Institute for Language, Logic and Information

SOME COMPLETE LOGICS FOR BRANCHED TIME

PART I
WELL-FOUNDED TIME, FORWARD LOOKING OPERATORS

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PART I
WELL-FOUNDED TIME, FORWARD LOOKING OPERATORS

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Abstract. Completeness proofs are given for tense logics for branched time under the additional assumption of well-foundedness, both with the sole operator F, and with the operators F and G combined. A relationship with intuitionistic logic is established.

key words: tense logic, branched time, intuitionistic logic

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§0. Introduction

Completeness proofs for tense logics for branched time tend to be difficult (cf. Burgess 1980). Adding well-foundedness (for future time) turns out to make things much easier. Therefore, even though this condition in itself does not seem particularly natural, it seems worthwile to work things out. In this first part we will concentrate on the forward looking operators F and G. Furthermore, we will look into the natural connection which exists with intuitionistic logic via the concept of bar (Beth-models). The latter connection is also one reason that we consider reflexive as well as irreflexive time structures. In Part **II** we will concentrate on branched time ordered as the integers. We thank Johan van Benthem, Krister Segerberg and Frank Veltman for their encouragement and their suggestions.

§1. The frames, the language

We consider frames $\mathbf{T} = \langle T, < \rangle$, where $<$ is a well-founded branched partial order, either reflexive or irreflexive. We call a partial order *branched* if it fulfills the following conditions:

(i) $\forall t,t' \, (t \neq t' \wedge t \not< t' \wedge t' \not< t \rightarrow \neg \exists t''(t<t'' \wedge t'<t''))$
(ii) $\forall t \exists t'(t<t')$
(iii) $\forall t(\exists t'(t<t' \wedge t \neq t') \rightarrow \exists t',t'' \, (t<t' \wedge t<t'' \wedge t' \neq t'' \wedge t' \not< t'' \wedge t'' \not< t'))$.

Condition (i) is the condition of *downward linear order* (DLO). We have decided to include condition (ii) which expresses socalled *succession*: each element has a successor. Inessential changes result if this condition is left out. Of course, condition (ii) is only effective in the irreflexive case. Also, in the irreflexive case, the branching condition (iii) can be stated in a simpler form:

$\forall t \exists t',t'' \, (t<t' \wedge t<t'' \wedge t' \neq t'' \wedge t' \not< t'' \wedge t'' \not< t')$,

which then also implies (ii).

The propositional language considered has, besides the propositional variables p, q, r, ... and the connectives $\wedge, \vee, \rightarrow, \leftrightarrow, \neg, \bot, \top$, the temporal modifier F, and the temporal modifier G may be added. (The other temporal modifiers will be added in Part II.) The forcing conditions are defined in the usual way for the connectives and for G, e.g.

$t \Vdash G\varphi$ iff $\forall t' > t\,(t' \Vdash \varphi)$.

For F the definition is as follows:

$t \Vdash F\varphi$ iff in each maximal chain $C \subseteq T$ containing t there is a $t' > t$ such that $t' \Vdash \varphi$.

§2. The systems WR-F, WI-F, WR-FG, WI-FG

The systems **WR**-F, **WI**-F, **WR**-FG, **WI**-FG are tense logics respecively for formulae with F only on reflexive frames, with F only on irreflexive frames, with both F and G on reflexive frames, and with both F and G on irreflexive frames. We give, for each of the systems, the axioms and rules added to classical tautologies.

Axiom system **WR**-F:
(a) $\varphi \to F\varphi$
(b) $F\bot \to \bot$
(c) $F\varphi \land F\psi \to F((\varphi \land F\psi) \lor (F\varphi \land \psi))$
(d) $FF\varphi \to F\varphi$
(R) $\varphi \to \psi \,/\, F\varphi \to F\psi$

Axiom system **WI**-F:
(b), (R) +
(c*) $F\varphi \land F\psi \to F((\varphi \land \psi) \lor (\varphi \land F\psi) \lor (F\varphi \land \psi))$
(d*) $F(F\varphi \lor \varphi) \to F\varphi$
(e) $F\top$

Axiom system **WR**-FG:
(a), (b), (c), (d) +
(r) $G(\varphi \to \psi) \to (F\varphi \to F\psi)$
S4-axioms for G
rule of nessitation (N) $\varphi \,/\, G\varphi$
(f) $G(\varphi \to G\psi) \land G(F\varphi \to \psi) \to (F\varphi \to G\psi)$

Axiom system **WI**-FG:
(b), (c*), (d), (e), (r), (f) +
$G\varphi \rightarrow GG\varphi$
$G(\varphi \rightarrow \psi) \rightarrow (G\varphi \rightarrow G\psi)$
rule of necessitation.

The validity of most of these schemata for their intended interpretation needs little clarification. Axiom (a) expresses reflexivity, (d) transitivity, (R) is a substitute for the rule of necessitation: it implies a version of the latter: $\vdash \varphi \Rightarrow \vdash \neg\varphi \rightarrow \bot \Rightarrow \vdash F\neg\varphi \rightarrow F\bot \Rightarrow \vdash \neg F\bot \rightarrow \neg F\neg\varphi \Rightarrow \vdash \neg F\neg\varphi$. The strengthening of (d) to (d*) in **WI**-F seems to be needed (see lemma 4.1). The systems **WR**-F and **WI**-F are non-normal, since the analogue of $G(\varphi \rightarrow \psi) \rightarrow (G\varphi \rightarrow G\psi)$ is not derivable. We will discuss the schemata (c) and (c*) now and leave (f) for later. For each of the systems the rule (R) or the rule of necessitation plus the axiom schema (r), plus in the FG-cases $G(\varphi \rightarrow \psi) \rightarrow (G\varphi \rightarrow G\psi)$ imply that a replacement lemma holds. It will be obvious that that in none of the systems the formula $F(\varphi \vee \psi) \rightarrow F\varphi \vee F\psi$ is derivable. This formula is needed in most standard completeness proofs for tense logics.

