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Levationis Laus

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Abstract. Praising the lifting-phenomenon, we present an expression-directed, but also variant-independent, approach to lineair resolution, eliminating most (if not all) problems involving substitutions in existing expositions and culminating in a strengthened form of lifting. Subsequently we give a twist to lifting ground derivations and obtain fast proofs for standard theorems, results on infinite derivations, greatest fixed points, and the question when these are reached after ω steps in the downward fixed point hierarchy.

1. Introduction.

The last part (section 4) of this paper centers around lifting ground derivations. As a key tool, we introduce the sheaf of all ground derivations which possess a lift in a given SLD search tree. There are three basic results on sheaves: on success, on finite height, and on well-foundedness. From these, we quickly deduce standard results such as the strongest form of completeness (involving computed answers as well as rule-independency), finite failure characterization, and soundness and completeness of negation as failure. We characterize the algebras J for which $T^J\downarrow = T^J\downarrow\omega$ (where $T^J = T_P^J$ is the operator on the space of models over J associated with an arbitrary program P) and we relate this condition to recursive saturation. As a corollary, short shrift is given to the result of [Blair/Brown 199?] that countable algebras satisfying $T^J\downarrow = T^J\downarrow\omega$ for any program-associated T^J exist.

However, for the proper exposure of these topics, we felt the need for an almost complete revision of the *usual formulation* of the syntax of lineair resolution. Some motivation for such an undertaking, independent of the contents of section 4, runs as follows.

Suppose that goals N and M are variants of each other and we have refutations for them which are *similar* : they select variant-atoms at corresponding places and apply variant-rules. Then if α and β are the two computed answers, the corresponding instances $N\alpha$ and $M\beta$ should be variants as well.

The usual formulation of lineair resolution makes results like this - obviously, a conditio sine qua non for a sensible theory - awkward to prove, to put it mildly. (Cf. the Variant Lemma, [Apt 1990] lemma 2.8.)

In our opinion, this is due to a number of artificialities in the usual formulation of resolution. The unification-algorithm is used only to unify expressions without common variables. (In particular, stressing the fact that unifiers always can be taken idempotent must be qualified a red herring.) The emphasis on most general unifiers (substitutions) instead of most general common instances

(expressions) brings along subtleties (e.g., the usual notion of "more general than" for substitutions behaves highly irregular) which easily cause mistakes. If θ is a most general unifier of an atom A in the goal N and the head of a program-rule, only the restriction $\theta|Var(N)$ of θ is relevant for the resolution-mechanism. Finally, the standardization-apart routine for obtaining computational generality is ad-hoc and usually introduces hosts of apparently unnecessary variables. It may be a useful trick for a machine-directed implementation of SLD-resolution, but for theoretical purposes it has drawbacks.
Consequently, the first half of this paper (section 3) presents an expression-directed and variant-independent reformulation of lineair resolution. Resolvents and derivations are defined simply by the property which we want them to have, which is maximal generality. Most general unification only recurs in an existence result. The (simple) proof of the main result - a generalized lifting-theorem - clearly shows exactly what is essential about the separation of variables in derivations. Thanks to the way matters are organized, most, if not all, existing problems with substitutions are eliminated.
To start with, section 2 contains some well-known preliminaries.
We refer to [Apt 1990] and [Lloyd 1987] for unexplained notions.
Thanks are due to Krzysztof Apt for discussion and comments improving the exposition with respect to the notion of *derivation.*

Remark. When an earlier version of this paper had been completed, Krzysztof Apt made available to me a copy of [Ko/Nadel 1990]. This work has hygienic aims similar to the ones of section 3 below, though it mainly deals with refutations, i.e., succesful, and hence finite, derivations. It signalizes a number of errors concerning substitutions in the existing literature. The remedy Ko and Nadel propose essentially consists in patching up the *usual formulation* of resolution; consequently, the rather formidable result is not too inviting. We insist that *both* simplicity and correctness can be, and hence, should be regained by focussing on expressions instead of substitutions.

2. Preliminaries on substitution and unification.

There are several possible definitons of *substitution.* The one that we shall use (the choice is not critical but we need to be definite first and consistent next) is the following: a *substitution* is a map α from a finite set Dom(α) of variables to the set of terms (built using some fixed set of function symbols). We use postfix notation in the context of substitutions: henceforth, $x\alpha$ denotes the value of α at x. Note that our definition does not imply $\alpha x \neq x$ for $x \in$ Dom(α). As usual, we may write α as $\{x_1/x_1\alpha,\ldots,x_n/x_n\alpha\}$ when Dom(α)=$\{x_1,\ldots,x_n\}$. Ran(α) denotes the set of variables occurring in terms $x\alpha$ for $x \in$ Dom(α).

Var(A) denotes the set of variables occurring in the expression A.

Notations such as $x\alpha$, $A\alpha$, $\beta\alpha$ do not imply a commitment as to whether, e.g., $x \in$ Dom(α), Var(A)$\subset$Dom(α) or Ran(β)$\subset$Dom(α) respectively: as usual, we put $x\alpha=x$ in case $x \notin$ Dom(α).

If α is a substitution and V a (finite) set of variables, then $\alpha|V$ - the *restriction* of α to V - denotes the substitution β with Dom(β)=V and $x\beta=x\alpha$ for $x \in$ V. Hence, Dom($\alpha|V$)=V, irrespective of the extent of Dom(α). For A an expression, $\alpha|A:=\alpha|$Var(A). ε is the identity-map on the set VAR of all variables. $\varepsilon|V$ is a substitution whenever V$\subset$VAR is finite.

2.1 Lemma. $A\alpha=A\beta$ iff $\alpha|A=\beta|A$.

α is a *renaming for* the expression A if $\alpha|A$ is injective and takes its values in VAR.

2.2 Lemma. If α is a renaming for A and B is any expression, then a renaming β for both A and B exists such that $\beta|A=\alpha|A$.

Proof. Suitably modify α on the variables in Var(B)\Var(A). ☒

B is an *instance* of A, notation: $B \leq A$, if $B=A\alpha$ for some substitution α.

If α is a renaming for A, $B:=A\alpha$ is called a *variant* of A; notation: $B \sim A$.

2.3 Lemma. $A \sim B$ iff $A \leq B$ and $B \leq A$.

A $\leq$-greatest element of K is called *most general in* K. By 2.3, most general elements (if they exist) are unique up to $\sim$.

(This notion of *most general* is not to be confused with the one in *most general unifier* which applies to substitutions instead of expressions.)

On the set of atoms, $\leq$ is a quasi-ordering (it is reflexive and transitive). Its quotient with respect to $\sim$ is a partial ordering. In this quotient, by most general unification, every two elements with a common lower bound have an infimum. (As a matter of fact, two elements with either a lower- or an upper bound have a supremum - but this is not needed in what follows.)

3. Resolution in logic programming.

I. Expressions.

A *clause* is considered here to be a multiset of atoms. (Logically, a clause K represents the first-order sentence $\exists\bigwedge K$ - the existential closure of the conjunction of the elements of K.)
If K and L are clauses and A is an atom then $K,L=K\cup L$ and $K,A=A,K=K\cup\{A\}$; if α is a substitution then $K\alpha:=\{A\alpha|A\in K\}$. $\square$ is the empty clause.
The reasons for considering multisets (the order of presentation does not count but repetitions do) are: first, simplicity of presentation; and second, even if α unifies A and B, $\{A,B\}\alpha=\{A\alpha,B\alpha\}$ can be considered to have *two* elements, corresponding to the two of $\{A,B\}$.
In definite logic programming, derivations are made up of ***goals***, i.e., expressions of the form $\leftarrow K$, where K is a clause. ($\leftarrow K$ represents $\neg\exists\bigwedge K$, or $\forall\bigvee\{\neg A|A\in K\}$.) Derivations are constructed using ***(program) rules*** which are expressions of the form $A\leftarrow K$ (representing $\forall(\bigwedge K\rightarrow A)$) where A is an atom and K a clause.

II. Resolvents.

Unrestricted resolution occurs in [Lloyd 1987].

3.1 Definition.

1. The pair $\alpha;M$ - where M is a goal and α is a substitution - is an **unrestricted resolvent (u.resolvent)** of the goal $N=\leftarrow K,A$ with respect to the atom A and the rule P if
   (i) $Dom(\alpha)=Var(N)$, and
   (ii) there is an instance $A\alpha\leftarrow L$ of P such that $M=\leftarrow K\alpha,L$.

In the situation described, A is called the **selected atom** and α the **specification.**

2. M is an **u.resolvent** of N (w.r.t. an atom and a rule) if a substitution α exists such that $\alpha;M$ is an u.resolvent in the sense of 1.

Example.
$\alpha;\leftarrow S(x)$ forms an u.resolvent of $\leftarrow R(x)$ w.r.t. the atom R(x) and the rule $R(x)\leftarrow S(y)$ for every α such that $Dom(\alpha)=\{x\}$.

