Institute for Language, Logic and Information

COMBINATORIAL PROPERTIES OF FINITE SEQUENCES WITH HIGH KOLMOGOROV COMPLEXITY

Ming Li
Paul M.B. Vitányi

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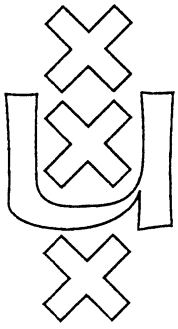

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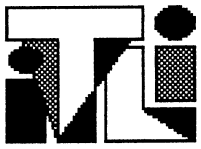

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ITLI Prepublication Series
for Computation and Complexity Theory
ISSN 0924-8374

Received June 1991

Combinatorial Properties of Finite Sequences with High Kolmogorov Complexity

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May 29, 1991

Abstract

We investigate to what extent finite binary sequences with high Kolmogorov complexity are normal (all blocks of equal length occur equally frequent), and the maximal length of all-zero or all-one runs which occur with certainty.

1 Introduction

Infinite sequences generated by a $(\frac{1}{2}, \frac{1}{2})$ Bernoulli process (flipping a fair coin) have the property that the relative frequency of zeros in an initial n-length segment goes to $\frac{1}{2}$ for n goes to infinity. A related statement can be made

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for finite sequences, in the sense that one can say that the majority of all sequences will have about one half zeros. However, whereas the earlier statement is a property about individual infinite random sequences, the classical theory of probability has no machinery to define or deal with individual finite random sequences.

In [7], Kolmogorov established a notion of complexity (self-information) of finite objects which is essentially finitary and combinatorial. Says Kolmogorov [8]: "Information theory must precede probability theory, and not be based on it. By the very essence of this discipline, the foundations of information theory must have a finite combinatorial character." It is the aim of this paper to derive randomness related combinatorial properties of high complexity finite binary sequences by combinatorial arguments, without any probabilistic assumptions at all.

It turns out to be quite natural to do combinatorial proofs by Kolmogorov complexity arguments, which are of themselves combinatorial in nature. We have demonstrated the utility of a Kolmogorov complexity method in combinatorial theory by proving several combinatorial lower bounds (like the 'coin-weighing' problem), [11]. Rather than doing combinatorics using Kolmogorov complexity, in this paper we are interested in combinatorial properties of individual finite binary sequences with high Kolmogorov complexity.

E. Borel (1909) has called an infinite sequence of zeros and ones 'normal' in the scale of two if, for each k, the frequency of occurrences of each block y of length k in the initial segment of length n goes to limit 2^{-k} for n grows unbounded, [6]. It is known that normality is not sufficient for randomness, since Champernowne's sequence

$$123456789101112\ldots$$

is normal in the scale of ten. On the other hand, it is universally agreed that a random infinite sequence must be normal. (If not, then some blocks occur more frequent than others, which can be used to obtain better than fair odds for prediction.)

While in the infinite case one considers limiting values of quantitive properties which hold for each individual sequence of a set of probability 1, in the finite case one considers the *expected* value of quantities over a set of all sequences of a given length.

We would like to obtain statements that *individual* random finite sequences have such-and-such quantitative properties in terms of their length.

But as the result of a sequence of n fair coin flips, *any* sequence of length n can turn up. This raises the question which subset of finite sequences can be regarded as genuinely random. In [12] the viewpoint is taken that finite sequences which satisfy all *effective* tests for randomness (known and unknown alike), are as random as we will ever be able to verify. This form of randomness of individual sequences turns out to be equivalent to such sequences having maximal Kolmogorov complexity. In the sequel we use 'complexity' in the sense of 'Kolmogorov complexity'.

We prove that each high complexity finite binary sequence is 'normal' in the sense that each binary block of length k occurs about equally frequent for k relatively small. In particular, this holds for $k = 1$. We quantify the 'about' and the 'relatively small' in this statement.

To distinguish individual random sequences obtained by flipping a physical coin from random sequences written down by human subjects, psychological tests [the correct reference is unknown to the authors] have shown that a consistent high classification score is reached by using the criterion that a real random sequence of length, say 40, contains a run of zeros or ones of length 6. In contrast, human subjects feel that short random sequences should not contain such long uniform runs.

We determine the maximal length of runs of zeros or ones which are *with certainty* contained in each high complexity finite sequence. We prove that each such sequence must contain a relatively long run of zeros.

The properties must be related to length of the sequence. In a sequence of length 1, or odd length, the number of zeros and ones cannot be equal. To apply such properties in mathematical arguments, it is often of importance that the precise extent to which such properties hold is known.