2.1 Lemma. Within the context of the reflexive frames, (c) is valid on the well-founded frames (likewise for (c*), within the context of the irreflexive frames).

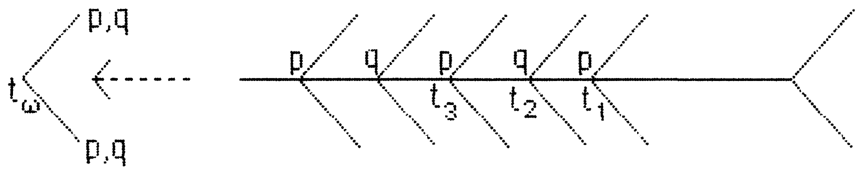
Proof. Let $\langle T, < \rangle$ be well-founded, $t \in T$. To show is that:

$$t \Vdash Fp \text{ and } t \Vdash Fq \Rightarrow t \Vdash F((p \wedge Fq) \vee (Fp \wedge q)).$$

Take an arbitrary maximal chain C through t. There are t', t" on C with t' > t, t" > t, such that t' ⊩ p and t" ⊩ q. Let t_c be the *minimal* node on C past t which forces p or q. If t_c forces p, then on any maximal chain through t_c (and therefore through t by DLO) there will be a node beyond t_c which forces q, since Fq is forced at t; so $t_c \Vdash Fq$. Similarly, if $t_c \Vdash q$, then $t_c \Vdash Fp$. So, $t_c \Vdash (p \wedge Fq) \vee (Fp \wedge q)$, and as C was arbitrary, $t \Vdash F((p \wedge Fq) \vee (Fp \wedge q))$. ⊠

2.2 Lemma. If $\langle T, < \rangle$ is a reflexive branched partial order containing a sequence $t_\omega < \ldots < t_2 < t_1 < t_0$, then there is a forcing relation on T such that $t_\omega \Vdash Fp \wedge Fq$, $t_\omega \nVdash F((p \wedge Fq) \vee (Fp \wedge q))$. (Similarly for irreflexive branched orders.)

Proof. Define ⊩ as in the picture below:



i.e., for $t \in T$, $t \Vdash q$ iff $t = t_{2n}$ for some $n \in \omega$, or $t_\omega < t$ and for all n, $t_n \not< t$,
$t \Vdash p$ iff $t = t_{2n+1}$ for some $n \in \omega$, or $t_\omega < t$ and for all n, $t_n \not< t$.
Clearly, the required conditions are fulfilled, because by the branching condition, neither $Fp \wedge q$, nor $p \wedge Fq$ is forced on any t_n. ⊠

2.3 Corollary. Within the context of the reflexive branched frames $Fp \wedge Fq \rightarrow F((p \wedge Fq) \vee (Fp \wedge q))$ characterizes the locally well-founded frames (i.e. for each $t \in T$, $\{t' \in T \mid t' > t\}$ is well-founded). (Similarly for (c^*), with respect to the irreflexive branched frames.)

2.4 Remark. The branchedness condition is not superfluous in lemma 2.2. Axiom schema (c) is valid on every reflexive linear ordering. In fact, and this is remarkable, under the standard semantical interpretation of F axiom scheme (c) *expresses* linearity (or to be more precise: nonbranchedness towards the future), and on linear orderings, the standard semantical interpretation and the one of this paper are, of course the same. On frames which are only required to be reflexive or DLO validity of (c) determines a more complex condition. As is semantically obvious, axiom schema (c) naturally generalizes to more than two propositional variables. This is spelled out in the next lemma.

2.5 Lemma. For $n > 1$, $\vdash_{\mathbf{WR}\text{-}F} \bigwedge_{i \le n} F\varphi_i \leftrightarrow F \bigvee_{i \le n}(\bigwedge_{j \ne i, j \le n}(F\varphi_j \wedge \varphi_i))$.
Proof. For n = 2, ⇒ is axiom (c).
⇐ (for n=2): $\vdash_{\mathbf{WR}\text{-}F} F\varphi_1 \wedge \varphi_2 \rightarrow F\varphi_1 \wedge F\varphi_2$ and
$\vdash_{\mathbf{WR}\text{-}F} \varphi_1 \wedge F\varphi_2 \rightarrow F\varphi_1 \wedge F\varphi_2$, by axiom (a). Therefore
$\vdash_{\mathbf{WR}\text{-}F} (F\varphi_1 \wedge \varphi_2) \vee (\varphi_1 \wedge F\varphi_2) \rightarrow F\varphi_1 \wedge F\varphi_2$. Now apply (R) and (d) to both

conjuncts.
Next we prove in **WR**-F, by induction on n: $n \Rightarrow n+1$.

