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Minimal Temporal Epistemic Logic[1]

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Abstract

In the study of nonmonotonic reasoning the main emphasis has been on static (declarative) aspects. Only recently there has been interest in the dynamic aspects of reasoning processes, particularly in artificial intelligence. We study the dynamics of reasoning processes by viewing them as a special class of processes, and by using temporal logic to specify reasoning processes and to reason about their properties, just as is common in theoretical computer science. In earlier work we have introduced a temporal epistemic logic and used it to specify a number of nonmonotonic reasoning processes. In the present paper we study this temporal formalism in more detail. It is composed of a base temporal epistemic logic with a preference relation on models, and an associated nonmonotonic inference relation in the style of Shoham, to account for the nonmonotonicity. We present an axiomatic proof system for the base logic and study decidability and complexity for both the base logic and the nonmonotonic inference relation based on it. Then we look at an interesting class of formulae, prove a representation result for it, and provide a link with the rule of Monotonicity.

Minimal Temporal Epistemic Logic

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1 Introduction

In theoretical computer science temporal logic has been widely recognized as a valuable tool for specifying processes and reasoning about their properties. In Artificial Intelligence this view is not very common, partly because (nonmonotonic) logic is usually thought of as a purely declarative notion. However, in nonmonotonic reasoning dynamic aspects of reasoning processes can be interesting to study, just as is common in computer science: there we often have declarative semantics next to procedural semantics of processes. One of the differences between the notion of process in computer science and in artificial intelligence lies in the nature of a state: in a computer it is composed of the values of the variables, in a reasoning process it consists of the facts which are believed at that time, so a state is an epistemic one.

A number of examples in which a temporal logic is used to specify reasoning processes can be found in [ET94], where such specifications are introduced for default logic (see [Re80]), classical inference systems and meta level architectures. Also, in [ET95] it is shown that there exists a large class of reasoning processes that can be specified in this temporal logic. Therefore it seems justified to study this temporal logic formalism in more detail, which will be done in the present paper.

In section 2 we will introduce the temporal logic which is the basis of the framework and in section 3 an extra restriction is imposed upon this logic. Section 4

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describes the notions of minimal models and minimal entailment, which will be studied in the rest of the paper. In section 5 decidability of this notion will be established and section 6 gives complexity results for both the base logic and minimal entailment. In section 7 we will look at a special class of formulae and prove a link with the rule of monotonicity. Section 8 gives conclusions and suggestions for further research.

2 Temporal Epistemic Logic

When designing a logic capable of describing the behaviour of reasoning processes over time, two important decisions have to be made: which temporal ontology is suited best for the purpose, and what is a state of a reasoning process. We view a reasoning process, performed by an agent for instance, as a stepwise process: the agent starts out with some initial facts (possibly none) and attempts to derive consequences by applying rules; a state where the agent has more knowledge results. It will then again try to derive new facts resulting in a next state, etcetera, possibly ad infinitum. This suggests a temporal ontology which is discrete and has a starting point. In theoretical computer science there has been much debate whether time should be linear or branching (towards the future) (see [BRR89]). The most important differences between these two approaches are that linear time logic has in general a lower complexity but also less expressivity than branching time logics . Although some results in [ET94] on specifying proof systems in temporal logic seem to suggest that sometimes the higher expressivity of branching time logic is needed, we will confine ourselves here to using linear time.

As suggested above, the important thing about the state of a reasoning agent at a particular moment, is the knowledge it has derived. Kripke semantics can be used to formalize such an *information state*. We will take propositional logic as the basic language in which the agent can describe its knowledge. A modal operator K will be used to denote the agent's knowledge. In principle the agent may perform (positive and negative) introspection which suggests an S5 logic to describe knowledge.

Definition 2.1 (Language of knowledge)
    Let P be a (finite or countably infinite) set of propositional atoms. The language $\mathcal{L}_{S5}$ is the smallest set closed under:
- if $p \in P$ then $p \in \mathcal{L}_{S5}$.
- if $\varphi, \psi \in \mathcal{L}_{S5}$ then $K\varphi, \varphi \wedge \psi, \neg \varphi \in \mathcal{L}_{S5}$

Furthermore, we introduce the following abbreviations:

$\varphi \vee \psi \equiv \neg(\neg \varphi \wedge \neg \psi), \varphi \rightarrow \psi \equiv \neg \varphi \vee \psi, M\varphi \equiv \neg K \neg \varphi, \top \equiv p \wedge \neg p, \bot \equiv \neg \top$

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If every atom occurring in a formula φ is in the scope of a **K** operator, we call φ *subjective*.

An example of a subjective formula is $\neg \mathbf{Kp} \wedge \mathbf{K(q \rightarrow p)}$, whereas $\mathbf{K(p \wedge q)} \vee \mathbf{s}$ is not subjective. In the rest of this paper we will be especially interested in subjective formulae since they describe (only) the knowledge and ignorance of the agent. As we want to talk about properties of the knowledge of the agent changing over time, this language will be temporalized below.

In the usual S5 semantics a model is a triple $(\mathbf{W}, \mathbf{R}, \pi)$ where $\mathbf{W}$ is a set of worlds, $\mathbf{R}$ is an equivalence relation on $\mathbf{W}$ and π is a function that assigns a propositional valuation to each world in $\mathbf{W}$. We may however (see [MH92]), in the case of one agent, restrict ourselves to *normal* S5-models, in which the relation is universal (each world is accessible from every other world), and worlds are identified with propositional valuations.

Definition 2.2 (S5 semantics)

A propositional valuation of signature $\mathbf{P}$ is a function from $\mathbf{P}$ into $\{0,1\}$ where 0 stands for false and 1 for true. The set of such valuations will be denoted $\mathbf{Mod(P)}$. A normal S5-model $\mathbf{M}$ is a non-empty set of valuations. The truth of an S5-formula φ in such a model, evaluated in a world $\mathbf{m} \in \mathbf{M}$, denoted $(\mathbf{M}, \mathbf{m}) \vDash_{S5} \varphi$, is defined inductively:

1. $(\mathbf{M}, \mathbf{m}) \vDash_{S5} \mathbf{p}$ $\Leftrightarrow$ $\mathbf{m(p)} = 1$, for $\mathbf{p} \in \mathbf{P}$
2. $(\mathbf{M}, \mathbf{m}) \vDash_{S5} \varphi \wedge \psi$ $\Leftrightarrow$ $(\mathbf{M}, \mathbf{m}) \vDash_{S5} \varphi$ and $(\mathbf{M}, \mathbf{m}) \vDash_{S5} \psi$
3. $(\mathbf{M}, \mathbf{m}) \vDash_{S5} \neg\varphi$ $\Leftrightarrow$ it is not the case that $(\mathbf{M}, \mathbf{m}) \vDash_{S5} \varphi$
4. $(\mathbf{M}, \mathbf{m}) \vDash_{S5} \mathbf{K}\varphi$ $\Leftrightarrow$ $(\mathbf{M}, \mathbf{m'}) \vDash_{S5} \varphi$ for every $\mathbf{m'} \in \mathbf{M}$

A pair $(\mathbf{M}, \mathbf{n})$ where $\mathbf{M}$ is a normal S5-model and $\mathbf{n} \in \mathbf{M}$ is called an *information state* and the set of such pairs is denoted by $\mathbf{IS}$.

It is easy to see that the truth of a subjective S5-formula in a model is independent of the world in which it is evaluated, so if we restrict ourselves to subjective formulae, the world $\mathbf{m}$ in which it is evaluated is often left out.

Remark

Note that an S5-formula is subjective if and only if it is equivalent to a formula of the form $\mathbf{K}\varphi$ with $\varphi \in \mathcal{L}_{S5}$.

Proof

If φ is subjective, it is equivalent to $K\varphi$. Of course a formula of the form $K\varphi$ is subjective.

Axiomatizations for S5 are known from the literature (e.g. [HM85]):

Definition 2.3 (Axiom system for S5)

The axiom system of **S5** consists of:

1. All instances of propositional tautologies
2. $K(\varphi \to \psi) \to (K\varphi \to K\psi)$ *(K)*
3. $K\varphi \to \varphi$
4. $K\varphi \to KK\varphi$ *(Positive Introspection)*
5. $\neg K\varphi \to K\neg K\varphi$ *(Negative Introspection)*

and the following two rules:

1. $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$ *(Modus Ponens)*

2. $\dfrac{\varphi}{K\varphi}$ *(Necessitation)*

If there is a proof for φ using this system, we will denote this by $\vdash_{S5} \varphi$.

It is well-known that this system is sound and complete with respect to the class of normal S5 models.

In order to describe past and future we will introduce temporal operators **P, H, F, G** and $\square$, standing for "sometimes in the past", "always in the past", "sometimes in the future", "always in the future" and "always" respectively. Note that we do not want to talk about the agent's knowledge of the future and past, but about the future and past of the agent's knowledge. Therefore temporal operators may never be within the scope of the epistemic **K** operator.

Definition 2.4 (Temporal epistemic language)

The language $\mathcal{L}_{TEL}$ is the smallest set closed under

- if $\varphi \in \mathcal{L}_{S5}$ then $\varphi \in \mathcal{L}_{TEL}$
- if $\alpha, \beta \in \mathcal{L}_{TEL}$ then $\alpha \wedge \beta, \neg\alpha, P\alpha, F\alpha \in \mathcal{L}_{TEL}$

Again the abbreviations for $\vee$, $\to$, $\top$ and $\bot$ are introduced, as well as:
$G\alpha \equiv \neg F(\neg\alpha)$, $H\alpha \equiv \neg P(\neg\alpha)$ and $\square\alpha \equiv H\alpha \wedge \alpha \wedge G\alpha$.

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If in the first clause we restrict ourselves to subjective formulae, we get the set of *subjective TEL formulae.*

In the rest of this paper we will be interested in subjective TEL formulae since they describe how the knowledge of the agent is changing over time. Based on the set of natural numbers as flow of time and the notion of information state as formalization of a state, the following semantics is introduced for temporal epistemic logic (TEL):

Definition 2.5 (Semantics of TEL)

A TEL-model is a function $\mathfrak{M}: \mathbb{N} \to \mathbf{IS}$. The truth of a formula $\varphi \in \mathcal{L}_{\mathbf{TEL}}$ in $\mathfrak{M}$ at time point $t \in \mathbb{N}$, denoted $(\mathfrak{M}, t) \vDash \varphi$, is defined inductively as follows:

1. $(\mathfrak{M}, t) \vDash \varphi$ $\Leftrightarrow$ $\mathfrak{M}(t) \vDash_{S5} \varphi$, if $\varphi \in \mathcal{L}_{S5}$
2. $(\mathfrak{M}, t) \vDash \varphi \wedge \psi$ $\Leftrightarrow$ $(\mathfrak{M}, t) \vDash \varphi$ and $(\mathfrak{M}, t) \vDash \psi$
3. $(\mathfrak{M}, t) \vDash \neg\varphi$ $\Leftrightarrow$ it is not the case that $(\mathfrak{M}, t) \vDash \varphi$
4. $(\mathfrak{M}, t) \vDash P\varphi$ $\Leftrightarrow$ $\exists s \in \mathbb{N}$ such that $s < t$ and $(\mathfrak{M}, s) \vDash \varphi$
5. $(\mathfrak{M}, t) \vDash F\varphi$ $\Leftrightarrow$ $\exists s \in \mathbb{N}$ such that $t < s$ and $(\mathfrak{M}, s) \vDash \varphi$

A formula φ is true in a model $\mathfrak{M}$, denoted $\mathfrak{M} \vDash \varphi$, if for all $t \in \mathbb{N}$, $(\mathfrak{M}, t) \vDash \varphi$. If φ is true in all models we write $\vDash \varphi$ (φ is *valid*), and we write $\psi \vDash \varphi$ (φ is a *semantical consequence* of ψ) if for all models $\mathfrak{M}$ and $t \in \mathbb{N}$, $(\mathfrak{M}, t) \vDash \psi$ implies $(\mathfrak{M}, t) \vDash \varphi$

For future use we give the following definition (here O^i stands for a sequence of O operators of length i):

Definition 2.6

For $i \in \mathbb{N}$ define $at_i := P^i(\top) \wedge H^{i+1}(\bot)$.

It is easy to see that $(\mathfrak{M}, j) \vDash at_i$ if and only if $i = j$.

We would like to find an axiom system for TEL. The idea is to use the axioms of an S5-system together with axioms for tense logic over the natural numbers. Instead of proving soundness and completeness for the resulting system from scratch, we will use results from [FG92] where a general method for temporalizing a given logic system is presented. In their notation, TEL would be T(S5). We cannot directly apply their results since they use the temporal operators Since and Until, but adaptation to our situation is easy. Our class of flows of time contains only the set of natural numbers. First we will

give an axiomatic system for propositional tense logic over the natural numbers (from [Go92]):

Definition 2.7 (Tense logic over the natural numbers)

The axiom system for tense logic over $\mathbb{N}$ consists of:

1.   All instances of propositional tautologies
2. $G(\varphi \to \psi) \to (G\varphi \to G\psi)$
3. $H(\varphi \to \psi) \to (H\varphi \to H\psi)$
4. $\varphi \to HF\varphi$ (C_P)
5. $\varphi \to GP\varphi$ (C_F)
6. $H\varphi \to HH\varphi$ (4_P)
7. $G\varphi \to GG\varphi$ (4_F)
8    $F(T)$ (D_F)
9. $G(G\varphi \to \varphi) \to (FG\varphi \to G\varphi)$ (Z_F)
10. $H(H\varphi \to \varphi) \to H\varphi$ (W_P)

and the following rules:

1. $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$ *(Modus Ponens)*

2. $\dfrac{\varphi}{G\varphi} \quad \dfrac{\varphi}{H\varphi}$ *(Necessitation)*

Using the axiom systems for S5 and tense logic, definition 2.6 of [FG92] allows us to give an axiomatization for TEL:

Definition 2.8 (Axiomatization for TEL)

The axiom system of **TEL** consists of:

1. The axioms 1-10 of definition 2.7
2. The inference rules 1 and 2 of definition 2.7
3. For every formula $\alpha \in \mathcal{L}_{S5}$, if $\vdash_{S5} \alpha$ then $\vdash_{\textbf{TEL}} \alpha$ *(Preserve)*

Using theorem 2.2 of [FG92], soundness of **S5** and tense logic over $\mathbb{N}$, we immediately have:

Theorem 2.9 (Soundness of TEL)

The axiom system **TEL** is sound.

Theorem 2.3 of [FG92] states that if the system to be temporalized is complete and the axiomatization of the logic with Since and Until is complete over a class of *linear* flows

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of time, then the "merged" axiomatization is complete for the temporalized logic. Our class of flows of time (consisting only of the natural numbers) is a subclass of the linear flows of time. A slight adaptation of their proof yields the same result for temporalizing over the temporal operators used in TEL. Therefore we have:

Theorem 2.10 (Completeness of TEL)
The axiom system **TEL** is complete.

Again borrowing from [FG92], theorem 3.1, and using the fact that both **S5** ([MH92]) and tense logic over the natural numbers ([SC85]) are decidable, we have:

Theorem 2.11 (Decidability of TEL)
The logic **TEL** is decidable.

In the next section we will impose an extra restriction on our models.

3 Conservativity

We want to use subjective temporal formulae for describing the behaviour of a reasoning agent. The reasoning will be assumed to be *conservative*, that is the agent's knowledge will increase as it is reasoning. Although the actual implementation of the reasoning behaviour may involve backtracking or the addition of extra assumptions which may later be retracted, we are interested only in the increase of knowledge over time: adding assumptions and later retracting them is assumed to be done in one step. This also presupposes a world which does not change. We will restrict ourselves to conservative behaviour here, though we agree that it may be worthwhile to investigate also non-conservative behaviour.

In the following we are only interested in subjective formulae, so we delete the world from the information state. We will study consequence relations between formulae, and it will turn out that these notions are independent of the propositional signature. Therefore the propositional signature will be assumed finite.

Definition 3.1 (Conservative models)
i) We define the *degree-of-information* ordering $\leq$ on information states as follows:

$$\text{for } M_1, M_2 \in IS, \qquad M_1 \leq M_2 \qquad \Leftrightarrow \qquad M_2 \subseteq M_1$$

We write $M_1 < M_2$ if $M_1 \leq M_2$ and $M_1 \neq M_2$.

ii) A TEL-model $\mathfrak{M}$ is called *conservative* if for all $s \in \mathbb{N}$:

$$\mathfrak{M}_s \leq \mathfrak{M}_{s+1}$$

iii) Validity and semantical consequence restricted to the class of conservative models will be denoted by $\vDash^c$.

