, $\exists t \in \Sigma_a^{\not\leq}(v), M, t \nvDash \neg E_a^{\leq}\neg\varphi$. By proposition 1.2.15 it follows that for some world $u \in t, M, u \nvDash \neg E_a^{\leq}\neg\varphi$, whence we infer $M, u \vDash E_a^{\leq}\neg\varphi$. From this it follows that $\forall t' \in \Sigma_a^{\leq}(u), M, t' \vDash \neg\varphi$.

Furthermore, as $u \in \sigma_a^{\not\leq}(v)$ we know that $\sigma_a^{\leq}(v) \subseteq \sigma_a^{\leq}(u)$.

For, as $u \in \sigma_a^{\not\leq}(v)$ we know that $\sigma_a^{\not\leq}(u) \subseteq \sigma_a^{\not\leq}(v)$, by condition 2 on IPNMs. Therefore, if $v \in \sigma_a^{\not\leq}(u)$ it must be the case that $v \in \sigma_a^{\not\leq}(v)$. However, we know that $v \in \sigma_a^{\leq}(v)$ by condition 3, and that $\sigma_a^{\leq}(v) \cap \sigma_a^{\not\leq}(v) = \emptyset$ by condition 4. This is a contradiction, whence $v \notin \sigma_a^{\not\leq}(u)$, but then as $u \in \sigma_a^{\not\leq}(v)$ and $\sigma_a(v) = \sigma_a^{\leq}(v) \cup \sigma_a^{\not\leq}(v)$ by condition 5 on IPNMs it follows that $u \in \sigma_a(v)$. So, by condition 6 we know $\sigma_a(u) = \sigma_a(v)$. So, using conditions 4 and 5 again, we can infer using the fact that $v \notin \sigma_a^{\not\leq}(u)$, that $v \in \sigma_a^{\leq}(u)$. So, from this it follows via condition 1 that $\Sigma_a^{\leq}(v) \subseteq \Sigma_a^{\leq}(u)$, whence $\sigma_a^{\leq}(v) \subseteq \sigma_a^{\leq}(u)$.

We observed above that $v \in \sigma_a^{\leq}(v)$, and so from this and the previous observation we know that $v \in \sigma_a^{\leq}(u)$. Now, we know that $v \in s$, and $M, s \vDash \varphi$, whence $M, v \vDash \varphi$ by persistence. But we also know that $\forall t' \in \Sigma_a^{\leq}(u), M, t' \vDash \neg\varphi$, and as $v \in \sigma_a^{\leq}(u)$, $\{v\} \in \Sigma_a^{\leq}(u)$, whence $M, v \vDash \neg\varphi$. Thus we have derived a contradiction.

2. $(\langle E_a^{\leq}\rangle(E_a^{\leq}\varphi \wedge E_a^{\not\leq}\psi) \vee \langle\not\leq_a\rangle(E_a^{\leq}\varphi \wedge E_a^{\not\leq}\psi)) \to$
$(E_a^{\leq}(\bigvee_{\alpha\in\mathcal{R}(\varphi)}\bigvee_{\beta\in\mathcal{R}(\psi)}(\alpha \vee \beta)) \wedge E_a^{\not\leq}(\bigvee_{\alpha\in\mathcal{R}(\varphi)}\bigvee_{\beta\in\mathcal{R}(\psi)}(\alpha \vee \beta)))$

Suppose it is the case that $M, w \vDash \langle E_a^{\leq}\rangle(E_a^{\leq}\varphi \wedge E_a^{\not\leq}\psi) \vee \langle\not\leq_a\rangle(E_a^{\leq}\varphi \wedge E_a^{\not\leq}\psi)$. As worlds behave classically this entails that either $M, w \vDash \langle E_a^{\leq}\rangle(E_a^{\leq}\varphi \wedge E_a^{\not\leq}\psi)$ or $M, w \vDash \langle\not\leq_a\rangle(E_a^{\leq}\varphi \wedge E_a^{\not\leq}\psi)$. Without loss of generality let us assume the former is the case.

This is shorthand for $M, w \vDash \neg E_a^{\leq}\neg(E_a^{\leq}\varphi \wedge E_a^{\not\leq}\psi)$. So, $\exists t \in \Sigma_a^{\leq}(w)$ such that $M, t \nvDash \neg(E_a^{\leq}\varphi \wedge E_a^{\not\leq}\psi)$. By proposition 1.2.15 it then follows that for some $v \in t, M, v \vDash E_a^{\leq}\varphi \wedge E_a^{\not\leq}\psi$.

We now need to show:
i. $(E_a^{\leq}(\bigvee_{\alpha\in\mathcal{R}(\varphi)}\bigvee_{\beta\in\mathcal{R}(\psi)}(\alpha \vee \beta)))$ and ii. $E_a^{\not\leq}(\bigvee_{\alpha\in\mathcal{R}(\varphi)}\bigvee_{\beta\in\mathcal{R}(\psi)}(\alpha \vee \beta))))$

We know from the above reasoning that for some $\{v\} \in \Sigma_a^{\leq}(w), M, v \vDash E_a^{\leq}\varphi \wedge E_a^{\not\leq}\psi$. As $\{v\} \in \Sigma_a^{\leq}(w)$ it follows that $v \in \sigma_a^{\leq}(w)$. So, by condition 5 on IPNMs we know $v \in \sigma_a(w)$ and therefore by the condition of introspection 1 on IPNMs we know $\Sigma_a(v) = \Sigma_a(w)$. Using this we show $M, w \vDash E_a(\bigvee_{\alpha\in\mathcal{R}(\varphi)}\bigvee_{\beta\in\mathcal{R}(\psi)}(\alpha \vee \beta))))$. From this and an application of axiom 7 the desired result will follow.

So, let $t \in \Sigma_a(w)$ be arbitrary. We know $\Sigma_a(w) = \Sigma_a(v)$, and so $t \in \Sigma_a(v)$. Let $u \in t$ be arbitrary, and note that as $t \in \Sigma_a(v), u \in \sigma_a(v)$. Now, as $u \in \sigma_a(v)$ we know it is the case that $u \in \sigma_a^{\leq}(v)$ or $u \in \sigma_a^{\not\leq}(v)$ by condition 5 on IPNMs. Without loss of generality suppose $u \in \sigma_a^{\leq}(v)$. Then, $\{u\} \in \wp(\sigma_a^{\leq}(v)) \cap \Sigma_a(v)$, whence

$M, u \vDash \varphi$. So, by theorem 6.2.7 we know that $M, u \vDash \alpha$ for some $\alpha \in \mathcal{R}(\varphi)$, whence $M, u \vDash \bigvee_{\alpha \in \mathcal{R}(\varphi)} \bigvee_{\beta \in \mathcal{R}(\psi)} (\alpha \vee \beta)$.

As $u \in \sigma_a(v)$ was arbitrary we know $\forall u \in \sigma_a(v), M, u \vDash \bigvee_{\alpha \in \mathcal{R}(\varphi)} \bigvee_{\beta \in \mathcal{R}(\psi)} (\alpha \vee \beta)$. So, by proposition 1.2.15 we have $\forall t \subseteq \sigma_a(v), M, t \vDash \bigvee_{\alpha \in \mathcal{R}(\varphi)} \bigvee_{\beta \in \mathcal{R}(\psi)} (\alpha \vee \beta)$. Therefore, via persistence and the fact that $\Sigma_a(v) \subseteq \wp(\sigma_a(v))$ we know that $\forall t \in \Sigma_a(v), M, t \vDash \bigvee_{\alpha \in \mathcal{R}(\varphi)} \bigvee_{\beta \in \mathcal{R}(\psi)} (\alpha \vee \beta)$. So, given we have established $\Sigma_a(v) = \Sigma_a(w)$ it is immediate that $M, w \vDash E_a(\bigvee_{\alpha \in \mathcal{R}(\varphi)} \bigvee_{\beta \in \mathcal{R}(\psi)} (\alpha \vee \beta))$

3. $\boldsymbol{E_a^{\not\leq} \bot \to (E_a^{\leq} \varphi \to E_a \varphi)}$

Suppose $M, w \vDash E_a^{\not\leq} \bot$ and $M, w \vDash E_a^{\leq} \varphi$. By the former assumption it must be the case that $\Sigma_a^{\not\leq}(w) = \{\emptyset\}$, whence $\sigma_a^{\not\leq}(w) = \emptyset$ and by condition 5 this entails $\sigma_a^{\leq}(w) = \sigma_a(w)$ and so by condition 7 it follows that $\Sigma_a^{\leq}(w) = \Sigma_a(w)$. So, as $\Sigma_a^{\leq}(w) \subseteq [\![\varphi]\!]$, $\Sigma_a(w) \subseteq [\![\varphi]\!]$, whence $M, w \vDash E_a \varphi$.