$$\begin{aligned}\bigwedge_{i\le n+1} F\varphi_i &\leftrightarrow \bigwedge_{i\le n} F\varphi_i \wedge F\varphi_{n+1}\\ &\leftrightarrow F\bigvee_{i\le n}(\bigwedge_{j\ne i, j\le n} F\varphi_j \wedge \varphi_i) \wedge F\varphi_{n+1} \text{ (ind. hyp.)}\\ &\leftrightarrow F((F\bigvee_{i\le n}(\bigwedge_{j\ne i, j\le n} F\varphi_j \wedge \varphi_i) \wedge \varphi_{n+1}) \vee\\ &\qquad (\bigvee_{i\le n}(\bigwedge_{j\ne i, j\le n} F\varphi_j \wedge \varphi_i) \wedge F\varphi_{n+1})) \quad (n=2)\\ &\leftrightarrow F((\bigwedge_{i\le n} F\varphi_i \wedge \varphi_{n+1}) \vee \bigvee_{i\le n}(\bigwedge_{j\ne i, j\le n+1} F\varphi_j \wedge \varphi_i)) \text{ (ind. hyp.)}\\ &\leftrightarrow F\bigvee_{i\le n+1}(\bigwedge_{j\ne i, j\le n+1} F\varphi_j \wedge \varphi_i). \quad \boxtimes\end{aligned}$$

A similar lemma holds for **WI**-F. We will return to it later.

§3. Completeness of WR-F

3.1 Definition. A set Δ of formulae is *adequate* if
(i) Δ is closed under the formation of subformulae,
(ii) if $\varphi \in \Delta$, φ is unnegated, then $\neg\varphi \in \Delta$,
(iii) $F\bot \in \Delta$.

In this section we work in a fixed Δ. We consider $\Gamma, \Gamma', \ldots \subseteq \Delta$ such that $\Gamma, \Gamma', \ldots$ are maximally consistent in Δ with respect to **WR**-F (we write for short: *maxcon* in Δ).

3.2 Lemma. If Γ maxcon in Δ, then:
(i) If $\neg\varphi \in \Delta$, then $\neg\varphi \in \Gamma$ iff $\varphi \notin \Gamma$,
(ii) If $\varphi \wedge \psi \in \Delta$, then $\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$,
(iii) If $\varphi \vee \psi \in \Delta$, then $\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$,
(iv) If $F\varphi \in \Delta$, then $\varphi \in \Gamma \Rightarrow F\varphi \in \Gamma$.

3.3 Definition. (a) $F\varphi$ is *yet to be barred in* Γ iff $F\varphi \in \Gamma$, $\varphi \notin \Gamma$.
(b) $F\varphi : \Gamma \Rightarrow \Gamma'$ iff (i) $F\varphi$ is yet to be barred in Γ,
(ii) $\varphi \in \Gamma'$,
(iii) For each $F\psi$ yet to be barred in Γ, $F\psi \in \Gamma'$.
(c) $\Gamma \Rightarrow \Gamma'$ iff $F\varphi : \Gamma \Rightarrow \Gamma'$ for some $F\varphi$.
(d) Γ *needs a bar* if some $F\varphi$ is yet to be barred in Γ.

3.4 Lemma. If Γ needs a bar and $\neg F\psi \in \Gamma$, then there is some Γ' such that $\Gamma \Rightarrow \Gamma'$, $\neg F\psi \in \Gamma$, $\neg\psi \in \Gamma'$.

Proof. Let $F\varphi_1,\ldots,F\varphi_n$ be all the formulae yet to be barred in Γ ($n \geq 1$). Assume that, for each Γ' such that $\Gamma \Rightarrow \Gamma'$, $F\psi \in \Gamma'$. Then, apparently, for each i ($1 \leq i \leq n$), $\{F\varphi_1,\ldots, F\varphi_{i-1}, \varphi_i, F\varphi_{i+1},\ldots,F\varphi_n, \neg F\psi\}$ is inconsistent, i.e. for each i, $\vdash_{\mathbf{WR}\text{-}F} (F\varphi_1 \wedge \ldots \wedge F\varphi_{i-1} \wedge \varphi_i \wedge F\varphi_{i+1} \wedge \ldots \wedge F\varphi_n) \rightarrow F\psi$. So, $\vdash_{\mathbf{WR}\text{-}F} \bigvee_{i=1\ldots n}(F\varphi_1 \wedge \ldots \wedge F\varphi_{i-1} \wedge \varphi_i \wedge F\varphi_{i+1} \wedge \ldots \wedge F\varphi_n) \rightarrow F\psi$. With rule (R): $F(\bigvee_{i=1\ldots n}(F\varphi_1 \wedge \ldots \wedge F\varphi_{i-1} \wedge \varphi_i \wedge F\varphi_{i+1} \wedge \ldots \wedge F\varphi_n)) \rightarrow F\psi$.
Now, using lemma 2.5, $\vdash_{\mathbf{WR}\text{-}F} (F\varphi_1 \wedge \ldots \wedge F\varphi_n) \rightarrow F\psi$ contradicting the consistency of Γ, since each of $F\varphi_1,\ldots,F\varphi_n, \neg F\psi \in \Gamma$. Finally $\psi \notin \Gamma$, since $\psi \rightarrow F\psi$ is an axiom of **WR**-F. ⊠

3.5 Definition. (a) $F\varphi$ is *yet to be barred in* $\{\Gamma_\alpha\}_{\alpha<\lambda}$, if for some $\beta<\lambda$, $F\varphi$ is yet to be barred in each Γ_α for $\beta \leq \alpha < \lambda$.
(b) $\{\Gamma_\alpha\}_{\alpha<\lambda}$ *needs a bar*, if some $F\varphi$ is yet to be barred in $\{\Gamma_\alpha\}_{\alpha<\lambda}$.
(c) $F\varphi: \{\Gamma_\alpha\}_{\alpha<\lambda} \Rightarrow \Gamma$ is defined by induction on λ (writing $\{\Gamma_\alpha\}_{\alpha<\mu} \Rightarrow \Gamma'$ for $F\psi: \{\Gamma_\alpha\}_{\alpha<\mu} \Rightarrow \Gamma'$ for some $F\psi$) by:

- (i) for each $\beta<\lambda$, $\Gamma_\beta \Rightarrow \Gamma_{\beta+1}$
- (ii) for each limit $\mu<\lambda$, $\{\Gamma_\alpha\}_{\alpha<\mu} \Rightarrow \Gamma_\mu$
- (iii) $F\varphi$ is yet to be barred in $\{\Gamma_\alpha\}_{\alpha<\lambda}$
- (iv) $\varphi \in \Gamma$
- (v) For each $F\psi$ yet to be barred in $\{\Gamma_\alpha\}_{\alpha<\lambda}$, $F\psi \in \Gamma$.

(d) $\{\Gamma_\alpha\}_{\alpha<\lambda}$ is called a *$\Rightarrow$-sequence* if $\{\Gamma_\alpha\}_{\alpha<\lambda}$ fulfills c(i) and c(ii).
(e) $\{\Gamma_\alpha\}_{\alpha<\lambda}$ is called a *$\neg F\psi$-sequence* if $\{\Gamma_\alpha\}_{\alpha<\lambda}$ is a $\Rightarrow$-sequence with $\neg F\psi \in \Gamma_\alpha$ for each $\alpha<\lambda$.

3.6 Lemma. Let λ be a limit, $\{\Gamma_\alpha\}_{\alpha<\lambda}$ a $\neg F\psi$-sequence which needs a bar, then, for some Γ with $\neg F\psi \in \Gamma$, $\neg\psi \in \Gamma$, $\{\Gamma_\alpha\}_{\alpha<\lambda} \Rightarrow \Gamma$.
Proof. Like lemma 3.4. ⊠

3.7 Definition (a) $\{\Gamma_\alpha\}_{\alpha<\beta+1} \Rightarrow \Gamma$, if $\Gamma_\beta \Rightarrow \Gamma$.
(b) $\{\Gamma_\alpha\}_{\alpha<\lambda}$ *uses* $F\varphi$, if for some $\gamma<\lambda$, $F\varphi: \{\Gamma_\alpha\}_{\alpha<\gamma} \Rightarrow \Gamma_\gamma$.
(c) $F\varphi$ is *used cofinally in* $\{\Gamma_\alpha\}_{\alpha<\lambda}$, if, for each $\gamma<\lambda$, $\{\Gamma_\alpha\}_{\gamma\leq\alpha<\lambda}$ uses $F\varphi$.

3.8 Lemma. If $\{\Gamma_\alpha\}_{\alpha<\lambda}$ uses $F\varphi$ cofinally, then it is impossible that $F\varphi: \{\Gamma_\alpha\}_{\alpha<\lambda} \Rightarrow \Gamma$.

Proof. Trivial. ⊠

3.9 Lemma. If $\{\Gamma_\alpha\}_{\alpha<\beta}$ is a $\Rightarrow$-sequence using only $F\psi_1,\ldots,F\psi_k$, then $\beta \leq \omega^{k-1}$.

Proof. By induction on k:

(a) k = 1: If $\{\Gamma_\alpha\}_{\alpha<\beta}$ uses only $F\psi$, then $F\psi$: $\Gamma_0 \Rightarrow \Gamma_1$. Then $\psi \in \Gamma$, so $F\psi$: $\Gamma_1 \Rightarrow \Gamma_2$ is impossible: $\beta \leq 1 = \omega^0$.

(b) k = 2: Let $\{\Gamma_\alpha\}_{\alpha<\omega}$ be a sequence in which only $F\psi_1, F\psi_2$ are used. Say $F\psi_1$: $\Gamma_0 \Rightarrow \Gamma_1$. Then it is only possible that $F\psi_2$: $\Gamma_1 \Rightarrow \Gamma_2$, $F\psi_1$: $\Gamma_2 \Rightarrow \Gamma_3$, etc. Both $F\psi_1$ and $F\psi_2$ are used cofinally in $\{\Gamma_\alpha\}_{\alpha<\omega}$ and neither can, by lemma 3.8, be used to extend the sequence. So, if $\{\Gamma_\alpha\}_{\alpha<\omega}$ is a $\Rightarrow$-sequence wich uses only $F\psi_1$ and $F\psi_2$, then $\beta \leq \omega$.

