Institute for Language, Logic and Information

TWO-DIMENSIONAL MODAL LOGICS

for Relational Algebras and Temporal Logic of Intervals

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ITLI Prepublication Series
LP-89-03

University of Amsterdam

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Received August 1989

1980 Mathematics Subject Classification: 03B45,03G15.
Key Words and Phrases: two-dimensional logic, relation algebras, temporal logics of intervals, completeness proofs

CONTENTS.

0. GENERAL INTRODUCTION.

The central idea of this paper is to look at both relation algebras and temporal logic of intervals from a viewpoint of two-dimensional modal logic, and try to find analogons and/or new results. Here we mean by a two-dimensional modal logic a system of modal logic in which (at least some of) the intended Kripke semantics is such that the set of possible worlds is (a subset of) a Cartesian product AxA for some set A: i.e. possible worlds are *pairs* of more basic objects. The theory of relation algebras was developed by Tarski et alii, in order to give an algebraic, variable-free treatment of binary relations. A temporal logic of intervals is a system of modal logic in which intervals are the entities where the truth of formulas is evaluated.
The report originated with the following very simple observation: if one views an interval as the ordered pair consisting of its beginning- and endpoint, a set of intervals can be seen as a binary relation (on the set of timepoints). The next idea was to use the well-known analogy between logics and algebras, or treatment of logics as algebras (e.g. proposition logic as Boolean algebra, predicate calculus in cylindric algebras). The immediate outcome was a striking similarity between an interval logic with a 'CHOP-operator' C - $\varphi C \psi$ holding at an interval if it can be chopped into two pieces at which φ resp. ψ hold - and the above mentioned relation algebras. In my opinion this similarity is shown most clearly in the perspective of two-dimensional modal logic.
Now one research line was to try and find applications of the theory of relation algebras for interval logic; in particular, the formulation of the most known principles holding for binary relations was a source of inspiration to finding *axioms for interval logic with CHOP*, and a natural deduction method of generating *all* principles holding for binary relations was the cornerstone for a similar system of interval logic.
On the other hand one might wonder whether the modal perspective on relation algebras sheds a new light on this field itself. As it turned out, two things proved to be useful: the concept of *correspondence* between modal and classical formulas, and the so-called 'inequality rule' of Gabbay which pops up like a deus ex machina every now and then in modal logic to make axiomatic systems complete.

In the next chapter we give our modal perspective on binary relations. First we give some basic facts and definitions concerning the relation algebraic approach to the subject; next we expose the above mentioned analogy of logics and algebras. The 'modal logic of relation algebras' which is the result of this analogy in the present case, is called CC and defined in section 1.3. In 1.4 we explore the expressive power of this logic and after that the recursive axiomatization with the inequality rule is given. We then treat an extension of CC with an "inequality operator", and in the last section of the first chapter we give two applications in the theory of relation algebras. Chapter two is on interval logic, starting with a short introduction to the subject. The 'CHOPPY interval temporal logic' CDT is defined in section 2.2, some results concerning expressibility come in 2.3, and after that an axiom system is treated, briefly. In the last section of this chapter we give a sound and complete deduction system for the logic. In Chapter III we discuss some related subjects and in the last chapter we give our conclusions.

1. RELATION ALGEBRAS AS TWODIMENSIONAL MODAL LOGIC.

1.1. Relation algebras.

We here state the facts we need on the theory of relation algebras. For an introduction to the subject the reader is referred to [HMTII], [JT] or [J2].

In the theory of binary relations one studies constant relations and operations on binary relations (over some unspecified set V). Examples are the Boolean operations and constants, the *identity* relation (Id) and the operations of *composition* (|), *converse* ($^{-1}$) and *reflexive transitive closure* (*), where

$Id = \{(x,x) | x \in V\}$

$R|S = \{(x,y) |$ there is a z with $(x,z) \in R, (z,y) \in S\}$

$R^{-1} = \{(x,y) | (y,x) \in R\}$

$R^* = \{(x,y) |$ there is a finite number of objects $x_0,\ldots,x_n$ with $x=x_0$, $y=x_n$ and $(x_i,x_{i+1}) \in R$ for $i = 0,\ldots,n-1\}$

Tarski [T] was the first one to suggest an *algebraic* approach to the subject. A *relation type algebra* is defined as a Boolean algebra with operators (by definition this is a Boolean algebra, extended with some normal/additive operators, cf. [JT]), in this case a binary operation ;, a unary $\breve{}$ and a constant 1'. The class *FRA* of *full relation algebras* consists of those relation type algebras that are isomorphic to an algebra of the form $(Sb(VxV), \cup, -_{VxV}, \emptyset, |, ^{-1}, Id)$, where Sb(VxV) is the powerset of VxV.

The question naturally arises as to study the variety *RRA* of *representable relation algebras* generated by the class of full relation algebras. The theory of universal algebra ([BS]) tells us that RRA = HSP(FRA) (where H, S and P denote the class operations of closing under homomorphisms, subalgebras and direct products) and by Birkhoffs theorem we have A in RRA iff all identities holding in FRA are true in A. Tarski proved that every representable relation algebra is a subalgebra of a direct product of full relation algebras, i.e. V(R) = SP(R). Some reflection shows that then every RRA can be embedded in an algebra of the form $(Sb(E), \cup, -_{VxV}, \emptyset, |, ^{-1}, Id)$ where E is an equivalence relation over the set V.

In order to enumerate the identities holding in RRA, Tarski suggested the following axiomatization, where we follow the literature in using the symbols ; for |, $\breve{}$ for $^{-1}$ and 1' for Id:

(1) axioms governing the Boolean part of the algebra.

(2) $(x+y);z = x;y + x;z$

(3) $(x+y)^{\smile} = x^{\smile} + y^{\smile}$
(4) $(x^{\smile})^{\smile} = x$
(5) $(x;y);z = x;(y;z)$
(6) $x;1' = x$
(7) $(x;y)^{\smile} = y^{\smile};x^{\smile}$
(8) $x^{\smile};-(x;y) \leqslant -y$.

A relation type algebra satisfying these axioms is called a *relation algebra*. It soon turned out, however, that the RA-axioms do not exhausively generate all valid principles governing binary relations. There are relation algebras that are not representable, as was first shown by Lyndon in [Ly]; perhaps the simplest, finite example was provided by MacKenzie and is given in [TG].

The question whether finitely many identities might be added to the RA-axioms in order to axiomatise RRA was answered negatively by by Monk in [Mo1], while in [Mo2] he showed that it is not even sufficient to add infinitely many axioms in a finite number of variables: infinitely many relation variables are needed. Explicit infinite axiomatizations are known, however, by Lyndon [Ly] or MacKenzie [McK] (cf [Mo2] for the related case of cylindric algebras), but these axiomatizations are not very appealing to our intuitions. Wadge [W] gave another way of recursively enumerating Id_{RRA}, by using a Gentzen type deduction method, in which object variables again are introduced in the proofs. (Maddux [M3] used this method to define varieties RA_{α} between RA and RRA, where the identities holding in RA_{α} being the ones provable in Wadges system by using only α different object variables.) In [M4] Maddux shows that by adding a 'non-logical' operator $^{\circ}$ to the algebras all identities holding in RRA° can be finitely axiomatized.