4. $\boldsymbol{\langle E_a^{\not\leq} \rangle \varphi \to E_a^{\leq} \langle E_a^{\not\leq} \rangle \varphi}$

Suppose $M, w \vDash \langle E_a^{\not\leq} \rangle \varphi$ while $M, w \nvDash E_a^{\leq} \langle E_a^{\not\leq} \rangle \varphi$. Then, by the latter $M, w \vDash \langle E_a^{\leq} \rangle E_a^{\not\leq} \neg \varphi$. And, by the former, we have that $\exists t \in \Sigma_a^{\leq}(w), M, t \vDash \varphi$ and by the previous inference we have that $\exists v \in \sigma_a^{\leq}(w)$,$M, v \vDash E_a^{\not\leq} \neg \varphi$, whence $\forall t' \in \Sigma_a^{\not\leq}(v)$, $M, t' \vDash \neg \varphi$.

As $v \in \sigma_a^{\leq}(w)$ we know by condition 1 on IPNMs that $\Sigma_a^{\leq}(v) \subseteq \Sigma_a^{\leq}(w)$, whence it is an easy argument to observe that $\Sigma_a^{\not\leq}(w) \subseteq \Sigma_a^{\not\leq}(v)$, given the conditions placed on IPNMs. Therefore $\forall t' \in \Sigma_a^{\not\leq}(v), M, t' \vDash \neg \varphi$ while for some $t \in \Sigma_a^{\not\leq}(v), M, t \vDash \varphi$, so given persistence we have derived a contradiction. $\square$

Chapter 8

Conclusion

In this thesis we have introduced and axiomatised two extensions of propositional inquisitive semantics: inquisitive conditional-doxastic logic and inquisitive plausibility logic. Moreover, we have shown both are sound and complete with respect to the same class of models; inquisitive plausibility models, which allow for an intuitive interpretation of the two logics.

The primary focus of the thesis is conditional-doxastic logic. This generalises conditional-doxastic logic, enabling the study of new propositional attitudes, grounded in familiar assumptions about the process of conditionalisation. For as we observed in chapter 3 (cf. corollary 3.3.11) the interpretation of conditionalisation captured by inquisitive conditional-doxastic logic occurs wholly at the level of declaratives. Therefore, no additional assumptions about the process of conditionalisation are made with the introduction of issues as our core semantic notion. We take the fact that the process of conditionalisation captured by conditional-doxastic logic can be straightforwardly generalised to an inquisitive setting to be the core conceptual achievement of this thesis.

In chapter 3 we gave a preliminary (formal) analysis of the modal operator termed 'considering,' and of the behaviour of (conditional) belief when extended to issues on inquisitive plausibility models. With the formal properties of these modalities established, we hope to explore their potential as formal represent propositional attitudes in future work, and other applications of the logic. We would also like to look at axiomatising the modalities under different assumptions about the semantic properties used to interpret the modalities (e.g., lifting the assumption of negative introspection).

Moreover, as a generalisation of conditional-doxastic logic, inquisitive conditional-doxastic logic forms a basis for extending the standard account of belief revision to an inquisitive setting, where agents may conditionalise on both inquisitive content as well as informative content.

However, the thesis has been largely technical, and we have not explored in any detail the consequences of this result, nor in general applications of ICDL. Still, following research presented by Ciardelli and Roelofsen (2014b) future work may build on ICDL to model issues in epistemic change, allowing greater expressive power when modelling, for example, the process of *contraction*—determining the information to give up when conditionalising on information which is inconsistent with an agents' current doxastic state. Appendix A shows how to axiomatise public announcements with respect to ICDL. One aspect of CDL not touched upon in detail is its relationship to the AGM principles of belief revision (cf. Baltag and Smets 2006, §3). So, given the relationship between CDL and AGM, and between CDL and ICDL, this suggests ICDL may be used as a semantic counterpart to syntactic theory of belief revision extended to interrogatives, akin to AGM theory.

However, in chapter 5 we observed that there are limitations to ICDL, in particular, while the logic is able to capture the issues agents consider when conditionalising on information, it is unable to capture the issues an agent holds unconditionally as the entertains modality of inquisitive epistemic logic is not definable in terms of the considers modality of ICDL. So, in order to obtain a logic in which the interaction between the issues an agent entertains, and those they consider conditional on further issues or information we explored inquisitive plausibility logic.

Inquisitive plausibility logic has sufficient expressive power to express both the entertains modality of IEL and the considers modality of ICDL, while being sound and complete to the same class of models as inquisitive conditional-doxastic logic. Moreover, we observed how IPL allows the expression of binary plausibility (or preference) operators, allowing one, for example, to express when an agent would prefer to resolve one interrogative over the other. However, as in the case of ICDL we did not consider any specific applications of IPL, which we leave to future work.

Furthermore, while inquisitive plausibility logic contained the expressive power we desired with respect to inquisitive plausibility models, we did not show that IPL was the *weakest* logic with this property. And, given significant distinction in expressive power between IPL and ICDL it seems safe to conjecture that there are weaker logics between IPL and ICDL whose axiomatisation leads to a more explicit account of the interaction between the entertains and considers modality. We also leave this to future work.

Below in figure 8.1 we diagram the logics mentioned in this thesis, in terms of their 'expressive power.' Given the length of this thesis we have left proofs establishing that logics with greater expressive power are conservative extensions of weaker logics with respect to the language of those logics.

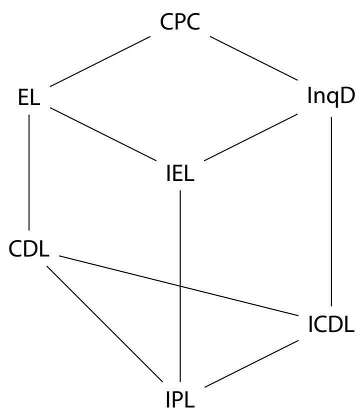


Figure 8.1: The logics mentioned in this thesis, ordered by expressive power.

Appendix A

Dynamics

We briefly show that ICDL and IPL can be extended to include dynamic modalities for public announcements. The central formal innovations required and used in this appendix can be found in Ciardelli (2015, Chap. 8), in particular §8.3.

The core insight from the chapters of conditional doxastic logic—that the fundamentals of belief revision takes place at the level of declaratives—is shown to continue, as the interrogative content of a proposition can be factored out of the revision process in the case of ICDL, and equivalent rational applies to IPL. This suggests that ICDL and IPL can be straightforwardly extended with other dynamic operations familiar from the transition of dynamic epistemic logic, in particular soft upgrades.

