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QUASI-INJECTIVE REDUCTIONS

Lane A. Hemachandra
Edith Spaan

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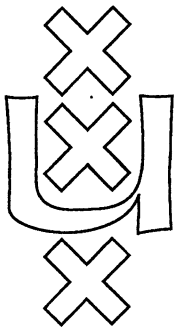

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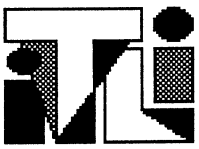

Instituut voor Taal, Logica en Informatie
Institute for Language, Logic and Information

Faculteit der Wiskunde en Informatica
(Department of Mathematics and Computer Science)
Plantage Muidergracht 24
1018TV Amsterdam

Faculteit der Wijsbegeerte
(Department of Philosophy)
Nieuwe Doelenstraat 15
1012CP Amsterdam

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Lane A. Hemachandra
Department of Computer Science
University of Rochester
Edith Spaan
Department of Mathematics and Computer Science
University of Amsterdam

ITLI Prepublication Series
for Computation and Complexity Theory
ISSN 0924-8374

Received August 1991

Quasi-Injective Reductions

Lane A. Hemachandra[*]
Department of Computer Science
University of Rochester
Rochester, NY 14627

Edith Spaan[†]
Faculteit der Wiskunde en Informatica
Universiteit van Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam

July 24, 1991

Abstract

A reduction is said to be quasi-injective if no element of the range is mapped to by infinitely many elements. Via two natural families of quasi-injective reductions, we study the connection between degree of injectivity and strength of reduction. In particular, we completely determine the relative strengths of polynomial-time $f(n)$-$\widehat{\text{to}}$-1 reductions, and of polynomial-time k-to-k' reductions.

1 Introduction

A many-one reduction may, in general, map infinitely many domain elements to the same range element. A one-to-one reduction (often referred to as an *injective* reduction) maps at most one domain element to a given element of the co-domain. In some settings, these different degrees of injectivity coincide; a famous example is provided by the Myhill Isomorphism Theorem (see, e.g., [Soa87]), which implies that all sets $\leq_m$-complete for the r.e. sets are indeed $\leq_{\text{1-to-1}}$-complete for the r.e. sets. In some other settings, it is not known whether differing degrees of injectivity coincide; the question of whether $\leq^p_m$-completeness and $\leq^p_{\text{1-to-1}}$-completeness coincide for NP remains a central unresolved problem, and is a weaker version of the Berman-Hartmanis Isomorphism Conjecture ([BH77], see also the survey [You90]). In this paper, we study the extent to which lack of injectivity gives power to polynomial-time reductions.

[*]Research supported in part by the National Science Foundation under grant CCR-8957604.
[†]Research supported in part by the NWO under grant SIR 13-785.

We focus on what we will call quasi-injective reductions—reductions that map at most a finite number of domain elements to a given element in their range. In particular, we study $f(n)$-$\widehat{\text{to}}$-1 reductions[1] and k-to-k' reductions. We observe that an $f(n)$-$\widehat{\text{to}}$-1 reduction is more powerful than a $g(n)$-$\widehat{\text{to}}$-1 reduction exactly when $f(n)$ is greater than $g(n)$ infinitely often. However, this result is definition-sensitive; it fails for $f(n)$-to-1 reductions. For the case of k-to-k' reductions—reductions for which no k' elements of the range are mapped to by more than k domain elements—we completely characterize when a c-to-d reduction is more powerful than an a-to-b reduction, namely when:

$$\left(\left\lfloor\frac{a}{b}\right\rfloor < \left\lfloor\frac{c}{d}\right\rfloor\right) \vee \left[\left(\left\lfloor\frac{a}{b}\right\rfloor = \left\lfloor\frac{c}{d}\right\rfloor\right) \wedge \left(a - b\left\lfloor\frac{a}{b}\right\rfloor < c - d\left\lfloor\frac{a}{b}\right\rfloor\right)\right].$$

2 Preliminaries

Σ will represent any fixed finite alphabet. Our reductions will in general be from Σ^* to Σ^*. However, at times we will use $\mathcal{N}$ instead of Σ^*, implicitly taking advantage of the standard nice correspondence between these two sets. Let $|x|$ denote the length of string x, and let $||S||$ denote the cardinality of set S. We will use the quantification symbol $(\exists_\infty x)$ to indicate "there exist infinitely many distinct x."