2 Kolmogorov Complexity

To make this paper self-contained we briefly review notions and properties needed in the sequel. We identify the natural numbers $\mathcal{N}$ and the finite binary sequences as

$$(0, \epsilon), (1, 0), (2, 1), (3, 00), (4, 01), \ldots,$$

where ϵ is the empty sequence. The *length* $l(x)$ of a natural number x is the number of bits in the corresponding binary sequence. For instance, $l(\epsilon) = 0$.

If A is a set, then $|A|$ denotes the *cardinality* of A. Let $< . >: \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ denote a standard computable bijective 'pairing' function. Throughout this paper, we will assume that $< x, y >= 1^{l(x)}0xy$.

Define $< x, y, z >$ by $< x, < y, z >>$.

We need some notions from the theory of algorithms, see [13]. Let $\phi_1, \phi_2, \ldots$ be a standard enumeration of the partial recursive functions. The (Kolmogorov) *complexity* of $x \in \mathcal{N}$, given y, is defined as

$$C(x|y) = \min\{l(< n, z >) : \phi_n(< y, z >) = x\}.$$

This means that $C(x|y)$ is the *minimal* number of bits in a description from which x can be effectively reconstructed, given y. The unconditional complexity is defined as $C(x) = C(x|\epsilon)$.

An alternative definition is as follows. Let

$$C_\psi(x|y) = \min\{l(z) : \psi(< y, z >) = x\} \tag{1}$$

be the conditional complexity of x given y with reference to decoding function ψ. Then $C(x|y) = C_\psi(x|y)$ for a universal partial recursive function ψ that satisfies $\psi(< y, n, z >) = \phi_n(< y, z >)$.

We will also make use of the *prefix* complexity $K(x)$, which denotes the shortest *self-delimiting* description. To this end, we consider so called *prefix* Turing machines, which have only 0's and 1's on their input tape, and thus cannot detect the end of the input. Instead we define an input as that part of the input tape which the machine has read when it halts. When $x \neq y$ are two such input, we clearly have that x cannot be a prefix of y, and hence the set of inputs forms what is called a *prefix code*. We define $K(x)$ similarly as above, with reference to a universal prefix machine that first reads 1^n0 from the input tape and then simulates prefix machine n on the rest of the input.

A survey is [10]. We need the following properties. Throughout 'log' denotes the binary logarithm. We often use $O(f(n)) = -O(f(n))$, so that $O(f(n))$ may denote a negative quantity. For each $x, y \in \mathcal{N}$ we have

$$C(x|y) \leq l(x) + O(1). \tag{2}$$

For each $y \in \mathcal{N}$ there is an $x \in \mathcal{N}$ of length n such that $C(x|y) \geq n$. In particular, we can set $y = \epsilon$. Such x's may be called *random*, since they are without regularities that can be used to compress the description. Intuitively,

the shortest effective description of x is x itself. In general, for each n and y, there are at least $2^n - 2^{n-c} + 1$ distinct x's of length n with

$$C(x|y) \geq n - c. \tag{3}$$

In some cases we want to encode x in *self-delimiting* form x', in order to be able to decompose $x'y$ into x and y. Good upper bounds on the prefix complexity of x are obtained by iterating the simple rule that a self-delimiting (s.d.) description of the length of x followed by x itself is a s.d. description of x. For example, $x' = 1^{l(x)}0x$ and $x'' = 1^{l(l(x))}0l(x)x$ are both s.d. descriptions for x, and this shows that $K(x) \leq 2l(x) + O(1)$ and $K(x) \leq l(x) + 2l(l(x)) + O(1)$.

Similarly, we can encode x in a self-delimiting form of its shortest program $p(x)$ ($l(p(x)) = C(x)$) in $2C(x)+1$ bits. Iterating this scheme, we can encode x as a selfdelimiting program of $C(x) + 2\log C(x) + 1$ bits, which shows that $K(x) \leq C(x) + 2\log C(x) + 1$, and so on.

3 Expectation versus Complexity

To derive our results, we often use a common pattern of argument. Following a suggestion of John Tromp, we can formulate it in the form of a general 'Tail Law'.

