Institute for Logic, Language and Computation

APPROXIMATION, SIMILARITY AND ROUGH CONSTRUCTIONS

Part I. Elementary Introduction

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Part I. Elementary Introduction

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APPROXIMATION, SIMILARITY AND ROUGH CONSTRUCTIONS
PART 1. ELEMENTARY INTRODUCTION

Introduction.

Around 1980 two formal concepts were formulated which may be seen as a formalization of the data base or attribute-value system. Namely R. Wille introduced the notion of concept lattices and Z. Pawlak formulated the idea of information system. In the sequel we concentrate on the investigations related to this second notion. Here the approach is based on Aristotle's idea of an attribute. We do not enter the area of philosophical considerations, mentioning only that in view of later developed formalizations, an attrribute was sometimes understood rather as a value of a function, say, then the attribute itself. The information system's methods are applied in medicine, industry, psychology, computer science etc., see Pawlak [28] for a good introductory presentation and Słowiński [48] for a collection of papers, cf. also Iwiński [11]. Theoretical investigations use Boolean, relational, topological and lattice-theoretical methods. The connections with cylindric algebras, hypergraphs, weak orders were examined, we can safely say that the set of information systems methods has grown up to the theory. In this paper, shortly speaking, we introduce and examine notions related to data analysis. Our considerations are also strongly related with rough sets and similar constructions. Some applications are also suggested.

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CHAPTER 1

APPROXIMATION

1. Introduction.

The notion of information system, which is a starting point of the present paper, was introduced by Pawlak in 1981, and since then it has been intensively investigated. In particular related notions of a nondeterministic information system and an approximation space were also examined. These notions are used to analyse computer and empirical data, being helpful in understanding indiscernibility and similarity of objects.

In sections 2, 3, 4 we recall basic notions and we give the short motivation for considering generalized approximation space. In section 5 we examine several approximation operations in view of lattice theory. In section 6 we introduce the notion of approximation algebra and we use this notion to characterize families of definable sets.

Throughout the paper we use the standard mathematical notation, in particular P(X) stands for the family of all subsets of the set X. A family $\mathbb{E} \subset P(X)$ such that $\bigcup \mathbb{E} = X$ is called a cover of X. Frequently we will consider the cover $\mathbb{E}$ whose elements are nonempty, pair wise disjoint subsets of X. In such a case it is called a partition of X. Any relation τ on a set **U** which is reflexive and symmetric is called the tolerance relation. A set $E \subseteq \mathbf{U}$ such that $E \times E \subseteq \tau$ and which is maximal with respect to inclusion is called a tolerance class.

2. Information system and approximation space. [27].

Throughout the paper **U** will be an arbitrary fixed set, called universe.

An **information system** is a quadruple $(\mathbf{U}, A, (V_a)_{a \in A}, f)$ where **U** is a set of objects, A stands for a set of attributes, V_a is a set of values of an attribute a, and $f: \mathbf{U} \times A \to \bigcup V_a$ is a function (called **information function**) such that $f(x,a) \in V_a$ for any $x \in \mathbf{U}$ and $a \in A$.

For every subset $B \subseteq A$ an indiscernibility relation $Ind(B) \subseteq \mathbf{U}^2$ is defined in the following way: for any x, y ∈ **U**

(1) x Ind(B)y iff f(x, a)=f(y, a) for every a ∈ B.

If $x \, Ind(B) y$ we say that x, y are indiscernible with respect to B.

Suppose that R is an equivalence relation in **U**. The pair (**U**,R) is called an **approximation space.** $[x]_R$ will stand for the equivalence class of the relation R determined by $x \in$ **U**.

Traditionally the equivalence classes of R are called **R-elementary sets.**

For any set $X \subseteq$ **U** its **lower** (resp. **upper**) **approximation** L(X) (resp. U(X)) is defined as follows:

$$L(X)=\{ x: [x]_R \subseteq X \}$$

(2)

$$U(X)=\{x: [x]_R \cap X \neq \emptyset \}$$

For brevity, we often write $\underline{X}$ instead of L(X) and $\overline{X}$ instead of U(X). L(X) is also denoted by $\underline{R}(X)$, and U(X) by $\bar{R}(X)$.

Let us recall that any set $X \subseteq$ **U** is called **definable** iff L(X)=U(X). Equivalently, X is definable iff L(X)=X iff U(X)=X iff X is a union of some R - elementary sets. Thus, the family Def(**U**,R) of all definable sets is a complete atomic Boolean algebra with the usual set operations, having as atoms the elementary sets. The family Def(**U**,R) is topology for **U** while the family of all elementary sets is a base for Def(**U**,R). L(X) (U(X)) is an interior (a closure) of X, respectively.

3. Nondeterministic information system.

Suppose we are given the information system $(\mathbf{U},A,(V_a)_{a\in A},f)$. It may happen that the information function f is not determined precisely i.e. the values of f are not settled uniquely. For instance, assume that one has to estimate the value of a light stimulus on a given measurement scale; then, an estimation is given by the interval in which we expect to find the actual value of the stimulus. Then it may be reasonable to consider a function F having as values the subsets of V.

Formally we define: The quadruple $(\mathbf{U},A,(V_a)_{a\in A},F)$ where F is an arbitrary function satisfying

$F: U \times A \to P(V)$ and $F(U \times \{a\}) \subseteq P(V_a)$ for any $a \in A$, is called a **nondeterministic information system** .

Now let $(U, A, (V_a)_{a \in A}, F)$ be the nondeterministic information system. For any subset $B \subseteq A$ we define a similarity of objects with respect to B in the following way: for any x, y $\in$ U

(3) $(x, y) \in sim(B)$ iff $\forall_{b \in B}\ F(x,b) \cap F(y,b) = \emptyset$.

The relation sim(B) is called **B-similarity** relation and, if $(x,y) \in sim(B)$ then we say that x, y are B-similar.
Some other tolerances in the system $(U, A, (V_a)_{a \in A}, F)$ are worth mentioning:

$(x, y) \in \Pi$ iff $_a\forall_{b \in B}$ $(F(x,a) \subseteq F(y,a)$ or $F(y,b) \subseteq F(x,b))$,
$(x, y) \in \Pi'$ iff $_b\forall_{\in B}$ $F(x,b) \cap F(y,b) \neq \emptyset$ and $\exists_{b \in B}$ $F(x,b) = F(y,b)$

Thus it seems to be desirable to examine systems with tolerance.

4. Approximation space.

As was mentioned above, the notion of approximation space has been defined by Pawlak as a pair (U,R) where R is an equivalence relation in U. Now, suppose that R is an arbitrary binary relation in U. Any set $E \subseteq U$ satisfying $E \times E \subseteq R$ and maximal with respect to inclusion will be called **R-elementary**.

Applying Kuratowski-Zorn lemma we shall prove the following

Lemma 1. Suppose R is a reflexive relation in U. Then the family **E** of all R-elementary sets is a cover of U.

Proof. Suppose x is an arbitrary element of U; we have $\{x\} \times \{x\} \subseteq R$ by reflexivity of R. Now consider any chain $\{E_\eta : \eta < \lambda\}$ of sets such that $E_\eta \times E_\eta \subseteq R$. We have

$$\bigcup_{\eta<\lambda} E_\eta \times \bigcup_{\eta<\lambda} E_\eta \subseteq R$$

Indeed, if $x \in \bigcup_{\eta<\lambda} E_\eta$ and $y \in \bigcup_{\eta<\lambda} E_\eta$ then $x \in E_\delta$ for some $\delta < \lambda$ and $y \in E_\gamma$ for some $\gamma < \lambda$. Since $E_\delta \subseteq E_\gamma$ or $E_\gamma \subseteq E_\delta$ we infer that $(x, y) \in E_\gamma \times E_\gamma$ or

$(x, y) \in E_\delta \times E_\delta$ hence finally $(x,y) \in R$. In other words $\bigcup_{\eta<\lambda} E_\eta$ is an upper bound of the chain $\{E_\eta : \eta<\lambda\}$. Therefore there exists a maximal set E satisfying $E \times E \subseteq R$ such that $x \in E$, in view of Kuratowski-Zorn lemma. Hence x belongs to some R-elementary set, as required.

As a consequence we obtain the well known

Corollary 1. Suppose τ is a tolerance in **U**. Then the family $\mathbb{E}(\tau)$ of all tolerance classes of τ is a cover of **U**.

This is our motivation to consider in what follows the space $(\mathbf{U}, \mathbb{E})$, where $\mathbb{E}$ is a cover of **U**. The pair $(\mathbf{U}, \mathbb{E})$ will be called generalized approximation space.

5. Approximation operations.

Suppose $(\mathbf{U}, \mathbb{E})$ is a generalized approximation space. Let us recall that the indiscernibility neighborhood of an element $x \in \mathbf{U}$ is the set

$$O_x^{\mathbb{E}} = \bigcup \{ E_t : x \in E_t \}.$$

For any element $x \in \mathbf{U}$, the set

$$I_x^{\mathbb{E}} = \{ y \in \mathbf{U} : \forall E_t \ (x \in E_t \Leftrightarrow y \in E_t) \}$$

will be called the **kernel of x** in view of its analogy to the notion used in the theory of tolerance relations. If no confusion is possible we shall write O_x and I_x instead of $O_x^{\mathbb{E}}$ and $I_x^{\mathbb{E}}$, respectively. Let J be the family of all the kernels of $(\mathbf{U}, \mathbb{E})$:

$$J(\mathbb{E}) = J = \{ I_x : x \in \mathbf{U} \}.$$

It is easy to verify that J is a partition; the equivalence relation determined by J will be denoted by I. If xIy then we say that x, y are **$\mathbb{E}$-inseparable**.