A.1 Updates on IPMS

Definition A.1.1 (Update). The update of an IPM $M = \langle W, \{\preceq_a^w\}_{a\in\mathcal{A}}^{w\in W}, \{\Sigma_a\}_{a\in\mathcal{A}}, V\rangle$ with a formula φ is the model: $M^\varphi := \langle W^\varphi, \{\preceq_a^w\}_{a\in\mathcal{A}}^{w\in W}, \{\Sigma_a^\varphi\}_{a\in\mathcal{A}}, V^\varphi\rangle$, defined as follows:

1. $W^\varphi = W \cap |\varphi|_M$
2. $\preceq_a^w = \preceq_a^w \cap (W^\varphi \times W^\varphi)$
3. For every $w \in W^\varphi$, $\Sigma_a^\varphi(w) = \Sigma_a(w) \cap [\![\varphi]\!]_M$
4. $V^\varphi = V_{\restriction W^\varphi}$

As in IPMS $\sigma_a(w) := \{v \mid v \preceq_a^w u \text{ for some } u\}$.

Proposition A.1.2. *For any model M, agent a, formula φ, and world $w \in W^\varphi$ we have:*

$$\sigma_a^\varphi(w) = \sigma_a(w) \cap |\varphi|_M$$

Proof. We observe $\{v \mid v \preceq_a^w u \text{ for some } u\} = (\{v \mid v \preceq_a^w u \text{ for some } u\} \cap |\varphi|_M)$. For, if $v \in \{v \mid v \preceq_a^w u \text{ for some } u\}$ then $v \in W \cap |\varphi|_M$ and $v \in \{v \mid v \preceq_a^w u \text{ for some } u\}$, whence $v \in \{v \mid v \preceq_a^w u \text{ for some } u\} \cap |\varphi|_M$. Similarly, if $v \in \{v \mid v \preceq_a^w u \text{ for some } u\} \cap |\varphi|_M$ then $v \preceq_a^w v$ and $v \in W^\varphi$, whence $\langle v, v\rangle \in W^\varphi \times W^\varphi$, and so $v \preceq_a^w v$, establishing $v \in \{v \mid v \preceq_a^w u \text{ for some } u\}$. ☐

Proposition A.1.3. *For any IPM M and any formula φ, M^φ is an IPM.*

Proof. First we observe that $\Sigma_a^\varphi(w)$ is a non-empty downward closed set of states. This is because $\Sigma_a^\varphi(w) = \Sigma_a(w) \cap [\![\varphi]\!]_M$, and the intersection of two non-empty downward closed set of states is itself a non-empty downward closed set of states.

Second, that $\preceq_a^w$ is a well-preorder follows from the fact that $\preceq_a^w$ is a restriction of the well-preorder $\leq_a^w$ to the elements of W^φ.

We now show that $\Sigma_a^\varphi(w)$ is an issue over $\sigma_a^\varphi(w) = \{v \mid v \preceq_a^w u \text{ for some } u\}$. Observe $\{v \mid v \preceq_a^w u \text{ for some } u\} = \{v \mid v \leq_a^w u \text{ for some } u\} \cap |\varphi|_M$. Therefore, as $\bigcup \Sigma_a(w) = \{v \mid v \leq_a^w u \text{ for some } u\}$, we know $\bigcup \Sigma_a(w) \cap |\varphi|_M = \{v \mid v \leq_a^w u \text{ for some } u\} \cap |\varphi|_M$ by proposition A.1.2. So, $\bigcup \Sigma_a^\varphi(w) = \{v \mid v \preceq_a^w u \text{ for some } u\}$.

Factivity follows from the fact that $\preceq_a^w$ is a restriction of $\leq_a^w$ to the elements of W^φ, as does introspection 2.

Finally we need to check that the updated maps Σ_a^φ satisfy the condition of introspection 1. So, suppose $v \in \sigma_a^\varphi(w)$. Therefore, $v \in \sigma_a(w)$, by proposition A.1.2, whence $\Sigma_a(w) = \Sigma_a(v)$. From this it follows that $\Sigma_a(w) \cap [\![\varphi]\!]_M = \Sigma_a(v) \cap [\![\varphi]\!]_M$, and so $\Sigma_a^\varphi(w) = \Sigma_a^\varphi(v)$. □

A.2 Inquisitive Dynamic Conditional-Doxastic Logic and Inquisitive Dynamic Plausibility Logic

The languages of inquisitive dynamic conditional doxastic logic, denoted by $\mathcal{L}^{\mathsf{IDCDL}}$, and inquisitive dynamic plausibility logic, $\mathcal{L}^{\mathsf{IDPL}}$, are given by enriching $\mathcal{L}^{\mathsf{ICDL}}$ and $\mathcal{L}^{\mathsf{IPL}}$ respectively by introducing dynamic modalities. These allow us to capture what is the case after a public announcement of an arbitrary formula φ has been performed. We begin by introducing the following clause:

- if $\varphi \in \mathcal{L}_! \cup \mathcal{L}_?$ and $\psi \in \mathcal{L}_\circ$, then $[\varphi]\psi \in \mathcal{L}_\circ$, where $\circ \in \{!, ?\}$

With syntax in place we define the following support condition, and its corresponding truth condition.

S: $M, s \models [\varphi]\psi$ iff $M^\varphi, s \cap |\varphi|_M \models \psi$
T: $M, w \models [\varphi]\psi$ iff $w \notin |\varphi|_M$ or $M^\varphi, w \models \psi$

Definition A.2.1 (Axioms and rules)**.** We enrich both ICDL and IPL with the following reduction axiom schemas.

1. $[\varphi]p \leftrightarrow (\varphi \to p)$
2. $[\varphi]\bot \leftrightarrow (\varphi \to \bot)$
3. $[\varphi]?\{\alpha_1, \dots, \alpha_n\} \leftrightarrow ?\{[\varphi]\alpha_1, \dots, [\varphi]\alpha_n\}$
4. $[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
5. $[\varphi](\psi \to \chi) \leftrightarrow ([\varphi]\psi \to [\varphi]\chi)$

To ICDL we add the additional axiom schema

6. $[\varphi]C_a^\psi \chi \leftrightarrow (\varphi \to (C_a^{\varphi \wedge [\varphi]\psi}[\varphi]\chi))$

And to IPL we add the following axiom schemas

7. $[\varphi]E_a\psi \leftrightarrow (\varphi \to E_a(\varphi \to [\varphi]\psi))$
8. $[\varphi]E_a^{\leq}\psi \leftrightarrow (\varphi \to E_a^{\leq}(\varphi \to [\varphi]\psi))$
9. $[\varphi]E_a^{\not\leq}\psi \leftrightarrow (\varphi \to E_a^{\not\leq}(\varphi \to [\varphi]\psi))$

In addition to the above reduction axioms we add the rule of general replacement of equivalents to ICDL/IPL.

$$\frac{\varphi \leftrightarrow \psi}{\chi \leftrightarrow \chi[{}^{\psi}/_{\varphi}]}$$

A.3 Results

Lemma A.3.1. *For any updated* IPM M^{φ}*, and formula* ψ*;*

$$M^{\varphi}, t \vDash \psi \textit{ iff } M^{\varphi}, t \cap |\varphi|_M \vDash \psi$$

Proof. For all $t \subseteq W^{\varphi}, t \subseteq |\varphi|_M$, therefore $t = t \cap |\varphi|_M$ From this the result is immediate. □

Lemma A.3.2. *For an* IPM M*, and its update with* φ*,* M^{φ}*,* $[\![\psi]\!]_{M^{\varphi}} = [\![[\varphi]\psi]\!]_M$.