The definition of the degree-of-information ordering is based on the observation that the more valuations one considers to be possible, the less knowledge (or information) one has. Note that for any conservative model $\mathfrak{M}$, time point $s \in \mathbb{N}$ and propositional formula φ: if $(\mathfrak{M}, s) \vDash K\varphi$, then for $t > s$ also $(\mathfrak{M}, t) \vDash K\varphi$. This means that whenever a formula is known, it will remain known in the future. The notion $\vDash^c$ is not compact: the set $\{P^i(\top) \mid i \in \mathbb{N}\}$ (where P^i stands for a sequence of i times P) is not satisfiable, whereas each finite subset is.

Proposition 3.2 (Axiomatization)

Let $C = \{\square(K\alpha \to G(K\alpha)) \mid \alpha \text{ a propositional formula}\}$, then for each TEL-model $\mathfrak{M}$:

$\mathfrak{M}$ is conservative $\Leftrightarrow$ $\mathfrak{M} \vDash C$ $\Leftrightarrow$ $(\mathfrak{M}, t) \vDash C$ for some $t \in \mathbb{N}$.

Furthermore, the axiom system **TELC**, consisting of **TEL** plus the axioms of C, is sound and complete with respect to the class of conservative TEL-models (TELC-models).

Proof

Let $\mathfrak{M}$ be conservative and let $t \in \mathbb{N}$. Suppose $(\mathfrak{M}, t) \vDash K\alpha$ and take $s > t$ arbitrary. Then for all $m \in \mathfrak{M}_t$, $m \vDash \alpha$. Take $m \in \mathfrak{M}_s$, then since $\mathfrak{M}$ is conservative we have $\mathfrak{M}_s \subseteq \mathfrak{M}_t$, so $m \in \mathfrak{M}_t$ and $m \vDash \alpha$. Therefore $(\mathfrak{M}, s) \vDash K\alpha$, and since s was arbitrary we have $(\mathfrak{M}, t) \vDash GK\alpha$, so $(\mathfrak{M}, t) \vDash K\alpha \to GK\alpha$. We have $(\mathfrak{M}, 0) \vDash \square(K\alpha \to GK\alpha)$.

Suppose $(\mathfrak{M}, t) \vDash C$ for some $t \in \mathbb{N}$, but $\mathfrak{M}$ is not conservative. Then there exists $s \in \mathbb{N}$ and $m \in \mathfrak{M}_{s+1}$ with $m \notin \mathfrak{M}_s$. Let φ_m be the conjunction of the literals that are true in m (i.e. $\varphi_m = \bigwedge\{p \in P \mid m \vDash p\} \wedge \bigwedge\{\neg p \mid m \nvDash p\}$; this is a formula since P was assumed finite). Then since $m \notin \mathfrak{M}_s$ and for all $m' \neq m$, $m' \vDash \neg\varphi_m$, we have $(\mathfrak{M}, s) \vDash K\neg\varphi_m$, but as $m \in \mathfrak{M}_{s+1}$ and $m \nvDash \neg\varphi_m$, $(\mathfrak{M}, s+1) \nvDash K\neg\varphi_m$, so $(\mathfrak{M}, s) \nvDash GK\neg\varphi_m$. Thus $(\mathfrak{M}, t) \nvDash \square(K\alpha \to GK\alpha)$, a contradiction.

The above shows that the axioms of C are sound. Now suppose $\vDash^c \varphi$, then we have for all TEL models $\mathfrak{M}$: if $\mathfrak{M}$ is conservative then $\mathfrak{M} \vDash \varphi$. Since there are only a finite number of non-equivalent propositional formulae for P, C can be taken to be finite and therefore we can take the conjunction of its elements. So if $(\mathfrak{M}, s) \vDash \bigwedge C$ then $\mathfrak{M}$ is conservative, so $\mathfrak{M} \vDash \varphi$ and therefore $(\mathfrak{M}, s) \vDash \varphi$. Thus we have $\bigwedge C \vDash \varphi$, and using the deduction lemma for TEL (which can be easily verified), $\vDash \bigwedge C \to \varphi$,

from which by the completeness of **TEL** it follows that $\vdash_{\textbf{TEL}} \wedge C \to \phi$. Since **TELC** contains **TEL** and the axioms of **C**, and has Modus Ponens as inference rule, we conclude $\vdash_{\textbf{TELC}} \phi$.

We also have that **TELC** is decidable:

Proposition 3.3 (Decidability of TELC)

The logic **TELC** is decidable.

Proof

Checking whether $\vdash_{\textbf{TELC}} \phi$ reduces to checking $\vdash_{\textbf{TEL}} \wedge C \to \phi$ where **C** is the set of rules $\Box(\textbf{K}\alpha \to G(\textbf{K}\alpha))$ for all non-equivalent propositional formulae α in the proposition letters of ϕ. This is decidable by theorem 2.11.

Using **TELC** as our base logic we will now consider minimal conservative models and minimal entailment.

4 Minimal models and minimal entailment

To describe the behaviour of a reasoning agent over time, we assume we have a finite number of subjective TEL formulae (or just a single one, the finite conjunction of these formulae). We are interested in the consequences of this description. It is for instance possible to describe the behaviour of an agent performing default reasoning by translating a default rule $(\alpha, \beta) / \gamma$ into the TEL-rule $\textbf{K}\alpha \wedge G(\neg \textbf{K} \neg \beta) \to G(\textbf{K}\gamma)$, as described in [ET93]. This description forces conclusions to be added in certain circumstances. However we want the knowledge of the agent to be minimal: only those facts which are prescribed by the description to be known, should be known, and no other facts. So we make the explicit assumption that "all the agent knows" is what is dictated by the description. Apart from the temporal aspect, this is similar in spirit to the theory of epistemic states of [HM84], introduced to formalize the notion of "only knowing ϕ". For a broader discussion of minimalization of models, see e.g. [Be90].

We will formalize this minimality by introducing a preference relation over TELC-models which favors models with as little propositional knowledge as possible. Formulae are assumed to be subjective.

Definition 4.1 (Minimal models and entailment)

i) We extend the degree of information ordering to TELC-models $\mathcal{M}, \mathcal{N}$:

$$\mathcal{M} \leq \mathcal{N} \qquad \Leftrightarrow \qquad \text{for all } s \in \mathbb{N}: \mathcal{M}_s \leq \mathcal{N}_s$$

We write $\mathcal{M} < \mathcal{N}$ if $\mathcal{M} \leq \mathcal{N}$ and $\mathcal{M} \neq \mathcal{N}$.

ii) A TELC-model $\mathcal{M}$ is a *minimal conservative model of* φ, denoted $\mathcal{M} \vDash_{\min} \varphi$, if $\mathcal{M} \vDash \varphi$ and for all conservative models $\mathcal{N}$, if $\mathcal{N} \vDash \varphi$ and $\mathcal{N} \leq \mathcal{M}$ then $\mathcal{N} = \mathcal{M}$.

iii) For TEL-formulae φ, ψ, we say φ *is a minimal conservative consequence of* ψ or ψ *minimally entails* φ, denoted $\psi \vDash_{\min}^c \varphi$, if for all minimal conservative models $\mathcal{M}$ of ψ, $\mathcal{M} \vDash \varphi$ holds.

For a subjective formula φ (which describes the reasoning of an agent) its minimal models represent the process of the agent's reasoning in time. We can then use minimal consequence to infer properties of this reasoning process.

Note that the notion of minimal entailment strengthens the notion of conservative entailment in the sense that $\varphi \vDash^c \psi$ implies $\varphi \vDash_{\min}^c \psi$. An easy example, even without temporal operators, shows that it is a proper extension: although $Kp \nvDash^c \neg Kq$ we do have $Kp \vDash_{\min}^c \neg Kq$.

For propositional formulae φ, ψ we can define a nonmonotonic entailment relation $\vdash$ by $\varphi \vdash \psi$ iff $K\varphi \vDash_{\min}^c K\psi$. For so called "honest" formulae φ this is exactly the entailment relation defined in [HM84] based on the notion of "only knowing φ".

Since we are working with a fixed propositional signature **P**, the above definition of minimal models and minimal entailment seems to depend on **P**, but this is not actually the case:

Proposition 4.2

The notion $\vDash_{\min}^c$ is independent of the propositional signature.

Proof

For a propositional signature **P** we write $\mathcal{L}_\mathbf{P}$ to denote the temporal language based on **P** and $\mathbf{P}\text{-}\vDash_{\min}^c$ to denote the associated notion of minimal conservative consequence. It is sufficient to show that for two signatures **P, Q** with $\mathbf{P} \subseteq \mathbf{Q}$ we have that for all formulae φ, ψ in $\mathcal{L}_\mathbf{P}$: $\varphi \ \mathbf{P}\text{-}\vDash_{\min}^c \psi$ if and only if $\varphi \ \mathbf{Q}\text{-}\vDash_{\min}^c \psi$.

Let **P, Q** be two propositional signatures with $\mathbf{P} \subseteq \mathbf{Q}$. For a propositional valuation **m** of signature **Q**, $\mathbf{m}|_\mathbf{P}$ denotes the restriction of **m** to atoms of **P**. Consider the following constructions:

- For a TEL-model $\mathcal{M}$ based on **Q**, we define its restriction to **P**, $\mathcal{M}|_\mathbf{P}$ by:

$$(\mathcal{M}|_\mathbf{P})_s = \{ \mathbf{m}|_\mathbf{P} : \mathbf{m} \in \mathcal{M}_s \}$$

- For a TEL-model $\mathcal{M}$ based on **P**, we define its extension to **Q**, $\mathcal{M}|^\mathbf{Q}$ by:

$$(\mathcal{M}|^\mathbf{Q})_s = \{ \mathbf{m} \in \mathbf{mod}(\mathbf{Q}) : \mathbf{m}|_\mathbf{P} \in \mathcal{M}_s \}$$

By induction on $\phi \in \mathcal{L}_\mathbf{P}$ it is easy to see that truth of ϕ at a point in time is preserved under these constructions.

Now suppose that $\mathcal{M}$ is a conservative TEL-model based on $\mathbf{Q}$ and $\mathcal{M} \models_{min} \phi$ (with the notion of $\models_{min}$ based on $\mathbf{Q}$). Then $\mathcal{M}|_\mathbf{P} \models_{min} \phi$ (with the notion of $\models_{min}$ based on $\mathbf{P}$): for suppose $\mathcal{N}$ is a conservative TEL-model based on $\mathbf{P}$ with $\mathcal{N} < \mathcal{M}|_\mathbf{P}$ and $\mathcal{N} \models \phi$, then (!) $\mathcal{N}|^\mathbf{Q} < \mathcal{M}$ and $\mathcal{N}|^\mathbf{Q} \models \phi$.

Conversely, suppose that $\mathcal{M}$ is a conservative TEL-model based on $\mathbf{P}$ and $\mathcal{M} \models_{min} \phi$. Then $\mathcal{M}|^\mathbf{Q} \models_{min} \phi$: for suppose $\mathcal{N}$ is a conservative TEL-model based on $\mathbf{Q}$ with $\mathcal{N} < \mathcal{M}|^\mathbf{Q}$ and $\mathcal{N} \models \phi$, then (!) $\mathcal{N}|_\mathbf{P} < \mathcal{M}$ and $\mathcal{N}|_\mathbf{P} \models \phi$.

It is now easy to see that $\phi \; \mathbf{P}\text{-}\models^c_{min} \psi$ if and only if $\phi \; \mathbf{Q}\text{-}\models^c_{min} \psi$.

As an example of the use of these notions it has been shown in [ET93] that it can capture default logic (see [Re80]). A default theory consists of a set of formulae, called the *axioms*, denoted by $\mathbf{W}$, and a set $\mathbf{D}$ of defaults of the form $(\alpha, \beta) / \gamma$, where α, β and γ are formulae, with the intended meaning: if you believe α and β is consistent with your beliefs, then you should also believe γ. The theory of Reiter then prescribes how, using the default rules, you can extend $\mathbf{W}$ to a set of formulae, called an *extension*. In general for a default theory there may be multiple extensions. If a formula ϕ is in all of these extensions, we call ϕ a *sceptical consequence* of the default theory.

Example 4.3 (Default logic)

Let a finite default theory $\Delta = \langle \mathbf{D}, \mathbf{W} \rangle$ be given and let

$\psi = \bigwedge \{ K\alpha \wedge G(\neg K \neg \beta) \to G(K\gamma) \mid (\alpha, \beta) / \gamma \in \mathbf{D} \} \wedge \bigwedge \{ K\alpha \mid \alpha \in \mathbf{W} \}$. Then ϕ is a sceptical consequence of Δ if and only if $\psi \models^c_{min} F(K\phi)$ (see [ET93]).

We are interested in the complexity of minimal entailment; we will first concentrate on the decidability.

5 Decidability of minimal entailment

The first question to be asked when investigating the complexity of a notion is whether it is decidable or not. The notion of minimal entailment will turn out to be decidable, but in order to prove that we will first need some lemmas.

Observation 5.1

A conservative TEL-model $\mathcal{M}$ consists of a sequence of normal S5-models. These models consist of a finite number of propositional valuations, since $\mathbf{P}$ is assumed to be finite. Furthermore the sequence is (not necessarily strictly) decreasing. Therefore there must exist a time point $s \in \mathbb{N}$ such that for all $t > s$: $\mathcal{M}_t = \mathcal{M}_s$. If s_0 is the smallest point for which this is true, we say that $\mathcal{M}$ *stabilizes* at s_0.

Since all TELC models stabilize, it is possible to write them down in finite space.

The idea in the proof of decidability is that for each formula ψ there is a number n_ψ such that a minimal model of ψ must stabilize before n_ψ. Then there are only a finite number of models to be checked, and since they stabilize, it is always possible to check whether a temporal formula holds in it. To obtain the n_ψ one reasons that if there exists a long enough sequence of identical states in a model, then it is possible to insert an extra (identical) state in it, without disturbing the truth of ψ. Since this enlarged model is smaller (with respect to $\leq$) than the original, the original model could not have been a minimal model of ψ. The length of such a sequence depends on the depth of nesting of temporal operators in ψ. We will now formalize these ideas.

Definition 5.2 (Depth)

The *depth of nesting* of temporal operators in a formula φ, depth(φ), is defined inductively as follows:
- depth(φ) = 0, if $\varphi \in \mathcal{L}_{S5}$
- depth($\alpha \wedge \beta$) = max{depth(α), depth(β)}
- depth($\neg \alpha$) = depth(α)
- depth($\mathbf{P}\alpha$) = depth($\mathbf{F}\alpha$) = depth(α) + 1

The first lemma states that in a sequence of identical states, formulae with small enough depth cannot discriminate between states in the middle of the sequence. All the present lemmas are also valid for non-subjective formulae.

Lemma 5.3

If $\mathcal{M}$ is a TEL-model such that for some $N \geq 1, s \geq N$:
$$\mathcal{M}_s = \mathcal{M}_{s+i} = \mathcal{M}_{s-i} \quad \text{for all } 1 \leq i \leq N,$$
then for all φ with depth(φ) < N and $1 \leq j \leq N$ - depth(φ):
$$(\mathcal{M}, s-j) \vDash \varphi \Leftrightarrow (\mathcal{M}, s) \vDash \varphi \Leftrightarrow (\mathcal{M}, s+j) \vDash \varphi.$$

Proof

By induction on φ, where the only interesting cases are the temporal operators (the abbreviation "i.h." stands for induction hypothesis):

- **Fα**   Let $1 \leq j \leq N - \text{depth}(F\alpha)$. The implications from right to left are trivial, so we will only prove $(\mathfrak{M}, s - j) \models F\alpha \Rightarrow (\mathfrak{M}, s + j) \models F\alpha$.

  - Suppose $(\mathfrak{M}, s - j) \models F\alpha$. There exists $n \in \mathbb{N}$, $n > s - j$ with $(\mathfrak{M}, n) \models \alpha$.

    If $n > s + j$ then $(\mathfrak{M}, s + j) \models F\alpha$, so suppose $s - j < n \leq s + j$.

    * If $n = s - k$ with $1 \leq k < j$ then $1 \leq k < j \leq N - \text{depth}(F\alpha) < N - \text{depth}(\alpha)$ and with i.h. we get $(\mathfrak{M}, s) \models \alpha$.

    * If $n = s$ then $(\mathfrak{M}, s) \models \alpha$.

    * If $n = s + k$ with $1 \leq k \leq j$ then $1 \leq k \leq j \leq N - \text{depth}(F\alpha) < N - \text{depth}(\alpha)$, so $(\mathfrak{M}, s) \models \alpha$.