Let FP denote the class of total functions computable in polynomial time. We will usually assume that such functions map from Σ^* to Σ^* (equivalently $\mathcal{N} \to \mathcal{N}$). However, in certain cases we will allow more flexible co-domains (such as $\Sigma^* \bigcup \{\text{YES}, \text{NO}\}$).

Recall the standard definition of many-one reductions: $A \leq_m^p B$ if there is a function $h \in \text{FP}$ such that $(\forall x \in \Sigma^*)[x \in A \iff h(x) \in B]$ [HU79]. We will call a finite-to-one reduction quasi-injective.

Definition 2.1 We will say that $h \in \text{FP}$ is *quasi-injective* if $(\forall y \in \Sigma^*)[\{x \in \Sigma^* \mid h(x) = y\}$ is a finite set].

Definitions 2.2 and 2.3 present the families of quasi-injective reductions that we will study.

Definition 2.2 We say that $A \leq_{f(n)\text{-to-}1}^p B$ if $A \leq_m^p B$ via a reduction $h \in \text{FP}$ satisfying $(\forall y \in \Sigma^*)[||\{x \in \Sigma^* \mid h(x) = y\}|| \leq f(|y|)]$.

Definition 2.3 We say that $A \leq_{k\text{-to-}k'}^p B$ if $A \leq_m^p B$ via a reduction $h \in \text{FP}$ satisfying $(\forall S \subseteq \Sigma^*)[||S|| \leq k' \Rightarrow ||\{x \in \Sigma^* \mid h(x) \in S\}|| \leq k]$.

[1]The hat indicates reductions that can simply state acceptance or rejection ([AS87], see also [DGHM89]). This paper will also discuss the structure of reductions that lack this ability.

In Definition 2.2, the special case $f(n) = 1$ ($\leq^p_{\text{1-to-1}}$ reducibility) has been extensively studied, and is related to issues of isomorphism, one-way functions, and cryptography [BH77,GS88,KMR90,You90]. In Definition 2.2, the special case $f(n) = n^{\mathcal{O}(1)}$ (polynomial-time polynomial-to-one reducibility) has been studied by Allender and Rubinstein, who related this notion to the P = FewP question [AR88].

Unfortunately, the standard definition of many-one reductions gives problems in certain settings. For example, $\Sigma^* \not\leq^p_m \emptyset$, though both sets are computationally trivial. More generally, a many-one reduction from A to B may "know" whether its input is a member of A, but may not be able to find an appropriate string (in B or $\overline{B}$) to map to. Ambos-Spies proposed dealing with this by allowing a many-one reduction from A to B to either reduce a given input to an appropriate output, or to directly proclaim whether its input is in A [AS87,DGHM89]. This is reflected in the definition below.

Definition 2.4

1. **[AS87]** We say that $A \leq^p_{\widehat{m}} B$ if there is a a reduction $h : \Sigma^* \to \Sigma^* \bigcup \{\text{YES}, \text{NO}\}$, $h \in \text{FP}$, satisfying $(\forall x \in \Sigma^*)[(h(x) = \text{YES} \Rightarrow x \in A) \wedge (h(x) = \text{NO} \Rightarrow x \notin A) \wedge (h(x) \in \Sigma^* \Rightarrow (h(x) \in B \iff x \in A))]$, where YES and NO are symbols not in Σ.

2. We say that $A \leq^p_{f(n)\text{-}\widehat{\text{to}}\text{-1}} B$ if $A \leq^p_{\widehat{m}} B$ via a reduction $h : \Sigma^* \to \Sigma^* \bigcup \{\text{YES}, \text{NO}\}$, $h \in \text{FP}$, satisfying $(\forall y \in \Sigma^*)[||\{x \in \Sigma^* \mid h(x) = y\}|| \leq f(|y|)]$.

Note that, in the latter part of the above definition, those strings x for which $h(x) \in \{\text{YES}, \text{NO}\}$ do not "count against" the injectivity restriction.

At times, we will want to argue that there are maps from A to B with certain quasi-injectivity properties, but that no such map can be computed quickly. The following notion will be useful; it should be compared with Goldsmith's related notion of sets to which both Σ^* and $\emptyset$ (and thus *every* set) reduce via reductions that are at most polynomially length-increasing (see [Gol89, Lemma 2.2.2]).