Consider the sample space $S = \{0,1\}^*$ with uniform probability $\Pr(x) = 2^{-2l(x)}$. Put $S^n = \{0,1\}^n$. Then, $\Pr(x|x \in S^n) = 2^{-n}$. Let $R : S \to \mathcal{Z}$, total recursive, be a function that (in our case) measures the deviation between some function g of x and a reference value $r(l(x))$ for all strings of the same length. We are interested in the relation between the complexity of a string x and this deviation. A natural choice of r would be the average $g(x)$ over S^n. Fix a class D of *deficiency* functions $\delta : \mathcal{N} \to \mathcal{N}$ for which $K(n|n-\delta(n)) = O(1)$. This is satisfied by every monotone sublinear recursive function that we are interested in. The complexity of R can be identified with the complexity of its index in the effective enumeration of recursive functions, which we can assume equals some constant plus (optionally) the complexity of its parameters.

Define the tail probability

$$p(R; n, m) = \Pr\{x \in S^n : |R(x)| \geq m\}.$$

Lemma 1 (Tail Lemma) *Let f be a function from $\mathcal{N} \times \mathcal{N}$ to $\mathcal{N}$ satisfying*

$$-\log p(R; n, f(n,k)) \geq K(R|n) + k + O(1).$$

Then for any $\delta \in D$, we have that all x with $C(x) > n - \delta(n)$ $(n = l(x))$, satisfy

$$|R(x)| < f(n, \delta(n)).$$

Proof. By contradiction. Assume that for some $\delta \in D$, there exists an n such that $A = \{x \in S^n : |R(x)| \geq f(n, \delta(n))\}$ is non-empty. We can describe such an $x \in A$ in the following way:

1. let s be a s.d. program for n given $n - \delta(n)$, of length $l(s) = K(n|n - \delta(n)) = O(1)$.
2. let q be a s.d. program for R given n, of length $l(q) = K(R|n)$.
3. let i be the index of x in an effective enumeration of A, from the x's with the highest $|R(x)|$'s down. From $|A| = 2^n \Pr(A) = 2^n p(R; n, f(n, \delta(n)))$ it follows that the length of the (not necessarily s.d.) description of i satisfies:

$$\begin{aligned} l(i) \leq \log |A| &= n + \log p(R; n, f(n, \delta(n))) \\ &\leq n - K(R|n) - \delta(n) - O(1). \end{aligned}$$

The string sqi has length at most $n - \delta(n) - O(1)$ and can be padded to a string z of length exactly $n - \delta(n) - c$, where c is a constant determined below. From z we can reconstruct x by first using its length plus c to compute n (and $\delta(n)$) from s, then use n to obtain R from q, and finally enumerate A to obtain the ith element. Note that we can enumerate A up to the ith element without using f at all, since we enumerate from the x's with the highest $|R(x)|$ down. So, if recursive function ψ embodies above procedure for reconstructing x, we have, by Equation 1,

$$C(x) \leq C_\psi(x) + c_\psi \leq n - \delta(n) - c + c_\psi.$$

Choosing $c = c_\psi$ finishes the proof. □

Corollary 1 (Tail Lemma Dual) *The exact same argument shows that for sufficiently random x, the deviation $|R(x)|$ is not too small. We thus obtain a Tail Lemma Dual starting from $q(R;n,m) = \Pr\{x \in S^n : |R(x)| \leq m\}$.*

4 Number of Zeros and Ones

Let x have length n. It is known that if $C(x|n) = n + O(1)$, then the number of zeros it contains is, [12],

$$\frac{n}{2} + O(\sqrt{n}).$$

4.1 Fixed Complexity

We analyse what complexity can say about the number of zeros and ones. Let $x = x_1 x_2 \ldots x_n$ and $\delta \in D$ a deficiency function. Suppose,

$$C(x) \geq n - \delta(n).$$

Let $R(x) = \sum x_i - \frac{n}{2}$ be the deviation in the number of ones in x. With $x \in \{0,1\}^n$ uniformly distributed, $\#ones(x) = \sum x_i$ is distributed according to the binomial distribution.

A general estimate of the tail probability of the binomial distribution, with s_n the number of successful outcomes in n experiments with probability of success $0 < p < 1$ and $q = 1 - p$, is given by Chernoff's bounds, [3, 2],

$$\Pr(|s_n - np| \geq m) \leq 2e^{-m^2/4npq}. \tag{4}$$

The tail probability $p(R;n,m)$ bounded by Equation 4 (with $R(x) = s_n - \frac{n}{2}$ and $p = q = 1/2$) yields:

$$-\log p(R;n,m) \geq \frac{m^2 \log e}{n} - 1$$

Clearly, R is a recursive function with $K(R|n) = O(1)$. Thus, choosing $f(n,k) = \sqrt{(k + O(1))n \ln 2}$, Lemma 1 gives us: all x with $C(x) > n - \delta(n)$ $(n = l(x))$, satisfy

$$|\#ones(x) - \frac{n}{2}| < \sqrt{(\delta(n) + O(1))n \ln 2}. \tag{5}$$

If the complexity of x satisfies that the conditional complexity $C(x|n) \geq n - \delta(n)$, clearly Equation 5 holds a fortiori.