Let -X stand for **U**-X. We say that two operations $G, G' : P(\mathbf{U}) \to P(\mathbf{U})$ are conjugated iff for any $X \subseteq \mathbf{U}$, the following condition is satisfied:

$$G(X) = -G'(-X).$$

Now we apply the introduced notions to define some special pairs of conjugated approximation operations in the space $(\mathbf{U}, \mathbb{E})$. When $\mathbb{E}$ is a partition of **U**, all those operations will coincide with the well known lower and upper approximation operations of

Pawlak. The motivation to consider pairs of conjugated operations comes from two sources: first, the operations G and G' may be used to define operators of necessity and possibility in a respective modal logic and second, in cases when G, G' are topological operations then in order to define the same topology on **U**, they have to be conjugated.

Let X be a subset of **U**. We will define the operations $\underline{E}_i$, as follows:

(4)

$$\underline{E}_1(X) = \{x:\ O_x \subseteq X\}$$

$$\bar{E}_1(X) = \bigcup \{E_t : E_t \cap X \neq \emptyset\}$$

$$\underline{E}_2(X) = \bigcup \{O_x : O_x \subseteq X\}$$

$$\bar{E}_2(X) = \{z: \forall y\ (z \in O_y \Rightarrow O_y \cap X \neq \emptyset\}$$

$$\underline{E}_3(X) = \bigcup \{E_t : E_t \subseteq X\}$$

$$\bar{E}_3(X) = \{y: \forall E_t\ (y \in E_t \Rightarrow E_t \cap X \neq \emptyset\}$$

$$\underline{E}_4(X) = \bigcup \{I_y : I_y \subseteq X\}$$

$$\bar{E}_4(X) = \bigcup \{I_y : I_y \cap X \neq \emptyset\}$$

First, let us observe that for i=2, 3, 4 the operations $\underline{E}_i$, $\bar{E}_i$ are idempotent i.e. for any $n \in \omega$, $X \subseteq \mathbf{U}$ and for i=2, 3, 4 the following conditions are valid:

(a) $(\underline{E}_i^n(X)) = \underline{E}_i(X)$

(b) $((\bar{E}_i)^n(X)) = \bar{E}_i(X)$

The situation is more complicated when we iterate operation $\underline{E}_1$ or $\bar{E}_1$. For any $X \subseteq \mathbf{U}$ we have the inclusions

$$\bar{E}_1(X) \subseteq (\bar{E}_1)^2(X) \subseteq (\bar{E}_1)^3(X) \ldots$$

but it may happen that the elements of this sequence are pair wise distinct. So, we introduce one more approximation operation $\bar{E}_0$ in the following manner:

(5) $$\bar{E}_0(X) = \bigcup_{i<\omega} (\bar{E}_1)^i(X)$$

We shall call $\bar{E}_0$ the **transitive closure operation** (by analogy to the terminology used in the theory of tolerance relations). The set $\bar{E}_0(\{x\})$ denoted by C_x will be called the **component** of x in **U**,

$$C_x \stackrel{def}{=} \bar{E}_0(\{x\})$$

It is easy to check that $\bar{E}_0(X)=\bigcup\{C_y: C_y \cap X \neq \emptyset\}$ and the conjugated operation $\underline{E}_0$ satisfies

$$\underline{E}_0(X) = \{x \in X: C_x \subseteq X\},$$

since the family $\mathbf{C}=\{C_x : x \in \mathbf{U}\}$ of all components in **U**, is a partition of **U**.

Let us also observe that the following inclusions hold:

$$\underline{E}_0 \le \underline{E}_1 \le \underline{E}_2 \le \underline{E}_3 \le \underline{E}_4 \le Id \le \bar{E}_4 \le \bar{E}_3 \le \bar{E}_2 \le \bar{E}_1 \le \bar{E}_0.$$

Now, to express the algebraic properties of the above operations, we recall some notions from lattice theory:

Let G be a mapping of P(**U**) into itself. We shall say that G is a **lower** (upper) **operation** on **U** iff for any $X \subseteq \mathbf{U}$, $G(X) \subseteq X$ ($G(X) \supseteq X$), respectively. (The upper operation is also called **extensive**)

The mapping G is said to be **monotonic** iff (if $X \subseteq Y$ then $G(X) \subseteq G(Y)$ for any $X, Y \subseteq \mathbf{U}$). Any monotonic and lower or monotonic and upper operation will be called an **approximation operation**. The most important examples of operations satisfying this definition are the lower $L=\underline{R}$ and upper $U=\bar{R}$ approximation operations of Pawlak

The mapping G is said to be **idempotent** iff for every $X \subseteq \mathbf{U}$, $G(X) = G(G(X))$. If G is an upper, monotonic and idempotent mapping then G is called a **closure mapping** and the pair (**U**, G) is called a **closure space**.

To summarize this section we recall that a closure operator H on the set **U** is an **algebraic** (resp. **topological**) closure operator if for every $X \subseteq \mathbf{U}$

$$H(X) = \bigcup \{ H(X') : X' \subseteq X \text{ and } X' \text{ is finite} \}$$

(resp. for every $X, Y \subseteq \mathbf{U}$ $H(X \cup Y) = H(X) \cup H(Y)$).

Theorem 1. Assume (**U**, **E**) is a generalized approximation space and $\underline{E}_i, \bar{E}_i$, i=0,...,4 are the approximation operations defined by (4). Then it holds:

(a) $\bar{E}_0, \bar{E}_4$ are topological algebraic closure operations;

(b) $\bar{E}_2, \bar{E}_3$ are closure operations;

(c) $\bar{E}_1$ is monotonic, extensive and it satisfies the condition

$$\overline{\mathbb{E}}_1(X \cup Y) = \overline{\mathbb{E}}_1(X) \cup \overline{\mathbb{E}}_1(Y), \text{ for any } X, Y \subseteq \mathbf{U} \text{ ;}$$

(d) $\overline{\mathbb{E}}_1$ is a topological closure operation iff $\{O_x : x \in \mathbf{U}\}$ is a partition of **U**.

Proof. It is easy to prove (a), (b) and (c). For a little bit more difficult (d) see [33].

6. Approximation algebra.

In applications it is often considered a family of all definable subsets of the universe **U**. To formulate definitions of these families in a unified way, we introduce the following approximation algebra:

An algebra $(P(\mathbf{U}), \{\underline{G}_i, \overline{G}_i : i \in I\})$ is called an **approximation algebra** on **U** if, for any X, Y ⊆ **U** and i∈I, it satisfies:

1) $\overline{G}_i : P(\mathbf{U}) \longrightarrow P(\mathbf{U})$

2) $X \subseteq \overline{G}_i$

3) $X \subseteq Y$ implies $\overline{G}_i(X) \subseteq \overline{G}_i(Y)$

4) $\underline{G}_i(X) = -\overline{G}_i(-X)$.

A subset X of **U** is called a **definable** subset with respect to $\{\overline{G}_i : i \in I_0 \subseteq I\}$ if for every $i \in I_0$ it holds $\overline{G}_i(X) = X$. In other words X is a fixed point of all $\overline{G}_i$, $i \in I_0$. Similarly, X is definable with respect to $\{\underline{G}_i : i \in I_0\}$ if $\forall_{i \in I_0} \underline{G}_i(X) = X$. The family of all definable sets with respect to $\{\underline{G}_i : i \in I_0\}$ will be denoted by $\mathrm{Def}(\mathbf{U}, \{\underline{G}_i : i \in I_0\})$, or in short by $\underline{\mathrm{Def}}(I_0)$. $\overline{\mathrm{Def}}(I_0) = \mathrm{Def}(\mathbf{U}, \{\overline{G}_i : i \in I_0\})$ denotes the family of all definable sets with respect to $\{\overline{G}_i : i \in I_0\}$.

Lemma 2. The family $\overline{\mathrm{Def}}(I_0)$ is closed on intersections i.e.

if $\forall_{t \in S} X_t \in \overline{\mathrm{Def}}(I_0)$ then $\bigcap_{t \in S} X_t \in \overline{\mathrm{Def}}(I_0)$.

Proof. Assume $i \in I_0$. It holds

$$\overline{G}_i\left(\bigcap_{t \in S} X_t\right) \subseteq \bigcap_{t \in S} \overline{G}_i(X_t) = \bigcap_{t \in S} X_t$$

in view of the monotonicity of $\bar{G}_i$ and the hypothesis. On the other hand $\bar{G}_i(\bigcap_{t\in S} X_t) \supseteq \bigcap_{t\in S} X_t$, since $\bar{G}_i$ is extensive. Finally, $\bar{G}_i(\bigcap_{t\in S} X_t) = \bigcap_{t\in S} X_t$.

Corollary 2. $\overline{\mathrm{Def}}(I_0)$ is a complete lattice with respect to set inclusion, and

$$\inf\{X_t : t\in S\} = \bigcap_{t\in S} X_t, \quad \sup\{X_t : t\in S\} = \bigcap\{X\in\overline{\mathrm{Def}}(I_0):\ X\supseteq X_t\ \ \forall t\in S\}$$

Proof. It is a consequence of Lemma 9, p. 184 in [7], or Theorem 4.2 p. 14 in [2].

Lemma 3. The family $\underline{\mathrm{Def}}(I_0)$ is closed on arbitrary unions.