Proof. $M^{\varphi}, s \vDash \psi$ iff $M^{\varphi}, s \cap |\varphi|_M \vDash \psi$, by lemma A.3.1, iff $M, s \vDash [\varphi]\psi$, by the support condition for the update operator. □

Corollary A.3.3. *Given an* IPM M*, and its update with* φ*,* M^{φ}*, for all* $w \in W^{\varphi}$*;*

$$\Sigma_a^{\varphi}(w) \cap [\![\psi]\!]_{M^{\varphi}} = \Sigma_a^{\varphi} \cap [\![[\varphi]\psi]\!]_M = \Sigma_a(w) \cap [\![\varphi \wedge [\varphi]\psi]\!]_M$$

Lemma A.3.4. *Given an* IPM M*, and its update with* φ*,* M^{φ}*, for all* $w \in W^{\varphi}$*;*

$$Min_{\preceq_a^w}(\Sigma_a^{\varphi}(w) \cap [\![\psi]\!]_{M^{\varphi}}) = Min_{\preceq_a^w}(\Sigma_a(w) \cap [\![\varphi \wedge [\varphi]\psi]\!]_M)$$

Proof. By the previous corollary $\Sigma_a^{\varphi}(w) \cap [\![\psi]\!]_{M^{\varphi}} = \Sigma_a(w) \cap [\![\varphi \wedge [\varphi]\psi]\!]_M$, and so $\sigma_a^{\varphi}(w) \cap |\psi|_{M^{\varphi}} = \sigma_a(w) \cap |\varphi \wedge [\varphi]\psi|_M$. From this it follows that $\preceq_a^w \restriction(\sigma_a^{\varphi}(w) \cap |\psi|_{M^{\varphi}}) = \preceq_a^w \restriction(\sigma_a(w) \cap |\varphi \wedge [\varphi]\psi|_M)$. So, the fragment of an agent's plausibility ordering $\mathrm{Min}_{\preceq_a^w}$ and $\mathrm{Min}_{\preceq_a^w}$ consider is the same. □

Definition A.3.5 (Resolutions)**.** The resolutions of a formula φ of IDCDL or IDPL are defined by augmenting the definitions of the resolutions for a formula of InqD (cf. def. 1.2.25) with the following clause:

– $\mathcal{R}([\varphi]\psi) = \{[\varphi]\alpha \mid \alpha \in \mathcal{R}(\psi)\}$

Proposition A.3.6 (Normal form for IDCDL/IDPL)**.**
For $\varphi \in \mathcal{L}^{IDCDL/IDPL}$*,* $\varphi \equiv ?\{\alpha_1, \ldots, \alpha_n\}$*, for* $\alpha_1, \ldots, \alpha_n \in \mathcal{R}(\varphi)$.

Proof. Via induction on the complexity of a formula.

We detail only the step for the dynamic modality, following Ciardelli (2015, p. 319). Let $\mathcal{R}(\psi) = \{\beta_1, \ldots, \beta_n\}$.

$$\begin{aligned} M, s \vDash [\varphi]\psi &\text{ iff } M^{\varphi}, s \cap |\varphi|_M \vDash \psi \\ &\text{ iff } M^{\varphi}, s \cap |\varphi|_M \vDash ?\{\beta_1, \ldots, \beta_n\} \\ &\text{ iff } M^{\varphi}, s \cap |\varphi|_M \vDash \beta_i \text{ for some } 1 \leq i \leq n \\ &\text{ iff } M, s \vDash [\varphi]\beta_i \text{ for some } 1 \leq i \leq n \\ &\text{ iff } M, s \vDash ?\{[\varphi]\beta_1, \ldots, [\varphi]\beta_n\} \end{aligned}$$

□

Theorem A.3.7 (Soundness of the reduction axioms)**.**

Proof.

Atomic sentences

$$\begin{aligned} M, w \vDash [\varphi]p &\text{ iff } M, w \nvDash \varphi \text{ or } M^{\varphi}, w \vDash p \\ &\text{ iff } M, w \nvDash \varphi \text{ or } M, w \vDash p \\ &\text{ iff } M, w \vDash \varphi \to p \end{aligned}$$

Falsum

$$\begin{aligned} M, w \vDash [\varphi]\bot &\text{ iff } M, w \nvDash \varphi \text{ or } M^{\varphi}, w \vDash \bot \\ &\text{ iff } M, w \nvDash \varphi \\ &\text{ iff } M, w \vDash \neg\varphi \end{aligned}$$

Interrogative operator

$$\begin{aligned} M, s \vDash [\varphi]?\{\alpha_1, \ldots, \alpha_n\} &\text{ iff } M^{\varphi}, s \cap |\varphi|_M \vDash ?\{\alpha_1, \ldots, \alpha_n\} \\ &\text{ iff } M^{\varphi}, s \cap |\varphi|_M \vDash \alpha_i \text{ for some } 1 \leq i \leq n \\ &\text{ iff } M, s \vDash [\varphi]\alpha_i \\ &\text{ iff } M, s \vDash ?\{[\varphi]\alpha_1, \ldots, [\varphi]\alpha_n\} \end{aligned}$$

Conjunction

$$\begin{aligned} M, s \vDash [\varphi](\psi \wedge \chi) &\text{ iff } M^{\varphi}, s \cap |\varphi|_M \vDash \psi \wedge \chi \\ &\text{ iff } M^{\varphi}, s \cap |\varphi|_M \vDash \varphi \text{ and } M^{\varphi}, s \cap |\varphi|_M \vDash \psi \\ &\text{ iff } M, s \vDash [\varphi]\psi \text{ and } M, s \vDash [\varphi]\chi \\ &\text{ iff } M, s \vDash [\varphi]\psi \wedge [\varphi]\chi \end{aligned}$$

Implication

$$\begin{aligned} M, s \vDash [\varphi](\psi \to \chi) &\text{ iff } M^{\varphi}, s \cap |\varphi|_M \vDash \psi \to \chi \\ &\text{ iff for any } t \subseteq s \cap |\varphi|_M,\ \text{if } M^{\varphi}, t \vDash \psi \text{ then } M^{\varphi}, t \vDash \chi \\ &\text{ iff for any } t \subseteq s,\ \text{if } M^{\varphi}, t \cap |\varphi|_M \vDash \psi \text{ then } M^{\varphi}, t \cap |\varphi|_M \vDash \chi \\ &\text{ iff for any } t \subseteq s,\ \text{if } M, t \vDash [\varphi]\psi \text{ then } M, t \vDash [\varphi]\chi \\ &\text{ iff } M, s \vDash [\varphi]\psi \to [\varphi]\chi \end{aligned}$$

C_a modality

$$\begin{aligned} &M, s \vDash [\varphi]C_a^{\psi}\chi \text{ iff } M^{\varphi}, s \cap |\varphi|_M \vDash C_a^{\psi}\chi \\ &\quad \text{iff } \forall w \in s \cap |\varphi|_M, \forall t \in \mathrm{Min}_{\leq_a^w}(\Sigma_a^{\varphi}(w) \cap [\![\psi]\!]_{M^{\varphi}}), M^{\varphi}, t \vDash \chi \\ &\quad \text{iff } \forall w \in s \cap |\varphi|_M, \forall t \in \mathrm{Min}_{\leq_a^w}(\Sigma_a^{\varphi}(w) \cap [\![\psi]\!]_{M^{\varphi}}), M^{\varphi}, t \cap |\varphi|_M \vDash \chi \\ &\quad \text{iff } \forall w \in s \cap |\varphi|_M, \forall t \in \mathrm{Min}_{\leq_a^w}(\Sigma_a^{\varphi}(w) \cap [\![\psi]\!]_{M^{\varphi}}), M, t \vDash [\varphi]\chi \\ &\quad \text{iff } \forall w \in s \cap |\varphi|_M, \forall t \in \mathrm{Min}_{\leq_a^w}(\Sigma_a(w) \cap [\![\varphi \wedge [\varphi]\psi]\!]_M), M, t \vDash [\varphi]\chi \\ &\quad \text{iff } \forall w \in s \cap |\varphi|_M, M, w \vDash C_a^{\varphi \wedge [\varphi]\psi}[\varphi]\chi \\ &\quad \text{iff } M, s \cap |\varphi|_M \vDash C_a^{\varphi \wedge [\varphi]\psi}[\varphi]\chi \\ &\quad \text{iff } M, s \vDash \varphi \to C_a^{\varphi \wedge [\varphi]\psi}[\varphi]\chi \end{aligned}$$

E_a modality Following Ciardelli (2015, p. 322) we split the proof into two steps.