3.2 Remarks.

1. In the context of the usual definition of resolvent involving a mgu θ, the restriction $\theta|N$ of θ to the initial goal N is the specification. We feel that the "remaining" part of θ really is irrelevant.
2. Ground-resolvents of ground-instances and resolvents (both in the usual sense and in the sense defined below) are examples of unrestricted resolvents: they form the two extreme cases of u.resolvents, so to speak. Below, we exploit the advantages of bringing ground-resolution and resolution under the common heading of unrestricted resolution.

Defining resolvents as most general unrestricted ones is not good enough if we want maximal computational generality.

3.3 Example. In that case, $\leftarrow\square$ would be resolvent of $\leftarrow R(x)$ and the rule $R(x)\leftarrow\square$, with a specification mapping x to some constant.
We need the notion of *resultant* (cf. [Apt 1990]) and make the lifting-property part of the definition of *resolvent*.

3.4 Definitions.

1. If $\alpha;\leftarrow D$ is an u.restricted resolvent of $\leftarrow C$, the implication $D \rightarrow C\alpha$ is called the associated **resultant**, notation: rest(C,α;D) or rest(←C,α;←D).

Logically speaking, the (universal closure of the) resultant is what is being proved by the u.resolution-step: it is a logical consequence of the rule involved. This fact is useful when proving soundness. Here, the *logical* content of the resultant is of motivational importance only.

2. A **resolvent** (of a clause, w.r.t. some rule and selected atom) is an u.resolvent (w.r.t. these same things) the associated resultant of which is most general.

In other words, α;M is a resolvent of the goal N w.r.t. A and the program-rule P if (next to being an u.resolvent) for every u.resolvent α';M' of N w.r.t. A and P, there exists σ such that both (i) $N\alpha' = N\alpha\sigma$ and (ii) $M' = M\sigma$.

Again, we'll say that M is a resolvent of N w.r.t. A and P if α exists such that α;M is a resolvent in the sense defined above.

Example. {x/x};←S(x) does not form a resolvent of ←R(x) w.r.t. the rule R(x)←S(y). However, ←S(x) *is* a resolvent: use the specification {x/y} instead.

3.5 Existence.

If a goal has an u.resolvent (w.r.t. a rule and a selected atom), then it has a resolvent (w.r.t. these things) as well.

Proof.

Via separation of variables and most general unification, as in the usual approach.

For completeness' sake, here are the details.

Suppose that the following is an unrestricted resolution-step:

$$N = \leftarrow K, A$$

$$P = B \leftarrow L,\ A\alpha = B\beta, \text{ specification } \alpha,$$

$$\leftarrow K\alpha, L\beta.$$

Let ξ be a renaming for P such that $Var(N) \cap Var(P\xi) = \emptyset$. Choose ξ^{-1} such that $P\xi\xi^{-1} = P$. Choose γ such that $Dom(\gamma) = Var(N) \cup Var(P\xi)$, $\gamma | N = \alpha | N$ and $\gamma | P\xi = (\xi^{-1}\beta) | P\xi$.

Since $A\gamma = A\alpha = B\beta = B\xi\xi^{-1}\beta = B\xi\gamma$, γ unifies A and Bξ. Let θ be a most general unifier for A and Bξ. Say, $\gamma = \theta\sigma$. The following now is an u.resolution-step as well:

$$N = \leftarrow K, A$$

$$P = B \leftarrow L,\ A\theta = B\xi\theta, \text{ specification } \theta | N,$$

$$M = \leftarrow K\theta, L\xi\theta.$$

Claim: θ|N;M is a resolvent of N w.r.t. A and P.

Proof: note that the definition of θ does not depend on α, β above. So, without loss of generality, it suffices to show that the resultant here is ≥ the one above. Now the resultant of the last step is $R := K\theta, L\xi\theta \rightarrow K\theta, A\theta$; the resultant above is $K\alpha, L\beta \rightarrow K\alpha, A\alpha$, and this is the σ -instance of R, since $(K,A)\theta\sigma = (K,A)\gamma = (K,A)\alpha$ and $L\xi\theta\sigma = L\xi\gamma = L\xi\xi^{-1}\beta = L\beta$. ⊠

Remark. The proof makes it obvious that every resolvent in the usual sense is a resolvent in our sense as well.

In what follows, 3.6-9 will not be needed.

Let A be an atom of the clause or goal K and ξ a substitution. We say that A and $A\xi$ are **corresponding** atoms of K resp. $K\xi$.

3.6 Lemma. Suppose that $\alpha;M$ is an u.resolvent of N w.r.t. $A\in N$ and P.

Assume that $M^-\leq M$, $N\leq N^+$, $P\leq P^+$, and $A^+\in N^+$corresponds to $A\in N$;

say, $M^-=M\delta$, $N=N^+\xi$.

Then 1. $(\xi\alpha\delta)|N^+;M^-$ is u.resolvent of N^+ w.r.t. A^+ and P^+;

2. $rest(N^+,\xi\alpha\delta;M^-)=rest(N,\alpha;M)\delta$.

Proof. Immediate verification. ⊠

3.7 Corollary. Under the same hypotheses as the previous lemma:

if δ is in fact a renaming for M and δ^* is a renaming for rest(N,α;M) such that $\delta^*|M=\delta\,|M$,

then $rest(N^+,\xi\alpha\delta^*;M^-)=rest(N,\alpha;M)\delta^*$;

and hence $rest(N^+,\xi\alpha\delta^*;M^-)\sim rest(N,\alpha;M)$.

Proof. Immediate. ⊠

N.B.1: Note that existence of such a δ^* follows by 2.2.

N.B.2: We need *not* have $rest(N^+,\xi\alpha\delta;M^-)\sim rest(N,\alpha;M)$.

3.8 Corollary. Under the same hypotheses as the previous corollary:

if $P\sim P^+$, $\xi=\varepsilon$ (or, more generally, if ξ is a renaming for N^+) and if $\alpha;M$ is in fact a resolvent of N, then $\xi\alpha\delta^*;M^-$ is a resolvent as well.

3.9 Unicity and variant-independency.

If M and M' are resolvents of N w.r.t. A and P then, by definition, $M\sim M'$.

More generally, suppose that $N'\sim N$, $P'\sim P$ and the atom $A'\in N'$ corresponds to $A\in N$.

If M is a resolvent of N w.r.t. A and the rule P, then the following are equivalent:

1. $M'\sim M$;
2. M' is a resolvent of N' w.r.t. A' and P'.

Proof.

1⇒2. This says that being a resolvent is variant-independent.

The result is immediate from the previous corollary.

2⇒1. This expresses that resolvents are unique, up to ~.

By the lemma, since $M\sim M$ trivially, it follows that M is u.resolvent of N' w.r.t. A' and P'. By maximality of resolvents, $M\leq M'$. Similarly, M' is u.resolvent of N w.r.t. A and P; hence $M'\leq M$. Combining: $M\sim M'$. ⊠

3.10 Lifting once.

Suppose that A is an atom of the goal N, γ is a substitution and P is a rule.

1. If $\alpha;M$ is u.resolvent of $N\gamma$ w.r.t. $A\gamma$ and P, then $(\gamma\alpha)|N;M$ is u.resolvent of N w.r.t. A and P with the same resultant.
2. Therefore, if $N\gamma$ has an u.resolvent w.r.t. $A\gamma$ and P then N has a resolvent w.r.t. A and P.
3. If R^- is resultant of a (u.)resolvent of $N\gamma$ w.r.t. $A\gamma$ and P, and R is resultant of a resolvent of N w.r.t. A and P, then $R^-\leq R$.

Proof.

1. This is a special case of 3.6.

2. From 1 and 3.5.

3. Immediate from 1 and resultant-maximality of resolvents. ⊠

III. Derivations.

In the sequel, λ always denotes an ordinal $\leq\omega$, measuring the length of a derivation.

3.11 Definitions.

1. The sequence Γ: N_0, $\alpha_{0,1}$; $N_1,\ldots$, $\alpha_{i,i+1}$; $N_{i+1},\ldots$ ($i<\lambda$) is an **u.derivation** of **length** λ relative to a program if, for every $i<\lambda$, $\alpha_{i,i+1}$;N_{i+1} is an u.resolvent of N_i w.r.t. some rule of the program.

We refer to the transition $N_i\Rightarrow\alpha_{i,i+1}$; N_{i+1} as the $(i+1)^{st}$ *step* of Γ (we start counting at 1).

2. The **resultant** of the (finite) u.derivation

Γ: $\leftarrow C_0,\alpha_0;\leftarrow C_1,\ldots,\alpha_n;\leftarrow C_{n+1}$; notation: rest($\Gamma$), is the implication $C_{n+1}\rightarrow C_0\alpha_0\ldots\alpha_n$.

(This is in accordance with the former definition for n=0.)

Suppose that a 1-1-correspondence between the atoms in the clause C and those in D is given such that corresponding atoms have the same relation symbol. Then if $\leftarrow C^*$ and $\leftarrow D^*$ are u.resolvents of $\leftarrow C$ resp. $\leftarrow D$ using corresponding atoms and the same rule, the correspondence is transformed naturally into one between the atoms of C^* and those of D^* such that, again, corresponding atoms have the same relation symbol.