Proof. If $X_t\in\underline{\mathrm{Def}}(I_0)$ for every $t\in S$, then

$$\bigcup_{t\in S} X_t = \bigcup_{t\in S} \underline{G}_i(X) \leq \underline{G}_i(\bigcup_{t\in S} X_t) \leq \bigcup_{t\in S} X_t, \text{ for } i\in I_0.$$

Hence

$$\underline{G}_i(\bigcup_{t\in S} X_t) = \bigcup_{t\in S} X_t, \text{ for } i\in I_0, \text{ i.e. } \bigcup_{t\in S} X_t\in \underline{\mathrm{Def}}(I_0).$$

Corollary 3. $\underline{\mathrm{Def}}(I_0)$ is a complete lattice with respect to set inclusion and $\sup\{X_t : t\in S\} = \bigcup_{t\in S} X_t$

$$\inf\{X_t : t\in S\} = \bigcup\{X\in\underline{\mathrm{Def}}(I_0):\ X \leq \bigcap_{t\in S} X_t\}.$$

Applying these lemmas to the approximation algebra $(P(U), \{\underline{E}_i, \bar{E}_i, i\in\{0...4\}\})$ we obtain:

Corollary 4. Assume that $\underline{E}_i, \bar{E}_i$, $i=0,\dots,4$, are approximation operations in the space (U, E). Then it holds:

(a) $\mathrm{Def}(U, \underline{E}_1) = \mathrm{Def}(U, \bar{E}_1) = \mathrm{Def}(U, \underline{E}_0) = \mathrm{Def}(U, \bar{E}_0)$ and

$\mathrm{Def}(U, \underline{E}_4) = \mathrm{Def}(U, \bar{E}_4)$ are fields of sets;

(b) $\mathrm{Def}(U, \bar{E}_2)$ and $\mathrm{Def}(U, \bar{E}_3)$ are complete lattices with respect to set inclusion and $\inf Y = \bigcap Y$, $\sup Y = \bar{E}_2(\bigcup Y)$ ($\sup Y = \bar{E}_3(\bigcup Y)$), for $Y \subseteq Def(U,\bar{E}_2)$, $(Y \subseteq Def(U,\bar{E}_3))$, respectively;

(c) $Def(U,\underline{E}_2)$, $Def(U,\underline{E}_3)$ are complete lattices with respect to set inclusion and

sup $Y = \bigcup Y$, inf $Y = \underline{E}_2(\bigcap Y)$, (inf $Y = \underline{E}_3(\bigcap Y)$), for $Y \subseteq Def(U,\underline{E}_2)$, $(Y \subseteq Def(U,\underline{E}_3))$ respectively.

CHAPTER 2
REDUCTS

In this chapter we formalize the notions of reduct and subreduct of the set of attributes, in a slightly different way than it is formulated in [18].

1. Reducts of attributes.

Let rel : $P(A) \rightarrow P(U\times U)$ be arbitrary function satisfying the condition: for every subset $B \subseteq A$

(1) $rel(B) = \bigcap \{rel(\{b\}) : b \in B\}$

$rel(\emptyset) = U\times U$.

In view of (1) we have

(2) $B \subseteq B' \Rightarrow rel(B) \supseteq rel(B')$

Instead of $rel(\{a\})$ we shall write often $rel(a)$.

Example 1:

(a) Assume that (U,A,V,f) is an information system. For $B \subseteq A$ we define $rel(B) = Ind(B)$;

(b) For a nondeterministic system (U,A,V,F) let us define for $B \subseteq A$: $rel(B) = sim(B)$;

(c) Assume that (U,A,V,f) is the information system and $\le$ is a linear order in the set V_a; for $b \in A$ we define

$rel(b) = \{(x,y) \in U\times U : f(x,b) \le f(y,b)\}$ and for $B \subseteq A$ let us set $rel(B) = \bigcap \{rel(b) : b \in B\}$;

(d) Consider the following table

	a	b	c	d
u1	1	0	3	8
u2	2	1	1	5

u3 | 3 | 1 | 2 | 7

Assume that $\leq$ is usual relation between real numbers, let us define rel like in the example 1(c) above. Then it holds

$$\mathrm{rel}(a)=\{(u1,u2),(u1,u3),(u2,u3)\}\cup Id,$$

$$\mathrm{rel}(b)=\{(u1,u2),(u1,u3)\}\cup Id, \quad \mathrm{rel}(\{a,b\})=\mathrm{rel}(b),$$

$$\mathrm{rel}(c)=\{(u2,u1),(u2,u3),(u3,u1)\}\cup Id,$$

$\mathrm{rel}(\{a,c\})=\{(u2,u3)\}\cup Id$, $\mathrm{rel}(\{a,c,d\})=\mathrm{rel}(\{a,c\})=\mathrm{rel}(\{a,d\})$. Then the reducts of the set $\{a,c,d\}$ are the sets $\{a,c\}$ i $\{a,d\}$ (the definition of the notion of reduct is given below).

Let $X \subseteq A$. We say that the set X jest **rel -independent** iff for every set $X' \subset X$ it holds $\mathrm{rel}(X') \neq \mathrm{rel}(X)$. Otherwise we say that X is **rel-dependent**. We say that $X' \subseteq X$ is **rel-reduct** of X iff $\mathrm{rel}(X') = \mathrm{rel}(X)$ and X' is independent.

Let us notice that the classical reduct (see [18]) is rel-reduct with respect to the function rel(B)=Ind(B). In the above example the rel-independent sets are the following sets: $\{a,b,c\}$, $\{a,b,d\}$, $\{a,b\}$, $\{a,d\}$, $\{a,c\}$, $\{b,c\}$, $\{b,d\}$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$.

Lemma 1. (see 3.1, 3.2, 4.1 in [18]).

(a) For every set $X \subseteq A$ there exists the reduct of X. If $\mathrm{rel}(X) \neq \mathrm{rel}(\emptyset)$ then the reducts of X are non empty sets.

(b) $X \subseteq A$ is independent iff for every $p \in X$ $\mathrm{rel}(X)\neq\mathrm{rel}(X-\{p\})$.

(c) If $X \subseteq A$ is independent and $X' \subseteq X$ then X' is also independent.

The set of all reducts of any subset $B \subseteq A$ will be denoted by Redrel(B). The set of all reducts of A will be denoted by Redrel. The set of independent subsets of B shall be denoted by Indrel(B). The set of independent subsets of A shall be denoted by Indrel. Assume $X \subseteq A$. An element $x \in X$ is called **dispensable** in X if $\mathrm{rel}(X) = \mathrm{rel}(X-\{x\})$. Otherwise x is called **indispensable** . The **core** of X is the following set (see 4.1 [18]):

$$\mathrm{Core}(X) = \{x \in X : x \text{ is indispensable in } X\}.$$

Theorem 1 (see 4.2 in [18]):

For every $X \subseteq A$ $Core(X) = \bigcap \{Q: Q \text{ is the reduct of } X\}$.

Proof.

Inclusion $\supseteq$ is obvious. We prove $\subseteq$. Let $p \in X$ be indispensable in X i.e. $rel(X - \{p\}) \neq rel(X)$. We will show that for every reduct $Q \subseteq X$, $p \in Q$. Assume on the contrary that Q is the reduct of X such that $p \notin Q$. Then

$\bigcap \{rel(q): q \in X\} = \bigcap \{rel(q): q \in Q\} \supseteq \bigcap \{rel(q): q \in X - \{p\}\}$, hence $rel(X) = rel(X - \{p\})$, which is contradictory to the choice of p.

2. Subreducts.

Let $X \subseteq A$. If $X' \subseteq X$ is maximal rel-independent subset of X, then it is called **rel-subreduct** of X. If moreover $rel(X') \neq rel(X)$ then X' is called the proper subreduct.

We denote the set of all rel-subreducts of X by Subr(X). Let $X \subseteq A$. The set $X_o = \{x \in X : rel(x) = rel(\emptyset)\}$ is called **null-part** of X (or zero part of X).

Lemma 3.

For every $X \subseteq A$ it holds:

(a) $X = X_o \cup \bigcup \{X' \subseteq X : X' \in Indrel(X)\}$

(b) $X = X_o \cup \bigcup \{X' : X' \in Subr(X)\}$

Proof:

Since every subreduct of X is independent the condition (a) follows from (b). To prove (b) let us notice that if $x \in X$ and $x \notin X_o$ then x belongs to some subreduct of X, because A (and as a consequence also X) is finite.

Lemma 4.

If $X \subseteq A$ and $rel(X) \neq rel(\emptyset)$ then there exists $Y \subseteq X$ s.t. $rel(Y) \supset rel(X)$ and $X = Y \cup \bigcup Redrel(X)$.

Proof:

If $X = \bigcup Redrel(X)$ then we take $Y = \emptyset$. If $X \neq \bigcup Redrel(X)$ then we put $Y = X - \bigcup Redrel(X)$. If $rel(Y) = rel(X)$ then a reduct of Y would be the reduct of X which is contradictory to the condition $Y = X - \bigcup Redrel(X)$.

Theorem 2.

If $X' \in Subr(X) - Redrel(X)$ then for every $z \in X-X'$ there exists $Z \subseteq X' \cup \{z\}$ such that

$\bigcap\{rel(x): x \in X' \cup \{z\} - Z\} \subseteq \bigcap\{rel(x): x \in Z\}$

Proof:

We have in view of the assumptions that $rel(X') \neq rel(X)$, X' is independent and for every X'', if $X' \subset X'' \subseteq X$ then X'' is dependent. In particular for every $z \in X - X'$ it holds $X' \subset X' \cup \{z\} \subseteq X$, therefore $X' \cup \{z\}$ is dependent, hence there exists a set $Z \subseteq X' \cup \{z\}$ such that

$$rel(X' \cup \{z\}) = rel(X' \cup \{z\} - Z),$$

in other words $\bigcap\{rel(x): x \in X' \cup \{z\}\} = \bigcap\{rel(x): x \in X' \cup \{z\} - Z\}$.