Some simple sufficient (yet not necessary) conditions for a RA to be representable are known: e.g. in [M1] it is shown that a relation algebra is in RRA if 1 is the sum of functional elements (elements x satisfying $x^{\smile};x \leqslant 1'$.)

1.2 CC: a modal logic of compositon and converse.

In this section we will define a modal logic, the associated algebra of which will have the type of relation algebras. Therefor this logic must have operators corresponding to the composition and converse operators of relation algebras, and a propositional constant for the identity element 1'.

Definition 1.

Let L be a set of propositional constants; the set of *CC-formulas in L* is inductively defined as follows:

(1) all atomic propositions δ,$\underline{tt}$,$\underline{ff}$,p_1,p_2,...are L-formulas ($p_i \in L$).

(2) if φ and ψ are L-formulas, then so are $\neg\varphi$ and $\varphi\wedge\psi$.

(3) if φ and ψ are L-formulas, then so are $\varphi\circ\psi$ and $\otimes\varphi$.

Definition 2.

We will use the following abbreviations, besides the usual ones for the Boolean connectives:

$\diamondminus\varphi$ for $\underline{tt}\circ\varphi$, $\diamondminus'\varphi$ for $\neg\delta\circ\varphi$, $\ominus\varphi$ for $\diamondminus(\varphi\wedge\delta)$

$\diamondbar\varphi$ for $\varphi\circ\underline{tt}$, $\diamondbar'\varphi$ for $\varphi\circ\neg\delta$, $\obar\varphi$ for $\diamondbar(\varphi\wedge\delta)$,

$\boxminus$, $\boxbar$, $\boxminus'$, $\boxbar'$ as the duals of $\diamondminus$, $\diamondbar$, $\diamondminus'$ and $\diamondbar'$, ($\boxminus\varphi\equiv\neg\diamondminus\neg\varphi$, etc.),

$h(\varphi)$ for $(\boxminus\varphi \wedge \boxbar'\boxminus\neg\varphi)$ and $v(\varphi)$ for $(\boxbar\varphi \wedge \boxminus'\boxbar\neg\varphi)$.

(cf. remark 4 for the intuitive meaning of these operators).

The *mirror image* $\mu(\varphi)$ of a formula φ is inductively defined as follows: for atomic propositions α, $\mu(\alpha)=\alpha$; $\mu(\neg\varphi) = \neg\mu(\varphi)$, $\mu(\varphi\wedge\psi) = \mu(\varphi)\wedge\mu(\psi)$, $\mu(\otimes\varphi) = \otimes\mu(\varphi)$ and $\mu(\varphi\circ\psi) = \mu(\psi)\circ\mu(\varphi)$.

Concerning the semantics for this logic, we have a choice. The 'most intended models' are two-dimensional:

Definition 3 Semantics.

A *(proper) model* for this logic is a pair M = (W,V), where W is a set called a *domain*, and V is a *valuation*, i.e. a map assigning a subset of WxW to each atomic proposition, such that $V(\underline{tt}) = W\times W$, $V(\underline{ff}) = \emptyset$, $V(\delta) = \{(x,y) \in W\times W \mid x=y\}$. Elements of WxW are called *worlds* of the model.

A *forcing relation* $\Vdash$ is inductively defined as follows:

(1) For atomic propositions α, $M,x,y \Vdash \alpha$ if $(x,y) \in V(\alpha)$,

(2) $M,x,y \Vdash \neg\varphi$ if $M,x,y \nVdash \varphi$,

$M,x,y \Vdash \varphi\wedge\psi$ if $M,x,y \Vdash \varphi$ and $M,x,y \Vdash \psi$,

(3) $M,x,y \Vdash \varphi\circ\psi$ if there is a z in W such that $M,x,z \Vdash \varphi$ and $M,z,y \Vdash \psi$,

$M,x,y \Vdash \otimes\varphi$ if $M,y,x \Vdash \varphi$.

If no confusion can arise concerning the valuation, we will write $x,y\Vdash\varphi$, etc.

A set of formulas Σ is *satisfiable in a model M* if there is a world (x,y) in M such that $M,x,y \Vdash \varphi$ for all φ in Σ; Σ is *satisfiable* if it is satisfiable in some model. A formula φ is *satisfiable (in a model M)* if $\{\varphi\}$ is. A formula is *valid /valid on a model M* (notation: $\models\varphi$ / $M\models\varphi$) if its negation is not satisfiable/ not satisfiable in M.

Remark 4.

Note that L-formulas, when interpreted in a plane W^2, have the following reading (viz. fig. 1, where CW denotes the current world):

⊗φ: φ holds at the point obtained by mirroring the current one in the diagonal;
⟠φ: somewhere on the same latitude, φ holds;
⊟φ: everywhere on the same latitude, φ holds;
⊖φ: on the diagonal point of this latitude, φ holds;
⟠'φ: somewhere else on this latitude, φ holds;
⌽φ: somewhere on this longititude, φ holds;
h(φ): φ holds only and everywhere on this latitude;
etc.

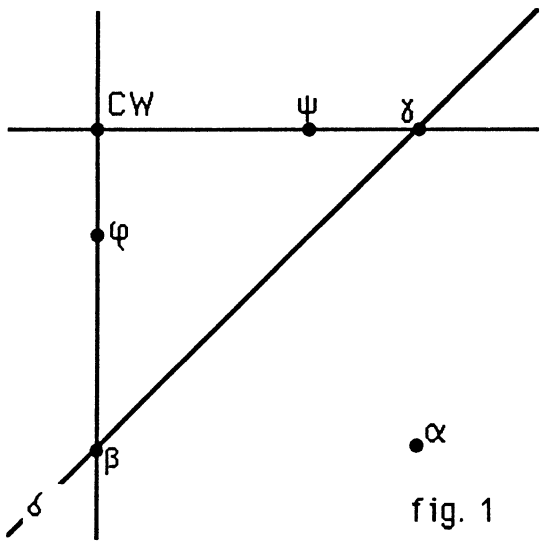


fig. 1

the following formulas hold in CW: φ∘ψ, ⊗α, ⊖β, ⌽γ, ⟠ψ, ⟠'ψ, etc.

It is also useful to have a more abstract, direct approach where the possible worlds are the basic entities themselves. In this case the presence of a n-ary modal operator gives rise to the existence of an n+1-ary 'accessibility' relation in the models:

Definition 5.

Let L_{cc} be the first order language with a triadic relation symbol C, a dyadic R and a monadic I.
ΠL_{cc} is the universal second order language of L_{cc}, i.e. formulas of the form $\forall P_1 \ldots \forall P_n \varphi(P_1,\ldots,P_n)$, where φ is a first order formulas in $L_{cc} \cup \{P_1,\ldots,P_n\}$.

Definition 6.

An *arrowframe* is a structure for L_{CC}, i.e. a pair $F = (W,*)$ where $*$ is an interpretation for the symbols C, R and I. (In the sequel we will blur the difference between syntax and semantics and write C for C*, etc.).