Definition 2.5

1. We say that $A \leq^{\textit{poly-length}}_{f(n)\text{-to-1}} B$ if there exist a function h and polynomial p satisfying:

   (a) $(\forall x \in \Sigma^*)[(x \in A \iff h(x) \in B) \wedge |h(x)| \leq p(|x|)]$ and

   (b) $(\forall y \in \Sigma^*)[||\{x \in \Sigma^* \mid h(x) = y\}|| \leq f(|y|)]$.

2. We say that $A \leq^{\textit{poly-length}}_{k\text{-to-}k'} B$ if there exist a function h and polynomial p satisfying:

   (a) $(\forall x \in \Sigma^*)[(x \in A \iff h(x) \in B) \wedge |h(x)| \leq p(|x|)]$ and

(b) $(\forall S \subseteq \Sigma^*)[||S|| \leq k' \Rightarrow ||\{x \in \Sigma^* \mid h(x) \in S\}|| \leq k]$.

As a final introductory note, we stress that we are comparing reductions between sets. If one looks at reductions of sets to classes, non-injective reductions can often be made injective. For example, it is easy to see the following, since if $A \in \mathrm{R}^p_{(n\mathcal{O}(1))\text{-to-}1}(\mathrm{SPARSE})$,[2] as certified by reduction f and sparse set S, then A is polynomial-time equivalent to the sparse set $\{\langle x, y\rangle \mid f(x) = y \wedge y \in S\}$.

Observation 2.6 $\mathrm{R}^p_{(n\mathcal{O}(1))\text{-to-}1}(\mathrm{SPARSE}) = \mathrm{R}^p_{1\text{-to-}1}(\mathrm{SPARSE}) \subseteq \mathrm{E}^p_{(n\mathcal{O}(1))\text{-}\widehat{\text{to}}\text{-}1}(\mathrm{SPARSE}) = \mathrm{E}^p_{1\text{-}\widehat{\text{to}}\text{-}1}(\mathrm{SPARSE})$.

3 Results

We first look at finite-to-one reductions, and then turn to the study of k-to-k' reductions. The following theorem completely characterizes whether $f(n)$-$\widehat{\text{to}}$-1 reducibility is more powerful than $g(n)$-$\widehat{\text{to}}$-1 reducibility.

Theorem 3.1 Let $f, g \in \mathrm{FP}$, $f, g : \mathcal{N} \to \mathcal{N}$.

$$(\exists_\infty n)[f(n) > g(n)]$$

$$\Longleftrightarrow$$

$$(\exists A, B)[A \leq^p_{f(n)\text{-}\widehat{\text{to}}\text{-}1} B \ \wedge \ A \not\leq^p_{g(n)\text{-}\widehat{\text{to}}\text{-}1} B].$$

The left to right direction of Theorem 3.1 also holds for the case of $f(n)$-to-1 reductions. However, to make the theorem non-trivial (e.g., to ban separating 2-to-1 reductions from 1-to-1 reductions via sets A and B with $||A|| = 2$ and $||B|| = 1$), we must note that the sets witnessing $A \not\leq^p_{g(n)\text{-to-}1} B$ also have the property that A plausibly might reduce to B: $A \leq^{\textit{poly-length}}_{g(n)\text{-to-}1} B$.

Theorem 3.2 Let $f, g \in \mathrm{FP}$, $f, g : \mathcal{N} \to \mathcal{N}$. If $(\exists_\infty n)[f(n) > g(n)]$, then $(\exists A, B)[A \leq^p_{f(n)\text{-to-}1} B$ and $A \leq^{\textit{poly-length}}_{g(n)\text{-to-}1} B$, yet $A \not\leq^p_{g(n)\text{-to-}1} B]$.

[2]Notation: SPARSE denotes the class of sparse sets. A set S is said to be *sparse* if there is a polynomial p such that $(\forall n)[||\{x \mid |x| = n \wedge x \in S\}|| \leq p(n)]$. $\mathrm{R}^p_\tau(\mathcal{C}) = \{L \mid (\exists A \in \mathcal{C})[L \leq^p_\tau A]\}$; we'll also use the notation $\mathrm{E}^p_\tau(\mathcal{C}) = \{L \mid (\exists A \in \mathcal{C})[L \leq^p_\tau A \ \wedge \ A \leq^p_\tau L]\}$. For discussion of reductions and equivalence to sparse sets, see [BK88,GW91,AH,AHOW,TB].