As a consequence

$\bigcap\{rel(x): x \in X' \cup \{z\} - Z\} \cap \bigcap\{rel(x): x \in Z\} = \bigcap\{rel(x): x \in X' \cup \{z\} - Z\}$

which implies that $\bigcap \{rel(x): x \in X' \cup \{z\} - Z\} c\subseteq \bigcap \{rel(x): x \in Z\}$. (intuitively speaking the set $X' \cup \{z\} - Z$ can better distinct (discern) the objects of the universe **U** then the set Z).

Corollary 1:

If $X' \in Subr(X) - Redrel(X)$ and $rel(x) = Ind(x)$ then for every $z \in X - X'$ there exist $Z \subseteq X' \cup \{z\}$ s.t. for every $u \in$ **U** $[u]_{Ind(X' \cup \{z\} - Z)} \subseteq [u]_{Ind(Z)}$

Example 2:

Assume that rel(B)=Ind(B) and consider the following information system

	a1	a2	a3	a4
u1	1	1	1	1
u2	0	1	1	1
u3	0	0	1	2
u4	0	0	0	3
u5	0	0	0	4

It holds : Redrel(A)={{a1,a4}} and Subr(A)={{a1,a4},{{a1,a2,a3}}.

Example 3:

Let **U**={u1,...,un+1}, A={a1,...,an} and $i \in \{1,...,n\}$ and let the system will be given by the following table

	a1	a2	a3	...	ai	...	an-1	an
u1	1	1	1		1		1	1
u2	0	1	1		1		1	1
u3	0	0	1		1		1	1
.	.	.	.		.		.	.
.	.	.	.		.		.	.
.	.	.	.		.		.	.
ui	0	0	0		1		1	1
.	.	.	.		0		1	1
.	.	.	.		.		.	2
.	.	.	.		.		.	3
un-1	0	0	0		0		1	n-i-2
un	0	0	0		0		0	n-(i+1)
un+1	0	0	0		0		0	n-i

It is easy to notice that for rel(A)=Ind(A) it holds:

Redrel(A)={{a1,...,ai,an}} and

Subr(A)={{a1,...,an-1},{a1,...,ai,an}}.

As a consequence of this example we see that the problem of finding the reducts and the subreducts of a given information system may force us to examine the subsets of A of arbitrary cardinality (smaller or equal to cardinality of A, of course). As a conclusion, there is no hope to invent an algorithm, which can give us all reducts (for every given information system) by examining only pairs (or pairs and triples etc.) of attributes .

In other words sometimes we have to investigate almost all subsets of the set of attributes.

CHAPTER 3
ROUGH CONSTRUCTIONS

1. Introduction

It is well known that if we apply to a set A two operations -the closure and the complement-in a fixed topological space $(\mathbf{U},\bar{\ })$ then the number of sets that can be obtained from A in this way is less or equal to 14 (Kuratowski [13]). This means that if we apply the closure and the complement operations to the sets A and B and we form an equality, then the number of relations defined with respect to these equalities in the family of all subsets of **U** has to be finite also. The equivalences of the similar kind are sometimes applied in computer science and data analysis, for example so called rough, bottom and top equality (see Nowotny, Pawlak [19]). In this paper we construct 18 relations (including rough top and bottom equalities) obtaining as a special case also topological rough sets (see Wiweger [45]).

2. Basic definitions

Assume that $(\mathbf{U}, {}^{-})$ is a topological space. If w is a finite sequence of ' and $^{-}$ (where ' means the complement operation) then we shall write w∈Word(', $^{-}$). In other words w is a word over the alphabet {' $^{-}$}

For $A \subseteq \mathbf{U}$ we define:

$A^{\emptyset} = A$

$A^{w} = A'$ for w='

$A^{w} = A^{-}$ for $w = {}^{-}$

and inductively for arbitrary w ∈ Word(', $^{-}$).

If R is an equivalence relation in **U** we denote by U and L Pawlak's closure and interior operations on **U**, i.e. for $A \subseteq \mathbf{U}$

$$U A = \bar{A} = \bigcup\{[x]: x \in A\}$$

where [x] is an equivalence class of x w.r.t. R.

L is defined to be conjugated to U, i.e. for each $A \subseteq U$ $LA = -U-A$.

In the sequel the operation $^{-}$ will be equal to U for some relation R. Usually we write $\bar{A}$, $\underline{A}$ instead of U A , L A , respectively. $(\mathbf{U}, {}^{-}) = (\mathbf{U}, \bar{R})$ is called approximation space.

Now let us define the following relations on P(**U**), let $A, B \subseteq \mathbf{U}$:

$$A \cong_1 B \text{ iff } A = B$$

$$A \cong_2 B \text{ iff } A = -B$$

$$A \cong_3 B \text{ iff } A = \underline{B}$$

$$A \cong_4 B \text{ iff } A = -\underline{B}$$

$$A \cong_5 B \text{ iff } A = \bar{B}$$

$A \cong_6 B$ iff $A = -\bar{B}$

$A \cong_7 B$ iff $\underline{A} = B$

$A \cong_8 B$ iff $\underline{A} = -B$

$A \cong_9 B$ iff $\underline{A} = \underline{B}$

$A \cong_{10} B$ iff $\underline{A} = -\underline{B}$

$A \cong_{11} B$ iff $\underline{A} = \bar{B}$

$A \cong_{12} B$ iff $\underline{A} = -\bar{B}$

$A \cong_{13} B$ iff $\bar{A} = B$

$A \cong_{14} B$ iff $\bar{A} = -B$

$A \cong_{15} B$ iff $\bar{A} = \underline{B}$

$A \cong_{16} B$ iff $\bar{A} = -\underline{B}$

$A \cong_{17} B$ iff $\bar{A} = \bar{B}$

$A \cong_{18} B$ iff $\bar{A} = -\bar{B}$

Let us observe that the first relation is the equivalence relation, the second is symmetric, the third is transitive, the fifth is transitive and similarly 13-th, finally 7-th, 9-th and 17-th are equivalences, 18-th is symmetric.

3. Main result

Our main theorem is the following

Theorem 1: If $(U,^{-}) = (U,U)$ is the approximation space for some equivalence relation R and $w1, w2 \in$ Word$(^{-},')$ then the relation $\equiv$ on P(U) defined by the condition

$$A \equiv B \text{ iff } A^{\sim\sim} = B^{\sim\sim}$$

is equal to one of the above 18 relations

Proof:

Lemma 1 (Kuratowski 1922)

Suppose that we apply to a set A the operations $^{-}$ and $'$. The number of sets that we obtain is less or equal to 14.

Lemma 2. If (U,U) is the approximation space and A' denotes the complement of A then there exist no more then 6 sets obtained by applying to the set A the operations of closure and of the complement. The following inclusions are generally valid among them:

$$A'^{-'} \subseteq A \subseteq A^{-}$$

$$A^{-'} \subseteq A' \subseteq A'^{-}$$

Lemma 3. The following equalities holds:

$$A'^{-'} = A'^{-,-} = A'^{-,-,-,}$$

$$A^{-,-,} = A^{-,-,-} = A^{-}$$

$$A^{-'} = A^{-,-} = A^{-,-,-,}$$

$$A'^{-} = A'^{-,-,} = A'^{-,-,-}$$

Now, in view of Kuratowski lemma we infer that the words w1,w2,in the equality $A^{w1} = B^{w2}$ may be reduced to words over the set

$$W=\{ '^{-'} , \varnothing , {}^{-}, {}^{-'} , ' , '^{-} \}.$$

Considering every pair of words w1,w2 belonging to W and examining the equalities $A^{w1} = B^{w2}$ it is easy to check that $A^{w1} = B^{w2}$ iff $A \cong_i B$ for some $i \in \{1...18\}$. Finally let us observe that all equivalences $\cong_1 ... \cong_{18}$ are different. The proof of the theorem is completed.

Remark. The general construction of this paper for arbitrary Kuratowski closure operation will be given in a forthcoming paper.

We leave open the problem of the description of all rough constructions defined with respect to the operations introduced in the paper [34].

It seems to be worth studying the structure of the algebras created from the family of all pairs (intA, clA) where A is included in a fixed topological space (U, cl), in particular it seems to be important to connect the properties of topological origin with those algebraical in spirit.

CHAPTER 4

ROUGH ALGEBRAS

In this chapter we formulate several algebras based on the family of rough sets. Our fundamental reference is the book of H.Rasiowa "An algebraic approach to non-classical logics" .

1. Basic definitions and results.

Let us recall that U is a fixed set, called in the sequel the universe, and R is an arbitrary equivalence relation on U. For every $x \in U$, by $[x]_R$ we shall denote the equivalence class of x. For every subset $a \subseteq U$ its lower approximation is the following set

(1) $\underline{a} = \{x\colon [x]_R \subseteq a\}$,

the upper approximation of a is defined in the following way

(2) $\overline{a} = \{x\colon [x]_R \cap a \neq \emptyset\}$.

In the family of all subsets of U the following relation is introduced: for $a, b \subseteq U$,

(3) $a \equiv b$ wtt $\underline{a} = \underline{b}$ i $\overline{a} = \overline{b}$.

The equivalence classes of $\equiv$ are called rough sets. [29].

$\mathcal{R} = \mathcal{R}(U,R)$ denotes the family of rough sets. In symbols

(4) $\mathcal{R} = \{[a]_{\equiv} : a \subseteq U\}$ or equivalently

(5) $\mathcal{R} = \{(\underline{a},\overline{a}) : a \subseteq U\}$.