3. We say that u.derivations Γ and Δ are **similar** (with respect to some initial correspondence between their initial goals) if they have the same length and, at corresponding places, apply the same rule and select atoms which correspond in the sense explained.

We are now ready for the definition of *derivation.*

4. A **derivation** of length λ relative to a program is an u.derivation Γ: N_0, $\alpha_{0,1}$; $N_1,\ldots$, $\alpha_{i,i+1}$; $N_{i+1},\ldots$ ($i<\lambda$) relative to that program such that the resultant of every finite subderivation $\Gamma_{i,j}$:= N_i, $\alpha_{i,i+1}$; $N_{i+1},\ldots$, $\alpha_{j-1,j}$; N_j ($0\leq i<j<\lambda$) is most general in the class of resultants of similar u.derivations starting from the same goal N_i.

Note that a derivation necessarily is a sequence of consecutive resolvents. (Just consider the one-step subderivations.) However, the converse may fail. There can be two reasons for a sequence of consecutive resultants not to form a derivation.

To make the following examples more illuminating, we first make the crucial

3.12 Definition. Suppose that α;M is (u.)resolvent of N. The variable x is **released** at the transition $N\Rightarrow\alpha$;M if $x\in Var(N\alpha)\backslash Var(M)=Ran(\alpha)\backslash Var(M)$.

Of course, variables can be released only when a rule $B\leftarrow L$ is applied for which $Var(B)\backslash Var(L)\neq\emptyset$.

3.13 Examples.

1. The sequence: $\leftarrow R(x,y)$, $\{x/x,y/y\}$;$\leftarrow L(y)$, $\{y/y\}$;$\leftarrow K(x,y)$ is an u.derivation using resolvents, given the rules $R(x,y)\leftarrow L(y)$ and $L(y)\leftarrow K(x,y)$. It shows that the resultant $K(x,y)\rightarrow R(x,y)$ follows logically from these rules. However, changing the specification $\{y/y\}$ to $\{y/z\}$, the more general resultant $K(z,y)\rightarrow R(x,y)$ is obtained.

The reason for not obtaining maximal generality here is the re-introduction of the variable x (released at the first step) in the last goal of the derivation.

2. The sequence: ←R(x,y), ←L(y), ←□ together with the specifications {x/x,y/y} and {y/x} forms a refutation using resolvents given the rules R(x,y)←L(y) and L(y)←□. It shows that the resultant □→R(x,x), i.e.: R(x,x), logically follows from the rules. But of course, the more general R(x,y) can be obtained replacing {y/x} by {y/y}.

The reason for not obtaining maximal generality here is the re-introduction of the variable x (released at the first step) by the second specification.

These examples suggest a simple syntactic criterion [†] on (u.)derivations, which in fact will turn out to be a necessary and sufficient condition for a sequence of resolvents to be a derivation.

3.14 Definition.

The (u.)derivation Δ: N_0, $\beta_{0,1}$; N_1, $\beta_{1,2}$; N_2,... of length λ is **variables-separated**, or is a †-(u.)derivation, if

[†] for every $i<\lambda$, for every variable x released by Δ at the (i+1)-st step $N_i \Rightarrow \beta_{i,i+1};N_{i+1}$ (which means that $x \in Ran(\beta_{i,i+1}) \setminus Var(N_{i+1})$), and for every j such that $i<j<\lambda$:

x does not occur in N_{j+1}, nor does it occur in $N_j\beta_{j,j+1}=Ran(\beta_{j,j+1})$.

In other words, a †-(u.)derivation is a sequence of successive (u.)resolvents in which a variable, once released, never is re-introduced, either by a goal or by a specification.

Remark. [Bol 1991] and [Doets 1990] note that every derivation in the usual sense in fact is a †-derivation.

Note that, in *constructing* derivations, we can trivially satisfy [†]:

3.15 Lemma. Suppose that Δ: N_0, $\beta_{0,1}$;N_1,...,$\beta_{n,n+1}$;N_{n+1} is a †-derivation.

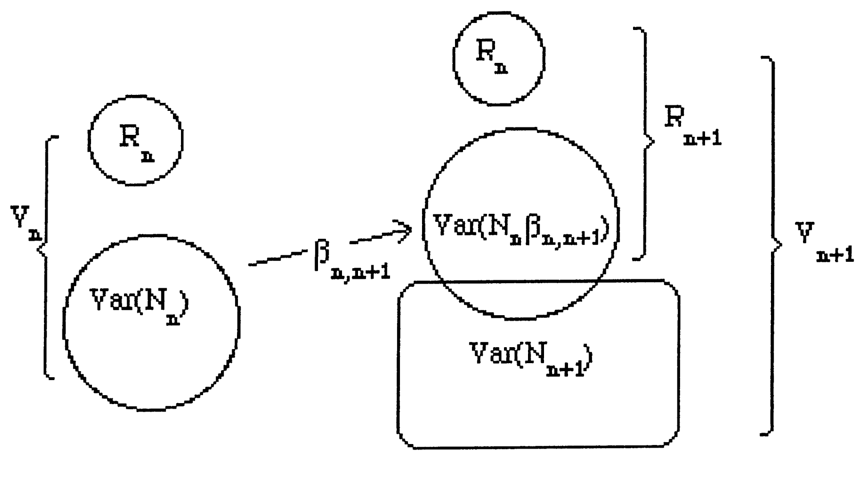
If N_{n+1} has a resolvent w.r.t. an atom and a rule then it has a resolvent β;N w.r.t. the same atom and rule such that N_0, $\beta_{0,1}$;N_1,...,$\beta_{n,n+1}$;N_{n+1}, β;N is a †-derivation.

The next definition brings some order in the ways of the variable.

3.16 Definition.

Suppose that Δ: N_0, $\beta_{0,1}$;N_1,...,$\beta_{n,n+1}$;N_{n+1},... is a †-(u.)derivation of length λ.

For each $n<\lambda$, define sets R_n and V_n of variables as follows.



R_n is the set of variables released in Δ at some step $\leq n$.

Thus: $R_0 = \emptyset$;

$R_{n+1} = R_n \cup (Var(N_n\beta_{n,n+1}) \setminus Var(N_{n+1}))$.

V_n is the set of variables "relevant" at stage n.

Thus: $V_n = Var(N_n) \cup R_n$;

or, equivalently: $V_0 = Var(N_0)$;

$V_{n+1} = Var(N_{n+1}) \cup Var(N_n\beta_{n,n+1}) \cup R_n$.

Note that, except for the last set V_{m+1} in case $\lambda = m+1 < \omega$, these sets can be defined in terms of the sequence of specifications of Δ, independently of the occurring goals: $Var(N_n\beta_{n,n+1}) = Ran(\beta_{n,n+1})$ anyway, and for $n<\lambda$ we always have $Var(N_n) = Dom(\beta_{n,n+1})$.

Therefore, these definitions make sense also with respect to, e.g., an infinite sequence of substitutions $\beta_{0,1}, \beta_{1,2}, \ldots$.

Observation. [†] exactly says that, for all $n<\lambda$: $R_n \cap (Var(N_{n+1}) \cup Var(N_n\beta_{n,n+1})) = \emptyset$.

We shall not use the fact that $V_n = Var(N_0\beta_{0,1}\cdots\beta_{n-1,n}, N_1\beta_{1,2}\cdots\beta_{n-1,n}, \ldots, N_{n-1}\beta_{n-1,n}, N_n)$.

Remark. A slightly different definition of u.derivation could run as follows: A sequence Δ: $N_0, \beta_{0,1}; N_1, \ldots$ such that for all i: $Dom(\beta_{i,i+1}) = V_i$, and $\beta_{i,i+1}|N_i; N_{i,i+1}$ is u.resolvent of N_i.

The proof of 3.19 below would become even simpler yet; but it is questionable whether the additional freedom in specifying variables would make much sense. (There is little point in going on transforming variables which do not occur in the goals any longer.)

IV. Lifting.

3.17 Definition. Suppose that $N_0 \leq M_0$ (say, $N_0 = M_0\sigma$).

If Γ: $N_0, \ldots$ is an unrestricted derivation and Δ: $M_0, \ldots$ is a derivation (resp., a †-derivation) which is similar to Γ (w.r.t. the initial correspondence between N_0 and M_0 mapping $A \in M_0$ to $A\sigma \in N_0$), then we say that Δ is a **lift** (resp. a **†-lift**) of Γ w.r.t. σ.

The proper handling of domains is critical for the following lemma (which otherwise is trivial and standard equipment in the context of commuting diagrams).

If $\underline{\theta}$: $\theta_{0,1}, \theta_{1,2}, \ldots, \theta_{i,i+1}, \ldots$ $(i<\lambda)$ is a sequence of substitutions of length λ, we use the notation $\theta_{i,j}$ for the composition $\theta_{i,i+1}\cdots\theta_{j-1,j}$ $(0 \leq i < j < \lambda)$.

3.18 Lemma.

Suppose that Δ: $N_0, \beta_{0,1}; N_1, \ldots, \beta_{n,n+1}; N_{n+1}, \ldots (n<\lambda)$ is an u.derivation of length λ.