4.2 Fixed Number of Zeros

It may be surprising at first glance, but there are no maximally complex sequences with about equal number of zeros and ones. Equal numbers of zeros and ones is a form of regularity, and therefore lack of complexity. Using the same notation as before, if $R(x) = O(1)$ then the randomness deficiency $\delta(n) = n - C(x)$ is relatively large. For instance,

$$\begin{aligned} q(R;n,m) &= \Pr\{x \in S^n : |R(x)| \leq m\} \\ &\leq (2m+1)2^{-n}\left(\begin{array}{c} n \\ n/2 \end{array}\right) = O(\frac{m}{\sqrt{n}}). \end{aligned}$$

Thus, setting $f(n,k) = 2^{-k-O(1)}\sqrt{n}$, the Tail Law Dual (Corollary 1) gives us: all x with $C(x) > n - \delta(n)$ $(n = l(x))$, satisfy

$$|\#ones(x) - \frac{n}{2}| > 2^{-\delta(n)-O(1)}\sqrt{n}.$$

Perhaps more interestingly, we can define

$$R'(x) = \#ones(x) - (\frac{n}{2} + j),$$

so that $K(R'|n)$ is about $K(j)$. Applying the Tail Law Dual with

$$f(n,k) = 2^{-k-K(j)-O(1)}\sqrt{n},$$

we then find that all x with $C(x) > n - \delta(n)$ satisfy

$$|\#ones(x) - (\frac{n}{2} + j)| > 2^{-\delta(n)-K(j)-O(1)}\sqrt{n}.$$

This means that for a random x having exactly $j + n/2$ ones, $K(j)$ must be at least about $\log n$.

5 Number of Blocks

An infinite binary sequence is called *normal* if each block of length k occurs with limiting frequency of 2^{-k}. This justifies our intuition, that a random infinite binary sequence contains about as many zeros as ones. But also, blocks 00, 01, 10, and 11 should appear about equally often. In general we expect that each block of length k occurs with about the same frequency. Can we find an analogue for finite binary sequences? We analyse these properties for high complexity finite binary sequences to obtain a quantification of a similar statement in terms of the length of the sequence and its complexity.

5.1 Fixed Complexity

Let $x = x_1 \dots x_n$ be a binary sequence of length n, and y a much smaller string of length l. Let $p = 2^{-l}$ and $\#y(x)$ be the number of (possibly overlapping) distinct occurrences of y in x. Put $R_y(x) = \#y(x) - np$. (So $R_1(x) = \sum x_i - n/2$.) For convenience, we assume that x 'wraps around' so that an occurrence of y starting at the end of x and continuing at the start also counts.

Theorem 1 *All x with $C(x) > n - \delta(n)$ satisfy*

$$|\#y(x) - np| < \sqrt{\alpha np},$$

with $\alpha = [K(y|n) + \log l + \delta(n) + O(1)](1-p)l4\ln 2$.

Proof. We prove by contradiction. Assume that n is divisible by l. (If it is not we can put x on a Procrustus bed to make its length divisible by l at the cost of having the above frequency estimate up to a $l/2$ additive term.) There are l ways of dividing (the ring) x into $N = n/l$ contiguous equal sized blocks, each of length l. For each such division $i \in \{0, 1, \dots, l-1\}$, let $R_{y,i}(x)$ be the number of (now nonoverlapping) occurrences of block y minus Np. Notice that $R_{y,i}(x)$ is the deviation from the expectation of a Bernoulli sequence of length N with probability of succes (a block matching y) p, for which we can use the Chernoff bound 4.

$$p(R_{y,i}; n, m) \leq 2e^{-m^2/4Np(1-p)}.$$

Taking the negative logarithm on both sides:

$$-\log p(R_{y,i}; n, m) \geq \frac{m^2 \log e}{4Np(1-p)} - 1. \tag{6}$$

Choose $m = f(n,k)$, such that

$$\frac{f(n,k)^2 \log e}{4Np(1-p)} = K(R_{y,i}|n) + k + O(1). \tag{7}$$

Equations 6, 7 enable us to apply the Tail Lemma 1. Application of the Tail Lemma yields that all x with $C(x) > n - \delta(n)$ satisfy $|R_{y,i}(x)| < f(n, \delta(n))$. Substitution of f according to Equation 7, with $K(R_{y,i}|n) = K(y,i|n)+O(1)$, gives:

$$|R_{y,i}(x)| < \sqrt{\frac{K(y,i|n) + \delta(n) + O(1)}{\log e} 4Np(1-p)}.$$

The theorem now follows by noting that $R_y(x) = \sum_{i=0}^{l-1} R_{y,i}(x)$, and $K(i|l) \leq \log l$ □

With $C(x|n, R_y) \geq n - \delta(n)$, Theorem 1 holds a fortiori.