In the family $\mathcal{R}$ we shall distinguish two constants:

$0 = (\emptyset, \emptyset)$ and $1 = (U,U)$.

Convention:

(6) $a = (\underline{a},\overline{a})$

Following Iwiński we introduce union and intersection operations:

$$(\underline{a},\overline{a}) \cup (\underline{b},\overline{b}) = (\underline{a} \cup \underline{b},\ \overline{a} \cup \overline{b})$$

(7)

$(\underline{a},\bar{a}) \cap (\underline{b},\bar{b}) = (\underline{a} \cap \underline{b},\ \bar{a} \cap \bar{b})$

It is easy to check that these operations are well defined, i.e.

$(\underline{a} \cup \underline{b},\ \bar{a} \cup \bar{b}) \in \mathcal{R}, \quad (\underline{a} \cap \underline{b},\ \bar{a} \cap \bar{b}) \in \mathcal{R}.$

Next, we introduce the one-argument operations $\underset{i}{\sim}, \underset{e}{\sim}, \underset{r}{\sim}$, called interior, exterior and rough complement, respectively:

$$(8) \qquad \begin{aligned} &\underset{i}{\sim}(\underline{a},\bar{a}) = (-\underline{a},-\underline{a}) \\ &\underset{e}{\sim}(\underline{a},\bar{a}) = (-\bar{a},-\bar{a}) \\ &\underset{r}{\sim}(\underline{a},\bar{a}) = (-\bar{a},-\underline{a}) \end{aligned}$$

As regards implication we introduce three operations

$\underset{i}{\longrightarrow}, \underset{e}{\longrightarrow}, \underset{r}{\longrightarrow}$ in the following way :

$$(9) \qquad \begin{aligned} &(\underline{a},\bar{a}) \underset{i}{\longrightarrow} (\underline{b},\bar{b}) = \underset{i}{\sim}(\underline{a},\bar{a}) \cup (\underline{b},\bar{b}) \\ &(\underline{a},\bar{a}) \underset{e}{\longrightarrow} (\underline{b},\bar{b}) = \underset{e}{\sim}(\underline{a},\bar{a}) \cup (\underline{b},\bar{b}) \\ &(\underline{a},\bar{a}) \underset{r}{\longrightarrow} (\underline{b},\bar{b}) = \underset{r}{\sim}(\underline{a},\bar{a}) \cup (\underline{b},\bar{b}) \end{aligned}$$

Finally we define several operations

$$\alpha,\beta^{\Rightarrow}$$

along the lines of the following schema:

$$(\underline{a},\bar{a}) \ {}_{\alpha\beta}{\Rightarrow} (\underline{b},\bar{b}) = ((\underline{a},\bar{a}) \underset{\alpha}{\longrightarrow} (\underline{b},\bar{b})) \cap (\underset{\beta}{\sim}(\underline{b},\bar{b}) \underset{\alpha}{\longrightarrow} \underset{\beta}{\sim}(\underline{a},\bar{a}))$$

where $\alpha,\beta \in \{i,e,r\}$.

Convention: if $\alpha = \beta$ then instead of ${}_{\alpha\beta}{\Rightarrow}$ we shall write ${}_{\alpha}{\Rightarrow}$.

Remark: In the sequel it would be interesting to examine the two argument operations ${}_{\alpha\beta\gamma\delta}{\Rightarrow}$ introduced in the following way:
for $\alpha,\beta,\gamma,\delta \in \{i,e,r\}$:

$$(\underline{a},\bar{a})_{\alpha\beta\gamma\delta}{\Rightarrow}(\underline{b},\bar{b}) = ((\underline{a},\bar{a}) \underset{\alpha}{\longrightarrow} (\underline{b},\bar{b})) \cap (\underset{\beta}{\sim}(\underline{b},\bar{b}) \underset{\gamma}{\longrightarrow} \underset{\delta}{\sim}(\underline{a},\bar{a}))$$

For each of the above implications we can introduce the following biconditionals:

$$(\underline{a},\bar{a}) \ {}_{\alpha\beta}{\Leftrightarrow}(\underline{b},\bar{b}) = ((\underline{a},\bar{a})_{\alpha\beta}{\Rightarrow}(\underline{b},\bar{b})) \cap ((\underline{b},\bar{b})_{\alpha\beta}{\Rightarrow}(\underline{a},\bar{a})).$$

In the lemmas below we shall use the above mentioned convention:

$a = (\underline{a},\bar{a})$.

Let us observe that in $\mathcal{R}$ there is a relative pseudo complement operation which we shall denote by ${}_{rp}{\Rightarrow}$.

Lemma 1.

For every $a, b \in \mathcal{R}$ it holds:

$$a \xrightarrow[i]{} b = (-\underline{a} \cup \underline{b},\ -\underline{a} \cup \bar{b})$$
$$a \xrightarrow[e]{} b = (-\bar{a} \cup \underline{b},\ -\bar{a} \cup \bar{b})$$
$$a \xrightarrow[r]{} b = (-\bar{a} \cup \underline{b},\ -\underline{a} \cup \bar{b})$$
$$a\ {}_{i}{\Rightarrow} b = (-\underline{a} \cup \underline{b},\ -\underline{a} \cup \underline{b})$$
$$a\ {}_{ie}{\Rightarrow} b = (-\bar{a} \cup \underline{b} \cup (-\underline{a}) \cap \bar{b},\ -\underline{a} \cup \bar{b})$$
$$a\ {}_{ei}{\Rightarrow} b = (-\bar{a} \cup \underline{b},\ -\bar{a} \cup \underline{b} \cup (-\underline{a}) \cap \bar{b})$$
$$a\ {}_{ri}{\Rightarrow} b = (-\bar{a} \cup \underline{b},\ -\underline{a} \cup \underline{b})$$
$$a\ {}_{rp}{\Rightarrow} b = (-\bar{a} \cup \underline{b} \cup (-\underline{a}) \cap \bar{b},\ -\bar{a} \cup \bar{b}) \qquad ///$$

Lemma 2.

$$a\ {}_{e}{\Rightarrow} b = a \xrightarrow[e]{} b = a \xrightarrow[e]{} b = a\ {}_{re}{\Rightarrow} b$$
$$a\ {}_{r}{\Rightarrow} b = a \xrightarrow[r]{} b$$
$$a\ {}_{ir}{\Rightarrow} b = a\ {}_{ie}{\Rightarrow} b \qquad ///$$

Let us recall that the algebra $(A, V, \Rightarrow)$ is called an implicative algebra if the following axioms are satisfied

(i_1) $a \Rightarrow a = V$

(i_2) $a \Rightarrow b = V$ i $b \Rightarrow c = V$ implies $a \Rightarrow c = V$

(i_3) $a \Rightarrow b = V$ i $b \Rightarrow a = V$ implies $a = b$

(i_4) $a \Rightarrow V = V$

Lemma 3.

In the table below we write 1 if the operation satisfies the corresponding axiom and 0 otherwise:

	$\xrightarrow[i]{}$	$\xrightarrow[e]{}$	$\xrightarrow[r]{}$	${}_{i}{\Rightarrow}$	${}_{ie}{\Rightarrow}$	${}_{ei}{\Rightarrow}$	${}_{ri}{\Rightarrow}$	${}_{rp}{\Rightarrow}$
(i_1)	1	0	0	1	1	0	0	1
(i_2)	1	1	1	1	1	1	1	1
(i_3)	0	1	1	0	1	1	1	1
(i_4)	1	1	1	1	1	1	1	1

///

Lemma 4.

If $1 = (U,U)$, and $\Rightarrow$ is one of the implications above then for all $a, b \in \mathcal{R}$, $a \Rightarrow b = 1$ and $a = 1$ implies $b = 1$.

Let us recall that an algebra $(A, 1, \cup, \cap, \sim)$ is **quasi-Boolean** if $(A, \cup, \cap)$ is a distributive lattice with 1 and $\sim$ satisfies the conditions:

(q1) $\sim\sim a = a$

(q2) $\sim(a \cup b) = \sim a \cap \sim b$

An algebra $\mathfrak{A} = (A, 1, \Rightarrow, \cup, \cap, ->, \sim, \neg)$ is **quasi-pseudo-Boolean** if the following hold

(qpB1) $(A, 1, \cup, \cap, \sim)$ is quasi-Boolean

(qpB2) a relation $<$ given by

$a < b$ iff $a -> b = 1$, is a quasi-order in A

(qpB3) $a \cap x < b$ iff $x < a -> b$

(qpB4) $a \Rightarrow b = (a -> b) \cap (\sim b -> \sim a)$

(qpB5) $a \Rightarrow b = 1$ iff $a \cap b = a$

(qpB6) $a < c$ i $b < c$ implies $a \cup b < c$

(qpB7) $a < b$ i $a < c$ implies $a < b \cap c$

(qpB8) $(a \cap \sim b) < \sim(a -> b)$

(qpB9) $\sim(a -> b) < (a \cap \sim b)$

(qpB10) $a < \sim \neg a$

(qpB11) $\sim \neg a < a$

(qpB12) $a \cap \sim a < b$

(qpB13) $\neg a = a -> \sim 1$

Theorem 1.

The algebras $(\mathcal{R}, 1, {}_{rp}\Rightarrow)$, $(\mathcal{R}, 1, {}_{ie}\Rightarrow)$ are implicative. $(\mathcal{R}, 1, {}_{rp}\Rightarrow)$ is a positive implicative algebra.

$(\mathcal{R}, 1, {}_{i}\Rightarrow)$ satisfies the axioms i_1, i_2, i_4 .

$(\mathcal{R}, \cup, \cap, {}_{rp}\Rightarrow)$ is a relatively pseudo-complemented lattice.