Let $\underline{\alpha}$: $\alpha_{0,1}, \alpha_{1,2}, \alpha_{2,3}, \ldots$ and $\underline{\sigma}$: $\sigma^0, \sigma^1, \sigma^2, \ldots$ be sequences of substitutions such that for all $i<\lambda$:

$$Dom(\sigma^i) = V_i, \text{ and}$$
$$(\sigma^i\alpha_{i,i+1})|V_i = (\beta_{i,i+1}\sigma^{i+1})|V_i.$$

Then for all i,j: if $i<j<\lambda$ then $(\sigma^i\alpha_{i,j})|V_i = (\beta_{i,j}\sigma^j)|V_i$.

Proof. Induction w.r.t. j-i.

j=i+1: This is the hypothesis.

j>i+1: Suppose that $x \in V_i$.

Then: $x\beta_{i,j+1}\sigma^{j+1} = x\beta_{i,i+1}\beta_{i+1,j+1}\sigma^{j+1}$

$= x\beta_{i,i+1}\sigma^{i+1}\alpha_{i+1,j+1}$, since $Var(x\beta_{i,i+1}) \subset V_{i+1}$, by induction hypothesis

$=x\sigma^i\alpha_{i,i+1}\alpha_{i+1,j+1}$, by hypothesis
$=x\sigma^i\alpha_{i,j+1}$. ⊠

The following (or rather, its corollary 3.21) is the main result of this section.

3.19 Lifting Often.

1. Every u.derivation Γ: $M_0=N_0\sigma,\ldots$ has a †-lift Δ: $N_0,\ldots$.
2. If Δ: N_0, $\beta_{0,1};N_1,\ldots$ is a †-lift of the u.derivation Γ: $M_0=N_0\sigma$, $\alpha_{o,1};M_1,\ldots$ of length λ, then there exists a (unique) sequence $\underline{\sigma}$: $\sigma^0,\sigma^1,\sigma^2,\ldots$ of substitutions such that:
    1. $\sigma^0=\sigma$;

    for all $j<1+\lambda$:

    2. $Dom(\sigma^j)=V_j$;
    3. $M_j=N_j\sigma^j$;

    and if i<j:

    4. $(\sigma^i\alpha_{i,j})|V_i=(\beta_{i,j}\sigma^j)|V_i$;
    5. $rest(\Gamma_{i,j})=rest(\Delta_{i,j})\sigma^j$.

(N.B. $\Gamma_{i,j}$ is the subderivation of Γ which derives M_j from M_i.)

Conditions 2-4 are visualized by the following diagram, commuting on the domains indicated:

$$\begin{array}{lllllll}
\Delta: & N_0 & \to & \beta_{0,1};N_1 & \to\ldots\to & \beta_{n-1,n};N_n & \to & \beta_{n,n+1};N_{n+1} \to\ldots \\
\underline{\sigma}: & \downarrow\sigma^0=\sigma & & \downarrow\sigma^1 & & \downarrow\sigma^n & & \downarrow\sigma^{n+1} \\
\Gamma: & M_0 & \to & \alpha_{0,1};M_1 & \to\ldots\to & \alpha_{n-1,n};M_n & \to & \alpha_{n,n+1};M_{n+1} \to\ldots
\end{array}$$

Proof. Part 1 easily follows from "lifting once" and 3.15. We concentrate on part 2.
Note that 5 is an immediate consequence of 3/4.
Suppose that $\sigma^0,\ldots,\sigma^n$ have been found, satisfying 1-4 for $j\leq n$.
By "lifting once", a substitution τ exists such that both $M_{n+1}=N_{n+1}\tau$ and $N_n\sigma^n\alpha_{n,n+1}=M_n\alpha_{n,n+1}=N_n\beta_{n,n+1}\tau$.
Since Δ satisfies [†], by observation 3.16 we have $R_n\cap Var(N_{n+1},N_n\beta_{n,n+1})=\emptyset$.
Therefore, we can define σ^{n+1} on $V_{n+1}=R_n\cup Var(N_{n+1},N_n\beta_{n,n+1})$ by:

$\sigma^{n+1}|\ Var(N_{n+1},N_n\beta_{n,n+1})=\tau|\ Var(N_{n+1},N_n\beta_{n,n+1})$;

$\sigma^{n+1}|\ R_n=(\sigma^n\alpha_{n,n+1})|\ R_n$.

It follows that $N_{n+1}\sigma^{n+1}=N_{n+1}\tau=M_{n+1}$, and $(\sigma^n\alpha_{n,n+1})|\ V_n=(\beta_{n,n+1}\sigma^{n+1})|V_n$: for $x\in Var(N_n)$, $x\beta_{n,n+1}\sigma^{n+1}=x\beta_{n,n+1}\tau=x\sigma^n\alpha_{n,n+1}$; and for $x\in R_n$, $x\beta_{n,n+1}\sigma^{n+1}=x\sigma^{n+1}=x\sigma^n\alpha_{n,n+1}$.
We have satisfied conditions 2,3 and 4 for i=n and j=n+1. To obtain 4 for $i<j\leq n+1$ generally, apply lemma 3.18. ⊠

Remarks.

1. Part 5 of 3.19.2 is what constitutes (a "local" version of) usual formulations of lifting. Note that 3/4 actually are somewhat stronger than 5 as they may involve more variables (cf. the end of 3.16).
2. A special case of 3.19.2 arises when Γ is ground (or even, if Γ is "J-ground", where J is an arbitrary algebra extending the the standard Herbrand-algebra of closed terms). The specifications

$\alpha_{i,i+1}$ of Γ then are empty and the σ^i transform easily into model-theoretic assignments into the Herbrand-algebra (resp. into J). This is the type of situation prominent in section 4.

3.20 Corollary. An u.derivation is a derivation iff it is a †-derivation.

Proof. Suppose that Δ is a †-derivation. Let Δ* be a finite subderivation of Δ. Then obviously, Δ* is a †-derivation as well. Hence by 3.19, if Γ is similar to Δ*, starting with the same goal, then rest(Γ)≤rest(Δ*). It follows that Δ is a derivation.

Conversely, suppose that Δ: $\leftarrow C_0, \alpha_0; \leftarrow C_1, \ldots$ is a derivation which is not a †-derivation. Say, $x \in Ran(\alpha_i) \setminus Var(C_{i+1})$ and j>i is such that either $x \in Var(C_{j+1})$ or $x \in Ran(\alpha_j)$. Consider the subderivation $\Gamma = \Delta_{i,j+1}$: $\leftarrow C_i, \alpha_i; \leftarrow C_{i+1}, \ldots, \alpha_j; \leftarrow C_{j+1}$. It is easy to see that, since Γ "unnecessarily" identifies variables, rest(Γ) is not most general among the resultants of similar derivations from the same goal; whence Δ cannot have been a derivation. ⊠

3.21 Generalized lifting theorem.

In 3.19.1/2, replace "†-lift" by "lift". The resulting statements continue to hold.

V. Miscellaneous.

Of course, a (u.)*refutation* is a (u.)derivation of the empty goal.

3.22 Corollary. If Nγ has an u.refutation by the program P in n steps via specifications $\alpha_0, \ldots, \alpha_n$, then N has a (similar) refutation by P in n steps via relevant $\beta_0, \ldots, \beta_n$ such that for some σ: $N\gamma\alpha_0 \ldots \alpha_n = N\beta_0 \ldots \beta_n\sigma$.

Proof. A suitable lift will do. ⊠

Remarks. Note that we *do not* assert the "global" result that for some σ: $\gamma\alpha_0 \ldots \alpha_n = \beta_0 \ldots \beta_n\sigma$. Cf. [Apt 1990], lemma 3.19, where this is claimed for the mgu's of which our specifications are restrictions, and then only for the case that the first u.refutation is a refutation. By [Ko/Nadel 1990], this will be false without further conditions.

As we suggested above, the sensibility of such a global result may be questioned, since the mgu's will have "irrelevant" parts. Proposition 3.2.1 of [Ko/Nadel 1990] is a version of our 3.22, but both statement and proof are much more involved. However, they also have a global result (corollary 3.2.1).

The following is a version of a result due to Lloyd and Shepherdson (cf. [Apt 1990] lemma 2.8) which, instead of being awkward to prove and dependent on the special way resolvents are constructed, now becomes a triviality:

3.23 The variant lemma. Suppose that $N_0 \sim M_0$.

If Δ: $N_0, \ldots$ and Γ: $M_0, \ldots$ are finite derivations which are similar, then rest(Δ) ~ rest(Γ).

Proof. Immediate from 3.21.2. ⊠

4. Sheaves of ground derivations and "reverse lifting".

Introduction.

Let ←C be a goal which is ground. By lifting, a resolvent of ←C in a sense represents all ground u.resolvents of ←C. More generally, an SLD search-tree for ←C represents a sheaf of ground u.derivations.