A *model for CC* is a pair (F,V), where F is an arrowframe and V is a *valuation*, i.e. a map assigning subsets of F to each atomic proposition of CC such that $V(\underline{tt}) = F$, $V(\underline{ff}) = \emptyset$, $V(\delta) = I$.

By induction on CC-formulas we define a *forcing relation* $\Vdash$. We only give the clauses for the modal operators:

$F,V,w \Vdash \delta$ if $F \vDash Iw$,

$F,V,w \Vdash \otimes\varphi$ if there is a v with Rwv and $F,V,v \Vdash \varphi$,

$F,V,w \Vdash \varphi\circ\psi$ if there are u,v in F with $F\vDash Cuvw$, $F,V,u \Vdash \varphi$ and $F,V,v \Vdash \psi$.

Concepts like *validity on an arrowframe* ($F\vDash_{CC}\varphi$) are defined in the usual way.

The *complex algebra* of an arrowframe F is defined as the relation-type algebra $CmF = (SbW, \cup, \cap, ^c, \emptyset, W, ;, ^\smallsmile, 1')$, where the operators $;$, $^\smallsmile$ and $1'$ are defined as:

$X;Y = \{z|$ there are x in X, y in Y with Cxyz$\}$,

$X^\smallsmile = \{y|$ there is an x in X with Rxy$\}$

$1' = I$.

Remark 7. It is in accordance with the above definition to view the proper models $M = (W,V)$ of definition 2.2 as models, by setting

$F = (W \times W, C, R, I)$ with

$C = \{ \langle (u,v),(v,w),(u,w)\rangle \mid u,v,w \in W\}$

$R = \{ \langle (u,v),(v,u)\rangle \mid u,v \in W\}$

$I = \{ \langle (u,u)\rangle \mid u \in W\}$.

We call F the *arrowframe over the set W*.

We will need some names for classes of frames:

Definition 8.

For a class of relation-type algebras K, let *ArK* be the class of arrowframes F for which CmF is in K.

$\mathfrak{C}2$, the class of *two-dimensional arrowframes*, is formed by those structures which are isomorphic to an arrowframe over a set.

Now that the syntax and semantics of this logic have been defined, we may look for some generalizations of basic constructions and facts we know from ordinary modal logic. As an example we mention the following:

Definition 9.
Call a relation Z between M = (F,V) and M' = (F',V') a *zigzag* if it satisfies the following conditions:
(1) if xZx', then x and x' force the same atomic propositions (including δ),
(2) if xZx' and Rxy, then there is a x' with xZx' and R'x'y', and analogously in the opposite direction,
(3) If xZx' and Cyvx, then there are y',v' with C'y'v'x', yZy' and vZv', and analogously in the opposite direction.
If we have a zigzag between M = (F,V) and M' = (F',V') which is functional and surjective, we call it a *zigzag morphism* from F onto F'.

By induction on the complexity of CC-formulas one easily proves the following:

Fact 10. For all CC-formulas φ:
(1) If Z is a zigzag between M and M' and wZw', then M,w $\Vdash \varphi$ iff M',w' $\Vdash \varphi$.
(2) If z is a zigzag morphism from F onto F', then F $\models \varphi \Rightarrow$ F' $\models \varphi$.

Just as in the case of ordinary modal logics and modal algebras (cf.[Gb]), notions like the above zigzagmorphism have a familiar relation algebraic counterpart (here taking subalgebras), but we will not go into that matter here.

1.3. CC: Characterizing frames and correspondence.

In this section we will show what additional structure we need to provide arrowframes with in order to let their complex algebras not only have the type of relation algebras, but be (representable, two-dimensional) relation algebras themselves. We also present our results concerning the language (CC or predicate logic) in which these properties can be expressed.
Recall that for a class of relation-type algebras K we defined ArK as the class of arrowframes of which the associated algebra is in K.

Starting with the clas RA, it is immediate from the setup that an arrowframe is in ArRA iff it validates the CC-versions of the RA-axioms. We will show that each of these axioms has a first-order correspondent as well, but first we give an example of such a correspondence: consider the condition that the binary relation R should be functional (each world should have exactly one R-successor). This property is easily expressible in predicate logic by the formula $\forall x \exists y\ (Rxy \wedge \forall y'\ (Rxy' \rightarrow y=y'))$, but one can also show that the class of 'R-functional arrowframes' is characterized by the CC-formula $\otimes\neg\varphi \leftrightarrow \neg\otimes\varphi$. In such a case we call such formulas *correspondents.* Now usually this functional behaviour of R is considered such a basic fact, that it is reflected in the definition of arrowframes: in this section we will take arrowframes to be structures of the type (W, C, f, I), with W, C, I as before and f a *function* from W into W.

<u>Theorem 2.</u> (Maddux)

There is a first order sentence φ_{RA} in L_{CC} such that

$F \models \varphi_{RA} \iff F$ is in FRA.

<u>Proof</u>

We will show that each of the RA-axioms, in CC-form, has a first order correspondent. The sentence φ_{RA} then is the conjunction of these correspondents.

<u>claim 1:</u> $F \models \otimes\otimes\varphi \leftrightarrow \varphi$ (RA4)

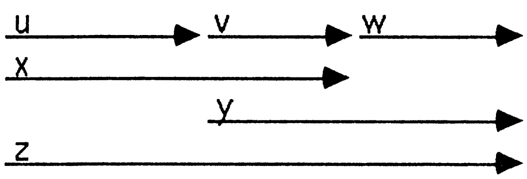
iff $F \models ffx=x$. (4)

<u>proof:</u> the $\Leftarrow$-direction is straightforward, for $\Rightarrow$, suppose $ffu \neq u$. Let V be a valuation with $V(p) = \{x\}$. Then $F,V,x \Vdash p \wedge \neg\otimes\otimes p$.

<u>claim 2:</u> $F \models \varphi\circ(\psi\circ\xi) \rightarrow (\varphi\circ\psi)\circ\xi$ (RA5)

iff $F \models Cvwy \wedge Cuyz \rightarrow \exists x\,(Cuvx \wedge Cxwz)$ (5)

<u>proof:</u> consider the following figure:



First suppose F is an arrowframe validating $\varphi\circ(\psi\circ\xi)\rightarrow(\varphi\circ\psi)\circ\xi$, and Cvwy and Cuyz hold. Let V be a valuation with $V(p)=\{u\}$, $V(q)=\{v\}$, $V(r)=\{w\}$. Then $F,V,z \Vdash p\circ(q\circ r)$, whence $F,V,z \Vdash (p\circ q)\circ r$. Written out this gives the existence of an x with $Cuvx \wedge Cxwz$, quod erat demonstrandum. On the other hand, suppose $F \models$ (5) and let $F,V,z \Vdash \varphi\circ(\psi\circ\xi)$. By the truth definition there are u,v,w and y such that Cvwy, Cuyz, u ⊩ φ, v ⊩ ψ, w ⊩ ξ. Using the existence of an arrow x as given by (1), we get F,V,z ⊩ (φ∘ψ)∘ξ.

claim 3: F ⊨ (φ∘ψ)∘ξ → φ∘(ψ∘ξ) (RA5')

iff F ⊨ Cuvx ∧ Cxwz → ∃y (Cvwy ∧ Cuyz) (5')

proof: like the previous one.