Theorems 3.1 and 3.2 are proven by direct diagonalizations, and are omitted. Though Theorem 3.2 shows that one direction of Theorem 3.1 holds for $f(n)$-to-1 reductions, the same claim cannot be made for the other direction.

Theorem 3.3 There are functions f, $g \in \mathrm{FP}$, $f, g : \mathcal{N} \to \mathcal{N}$, and sets A and B such that:

1. $f(n) \leq g(n)$ almost everywhere,
2. $A \leq^{p}_{f(n)\text{-to-}1} B$, and
3. $A \leq^{\textit{poly-length}}_{g(n)\text{-to-}1} B$, yet
4. $A \not\leq^{p}_{g(n)\text{-to-}1} B$.

The proof of Theorem 3.3 is a simplified version of the proof of Theorem 3.4, and thus is omitted. However, it should be noted that the counter-example described in the above theorem can be easily taken to be the case where $f(0) = 2$, $g(0) = 1$, and $(\forall n \geq 1)[f(n) = g(n) = 1]$.

Now we turn to our second family of quasi-injective reductions—k-to-k' reductions. We will refer to $\left\lfloor \frac{k}{k'} \right\rfloor$ as the *base* of the k-to-k' reduction, and $k - k'\left\lfloor \frac{k}{k'} \right\rfloor$ as the *excess* of the reduction. Intuitively, the base indicates the level of non-injectivity that can be tolerated infinitely often, and the excess indicates a cap on the total amount of non-injectivity beyond the base level. As an example, a 4-to-2 reduction can be (at most) 2-to-1 everywhere, or it can be (at most) 3-to-1 on one range point and (at most) 1-to-1 elsewhere.

For k-to-k' reductions with large values of k and k', the range of possibilities expands dramatically. Nonetheless, the notions of base and excess offer a complete characterization of the relative strength of k-to-k' reductions. In the sense made formal by Theorem 3.4, a c-to-d reduction is more flexible than an a-to-b reduction exactly when the former has a larger base, or, in the case of identical bases, when the former has a larger excess.

Theorem 3.4 Let $a, b, c, d \in \{1, 2, 3, \ldots\}$, $c \geq d$, $a \geq b$.

$$\left(\left\lfloor \frac{a}{b} \right\rfloor < \left\lfloor \frac{c}{d} \right\rfloor\right) \vee \left[\left(\left\lfloor \frac{a}{b} \right\rfloor = \left\lfloor \frac{c}{d} \right\rfloor\right) \wedge \left(a - b\left\lfloor \frac{a}{b} \right\rfloor < c - d\left\lfloor \frac{a}{b} \right\rfloor\right)\right]$$

$$\Longleftrightarrow$$

$$(\exists A, B)[A \leq^{p}_{c\text{-to-}d} B \text{ and } A \leq^{\textit{poly-length}}_{a\text{-to-}b} B, \text{ yet } A \not\leq^{p}_{a\text{-to-}b} B].$$

Indeed, in the case where $(\lfloor \frac{a}{b} \rfloor < \lfloor \frac{c}{d} \rfloor) \vee [(\lfloor \frac{a}{b} \rfloor = \lfloor \frac{c}{d} \rfloor) \wedge (a - b\lfloor \frac{a}{b} \rfloor < c - d\lfloor \frac{a}{b} \rfloor)]$ does not hold, it follows that $(\forall A, B : B \text{ and } \overline{B} \text{ are infinite})\,[A \leq^{p}_{c\text{-to-}d} B \Rightarrow A \leq^{p}_{a\text{-to-}b} B]$.

Before proving Theorem 3.4, we state and prove a useful lemma.

Lemma 3.5 If g is an a-to-b reduction and $||S|| \geq b$, then $||g^{-1}(S)|| \leq a + (||S|| - b)\lfloor \frac{a}{b} \rfloor$.