    So we have $(\mathfrak{M}, s) \models \alpha$ and $1 \leq j + 1 \leq N - (\text{depth}(F\alpha) - 1) = N - \text{depth}(\alpha)$, so by i.h. we have $(\mathfrak{M}, s + (j + 1)) \models \alpha$ so $(\mathfrak{M}, s + j) \models F\alpha$.

- **Pα**   Analogous to Fα.

We will use this lemma with $j = 1$ and $N = \text{depth}(\varphi) + 1$. The following example shows that we really need that many identical states:
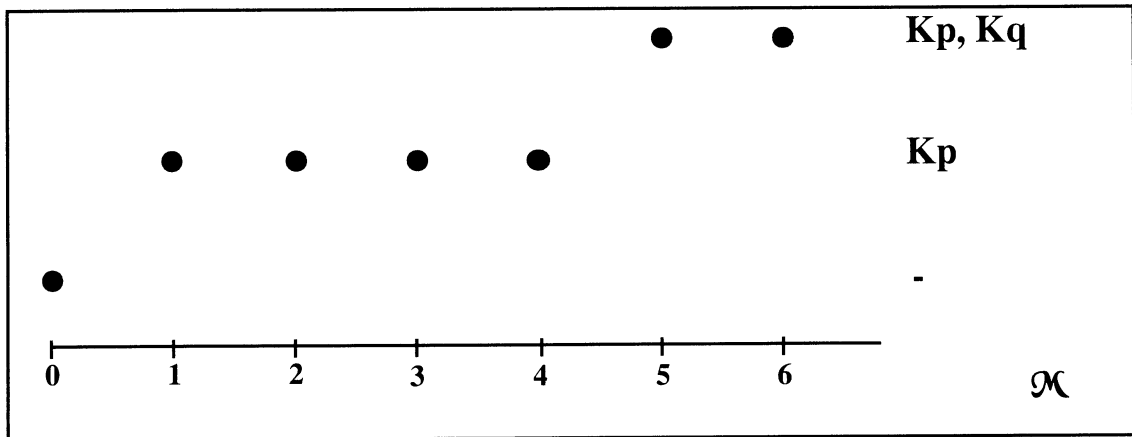
This picture represents the model in which nothing is known at time point 0, p is known from time point 1 onwards, and q is known from time point 5. We have $(\mathfrak{M}, 3 + 1) \models G(Kq)$ but $(\mathfrak{M}, 3) \not\models G(Kq)$ (we need an extra Kp state between 4 and 5); also $(\mathfrak{M}, 2 - 1) \models H(\neg Kp)$ but $(\mathfrak{M}, 2) \not\models H(\neg Kp)$ (we need an extra Kp state between 0 and 1).

The next lemma shows that if we have a sequence of identical states, a middle state can be duplicated or removed without changing the truth of formulae with sufficiently small depth of operator-nesting:

Lemma 5.4

Let $\mathcal{M}$ be a model as in lemma 5.3. Define $f: \mathbb{N} \to \mathbb{N}$ as follows:

$$f(n) = \begin{array}{ll} n & \text{if } n \le s \\ n-1 & \text{if } n > s \end{array}$$

and let $\mathcal{N}$ be a model satisfying $\mathcal{N}_i = \mathcal{M}_{f(i)}$ for all $i \in \mathbb{N}$.

Then for all formulae φ with $\text{depth}(\varphi) \le N$ we have:

$$(\mathcal{N}, i) \vDash \varphi \quad \Leftrightarrow \quad (\mathcal{M}, f(i)) \vDash \varphi \quad \text{for all } i \in \mathbb{N}.$$

Proof

By induction on φ, where the only non-trivial cases are the operators (for which we will take H and G):

- $H\varphi$ Suppose $(\mathcal{N}, i) \vDash H\varphi$. Take $k < f(i)$. Then there exists $t < i$ such that $f(t) = k$ and then $(\mathcal{N}, t) \vDash \varphi$, so by i.h. $(\mathcal{M}, k) \vDash \varphi$. Thus $(\mathcal{M}, f(i)) \vDash H\varphi$.

  Suppose $(\mathcal{M}, f(i)) \vDash H\varphi$.

  - If $i \le s$: Take $k < i$ then $f(k) < f(i)$, so $(\mathcal{M}, f(k)) \vDash \varphi$ and by i.h. $(\mathcal{N}, k) \vDash \varphi$. We have $(\mathcal{N}, i) \vDash H\varphi$.

  - If $i \ge s + 1$: Take $k < i$;

    * If $k \ne s$ then $f(k) < f(i)$, so $(\mathcal{M}, f(k)) \vDash \varphi$ and by i.h. $(\mathcal{N}, k) \vDash \varphi$.

    * If $k = s$ then $s - 1 < f(i)$, so $(\mathcal{M}, s-1) \vDash \varphi$. As $\text{depth}(H\varphi) \le N$ we have $1 \le 1 \le N - \text{depth}(\varphi)$ and by lemma 5.3 we have $(\mathcal{M}, s) \vDash \varphi$ and by i.h. $(\mathcal{N}, s) \vDash \varphi$, or $(\mathcal{N}, k) \vDash \varphi$

  So we have $(\mathcal{N}, i) \vDash H\varphi$.

- $G\varphi$ Analogous.

The following picture sketches the situation with $N = 2$:
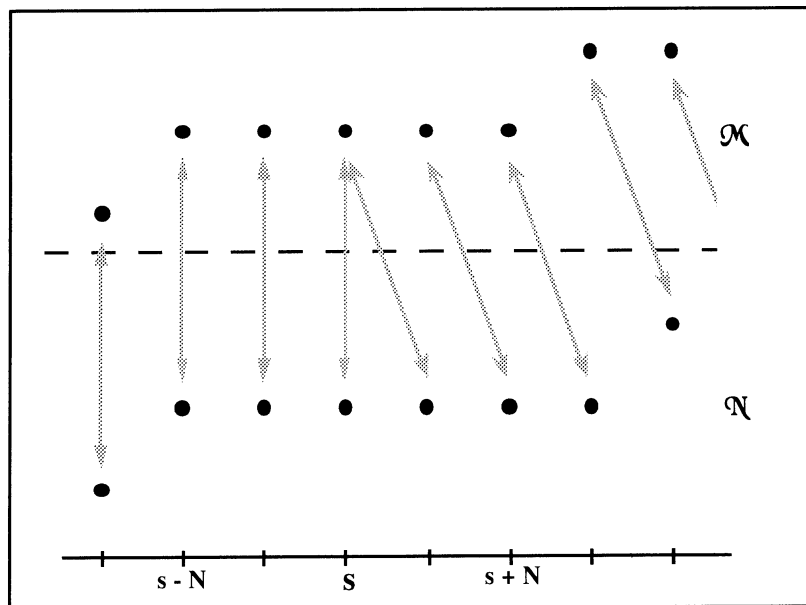
Another way of proving this lemma is to show that there exist bisimulations up to N between these two models. The main use of the lemma lies in the possibility of enlarging or reducing sequences of identical states in a model without disturbing truth of formulae with sufficiently small depth of nesting.

Observation 5.5

For the models $\mathcal{M}$, $\mathcal{N}$ of lemma 5.4 the following holds: if $\mathcal{M}$ is conservative then $\mathcal{N}$ is conservative and vice versa, $\mathcal{N} \leq \mathcal{M}$, and if there exists $t \geq s + N$ such that $\mathcal{M}_t < \mathcal{M}_{t+1}$ then $\mathcal{N} < \mathcal{M}$.

Proof

Take $s \in \mathbb{N}$, then $\mathcal{N}_s = \mathcal{M}_{f(s)}$. Since $f(s) \leq s$ and $\mathcal{M}$ is conservative we have $\mathcal{M}_{f(s)} \leq \mathcal{M}_s$ so $\mathcal{N}_s \leq \mathcal{M}_s$. If there exists $t \geq s + N$ such that $\mathcal{M}_t < \mathcal{M}_{t+1}$ then $\mathcal{N}_{t+1} = \mathcal{M}_{f(t+1)} = \mathcal{M}_t < \mathcal{M}_{t+1}$.

This observation and the previous lemma allow us to conclude that for each formula there is a time point such that the minimal models of the formula must stabilize before this point. From now on we will again restrict ourselves to subjective formulae.

Lemma 5.6

Suppose the propositional signature P consists of n atoms. If a conservative model $\mathcal{M}$ of signature P is a minimal model of a subjective formula φ then it stabilizes on or before time point $(2^n - 1) \cdot 2 \cdot \text{depth}(\varphi)$.

Proof

First we will show that a minimal model $\mathcal{M}$ of φ cannot have more than $2 \cdot \text{depth}(\varphi)$ successive identical states before it stabilizes. Suppose $\mathcal{M} \vDash_{\min} \varphi$ and it has at least $2 \cdot \text{depth}(\varphi) + 1$ successive identical states before it stabilizes. So there exists $s \geq \text{depth}(\varphi)$ such that $\mathcal{M}_s = \mathcal{M}_{s+i} = \mathcal{M}_{s-i}$ for all $1 \leq i \leq \text{depth}(\varphi)$, and $t \geq s + \text{depth}(\varphi)$ such that $\mathcal{M}_t < \mathcal{M}_{t+1}$. Now consider the model $\mathcal{N}$ as described in lemma 5.4. Since $\mathcal{M} \vDash \varphi$ we have $\mathcal{N} \vDash \varphi$, and by observation 5.5 we have $\mathcal{N} < \mathcal{M}$. Therefore $\mathcal{M}$ cannot be a minimal model of φ.

As P has n atoms, there exist 2^n different propositional models. Since a conservative model $\mathcal{M}$ consists of a decreasing sequence of (non-empty) sets of propositional models, there are at most $2^n - 1$ points s such that $\mathcal{M}_s < \mathcal{M}_{s+1}$. If $\mathcal{M}$ is a minimal model of φ then there can be at most $2 \cdot \text{depth}(\varphi)$ successive identical states before it stabilizes, and therefore $\mathcal{M}$ must stabilize on or before time point $(2^n - 1) \cdot 2 \cdot \text{depth}(\varphi)$.

Lemma 5.7

For a conservative model $\mathcal{M}$, $s \in \mathbb{N}$ and a formula φ it is decidable whether $(\mathcal{M}, s) \models \varphi$.

Proof

Suppose we have a conservative model $\mathcal{M}$ and $s \in \mathbb{N}$. By Observation 5.1, $\mathcal{M}$ stabilizes at some point s_0. It is easily seen from lemma 5.3 that for a formula φ we have $(\mathcal{M}, t) \models \varphi \Leftrightarrow (\mathcal{M}, u) \models \varphi$ for all $t, u \geq s_0 + \text{depth}(\varphi)$. Then use induction on φ.

Most importantly, it is decidable if a model is a minimal model of a subjective formula:

Lemma 5.8

For a conservative model $\mathcal{M}$ and a subjective formula φ it is decidable whether $\mathcal{M} \models_{\min} \varphi$.

Proof

First, we need to check whether $\mathcal{M} \models \varphi$, which is equivalent to checking $(\mathcal{M}, 0) \models \Box\varphi$, decidable by lemma 5.7. Suppose $\mathbf{P}$ has n atoms. If $\mathcal{M}$ stabilizes after time point $(2^n - 1) \cdot 2 \cdot \text{depth}(\varphi)$ it is not a minimal model of φ by lemma 5.6. So suppose $\mathcal{M} \models \varphi$ and $\mathcal{M}$ stabilizes on or before time point $(2^n - 1) \cdot 2 \cdot \text{depth}(\varphi)$.

In order to check whether $\mathcal{M} \models_{\min} \varphi$ we have to see if there exists a conservative model smaller than $\mathcal{M}$ which satisfies φ. Of course in general there are an infinite number of conservative models smaller than $\mathcal{M}$, but we will show that we only have to consider models which stabilize not later than time point $(2^n - 1) \cdot (4 \cdot \text{depth}(\varphi) + 1)$. In other words, we will show that if there exists a conservative model smaller than $\mathcal{M}$ satisfying φ, there also exists such a model which stabilizes on or before point $(2^n - 1) \cdot (4 \cdot \text{depth}(\varphi) + 1)$. The converse of this statement is of course trivial.

Suppose we have a conservative model $\mathcal{N}$ with $\mathcal{N} < \mathcal{M}$ and $\mathcal{N} \models \varphi$, and let s be the stabilizing point of $\mathcal{N}$. If $s \leq (2^n - 1) \cdot (4 \cdot \text{depth}(\varphi) + 1)$ then we are done, so suppose not. Now consider the following procedure for constructing a model $\mathcal{N}'$: if there exists a sequence of more than $2 \cdot \text{depth}(\varphi) + 1$ successive identical states in $\mathcal{N}$ between time points $(2^n - 1) \cdot 2 \cdot \text{depth}(\varphi)$ and s then we delete as many points from this sequence until it has length $2 \cdot \text{depth}(\varphi) + 1$. Lemma 5.4 ensures that we can do this without disturbing the truth of φ. It is also easy to see that the result is conservative and still (strictly) smaller than $\mathcal{M}$. Let $\mathcal{N}'$ be the model which results from applying this procedure for every such sequence. Then $\mathcal{N}' \models \varphi$ and $\mathcal{N}' < \mathcal{M}$. Let s' be the stabilizing point of $\mathcal{N}'$. Then in $\mathcal{N}'$ there are at most $2^n - 1$ points t with $(2^n - 1) \cdot 2 \cdot \text{depth}(\varphi) \leq t < s$ and $\mathcal{N}'_t < \mathcal{N}'_{t+1}$. Between such points there are at most

2·depth(φ) + 1 identical states and therefore

$s \leq (2^n - 1) \cdot 2 \cdot \text{depth}(\varphi) + (2^n - 1) \cdot (2 \cdot \text{depth}(\varphi) + 1) = (2^n - 1) \cdot (4 \cdot \text{depth}(\varphi) + 1)$.

It is easy to see that, given the finite signature, there are only a finite number of conservative models which stabilize not later than time point $(2^n - 1) \cdot (4 \cdot \text{depth}(\varphi) + 1)$. For each such model $\mathcal{N}$ we can check whether $\mathcal{N} < \mathcal{M}$ (only the first $(2^n - 1) \cdot (4 \cdot \text{depth}(\varphi) + 1)$ time points have to be considered), and we can check if $\mathcal{N} \models \varphi$ (again decidable). If we find such a model then $\mathcal{M} \nvDash_{min} \varphi$, otherwise $\mathcal{M} \vDash_{min} \varphi$.

Now we are ready to prove decidability of minimal entailment:

Theorem 5.9 (Decidability of minimal entailment)
    For two subjective formulae φ, ψ it is decidable whether $\varphi \vDash^c_{min} \psi$.
Proof
    We can take the signature **P** to consist of the atoms occurring in φ and ψ. Suppose there are n such atoms. Then lemma 5.6 states that we only have to consider models which stabilize not later than time point $(2^n - 1) \cdot 2 \cdot \text{depth}(\varphi)$, and since the signature is finite, there are finitely many such models. For each such model $\mathcal{M}$ it is decidable by lemma 5.8 whether $\mathcal{M} \vDash_{min} \varphi$. Now we only have to check for each of these (finitely many) minimal models $\mathcal{M}$ of φ whether $\mathcal{M} \vDash \psi$, decidable by lemma 5.7.

Of course the procedure given in the proof will be very inefficient.
Having established that both **TELC** and minimal entailment are decidable, in the next section we will look at the complexity of these notions, and in particular whether the minimalization process has a structural impact on complexity.

6 Complexity

In order to study the complexity we will first look at satisfiability of TELC. We restrict ourselves to satisfiability of subjective formulae in time point 0.

Definition 6.1 (TELC(0)-SAT)
    A subjective formula φ is in **TELC(0)-SAT** if there exists a TELC-model $\mathcal{M}$ such that $(\mathcal{M}, 0) \vDash \varphi$.

Remark 6.2

It is easy to see that **TELC(0)-SAT** is polynomially reducible (and vice versa) to satisfiability (in any time point): φ is satisfiable iff $\varphi \vee \mathbf{F}\varphi$ is in **TELC(0)-SAT**, and φ is in **TELC(0)-SAT** iff $\Box(\mathbf{at_0} \rightarrow \varphi)$ is satisfiable.

Definition 6.3 (Size of a TELC-model)

For a TELC-model $\mathcal{M}$ we call its stabilizing point the size of $\mathcal{M}$, denoted size($\mathcal{M}$).