(Remark. In [M2] varieties are defined of relation-type algebras satisfying weaker versions of the asociativity axiom. For all these versions similar correspondences hold, as Maddux shows.)

claim 4: F ⊨ φ → φ∘δ (RA6)

iff F ⊨ ∀x∃y (Iy ∧ Cxyx) (6)

proof: ⇐ is immediate, for ⇒, suppose for u∈W there is no v with Cuvu. Take a V with V(p) = {u}, then F,V,u ⊩ p ∧ ¬(p∘δ).

claim 5: F ⊨ φ∘δ → φ (RA6')

iff F ⊨ ∀x (Cxyz ∧ Iy → x=z) (6')

claim 6: F ⊨ ⊗(φ∘ψ) ↔ ⊗ψ∘⊗φ (RA7)

iff F ⊨ Cuvw ↔ Cf(v)f(u)f(w) (7)

claim 7: F ⊨ ⊗φ∘¬(φ∘ψ) → ¬ψ (RA8)

iff F ⊨ Cuvw ↔ Cf(u)wv (8)

(assuming the function f satisfies ∀x ffx=x.).

proof:

Let F,V,w ⊩ ⊗φ∘¬(φ∘ψ), so there are u,v with f(u) ⊩ φ and v ⊩ ¬(φ∘ψ). Now suppose w ⊩ ψ, then as Cf(u)wv, v ⊩ φ∘ψ, which is a contradiction, so w ⊩ ψ. For the other direction, let F ⊨ ⊗φ∘¬(φ∘ψ) → ¬ψ and assume (u,v,w) is a triple in C. Let V be a valuation with V(p)={f(u)}. V(q)={w}. We claim that v ⊩ p∘q, whence Cf(u)wv holds in F. For, suppose otherwise, i.e. v ⊩ ¬(p∘q); then w ⊩ ⊗p∘¬(p∘q), so w ⊩ ¬q: a contradiction.

(Remark. We may also characterize this property by the CC-formula (φ ∧ ψ∘ξ) → ψ ∘ (ξ ∧ ⊗ψ∘φ).)

The sentence φ_{RA} is of course the conjunction of the (universal closure of the) L_{CC}-formulas 4,5,5',6,6',7 and 8.

□Theorem 1.3.2.

Note that in in L_{CC} we now have the following equivalent characterizations for Cuvw (provided f is in ArRA):

Cuvw	Cvfwfu	Cwfvfu	
Cfuwv	Cfvfufw	CFwuv	(9)

In general, though, CC-formulas will have a universal monadic second-order equivalent on the frame level. We use some standard ([vB3]) correspondence-theoretic means to establish this:

Definition 3.

(1) Let θ be the follwing translation of CC-formulas into first order formulas in $L_{cc} \cup \{P_1, P_2, ...\}$:

(i) $\theta(p_i) = P_i x$ for propositional constants p_i,

$\theta(\delta) = Ix$,

$\theta(\underline{tt}) = Ix \vee \neg Ix$

$\theta(\underline{ff}) = Ix \wedge \neg Ix$,

(ii) $\theta(\varphi \wedge \psi) = \theta(\varphi) \wedge \theta(\psi)$,

$\theta(\neg\varphi) = \neg\theta(\varphi)$,

(iii) $\theta(\varphi \circ \psi) = \exists yz\, (Cyzx \wedge \theta(\varphi)\,[y/x] \wedge \theta(\psi)[z/x]$,

$\theta(\otimes\varphi) = \theta(\varphi)\,[fx/x]$.

(2) Θ is the translation of CC-formulas into ΠL_{cc}-formulas, defined by the following procedure: let $p_1, ... p_n$ be the propositional constants in φ, then $\Theta(\varphi) = \forall P_1 ... \forall P_n \forall x \theta(\varphi)$.

The following lemma expresses the correspondence of φ and $\theta(\varphi)$ on the level of models, and of φ and $\Theta(\varphi)$ on the level of frames:

Lemma 4.

(1) For all models $M = (F,V)$ and worlds u in F:

$$M \models_{cc} \varphi\,[u] \Leftrightarrow M \models \theta(\varphi)\,[u].$$

(2) For all frames **F**:

$$\mathbf{F} \models_{cc} \varphi \Leftrightarrow \mathbf{F} \models \Theta(\varphi).$$

Proof.

(1) Straightforward by induction on the complexity of φ.

(2) By (1): the universal second-order quantification in $\mathbf{F} \models \Theta(\varphi)$ is the classical counterpart of the informal quantification over valuations in $\mathbf{F} \models_{cc} \varphi$.

□Lemma 1.3.4.

The fact that in the above examples this second-order formula reduces to a first-order one, is due to their syntactical form, falling under an appropriate generalization of Sahlqvists Theorem ([Sa], [vB2]]), of which we only give the following case.

Definition 5.

A proposition letter p *occurs positively* in p; it *occurs negatively* in $\neg\varphi$ if positively in φ and the other way around; if p occurs positively (negatively) in φ or ψ, then it does so in $\varphi\wedge\psi$, $\varphi\vee\psi$, $\otimes\varphi$ and $\varphi\circ\psi$. A formula is *positive* if all occurrences of proposition variables are positive.

The set UD of *universally dominated* formulas is the smallest set containing the atomic formulas which is closed under: if φ and ψ $\in$ UD, then so are $\varphi\wedge\psi$ and $\neg(\neg\varphi\circ\neg\varphi)$. The set of *existentially dominated* formulas is the smallest set containing UD which is closed under the operators $\vee$, $\otimes$ and $\circ$.
A formula is in *Sahlqvist form*, or *a Sahlqvist formula*, if it is of the form $\varphi \equiv \psi\rightarrow\xi$, where φ is existentially dominated and ξ is positive.

Theorem 6.
Let φ be a formula in Sahlqvist form. Then φ has an effectively producable first order correspondent φ^* with $F\models_{cc}\varphi \Leftrightarrow F\models\varphi^*$.

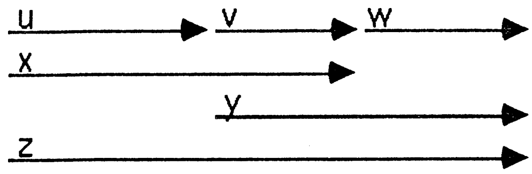
Proof.
Using the substitution method of [vB2], chapter IX. □1.3.6

Turning to the class ArRRA, we know that this class has a CC-characterization (as RRA is a variety) and therefor a ΠL_{cc}-characterization. It is as yet an open question whether ArRRA has a first-order characterizaton as well.