5.2 Fixed Number of Blocks

Similar to the analysis of blocks of length 1, the complexity drops below its maximum in case some block y of length l occurs in one of the l block divisions, say i, with frequency exactly pN ($p = 1/2^l$). Then we can point out x by giving n, y, i and its index in a set of cardinality

$$\binom{N}{pN} (2^l - 1)^{N-pN} = O(\frac{2^{Nl}}{\sqrt{Np(1-p)}}).$$

Therefore,

$$C(x|n,y) \leq n - \frac{1}{2}\log n + \frac{1}{2}(l + 3\log l) + O(1).$$

6 Length of Runs

It is known from probability theory, that in a randomly generated finite sequence the *expectancy* of the length of the longest run of zeros or ones is pretty high. For each individual finite sequence with high Kolmogorov complexity we are *certain* that it contains each block up to a certain length (like a run of zeros).

Theorem 2 *Let x of length n satisfy $C(x) \geq n - \delta(n)$. Then x contains all blocks y of length*

$$l = \log n - \log\log n - \log(\delta(n) + \log n) - O(1).$$

Proof. We are sure that y occurs at least once in x, if $\sqrt{\alpha np}$ in Theorem 1 is at most np. This is the case if $\alpha \leq np$, that is:

$$\frac{K(y|n) + \log l + \delta(n) + O(1)}{\log e} 4l \leq np.$$

Substitute $K(y|n) \leq l + 2\log l + O(1)$ (since $K(y|n) \leq K(y)$), and $p = 2^{-l}$ with l set at

$$l = \log n - \log(3\delta(n)\log n + 3\log^2 n),$$

(which equals l in the statement of the theorem up to an additive constant). The result is

$$\frac{l + 3\log l + \delta(n) + O(1)}{\log e} 4l \leq 3(\delta(n)\log n + \log^2 n),$$

and it is easy to see that this holds for sufficiently large n. □

Corollary 2 *If $\delta(n) = O(\log n)$ then each block of length $\log n - 2\log\log n - O(1)$ is contained in x.*

Corollary 3 *Analysing the proof of Theorem 2 we can improve this in case $K(y|n)$ is low. If $\delta(n) = O(\log\log n)$, then for each $\epsilon > 0$ and n large enough, x contains an all-zero run y (for which $K(y|n) = O(\log l)$) of length $l = \log n - (1+\epsilon)\log\log n + O(1)$.*

Corollary 4 (improving [2]) *Since there are* $2^n(1 - O(1/\log n))$ *strings* x *of length* n *with* $C(x) \geq n - \log\log n + O(1)$, *the* expected length of the longest run of consecutive zeros *if we flip a fair coin* n *times, is at least* l *as in Corollary 3. This improves the lower bound of* $\log n - 2\log\log n$ *cited in [2] by a* $\log\log n$ *additive term.*

We show in what sense Theorem 2 is sharp. Let $x = uvw$, $l(x) = n$ and $C(x) \geq n - \delta(n)$. We can describe x by giving

1. A description of v in $K(v)$ bits.
2. The literal representation of uw.
3. A description of $l(u)$ in $\log n + \log\log n + 2\log\log\log n + O(1)$

Then, since we can find n by $n = l(v) + l(uw)$,

$$\begin{aligned} C(x) &\leq n - l(v) + K(v) + \log n \\ &+ (1 + o(1))\log\log n + O(1). \end{aligned} \tag{8}$$

Substitute $C(x) = n - \delta(n)$ and $K(v) = o(\log\log n)$ (choose v to be very regular) in Equation 8 to obtain:

$$l(v) \leq \delta(n) + \log n + (1 + o(1))\log\log n.$$

This means that, for instance, for each $\epsilon > 0$, no maximally complex string x with $C(x) = n + O(1)$ contains a run of zeros (or the initial binary digits of π) of length $\log n + (1 + \epsilon)\log\log n$ for n large enough and regular enough. By Corollary 3, on the other hand, such a string x *must* contain a run of zeros of length $\log n - (1 + \epsilon)\log\log n + O(1)$.

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