Proof of Lemma 3.5

Let S' be a subset of S of size b such that $(\forall m \in S')(\forall m' \in S - S')[||g^{-1}(m)|| \geq ||g^{-1}(m')||]$. Since $||g^{-1}(S')|| \leq a$, there is an element m of S' such that $||g^{-1}(m)|| \leq \lfloor \frac{a}{b} \rfloor$. So $||g^{-1}(m')|| \leq \lfloor \frac{a}{b} \rfloor$ for all $m' \in S - S'$. It follows that $||g^{-1}(S)|| = ||g^{-1}(S')|| + ||g^{-1}(S - S')|| \leq a + (||S|| - b)\lfloor \frac{a}{b} \rfloor$. ∎

Proof of Theorem 3.4

($\Rightarrow$) Suppose $(\lfloor \frac{a}{b} \rfloor < \lfloor \frac{c}{d} \rfloor) \vee [(\lfloor \frac{a}{b} \rfloor = \lfloor \frac{c}{d} \rfloor) \wedge (a - b\lfloor \frac{a}{b} \rfloor < c - d\lfloor \frac{a}{b} \rfloor)]$. It follows immediately that there is an $m_0 \geq b$ such that $a + (m_0 - b)\lfloor \frac{a}{b} \rfloor < c + (m_0 - d)\lfloor \frac{c}{d} \rfloor$. Let $f : \mathcal{N} \to \mathcal{N}$ satisfy:

- $n \leq n' \Rightarrow f(n) \leq f(n')$,
- $||f^{-1}(0)|| = c + (1 - d)\lfloor \frac{c}{d} \rfloor$,
- $||f^{-1}(m)|| = \lfloor \frac{c}{d} \rfloor$ for $1 \leq m < m_0$, and
- $||f^{-1}(m)|| = \lfloor \frac{a}{b} \rfloor$ for $m \geq m_0$.

Clearly, f is uniquely defined and is computable in time polynomial in the size of its input. Since $\lfloor \frac{a}{b} \rfloor \leq \lfloor \frac{c}{d} \rfloor \leq c + (1 - d)\lfloor \frac{c}{d} \rfloor$, for any set S of size d it holds that $||f^{-1}(S)|| \leq c + (1 - d)\lfloor \frac{c}{d} \rfloor + (d - 1)\lfloor \frac{c}{d} \rfloor = c$. Thus, for any set B, $f^{-1}(B) \leq^p_{c\text{-to-}d} B$, as certified by polynomial-time reduction f. We will construct $B = \bigcup_{i \geq 0} B_i$ in stages so that $f^{-1}(B) \not\leq^p_{a\text{-to-}b} B$, and $f^{-1}(B) \leq^{poly\text{-}length}_{a\text{-to-}b} B$. Let $\sigma_1, \sigma_2, \cdots$ be an enumeration of all polynomial-time a-to-b reductions.[3]

Stage 0: Set $B_0 = \{0, \ldots, m_0\}$.

Stage i: Let m be the largest element of B_{i-1}.
Choose n such that $f(n) \in B_{i-1}$ and $\sigma_i(n) \notin B_{i-1}$.
If $\sigma_i(n) > m$ then set $B_i = B_{i-1} \cup \{\sigma_i(n) + 1\} \cup \{j \mid m + 1 \leq j \leq \sigma_i(n) - 1\}$,
else set $B_i = B_{i-1}$.

If n can be chosen as described above, then $f^{-1}(B) \not\leq^p_{a\text{-to-}b} B$. Furthermore, $f^{-1}(B) \leq^{poly\text{-}length}_{a\text{-to-}b} B$, by the recursive reduction that is identical to f on elements not in $f^{-1}(B)$,[4] and that maps the nth element of $f^{-1}(B)$ to the $(n/\lfloor \frac{a}{b} \rfloor)$th element of B. This

[3] One can construct a list of exponential-time machines that enumerate the $\leq^p_{a\text{-to-}b}$ reductions. Thus, this step is effective; it is not hard to see that the sets B and $f^{-1}(B)$ will be recursive.

[4] Note that all such elements are greater than m_0.

reduction is *poly-length*, since f is $\lfloor \frac{a}{b} \rfloor$-to-1 almost everywhere, and no two consecutive elements are in $\overline{B}$.