Finally we show that the class $\mathfrak{C}2$ of two-dimensional frames has a first-order characterization. That this class does not have a CC-characterization is immediate from the fact that the class of full relation algebras is not a variety. It is easy to show that $\mathfrak{C}2$ is not closed under taking zigzag-morphic images: let F be the arrowframe on the integers $\mathbb{Z}$. Define $F' = (W,C,R,I)$ with $W = \mathbb{Z}$, $C = \{\langle x,y,z\rangle \mid z = x+y\}$, $R = \{\langle x,-x\rangle \mid x\in\mathbb{Z}\}$ and $I = \{0\}$, and let g be the map from F onto F' defined by $g(x,y) = y-x$. It is the straightforward to verify that g is a zigzagmorphism while F' clearly is not twodimensional.

Theorem 7. An arrowframe F is two-dimensional
iff $F \models \varphi_{RA} \wedge \forall uw\, \exists! vxy\, (Cuvx \wedge Cvyw)$ (10)

Proof.
We only prove the direction from right to left:
Let F be an arrowframe in FRA satisfying the above condition (10) for C. We will show that F is isomorphic to a proper frame.
First, as $F \models (5)$, one easily verifies that
claim 1. $F \models \forall uw\, \exists! vxyz\, (Cuvx \wedge Cvyw \wedge Cvwy \wedge Cxwz)$, (11)

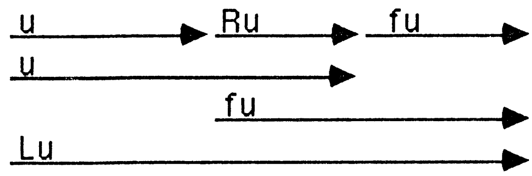


An arrow in a two-dimensional frame will be a pair (a,b). We now define the arrowframe-counterparts Lu and Ru of (a,a) and (b,b):

claim 2. F ⊨ ∀u∃!Lu (ILu ∧ CLuuu) (12)

F ⊨ ∀u∃!Ru (IRu ∧ CuRuu) (13)

Existence of Lu and Ru follows from (3), unicity from (9) and (11), cf.



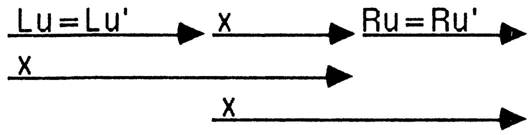
We now define g: W ↦ IxI by g(u) = (Lu,Ru) and prove that g is an isomorphism:

claim 3. g is surjective:

Let u and w be in I. By (11) there are v,x,y and z as in the figure. By Cuvx, Iu, (6') and (9') we get v=x and by Cvwy, Iw and (6'): v=y. So we have Cuvv and Cvwv, whence u and w satisfy the definition of Lv and Rv, so (u,w) = g(v).

claim 4. g is injective.

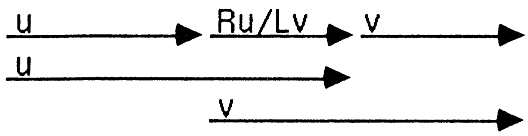
Suppose Lu=Lu' and Ru=Ru'. By the proof of the previous claim both x=u and x=u' satisfy



whence by (10) u=u'.

claim 5. g preserves C, f and I.

For C, suppose Cuvw. We will not prove Lu=Lw or Rv=Rw, only Ru=Lv: both Ru and Lu satisfy



so by (10) Ru=Lv.

claim 6. g antipreserves C, f and I.

This proof is left to the reader.

□Theorem 1.3.7.

1.4 Completeness.

In this section we will give a recursive axiomatization of all CC-formulas valid on the class of twodimensional frames. The axiom system consists of the axioms for Relation Algebras and three derivation rules, of which the third one, IR, is a bit odd and worth some discussion. For the moment we only mention that the rule originates with Dov Gabbay in [G], and that we believe that in this proof we have the first example of a setting in which applying the rule really adds new theorems to a logic.

<u>Definition 1.</u> The modal logic *ACC* consists of

(1) the following *axioms.* :

1. all propositional tautologies.
2. $\varphi C(\psi C\chi) \leftrightarrow (\varphi C\psi)C\chi$
3. $\otimes(\varphi C\psi) \leftrightarrow \otimes\psi C\otimes\varphi)$
4. $\varphi C\delta \leftrightarrow \delta C\varphi \leftrightarrow \varphi$
5. $\otimes\otimes\varphi \leftrightarrow \varphi$
6. $(\varphi\vee\psi)C\chi \leftrightarrow \varphi C\chi \vee \psi C\chi$
7. $\otimes(\varphi\vee\psi) \leftrightarrow \otimes\varphi\vee\otimes\psi$
8. $\otimes\varphi C\neg(\varphi C\psi) \rightarrow \neg\psi$.

(2) the rules of inference

MP. Modus Ponens:
to infer φ from ψ and $\psi\rightarrow\varphi$.

N. Necessitation:
to infer $\otimes\varphi$, $\neg(\neg\varphi C\psi)$ and $\neg(\psi C\neg\varphi)$ from φ.

IR. Redundancy of distinguishing properties:
to infer φ from $h(p)\rightarrow\varphi$, provided p does not occur in φ.

A *deduction* in ACC is a finite string of formulas each of which is either an axiom or follows from earlier formulas by a rule of inference.
A formula φ is a *thesis* of ACC (notation: ACC $\vdash\varphi$ or $\vdash\varphi$ if no confusion arises) if it appears as the last item of a deduction.
A formula φ is a *consequence* of a set Γ of formulas, notation $\Gamma\vdash\varphi$, if there are formulas $\gamma_1,\dots,\gamma_n$ in Γ such that $\vdash(\gamma_1\wedge\dots\wedge\gamma_n)\rightarrow\varphi$.
A set of formulas Γ is *consistent* if <u>ff</u> is not a consequence of Γ; a *maximal consistent* set (short: *MCS*) is a consistent set which has no consistent extension.

We will now prove the soundness and completeness of this system for the class of models of definition 2. As usual, one side of this proof is easy:

Theorem 2. (SOUNDNESS)

Every thesis is valid in $\mathfrak{C}2$.

Proof:

This is a routine check for all axioms. As an example we verify the validity of axiom 8:

Suppose $M,x,y \Vdash \otimes\varphi\circ\neg(\varphi\circ\psi)$. By the truth definition, this implies the existence of a z in M satisfying $M,z,x \Vdash \varphi$ and $M,z,y \Vdash \neg(\varphi\circ\psi)$, whence for all w, $M,z,w \Vdash \varphi$ implies $M,w,y \nVdash \psi$. Taking w=x, we obtain $M,x,y \nVdash \psi$.

The theorem then follows by the observation that the rules of inference preserve validity. We only prove this for R3: suppose $\nvDash \varphi$, then there is a model M=(W,V) and a world (s,t) in M such that $M \nVdash \varphi$ [s,t]. Let V' be the valuation which differs from V only with respect to p: set $V'(p) = \{[x,y] \in W\times W \mid y=t\}$. Then (W,V') makes both h(p) and $\neg\varphi$ true in [s,t]; hence we have $\nvDash h(p)\rightarrow\varphi$.

□Thm 2.

The completeness part of the proof is the place where the action is. As usual, to prove that every valid formula is a thesis, we use contraposition and show that every negation of a non-thesis, or more generally every consistent set, is satisfiable in some twodimensional model. We should mention here that in section 3.4 we give an analogous yet simpler proof for the {⊖, ⦶, ⊟, ◫, ⊗}-fragment of CC; a reader having difficulties with understanding this section is advised to read 3.4 first.