$(\mathcal{R}, 1, \cup, \cap, \sim_r)$ is a quasi - Boolean algebra.

$(\mathcal{R}, 1, \cup, \cap, {}_{rp}\Rightarrow, \sim_e)$ is a contrapositionally complemented lattice.

$(\mathcal{R}, 1, \cup, \cap, {}_{rp}\Rightarrow, \sim_i)$, $(\mathcal{R}, 1, {}_{rp}\Rightarrow, \cup, \cap, \sim_e)$,

$(\mathcal{R}, 1, \cup, \cap, {}_{rp}\Rightarrow, \sim_r)$ are semi complemented lattices.

$(\mathcal{R}, 1, {}_{rp}\Rightarrow, \cup, \cap, \sim_e)$ is a pseudo - Boolean algebra.

$(\mathcal{R}, 1, {}_{ie}\Rightarrow, \cup, \cap, \underset{i}{\longrightarrow}, \sim_r, \sim_i)$ is a quasi - pseudo - Boolean algebra.

$(\mathcal{R}, 1, {}_{rp}\Rightarrow, \cup, \cap, \underset{i}{\longrightarrow}, \sim_r, \sim_i)$ satisfies the axioms of the quasi-pseudo-

Boolean algebra, with the exception of the following condition:

$a \Rightarrow b = (a \longrightarrow b) \cap (\sim b \longrightarrow \sim a)$.

Proof

We will show for example that

$\mathfrak{R} = (\mathcal{R}, 1, {}_{lr}\Rightarrow, \cup, \cap, \underset{l}{-->}, \underset{r}{\sim}, \underset{l}{\sim})$

is the quasi-pseudo-Boolean algebra.

(qpB1): it is easy to check that $(R, \cup, \cap)$ is a distributive lattice with $1=(U,U)$ and that $\underset{r}{\sim}$ satisfies conditions q1 and q2.

(qpB2) : $a<b$ iff $a\underset{l}{-->}b=1$ iff $\underset{l}{\sim} a \cup b=1$ iff $\underset{l}{\sim}(\underline{a},\bar{a}) \cup (\underline{b},\bar{b})=1$ iff $(-\underline{a},-\underline{a}) \cup (\underline{b},\bar{b})=1$ iff $-\underline{a} \cup \underline{b}=U$ iff $\underline{a} \subseteq \underline{b}$. Therefore

(*) $a<b$ iff $\underline{a} \subseteq \underline{b}$, which implies that the relation $<$ reflexive and transitive.

(qpB3) : $a \cap x<b$ iff $\underline{a} \cap \underline{x} \subseteq \underline{b}$ iff $\underline{x} \subseteq -\underline{a} \cup \underline{b}$ iff $x<a\underset{l}{-->}b$

(qpB4) : it follows from the definition of ${}_{lr}\Rightarrow$

(qpB5) : $a \;{}_{lr}\Rightarrow b=1$ iff $(-\bar{a} \cup \underline{b} \cup (-\underline{a})\cap \bar{b}, -\underline{a} \cup \bar{b})=1$ iff $-\bar{a} \cup \underline{b} \cup (-\underline{a})\cap\bar{b}=U$ and $-\underline{a} \cup \bar{b}=U$.

We now prove the following

Lemma 1:

$-\bar{a} \cup \underline{b} \cup (-\underline{a} \cap \bar{b})=U$ iff $\underline{a} \subseteq \underline{b}$ and $\bar{a} \subseteq \bar{b}$

Proof:

We shall prove the implication to the right.

Assume that $-\bar{a} \cup \underline{b} \cup (-\underline{a} \cap \bar{b})=U$ and that it is not true that $\underline{a} \subseteq \underline{b}$. There exists $x \in U$ s.t. $x \in \underline{a}$ i $x \notin \underline{b}$. Then $x \notin -\bar{a}$, $x \notin \underline{b}$ i $x \notin -\underline{a} \cap \bar{b}$, contrary to the assumption. Now let us assume that $\underline{a} \subseteq \underline{b}$ and not $\bar{a} \subseteq \bar{b}$. Therefore there exists $x \in U$ such that

$x \in \bar{a}$ i $x \notin \bar{b}$. Then $x \notin -\bar{a}$, $x \notin \underline{b}$ and $x \notin -\underline{a} \cap \bar{b}$ and again this is contrary to $-\bar{a} \cup \underline{b} \cup (-\underline{a} \cap \bar{b})=U$, the proof of the lemma is completed.

In view of the lemma $\underline{a} \subseteq \underline{b}$ and $\bar{a} \subseteq \bar{b}$ i.e. $a \cap b = a$.

(qpB6): $a<c$ i $b<c$ iff $\underline{a} \subseteq \underline{c}$ i $\underline{b} \subseteq \underline{c}$ i.e. $\underline{a} \cup \underline{b} \subseteq \underline{c}$ iff $a \cup b < c$

Analogously by definition of $\cap$ and by (*) it holds (qpB7).

(qpB8) : $a \cap \sim_r b < \sim_r(a \dashrightarrow_i b)$ iff $a \cap \sim_r b \subseteq \sim_r (a \dashrightarrow_i b)$ iff

$\underline{a} \cap -\bar{b} \subseteq \sim_r(-\underline{a} \cup \underline{b}\ ,\ -\underline{a} \cup \bar{b})$ iff $\underline{a} \cap -\bar{b} \subseteq -(-\underline{a} \cup \bar{b})$ iff

$\underline{a} \cap -\bar{b} \subseteq \underline{a} \cap -\bar{b}$

(qpB9) : $\sim_r(a \dashrightarrow_i b) < (a \cap \sim_r b)$

$\sim_r(a \dashrightarrow_i b) = \sim_r(\sim_i a \cup b) = \sim_r(-\underline{a} \cup \underline{b}\ ,\ -\underline{a} \cup \bar{b}) = (\underline{a} \cap -\bar{b}\ ,\underline{a} \cap -\underline{b})$

hence $\sim_r(a \dashrightarrow_i b) = \underline{a} \cap -\bar{b} = (a \cap \sim_r b)$. Hence thesis by (*).

(qpB10) : $a < \sim_r \sim_i a$ iff $\underline{a} \subseteq \sim_r \sim_i a$ iff $\underline{a} \subseteq \sim_r(-\underline{a},-\underline{a})$ iff $\underline{a} \subseteq (\underline{a},\underline{a})$ iff $\underline{a} \subseteq \underline{a}$

(qpB11) : the proof similar to the above.

(qpB12) : $a \cap \sim_r a < b$ iff $(\underline{a},\bar{a}) \cap (-\bar{a},-\underline{a}) < b$ iff $(\emptyset,\bar{a}-\underline{a}) <(\underline{b},\bar{b})$ iff $\emptyset \subseteq \underline{b}$

(qpB13) : $\sim_i a = a \dashrightarrow_i \sim_r 1$

$\sim_i a = \sim_i (\underline{a},\bar{a}) = (-\underline{a},-\underline{a})$

$\sim_r 1 = 0$

$a \dashrightarrow_i 0 = (-\underline{a},-\underline{a}) \cup (\emptyset,\emptyset) = (-\underline{a},-\underline{a})$///

2. Selected algebras of rough sets. Representation theorems.

We recall that the family of rough sets determined by the relation R in U is denoted by $\mathcal{R} = \{(\underline{a},\bar{a}) : a \subseteq U\}$. According to this notation the family of one element equivalence classes of R will be denoted as follows [9] :

(1) $Ati = \{[x]_R : \text{card } [x]_R = 1\}$

The family of equivalence classes having cardinality greater then one is denoted by

(2) $Atb = \{[x]_R : \text{card } [x]_R > 1\}$.

The elements of the sets Ati and Atb are called individual and boundary atoms respectively. Finally by Atn we denote the set of all n-element atoms:

(3) $\mathrm{Atn} = \{[x]_R : \mathrm{card}\ [x]_R = n\}$

The set of all atoms At is the union of Ati and Atb.

By Def($\mathcal{R}$) we shall denote the family of all definable rough sets, ie.

(4) $\mathrm{Def}(\mathcal{R}) = \{(\underline{a},\bar{a}) \in \mathcal{R} : \underline{a} = \bar{a}\}$.

The definable sets are also called the exact sets (see Pawlak [30] and Iwiński [9]). The family of all externally non-definable sets shall be denoted by End($\mathcal{R}$):

(5) $\mathrm{Znd}(\mathcal{R}) = \{(\underline{a},\bar{a}) : \bar{a} = U\}$.

By $\mathfrak{B}_0, \mathfrak{C}_0, \mathfrak{U}_0$ we shall denote the following algebras (Rasiowa [36])

(6) $\mathfrak{B}_0 = (B_0, \mathbf{1}, \cup, \cap, \sim)$, where $B_0 = \{\mathbf{0}, \mathbf{1}\}$,

(7) $\mathfrak{C}_0 = (C_0, \mathbf{1}, \cup, \cap, \sim)$, where $C_0 = \{\mathbf{0}, 1, \mathbf{1}\}$,

These are subalgebras of the algebra

(8) $\mathfrak{U}_0 = (A_0, \mathbf{1}, \cup, \cap, \sim)$, where $A_0 = \{\mathbf{0}, 1, r, \mathbf{1}\}$

and the operations $\cup, \cap, \sim$ are defined in the following way:

$\mathbf{0} \cup x = x \cup \mathbf{0} = x$, $\mathbf{1} \cup x = x \cup \mathbf{1} = \mathbf{1}$ for every $x \in A_0$,

$1 \cup 1 = 1$, $r \cup r = r$, $1 \cup r = \mathbf{1}$,

(9) $\mathbf{0} \cap x = x \cap \mathbf{0} = 0$, $\mathbf{1} \cap x = x \cap \mathbf{1} = x$, for each $x \in A_0$,

$1 \cap 1 = 1$, $r \cap r = r$, $1 \cap r = \mathbf{0}$

$\sim\mathbf{0} = \mathbf{1}$, $\sim 1 = 1$, $\sim r = r$, $\sim\mathbf{1} = \mathbf{0}$.