The main virtue of an SLD search-tree is that it is finitely splitting. On the negative side, search trees are not able to distinguish between atoms in $T\!\downarrow\!\backslash T\!\uparrow$ and $T\!\downarrow\!\omega\backslash T\!\downarrow$: the corresponding goals will generate infinite failing trees always. The sheaves of ground u.derivations they represent have an opposite behaviour in these respects: generally, these will be *not* finitely splitting, but *do* distinguish such goals, as we shall show - and make use of - below.

Some of our basic results on sheaves are obtained by what might be called *reverse lifting*, that is: constructing ground derivations which have a lift in an SLD search tree given in advance.

We first formally introduce sheaves of ground u.derivations represented by search trees (see 4.3 below). However, we generalize the discussion, allowing for non-standard algebras as well. Cf. [Apt 1990] for more detail.

Algebras.

An *algebra* (or *pre-interpretation*) is a model of the *free equality axioms* interpreting all function symbols (but no relation symbol). The canonical Herbrand algebra over the Herbrand universe HU of closed terms forms but one example. (Algebras satisfying a suitable *domain-closure axiom* are nothing but elementary equivalents of HU.) Within isomorphism, every algebra extends HU. It is *non-standard* if it extends HU properly.

There are *two* types of non-standardness (which may coexist). First, we can extend HU with extra generators: let V be any set of variables, and consider the canonical algebra of all terms over V. This simple type of algebra spoils the domain-closure axiom, but we find a use for it below.

Second, we can extend HU with "non-well-founded" elements. A simple example is the successor structure on a type-$\omega+\zeta$ ordering, which is an elementary equivalent of the standard algebra (**N**,0,S) of the natural numbers (one individual constant, one unary function symbol).

We shall deal with this type of extension extensively.

For the sequel, let an arbitrary algebra J be fixed.

Operators.

Extend the language, admitting all elements of J as new individual constants.

Since we may assume $J\supset HU$ and elements of HU are (named by) closed terms, it in fact suffices to add constants for elements in $J\backslash HU$ only.

HB^J, the *Herbrand base* over J, is the set of all atomic formulas $R(a_1,\ldots,a_n)$ where $a_1,\ldots,a_n$ are either new constants from $J\backslash HU$ or closed terms from HU. Note that this definition makes HB^J extend the standard Herbrand base HB of closed atoms of the old language.

A *model* over J is identified with a subset of HB^J.

Every program P induces a monotone operator $T^J = T_P{}^J$ over HB^J. $T^J\!\uparrow$ is the least fixed point of T^J, $T^J\!\downarrow$ its greatest fixed point, and $T^J\!\downarrow\!\omega$ the ω-th approximation thereof.

J-ground resolution.

A J-*assignment* is a map from variables to elements of J.

If t is a term and σ is a J-assignment such that $Var(t) \subset Dom(\sigma)$, then $t\sigma = t^J[\sigma]$ denotes the value of t in J under σ. If $A=R(t_1,\ldots,t_n)$ is an atom then $A\sigma := R(t_1\sigma,\ldots, t_n\sigma) \in HB^J$; if K is a goal then $K\sigma := \{A\sigma | A \in K\}$. Atoms and goals of these forms are called J-**ground**.

We sometimes write $L \leq K$ also if $L=K\sigma$ for some J-assignment σ.

The J-ground goal M is called J-**ground resolvent** of the J-ground goal $N=(\leftarrow A,K)\sigma$ w.r.t. $A\sigma$ and the rule P if for some L: $M=\leftarrow L,K\sigma$ and $(A\sigma \leftarrow L) \leq P$; of course, we put rest(M,N) equal to the implication $L,K\sigma \rightarrow A\sigma,K\sigma$.

The material of the previous section can be easily transferred to the present context:

3.10A Lifting a J-ground resolution step.

2. If the J-ground goal $N\sigma$ has a J-ground resolvent w.r.t. $A\sigma$ and P then N has a resolvent w.r.t. A and P.
3. If R^- is resultant of a J-ground resolvent of the J-ground goal $N\sigma$ w.r.t. $A\sigma$ and P, and R is resultant of a resolvent of N w.r.t. A and P, then $R^- \leq R$.

Proof. Similar to 3.5, by noting that the Martelli-Montanari-algorithm "works over J" as well as over the canonical Herbrand-algebra, since J is assumed to satisfy the free equality-axioms; cf. [Apt 1990], corollary 5.22. I.e.: if A unifies over J with the head B of P (that is: if $A\sigma=B\tau$ for some J-assignments σ and τ) then A unifies with B over the standard Herbrand algebra as well (that is: $A\alpha=B\beta$ for some substitutions α and β). In fact, if $A\sigma=B'\sigma$ for some J-assignment σ (B' the head of a variables-separated variant P' of P) then $A\theta=B'\theta$ for an algorithmically produced mgu θ with the property that $\sigma=\theta\sigma$ on *all* variables, etc. ⊠

Realizing a sequence of substitutions.

Suppose that $\underline{\theta}$: $\theta_{0,1}, \theta_{1,2}, \theta_{2,3} \ldots$ is a sequence of consecutive specifications, which may be thought of as taken from a derivation Γ: $N_0, \theta_{0,1};N_1, \theta_{1,2};N_2,\ldots$ of any length $\lambda \leq \omega$.

As before, R_n is the set of variables released by Γ at some stage $\leq n$; V_n is the set of variables relevant at stage n. Again, we use the notation $\theta_{i,j}$ for the composition $\theta_{i,i+1}\ldots\theta_{j-1,j}$ $(i<j<\lambda)$.

The following notion will be central to this section.

4.1 Definition. The sequence of J-assignments $\underline{\sigma}$: $\sigma^0, \sigma^1, \sigma^2, \ldots$ of length λ **realizes** $\underline{\theta}$ in the algebra J if for all $j<\lambda$: $Dom(\sigma^j)=V_j$ and $\sigma^i|V_i=(\theta_{i,j}\sigma^j)|V_i$ whenever $i<j$.

Example: if $\underline{\sigma}$: $\sigma^0, \ldots$ realizes the constant sequence of infinite length $\{x/Sx\},\{x/Sx\},\{x/Sx\}, \ldots$ in J, then we must have $\sigma^0(x)=S\,\sigma^1(x)=SS\,\sigma^2(x)= \ldots =SSS \ldots$ in J.

Lifting a J-ground derivation.

A J-ground **derivation** is nothing but a (finite or infinite) sequence of successive J-ground resolvents.

The notions of similarity, lift and resultant are easily adapted to this situation.

3.19A Lifting theorem for J-ground derivations.

1. Every J-ground derivation Γ: $N\sigma$,... has a lift Δ: N,... .
2. If Δ: N_0, $\theta_{0,1}$;N_1, ... lifts the J-ground derivation Γ: $M_0=N_0\sigma$, M_1,...,
   then a (unique) sequence $\underline{\sigma}$: $\sigma^0=\sigma|N_0$, σ^1, σ^2,... exists such that for all j:
   (i) $Dom(\sigma^j)=V_j$
   (ii) $M_j=N_j\sigma^j$
   (iii) $\underline{\sigma}$ realizes $\underline{\theta}$: $\theta_{0,1}$, $\theta_{1,2}$,... .

Proof. Compare 3.19. ⊠

A realization of a sequence of substitutions transforms any derivation involving it into a J-ground derivation:

4.2 Lemma. Suppose that Δ: N_0, $\theta_{0,1}$;N_1, $\theta_{1,2}$;N_2,... is a (finite or infinite) derivation, and $\underline{\theta}$ is the associated sequence $\theta_{0,1}$, $\theta_{1,2}$,... of specifications.
If $\underline{\sigma}$: σ^0, σ^1,... realizes $\underline{\theta}$ in J, then Γ: $N_0\sigma^0$, $N_1\sigma^1$,... is a J-ground-derivation of which Δ is a lift.

Proof.
By inspection. Suppose that the following is a resolution-step w.r.t. A and P:

N: $\leftarrow$A,K

$(A\theta\leftarrow L)\leq P$, specification θ

M: $\leftarrow L,K\theta$.

Assume that σ and τ are J-assignments such that $\sigma=\theta\tau$ on Var(N), i.e, that (σ,τ) realizes θ.
Then the following is a J-ground resolution-step:

$N\sigma$: $\leftarrow A\sigma,K\sigma$

$A\sigma\leftarrow L\tau=(A\theta\leftarrow L)\tau\leq P$

$M\tau$: $\leftarrow L\tau,K\theta\tau$. ⊠

Remark. By 3.19A and 4.2, for a given derivation Δ: N,... and J-assigment σ:Var(N)$\to$J, realizing sequences for (the specifications of) Δ starting with σ on the one hand and J-ground derivations Γ: $N\sigma$,... starting from $N\sigma$ on the other are tantamount.

Sheaves of realizations.

The following defines our main tool: the sheaf of J-realizations (or J-ground derivations) represented by an SLD search-tree.