First we establish $M^\varphi, w \vDash E_a\psi$ iff $M, w \vDash E_a(\varphi \to [\varphi]\psi)$.

$$\begin{aligned}
M^\varphi, w \vDash E_a\psi \text{ iff } & \forall s \in \Sigma_a^\varphi(w), M^\varphi, s \vDash \psi \\
\text{iff } & \forall t \in \Sigma_a(w), \forall \alpha_i \in \mathcal{R}(\varphi), M^\varphi, t \cap |\alpha_i|_M \vDash \psi \\
\text{iff } & \forall t \in \Sigma_a(w), \forall \alpha_i, M^\varphi, t \cap |\alpha_i|_M \cap |\varphi|_M \vDash \psi \\
\text{iff } & \forall t \in \Sigma_a(w), \forall \alpha_i \in \mathcal{R}(\varphi), M^\varphi, t \cap |\alpha_i|_M \vDash [\varphi]\psi \\
\text{iff } & \forall t \in \Sigma_a(w), \forall \alpha_i \in \mathcal{R}(\varphi), M^\varphi, t \vDash \alpha_i \to [\varphi]\psi \\
\text{iff } & \forall t \in \Sigma_a(w), \forall \alpha_i \in \mathcal{R}(\varphi), M^\varphi, t \vDash (\alpha_1 \to [\varphi]\psi) \wedge \cdots \wedge (\alpha_n \to [\varphi]\psi) \\
\text{iff } & \forall t \in \Sigma_a(w), \forall \alpha_i \in \mathcal{R}(\varphi), M^\varphi, t \vDash ?\{\alpha_1, \dots, \alpha_n\} \to [\varphi]\psi) \\
\text{iff } & \forall t \in \Sigma_a(w), \forall \alpha_i \in \mathcal{R}(\varphi), M^\varphi, t \vDash \varphi \to [\varphi]\psi) \\
\text{iff } & M, w \vDash E_a(\varphi \to [\varphi]\psi)
\end{aligned}$$

Second we establish $M, w \vDash [\varphi]E_a\psi$ iff $M, w \vDash \varphi \to E_a(\varphi \to [\varphi]\psi)$

$$\begin{aligned}
M, w \vDash [\varphi]E_a\psi \text{ iff } & M, w \vDash \varphi \Rightarrow M^\varphi, w \vDash E_a\psi \\
\text{iff } & M, w \vDash \varphi \Rightarrow M, w \vDash E_a(\varphi \to [\varphi]\psi) \\
\text{iff } & M, w \vDash \varphi \to E_a(\varphi \to [\varphi]\psi)
\end{aligned}$$

$E_a^{\leq}$ modality Analogous to the case of the entertains modality.

$E_a^{\nleq}$ modality Analogous to the case of the entertains modality. □

Corollary A.3.8. *For any formula $\varphi \in$ IDCDL/IDPL there exists a provably equivalent formula $\varphi^\rho \in$ IDCDL/IDPL.*

Corollary A.3.9. *IDCDL has the same expressive power as ICDL and IDPL has the same expressive power as IPL.*

Theorem A.3.10 (Completeness for IDCDL). *IDCDL/IDPL is (weakly) complete with respect to IPMs, and by implication inquisitive conditional doxastic models.*

Proof. Suppose $\vDash \psi$, by corollary A.3.8 and the soundness of IDCDL/IDPL it follows that $\vDash \psi^\rho$. The completeness of the derivation of IDCDL/IDPL for ICDL/IPL gives $\vdash \psi^\rho$, therefore a second application of corollary A.3.8 gives us $\vdash \psi$. □

Corollary A.3.11 (Decidability). *IDCDL/IDPL is decidable.*

Cheat Sheet

We restate a number of definitions, facts, propositions, lemmas, and corollaries used.

Models and support conditions

Definition 3.3.1 (Inquisitive plausibility models)**.**

An inquisitive plausibility model M for a set At of atomic formulas and a set $\mathcal{A}$ of agents,[1] is a tuple: $\langle W, \{\leq_a^w\}_{a\in\mathcal{A}, w\in W}, \{\Sigma_a\}_{a\in\mathcal{A}}, V\rangle$, where:

1. W is a set of possible worlds
2. $\leq_a^w$ is a well-preorder over a subset of W
3. $\Sigma_a(w)$ is an issue over $\sigma_a(w)$, where $\sigma_a(w) := \{v \mid \exists u\colon v \leq_a^w u\}$
4. $V\colon W \to \wp(\mathsf{At})$ is a valuation map, stating for each $w \in W$ the atomic formulas true at w

And the following conditions are satisfied:

Factivity	$w \in \sigma_a(w)$, for all $w \in W$
Introspection 1	if $v \in \sigma_a(w)$, then $\Sigma_a(w) = \Sigma_a(v)$
Introspection 2	if $v \in \sigma_a(w)$, then $x \leq_a^v y$ if and only if $x \leq_a^w y$

Definitions 1.2.5, 3.3.9, and 7.1.2. Let M be an IPM and s an information state:

Common InqD conditions

1. $M, s \vDash p$ iff $p \in V(w)$ for all $w \in s$
2. $M, s \vDash \bot$ iff $s = \emptyset$
3. $M, s \vDash ?\{\alpha_1, \ldots, \alpha_n\}$ iff $M, s \vDash \alpha_1$ or $\ldots$ or $M, s \vDash \alpha_n$
4. $M, s \vDash \varphi \wedge \psi$ iff $M, s \vDash \varphi$ and $M, s \vDash \psi$
5. $M, s \vDash \varphi \to \psi$ iff $\forall t \subseteq s,$ if $M, t \vDash \varphi$ then $M, t \vDash \psi$

ICDL conditions

6. $M, s \vDash K_a\varphi$ iff $\forall w \in s, M, \sigma_a(w) \vDash \varphi$
7. $M, s \vDash C_a^{\psi}\varphi$ iff $\forall w \in s\colon \forall t \in \mathrm{Min}_{\leq_a^w}([\![\psi]\!] \cap \Sigma_a(w)), M, t \vDash \varphi$
8. $M, s \vDash B_a^{\psi}\varphi$ iff $\forall w \in s\colon \forall t \in \mathrm{Min}_{\leq_a^w}[\![\psi]\!], M, t \vDash \varphi$

where:

a. $\mathrm{Min}_{\leq_a^w}|\psi| := \{v \in |\psi| \mid v \leq_a^w u$ for all $u \in |\psi|\}$ and
b. $\mathrm{Min}_{\leq_a^w}[\![\psi]\!] := \{s \in \psi\psi \mid s \subseteq \mathrm{Min}_{\leq_a^w}|\psi|\}$

[1] We assume the set of agents is finite. However, modalities for common knowledge, belief and so on will not be explored in this thesis, and so there is no technical need for this assumption.

IPL conditions

3. $M, s \vDash E_a\varphi$ iff $\forall w \in s$: $\forall t \in \Sigma_a(w), M, t \vDash \varphi$
4. $M, s \vDash E_a^{\leq}\varphi$ iff $\forall w \in s$: $\forall t \in \Sigma_a^{\leq}(w), M, t \vDash \varphi$
5. $M, s \vDash E_a^{\not\leq}\varphi$ iff $\forall w \in s$: $\forall t \in \Sigma_a^{\not\leq}(w), M, t \vDash \varphi$

where:

a. $\Sigma_a^{\leq}(w) := \wp(\sigma_a^{\leq}(w)) \cap \Sigma_a(w)$, and $\sigma_a^{\leq}(w) := \{v \mid v \leq_a^w w\}$
b. $\Sigma_a^{\not\leq}(w) := \wp(\sigma_a^{\not\leq}(w)) \cap \Sigma_a(w)$, and $\sigma_a^{\not\leq}(w) := \{v \in \sigma_a(w) \mid v \not\leq_a^w w\}$

Definition 4.1.1 (Inquisitive conditional-doxastic models).
An inquisitive conditional-doxastic model for a set At of atomic formulas and a set $\mathcal{A}$ of agents, is a tuple: $\langle W, \{\mathcal{S}_a^P\}_{a\in\mathcal{A}, P\in I}, V\rangle$, where:

- W is a set of possible worlds
- I is the set of all issues over W
- $V: W \to \wp(\mathsf{At})$ is a valuation map, stating for each $w \in W$ the atomic formulas true at w
- $\mathcal{S}_a^P$ is a map $W \to \mathscr{I}$ associating to each world an issue, $\mathcal{S}_a^P(w)$, satisfying the following conditions:

Safety	if $w \in \lvert P\rvert$ then $\mathcal{S}_a^P(w) \neq \{\emptyset\}$, where $\lvert P\rvert := \bigcup P$
Introspection	if $v \in s_a^P(w)$, then $\mathcal{S}_a^Q(w) = \mathcal{S}_a^Q(v)$
Adjustment	$\mathcal{S}_a^P(w) \subseteq P$
Success	$\mathcal{S}_a^P(w) \neq \{\emptyset\}$, if $\mathcal{S}_a^Q(w) \cap P \neq \{\emptyset\}$
Minimality	$\mathcal{S}_a^{P\cap Q}(w) = \mathcal{S}_a^P(w) \cap Q$, if $\mathcal{S}_a^P(w) \cap Q \neq \{\emptyset\}$

Definitions 1.2.5, 4.1.2, and 4.1.3. Let M be an ICDM and s an information state:

1. $M, s \vDash p$ iff $p \in V(w)$ for all $w \in s$
2. $M, s \vDash \bot$ iff $s = \emptyset$
3. $M, s \vDash ?\{\alpha_1, \ldots, \alpha_n\}$ iff $M, s \vDash \alpha_1$ or $\ldots$ or $M, s \vDash \alpha_n$
4. $M, s \vDash \varphi \wedge \psi$ iff $M, s \vDash \varphi$ and $M, s \vDash \psi$
5. $M, s \vDash \varphi \to \psi$ iff $\forall t \subseteq s$, if $M, t \vDash \varphi$ then $M, t \vDash \psi$
6. $M, s \vDash C_a^{\psi}\varphi$ iff $\forall w \in s$ and $\forall t \in \mathcal{S}_a^{\psi}(w), M, t \vDash \varphi$
7. $M, s \vDash B_a^{\psi}\varphi$ iff $\forall w \in s$ and $\forall t \in (\wp(s_a^{\psi}(w)) \cap [\![\psi]\!]), M, t \vDash \varphi$

Definition 7.2.1 (Inquisitive plausibility neighbourhood models).
An Inquisitive plausibility neighbourhood model is a tuple: $\langle W, \{\Sigma_a^{\leq}\}, \{\Sigma_a^{\not\leq}\}, \{\Sigma_a\}, V\rangle$, where:

- W is a set of possible worlds
- $V: W \to \wp(\mathsf{At})$ is a valuation map, stating for each $w \in W$ the atomic formulas true at w
- Each $\Sigma_a^{\square}$ for $\square \in \{\leq, \not\leq, \cdot\}$ is a map $W \to \Pi$ associating to each world an issue in accordance with the following conditions:
  1. if $v \in \sigma_a^{\leq}(w)$, then $\Sigma_a^{\leq}(v) \subseteq \Sigma_a^{\leq}(w)$, where $\sigma_a^{\leq}(w) := \bigcup\Sigma_a^{\leq}(w)$
  2. if $u \in \sigma_a^{\not\leq}(w)$, then $\Sigma_a^{\not\leq}(u) \subseteq \Sigma_a^{\not\leq}(w)$, where $\sigma_a^{\not\leq}(w) := \bigcup\Sigma_a^{\not\leq}(w)$
  3. $w \in \sigma_a^{\leq}(w)$
  4. $\sigma_a^{\leq}(w) \cap \sigma_a^{\not\leq}(w) = \emptyset$
  5. $\sigma_a^{\leq}(w) \cup \sigma_a^{\not\leq}(w) = \sigma_a(w)$

6. if $v \in \sigma_a(w)$, then $\Sigma_a(w) = \Sigma_a(v)$, where $\sigma_a(w) := \bigcup \Sigma_a(w)$
7. $\Sigma_a^{\leq}(w) = (\wp(\sigma_a^{\leq}(w)) \cap \Sigma_a(w))$
8. $\Sigma_a^{\not\leq}(w) = (\wp(\sigma_a^{\not\leq}(w)) \cap \Sigma_a(w))$

Definitions 1.2.5 and 7.2.3. Let M be an IPNM and s an information state:

1. $M, s \vDash p$ iff $p \in V(w)$ for all $w \in s$
2. $M, s \vDash \bot$ iff $s = \emptyset$
3. $M, s \vDash ?\{\alpha_1, \ldots, \alpha_n\}$ iff $M, s \vDash \alpha_1$ or $\ldots$ or $M, s \vDash \alpha_n$
4. $M, s \vDash \varphi \wedge \psi$ iff $M, s \vDash \varphi$ and $M, s \vDash \psi$
5. $M, s \vDash \varphi \to \psi$ iff $\forall t \subseteq s$, if $M, t \vDash \varphi$ then $M, t \vDash \psi$
6. $M, s \vDash E_a\varphi$ iff $\forall w \in s \colon \forall t \in \Sigma_a(w), M, t \vDash \varphi$
7. $M, s \vDash E_a^{\leq}\varphi$ iff $\forall w \in s \colon \forall t \in \Sigma_a^{\leq}(w), M, t \vDash \varphi$
8. $M, s \vDash E_a^{\not\leq}\varphi$ iff $\forall w \in s \colon \forall t \in \Sigma_a^{\not\leq}(w), M, t \vDash \varphi$

Definitions Relating to Models

Definition 1.2.3 (States and issues). Let $M = \langle W, V \rangle$ be an InqD model.

- An *information state* is a set $s \subseteq W$ of possible worlds.[2]
- An *issue* is a non-empty set I of information states which is downward closed: if $s \in I$ and $t \subseteq s$, then $t \in I$.
- Given a model M we denote by $\mathscr{I}_M$ the set of all issues over W. We will suppress the subscript M when the set of possible worlds is given by context.

Definition 1.2.7 (Inquisitive propositions). For an InqD model M and formula φ, $[\![\varphi]\!]_M := \{s \mid M, s \vDash \varphi\}$ denotes the proposition expressed by φ.

Definition 1.2.13 (Truth set). We define the *truth-set* of a formula φ in a model M as the set of all worlds in M in which φ is true: $|\varphi|_M := \{w \in W \mid M, w \vDash \varphi\}$.

Definition 1.2.20. For any issue P, $!P := \wp(|P|)$.

Definition 1.2.23 (Entailment).

$\Phi \vDash \psi$ iff for any model M and state s, if $M, s \vDash \Phi$ then $M, s \vDash \psi$.

Definition 3.3.5 ($\mathrm{Min}_{\leq_a^w} P$). For any issue P, $\mathrm{Min}_{\leq_a^w} P := \{s \in P \mid s \subseteq \mathrm{Min}_{\leq_a^w}|P|\}$, where $\mathrm{Min}_{\leq_a^w}|P| := \{v \in |P| \mid v \leq_a^w u$ for all $u \in |P|\}$.

Properties of Models

Fact 1.2.6 (Persistence). *For all formulas φ, if $M, s \vDash \varphi$ and $t \subseteq s$, then $M, t \vDash \varphi$.*

Proposition 1.2.15. $M, s \vDash \alpha$ *iff* $\forall w \in s, M, w \vDash \alpha$.