The algebras $\mathfrak{U}_0, \mathfrak{B}_0, \mathfrak{C}_0$ are quasi - Boolean.

The Quasi Boolean algebra $\mathbb{R} = (\mathcal{R}, \mathbf{1}, \cup, \cap, \sim_r)$

It is well-known that every quasi Boolean algebra is isomorphic with a subalgebra of the product $\prod_{t \in T} \mathfrak{U}_t$, where T is a set of indices and $\mathfrak{U}_t = \mathfrak{U}_0$, for all t. The following seems to be a natural question: is the algebra $\mathfrak{U}_0$ isomorphic to some algebra of rough sets $\mathcal{R}$?

The answer is in the negative, since in $(\mathcal{R}, \cup, \cap, \mathbf{1}, \sim_r)$ for every $a \in \mathcal{R}$ does not hold $a = \sim a$, and in $\mathfrak{U}_0$ it holds $\sim 1 = 1$.

So the class of all quasi-Boolean algebras of rough sets is narrower then the class of all quasi Boolean algebras. We now prove the representation theorem for this class.

Theorem 2.

Every quasi-Boolean algebra of rough sets $(\mathcal{R}, 1, \cup, \cap, \underset{r}{\sim})$ is isomorphic with a product $\prod_{t\in T} \mathfrak{U}_t$, T is a set of indices and $\mathfrak{U}_t = \mathfrak{B}_0$ or $\mathfrak{U}_t = \mathfrak{C}_0$, for each $t \in T$.

Proof:

Let T be an indexing set of the family of all equivalence classes of the relation R, for instance T = At. If the atom $i \in T$ is an individual atom, then we put $\mathfrak{U}_i = \mathfrak{B}_0$. On the other hand if $b \in T$ is a boundary atom then we take $\mathfrak{U}_b = \mathfrak{C}_0$. Let us take $\prod_{t\in T} \mathfrak{U}_t$.

We shall prove that the algebra $(\mathcal{R}, 1, \cup, \cap, \underset{r}{\sim})$ is isomorphic to the algebra $\prod_{t\in T} \mathfrak{U}_t$.

Let us assume that $(\underline{a}, \bar{a}) \in \mathcal{R}$. We define the isomorphism $f: \mathbb{R} \longrightarrow \prod_{t\in T} \mathfrak{U}_t$ as follows

(10) $f((\underline{a}, \bar{a})) = (x_t)_{t\in T} \in \prod_{t\in T} U_t$ iff

the following conditions are satisfied:

1^0 if $t \in Ati$ and $t \le \underline{a}$ then $x_t = 1$

2^0 if $t \in Ati$ and $\neg(t \le \underline{a})$ then $x_t = 0$

3^0 if $t \in Atb$ and $\neg(t \le \underline{a})$ and $\neg(t \le \bar{a})$ then $x_t = 0$

4^0 if $t \in Atb$ and $\neg(t \le \underline{a})$ and $t \le \bar{a}$ then $x_t = 1$

5^0 if $t \in Atb$ and $t \le \underline{a}$ then $x_t = 1$

It is straightforward to check that f is 1-1 and onto. It is also easy to check that for arbitrary $a = (\underline{a}, \bar{a})$, $b = (\underline{b}, \bar{b})$, it holds $f(a \cup b) = f(a) \cup f(b)$, and $f(a \cap b) = f(a) \cap f(b)$.

We shall see that

(11) $f(\underset{r}{\sim} a) = \sim f(a)$

We have

$f(\underset{r}{\sim}(\underline{a}, \bar{a})) = f((-\bar{a}, -\underline{a})) = (x_t)_{t\in T}$

On the other hand, if

$f(a) = f(\underline{a}, \bar{a}) = (y_t)_{t\in T}$ then

$\sim f(a) = (\sim y_t)_{t\in T} = (z_t)_{t\in T}$

1^0 If $t \in Ati$ and $t \le \underline{a}$ then $\neg(t \le -\bar{a})$, therefore $x_t = 0$.

On the other hand $y_t = 1$ so $\sim y_t = 0$ i.e. $z_t = 0$. In consequence $x_t = z_t$.

2^0 If $t \in$ Ati and $\neg(t \le \underline{a})$ then $t \le -\bar{a}$ hence $x_t = 1$. On the other hand $y_t = 0$ so $z_t = 1$ i.e. $x_t = z_t$.

3^0 If $t \in$ Atb and $\neg(t \le \underline{a})$ and $\neg(t \le \bar{a})$ then $t \le -\bar{a}$, in consequence $x_t = 1$. On the other hand $y_t = 0$ so $z_t = \sim y_t = 1$. Therefore $x_t = z_t$.

Analogously we can check the equality $x_t = z_t$ in case 4^0 and 5^0.

Corollary 1.

Every lattice $(\mathcal{R}, \cup, \cap)$ is isomorphic with a product of a two and three element chains.

Corollary 2.

Every algebra $\mathbb{R}$ quasi - Boolean of rough sets is isomorphic to the product of some algebras $\mathfrak{U} = (\{0,1\},+,\cdot,1,\sim)$ $\mathfrak{U}' = (\{0,1/2,1\},+,\cdot,1,\sim)$ where

$x_i + x_j = \max(x_i, x_j)$,
$x_i \cdot x_j = \min(x_i, x_j)$, **for** $x_i, x_j \in \{0, 1/2, 1\}$,
$\sim 0 = 1$, $\sim 1 = 0$.

Second representation theorem

It is known that every quasi-Boolean algebra is isomorphic to a quasi field of sets ([26]). In view of this representation and changing slightly the definition of the family of rough sets we obtain the second representation theorem.

Let us assume that

(12) $X = X_i \cup X_l \cup X_r$, where sets X_i, X_l, X_r are disjoint, X_i - is called a set of individual atoms and $X_b = X_l \cup X_r$ is called borderline atoms set.

Let us assume that $\sim^0$, $\sim^0 : X_b \xrightarrow[\text{on}]{1-1} X_b$,

$\sim^0 : X_r \xrightarrow[\text{on}]{1-1} X_l$

and

$\sim^0 : X_l \xrightarrow[\text{on}]{1-1} X_r$, $\sim^0(x_l) = x_r$, $\sim^0(x_r) = x_l$.

A set $Y \subseteq X$ will be represented as a triple (Y_i, Y_l, Y_r), where $Y_i = X_i \cap Y$ $Y_l = X_l \cap Y$, $Y_r = X_r \cap Y$.

We define $\cup$ and $\cap$ componentwise. The lower and upper approximations of a set Y may be defined as follows

(13) $\underline{Y} = Y_1 \cup \{y: y \in Y \text{ and } \sim^0(y) \in Y\}$

$\overline{Y} = Y_1 \cup \{y: y \in Y \text{ or } \sim^0(y) \in Y\}$

Denoting by Y_1^* the image of Y_1 i.e.
$Y_1^* = \sim(Y_1) = \{y_r: y_r = \sim^0(y_1) \text{ and } y_1 \in Y_1\}$
and
$Y_r^* = \sim(Y_r)$,
we have
$\underline{Y} = Y_1 \cup Y_1 \cap Y_r^* \cup Y_r \cap Y_1^*$

$\overline{Y} = Y_1 \cup Y_1 \cup Y_r \cup Y_1^* \cup Y_r^*$.

There are two possibilities to define $\sim$. If $\sim$ corresponds to $\sim_r$ i.e.
$\sim_r(Y_1, Y_1, Y_r) = (-Y_1, -(Y_1 \cup Y_r^*), -(Y_r \cup Y_1^*))$
then the algebra $\mathbb{R}^3 = (\mathcal{R}^3, 1, \cup, \cap, \sim_r)$, where $\mathcal{R}^3$ denotes the family of the triples (Y_1, Y_1, Y_r) for $Y \subseteq X$, and $\mathbf{1} = (X_1, X_1, X_r)$, moreover the complement is defined componentwise with respect to X_1, X_1, X_r, respectively, is isomorphic with a quasi Boolean algebra of rough sets.
Second possibility is the following:
$\sim$ for subsets of X corresponds to $\sim^1$ given by an inwolution g of X, $\sim^1 Y = X - g(Y)$, for $Y \subseteq X$. Then $\sim$ is defined as follows

(15) $\sim(Y_1, Y_1, Y_r) = (-Y_1, -Y_r^*, -Y_1^*)$.

We prove now that $\mathbb{R}^3 = (\mathcal{R}^3, \mathbf{1}, \cup, \cap, \sim)$ is a representative example of quasi - Boolean algebras, which means that every quasi-Boolean algebra is isomorphic to an algebra $\mathbb{R}^3$. Let us denote: $\mathbf{0} = (\emptyset, \emptyset, \emptyset)$.

Lemma 5.
An algebra $\mathbb{R}^3 = (\mathcal{R}^3, \cup, \cap, \mathbf{0}, \mathbf{1}, \sim)$ is isomorphic to a quasi-field $(Q(X), \cup, \cap, \emptyset, X, \sim)$, where $\sim Y = X - g(Y)$ for an involution $g: X \longrightarrow X$.
Proof:
Let us define the inwolution $g: X \longrightarrow X$ as follows : $g(x_1) = x_1$ for

$x_l \in X_l$, $g(x_l) = \sim^0(x_l)$ i.e. $g(x_l) = x_r$, for $x_l \in X_l$,
$g(x_r) = \sim^0(x_r) = x_l$, if $x_r \in X_r$.
Then $f: \mathcal{R}^3 \longrightarrow Q(X)$ given by $f(Y_i, Y_l, Y_r) = Y_i \cup Y_l \cup Y_r$,
is the isomorphism.