(c) Finally, assume the lemma holds for k (where $k \geq 2$), and let $\{\Gamma_\alpha\}_{\alpha<\omega^k}$ use only $F\psi_1,\ldots,F\psi_{k+1}$. It is, by lemma 3.8, sufficient to show that $\{\Gamma_\alpha\}_{\alpha<\omega^k}$ uses each $F\psi_i$ $(1 \leq i \leq k+1)$ cofinally. For this purpose we divide ω^k up as follows: $\omega^k = \Sigma_{j<\omega} A_j$, where $A_j = \{\alpha \mid \omega^{k-1}.j \leq \alpha < \omega^{k-1}.(j+1)\}$. For each j, the ordertype of A_j is ω^{k-1}, so, by induction hypothesis, each member of a subset of cardinality k of $\{F\psi_1, \ldots, F\psi_{k+1}\}$ is used cofinally in A_j. (For, if not, there would be some $\gamma \in A_j$ such that in $\{\alpha \mid \alpha \in A_j, \alpha \geq \gamma\}$ only k-1 of $F\psi_1,\ldots,F\psi_{k+1}$ would be used, whereas $\{\alpha \mid \alpha \in A_j, \alpha \geq \gamma\}$ has ordertype ω^{k-1}.) Since there are only a finite number of such subsets at least one of them has to occur an infinite number of times. Without loss of generality we can assume this to be $\{F\psi_1,\ldots,F\psi_k\}$. This immediately implies that each of $F\psi_1,\ldots,F\psi_k$ is used cofinally in $\{\Gamma_\alpha\}_{\alpha<\omega^k}$. But the same holds for $F\psi_{k+1}$, since, for each A_j in which $F\psi_1,\ldots,F\psi_k$ are used cofinally, $F\psi_{k+1}$: $\{\Gamma_\alpha \mid \alpha \in A_j\} \Rightarrow \Gamma_\delta$, where $\delta = \cup A_j$. So, all of $F\psi_1,\ldots,F\psi_k$ are used cofinally in $\{\Gamma_\alpha\}_{\alpha<\omega^k}$ which was to be proved. ⊠

3.10 Lemma. Each $\Rightarrow$-sequence has length $<\omega^\omega$.

Proof. Immediate from lemma 3.9; in fact the length is $<\omega^{|\Delta|-1}$. ⊠

It is not clear at present if the bound ω^ω is the best possible one. In the linear case it can be shown that ω^2 will do (cf. van Benthem 1986).

3.11 Theorem. Axiom system **WR**-F is complete with respect to well-founded branched reflexive frames.

Proof. Assume $\nvdash_{\mathbf{WR}\text{-F}} \theta$. We construct a countermodel to θ. Let Δ be the

set of subformulae of θ and $F\bot$ together with the negations of such formulae. Then Δ is adequate. Since $\nvdash_{\mathbf{WR}\text{-F}} \theta$, there is a maxcon subset of Δ containing $\neg\theta$. For the frame of the model we now take the set of all $\Rightarrow$-sequences with $\Gamma_0 = \Gamma$. By lemma 3.10, this is indeed a set. For the nodes t, t' of this frame we define t<t' if t is a (not necessarily proper) initial segment of t'. Finally, $\{\Gamma_\alpha\}_{\alpha\leq\beta} \Vdash p$, for p a propositional letter, if $p\in\Gamma_\beta$. We now show, by induction on the length of φ that, for each $\varphi\in\Delta$, $\{\Gamma_\alpha\}_{\alpha\leq\beta} \Vdash \varphi$ iff $\varphi\in\Gamma_\beta$. The only interesting case is where φ is $F\psi$:
$\Longrightarrow$: Assume $F\psi\notin\Gamma_\beta$, i.e. $\neg F\psi\in\Gamma_\beta$. It is clearly sufficient to show that there is a Δ-maximal $\neg F\psi$-sequence with Γ_β as its first element. But the existence of such a sequence follows from the lemmas 3.6 and 3.10.
$\Longleftarrow$: Assume $F\psi\in\Gamma_\beta$. Let $\{\Gamma_\alpha\}_{\alpha<\gamma}$ be an arbitrary Δ-maximal $\Rightarrow$-sequence with Γ_β as its first element. It is sufficient to show that $F\psi$ is used in this sequence. This is clear though from the way these sequences are constructed. Since $F\psi$ is yet to be barred in Γ_β, $F\psi$ will remain a member of each Γ_α as long as $F\psi$ is not used. This implies that, if $F\psi$ is not used at all, $F\psi$ is yet to be barred in $\{\Gamma_\alpha\}_{\alpha<\gamma}$. But then $\{\Gamma_\alpha\}_{\alpha<\gamma}$ needs a bar which by lemma 3.6 contradicts the maximality of $\{\Gamma_\alpha\}_{\alpha<\gamma}$.

It may be that the resulting frame is not branched. That is, of course easy to repair. If from some point on everything is linearly ordered, we make it into an infinite binary branching order instead and copy all the forcing relations on each "level". ⊠

§4 Completeness of WI-F

We will just give the changes in the definitions of §3, as well as some hints for the simple changes in the proofs needed to give a completenessproof for **WI**-F.

4.1 Lemma (analogue of lemma 2.5).
For $n>1$, $\vdash_{\mathbf{WI}\text{-F}} \bigwedge_{i=1\ldots n} F\varphi_i \leftrightarrow F\bigvee_{X\subset\{1\ldots n\}}(\bigwedge_{i\notin X}\varphi_i \wedge \bigwedge_{j\in X} F\varphi_j)$.
Proof. The induction is like in lemma 2.5. The $\Longleftarrow$-step (for n=2) is slightly different:
$\vdash_{\mathbf{WI}\text{-F}} (F\varphi_1\wedge\varphi_2)\vee(\varphi_1\wedge F\varphi_2)\vee(\varphi_1\wedge\varphi_2)\rightarrow(F\varphi_1\vee\varphi_1)\wedge(F\varphi_2\vee\varphi_2)$.
Therefore, by (R),
$\vdash_{\mathbf{WI}\text{-F}} F((F\varphi_1\wedge\varphi_2)\vee(\varphi_1\wedge F\varphi_2)\vee(\varphi_1\wedge\varphi_2))\rightarrow F(F\varphi_1\vee\varphi_1)\wedge F(F\varphi_2\vee\varphi_2)$.