Proposition 1.2.26. *For any M, s and φ; $M, s \vDash \varphi$ iff $M, s \vDash \alpha$, for some $\alpha \in \mathcal{R}(\varphi)$.*

Resolutions

Definition 3.3.4 (Resolutions for ICDL). The set $\mathcal{R}(\varphi)$ of resolutions for a given formula φ is defined inductively by:

[2]Given a state s we write $s^{\downarrow}$ for its downward closure. For example, $\{w, v\}^{\downarrow} = \{\{w, v\}, \{w\}, \{v\}, \emptyset\}$

- $\mathcal{R}(\alpha) = \{\alpha\}$
- $\mathcal{R}(?\{\alpha_1, \ldots, \alpha_n\}) = \{\alpha_1, \ldots, \alpha_n\}$
- $\mathcal{R}(\mu \wedge \nu) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\mu)$ and $\beta \in \mathcal{R}(\nu)\}$
- $\mathcal{R}(\varphi \to \mu) = \{\bigwedge_{\alpha \in \mathcal{R}(\varphi)}(\alpha \to f(\alpha)) \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\mu)\}$

Axioms and Rules

Conjunction

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \frac{\varphi \wedge \psi}{\varphi} \qquad \frac{\varphi \wedge \psi}{\psi}$$

Implication

$$\frac{\begin{array}{c}[\varphi] \\ \vdots \\ \psi\end{array}}{(\varphi \to \psi)} \qquad \frac{(\varphi \to \psi) \quad \varphi}{\psi}$$

Interrogative

$$\frac{a_i}{?\{\alpha_1, \ldots, \alpha_n\}} \qquad \frac{\begin{array}{c}[\alpha_1] \\ \vdots \\ \varphi\end{array} \quad \ldots \quad \begin{array}{c}[\alpha_n] \\ \vdots \\ \varphi\end{array} \quad ?\{\alpha_1, \ldots, \alpha_n\}}{\varphi}$$

Kreisel-Putnam

$$\frac{\alpha \to ?\{\beta_1, \ldots, \beta_n\}}{?\{\alpha \to \beta_1, \ldots, \alpha \to \beta_n\}}$$

Falsum

$$\frac{\bot}{\varphi}$$

Declarative double negation

$$\frac{\neg\neg\alpha}{\alpha}$$

The natural deduction system of InqD.

Definition A.3.12 (Axioms and rules of ICDL)**.**

Considers

1. $C_a^{\psi}(\varphi \to \chi) \to (C_a^{\psi}\varphi \to C_a^{\psi}\chi)$
2. $C_a^{\varphi}\neg\varphi \to \neg\varphi$
3. $C_a^{\varphi}\varphi$
4. i. $C_a^{\psi}\varphi \to C_a^{\chi}C_a^{\psi}\varphi$ and ii. $\neg C_a^{\psi}\varphi \to C_a^{\chi}\neg C_a^{\psi}\varphi$
5. $\neg C^{\psi}\neg\varphi \to (C_a^{\psi\wedge\varphi}\chi \leftrightarrow C_a^{\psi}(\varphi \to \chi))$

And when ICDL is enriched with B and K modalities:

Belief

6. $B_a^{\psi}\alpha \leftrightarrow C_a^{\psi}\alpha$
7. $B_a^{\psi}\varphi \leftrightarrow \bigwedge_{\alpha \in \mathcal{R}(\psi)}(\neg B_a^{\psi}\neg\alpha \to \bigvee_{\beta \in \mathcal{R}(\varphi)} B_a^{\alpha}\beta)$

Knowledge

8. $K_a\varphi \leftrightarrow \bigvee_{\alpha \in \mathcal{R}(\varphi)} B_a^{\neg\alpha}\alpha$

In conjunction with the rules of inference of InqD we add Necessitation and Replacement of Equivalents:

$$\frac{\begin{array}{c}\emptyset\\ \vdots\\ \varphi\end{array}}{C_a^{\psi}\varphi} \qquad \frac{\varphi \leftrightarrow \psi}{C_a^{\varphi}\chi \leftrightarrow C_a^{\psi}\chi}$$

$$\frac{\begin{array}{c}\emptyset\\ \vdots\\ \varphi\end{array}}{C_a^{\psi}\varphi}$$

Definition A.3.13. Axioms and rules of IPL

1. $E_a(\varphi \to \psi) \to (E_a\varphi \to E_a\psi)$
2. $E_a^{\leq}(\varphi \to \psi) \to (E_a^{\leq}\varphi \to E_a^{\leq}\psi)$
3. $E_a^{\nleq}(\varphi \to \psi) \to (E_a^{\nleq}\varphi \to E_a^{\nleq}\psi)$
4. $E_a^{\leq}\varphi \to E_a^{\leq}E_a^{\leq}\varphi$
5. $E_a^{\nleq}\varphi \to E_a^{\nleq}E_a^{\nleq}\varphi$
6. i. $E_a\varphi \to E_aE_a\varphi$ and ii. $\neg E_a\varphi \to E_a\neg E_a\varphi$
7. $E_a\varphi \to (E_a^{\leq}\varphi \wedge E_a^{\nleq}\varphi)$
8. $(E_a^{\leq}\varphi \wedge E_a^{\nleq}\psi) \to E_a(\bigvee_{\alpha\in\mathcal{R}(\varphi)}\bigvee_{\beta\in\mathcal{R}(\psi)}(\alpha \vee \beta))$
9. $E_a^{\leq}\alpha \to \alpha$
10. $(E_a^{\leq}\varphi \wedge E_a^{\nleq}\neg\alpha) \to E_a(\alpha \to \varphi)$
11. $(E_a^{\nleq}\varphi \wedge E_a^{\leq}\neg\alpha) \to E_a(\alpha \to \varphi)$

In conjunction with the rules of inference of InqD we add Necessitation:

$$\frac{\begin{array}{c}\emptyset\\ \vdots\\ \varphi\end{array}}{E_a\varphi}$$

The Canonical Model of ICDL

Lemma 6.2.3. *For any φ, $\varphi \dashv\vdash ?\mathcal{R}(\varphi)$.*

Definition 6.2.10 ($\mathfrak{F}$)**.** We define five successive sets based on a set of formulas $\mathcal{F}$.

1. We define $\mathcal{F}^{\mathsf{I}}$ to be the smallest set satisfying the following:
   (a) $\mathcal{F} \subseteq \mathcal{F}^{\mathsf{I}}$,
   (b) if $\varphi \in \mathcal{F}^{\mathsf{I}}$ and $\psi \in sub(\varphi)$, then $\psi \in \mathcal{F}^{\mathsf{I}}$,
   (c) if $\varphi \in \mathcal{F}^{\mathsf{I}}$, and $\alpha \in \mathcal{R}(\varphi)$, then $\alpha \in \mathcal{F}^{\mathsf{I}}$,
   (d) if $\alpha \in \mathcal{F}^{\mathsf{I}}$, then $\sim\alpha \in \mathcal{F}^{\mathsf{I}}$.[3]
2. We define $\mathcal{F}^{\mathsf{II}}$ to be the smallest set satisfying the following:
   (e) $\mathcal{F}^{\mathsf{I}} \subseteq \mathcal{F}^{\mathsf{II}}$,
   (f) if $\alpha_1, \ldots, \alpha_n \in \mathcal{F}^{\mathsf{I}}$ and are distinct then $\alpha_1 \wedge \cdots \wedge \alpha_n \in \mathcal{F}^{\mathsf{II}}$.
3. We define $\mathcal{F}^{\mathsf{III}}$ to be the smallest set satisfying the following:

[3] Recall $\sim\alpha := \beta$ if α is of the form $\neg\beta$, and $\neg\alpha$ otherwise.

(g) $\mathcal{F}^{\mathsf{II}} \subseteq \mathcal{F}^{\mathsf{III}}$,

(h) if $\alpha_1, \ldots, \alpha_n \in \mathcal{F}^{\mathsf{II}}$ and are distinct then $\alpha_1 \vee \cdots \vee \alpha_n \in \mathcal{F}^{\mathsf{III}}$.