Definition 6.4 (Subformula)

Let Subf(φ) denote the subformulae of φ, where maximal S5-subformulae of φ are not further decomposed, and let SubfS5(φ) denote the set of subformulae of φ which are in $\mathcal{L}_{S5}$.

We give an example to clarify this definition: $\text{Subf}(\mathbf{G(Kp \wedge Kq)}) = \{\mathbf{G(Kp \wedge Kq), Kp \wedge Kq}\}$ and $\text{SubfS5}(\mathbf{G(Kp \wedge Kq)}) = \{\mathbf{Kp \wedge Kq, Kp, Kq, p, q}\}$. So $\text{Subf}(\varphi) \cup \text{SubfS5}(\varphi)$ is the set of all subformulae of φ.

First we will prove a small-model theorem for TELC. Let length(φ) denote the length of the formula φ as a string.

Lemma 6.5 (Small model theorem)

If a subjective formula φ is in **TELC(0)-SAT** then there exists a TELC-model $\mathcal{M}$ such that $(\mathcal{M}, 0) \models \varphi$, $\text{size}(\mathcal{M}) \leq 4 \cdot (\text{length}(\varphi))^2$, and for all $i \in \mathbb{N}$ the S5-model $\mathcal{M}_i$ contains less then $2 \cdot \text{length}(\varphi)$ valuations.

Proof

Suppose for some TELC-model $\mathcal{N}$ we have $(\mathcal{N}, 0) \models \varphi$ and let $s_{\mathcal{N}}$ be the stabilizing point of $\mathcal{N}$.

Let $\mathcal{L}_0$ denote the propositional language based on **P**.

Now let $A = \{\psi, \neg\psi \mid \psi \in \mathcal{L}_0, \psi \in \text{SubfS5}(\varphi)\}$ and for $i \in \mathbb{N}$:

$B(i) = \{\mathbf{K}\psi \mid \psi \in A, \mathcal{N}_i \models \mathbf{K}\psi\} \cup \{\neg\mathbf{K}\psi \mid \psi \in A, \mathcal{N}_i \not\models \mathbf{K}\psi\}$. Based on these sets we will define a TELC-model $\mathcal{N}'$:

- For each $\neg\mathbf{K}\psi \in B(s_{\mathcal{N}})$ choose a valuation $m \in \text{Mod}(\mathbf{P})$ such that $m \not\models \psi$ and $m \models \alpha$ for each $\mathbf{K}\alpha \in B(s_{\mathcal{N}})$ (such a valuation exists since $(\mathcal{N}, s_{\mathcal{N}}) \not\models \mathbf{K}\psi$ and $(\mathcal{N}, s_{\mathcal{N}}) \models \mathbf{K}\alpha$ for each $\mathbf{K}\alpha \in B(s_{\mathcal{N}})$). Let M be the set of these valuations. We have $M \models B(s_{\mathcal{N}})$. If there are no formulae $\neg\mathbf{K}\psi \in B(s_{\mathcal{N}})$ then choose any valuation m with $m \models \alpha$ for each $\mathbf{K}\alpha \in B(s_{\mathcal{N}})$ (which again exists). Set $\mathcal{N}'_j = M$ for all $j \geq s_{\mathcal{N}}$. It easy to verify that $\mathcal{N}'_j \models B(j)$ for all $j \geq s_{\mathcal{N}}$.

- Now using induction on $s_{\mathcal{N}} > j \geq 0$:

Let $B(j) \setminus B(j+1) = \{ \neg K\psi_1 , \dots \neg K\psi_n \}$ (because $\mathcal{N}$ is conservative there will be no formulae $K\psi$ in this set!). For $k = 1 \dots n$ choose a valuation m_k with $m_k \nvDash \psi_k$ and $m \vDash \alpha$ for each $K\alpha \in B(j)$ (again such valuations exist). Let $\mathcal{N}_j = \mathcal{N}_{j+1} \cup \{m_1, \dots, m_k\}$. It is again easy to verify that $\mathcal{N}_j \vDash B(j)$.

The resulting model $\mathcal{N}$ has the following properties:

1. $\mathcal{N}$ is a TELC-model.

2. $\mathcal{N}_j \vDash B(j)$ for all $j \in \mathbb{N}$

3. The number of valuations of $\mathcal{N}_j$ is smaller than the number of elements in A ($\leq 2 \cdot \mathrm{length}(\varphi)$).

4. $(\mathcal{N}, 0) \vDash \varphi$: Take $\psi \in \mathrm{Subf}(\varphi) \cap \mathcal{L}_{S5}$ (which must be subjective!). Then using a normal form described in [MH92] it is easy to see that ψ is equivalent to a formula $\psi' = \delta_1 \vee \dots \vee \delta_m$ with for $i = 1 \dots m : \delta_i = K\varphi_{1,i} \wedge \dots \wedge K\varphi_{k(i),i} \wedge \neg K\psi_{1,i} \wedge \dots \wedge \neg K\psi_{l(i),i}$ with $\varphi_{jk}, \psi_{jk} \in A$. So using 2. we have:

$$\mathcal{N}_i \vDash K\varphi_{jk} \quad \Leftrightarrow \quad \mathcal{N}_i \vDash K\varphi_{jk} \quad \text{and}$$
$$\mathcal{N}_i \vDash \neg K\psi_{jk} \quad \Leftrightarrow \quad \mathcal{N}_i \vDash \neg K\psi_{jk} \quad \text{so}$$

$\mathcal{N}_i \vDash \psi' \Leftrightarrow \mathcal{N}_i \vDash \psi'$ so $\mathcal{N}_i \vDash \psi \Leftrightarrow \mathcal{N}_i \vDash \psi$. An easy induction gives:
for all $i \in \mathbb{N}$, for all $\psi \in \mathrm{Subf}(\varphi)$: $(\mathcal{N}, i) \vDash \psi \Leftrightarrow (\mathcal{N}, i) \vDash \psi$ and therefore $(\mathcal{N}, 0) \vDash \varphi$.

5. The number of i for which $\mathcal{N}_i < \mathcal{N}_{i+1}$ is less then $2 \cdot \mathrm{length}(\varphi)$: real updates occur at most once for each $\neg K\psi$ with $\psi \in A$ and A contains at most $2 \cdot \mathrm{length}(\varphi)$ elements.

Now construct the model $\mathcal{M}$ as follows:

for each sequence of more than $2 \cdot \mathrm{depth}(\varphi) + 1$ identical states in $\mathcal{N}$, before its stabilizing point, delete (as many) states from this sequence until it has length $2 \cdot \mathrm{depth}(\varphi) + 1$. Let $\mathcal{M}$ be the resulting model. Now lemma 5.4 ensures that $(\mathcal{M}, 0) \vDash \varphi$. Furthermore $2 \cdot \mathrm{depth}(\varphi) + 1 \leq 2 \cdot \mathrm{length}(\varphi)$ so that: $\mathrm{size}(\mathcal{M}) \leq (2 \cdot \mathrm{length}(\varphi))^2$.

With this lemma we can show that **TELC(0)-SAT** is in NP, using methods similar to those in e.g. [SC85], [La77]:

Theorem 6.6

    TELC(0)-SAT is in NP.

Proof

For a subjective formula φ we present the following nondeterministic algorithm to verify if φ is in **TELC(0)-SAT**. A nondeterministic Turing machine (M) guesses $4 \cdot (\mathrm{length}(\varphi))^2$ Kripke models $\mathcal{M}_i$ with each less than $2 \cdot \mathrm{length}(\varphi)$ valuations, such that $\mathcal{M}_i \supseteq \mathcal{M}_{i+1}$. $\mathcal{M}$ will be this model, remaining constant after time point $4 \cdot (\mathrm{length}(\varphi))^2$. Then it verifies if $(\mathcal{M}, 0) \vDash \varphi$ as follows: for each

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$i \in \{0, ..., 4 \cdot (\text{length}(\varphi))^2 + \text{length}(\varphi) \}$ M maintains a set label(i) which is initialized to the empty set and at the end will contain the subformulae of φ true at time point i. Now for each $\psi \in \text{Subf}(\varphi)$ (increasing in the length of ψ) and for each $i \in \{0, ..., 4 \cdot (\text{length}(\varphi))^2 + \text{length}(\varphi) \}$ update label(i) as follows:

1. Add $\psi \in \mathcal{L}_{S5}$ to label(i) iff $\mathcal{M}_i \vDash \psi$ (this can be checked in time polynomial in the number of states in $\mathcal{M}_i$, using a labelling algorithm similar to the one described here, see e.g. [HM85]).

2. Add $\neg\psi$ to label(i) iff $\psi \notin$ label(i).

3. Add $\alpha \wedge \beta$ to label(i) iff $\alpha \in$ label(i) and $\beta \in$ label(i).

4. Add $\mathbf{F}\alpha$ to label(i) iff $\alpha \in$ label(j) for some $j > i$
    (If $i = 4 \cdot (\text{length}(\varphi))^2 + \text{length}(\varphi)$ then add $\mathbf{F}\alpha$ to label(i) iff $\alpha \in$ label(i)).

5. Add $\mathbf{P}\alpha$ to label(i) iff $\alpha \in$ label(j) for some $j < i$.

Now we have $(\mathcal{M}, 0) \vDash \varphi$ iff $\varphi \in$ label(0) at the end of this procedure. It is easy to verify that this algorithm works properly in time polynomial in length(φ). Lemma 6.5 ensures that there is a guess for which M halts in an accepting state iff φ is in **TELC(0)-SAT**.

This gives us:

Corollary 6.7
TELC satisfiablity is NP-complete.

Proof
The reduction given in remark 6.2 ensures that TELC satisfiablity is in NP, and clearly a propositional formula φ is satisfiable iff $\mathbf{M}\varphi$ is TELC satisfiable, and as satisfiability of propositional formulae is NP-complete, TELC satisfiability is also NP-complete.

We would like to show that the minimalization of models makes the consequence relation more complex, and we can do this using the reduction of sceptical consequence in default logic to minimal consequence, as described in example 3.5.

Proposition 6.8
Minimal consequence is Π_2^P-hard.

Proof
The reduction of example 3.5 is clearly polynomial, and sceptical consequence in default logic is Π_2^P-complete ([Gl92], [St92], see also [PS92]).

So, minimal consequence is harder than TELC-consequence (which is Π_1^P-complete, or co-NP-complete), provided that the polynomial hierarchy does not collapse (see [Jo90]).

In [ET95] a sublanguage of the subjective part of $\mathcal{L}_{\textbf{TEL}}$ is proposed as a specification language for (conservative) reasoning processes and it is shown that this language is suited for this task. We will now look at the complexity of minimal entailment restricted to this language. Let $H_0\varphi$ be an abbreviation for $(at_0 \rightarrow \varphi)$.

Definition 6.9

The language $\mathcal{L}'$ is the smallest set such that:

1. If $\alpha \in \mathcal{L}_0$ then $K\alpha \in \mathcal{L}'$
2. If $\alpha, \beta, \gamma, \psi$ and $\varphi \in \mathcal{L}_0$ then $H_0(K\alpha) \wedge H_0(\neg K\beta) \wedge K\gamma \wedge G(\neg K(\neg \psi)) \rightarrow G(K\varphi) \in \mathcal{L}'$
3. If $\varphi, \psi \in \mathcal{L}'$ then $\varphi \wedge \psi \in \mathcal{L}'$

For $\varphi \in \mathcal{L}'$ and $\psi = F(K\alpha)$ with $\alpha \in \mathcal{L}_0$ we define $\varphi \vDash'^c_{min} \psi$ iff $\varphi \vDash^c_{min} \psi$.

Since we can reduce default logic to this fragment, $\vDash'^c_{min}$ is Π_2^P-hard. However, it is no harder than that:

Proposition 6.10

$\vDash'^c_{min}$ is Π_2^P-complete.

Proof

It is easy to describe a nondeterministic Turing machine M with access to an NP-oracle for determining whether *not* $\varphi \vDash'^c_{min} \psi$ (similar to the proofs in [St92], [PS92] or [Gl92]). A minimal model of φ can have no identical states before it stabilizes. For each conjunct $H_0(K\alpha) \wedge H_0(\neg K\beta) \wedge K\gamma \wedge G(\neg K(\neg \delta)) \rightarrow G(K\varepsilon)$ in φ, M guesses a time point $i \geq 1$ but less than the number n of these conjuncts, from which time onwards ε will be assumed to hold (or it guesses that ε will never hold). Denote for $i \in \{ 0, ..., n \}$, the set of formulae assumed to hold at i plus the formulae α for which there is a conjunct $K\alpha$ in φ, by $A(i)$. Then M uses the NP-oracle to perform the following:

1. Let $f(\varepsilon)$ be the point from which ε is assumed to hold (so $f(\varepsilon) \in \{ 1, ..., n, \infty \}$). Now it checks for all $i \in \{1, ..., n\}$ if $\{ K\varepsilon \mid f(\varepsilon) \leq i \} \cup \{ \neg K\varepsilon \mid f(\varepsilon) > i \}$ is S5-satisfiable (note that S5-satisfiability is in NP). If not, it halts in a rejecting state (the guess does not induce a TELC-model).

2. For each conjunct $H_0(K\alpha) \wedge H_0(\neg K\beta) \wedge K\gamma \wedge G(\neg K(\neg \delta)) \rightarrow G(K\varepsilon)$ and for each time point $i \in \{0, ..., n\}$ it computes whether $A(0) \vDash \alpha$, whether $A(0) \nvDash \beta$, whether $A(i) \vDash \gamma$ and whether for no $i < j \leq n$, $A(j) \vDash \delta$. If this is true for no time point then it checks

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whether ε is assumed never to hold; otherwise it takes the first such point and checks whether ε is assumed to hold from the next time point on. If these conditions are violated then M halts in a rejecting state (the guess does not induce a minimal model of φ).

3. It checks if $A(n) \vDash \chi$ (when $\psi = F(K\chi)$). If this is the case then in this minimal model of φ, ψ holds, so M halts in a rejecting state (the guess does not induce a minimal model of φ in which ψ fails). Otherwise it halts in an accepting state (the guess induces a minimal model of φ in which ψ does not hold).

This nondeterministic algorithm is polynomial in φ (using an NP-oracle for propositional consequence and S5-satisfiability) so the converse of $\vDash'^c_{min}$ is in Σ^P_2 which implies that $\vDash'^c_{min}$ is in Π^P_2. Together with Π^P_2-hardness this gives the desired result.

Apart from default logic, sceptical consequence relations of many other well-known monotonic logics such as McDermott and Doyle's nonmonotonic logic, autoepistemic logic and nonmonotonic logic N are Π^P_2-complete ([Gl92]) which means that we can reduce these relations to minimal consequence (or even $\vDash'^c_{min}$), using a polynomial reduction. Further research is needed to find these reductions.

We would also like to have an upper bound on the complexity of minimal consequence. In order to get this, we need to sharpen some previous lemmas.

Definition 6.11

For a subjective formula φ, define $A(\varphi) = \{ \psi, \neg\psi \mid \psi \in \mathcal{L}_0 \cap \text{SubfS5}(\varphi) \}$. A TELC-model $\mathcal{M}$ of φ is called *based on* φ (abbreviated bo(φ)) if there exist sets $A(i)$ for each $i \in \mathbb{N}$ with $A(0) \subseteq A(1) \subseteq ... \subseteq A(\varphi)$ and $\mathcal{M}_i = \text{Mod}(A(i)) = \{ m \in \text{Mod}(P) \mid m \vDash A(i) \}$.

Lemma 6.12

If $\mathcal{M} \vDash_{min} \varphi$ then $\mathcal{M}$ is bo(φ) and $\text{size}(\mathcal{M}) \leq 4 \cdot (\text{length}(\varphi))^2$.

Proof

Suppose $\mathcal{M}$ is not based on φ. Define $A(i) = \{ \alpha, \neg\alpha \mid \alpha \in A(\varphi)$ and $\mathcal{M}_i \vDash K\alpha \}$ and let $\mathcal{N}_i = \text{Mod}(A(i))$. Clearly $A(0) \subseteq A(1) \subseteq ... \subseteq A(\varphi)$ so $\mathcal{N}$ is a TELC-model and $\mathcal{N} < \mathcal{M}$. Furthermore for all $\alpha \in \mathcal{L}_0 \cap \text{SubfS5}(\varphi)$ we have $\mathcal{M}_i \vDash K\alpha \Leftrightarrow \mathcal{N}_i \vDash K\alpha$ and $\mathcal{M}_i \vDash M\alpha \Leftrightarrow \mathcal{N}_i \vDash M\alpha$, so using the same argument as in the proof of lemma 6.4 we have $\mathcal{N} \vDash \varphi$. This contradicts the assumption, so $\mathcal{M}$ is based on φ. But then the number of "updates" cannot be larger than the number of elements of $A(\varphi)$ and in-

between such updates there cannot be sequences of identical states longer than $2 \cdot \text{depth}(\varphi) + 1$ so $\text{size}(\mathcal{M}) \leq 4 \cdot (\text{length}(\varphi))^2$.