4.3 Definition. Suppose that **B** is an SLD search-tree for the goal N_0 and σ:Var(N_0)$\to$J.
SH(**B**,σ)=SH(**B**,J,σ), the **sheaf of realizations** over J and σ represented by **B** is the set of pairs (Δ, $\underline{\sigma}$) such that for some n:

1. Δ: N_0, $\theta_{0,1}$;N_1,...,$\theta_{n-1,n}$;N_n is (initial of) a derivation through **B**
2. $\underline{\sigma}$: σ^0,..., σ^n is a sequence of J-assignments with $\sigma^0=\sigma$
3. $\underline{\sigma}$ realizes $\underline{\theta}$: $\theta_{0,1}$,...,$\theta_{n-1,n}$.

We make SH(**B**,σ) into a tree by putting (Γ,$\underline{\tau}$) before (Δ,$\underline{\sigma}$) iff both Γ extends Δ and $\underline{\tau}$ extends $\underline{\sigma}$.

By the remark following 3.19A and 4.2, we could have defined SH(**B**,σ) equivalently as a sheaf of J-ground derivations, i.e., as the set of (Δ,Γ) where Γ: $N_0\sigma,\dots$ is a finite J-ground derivation of which the lift Δ: $N_0,\dots$ is in **B**.

NOTE. In the sequel, we shall take the viewpoint which suits best for the situation at hand.

If **B** is infinite, then, by König's lemma, it necessarily is non-well-founded.
Nevertheless, J need not at all realize any of the infinite sequences of specifications going with an infinite derivation through **B** and hence SH(**B**,σ) may very well be well-founded.
The complete picture is given by 4.4/6 below.

The *height* of a well-founded tree is measured by an ordinal. (By results of [Blair 1982], greatest fixed-point hierarchies and hence, sheaf-trees over J=HU, can have any height $\leq\omega_1^{CK}$.)
To have *infinite height* means to have either ordinal height $\geq\omega$ or being non-well-founded.

Reverse lifting.
A derivation can be looked at as the result of playing a game in which two players I and II move alternatingly, I selecting an atom in the goal at hand and II picking an applicable rule, after which a resolvent w.r.t. atom and rule emerges which forms a next position in the game.
Thus, an SLD search-tree can be seen as the space of all plays (given some initial goal) in which I uses a specific *selection rule*, which is nothing but a strategy for the first player.
The first halves of the basic results 4.4/6 below are obtained by an argument for which we coin the name *reverse lifting*. A selection rule/strategy for the first player is given by means of a search tree. The second player counters by lifting a shadow-J-ground derivation which he has to construct simultaneously. Hence, we could view the procedure a producing a ground derivation from its lift in the tree (instead of the other way around).

4.4 Success Theorem.

Let **B** be an SLD search-tree for the goal $N_0=\leftarrow C_0$ and assume σ:Var$(C_0)\rightarrow$J.

The following are equivalent:

1. $C_0\sigma\subset T^J\uparrow$.
2. SH(**B**,σ) contains an element $(\Delta,\underline{\sigma})$ of which the first component Δ is a refutation.

Remark. If Δ: $\leftarrow C_0$, $\theta_{0,1};\leftarrow C_1,\dots,\theta_{n-1,n};\leftarrow C_n$ is a refutation, i.e., $C_n=\square$, then clearly rest(Δ) can be identified with $C_0\theta_{0,n}$.
If in condition 2, $\underline{\sigma}$ is: $\sigma^0,\dots,\sigma^n$, then $C_0\sigma=C_0\sigma^0=C_0\theta_{0,n}\sigma^n$; i.e.: $C_0\sigma\leq$rest(Δ).

The following corollary is mentioned by [Apt 1990] section 3.11 as the "strongest completeness result" of the subject. Our proof employs the simplest type of non-standard algebra. (The proof in [Apt 1990] for the weaker 3.18 uses "fresh" individual constants and can be looked at as a proof-theoretic counterpart of our model-theoretic argument.)

4.5 Strong Completeness. (Clark)

If $\forall(C\theta)$ logically follows from P, then every SLD search tree for $\leftarrow$C contains a refutation Δ such that $C\theta\leq$rest(Δ).

Proof. Let V:=Var($C\theta$). Let J be the canonical algebra of all terms over V. If $\forall(C\theta)$ logically follows from P then, since $T^J\uparrow$ is a model of P, $C\theta\subset T^J\uparrow$. The result now is immediate from the $1\Rightarrow2$-half of the theorem. ⊠

Proof of 4.4.
$2\Rightarrow1$. Suppose that **B** contains a refutation Δ such that $C_0\sigma\leq rest(\Delta)$.
Since $T^J\uparrow$ is a model of P, $\forall rest(\Delta)$ logically follows from P, and (since $C_0\sigma\leq rest(\Delta)$) $C_0\sigma$ logically follows from $\forall rest(\Delta)$, we must have $C_0\sigma\subset T^J\uparrow$.
$1\Rightarrow2$.
The proof is by reverse lifting and uses induction along a certain well-founded relation between finite multisets $\subset T^J\uparrow$.
Verifying well-foundedness of the relation can be done most easily by employing a map to ordinals $<\omega^\omega$.
The **rank** of $A\in T^J\uparrow$, notation: r(A); is the least n such that $A\in T^J\uparrow(n+1)$. Let $K\subset T^J\uparrow$ be a finite multiset. Define $R:=\{r(A)|A\in K\}$. R is finite and may be enumerated as $R=\{n(i)|i<k\}$, where $n(0)>n(1)>\ldots>n(k-1)$. Put m(i) equal to the number of $A\in K$ with rank n(i) ($i<k$).
Finally, define the **rank** of the clause K by: $r(K)=\omega^{n(0)}\cdot m(0)+\ldots+\omega^{n(k-1)}\cdot m(k-1)$.
(Of course, $r(\Box)=0$.)
The point of this assigning ordinal-values to multisets $K\subset T^J\uparrow$ is the
Claim: If $\leftarrow L$ is a J-ground resolvent of $\leftarrow K$, $K\subset T^J\uparrow$, then $r(L)<r(K)$.
Proof: clear.
Now suppose that $C_0\sigma\subset T^J\uparrow$. Construct a J-ground derivation $\leftarrow D_0, \leftarrow D_1,\ldots$ with $D_0=C_0\sigma$, $D_i\subset T^J\uparrow$, using the selection rule of **B**, i.e., such that it has a lift Δ: $\leftarrow C_0, \theta_{0,1};\leftarrow C_1,\ldots$ in **B**.
As long as D_i is non-empty, the construction must continue by 3.10A2. In detail: supposing $D_i\subset T^J\uparrow$ and $C_i\in$ **B** constructed and non-empty, let A be the atom of D_i corresponding to the atom in C_i which is selected by **B**. Choose a J-ground resolvent $\leftarrow D_{i+1}$ of $\leftarrow D_i$ w.r.t. A (D_{i+1} exists since $A\in D_i\subset T^J\uparrow=T^J(T^J\uparrow)$). Let $\theta_{i,i+1};\leftarrow C_{i+1}$ be its lift in **B**.
Therefore, by the claim, we cannot fail to reach empty goals (in the J-ground derivation and, hence, in Δ) eventually. ⊠

The infinite derivation Γ: $\leftarrow C_0, \theta_{0,1};\leftarrow C_1,\ldots$ is *fair* if for all i and $A\in C_i$ there exists $j\geq i$ such that the *descendant* $A\theta_{i,j}$ of A in C_j is selected in $\leftarrow C_j$.
The SLD search-tree **B** is *fair* if all its infinite branches are fair.
A selection rule is *fair* if every infinite derivation produced by it is fair.

4.6 Failure Theorem.

Let **B** be an SLD search-tree for the goal $N_0=\leftarrow C_0$ and assume $\sigma:Var(C_0)\to J$.

1. On (in)finite height.
   a. Suppose not $C_0\sigma\subset T^J\uparrow$.
      If $C_0\sigma\subset T^J\downarrow\omega$ then SH(**B**,σ) has infinite height.
   b. Suppose that **B** is fair.
      Conversely: if SH(**B**,σ) has infinite height then $C_0\sigma\subset T^J\downarrow\omega$.
2. On well-foundedness.

a. Suppose not $C_0\sigma \subset T^J\uparrow$.

If $C_0\sigma \subset T^J\downarrow$ then SH(**B**,σ) is not well-founded.

b. Suppose that **B** is fair.

Conversely: if SH(**B**,σ) is not well-founded then $C_0\sigma \subset T^J\downarrow$.

Remarks.

1.It is easy to deduce the theorem on the characterization of finite failure ([Apt 1990], theorem 5.6) from 4.6.1. Just take J=HU, A a ground atom, C_0=A and σ empty.

In fact, if $A \in T\downarrow\omega$ and **B** fails, then (by 4.4) $A \notin T\uparrow$, whence, by 4.6.1a, SH(**B**,σ) has infinite height, and it follows that **B** must be infinite.

Also, if $A \notin T\downarrow\omega$ then, by 4.6.1b, if **B** is fair, SH(**B**,σ) has finite height; whence **B** is finite.

2. Similarly, 4.6.2 generalizes the theorem on soundness and completeness of negation as failure. For completeness, see below.