Now apply (d*) to both conjuncts. ⊠

4.2 Definition. In the definition of Δ being *adequate* (definition 3.1) Δ is now required to also contain FT.

4.3 Definition (replacing definition 3.3).
(a) $F\varphi$ is *yet to be barred in* Γ, if $F\varphi \in \Gamma$.
(b) $F\varphi_1,\ldots,F\varphi_k: \Gamma \Rightarrow \Gamma'$ iff (i) $F\varphi_1,\ldots,F\varphi_k$ are yet to be barred in Γ,
(ii) $\varphi_1,\ldots,\varphi_k \in \Gamma'$,
(iii) For each $F\psi$ yet to be barred in Γ, not among $F\varphi_1,\ldots,F\varphi_k$, $F\psi \in \Gamma'$.
(c) $\Gamma \Rightarrow \Gamma'$ if $F\varphi_1,\ldots,F\varphi_k: \Gamma \Rightarrow \Gamma'$ for some $F\varphi_1,\ldots,F\varphi_k$.

4.4 Lemma. Like lemma 3.4.

4.5 Definition (*yet to be barred, needs a bar,* $\Rightarrow$*-sequence,* $\neg F\psi$*-sequence*). Obtained by obvious changes from definition 3.5.

4.6 Lemma. Like lemma 3.6.

4.7 Definition. Obtained by the obvious changes from definition 3.7 ($\{\Gamma_\alpha\}_{\alpha<\lambda}$ *uses* $F\varphi_i$, if for some $\gamma<\lambda$, $F\varphi_1,\ldots,F\varphi_k$: $\{\Gamma_\alpha\}_{\alpha<\gamma} \Rightarrow \Gamma_\gamma$).

4.8 Lemma. Like lemmas 3.8.

4.9 Lemma. If $\{\Gamma_\alpha\}_{\alpha<\beta}$ is a $\Rightarrow$-sequence which only uses $F\varphi_1,\ldots,F\varphi_k$, then $\beta \leq \omega^k$.
Proof. (a) k = 1. Now $F\psi$: $\Gamma_0 \Rightarrow \Gamma_1 \Rightarrow \Gamma_2$ is of course possible, but, if $\{\Gamma_\alpha\}_{\alpha<\omega}$ uses only $F\psi$, then clearly $F\psi$ is used cofinally in $\{\Gamma_\alpha\}_{\alpha<\beta}$. so $\{\Gamma_\alpha\}_{\alpha<\omega} \Rightarrow \Gamma$ is impossible. Steps (b) and (c) can be amalgamated in this case and present no problem. ⊠

4.10 Theorem (*Completeness of* ***WI****-F*). Axiom system **WI**-F is complete with respect to the well-founded branched irreflexive frames.
Proof. The proof is just slightly different from that of theorem 3.11. The only noteworthy distinction is that we take FT to be in Δ. This entails that each Γ needs a bar (though not each $\{\Gamma_\alpha\}_{\alpha<\lambda}$!). ⊠

§5 Completeness of WR-FG and WI-FG

In the proof of the completeness of **WR**-FG we just note the changes and additions to be made to the treatment of **WR**-F in §3. The completeness of the system **WI**-FG is then left to the reader. We remark that in **WR**-FG, if Γ is maxcon, then $G\varphi \in \Gamma$ implies $\varphi \in \Gamma$. We therefore add to the definition of $F\varphi: \Gamma \Rightarrow \Gamma'$ (Def 3.3.(b)) as a fourth clause that, if $G\varphi \in \Gamma$, then $G\varphi \in \Gamma'$ and $\varphi \in \Gamma'$. A similar change is of course to be made in Def. 3.5(c).

We note that the rule (R) is now a derived rule. To be able to take care of the case that $\neg G\varphi \in \Gamma$ we also define:

5.1 Definition.

(a) $G\varphi: \Gamma \Rightarrow \Gamma'$ iff

(i) $\neg G\varphi \in \Gamma$, $\varphi \in \Gamma$

(ii) $\varphi \notin \Gamma'$

(iii) if $G\psi \in \Gamma$, then $G\psi \in \Gamma'$

(iv) if $F\psi$ yet to be barred in Γ, then $F\psi \in \Gamma'$.

(b) $\Gamma \Rightarrow \Gamma'$ iff for some φ, $F\varphi: \Gamma \Rightarrow \Gamma'$ or $G\varphi: \Gamma \Rightarrow \Gamma'$.

The basic lemma is in this case:

5.2 Lemma. If $\neg G\varphi, \varphi \in \Gamma$, then there is a Γ' such that either (i) or (ii),

(i) $G\varphi: \Gamma \Rightarrow \Gamma'$

(ii) for some $F\psi$, $F\psi: \Gamma \Rightarrow \Gamma'$ and $\neg G\varphi \in \Gamma'$.