Fact 3.

(1)The following are theses of ACC, as are their mirror images:

1. $\boxminus(\varphi\rightarrow\psi) \rightarrow (\boxminus\varphi\rightarrow\boxminus\psi)$
2. $\boxminus\varphi\rightarrow\boxminus\boxminus\varphi$
3. $\varphi\rightarrow\boxminus\diamondminus\varphi$
4. $\boxminus\varphi\rightarrow\varphi$
5. $\ominus(\varphi\rightarrow\psi) \rightarrow (\ominus\varphi\rightarrow\ominus\psi)$
6. $\ominus\neg\varphi\leftrightarrow\neg\ominus\varphi$
7. $\ominus\ominus\varphi\leftrightarrow\ominus\varphi$
8. $\otimes(\varphi\rightarrow\psi) \rightarrow (\otimes\varphi\rightarrow\otimes\psi)$
9. $\otimes\neg\varphi\leftrightarrow\neg\otimes\varphi$

10. $\varphi \leftrightarrow \otimes\otimes\varphi$
11. $\boxminus\varphi \rightarrow \ominus\varphi$
12. $\boxminus\otimes\varphi \rightarrow \otimes\boxbar\varphi$
13. $\diamondminus\diamondbar\varphi \rightarrow \diamondbar\diamondminus\varphi$
14. $\diamondbar\boxminus\varphi \rightarrow \boxminus\diamondbar\varphi$
15. $\varphi\circ\psi \rightarrow (\varphi \vee \varphi')\circ(\psi \vee \psi')$
16. $(\psi \wedge \ominus\sigma)\circ\chi \rightarrow \psi\circ(\chi \wedge \obar\sigma)$
17. $(\delta \wedge \diamondminus\varphi \wedge \diamondbar\psi) \rightarrow \diamondminus(\varphi \wedge \diamondbar(\varphi\circ\psi))$
18. $\diamondminus\varphi \leftrightarrow \varphi \vee \diamondminus'\varphi$
19. $\diamondminus(v(\varphi) \wedge \psi) \leftrightarrow \boxminus(v(\varphi) \rightarrow \psi)$

(2) If φ is a thesis of ACC, then so are $\boxminus\varphi$ and $\boxbar\varphi$.

<u>Definition 4.</u>

For an MCS Σ we define

$\obar\Sigma = \{\varphi \mid \obar\varphi \in \Sigma\}$,

$\ominus\Sigma = \{\varphi \mid \ominus\varphi \in \Sigma\}$,

$\otimes\Sigma = \{\varphi \mid \otimes\varphi \in \Sigma\} \cup \{\otimes\varphi \mid \varphi \in \Sigma\}$.

An MCS Δ is called *on the diagonal* if $\delta \in \Delta$.

<u>Lemma 5.</u>

(1) Every consistent set has a maximal consistent extension.

(2) If Σ is an MCS, then so are $\obar\Sigma$, $\ominus\Sigma$ and $\otimes\Sigma$. Furthermore $\delta \in \obar\Sigma$ and $\delta \in \ominus\Sigma$.

<u>Proof.</u>

(1) By a standard Lindenbaum construction.

(2) By the theses 7 and 10.

□Lemma 5.

<u>Definition 6.</u>

Γ *is on a row with* Δ, notation $\Gamma ▭ \Delta$, if $\{\varphi \mid \boxminus\varphi \in \Gamma\} \subseteq \Delta$.

Likewise we define: Γ *is in a column with* Δ, and write $\Gamma ▯ \Delta$.

<u>Lemma 7.</u>

(1) $\Gamma ▭ \Delta \Leftrightarrow$ for all $\varphi \in \Delta$, $\diamondminus\varphi \in \Gamma$.

(2) ▭ and ▯ are equivalence relations

(3) For all MCSs Γ, $\ominus\Gamma ▭ \Gamma$.

(4) For all MCSs Γ and Δ: $\Gamma ▭ \Delta \Rightarrow \otimes\Gamma ▯ \otimes\Delta$.

(5) If $\diamondminus\varphi$ is in Γ, then there is a Σ with $\varphi \in \Sigma$ and $\Sigma ▭ \Gamma$

(6) If $\Gamma ▭ \Delta$, $\delta \in \Gamma$ implies $\Gamma = \ominus\Delta$.

(7) If $\Gamma ▭ \Delta$, $v(\varphi) \in \Gamma$ and $v(\varphi) \in \Delta$ then $\Gamma = \Delta$.

<u>Proof.</u>

(1) Standard.

(2) Standard, as $\boxminus$ and $\boxbar$ are S5-modalities.

(3) Straightforward, by definition of $\ominus$.

(4) Idem, by 3.13.

(5) By showing that, under the given conditions, $\{\varphi\} \cup \{\psi \mid \boxminus\psi \in \Gamma\}$ is consistent. Σ can be any maximal extension of this set.

(6) For $\gamma \in \Gamma$, $\gamma \wedge \delta \in \Gamma \Rightarrow \diamondminus(\gamma \wedge \delta) \in \Delta \Rightarrow \neg\ominus\neg\gamma \in \Delta \Rightarrow \ominus\gamma \in \Delta \Rightarrow \gamma \in \ominus\Delta$, so $\Gamma \subseteq \ominus\Delta$. On the other hand, $\varphi \in \ominus\Delta$ implies $\boxminus(\delta \rightarrow \varphi) \in \Delta \Rightarrow \delta \rightarrow \varphi \in \Gamma \Rightarrow \varphi \in \Gamma$. So $\ominus\Delta \subseteq \Gamma$.

(7) Suppose $\gamma \in \Gamma \Rightarrow \diamondminus(v(\varphi) \wedge \gamma) \in \Delta \Rightarrow$ (3.19) $\boxminus(v(\varphi) \rightarrow \gamma) \in \Delta \Rightarrow v(\varphi) \rightarrow \gamma \in \Delta \Rightarrow \gamma \in \Delta$.

□Lemma 7.

<u>Definition 8.</u>

Γ, Π and Σ *form a triangle* if for all $\varphi \in \Pi$ and $\psi \in \Sigma$ we have $\varphi \circ \psi$ in Γ.

<u>Lemma 9.</u> For MCSs Γ, Π and Σ the following propositions hold:

(1) Γ, $\Phi\Gamma$ and Γ form a triangle.

(2) If Γ, Π and Σ form a triangle then $\ominus\Pi = \Phi\Sigma$.

(3) If Γ, Π and Σ form a triangle, then $\Gamma - \Pi$ and $\Gamma \| \Sigma$.

(4) If Γ, Π and Σ form a triangle, then so do Σ, $\otimes\Pi$ and Γ.

(5) If Γ, Π and Σ form a triangle, then so do $\otimes\Gamma$, $\otimes\Sigma$ and $\otimes\Pi$.

(6) For any L-MCS Γ and formula $\varphi \circ \psi$ in Γ, there are L-MCSs Π and Σ such that $\varphi \in \Pi$, $\psi \in \Sigma$ and Γ, Π and Σ form a triangle.