Soundness asserts that every ground atom in the finite failure set of the program P is refuted by the completion Comp(P) of P. To see this, assume the model M of Comp(P) satisfies the ground atom A. Let J be the underlying algebra of M. Clearly then, $M \subset T^J\downarrow$; whence $A \in T^J\downarrow$. Apply the following lemma with σ empty (also, compare [Apt 1990], lemma 6.2):

Lemma. If $A\sigma \in T^J\downarrow$ then no search tree for ←A is finite.

Proof. Assume $A\sigma \in T^J\downarrow$ and let **B** be any search tree for ←A. By 4.6.2a, SH(**B**,J,σ) has an infinite branch, whence **B** cannot be finite. ⊠

Proof of 4.6. We begin with 2a/b since there, the argument is slightly simpler.

2a. Reverse lifting. We need to show that SH(**B**,σ) has an infinite branch.

Simultaneously, produce the infinite derivation $\Delta = \leftarrow C_0, \theta_{0,1}; \leftarrow C_1, \ldots$ through **B** and a J-ground derivation $\Gamma: \leftarrow D_0, \leftarrow D_1, \ldots$ (where $D_0 = C_0\sigma$), of which Δ is a lift, such that, for all i: $D_i \subset T^J\downarrow$.

To start with, by hypothesis: $D_0 \subset T^J\downarrow$.

Suppose that Δ and Γ have been constructed up to and including stage i such that $D_i \subset T^J\downarrow$.

Let **B** select the atom A in C_i. Let $A\sigma^i$ be the atom of D_i corresponding to A.

Since $A\sigma^i \in D_i \subset T^J\downarrow = T^J(T^J\downarrow)$, $\leftarrow D_i$ has a J-ground resolvent $\leftarrow D_{i+1}$ w.r.t. $A\sigma^i$.

By 3.10A, this can be lifted to a resolvent $\theta_{i,i+1}; \leftarrow C_{i+1}$ of $\leftarrow C_i$ in **B**.

2b. Suppose that $\Delta = \leftarrow C_0, \theta_{0,1}; \leftarrow C_1, \ldots$ is an infinite derivation through **B** realized by the sequence $\underline{\sigma}$: $\sigma^0 = \sigma, \ldots$ in J. Define the model X over J by: $X := \{A\sigma^i | A \in C_i\}$.

It suffices to prove the

Claim. $X \subset T^J(X)$.

For then $X \subset T^J\downarrow$ (since $T^J\downarrow$ is the greatest model with this property), and the required result follows.

To prove the claim, suppose that $A\sigma^i \in X$, i.e., $A \in C_i$. We have to show that $A\sigma^i \in T^J(X)$.

Since Δ is fair, at some place j≥i the descendant of A at that place, which is $A\theta_{i,j}$, must be selected.

Then for some instance B←K of a P-rule, $A\theta_{i,j+1} = A\theta_{i,j}\theta_{j,j+1} = B$ and K is part of C_{j+1}.

By definition of X we have $K\sigma^{j+1} \subset X$.

Therefore, $B\sigma^{j+1} \in T^J(X)$. However, $A\sigma^i = A\theta_{i,j+1}\sigma^{j+1} = B\sigma^{j+1}$.

1a. Reverse lifting. Fix n. We want to show that SH($\mathbf{B}$,σ) has a branch of length at least n. Simultaneously, produce $\Delta=\leftarrow C_0, \theta_{0,1};\leftarrow C_1,\ldots,\theta_{n-1,n};\leftarrow C_n$ through **B** and and a J-ground derivation Γ: $\leftarrow D_0, \leftarrow D_1,\ldots$ (where $D_0=C_0\sigma$), of which Δ is a lift, such that:
[*] for all $j\leq n$: $D_j\subset T^J\!\downarrow\!(n-j)$.
By assumption, $D_0=C_0\sigma\subset T^J\!\downarrow\!\omega\subset T^J\!\downarrow\! n$; which means that we satisfy the requirement [*] for n=0.
Suppose that Δ and Γ have been constructed up to and including index i<n, satisfying [*] for n=i.
If $C_i=\Box$ then $D_i=\Box$ as well and we have a J-ground refutation of $\leftarrow C_0\sigma$, whence $C_0\sigma\subset T^J\!\uparrow$, contrary to hypothesis. Therefore, $C_i\neq\Box$.
Suppose then, that **B** selects the atom A in C_i. Let $A\sigma^i$ in D_i correspond to A.
Since $A\sigma^i\in D_i\in T^J\!\downarrow\!(n-i)$, n-i>0, $\leftarrow D_i$ has a J-ground resolvent $\leftarrow D_{i+1}$ w.r.t. $A\sigma^i$.
By 3.10A, $\leftarrow C_i$ has a resolvent $\theta_{i,i+1};\leftarrow C_{i+1}$ w.r.t. A in **B**, as required.

1b. Let **B*** be the subtree of **B** consisting of all derivations Γ which possess a realization $\underline{\sigma}$ in J with σ as first element, i.e., such that $(\Gamma,\underline{\sigma})\in$ SH($\mathbf{B}$,σ). By assumption, **B*** has infinite height. Since **B*** is finitely splitting, by König's lemma, it must have an infinite branch
$\Delta=\leftarrow C_0, \theta_{0,1};\leftarrow C_1,\ldots$.
Fix n. We show that $C_0\sigma\subset T^J\!\downarrow\! n$.
By fairness, choose indices m(0)=0<m(1)<...<m(n) (in *that* order) such that (a descendant of) every atom in $C_{m(i)}$ has been selected in Δ at or before stage m(i+1) (i<n).
Choose $\underline{\sigma}$: $\sigma^0=\sigma,\ldots,\sigma^{m(n)}$ realizing $\underline{\theta}$: $\theta_{0,1},\ldots,\theta_{m(n)-1,m(n)}$.
It follows that, for $i\leq n$, $C_{m(i)}\sigma^{m(i)}\subset T^J\!\downarrow\!(n-i)$ (induction w.r.t. n-i); whence, in particular, $C_0\sigma^0\subset T^J\!\downarrow\! n$. ⊠

Remarks.
1. The proof of 2b constitutes, in simplified form, half of the usual argument for the so-called *completeness of negation as failure* - cf. 4.8. (The other half consists in the construction af an algebra realizing the sequence of specifications of Δ; cf. the remark following 4.7.)
2. There is an obvious connection between 4.6.2 and the Kleene-Spector analysis of $\Pi^1{}_1$-sets, cf., for instance [Doets 199?].
3. [Doets 1990] contains refinements of 4.6.2 connecting the ordinal height of sheaf-trees to the drop-out stage in the greatest fixed-point hierarchy. (The connection is not completely exact, due to the fact that in a derivation, we can only resolve one atom at the time.)

The interplay of search trees and sheaves: $T^J\!\downarrow$ and $T^J\!\downarrow\!\omega$.
The following is a compactness phenomenon for the present context.
4.7 Lemma. Let **B** be an SLD search-tree for the goal $N_0=\leftarrow C_0$ and suppose σ:Var(C_0)$\rightarrow$J.

If SH($\mathbf{B}$,J,σ) has infinite height, then for some $I\supset J$, SH($\mathbf{B}$,I,σ) is not well-founded.

Proof. Suppose that SH($\mathbf{B}$,J,σ) has infinite height. Argue as in the proof of 4.6.1b above. Let **B*** be the subtree of **B** consisting of all Γ which possess a realization $\underline{\sigma}$ in J starting with σ, i.e., such that $(\Gamma,\underline{\sigma})\in$ SH($\mathbf{B}$,σ). By assumption, **B*** has infinite height. Since **B*** is finitely splitting, by König's lemma, it must have an infinite branch $\Delta=\leftarrow C_0, \theta_{0,1};\leftarrow C_1,\ldots$. By assumption on **B***, every finite sequence $\theta_{0,1},\ldots,\theta_{n-1,n}$ is realized in J by a sequence starting with σ. By the compactness theorem for first-order logic, the infinite sequence $\theta_{0,1},\ldots$ is realized in some elementary extension I of J by a sequence starting with σ. But then, SH($\mathbf{B}$,I,σ) is not well-founded. ⊠

Remark. For J=HU, there is an easy, "algebraic" proof for 4.7, not using the compactness theorem. In that case, I is obtained from J=HU by the adjunction of non-standard elements, the exact nature of which is prescribed by the sequence $\theta_{0,1},\ldots$.

The next result generalizes the theorem on completeness of negation as failure.

4.8 Corollary. Let A be an atom and σ:Var(A)→J.

If $A\sigma \in T^J{\downarrow}\omega$ then for some $I \supset J$: $A\sigma \in T^I{\downarrow}$.