Proof. Assume $F\psi_1, \ldots, F\psi_k$ are all the formulae yet to be barred in Γ and $G\theta_1, \ldots, G\theta_m$ all the G-formulae in Γ. Just to make writing and reading easier, assume $k=m=2$. Suppose $\neg G\varphi, \varphi \in \Gamma$ and neither (i) nor (ii) applies. Then

(i) $\vdash_{\mathbf{WR}\text{-FG}} (G\theta_1 \wedge G\theta_2) \wedge (F\psi_1 \wedge F\psi_2) \rightarrow \varphi$

(ii) $\vdash_{\mathbf{WR}\text{-FG}} (G\theta_1 \wedge G\theta_2) \wedge (F\psi_1 \wedge \psi_2) \rightarrow G\varphi$ and
$\vdash_{\mathbf{WR}\text{-FG}} (G\theta_1 \wedge G\theta_2) \wedge (\psi_1 \wedge F\psi_2) \rightarrow G\varphi$, so

$$\vdash_{\mathbf{WR}\text{-FG}} (G\theta_1 \wedge G\theta_2) \wedge ((F\psi_1 \wedge \psi_2) \vee (\psi_1 \wedge F\psi_2)) \to G\varphi.$$

From (i) and lemma 2.5:

$$\vdash_{\mathbf{WR}\text{-FG}} (G\theta_1 \wedge G\theta_2) \wedge F((F\psi_1 \wedge \psi_2) \vee (\psi_1 \wedge F\psi_2)) \to \varphi,$$

and hence,

$$\vdash_{\mathbf{WR}\text{-FG}} G(\theta_1 \wedge \theta_2) \to G(\ F((F\psi_1 \wedge \psi_2) \vee (\psi_1 \wedge F\psi_2)) \to \varphi\ ),$$

and from (ii),

$$\vdash_{\mathbf{WR}\text{-FG}} G(\theta_1 \wedge \theta_2) \wedge ((F\psi_1 \wedge \psi_2) \vee (\psi_1 \wedge F\psi_2)) \to G\varphi,$$

and hence,

$$\vdash_{\mathbf{WR}\text{-FG}} G(\theta_1 \wedge \theta_2) \to G((F\psi_1 \wedge \psi_2) \vee (\psi_1 \wedge F\psi_2) \to G\varphi\ ).$$

Now apply Axiom (f) to obtain,

$$\vdash_{\mathbf{WR}\text{-FG}} G(\theta_1 \wedge \theta_2) \to G((F\psi_1 \wedge \psi_2) \vee (\psi_1 \wedge F\psi_2) \to G\varphi\ ),$$

whence finally, by a second application of lemma 2.5,

$$\vdash_{\mathbf{WR}\text{-FG}} (G\theta_1 \wedge G\theta_2) \wedge (F\psi_1 \wedge F\psi_2) \to G\varphi,$$

contradicting the consistency of Γ. ⊠

5.3 Definition

$G\varphi: \{\Gamma_\alpha\}_{\alpha<\lambda} \Rightarrow \Gamma$ iff

(i) for some $\beta<\lambda$, $\neg G\varphi, \varphi \in \Gamma_\alpha$ for each α $(\beta \leq \alpha < \lambda)$ ($\{\Gamma_\alpha\}_{\alpha<\lambda}$ *needs a block* (for $G\varphi$))

(ii) $\varphi \notin \Gamma$

(iii) if for some $\beta<\lambda$, $G\psi \in \Gamma_\alpha$ for each $\alpha\,(\beta \leq \alpha < \lambda)$, then $G\psi \in \Gamma$,

(iv) if $F\psi$ yet to be barred in $\{\Gamma_\alpha\}_{\alpha<\lambda}$, then $F\psi \in \Gamma$.

5.4 Lemma. If $\{\Gamma_\alpha\}_{\alpha<\lambda}$ *needs a block* for $G\varphi$, then there is a Γ such that either (i) or (ii),

(i) $G\varphi: \{\Gamma_\alpha\}_{\alpha<\lambda} \Rightarrow \Gamma$

(ii) for some $F\psi$, $F\psi: \{\Gamma_\alpha\}_{\alpha<\lambda} \Rightarrow \Gamma$ and $\neg G\varphi \in \Gamma$.

Proof. Obvious. ⊠

It is now clear that any $\Rightarrow$-sequence which needs a bar or a block can be extended. As in §3 the length of $\Rightarrow$-sequences is bouded by ω^ω. In the proof of the completeness theorem itself the only interesting new case to be considered is, where $\neg G\varphi, \varphi \in \Gamma$. Now, if in such a circumstance case (i) of lemma 5.1 applies, we are of course done immediately. If it doesn't, case (ii) applies and we get $\Gamma \Rightarrow \Gamma'$ with $\neg G\varphi, \varphi \in \Gamma'$. Continuing in this way, if we keep only getting case (ii), also at limit stages, we remain all the time in a $\Rightarrow$-sequence $\{\Gamma_\alpha\}_{\alpha<\lambda}$ with $\Gamma_0 = \Gamma$ and $\neg G\varphi, \varphi \in \Gamma_\alpha$ for each $\alpha < \lambda$. This will come to a stop, since the $\Rightarrow$-sequences are bounded, and at some stage no F-formula can be used to extend the sequence: case (i) will apply.

§6 The relationship of WR-FG with intuitionistic logic

In this section it will be indicated how intuitionistic logic can be interpreted in **WR**-FG. This is done by first looking at the combination FG as a modal operator $\Box$, and then interpreting intuitionistic logic in the resulting modal logic **S4**$^-$, which is a little weaker than **S4**. It is a well-known fact that for the Gödel-translation of intuitionistic logic into modal logic the full strength of **S4** is not needed. The present sytem, **S4**$^-$, is suggested by the Beth-models for intuitionistic logic in which the forcing conditions are somewhat different than in Kripke-models.