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Lemma 6.
Every quasi - field $(Q(X), \cup, \cap, \emptyset, X, \sim)$ defined by inwolution $g: X \longrightarrow X$, is isomorphic to a subalgebra of $\mathbb{R}^3 = (\mathcal{R}^3, \cup, \cap, \mathbf{0}, \mathbf{1}, \sim)$.
Proof:
First we define a partition of X by X_i and X_b, namely
$X_i = \{x \in X: g(x) = x\}$
$X_b = \{x \in X: g(x) \neq x\}$
Since $g(g(x)) = x$, therefore we can divide X_b as an union $X_l \cup X_r$ such that $g(X_l) = X_r$ and $g(X_r) = X_l$, $X_l \cap X_r = \emptyset$. We put also $\sim^0(x) = g(x)$, for every $x \in X$. The operations in $\mathbb{R}^3$ are defined as before. The function f s.t. for $Y \in Q(X)$
$f(Y) = (Y \cap X_i, Y \cap X_l, Y \cap X_r)$
is the isomorphism needed.

Theorem 3.
Every quasi - Boolean algebra $\mathfrak{U} = (A, \mathbf{1}, \cup, \cap, \sim)$ is isomorphic to a subalgebra of $\mathbb{R}^3 = (\mathcal{R}^3, \mathbf{1}, \cup, \cap, \sim)$.
Proof:
By Lemma 1, Lemma 2, and the reprezentation theorem for quasi - Boolean algebras.

CHAPTER 5
INDISCERNIBILITY AND SIMILARITY

The point of departure for this chapter is the standard connection between relations and operations of taking the relational image of sets. We shall give abstract c h aracterizations of similarity and indiscernibility operations, derived from information systems.

1. Basic definitions.

Let us recall definitions of indiscernibility, similarity and informational inclusion -relations, which will play the main role in the sequel. Let $x, y \in U$ and $P \subseteq A$, we define:

Ind(P)xy iff $\forall a \in P\ f(x,a) = f(y,a)$

ind(P)xy iff $\exists a \in P\ f(x,a) = f(y,a)$

Sim(P)xy iff $\forall a \in P\ f(x,a) \cap f(y,a) \neq \emptyset$

sim(P)xy iff $\exists a \in P\ f(x,a) \cap f(y,a) \neq \emptyset$

Con(P)xy iff $\forall a \in P\ f(x,a) \subset f(y,a)$

con(P)xy iff $\exists a \in P\ f(x,a) \subset f(y,a)$

Inc(P)xy iff $\forall a \in P\ f(x,a) \subseteq f(y,a)$

inc(P)xy iff $\exists a \in P\ f(x,a) \subseteq f(y,a)$

The relations are called respectively: indiscernibility, weak indiscernibility, similarity, weak similarity, contains, weak contains, inclusion and weak inclusion.

For parametrized relations we introduce the following convenient notation: if $P \subseteq A$ and for every $a \in P$ $R(a) \subseteq U^n$ then

$R(P^\wedge)x_1 \ldots x_n$ iff $\forall\ a \in P\ R(a)x_1 \ldots x_n$

$R(P^\vee)x_1 \ldots x_n$ iff $\exists\ a \in P\ R(a)x_1 \ldots x_n$

$R(P^m)x_1 \ldots x_n$ iff $\exists\ p_1 \ldots p_m \in P$ $R(p_1)x_1 \ldots x_n \wedge \ldots \wedge R(p_m)x_1 \ldots x_n$

$R(P^{!m})x_1 \ldots x_n$ iff $\exists !\, p_1 \ldots p_m \in P$ $\forall i < m+1\ R(p_i)x_1 \ldots x_n$

Convention: If $P = \emptyset$ then $R(P^\wedge) = U^2$ and $R(P^\vee) = \emptyset$

If in a Boolean algebra **B** there exist $\inf\{R^a y : a \in P\}$ and $\sup\{R^a y : a \in P\}$ then we write $R^P y = \inf\{R^a y : a \in P\}$ and $R_P y = \sup\{R^a y : a \in P\}$.

Lemma 1:

Assume that **B** is a complete Boolean algebra and for all $x, y \in At\mathbf{B}$ and every $a \in P$ it holds: $xR(a)y$ iff $x \le R^P y$. Then :

a) $xR(P^\wedge)y$ iff $x \le R^P\ y$

b) $xR(P^\vee)y$ iff $x \le R_P\ y$.

Now, for R(a)=Ind(P), Sim(P), Con(a), Inc(a) we have relational systems $(\mathbf{U}, \{R(P^\wedge) : P \subseteq A\})$ and $(\mathbf{U}, \{R(P^\vee) : P \subseteq A\})$, respectively. With each of these systems we relate corresponding algebra in the usual way, considering the image operation R^* for every relation R. Our next aim is to give abstract characterizations of the algebras described above.

2. Indiscernibility algebra

The following class of algebras was introduced by S.D.Comer [3].
A Boolean algebra with operators $(B, \{I^P: P\subseteq A\})$ satisfying the conditions:

1) B is complete atomic Boolean algebra
2) $I^P 0=0$
3) $I\ x\geq x$
4) $I^P(x\ I^P y)=I^P x\quad I^P y$
5) $x\neq 0$ implies $I^{\emptyset}x=1$
6) $I^{P\cup Q}\ x = I^P x\ I^Q x$, for $x\in AtB$

is called indiscernibility algebra of the A-type.
It is reduced if $I^A x = x$ for all $x\in B$.

Theorem 1:
If B is a reduced indiscernibility algebra of A-type with A-finite, then there exists information system which indiscernibility algebra is isomorphic to B.
Proof: see Comer [3].

3. Weak indiscernibility algebras

We introduce a class of algebras corresponding to weak indiscernibility relations.
A Boolean algebra with operators $(B, \{I^P: P\subseteq A\})$ satisfying the conditions:

1) B is complete atomic Boolean algebra
2) $I_P 0=0$
3) $I_P x\geq x$
4) $I_P(x\cdot I_P y)=I_P x\cdot I_P y$
5) $x\neq 0$ implies $I_{\emptyset}x=0$
6) $I_{P\cup Q}\ x = I_P x+I_Q x$, for $x\in AtB$

will be called weak indiscernibility algebra of the A-type.

Theorem 2:
If B is a reduced weak indiscernibility algebra of A-type with A-finite, then there exists information system in which derived weak indiscernibility algebra is isomorphic to B.
Proof:

We define the information system as follows:

$U = AtB$, $V = P(U)$, $f(a,x) = h(I_a(x))$, where $hx = \{y \in AtB : y \le x\}$.

The details are presented in Pomykała [36].

4. Similarity algebras

We define (abstract) similarity algebra of A-type as a Boolean algebra with operators $(B, \{S^P : P \subseteq A\})$ satisfying the conditions:

1) $B = (B, +, \cdot, -, 0, 1)$ is a complete atomic Boolean algebra,
2) $\forall P \subseteq A\ \forall x,y \in AtB\ (y \le S^P x \text{ iff } x \le S^P y)$
3) $\forall P,Q \subseteq A\ \forall x \in AtB\ S^{P \cup Q} x = S^P x \cdot S^P x$
4) $x \ne 0$ implies $S^{\emptyset} x = 1$.

Lemma 2:

Condition 2) is equivalent to the following:

$\forall P \subseteq A\ \forall x,y \in AtB\ (y \cdot S^P x = 0 \text{ iff } S^P y \cdot x = 0.$

Theorem 3:

The algebra derived from similarity relational system $(U, Sim(P) : P \subseteq A)$ in an information system satisfies conditions 1)-4).

If **B** is a similarity algebra of A-type with A-finite, then there exists information system in which similarity algebra is isomorphic to **B**.

Proof:

Given **B** we define: $U = AtB$, A is a type of **B**, $f(x,a) = \{\{x,y\} : x \le S^a y\}$.

The details are presented in [36].

5. Weak similarity algebras

We define **weak** similarity algebra of A-type as a Boolean algebra with operators $(B, \{S_P : P \subseteq A\})$ satisfying the conditions:

1) $B = (B, +, \cdot, -, 0, 1)$ is a complete atomic Boolean algebra,
2) $\forall P \subseteq A\ \forall x,y \in AtB\ (y \le S_P x \text{ iff } x \le S_P y)$
3) $\forall P,Q \subseteq A\ \forall x \in AtB\ S_{P \cup Q} x = S_P x + S_Q x$
4) $x \ne 0$ implies $S_{\emptyset} x = 0$.

Theorem 4:

The algebra derived from weak similarity system $(U, sim(P): P \leq A)$ in an information system satisfies conditions 1)-4).

If B is a similarity algebra of A-type with A-finite, then there exists information system with weak similarity algebra isomorphic to B.

Proof:

Given B we define: U=AtB, A is a type of B, $f(x,a) = \{\{x,y\}: x \leq S_a y\}$.

Final remarks

In the part 2 of the paper we relate our considerations to selected papers from Amsterdam School of Logic. We shall express our gratitude for many people from Warsaw and Amsterdam at the end of paper. The second part will be titled: Logical Systems. It will contain four chapters:

Ch.1. Modal approximation logic

Ch.2. Many sorted logic of information systems

Ch.3. Logic of rough constructions

Ch.4. Cover space.

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