Notice that a model $\mathcal{M}$ based on φ can equivalently be described by giving for each formula in $A(\varphi)$ the time point at which it is known in $\mathcal{M}$, or "infinity" if this is never the case. We have a similar result for models which refute that $\mathcal{M}$ is a minimal model of φ:

Lemma 6.13

If $\mathcal{M} \models \varphi$ but $\mathcal{M} \not\models_{\min} \varphi$ then there exists a TELC-model $\mathcal{N}$ such that $\mathcal{N} < \mathcal{M}, \mathcal{N} \models \varphi$ and $\mathcal{N}$ is based on φ with $\text{size}(\mathcal{N}) \leq \text{size}(\mathcal{M}) + 4 \cdot (\text{length}(\varphi))^2$.

Proof

Suppose $\mathcal{M} \models \varphi$ but $\mathcal{M} \not\models_{\min} \varphi$ then there is a TELC-model $\mathcal{M}$ with $\mathcal{M} < \mathcal{M}$ and $\mathcal{M} \models \varphi$. In the same way as in the proof of lemma 6.12 we can make a model $\mathcal{M}'$ which is a model of φ based on φ and $\mathcal{M}' \leq \mathcal{M}$. Now from any sequence of identical states in $\mathcal{M}'$ after $\text{size}(\mathcal{M})$ but before $\text{size}(\mathcal{M}')$ with length more than $2 \cdot \text{depth}(\varphi) + 1$ we can delete states until it has length $2 \cdot \text{depth}(\varphi) + 1$. Let $\mathcal{N}$ be the resulting model (this construction is the same as the one used in the proof of lemma 5.8). So we have $\mathcal{N} < \mathcal{M}, \mathcal{N} \models \varphi$ and $\mathcal{N}$ is based on φ. Furthermore, $\mathcal{N}$ has less than $2 \cdot \text{length}(\varphi)$ updates, and sequences between $\text{size}(\mathcal{M})$ and $\text{size}(\mathcal{N})$ have length no greater than $2 \cdot \text{depth}(\varphi) + 1$, so $\text{size}(\mathcal{N}) \leq \text{size}(\mathcal{M}) + 2 \cdot \text{length}(\varphi) \cdot 2 \cdot \text{length}(\varphi) = \text{size}(\mathcal{M}) + 4 \cdot (\text{length}(\varphi))^2$.

Lemma 6.14

Deciding for a formula φ and a model $\mathcal{M}$ based on φ whether $\mathcal{M} \models_{\min} \varphi$ is in Π_2^P.

Proof

We assume the model $\mathcal{M}$ encoded as described in the remark after lemma 4.18: there is a function $f : A(\varphi) \to \mathbb{N} \cup \{\infty\}$ such that $f(\alpha)$ gives the time point from which α is known. We will show that deciding whether $\mathcal{M} \not\models_{\min} \varphi$ is in Σ_2^P by describing a nondeterministic Turing machine M with access to an NP-oracle. Let $\text{size}(\mathcal{M}) = \max f[A(\varphi)] \setminus \{\infty\}$ (if $f[A(\varphi)] = \{\infty\}$, then let $\text{size}(\mathcal{M}) = 0$). First we check if $\text{size}(\mathcal{M}) \leq 4 \cdot (\text{length}(\varphi))^2$; if not we halt in an accepting state. Otherwise we use a labelling algorithm as described earlier to check if $\mathcal{M} \models \varphi$. The range of time points we have to check is from 0 to $\text{size}(\mathcal{M}) + \text{length}(\varphi)$. The subformulae in $\text{Subf}(\varphi) \cap \mathcal{L}_{S5}$ are treated as follows: for such a formula α and time point i it is checked (using the NP-oracle) if $\{ K\varepsilon \mid f(\varepsilon) \leq i \} \cup \{ \neg K\varepsilon \mid f(\varepsilon) > i \} \models_{S5} \alpha$. If so, α is added to $\text{label}(i)$, otherwise not. If $\mathcal{M} \not\models \varphi$, M halts in an accepting state (certainly $\mathcal{M} \not\models_{\min} \varphi$).

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Otherwise M guesses a TELC-model $\mathfrak{N}$ by guessing a function $g : A(\varphi) \rightarrow \mathbb{N} \cup \{\infty\}$ such that:

1. $f(\varepsilon) \leq g(\varepsilon)$

2. either $g(\varepsilon) \leq size(\mathfrak{M}) + 4 \cdot (length(\varphi))^2$ or $g(\varepsilon) = \infty$

3. For at least one $\varepsilon \in A(\varphi)$ we have $g(\varepsilon) > f(\varepsilon)$

Then we know that g induces a TELC-model $\mathfrak{N}$ with $\mathfrak{N} < \mathfrak{M}$ (if such a guess is not possible then we halt in a rejecting state because $\mathfrak{M} \models_{min} \varphi$). Next we use the labelling algorithm to check if $\mathfrak{N} \models \varphi$; if not we halt in a rejecting state, otherwise in an accepting state: $\mathfrak{N}$ is a smaller model of φ. It is clear that the algorithm works in polynomial time (using the NP-oracle). Lemma 6.13 ensures that there is a guess for which M halts in an accepting state iff $\mathfrak{M} \not\models_{min} \varphi$. Thus deciding if $\mathfrak{M} \not\models_{min} \varphi$ is in Σ_2^P so the complement is in Π_2^P.

Theorem 6.15

Deciding whether $\varphi \models_{min}^c \psi$ is in Π_3^P.

Proof

We will show that deciding whether *not* $\varphi \models_{min}^c \psi$ is in Σ_3^P by giving a nondeterministic Turing machine M with access to a Π_2^P-oracle. First M guesses a TELC-model $\mathfrak{M}$ based on φ by guessing a function $f : A(\varphi) \rightarrow \mathbb{N} \cup \{\infty\}$ such that either $g(\varepsilon) \leq 4 \cdot (length(\varphi))^2$ or $g(\varepsilon) = \infty$. Then it checks for $i \in \{0, .., 4 \cdot (length(\varphi))^2\}$ whether $\{ K\varepsilon \mid f(\varepsilon) \leq i \} \cup \{ \neg K\varepsilon \mid f(\varepsilon) > i \}$ is S5-consistent, using the oracle. If not it halts in a rejecting state (f does not induce a TELC-model). Now it uses the Π_2^P-oracle to determine if $\mathfrak{M} \models_{min} \varphi$. If not it halts in a rejecting state. Otherwise it uses a labelling algorithm to check if $\mathfrak{M} \models \psi$ (as in the proof of the previous lemma, using the Π_2^P-oracle for S5 consequence); if this is true M halts in a rejecting state, otherwise in an accepting state. The algorithm works in polynomial time, and lemma 6.12 ensures there is a guess for which M halts in an accepting state iff *not* $\varphi \models_{min}^c \psi$. So as this is in Σ_3^P, the complement is in Π_3^P.

7 Downward persistence

The entailment relation we have defined is a non-monotonic one, which means that one can have that $\alpha \models_{min}^c \gamma$ but not $\alpha \wedge \beta \models_{min}^c \gamma$ for some formulae α, β and γ (see also the appendix). We are interested in the class of formulae β which can be added to the premises without disturbing any of the conclusions. It will turn out that this is the class of downward persistent formulae. In the rest of this chapter we will investigate the class of

formulae which are preserved under decreasing or increasing (with respect to $\leq$) the models. Since our logic is essentially a temporalized version of S5, we will first look at S5 formulae preserved under going to larger and smaller models.

Definition 7.1 (Preservation under supermodels)

    i) An S5 formula φ is *preserved under supermodels* if for any two S5 models **M, N** such that $N \subseteq M$, and $m \in N$: if $(N, m) \models_{S5} \varphi$ then $(M, m) \models_{S5} \varphi$.

    ii) Define the class of S5 formulae **DIAM** by:

$$\text{DIAM} ::= p \mid \neg p \mid \text{DIAM} \wedge \text{DIAM} \mid \text{DIAM} \vee \text{DIAM} \mid M(\text{DIAM})$$

We want to prove that formulae in this class are the only ones (up to equivalence) which are preserved under supermodels:

Theorem 7.2

    An S5 formula φ is preserved under supermodels if and only if it is S5-equivalent to a formula in **DIAM**.

Proof

    It is easy to see that a formula equivalent to one in **DIAM** is preserved under supermodels. Now let φ be preserved under supermodels. Suppose $\text{Mod}(P) = \{ m_1, ..., m_n \}$. For $i = 1 .. n$ define $A(i) = \min \{ N \subseteq \text{Mod}(P) \mid (N, m_i) \models_{S5} \varphi \}$, where for a set $\mathfrak{B}$ of S5-models, $\min \mathfrak{B} = \{ N \in \mathfrak{B} \mid$ there is no $M \in \mathfrak{B}$ such that M is a proper subset of $N \}$. Define for $j = 1 .. n$:

$\alpha_j := \bigwedge \{ p \mid p \in P, m_j \models p \} \wedge \bigwedge \{ \neg p \mid p \in P, m_j \not\models p \}$ and for an S5-model M,

$\varphi_M = \bigwedge \{ M\alpha_j \mid j = 1 .. n$ and $m_j \in M \}$. It is easy to see that for an S5-model N: $M \subseteq N$ iff $(N, m) \models_{S5} \varphi_M$ for some/all $m \in N$. Now define for $j = 1 .. n$:

$\psi_j = \alpha_j \wedge \bigvee \{ \varphi_M \mid M \in A(j) \}$ if there exists and S5-model N with $(N, m_j) \models_{S5} \varphi$,

    $\perp$ otherwise

Note that $\perp$ is equivalent to $M(p \wedge \neg p)$. Now let $\psi = \bigvee \{ \psi_j \mid j = 1 .. n \}$. Then ψ is in **DIAM**, and ψ is equivalent to φ:

Suppose $(N, m_i) \models_{S5} \varphi$. Then there exists an $M \in A(i)$ with $M \subseteq N$, so $(N, m_i) \models_{S5} \varphi_M$ and $(N, m_i) \models_{S5} \alpha_i$, so $(N, m_i) \models_{S5} \psi_i$ and $(N, m_i) \models_{S5} \psi$.

Suppose that $(N, m_i) \models_{S5} \psi$. Then there exists a j such that $(N, m_i) \models_{S5} \psi_j$ but then $i = j$ and there exists $M \in A(i)$ such that $(N, m_i) \models_{S5} \varphi_M$, so $M \subseteq N$ and $(M, m_i) \models_{S5} \varphi$, but since φ is preserved under supermodels we have $(N, m_i) \models_{S5} \varphi$.

We are also interested in formulae preserved under taking submodels:

Definition 7.3 (Preservation under submodels)

i) An S5 formula φ is *preserved under submodels* if for any two S5 models $\mathbf{M, N}$ such that $\mathbf{N} \subseteq \mathbf{M}$, and $\mathbf{m} \in \mathbf{N}$: if $(\mathbf{M, m}) \vDash_{S5} \varphi$ then $(\mathbf{N, m}) \vDash_{S5} \varphi$.

ii) Define the class of S5 formulae **BOX** by:

$$\mathbf{BOX ::= p \mid \neg p \mid BOX \wedge BOX \mid BOX \vee BOX \mid K(BOX)}$$

We have:

Proposition 7.4

An S5 formula φ is preserved under submodels if and only if it is equivalent to a formula in **BOX**.

Proof

Easy.

Now we are ready to use these results to get a preservation result for TELC formulae. As we were interested in downward persistent formulae because of the link with the rule of Monotonicity for minimal consequence, the definition of downward persistence should use the corresponding notion of satisfaction of a formula in a model ($\mathcal{M} \vDash \varphi$). Also the notion of equivalence between formulae should be based on this notion.

Definition 7.5 (Upward and downward persistence)

i) A subjective TEL formula φ is called

downward persistent (dp) if for all TELC models $\mathcal{M}, \mathcal{N}$:

if $\mathcal{M} \leq \mathcal{N}$ and $\mathcal{N} \vDash \varphi$ then $\mathcal{M} \vDash \varphi$.

upward persistent (up) if for all TELC models $\mathcal{M}, \mathcal{N}$:

if $\mathcal{M} \leq \mathcal{N}$ and $\mathcal{M} \vDash \varphi$ then $\mathcal{N} \vDash \varphi$.

ii) Define $\mathbf{DP := DIAM \mid DP \wedge DP \mid DP \vee DP \mid F(DP) \mid G(DP) \mid P(DP) \mid H(DP)}$

$\mathbf{UP := BOX \mid UP \wedge UP \mid UP \vee UP \mid F(UP) \mid G(UP) \mid P(UP) \mid H(UP)}$

iii) For two subjective TEL formulae φ, ψ:

$\varphi \sim \psi \quad \Leftrightarrow \quad$ for all TELC models $\mathcal{M}$: $\mathcal{M} \vDash \varphi \Leftrightarrow \mathcal{M} \vDash \psi$

We can link the notion of $\sim$ with the notion $\vDash^c$: if we denote $\varphi \vDash^c \psi$ & $\psi \vDash^c \varphi$ by $\varphi \equiv^c \psi$ then: $\varphi \sim \psi \Leftrightarrow \Box \varphi \equiv^c \Box \psi$. This implies that $\sim$ is decidable.

Now we are ready to prove:

Theorem 7.6

A subjective TEL formula φ is downward persistent if and only if it is equivalent (in the sense of $\sim$) to a subjective formula in **DP**.

Proof

For a subjective (!) formula φ in **DP** one can easily prove that for all TELC models $\mathcal{M}, \mathcal{N}$ and $i \in \mathbb{N}$: if $\mathcal{M} \leq \mathcal{N}$ and $(\mathcal{N}, i) \vDash \varphi$ then $(\mathcal{M}, i) \vDash \varphi$. This implies that a formula equivalent (in the sense of $\sim$) to one in **DP** is dp.

Suppose φ is a subjective dp formula. We will construct its equivalent in **DP**. If there is no TELC model $\mathcal{M}$ such that $\mathcal{M} \vDash \varphi$ then φ is equivalent to $\bot$. Note that $\bot$ is equivalent to $M(p \wedge \neg p)$ which is a subjective formula in **DP**. Suppose we have a propositional signature **P** with m atoms. For a set of TELC-models $\mathcal{B}$ define $\max \mathcal{B} = \{ \mathcal{M} \in \mathcal{B} \mid \text{there is no } \mathcal{N} \in \mathcal{B} \text{ with } \mathcal{M} < \mathcal{N} \}$. If there is a TELC model $\mathcal{M}$ such that $\mathcal{M} \vDash \varphi$, then we define $\mathcal{A} = \max \{ \mathcal{M} \mid \mathcal{M} \vDash \varphi \}$. Suppose $\mathcal{M} \vDash \varphi$ and $\mathcal{M}$ stabilizes after time point $(2^m - 1) \cdot (2 \cdot \text{depth}(\varphi) + 1)$. Then we can delete points in sequences of more than $(2 \cdot \text{depth}(\varphi) + 1)$ identical states before the stabilizing point, without disturbing the truth of φ. If we do this for each such a sequence we end up with a model of φ which is larger (with respect to $\leq$) than $\mathcal{M}$ and stabilizes not later than $(2^m - 1) \cdot (2 \cdot \text{depth}(\varphi) + 1)$. Thus:

$\mathcal{A} = \max \{ \mathcal{M} \mid \mathcal{M} \vDash \varphi \text{ and } \mathcal{M} \text{ stabilizes not later than } (2^m - 1) \cdot (2 \cdot \text{depth}(\varphi) + 1) \}$. As the set we take the maximal elements of is non-empty and finite and the relation $<$ on TELC-models is transitive and irreflexive, $\mathcal{A}$ is non-empty and finite. Note that the argument used here (for maximal models) is similar to the one used for minimal models in the proof of lemma 5.6: there the idea was that a model which is too long can be enlarged (yielding a smaller model w.r.t. $\leq$), whereas here the idea is that if a model is too long, it can be reduced (yielding a bigger model w.r.t. $\leq$).