Suggestive is, for example, that in Beth-models, $t \Vdash \varphi \vee \psi \iff$ in each maximal chain $C \subseteq T$ containing t there is a $t' > t$ ($>$ being reflexive) such that $t' \Vdash \varphi$ or $t' \Vdash \psi$. We first introduce the modal logic **S4**$^-$ with the axiomatization:

$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
$\Box\Box\varphi \to \Box\varphi$
$\Box\varphi \to \Box\Box\varphi$
$\Box\bot \to \bot$
$\varphi / \Box\varphi$

This system occurs in Segerberg (1971) under the name **KAD4**. We translate modal formulae φ into φ^{FG}, by simply replacing $\Box$ with FG. Formulae ψ from intuitionistic propositional logic **IPC** are translated into modal formulae $\psi^{\Box}$ by the standard Gödel-translation procedure for translating **IPC** into **S4** (Gödel, 1933):

6.1 Definition.
(a) $p^{\Box} = \Box p$ for propositional formulae p (including $\bot$)
(b) $(\varphi \wedge \psi)^{\Box} = \varphi^{\Box} \wedge \psi^{\Box}$, (c) $(\varphi \vee \psi)^{\Box} = \varphi^{\Box} \vee \psi^{\Box}$
(d) $(\varphi \to \psi)^{\Box} = \Box(\varphi^{\Box} \to \psi^{\Box})$

6.2 Proposition. $\vdash_{IPC}\varphi \Longleftrightarrow \vdash_{S4^-} \varphi^{\Box}$.

Proof. $\Longleftarrow$: Since **S4**$^-$ is a subsystem of **S4** this follows immediately from the Gödel result. (Of course, in a direct semantic proof this is where the Beth-models would come in; we will not even need them.)

$\Longrightarrow$: Just a matter of checking that the proofs of the translations of the axioms can be executed in **S4**$^-$. ⊠

The most straightforward way of getting the translation of **S4**$^-$ into **WR**-FG is to prove a completeness theorem for **S4**$^-$ with respect to reflexive well-founded branched models with the following non-standard definition of forcing for $\Box$-formulae:

$$t \Vdash \Box\varphi \Longleftrightarrow \text{for each complete chain } C \subseteq T \text{ containing } t \text{ there is a } t' > t \text{ such that, for all } t'' > t',\ t'' \vdash \varphi.$$

The method of proof is suggested by Kripke's (1965) method in comparing Kripke-semantics with Beth-models.

6.3 Theorem. **S4**$^-$ is complete with respect to the reflexive well-founded branched models.

Proof. Checking the validity of the axioms is again a routine matter. We did not succeed in proving the completeness part of the theorem directly, but luckily it does follow from the completeness theorem that Segerberg (1970) proved for **S4**$^-$ (under the name **KAD4**) with respect to standard Kripke-models. He proved completeness of **S4**$^-$ for *pseudo-dense* (i.e. for all $k,k' \in T$, such that $k<k'$ there is a $k'' \in T$ such that $k<k''<k'$), successive, transitive Kripke-models. So, let us assume that **T** is such a structure, with a forcing relation $\Vdash$ defined on it. Define T' to be the set of all non-empty finite sequences $\langle t_1,\dots,t_n\rangle$ over T such that $t_1 < \dots < t_n$. For two such sequences u and v, $u <' v$ is defined to hold if u is a (not necesssarily proper) initial segment of v. It is clear that $<'$ is a reflexive and transitive relation. To prove completeness for **S4**$^-$ it is clearly sufficient to show that, if we define $\langle t_1,\dots,t_n\rangle \Vdash' p$ iff $t_n \Vdash p$ for propositional variables p, then

$\langle t_1,\dots,t_n\rangle \Vdash' \varphi$ iff $t_n \Vdash \varphi$ for all formulae φ. As $\vee$ and $\neg$ are treated the same in both cases we will just have to check $\Box$.

First assume $t_n \Vdash \Box\varphi$. This means that, for each $t' > t_n$, $t' \Vdash \varphi$. By induction hypothesis, this implies that, for any proper extension $u = \langle t_1, \ldots, t_n, t_{n+1}, \ldots, t_{n+m}\rangle$ of $\langle t_1, \ldots, t_n\rangle$, $u \Vdash' \varphi$. So, indeed $\langle t_1, \ldots, t_n\rangle \Vdash' \Box\varphi$.

Next assume $t_n \nVdash \Box\varphi$. This means that, for some $t' > t_n$, $t' \nVdash \varphi$. From the fact that $<$ is pseudo-dense it follows then that there is a sequence $s_0, s_1, \ldots$ such that $t_n < s_0 < s_1 < \ldots < t'$. Consider the following chain from $\langle t_1, \ldots, t_n\rangle$ in T': $\langle t_1, \ldots, t_n\rangle$, $\langle t_1, \ldots, t_n, s_0\rangle$, $\langle t_1, \ldots, t_n, s_0, s_1\rangle, \ldots$.
Any element $\langle t_1, \ldots, t_n, s_0, s_1, \ldots, s_m\rangle$ from this (maximal) chain can be extended to $v_m = \langle t_1, \ldots, t_n, s_0, s_1, \ldots, s_m, t'\rangle$, and v_m, by induction hypothesis, does not force φ in T'. Clearly, therefore, $\langle t_1, \ldots, t_n\rangle$ does not force $\Box\varphi$ in T'. ⊠

6.4 Theorem. (a) $\vdash_{\mathbf{S4}^-} \varphi \Longleftrightarrow \vdash_{\mathbf{WR}\text{-FG}} \varphi^{FG}$, (b) $\vdash_{IPC} \varphi \Longleftrightarrow \vdash_{\mathbf{WR}\text{-FG}} \varphi^{\Box FG}$.
Proof. Immediate from the preceding. ⊠

It is to be noted that the fragment of **WR**-FG into which **S4**⁻ is translated needs no models of order type larger than ω.

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