Proof. Assume $A\sigma \in T^J{\downarrow}\omega$. If $A\sigma \in T^J{\uparrow}$, then we may take I=J; so assume $A\sigma \notin T^J{\uparrow}$. Let **B** be a fair SLD search-tree for ←A. By 4.6.1a, SH(**B**,J,σ) has infinite height. By the lemma, for some $I \supset J$, SH(**B**,I,σ) is not well-founded. By 4.6.2b, $A\sigma \in T^I{\downarrow}$. ⊠

4.9 Lemma. If $J \subset I$ then $T^J{\downarrow} \subset T^I{\downarrow}$.

Proof. It suffuces to show that $T^J{\downarrow} \subset T^I(T^J{\downarrow})$, since $T^I{\downarrow}$ is the largest set satisfying this inclusion. Now obviously for all $X \subset HB^J$: $T^J(X) \subset T^I(X)$; hence in particular $T^J{\downarrow} = T^J(T^J{\downarrow}) \subset T^I(T^J{\downarrow})$. ⊠

The following is the main result of [Blair/Brown 199?]. Note that the proof below avoids the relativized resolution-machinery of that paper.

4.10 Theorem. There exist countable J such that $T^J{\downarrow} = T^J{\downarrow}\omega$ for all program-associated T^J.

Proof. Construct a sequence of countable algebras $J_0 = HU \subset J_1 \subset J_2 \subset \ldots$ such that for all atoms A, program-associated $T=T_P$, and σ:Var(A)→J_n: if $A\sigma \in T^{J_n}{\downarrow}\omega$ then $A\sigma \in T^{J_{n+1}}{\downarrow}$.

J_n constructed, J_{n+1} is taken to be the union of a countable sequence of countable algebras obtained from 4.8 - note that the number of atoms and programs is countable and the number of J_n-assignments to be considered is countable if J_n is countable. By 4.9, once an instance $A\sigma$ is manoeuvred into a greatest fixed point, it remains so upon extending the algebra any further. Now, just let J be the union of the J_n. ⊠

We have the following easy characterization of the algebras J for which $T^J{\downarrow} = T^J{\downarrow}\omega$ - $T^J = T_P{}^J$, P any program.

4.11 Theorem. Let ρ be any fair selection rule and $T^J = T_P{}^J$, P any program.

The following are equivalent:

1. $T^J{\downarrow} = T^J{\downarrow}\omega$
2. every well-founded sheaf-tree SH(**B**,J,σ), where **B** is a failing ρ-produced search-tree for an atomic goal ←A, has finite height.

Proof.

1⇒2. Suppose that the search-tree **B** for ←A fails, is ρ-produced, and SH(**B**,J,σ) is well-founded of infinite height. If $A\sigma \in T^J{\uparrow}$ then, by 4.4, it follows that **B** cannot fail, contrary to assumption. Therefore, $A\sigma \notin T^J{\uparrow}$. By 4.6.2a, $A\sigma \notin T^J{\downarrow}$. By 1., $A\sigma \notin T^J{\downarrow}\omega$, contradicting 4.6.1b.

2⇒1. Suppose that $A\sigma \in T^J{\downarrow}\omega$. We may assume that $A\sigma \notin T^J{\uparrow}$, since $T^J{\uparrow} \subset T^J{\downarrow}$. By 4.6.1a, SH(**B**,J,$\sigma$) has infinite height, **B** the ρ-produced search-tree for ←A. By 2., SH(**B**,J,σ) has an infinite branch. By 4.6.2b, $A\sigma \in T^J{\downarrow}$. ⊠

The selection rule ρ is *lifting-invariant* if for all derivations Γ and Δ: Γ is ρ-produced iff Δ is ρ-produced whenever Δ is a lift of Γ.

4.12 Lemma.

There exists a (primitive) recursive selection rule which is both fair and lifting-invariant.

Proof.

The rule keeps a stack of atoms to be selected and uses it first-in first-out. Thus:

derivation:	stack:
$\leftarrow$A,B,K	$\rightarrow$A,B,K
α; $\leftarrow$L,Bα,Kα	$\rightarrow$Bα,Kα,L
β; $\leftarrow$Lβ,M,K$\alpha\beta$	$\rightarrow$K$\alpha\beta$,Lβ,M
etc.	etc.

Note that the stack is a *list* : in the stack, order *and* repetitions count.

Fairness can be obtained automatically by just replacing the successive goals by the corresponding stacks, i.e., using the Prolog left-most selection rule in combination with a repeated moving-around of the atoms of the goals. ⊠

Remark. The neat trick of obtaining fairness by spinning around constituents in goals is due to Löb ([Löb 197-]) and was used by him to construct a notion of derivation for first-order logic with an incredibly fast and elegant proof of completeness.

Of course, as does logic programming, Löb's syntax needs a disjunction of indefinite arity, applying to arbitrarily many arguments - i.e., it works with "generalized goals", so to speak.

Viewing (as we do) search trees as sets of finite derivations closed under "initial-of", obviously, search trees produced by recursive selection rules will be recursive. (If we consider search trees as sets of resolvents, recursively produced trees will be recursively enumerable only.)

Together with the easy existence theorem for countable recursively saturated models (cf. [Chang/Keisler 1990] for elementary facts on recursive saturation), the following produces a shortcut to the Blair/Brown theorem 4.10.

([Doets 1990] notes that 4.14 follows from the two facts (i) the least admissible set above a recursively saturated structure J has height O(J)=ω, and (ii) ("Gandy's theorem") elementary inductive definitions over J close at O(J) at the latest. See [Barwise 1975].)

4.13 Theorem.

If J is recursively saturated, then every well-founded sheaf-tree SH(**B**,J,σ) over a recursive search-tree **B** has finite height.

4.14 Corollary.

If J is recursively saturated, then for every program-induced operator T^J over it: $T^J\downarrow = T^J\downarrow\omega$.

Proof of 4.12. Take the recursive fair selection rule of 4.12. Recursive rules produce recursive search trees. Therefore, the result follows by 4.11 and 4.13. ⊠

Proof of 4.13. Let **B** be a recursive search tree. Suppose that the sheaf-tree SH(**B**,J,σ^0) has infinite height.

Clearly then, for every n:

there exists σ^1 such that for some resolvent $\theta_{0,1};N_1$ of N_0 in **B**:

(σ^0,σ^1) realizes $\theta_{0,1}$ (i.e., $\sigma^0=\theta_{0,1}\sigma^1$ on $Var(N_0)$) and $SH(\mathbf{B}(N_1),J,\sigma_1)$ has height $\geq n$.

Here, $\mathbf{B}(N_1)$ is the subtree of **B** with top N_1.

Trivially then also, for every n:

there exists σ^1 such that, for all $m<n$:

[[m]]: for some resolvent $\theta_{0,1};N_1$ of N_0 in **B**:

(σ^0,σ^1) realizes $\theta_{0,1}$ and $SH(\mathbf{B}(N_1),J,\sigma_1|N_1)$ has height $\geq m$.

It is not hard to see (since **B** is finitely splitting and recursive) that condition [[m]] can be expressed by a first-order formula (which, in fact, does not depend on J) computable in m. Therefore, by recursive saturation of J:

there exists σ^1 such that for all m, condition [[m]] is satisfied.

Fix such a σ^1. Since **B** has only finitely many resolvents $\theta_{0,1};N_1$ of N_0, by the pigeon-hole principle:

there exists a resolvent $\theta_{0,1};N_1$ of N_0 in **B** such that, for all m:

(σ^0,σ^1) realizes $\theta_{0,1}$ and $SH(\mathbf{B}(N_1),J,\sigma_1|N_1)$ has height $\geq m$.

I.e., $SH(\mathbf{B}(N_1),J,\sigma^1|N_1)$ has infinite height, and we can repeat the argument to find a resolvent $\theta_{1,2};N_2$ of N_1 in **B** and σ^2 such that $(\sigma^0, \sigma^1, \sigma^2)$ realizes $(\theta_{0,1}, \theta_{1,2})$ and $SH(\mathbf{B}(N_2),J,\sigma^2|N_2)$ has infinite height, etc.; eventually constructing our infinite branch through $SH(\mathbf{B},J,\sigma^0)$. ⊠

Note that the recursive types involved in the proof of 4.13 only contain formulas built using $\wedge$, $\vee$ and $\exists$. In particular, $\neg$ and $\forall$ are not needed. This suggests a

4.15 Question. Is the condition: for all programs, $T^J\downarrow=T^J\downarrow\omega$, equivalent with saturation w.r.t. recursive types of *positive* formulas?

4.16 Example. Let J be the non-standard algebra which is the successor-structure on a type-$\omega+\zeta$ ordering for the language with one individual constant 0 and one unary function symbol S mentioned before. J is far from being recursively saturated (this would need infinitely many type-ζ constituents), but it does satisfy $T^J\downarrow=T^J\downarrow\omega$ for all T^J.

4.17 Remark. By an example of [Kleene 1952], we cannot expect the branch through **B** constructed in the proof of 4.14 to be recursive: there are recursive search trees without recursive branches. An element of an algebra has a (by the free equality axioms, unique) parsing tree. It follows that a recursively saturated algebra for at least two operations or one binary operation (this excludes algebras of the 4.16-type) has elements with a non-recursive parsing tree. (It is easy to see that, if $T^J\downarrow=T^J\downarrow\omega$ for all T^J, all recursive functions will be represented as parsing trees of elements of J; cf. [Blair/Brown 199?].)

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