Suppose $\text{Mod}(\mathbf{P}) = \{ m_1, ..., m_n \}$ (with of course $n = 2^m$). Again define for $j = 1 .. n$:

$\alpha_j := \bigwedge \{ p \mid p \in \mathbf{P}, m_j \vDash p \} \wedge \bigwedge \{ \neg p \mid p \in \mathbf{P}, m_j \nvDash p \}$. Now define for $i = 1 ... n$ and for a TELC-model $\mathcal{M}$:

$n(i, \mathcal{M}) = \sup \{ j \in \mathbb{N} \mid m_i \in \mathcal{M}_j \}$ where $\sup \varnothing = -\infty$

Let $\psi(i, \mathcal{M}) = $
- $\Box (at_{n(i, \mathcal{M})} \to M\alpha_i)$ if $n(i, \mathcal{M}) \in \mathbb{N}$
- $\Box (M\alpha_i)$ if $n(i, \mathcal{M}) = \infty$
- $\top$ if $n(i, \mathcal{M}) = -\infty$

(Note that $\top$ is equivalent to $M(p \vee \neg p)$)

Furthermore, define $\psi_{\mathcal{M}} = \bigwedge \{ \psi(i, \mathcal{M}) \mid i = 1 ... n \}$. Now it can easily be proven that $\mathcal{N} \vDash \psi_{\mathcal{M}} \Leftrightarrow \mathcal{N} \leq \mathcal{M}$: the formulae $\psi(i, \mathcal{M})$ make sure that the valuation m_i is in $\mathcal{N}_t$ at least until the last time point s for which m_i is in $\mathcal{M}_s$. Finally, define:

$\psi = \bigvee \{ \psi_{\mathcal{M}} \mid \mathcal{M} \in \mathcal{A} \}$. Then ψ is in **DP** and $\varphi \sim \psi$:

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- Suppose $\mathcal{M} \vDash \varphi$. Then there exists $\mathcal{N} \in \mathcal{A}$ with $\mathcal{M} \leq \mathcal{N}$ (!), so $\mathcal{M} \vDash \psi_{\mathcal{N}}$ and $\mathcal{M} \vDash \psi$.

- Suppose $\mathcal{M} \vDash \psi$. Then there exists $\mathcal{N} \in \mathcal{A}$ with $\mathcal{M} \vDash \psi_{\mathcal{N}}$, so $\mathcal{M} \leq \mathcal{N}$ and as $\mathcal{N} \in \mathcal{A}$ we have $\mathcal{N} \vDash \varphi$, and φ was dp, so $\mathcal{M} \vDash \varphi$.

As in the case of S5-formulae we have:

Proposition 7.7

A subjective TEL formula φ is upward persistent if and only if it is equivalent (in the sense of ~) to a subjective formula in **UP**.

Proof

If φ is up then $\neg\Box\varphi$ is dp so by the previous theorem $\neg\Box\varphi \sim \psi$ for some $\psi \in$ **DP**. Then $\varphi \sim \neg\Box\psi$ and $\neg\Box\psi$ is equivalent to some formula in **UP**.

Furthermore, the property of downward persistence is decidable:

Proposition 7.8

For a subjective formula φ it is decidable whether φ is dp.

Proof

Suppose **P** contains n propositional atoms. We will prove that φ is dp iff for all TELC models $\mathcal{M}, \mathcal{N}$ with $\text{size}(\mathcal{M}) \leq (2^n - 1)\cdot(2\cdot\text{depth}(\varphi) + 1)$, $\text{size}(\mathcal{N}) \leq 2\cdot(2^n - 1)\cdot(2\cdot\text{depth}(\varphi) + 1)$: if $\mathcal{N} \leq \mathcal{M}$ and $\mathcal{M} \vDash \varphi$ then $\mathcal{N} \vDash \varphi$. This implies the decidability of dp.

Suppose φ is not dp, then there exist TELC-models $\mathcal{M}, \mathcal{N}$ with $\mathcal{N} < \mathcal{M}$, $\mathcal{M} \vDash \varphi$ and $\mathcal{N} \nvDash \varphi$. Now we construct a TELC-model $\mathcal{M}$ by deleting points from sequences of more than $2\cdot\text{depth}(\varphi) + 1$ identical states before the stabilizing point from $\mathcal{M}$ until each such sequence is at exactly $2\cdot\text{depth}(\varphi) + 1$ states long. Then $\text{size}(\mathcal{M}) \leq (2^n - 1)\cdot(2\cdot\text{depth}(\varphi) + 1)$, $\mathcal{N} < \mathcal{M}$ and $\mathcal{M} \vDash \varphi$ (by lemma 5.4). Now we construct a model $\mathcal{N}'$ using the following procedure: for each sequence of identical states in $\mathcal{N}$ after time point $(2^n - 1)\cdot(2\cdot\text{depth}(\varphi) + 1)$ but before the stabilizing point of $\mathcal{N}$ of length more than $(2\cdot\text{depth}(\varphi) + 1)$ points, we delete points until each such sequence has length $(2\cdot\text{depth}(\varphi) + 1)$. Then $\text{size}(\mathcal{N}) \leq 2\cdot(2^n - 1)\cdot(2\cdot\text{depth}(\varphi) + 1)$, $\mathcal{N} \nvDash \varphi$ (lemma 5.4), and it is easily checked that $\mathcal{N}' < \mathcal{M}$.

Similarly it is decidable whether a formula is up, and this gives us another way of verifying **TELC** theorems since $\vdash_{\textbf{TELC}} \varphi \Leftrightarrow \mathcal{M}^{ti} \vDash \varphi$ and φ is up, where $\mathcal{M}^{ti}$ is the *totally ignorant* model defined by $\mathcal{M}^{ti}_s = \text{Mod}(\textbf{P})$ for all s (note that for all

TELC-models $\mathfrak{N}$ we have $\mathfrak{M}^{ti} \leq \mathfrak{N}$; use soundness and completeness of **TELC**). Since **TELC**-theoremhood is co-NP-complete, we have as an immediate consequence:

Corollary 7.9

Upward persistence for subjective formulae is co-NP-hard.

For a valuation $\mathbf{m} \in \mathbf{Mod(P)}$ we can define the TELC-model $\mathfrak{M}^{\mathbf{m}}$ by $(\mathfrak{M}^{\mathbf{m}})_t = \{\mathbf{m}\}$ for all t. It is easy to see that such a model is maximal in the ordering $\leq$, and this gives us another way of checking **TELC** theorems since $\vdash_{\mathbf{TELC}} \varphi \Leftrightarrow \varphi$ is dp and $\mathfrak{M}^{\mathbf{m}} \vDash \varphi$ for all $\mathbf{m} \in \mathbf{Mod(P)}$. Furthermore we have: φ up and dp $\Leftrightarrow \vdash_{\mathbf{TELC}} \varphi$ or $\varphi \sim \bot$, which gives us:

Corollary 7.10

Checking whether a subjective formula is downward and upward persistent is co-NP-complete.

One of the reasons we were interested in formulae preserved under shrinking models was the link to monotonicity, which we can now prove:

Proposition 7.11

If a formula β is downward persistent then for all formulae α, γ:

If $\alpha \vDash_{\mathbf{min}}^{c} \gamma$ then $\alpha \wedge \beta \vDash_{\mathbf{min}}^{c} \gamma$

Proof

Suppose β is downward persistent and that for two formula α, γ we have $\alpha \vDash_{\mathbf{min}}^{c} \gamma$. Take a minimal model $\mathfrak{M}$ of $\alpha \wedge \beta$, then $\mathfrak{M} \vDash \alpha \wedge \beta$ so $\mathfrak{M} \vDash \alpha$, but $\mathfrak{M}$ is also minimal with respect to this property, for suppose $\mathfrak{N} \leq \mathfrak{M}$ and $\mathfrak{N} \vDash \alpha$, then since β is downward persistent, we also have $\mathfrak{N} \vDash \beta$, so $\mathfrak{N} \vDash \alpha \wedge \beta$, but since $\mathfrak{M}$ was a minimal model of $\alpha \wedge \beta$ we must have $\mathfrak{N} = \mathfrak{M}$. So $\mathfrak{M}$ is a minimal model of α so $\mathfrak{M} \vDash \gamma$. We have proved that $\alpha \wedge \beta \vDash_{\mathbf{min}}^{c} \gamma$.

We have given a syntactical characterization of downward persistent formulae, and the link with monotonicity, but it is also possible to characterize the downward persistent formulae using monotonicity (referring only to minimal entailment):

Proposition 7.12

A formula φ is downward persistent if and only if:

$\forall \alpha, \beta$: $\quad \alpha \vDash_{\mathbf{min}}^{c} \beta \quad \Rightarrow \quad \alpha \wedge \varphi \vDash_{\mathbf{min}}^{c} \beta$

Proof

The "only if" part is Proposition 7.11.

Suppose φ is not dp, then there exist TELC-models $\mathcal{M}, \mathcal{N}$ such that $\mathcal{N} < \mathcal{M}, \mathcal{M} \models \varphi$ but $\mathcal{N} \not\models \varphi$. For a TELC-model $\mathfrak{Q}$, define (using notation from the proof of theorem 7.6):

$$m(i, \mathfrak{Q}) = \min\{\, j \in \mathbb{N} \mid m_i \not\in \mathcal{M}_j \,\} \qquad \text{where } \min \varnothing = \infty, \text{ and}$$

$$\psi^{\mathfrak{Q}} = \{\, \Box(\, at_{m(i, \mathfrak{Q})} \to K(\neg\alpha_i)) \mid i = 1 \,..\, n,\ m(i, \mathfrak{Q}) < \infty \,\}$$

It is easy to see that for a TELC-model $\mathcal{K}$, $\mathcal{K} \models \psi^{\mathfrak{Q}}$ iff $\mathcal{K} \ge \mathfrak{Q}$. Now take $\alpha = (\psi^{\mathcal{N}} \wedge (\Box\varphi \to \psi^{\mathcal{M}}))$ and $\beta = \Diamond(\neg\varphi)$. Any TELC-model $\mathfrak{Q}$ of α has to satisfy $\mathfrak{Q} \ge \mathcal{N}$ and $\mathcal{N} \models \alpha$ ($\mathcal{N} \models \Box\varphi \to \psi^{\mathcal{M}}$ since $(\mathcal{N}, i) \not\models \Box\varphi$ for all $i \in \mathbb{N}$). Therefore $\mathcal{N} \models_{min} \alpha$ and it is the only minimal model of α. Since $\mathcal{N} \models \Diamond(\neg\varphi)$ we have $\alpha \models^c_{min} \beta$. Any TELC-model $\mathfrak{Q}$ of $\alpha \wedge \varphi$ has $\mathfrak{Q} \models \Box\varphi$ so $\mathfrak{Q} \models \psi^{\mathcal{M}}$ which implies $\mathfrak{Q} \ge \mathcal{M}$. Also $\mathcal{M} \models \alpha \wedge \varphi$ (since $\mathcal{N} < \mathcal{M}$) so $\mathcal{M}$ is the unique minimal model of $\alpha \wedge \varphi$, but $\mathcal{M} \not\models \beta$ and therefore we do not have $\alpha \wedge \varphi \models^c_{min} \beta$.

So this proposition says that a formula is downward persistent if and only if you can always be sure that adding this formula to your knowledge does not disturb any consequences.

8 Conclusions and further research

The logic TELC was proposed to describe the behaviour of a conservative reasoning agent. This logic was shown to be decidable, and a sound and complete axiomatization was given. Based on this logic we defined a notion of minimal entailment and studied the decidability and complexity. Furthermore, a syntactical characterization of formulae preserved under going to smaller models was presented and a link with monotonicity was given. As minimal entailment restricted to a sublanguage was shown to be decidable and complete for the complexity class where most nonmonotonic formalisms reside, a program for deciding minimal entailment can be used as a general "theorem prover" for these formalisms. The translation of default logic into TEL is already known ([ET93]); Further work is needed to find the translations for other nonmonotonic logics such as Autoepistemic Logic.

Although a decision procedure is sketched for minimal entailment, we would also like to have an axiomatization, although this might not be easy: it would immediately yield an axiomatization for default logic, which has not been given before.

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We have characterized the downward persistent formulae. We would like to find a similar result for the class of formulae which have no minimal models (like $F(Kp)$). These are the formulae which are in a sense not "honest" since they do not describe the reasoning behaviour of an agent properly.

The use of S5 as the logic to describe the knowledge of the agent at any point in time (allowing negative introspection) is not always realistic. If we use another modal logic such as S4, many results in this paper would have to be re-examined; in particular the complexity might be higher. A number of constructions used in the proofs will no longer work, and we might have to use methods like those in for instance [ABN95].

It would also be interesting to lift the restriction of conservativeness. This plays an important role in many of the proofs, but does not allow retraction, needed for belief revision (see for instance [AGM85]). In the non-conservative case, we would also like to extend the language with operators like Next, Since and Until.

Acknowledgements

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Appendix A

We have studied the notion of minimal entailment as a logical formalism. In this appendix we will look at minimal entailment as a nonmonotonic relation and study it using a number of rules as proposed in [GM94]. As $\vDash_{min}^c$ is not a cumulative relation (see [KLM90]) and therefore does not fit into the framework described in [KLM90] we will define a class in which it fits and try to give a characterization of this class.

A.1 Inference Rules

We would like to study the behaviour of our notion $\vDash_{min}^c$ in more detail, and we will do this by looking at inference rules for nonmonotonic inference relations proposed in the literature, specifically those mentioned in [GM94]. We will first list these rules, substituting our notion $\vDash_{min}^c$ for the nonmonotonic inference relation and the consequence relation $\vDash^c$ of **TELC** for the classical inference relation. Here $\vDash_{min}^c \alpha$ is an abbreviation for $\top \vDash_{min}^c \alpha$ where $\top$ is the constant true.

Inference rules

1. Supraclassicality: If $\alpha \vDash \gamma$ then $\alpha \vDash_{min}^c \gamma$
2. Left Logical Equivalence: If $\vDash \alpha \leftrightarrow \beta$ and $\alpha \vDash_{min}^c \gamma$ then $\beta \vDash_{min}^c \gamma$
3. Right Weakening: If $\vDash \beta \to \gamma$ and $\alpha \vDash_{min}^c \beta$ then $\alpha \vDash_{min}^c \gamma$
4. And: If $\alpha \vDash_{min}^c \beta$ and $\alpha \vDash_{min}^c \gamma$ then $\alpha \vDash_{min}^c \beta \wedge \gamma$
5. Weak Conditionalization: If $\alpha \vDash_{min}^c \beta$ then $\vDash_{min}^c \alpha \to \beta$
6. Conditionalization: If $\alpha \wedge \beta \vDash_{min}^c \gamma$ then $\alpha \vDash_{min}^c \beta \to \gamma$
7. Weak Rational Monotony: If $\nvDash_{min}^c \neg \alpha$ and $\vDash_{min}^c \alpha \to \beta$ then $\alpha \vDash_{min}^c \beta$
8. Rational Monotony: If $\alpha \nvDash_{min}^c \neg \beta$ and $\alpha \vDash_{min}^c \gamma$ then $\alpha \wedge \beta \vDash_{min}^c \gamma$
9. Cautious Monotony: If $\alpha \vDash_{min}^c \beta$ and $\alpha \vDash_{min}^c \gamma$ then $\alpha \wedge \beta \vDash_{min}^c \gamma$
10. Cut: If $\alpha \vDash_{min}^c \beta$ and $\alpha \wedge \beta \vDash_{min}^c \gamma$ then $\alpha \vDash_{min}^c \gamma$
11. Consistency Preservation: If $\alpha \vDash_{min}^c \bot$ then $\alpha \vDash \bot$
12. Cumulativity: If $\alpha \vDash_{min}^c \beta$ and $\beta \vDash \alpha$ then ($\alpha \vDash_{min}^c \gamma \Leftrightarrow \beta \vDash_{min}^c \gamma$)
13. Reciprocity: If $\alpha \vDash_{min}^c \beta$ and $\beta \vDash_{min}^c \alpha$ then ($\alpha \vDash_{min}^c \gamma \Leftrightarrow \beta \vDash_{min}^c \gamma$)
14. Or: If $\alpha \vDash_{min}^c \gamma$ and $\beta \vDash_{min}^c \gamma$ then $\alpha \vee \beta \vDash_{min}^c \gamma$
15. Disjunctive Rationality: If $\alpha \vee \beta \vDash_{min}^c \gamma$ then $\alpha \vDash_{min}^c \gamma$ or $\beta \vDash_{min}^c \gamma$

Proposition A.1.1 (Inference rules for $\vdash^c_{min}$)

Of the rules stated above, $\vdash^c_{min}$ satisfies only Supraclassicality, Left Logical Equivalence, Right Weakening, And and Cut.