<u>Proof.</u>

A glance at the following figure may help to understand the meaning of the above.

Γ	Σ	$\circ$
Π	$\ominus\Pi = \Phi\Sigma$	$\otimes\Sigma$
$\circ$	$\otimes\Pi$	$\otimes\Gamma$

(1) Straightforward.

(2) Assume $\varphi \in \ominus\Pi$, $\neg\varphi \in \Phi\Sigma$. Then $\ominus\varphi \in \Pi$, $\Phi\varphi \in \Sigma \Rightarrow \ominus\varphi \circ \Phi\neg\varphi \in \Gamma$. But then $(\ominus\varphi \wedge \ominus\neg\varphi) \circ \Phi\neg\varphi \in \Gamma$ by thesis 16, which is impossible.

(3) Straightforward by the definitions.

(4) Let Γ, Π and Σ form a triangle and $\pi' \in \otimes\Pi$, $\gamma \in \Gamma$. If $\pi' \circ \gamma \notin \Sigma$, then $\neg(\pi' \circ \gamma) \in \Sigma$, as Σ is an MCS. Hence $\otimes\pi' \circ \neg(\pi' \circ \gamma)$ is in Γ, contradicting its consistency by axiom 8.

(5) If $\sigma' \in \otimes\Sigma$, $\pi' \in \otimes\Pi$, then $\otimes\sigma \in \Sigma$, $\otimes\pi' \in \Pi$. This gives $\otimes\pi' \circ \otimes\sigma' \in \Gamma \Rightarrow \otimes(\sigma' \circ \pi') \in \Gamma \Rightarrow \sigma' \circ \pi' \in \otimes\Gamma$.

(6) Let $\varphi_1, \varphi_2, \ldots$ be an enumeration of L-formulas. We will define, in a Lindenbaum-like construction, sequences of sets of L-formulas $\Pi_0 \subseteq \Pi_1 \subseteq \ldots$, $\Sigma_0 \subseteq \Sigma_1 \subseteq \ldots$, such that all Π_n, Σ_n are finite and consistent, Π_{n+1} is either $\Pi_n \cup \{\varphi_n\}$ or $\Pi_n \cup \{\neg\varphi_n\}$, Σ_{n+1} either $\Sigma_n \cup \{\varphi_n\}$ or $\Sigma_n \cup \{\varphi_n\}$, and $(\bigwedge\Pi_n) \circ (\bigwedge\Sigma_n)$ is in Γ for all n.
The key observation for the induction step is the following (with $\pi_n = \bigwedge\Pi_n$, $\sigma_n = \bigwedge\Sigma_n$): as $\pi_n \circ \sigma_n$ is in Γ we have $((\pi_n \wedge \varphi_n) \vee (\pi_n \wedge \neg\varphi_n)) \circ ((\sigma_n \wedge \varphi_n) \vee (\sigma_n \wedge \neg\varphi_n))$ in Γ, and so by axiom 6 (at least) one of the following is in Γ:
$(\pi_n \wedge \varphi_n) \circ (\sigma_n \wedge \varphi_n)$, $\quad (\pi_n \wedge \varphi_n) \circ (\sigma_n \wedge \neg\varphi_n)$,
$(\pi_n \wedge \neg\varphi_n) \circ (\sigma_n \wedge \varphi_n)$ or $\quad (\pi_n \wedge \neg\varphi_n) \circ (\sigma_n \wedge \neg\varphi_n)$.
If for instance the second is the case, we take $\Pi_{n+1} = \Pi_n \cup \{\varphi_n\}$, $\Sigma_{n+1} = \Sigma_n \cup \{\neg\varphi_n\}$, etc. It is then straightforward to prove that Γ, $\bigcup_{n\in\omega}\Pi_n$ and $\bigcup_{n\in\omega}\Sigma_n$ form a triangle.

□Lemma 10.

As was mentioned before, we will prove completeness by constructing for a given MCS Γ, a model M in which Γ is satisfied. In every finite stage of the construction we will be dealing with a finite approximation of M called a matrix:

<u>Definition 11.</u>

For n a natural number and L a set of propositional variables we define an L-*matrix* of *size* n to be a pair $\lambda = (\mathcal{I}, \Lambda)$, where
$\mathcal{I}$ is a set of n elements called the *domain* of λ, and
Λ is a coherent L-chronicle on $\mathcal{I}$.
An *L-chronicle* on a set $\mathcal{I}$ is a map which assigns an L-MCS to each pair (v,w) of $\mathcal{I}$. A chronicle Λ is *coherent* if for all $u,v,w \in \mathcal{I}$:

(1) $\Lambda(u,v)$, $\Lambda(w,v)$ and $\Lambda(u,w)$ form a triangle.
(2) $\Lambda(v,u) = \otimes\Lambda(u,v)$.
(3) $\delta \in \Lambda(u,v) \Leftrightarrow u=v$.

Note: usually the domain $\mathcal{I}$ of λ will be formed by a natural number $n = \{0,1,\ldots,n-1\}$, and informally we will speak of MCSs *on the i-th row*, etc.
If λ, λ' are L-,L'-matrices of size n, n', then λ' is said to *extend* λ (notation: $\lambda \subseteq \lambda'$) if $n \leqslant n'$, $\mathcal{I} \subseteq \mathcal{I}'$ and for all u,v in $\mathcal{I}$: $\Lambda(u,v) \subseteq \Lambda'(u,v)$.

A matrix is *maximally distinguishing* if there are, for every $u \in \mathcal{L}$, formulas φ and ψ such that $h(\varphi) \in \Lambda(v,u)$ for every v, and $v(\psi) \in \Lambda(u,v)$ for every v.

A matrix is only an approximation of a model; it will not be perfect:

Definition 12.

An *L-defect* of a matrix $\lambda = (\mathcal{L},\Lambda)$ is a quadruple (v,w,φ,ψ) with $v,w \in \mathcal{L}$, φ and ψ L-formulas such that $\varphi\circ\psi \in \Lambda(v,w)$, but for no u in $\mathcal{L}$ one finds $\varphi \in \Lambda(v,u)$ and $\psi \in \Lambda(u,w)$.

We will prove in lemma 20 that we can repair every defect of a maximally distinguishing matrix λ by adding new points to λ, together with new Λ-images. First we need the following lemmas, of which the first one expresses the fact that if one of the Λ-images in a matrix is enlarged by formulas in new propositional constants, then this extension can be carried over to the whole matrix.

Lemma 13.

Let $\lambda = (\mathcal{L},\Lambda)$ be a maximally distinguishing L-matrix, $u,v \in \mathcal{L}$, $L' \supset L$ and Γ a consistent L'-set with $\Gamma \supset \Lambda(u,v)$. Then there is an L'-matrix $\lambda' = (\mathcal{L},\Lambda')$ extending λ such that $\Lambda'(u,v) \supset \Gamma$.

Proof.

The lemma is proved by induction on $|\lambda|$, the size of λ. Without loss of generality we may assume $\mathcal{L} = \{0,1,\dots,|\lambda|-1\}$, $u=0$ and $v=1$.