4. We define $\mathcal{F}^{\mathsf{IV}}$ to be the smallest set satisfying the following:

(i) $\mathcal{F}^{\mathsf{III}} \subseteq \mathcal{F}^{\mathsf{IV}}$,

(j) if $\alpha_1, \ldots, \alpha_n \in \mathcal{F}^{\mathsf{III}}$ and are distinct then: $?\{\alpha_1, \ldots, \alpha_n\} \in \mathcal{F}^{\mathsf{IV}}$,

(k) if $\alpha_1, \ldots, \alpha_n \in \mathcal{F}^{\mathsf{III}}$ and are distinct then $\neg(?\{\alpha_1, \ldots, \alpha_n\}) \in \mathcal{F}^{\mathsf{IV}}$.

5. We define $\mathfrak{F}$ to be the smallest set satisfying the following:

(l) $\mathcal{F}^{\mathsf{IV}} \subseteq \mathfrak{F}$,

(m) if $\alpha \in \mathcal{F}^{\mathsf{IV}}$, then $\sim\alpha \in \mathfrak{F}$.

Definition 6.2.12 ($\mathfrak{D}$). $\mathfrak{D} := \{\alpha \mid \alpha \in \mathfrak{F}\}$.

Definition 6.2.16 (Nuclei). For a set of formulas $\mathcal{F}$ we define a set of declaratives N to be an nucleus over $\mathcal{F}$ if it is a maximally consistent theory of declaratives in $\mathfrak{D}$. So, N is an nucleus over $\mathcal{F}$ if *a)* A is a set of declaratives, *b)* A is consistent, *c)* $A \subseteq \mathfrak{D}$, and *d)* if $A \subset B \subseteq \mathfrak{D}$, then B is inconsistent. Let $\mathrm{Nu}(\mathcal{F})$ be the set of all nuclei over $\mathcal{F}$.

Definition 6.2.19 (Atoms). Let $\alpha_1, \ldots, \alpha_i, \ldots$ be an enumeration of the declaratives of $\mathcal{L}^{\mathsf{ICDL}}$. We define an atom A relative to a nucleus N as the union of a chain of $\mathcal{L}^{\mathsf{ICDL}}$-consistent sets as follows:

$$A_0 = N$$
$$A_{n+1} = \begin{cases} A_n \cup \{\alpha_n\}, & \text{if } A_n \vdash \alpha_n \\ A_n \cup \{\neg\alpha_n\}, & \text{otherwise} \end{cases}$$
$$A = \bigcup_{n \geq 0} A_n.$$

Lemma 6.2.21. *If $\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathfrak{F}$ and $\{\alpha_1, \ldots, \alpha_n\}$ is consistent, there is an atom $A \in At(\mathcal{F})$ such that $\{\alpha_1, \ldots, \alpha_n\} \subseteq A$.*

Proposition 6.2.22 (Deduction of declaratives).
For a set of formulas $\mathcal{F}$ and every A in $At(\mathcal{F})$, if $A \vdash \beta$, then $\beta \in A$.

Definitions 6.2.25, 6.2.28, and 6.2.33.

1. $\gamma_A := \bigwedge_{\alpha \in A\restriction_{\mathcal{F}'}} \alpha$
2. $\gamma_S := \bigvee_{A \in S} \gamma_A$, where $\gamma_\emptyset := \bot$.
3. $\chi_P := ?\{\gamma_S \mid S \in P\}$

Lemma 6.2.34. $\bigcap S \vdash \chi_P \iff S \in P$

Definition 6.3.1 (Canonical model over $\mathcal{F}$). Let $\mathcal{F}$ be a finite set of formulas. The canonical model over $\mathcal{F}$ is the tuple: $M^{\mathcal{F}} = \langle \mathrm{At}(\mathcal{F}), \{\mathcal{S}_a^P\}_{a \in \mathcal{A}, P \in \mathcal{I}}, V \rangle$, defined as follows:

- $\mathrm{At}(\mathcal{F})$ is the set of atoms over $\mathcal{F}$
- $V(A) = \{p \in \mathsf{At} \mid p \in A\}$
- For every issue $P \in \mathcal{I}$, $\mathcal{S}_a^P(w)$ is the set of states $S \subseteq \mathrm{At}(\mathcal{F})$ defined by:

$$S \in \mathcal{S}_a^P(A) \iff \bigcap S \vdash \varphi \text{ whenever } A \vdash C_a^{\chi_P} \varphi$$

Lemma 6.3.4 (Support lemma). *For a set of formulas $\mathcal{F}$ and the canonical model over $\mathcal{F}$, $M^{\mathcal{F}}$, for any $S \subseteq At(\mathcal{F})$ and any $\varphi \in \mathfrak{F}$,*

$$M^{\mathcal{F}}, S \vDash \varphi \iff \bigcap S \vdash \varphi$$

Mappings

Definition 5.1.2 (Map from ICDMS to IPMS).
Given an arbitrary ICDM, $M = \langle W, \{\mathcal{S}_a^P\}_{a \in \mathcal{A}, P \in \mathscr{I}}, V\rangle$, we define a map $M \mapsto M^\sharp$, where $M^\sharp = \langle W^\sharp, \{\leq_a^w\}_{a \in \mathcal{A}, w \in W}, \{\Sigma_a\}_{a \in \mathcal{A}}, V^\sharp\rangle$ is constructed in the following way:

1. $W^\sharp := W$
2. $v \leq_a^w u$ if $v \in s_a^{\{\{v\},\{u\}\}^\downarrow}(w)$
3. $\Sigma_a := \mathcal{S}_a(w)$, where $\mathcal{S}_a(w) := \bigcup_{Q \in \mathscr{I}} \mathcal{S}_a^Q(w)$
4. $V^\sharp := V$

Definition 5.2.2 (Map from IPMS to ICDMS).
Given an arbitrary inquisitive plausibility model, $M = \langle W, \{\leq_a\}_{a \in \mathcal{A}}, \{\Sigma_a\}_{a \in \mathcal{A}}, V\rangle$, we define a map $M \mapsto M^\flat$, where $M^\flat = \langle W, \{\mathcal{S}_a^P\}_{a \in \mathcal{A}, P \in \mathscr{I}}, V\rangle$ is constructed in the following way:

1. $W^\flat := W$
2. $\mathcal{S}_a^P(w) := \mathrm{Min}_{\leq_a^w}(\Sigma_a(w) \cap P)$
3. $V^\flat := V$

Definition 7.5.2 (Map from IPNMS to IPMS).
Given an arbitrary IPNM, $M = \langle W, \{\Sigma_a^{\leq}\}, \{\Sigma_a^{\not\leq}\}, \{\Sigma_a\}, V\rangle$ we define a map $M \mapsto M^\sharp$, where $M^\sharp = \langle W^\sharp, \{\leq_a^w\}_{a \in \mathcal{A}, w \in W^\sharp}, \{\Sigma_a\}_{a \in \mathcal{A}}, V^\sharp\rangle$ is constructed in the following way:

1. $W^\sharp := W$
2. $v \leq_a^w u$ if $v, u \in \sigma_a(w)$ and $v \in \sigma_a^{\leq}(u)$
3. $\Sigma_a^\sharp := \Sigma_a$
4. $V^\sharp := V$

Definition 7.6.2 (Map from IPMS to IPNMS).
Given an arbitrary IPM, $M = \langle W, \{\leq_a^w\}_{a \in \mathcal{A}, w \in W}, \{\Sigma_a\}_{a \in \mathcal{A}}, V\rangle$ we define a map $M \mapsto M^\flat$, where $M^\flat = \langle W^\flat, \{\Sigma_a^{\leq,\flat}\}, \{\Sigma_a^{\not\leq,\flat}\}, \{\Sigma_a^\flat\}, V^\flat\rangle$ is constructed in the following way:

1. $W^\flat := W$
2. $\Sigma_a^{\leq,\flat}(w) = \wp(\sigma_a^{\leq}(w)) \cap \Sigma_a(w)$
3. $\Sigma_a^{\not\leq,\flat}(w) = \wp(\sigma_a^{\not\leq}(w)) \cap \Sigma_a(w)$
4. $\Sigma_a^\flat := \Sigma_a$
5. $V^\flat := V$

List of Theorems, Lemmas, etc.

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