Proof

We will first prove that the above rules are satisfied:

- Supraclassicality: Any minimal model of α is a model of α and with $\alpha \models \gamma$ also a model of γ.

- Left Logical Equivalence: Suppose $\mathcal{M} \models_{min} \beta$. Then $\mathcal{M} \models_{min} \alpha$: $\mathcal{M}$ is a model of β and, as $\models \alpha \leftrightarrow \beta$, also a model of α. If $\mathcal{N} < \mathcal{M}$ and $\mathcal{N} \models \alpha$ then by $\models \alpha \leftrightarrow \beta$ also $\mathcal{N} \models \beta$, which is impossible since $\mathcal{M} \models_{min} \beta$. As $\mathcal{M} \models_{min} \alpha$ and $\alpha \vdash^c_{min} \gamma$ we have $\mathcal{M} \models \gamma$. Therefore $\beta \vdash^c_{min} \gamma$.

- Right Weakening: Suppose $\mathcal{M} \models_{min} \alpha$ then $\mathcal{M} \models \beta$ and using $\models \beta \rightarrow \gamma$ also $\mathcal{M} \models \gamma$. We have $\alpha \vdash^c_{min} \gamma$.

- And: Suppose $\mathcal{M} \models_{min} \alpha$. Then $\mathcal{M} \models \beta$ and $\mathcal{M} \models \gamma$ so $\mathcal{M} \models \beta \wedge \gamma$, so $\alpha \vdash^c_{min} \beta \wedge \gamma$.

- Cut: Suppose $\mathcal{M} \models_{min} \alpha$. Then $\mathcal{M} \models \alpha$ and by $\alpha \vdash^c_{min} \beta$ also $\mathcal{M} \models \beta$, so $\mathcal{M} \models \alpha \wedge \beta$ and even $\mathcal{M} \models_{min} \alpha \wedge \beta$, for if $\mathcal{N} < \mathcal{M}$ and $\mathcal{N} \models \alpha \wedge \beta$, then $\mathcal{N} \models \alpha$, which is impossible since $\mathcal{M} \models_{min} \alpha$. Therefore $\mathcal{M} \models \gamma$, so $\alpha \vdash^c_{min} \gamma$.

Now for the rules which are not satisfied:

- Weak Conditionalization: Take $\alpha = P(\neg Ka)$ and $\beta = Ka$ (with some $a \in P$). Since there is no model $\mathcal{M}$ such that $\mathcal{M} \models \alpha$, there are no minimal models of α, so $\alpha \vdash^c_{min} \beta$. Define the totally ignorant model $\mathcal{M}^{ti}$ by $\mathcal{M}^{ti}_s = Mod(P)$ for all $s \in \mathbb{N}$. Then it is easy to see that this is the (only) minimal model of T. Furthermore $(\mathcal{M}^{ti}, 1) \models \alpha$, but $(\mathcal{M}^{ti}, 1) \not\models \beta$, so $\mathcal{M}^{ti} \not\models \alpha \rightarrow \beta$, so $\not\vdash^c_{min} \alpha \rightarrow \beta$.

- Conditionalization: Number 5. is a special case of this one (take $\alpha = T$).

- Weak Rational Monotony: Take $\alpha = H(Ka)$, $\beta = \neg Ka$, then $(\mathcal{M}^{ti}, 0) \models \alpha$ so $\mathcal{M}^{ti} \not\models \neg \alpha$ so $\not\vdash^c_{min} \neg \alpha$. Also $\mathcal{M}^{ti} \models \beta$, so $\vdash^c_{min} \alpha \rightarrow \beta$. Define $\mathcal{M}$ by $\mathcal{M}_t = \{ m \in Mod(P) \mid m \models a \}$. Then $\mathcal{M}$ is a minimal model of α but not a model of β.

- Rational Monotony: Take $\alpha = Ka$, $\beta = Kb \vee H(\perp)$, $\gamma = \neg F(Kb)$.

The details are left to the reader from here on.

- Cautious Monotony: Take $\alpha = F(Ka)$, $\beta = Ka$, $\gamma = \neg Ka$.

- Consistency Preservation: Take $\alpha = F(Ka)$.

- Cumulativity: Take $\alpha = F(Ka)$, $\beta = G(Ka)$ and $\gamma = \perp$.

- Reciprocity: Take $\alpha = F(Ka)$, $\beta = Ka$ and $\gamma = \perp$.

- Or: Take $\alpha = P(T)$, $\beta = \neg P(T)$ and $\gamma = \perp$.

- Disjunctive Rationality: Take $\alpha = H(\perp) \vee Ka$, $\beta = P(T) \vee Ka$, $\gamma = \neg Ka$.

These results indicate that our consequence operator is not a very well-behaved notion and one of the reasons for this is that there is an implicit modal operator in the definition of minimal consequence. We take the minimal elements of the set of models of ψ (meaning that *always* ψ must hold), and check if they are all models of ϕ (again meaning that *always* ϕ must hold). This is a source of counterexamples, for instance for rule 14 (Or): although there are no models where always $P(T)$ holds, nor models where always $\neg P(T)$ holds, there are models (in fact all models) where always $P(T) \vee \neg P(T)$ holds. In order to get a better behaved notion, it would therefore make sense to remove this implicit modal operator (as is argued on p. 190 of [KLM90]):

Definition A.1.2 (Minimal anchored models and entailment)

    i) A TELC-model $\mathcal{M}$ is a *minimal (conservative) anchored model (minimal a-model) of* ϕ, denoted $\mathcal{M} \vDash^a_{min} \phi$, if $(\mathcal{M}, 0) \vDash \phi$ and for all conservative models $\mathcal{N}$, if $(\mathcal{N}, 0) \vDash \phi$ and $\mathcal{N} \leq \mathcal{M}$ then $\mathcal{N} = \mathcal{M}$.

    ii) For TEL-formulae ϕ, ψ, we say ϕ is a *minimal (conservative) anchored consequence of* ψ or ψ *minimally anchoredly entails* ϕ, denoted $\psi \vDash^{c, a}_{min} \phi$ if for all minimal (conservative) a-models $\mathcal{M}$ of ψ it holds $(\mathcal{M}, 0) \vDash \phi$.

Notice that whereas the old notion was not a "preferential logic" in the sense of Shoham [Sh88], the new one is. This new definition is also more in line with the definition of classical consequence. Fortunately, these two alternative notions of entailment can be related easily:

Proposition A.1.3

    For all formulae ϕ, ψ we have:

$$\psi \vDash^c_{min} \phi \quad\quad \Leftrightarrow \quad\quad \Box\psi \vDash^{c, a}_{min} \Box\phi$$

$$\psi \vDash^{c, a}_{min} \phi \quad\quad \Leftrightarrow \quad\quad (H(\bot) \to \psi) \vDash^c_{min} (H(\bot) \to \phi)$$

As these reductions are polynomial, the new notion of minimal (conservative) anchored consequence inherits the properties of minimal (conservative) consequence described in the sections 4, 5 and 6 regarding decidability and complexity.

Let us look at the rules satisfied by this new notion:

Proposition A.1.4 (Inference rules for $\vDash^{c, a}_{min}$)

    Of the rules stated above, $\vDash^{c, a}_{min}$ satisfies only Supraclassicality, Left Logical Equivalence, Right Weakening, And, Weak Conditionalization, Conditionalization, Weak Rational Monotony, Cut and Or.

Proof

We will again first prove the rules which are satisfied:

- Supraclassicality, Left Logical Equivalence, Right Weakening and And are easy.

- Weak Conditionalization is implied by Conditionalization.

- Conditionalization is implied by Supraclassicality, Right Weakening, Left Logical Equivalence and Or ([KLM90], Lemma 5.2)

- Weak Rational Monotony: Suppose $\nvdash_{min}^{c,\,a} \alpha$. Then $(\mathcal{M}^{ti}, 0) \nvDash \neg\alpha$ so $(\mathcal{M}^{ti}, 0) \vDash \alpha$ and $(\mathcal{M}^{ti}, 0) \vDash \alpha \to \beta$ so $(\mathcal{M}^{ti}, 0) \vDash \beta$. Since $\mathcal{M}^{ti}$ is the only minimal a-model of α we have $\alpha \vdash_{min}^{c,\,a} \beta$.

- Cut is implied by Right Weakening, Conditionalization and And ([KLM90], Lemma 5.3).

- Or: Suppose $\mathcal{M} \vDash^{a}_{min} \alpha \vee \beta$. Then $(\mathcal{M}, 0) \vDash \alpha$ or $(\mathcal{M}, 0) \vDash \beta$ and even (!) $\mathcal{M} \vDash^{a}_{min} \alpha$ or $\mathcal{M} \vDash^{a}_{min} \beta$, therefore $(\mathcal{M}, 0) \vDash \gamma$, so $\alpha \vee \beta \vdash_{min}^{c,\,a} \gamma$.

Now the rules which are not satisfied:

- Rational Monotony: Take $\alpha = Ka \vee Kb$, $\beta = Kb \to K\neg a$ and $\gamma = \neg K\neg a$. Consider the following three models $\mathcal{M}^1, \mathcal{M}^2$ and $\mathcal{M}^3$, defined by $\mathcal{M}^1_t = \{ m \mid m \vDash a \}$ for all t, $\mathcal{M}^2_t = \{ m \mid m \vDash b \}$ for all t and $\mathcal{M}^3_t = \{ m \mid m \vDash \neg a \wedge b \}$ for all t. Then $\mathcal{M}^1 \vDash^{a}_{min} \alpha$ and $(\mathcal{M}^1, 0) \vDash \beta$ so $\alpha \nvdash_{min}^{c,\,a} \neg\beta$. Furthermore, $\mathcal{M}^1$ and $\mathcal{M}^2$ are the only two minimal a-models of α, and $(\mathcal{M}^1, 0) \vDash \gamma$ and $(\mathcal{M}^2, 0) \vDash \gamma$ so $\alpha \vdash_{min}^{c,\,a} \gamma$. But $\mathcal{M}^3$ is a minimal a-model of $\alpha \wedge \beta$ with $(\mathcal{M}^3, 0) \vDash K\neg a$, so $(\mathcal{M}^3, 0) \nvDash \gamma$, so $\alpha \wedge \beta \nvdash_{min}^{c,\,a} \gamma$.

- Cautious Monotony: Take $\alpha = F(Kp)$, $\beta = Kp$ and $\gamma = \perp$.

- Consistency Preservation: Take $\alpha = F(Kp)$, this has no minimal a-model (See also 9.).

- Cumulativity: Take $\alpha = F(Kp)$, $\beta = Kp$ and $\gamma = \perp$.

- Reciprocity: Take $\alpha = F(Kp)$, $\beta = Kp$ and $\gamma = \perp$.

- Disjunctive Rationality: Take $\alpha = Ka \vee F(Kb)$, $\beta = Kb \vee F(Ka)$ and $\gamma = \perp$.

So although minimal anchored consequence behaves better then minimal consequence, it still does not satisfy all the rules, and the most important reason for that is that Consistency Preservation does not hold: there are formulae which are classically satisfiable but have no minimal model. A formula like $F(Kp)$ is in a sense not honest, because it says that *sometimes* we will know p, but does not (by itself) specify *when* we will know p. The tuple of the set of conservative models with our preference relation does not satisfy the condition of *smoothness* (or *well-foundedness* or *non-stopperedness*) (see [KLM90]), so it is not a cumulative model, and therefore the consequence relation

based on it cannot be supposed to be cumulative (see the representation result in [KLM90] for cumulative relations).

As said before, with our new notion of anchored consequence we have a preferential logic in Shoham's sense. In the above, we have interpreted $\vDash_{min}^{c,(a)} \alpha$ as $\top \vDash_{min}^{c,(a)} \alpha$ and (since $\mathfrak{M}^{ti}$ is the only minimal model of $\top$), this essentially reduces $\vDash_{min}^{c,(a)} \alpha$ to $\mathfrak{M}^{ti} \vDash \alpha$, respectively $(\mathfrak{M}^{ti}, 0) \vDash \alpha$. In Shoham's book ([Sh88]), another definition is given: $\vDash_{min}^{c,(a)} \alpha$ iff $\neg\alpha$ is not preferentially satisfiable (i.e. has no minimal model). If we use this definition, we get strange results with the non-anchored case: since neither $P(\top)$ nor $\neg P(\top)$ are preferentially satisfiable we would get both $\vDash_{min}^{c} P(\top)$ and $\vDash_{min}^{c} \neg P(\top)$. The only two rules affected by this change are Weak Conditionalization and Weak Rational Monotony and they are still not met (counterexamples: $\alpha = \mathbf{Ka}$, $\beta = \neg\mathbf{Ka}$ and $\alpha = \mathbf{Ka}$, $\beta = P(\top)$). In the case of anchored minimal consequence we can never have $\vDash_{min}^{c,a} \varphi$ and $\vDash_{min}^{c,a} \neg\varphi$: for the model $\mathfrak{M}^{ti}$, either $(\mathfrak{M}^{ti}, 0) \vDash \varphi$ or $(\mathfrak{M}^{ti}, 0) \vDash \neg\varphi$, and since $\mathfrak{M}^{ti}$ is the smallest model of all, it is either a minimal a-model of φ (in the first case) or a minimal a-model of $\neg\varphi$, so either $\nvDash_{min}^{c,a} \varphi$ or $\nvDash_{min}^{c,a} \neg\varphi$. The effect on the rules is that Weak Conditionalization no longer holds ($\alpha = F(\mathbf{Ka})$ and $\beta = \neg\mathbf{Ka}$) and Weak Rational Monotony still holds. (Suppose $\alpha \nvDash_{min}^{c,a} \beta$ then there exists $\mathfrak{M}$ such that $\mathfrak{M} \vDash^{a}_{min} \alpha$ and $(\mathfrak{M}, 0) \nvDash \beta$ so $(\mathfrak{M}, 0) \vDash \neg\beta$. So $(\mathfrak{M}, 0) \vDash \alpha \wedge \neg\beta$ and even $\mathfrak{M} \vDash^{a}_{min} \alpha \wedge \neg\beta$, so α and $\alpha \wedge \neg\beta$ are preferentially satisfiable so $\nvDash_{min}^{c,a} \neg \alpha$ and $\nvDash_{min}^{c,a} \alpha \rightarrow \beta$. Note that the condition $\nvDash_{min}^{c,a} \neg\alpha$ is not even needed.)

The consequence relations which are studied in [KLM90] all satisfy the rule of Cautious Monotonicity, and it is argued there (but also by others, e.g. Gabbay) that a system which does not satisfy it, should not be considered a logical system. But even there it is said: "This appreciation probably only reflects the fact that, so far, we do not know anything interesting about weaker systems" ([KLM90], p.176). Since we have defined a consequence relation which does not satisfy Cautious Monotonicity, we are interested in such systems, and in the next chapter we will look for representation results analogous to those in [KLM90] for systems which do not satisfy this rule.

A.2 Non-cumulative preferential reasoning

In the previous chapter we have seen which rules are satisfied by minimal anchored consequence. Of these rules, (Weak) Conditionalization is implied by Supraclassicality, Right Weakening , Left Logical Equivalence and Or ([KLM90], p. 191); furthermore, Right Weakening, Conditionalization and And imply Cut ([KLM90], p. 191). We will therefore focus on the rules Supraclassicality, Left Logical Equivalence, Right

Weakening, And, Weak Rational Monotony and Or. First we will look at the first four of these, after which we will add the Or rule.

The rules of Supraclassicality, Left Logical Equivalence, Right Weakening and And imply that the set of consequences of a formula α is a deductively closed set of formulae containing α, and that this set is the same for formulae equivalent to α. We can give semantics to such consequence relations, borrowing from the theory of nonmonotonic model operators (see e.g. [DH94]). In a way analogous to the techniques in [KLM90] we define *inference models* to base consequence relations on. We make the same assumptions as in [KLM90]:

Assumptions

- we have a language, L, of well-formed formulae, closed under the classical propositional connectives.
- the semantics of this language is given by a set $\mathcal{U}$, the elements of which will be called worlds, and a binary relation $\models$ of satisfaction between worlds and formulae, which satisfies, for $\alpha, \beta \in L, u \in \mathcal{U}$:

  (1) $u \models \neg\alpha$ iff $u \not\models \alpha$

  (2) $u \models \alpha \vee \beta$ iff $u \models \alpha$ or $u \models \beta$
- we have compactness: A set of formulae is satisfiable if all of its finite subsets are.

Note that we do not have compactness for TELC.