To prove that n can be chosen as specified above, it suffices to show that for all $i \geq 1$, $||\sigma_i^{-1}(B_{i-1})|| < ||f^{-1}(B_{i-1})||$. Since $||B_{i-1}|| \geq m_0 \geq b$, it follows from Lemma 3.5 and our choice of m_0 that:

$$\begin{aligned} ||\sigma_i^{-1}(B_{i-1})|| &\leq a + (||B_{i-1}|| - b)\lfloor \tfrac{a}{b} \rfloor \\ &= a + (m_0 - b)\lfloor \tfrac{a}{b} \rfloor + (||B_{i-1}|| - m_0)\lfloor \tfrac{a}{b} \rfloor \\ &< c + (m_0 - d)\lfloor \tfrac{c}{d} \rfloor + (||B_{i-1}|| - m_0)\lfloor \tfrac{a}{b} \rfloor \\ &= ||f^{-1}(B_{i-1})||. \end{aligned}$$

($\Leftarrow$) Suppose $(\lfloor \frac{a}{b} \rfloor > \lfloor \frac{c}{d} \rfloor) \vee [(\lfloor \frac{a}{b} \rfloor = \lfloor \frac{c}{d} \rfloor) \wedge (a - b\lfloor \frac{a}{b} \rfloor \geq c - d\lfloor \frac{a}{b} \rfloor)]$. We will prove that $(\forall A, B : B \text{ and } \overline{B} \text{ are infinite})\, [A \leq^p_{c\text{-to-}d} B \Rightarrow A \leq^p_{a\text{-to-}b} B]$. This proves the theorem, since for any sets A and B, if $A \not\leq^p_{a\text{-to-}b} B$ and $A \leq^{poly\text{-}length}_{a\text{-to-}b} B$, then B and $\overline{B}$ are infinite.

Suppose $A \leq^p_{c\text{-to-}d} B$ via reduction f, B and $\overline{B}$ are infinite, and f is not a-to-b. Since f is $\lfloor \frac{c}{d} \rfloor$-to-1 (and therefore $\lfloor \frac{a}{b} \rfloor$-to-1) almost everywhere and finite-to-one everywhere, there exists a finite set X such that $(\forall m \in \mathcal{N})[||f^{-1}(m) - X|| \leq \lfloor \frac{a}{b} \rfloor]$. To transform f to an $\lfloor \frac{a}{b} \rfloor$-to-1 reduction from A to B, we need only change f on the (finite number of) elements in X. Since B and $\overline{B}$ are both infinite, it suffices to show that f is $(\lfloor \frac{a}{b} \rfloor - 1)$-to-1 almost everywhere.

This clearly is the case when $\lfloor \frac{a}{b} \rfloor > \lfloor \frac{c}{d} \rfloor$. In the case where $(\lfloor \frac{a}{b} \rfloor = \lfloor \frac{c}{d} \rfloor) \wedge (a - b\lfloor \frac{a}{b} \rfloor \geq c - d\lfloor \frac{a}{b} \rfloor)$ and $b < d$, let Y be such that $||Y|| = b$ and $||f^{-1}(Y)|| > a$. If—as will be the case if f is not $(\lfloor \frac{a}{b} \rfloor - 1)$-to-1 almost everywhere—for infinitely many m it holds that $||f^{-1}(m)|| \geq \lfloor \frac{a}{b} \rfloor$, then there exists a set $Y' \supsetneq Y$ such that $||Y'|| = d$ and $c \geq ||f^{-1}(Y')|| > a + (d - b)\lfloor \frac{a}{b} \rfloor$. Then $c - d\lfloor \frac{a}{b} \rfloor > a - b\lfloor \frac{a}{b} \rfloor$, which contradicts our assumptions. Therefore, in this case f is $(\lfloor \frac{a}{b} \rfloor - 1)$-to-1 almost everywhere, as desired. In the final case, $(\lfloor \frac{a}{b} \rfloor = \lfloor \frac{c}{d} \rfloor) \wedge (a - b\lfloor \frac{a}{b} \rfloor \geq c - d\lfloor \frac{a}{b} \rfloor)$ and $b \geq d$. In this case, f is already an a-to-b reduction, since by Lemma 3.5, for any subset Y of size b, $||f^{-1}(Y)|| \leq c + (b - d)\lfloor \frac{c}{d} \rfloor \leq a$. This contradicts our assumption that f is not a-to-b. ∎

Acknowledgments

We thank William Gasarch and Yenjo Han for helpful conversations and comments.

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