Definition A.2.1 (Inference models)

An *inference model* W is a triple $<S, l, P>$ where S is a set (the set of *states*), l is a function $l: S \rightarrow \mathcal{U}$ which assigns a world to every state and P is a function $P: L \rightarrow \mathcal{P}(S)$ which assigns to every formula a set of states (the *intended* states) such that:

1. $P(\alpha) \subseteq \{ s \in S \mid l(s) \models \alpha \}$
2. $\models \alpha \leftrightarrow \beta$ implies $P(\alpha) = P(\beta)$

Definition A.2.2 (Consequence relation of an inference model)

Given an inference model $W = <S, l, P>$ we define the associated consequence relation $\mathrel{\vdash}_W$ by: $\alpha \mathrel{\vdash}_W \beta$ iff $P(\alpha) \models \beta$, where $P(\alpha) \models \beta$ iff for all $s \in P(\alpha)$ we have $l(s) \models \beta$.

Before we prove a representation theorem, we need one more definition from [KLM90]:

Definition A.2.3 (**Normal world**)

A world $u \in \mathcal{U}$ is called a *normal world for* α with respect to a consequence relation $\vdash\!\!\!\sim$ if for all $\beta \in L$ for which $\alpha \vdash\!\!\!\sim \beta$ we have $u \vDash \beta$.

So a normal world for α satisfies all consequences of α. Now we will show the representation result for consequence relations satisfying the above mentioned four rules:

Proposition A.2.4

A consequence relation $\vdash\!\!\!\sim$ satisfies Supraclassicality, Left Logical Equivalence, Right Weakening and And if and only if $\vdash\!\!\!\sim \; = \; \vdash\!\!\!\sim_W$ for some inference model W.

Proof

The proof from right to left is easy and left to the reader. For the other direction define $W = \langle S, l, P \rangle$ with $S = \mathcal{U}$, l is the identity function and for all α:
$P(\alpha) = \{\, m \in \mathcal{U} \mid m$ is a normal world for α with respect to $\vdash\!\!\!\sim \,\}$. Since $\alpha \vDash \alpha$, by supraclassicality we have that $\alpha \vdash\!\!\!\sim \alpha$ so $P(\alpha) \subseteq \{\, s \in S \mid l(s) \vDash \alpha \,\}$. Furthermore by Left Logical Equivalence we have that if $\vDash \alpha \leftrightarrow \beta$ then $\alpha \vdash\!\!\!\sim \gamma$ iff $\beta \vdash\!\!\!\sim \gamma$ so a world m is a normal world for α iff it is a normal world for β, so $P(\alpha) = P(\beta)$. Thus, W is an inference model.

Now we have to prove that for all $\alpha, \beta \in L$: $\alpha \vdash\!\!\!\sim \beta$ iff $\alpha \vdash\!\!\!\sim_W \beta$.

- Suppose $\alpha \vdash\!\!\!\sim \beta$ and choose $m \in P(\alpha)$ then m is normal for α so $m \vDash \beta$. So $P(\alpha) \vDash \beta$, so $\alpha \vdash\!\!\!\sim_W \beta$.

- Suppose $\alpha \not\vdash\!\!\!\sim \beta$. We will prove that $\{\neg\beta\} \cup \{\gamma \mid \alpha \vdash\!\!\!\sim \gamma\}$ is satisfiable. Suppose not, then by the compactness assumption there is a finite set $D \subseteq \{\gamma \mid \alpha \vdash\!\!\!\sim \gamma\}$ such that $D \cup \{\neg\beta\}$ is unsatisfiable, so we have $\vDash \wedge D \rightarrow \beta$. But for all $\gamma \in D$ we have $\alpha \vdash\!\!\!\sim \gamma$ and so, using the And rule, we have $\alpha \vdash\!\!\!\sim \wedge D$ and with Right Weakening we have $\alpha \vdash\!\!\!\sim \beta$ contrary to our assumption. Thus there is an $m \in \mathcal{U}$ such that $m \vDash \{\neg\beta\} \cup \{\gamma \mid \alpha \vdash\!\!\!\sim \gamma\}$, but then m is a normal world for α, so $m \in P(\alpha)$ and $m \not\vDash \beta$, so $P(\alpha) \not\vDash \beta$ so $\alpha \not\vdash\!\!\!\sim_W \beta$ which was to be proven.

Now we want to look at consequence relations which satisfy in addition to the above four rules also the rule Or, and we will see that we can get a similar representation result. For this, we will first need a small lemma:

Lemma A.2.5

If a consequence relation $\vdash\!\!\!\sim$ satisfies Supraclassicality, Left Logical Equivalence, Right Weakening, And and Or, then for any α, β and γ:

If $\vDash \alpha \rightarrow \beta$ and $\alpha \vdash\!\!\!\sim \gamma$ then $\beta \vdash\!\!\!\sim \alpha \rightarrow \gamma$.

Proof

From $\vDash \alpha \to \beta$ we have $\vDash \alpha \leftrightarrow (\beta \wedge \alpha)$ and with Left Logical Equivalence we have $\beta \wedge \alpha \mathrel{\vdash\!\!\!-} \gamma$. Using Conditionalization (which follows from the other rules), we get $\beta \mathrel{\vdash\!\!\!-} \alpha \to \gamma$.

Proposition A.2.6

A consequence relation $\mathrel{\vdash\!\!\!-}$ satisfies Supraclassicality, Left Logical Equivalence, Right Weakening, And and Or if and only if $\mathrel{\vdash\!\!\!-} = \mathrel{\vdash\!\!\!-}_W$ for some inference model $W = \langle S, l, P \rangle$ with P such that for all α, β:

If $\vDash \alpha \to \beta$ then $P(\beta) \cap \{ s \in S \mid l(s) \vDash \alpha \} \subseteq P(\alpha)$.

Proof

- Going from right to left we have to prove that $\mathrel{\vdash\!\!\!-}_W$ satisfies Or: suppose $\alpha \mathrel{\vdash\!\!\!-}_W \gamma$ and $\beta \mathrel{\vdash\!\!\!-}_W \gamma$. Choose $s \in P(\alpha \vee \beta)$. Then $l(s) \vDash \alpha \vee \beta$, so $l(s) \vDash \alpha$ or $l(s) \vDash \beta$. Suppose $l(s) \vDash \alpha$. As $\vDash \alpha \to (\alpha \vee \beta)$ we have $s \in P(\alpha)$ and as $\alpha \mathrel{\vdash\!\!\!-}_W \gamma$ we have $l(s) \vDash \gamma$. So $P(\alpha \vee \beta) \vDash \gamma$, so $\alpha \vee \beta \mathrel{\vdash\!\!\!-}_W \gamma$ as required.

- For the other direction we take the same definition for W as in the previous proposition, and we only have to prove the extra condition. So suppose $\vDash \alpha \to \beta$ and take $m \in P(\beta) \cap \{ s \in S \mid l(s) \vDash \alpha \}$. Suppose that $\alpha \mathrel{\vdash\!\!\!-} \gamma$. Then lemma A.2.5 gives us that $\beta \mathrel{\vdash\!\!\!-} \alpha \to \gamma$ and as m is a normal world for β we have $m \vDash \alpha \to \gamma$, but as $m \in \{ s \in S \mid l(s) \vDash \alpha \}$ we have $m \vDash \alpha$ and so $m \vDash \gamma$. We have proved that m is a normal α world, so $m \in P(\alpha)$.

So we have a representation result, but it uses a rather strange condition. We would like to use a preference relation on worlds instead of a model operator, as in [KLM90].

Definition A.2.7 (Non-cumulative preference model)

A *non-cumulative preference model* W is a triple $\langle S, l, \prec \rangle$ where S is a set (the set of *states*), l is a function $l: S \to \mathcal{U}$ which assigns a world to every state and $\prec$ is a binary relation on S.

Definition A.2.8 (Consequence relation of a non-cumulative preference model)

Given a non-cumulative preference model $W = \langle S, l, \prec \rangle$ we define the associated consequence relation $\mathrel{\vdash\!\!\!-}_W$ by: $\alpha \mathrel{\vdash\!\!\!-}_W \beta$ iff $m \vDash \beta$ for all $m \in \min_R \{ s \in S \mid l(s) \vDash \alpha \}$ where $\min_R A = \{ a \in A \mid$ there exists no $b \in A$ such that $bRa \}$.

We want to prove that the class of these consequence relations is exactly the class of consequence relations which satisfy the basic four rules and the Or rule. First we show

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that the class of consequence relations of a non-cumulative preference model satisfies the five rules:

Lemma A.2.9 (Soundness)

For any non-cumulative preference model **W**, its associated consequence relation satisfies Supraclassicality, Left Logical Equivalence, Right Weakening, And and Or.

Proof

Let $W = \langle S, l, \prec \rangle$. We will show that $W' = \langle S, l, P \rangle$ with $P(\alpha) = \min_R \{ s \in S \mid l(s) \vDash \alpha \}$ defines an inference model which satisfies the condition in proposition A.2.6. It is then easy to see that $\vdash_W = \vdash_{W'}$ and that W' is an inference model. Now suppose $\vDash \alpha \to \beta$ and take $s \in P(\beta) \cap \{ s \in S \mid l(s) \vDash \alpha \}$, so $l(s) \vDash \alpha$. Suppose $s \notin P(\alpha)$. Then there exists $t \in S$ such that $l(t) \vDash \alpha$ and tRs, but since $\vDash \alpha \to \beta$ we have $l(t) \vDash \beta$ and tRs, which contradicts the assumption that $s \in P(\beta)$. So $s \in P(\alpha)$ and the condition is satisfied, so proposition A.2.6 gives that $\vdash_{W'}$ satisfies the five rules.

We now intend to show that for any consequence relation satisfying the five rules we can define a non-cumulative preference model with a consequence relation identical to the one we started with:

Definition A.2.10

Given a consequence relation $\vdash$ which satisfies Supraclassicality, Left Logical Equivalence, Right Weakening, And and Or we define its associated non-cumulative preference model $W = \langle S, l, \prec \rangle$ by:

$S = \{ \langle m, \alpha \rangle \mid m \in \mathcal{U}, \alpha \in L \}$,

$l(\langle m, \alpha \rangle) = m$ and

$\langle n, \alpha \rangle \prec \langle m, \beta \rangle$ iff $m \vDash \alpha$, m not normal for α w.r.t. $\vdash$, $n \vDash \alpha$ and
               (if m is normal for β w.r.t. $\vdash$ then $n \nVdash \beta$).

We want to show that the consequence relation based on this model is identical to the one we started with:

Lemma A.2.11

For a consequence relation $\vdash$ which satisfies Supraclassicality, Left Logical Equivalence, Right Weakening, And and Or, its associated non-cumulative preference model W induces a consequence relation $\vdash_W$ with $\vdash = \vdash_W$.

Proof

Again defining $P(\alpha) = \min_{\prec} \{ s \in S \mid l(s) \vDash \alpha \}$ we want to show that
$l[P(\alpha)] = \{ m \in \mathcal{U} \mid m$ is normal for α w.r.t. $\vdash \}$. Suppose we have $\langle m, \gamma \rangle$ with $m \vDash \alpha$
and m is not normal for α. We distinguish two cases:

- m is not normal for γ, then we have $\langle m, \alpha \rangle \prec \langle m, \gamma \rangle$, so $\langle m, \gamma \rangle \notin P(\alpha)$.
- m is normal for γ. Now we want to find a world n such that $n \vDash \alpha$ but $n \nvDash \gamma$. So
suppose $\vDash \alpha \to \gamma$. Then as m is not normal for α there exists a β such that $\alpha \vdash \beta$
and $m \nvDash \beta$. Then with $\vDash \alpha \to \gamma$ we have $\vDash \alpha \leftrightarrow (\gamma \wedge \alpha)$ and with Left Logical
Equivalence we have $\gamma \wedge \alpha \vdash \beta$ and with Conditionalization (which follows from the
five rules) we have $\gamma \vdash \alpha \to \beta$, but as m is normal for γ we have $m \vDash \alpha \to \beta$ and
$m \vDash \alpha$ so $m \vDash \beta$, in contradiction with our assumption. Therefore we have $\nvDash \alpha \to \gamma$, so
there exists an n with $n \vDash \alpha$ and $n \nvDash \gamma$. Then we have $\langle n, \alpha \rangle \prec \langle m, \gamma \rangle$ and therefore
$\langle m, \gamma \rangle \notin P(\alpha)$.

So if $m \in l[P(\alpha)]$ then m is normal for α.

Now suppose m is normal for α. Then $l(\langle m, \alpha \rangle) \vDash \alpha$ (using Supraclassicality).
Furthermore, if we have $\langle n, \gamma \rangle \prec \langle m, \alpha \rangle$, then as m is normal for α we must have
$n \nvDash \alpha$, so $\langle m, \alpha \rangle \in P(\alpha)$, so $m \in l[P(\alpha)]$.

We will now use a lemma from [KLM90], stating:

If $\vdash$ satisfies Reflexivity, Right Weakening and And (where Reflexivity means that
$\alpha \vdash \alpha$ for all α) then all normal worlds for α satisfy β iff $\alpha \vdash \beta$.

So then we get:

$\alpha \vdash_W \beta$ iff $l[P(\alpha)] \vDash \beta$ iff $\{ m \in \mathcal{U} \mid m$ is normal for α w.r.t. $\vdash \} \vDash \beta$ iff $\alpha \vdash \beta$.
Indeed $\vdash = \vdash_W$ as required.

Theorem A.2.12

A consequence relation $\vdash$ satisfies Supraclassicality, Left Logical Equivalence,
Right Weakening, And and Or if and only if $\vdash = \vdash_W$ for some non-cumulative
preference model $W = \langle S, l, \prec \rangle$.

Proof

Follows immediately from lemma A.2.11 and lemma A.2.9

We can even restrict the relation $\prec$ to an irreflexive one. Each reflexive point s in a
non-cumulative preference model $W = \langle S, l, \prec \rangle$ can be replaced by an infinite sequence
(s_i) of states labelled with the model $l(s)$ and $s_i \prec s_{i+1}$. Each element $s \prec t$ has to be
replaced by $s_i \prec t$ for all i, and the same for elements $t \prec s$. It is easy to see that $\vdash_W$ is
not affected by these changes.

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We would like to find a similar representation result when the rule of Weak Rational Monotony is included, and we would like to find other rules (possibly including temporal operators) to be able to characterize our notion of minimal consequence further.

A.3 Downward persistence for minimal anchored entailment

As said before, decidability and complexity results transfer from minimal entailment to minimal anchored entailment. We will now look at the persistence results of section 7. First we have to change the notion of satisfaction of a formula in a model (now $(\mathcal{M}, 0) \models \varphi$) and equivalence between formulae.

Definition A.3.1 (Upward and downward persistence)

    i) A subjective TEL formula φ is called

        downward persistent[2] (dp²) if for all TELC models $\mathcal{M}, \mathcal{N}$:

            if $\mathcal{M} \leq \mathcal{N}$ and $(\mathcal{N}, 0) \models \varphi$ then $(\mathcal{M}, 0) \models \varphi$.

        upward persistent[2] (up²) if for all TELC models $\mathcal{M}, \mathcal{N}$:

            if $\mathcal{M} \leq \mathcal{N}$ and $(\mathcal{M}, 0) \models \varphi$ then $(\mathcal{N}, 0) \models \varphi$.

    ii) For two subjective TEL formulae φ, ψ:

$$\varphi \sim^2 \psi \quad \Leftrightarrow \quad \text{for all TELC models } \mathcal{M}: \quad (\mathcal{M}, 0) \models \varphi \Leftrightarrow (\mathcal{M}, 0) \models \psi.$$

Again we can link the notion of $\sim^2$ with the notion $\models^c$: $\varphi \sim^2 \psi \Leftrightarrow \Box(at_0 \to \varphi) \equiv^c \Box(at_0 \to \psi)$. So also $\sim^2$ is decidable.

Theorem 7.6 and proposition 7.7 also transfer; the proofs are similar:

Theorem A.3.2

    A subjective TEL formula φ is downward persistent[2] (upward persistent[2]) if and only if it is equivalent (in the sense of $\sim^2$) to a subjective formula in **DP** (**UP**).

The link with monotonicity again holds, but now for minimal anchored entailment. The proof is similar to that of proposition 7.10:

Proposition A.3.3

    A formula φ is downward persistent[2] if and only if:

$$\forall \alpha, \beta: \quad \alpha \models^{c,\,a}_{min} \beta \quad \Rightarrow \quad \alpha \wedge \varphi \models^{c,\,a}_{min} \beta$$

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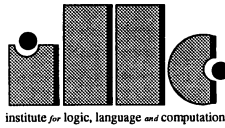
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