For $|\lambda|=2$ we take $\Lambda(0,1)$ to be L'-MCS extending Γ (such a set exists by lemma 7.1). Then we define $\Lambda(1,0)=\otimes\Lambda(0,1)$, $\Lambda(0,0)=\oplus\Lambda(0,1)$ and $\Lambda(1,1)=\ominus\Lambda(0,1)$. It is left to the reader to verify that λ' indeed is a matrix.

If $|\lambda|=n+1$, by the induction hypothesis we have an L'-matrix $\lambda'' = (\{0,1,\dots,n-1\},\Lambda')$ extending λ and such that $\Lambda'(0,1) \supset \Gamma$. We must complete the construction by adding L'-formulas to the L-MCSs $\Lambda(i,j)$ where i or j equals n (viz. the following figure).

Let φ_n be the formula for which $h(\varphi_n) \in \Lambda(i,n)$, for every i.
Define, for $0 \leqslant i \leqslant n-1$:
$\Lambda'(i,n) = \{\varphi |\ \Phi(\varphi \wedge h(\varphi_n)) \in \Lambda'(i,j)\}$ and $\Lambda'(n,i) = \otimes\Lambda'(i,n)$
$\Lambda'(n,n) = \oplus\Lambda'(n,j)$.

(Note that we need not specify j in this definition as ◫ is an S5-modality and $\Lambda(i,j) \Vert \Lambda(i,k)$.).
λ' itself is then defined in the obvious way.

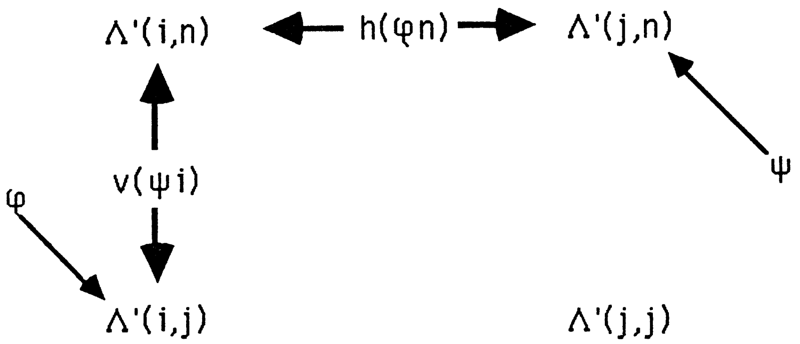
n	$\Lambda(0,n)$	$\Lambda(1,n)$	—	$\Lambda(i,n)$	—	$\Lambda(n-1,n)$	$\Lambda(n,n)$
n-1	$\Lambda'(0,n-1)$	$\Lambda'(1,n-1)$	—	$\Lambda(i,n-1)$	—	$\Lambda'(n-1,n-1)$	$\Lambda(n,n-1)$
	$\mid$	$\mid$		$\mid$		$\mid$	$\mid$
i	$\Lambda'(0,i)$	$\Lambda'(1,i)$	—	$\Lambda(i,i)$	—	$\Lambda'(n-1,i)$	$\Lambda(n,i)$
	$\mid$	$\mid$		$\mid$		$\mid$	$\mid$
1	$\Lambda'(0,1)$	$\Lambda'(1,1)$	—	$\Lambda'(i,1)$	—	$\Lambda'(n-1,1)$	$\Lambda(n,1)$
0	$\Lambda'(0,0)$	$\Lambda'(1,0)$	—	$\Lambda'(i,0)$	—	$\Lambda'(n-1,0)$	$\Lambda(n,0)$
	0	1		i		n-1	n

Claim 1: $\Lambda'(i,n) \supseteq \Lambda(i,n)$.
Proof: Straightforward. □

Claim 2: $\Lambda'(i,n)$, $\Lambda'(n,i)$ and $\Lambda'(n,n)$ are maximal consistent.
Proof: We first proof the maximality of $\Lambda'(i,n)$: as ⟐$h(\varphi_n) \in \Lambda'(i,j)$, at least one of ⟐$(\varphi \wedge h(\varphi_n))$, ⟐$(\neg\varphi \wedge h(\varphi_n))$ is in $\Lambda'(i,j)$, whence one of $\varphi, \neg\varphi$ is in $\Lambda'(i,n)$.
For consistency, suppose both φ and $\neg\varphi$ are in $\Lambda'(i,n)$. Then both ⟐$(\varphi \wedge h(\varphi_n))$ and ⟐$(\neg\varphi \wedge h(\varphi_n))$ are in $\Lambda'(i,j)$. By lemma 9.5 there are MCSs Σ and Π with $\Sigma \Vert \Gamma$, $\Pi \Vert \Gamma$, $\varphi \wedge h(\varphi_n)$ in Σ and $\neg\varphi \wedge h(\varphi_n)$ in Π. This implies $\Sigma \Vert \Pi$, as $\Vert$ is an equivalence relation, and $\Sigma \neq \Pi$, as φ is in Σ and not in Π. But by lemma 9.7 this contradicts the fact that $h(\varphi_n) \in \Sigma$ and $h(\varphi_n) \in \Pi$. □

Claim 3: λ' is coherent.
Proof. We prove that λ' satisfies (2), (1) and (3) of definition 16.
The fact that $\Lambda'(j,i) = \otimes\Lambda'(i,j)$ is immediate, either by induction hypothesis or by definition of $\Lambda'(n,i)$. The proofs for the other conditions in (2) are likewise simple.

For the triangle property (1), it suffices to prove that every triple of ℓ-elements involving n forms a coherent triangle. So first suppose $\psi \in \Lambda'(i,j)$, $\varphi \in \Lambda'(j,n)$. We must show $\psi \circ \varphi \in \Lambda'(i,n)$. As λ was maximally distinguishing, there are formulas φ_n and ψ_i such that $v(\psi_i)$ is in every MCS of the i-th column and $h(\varphi_n)$ in every MCS of the n-th row (cf. the figure)



Now we can establish the following facts:

$(v(\psi_i)\wedge\psi) \in \Lambda'(i,j)$, $(h(\varphi_n)\wedge\varphi) \in \Lambda'(j,n)$

$\Rightarrow$ $\ominus(v(\psi_i)\wedge\psi) \in \Lambda'(j,j)$ and $\Phi(\varphi\wedge h(\varphi_n)) \in \Lambda'(j,j)$ (16.1, 12.3)

$\Rightarrow$ $\ominus(\, v(\psi_i)\wedge\psi \wedge \Phi(\, (v(\psi_i)\wedge\psi)\circ(\varphi\wedge h(\varphi_n)\,) \in \Lambda'(j,j)$ (6.17)

$\Rightarrow$ $\ominus(\, v(\psi_i) \wedge \Phi(\, \psi\circ(\varphi\wedge h(\varphi_n)\,) \in \Lambda'(j,j)$ (prop. logic)

$\Rightarrow$ $\boxminus(v(\psi_i)\rightarrow \Phi(\psi\circ(\varphi\wedge h(\varphi_n))\,) \in \Lambda'(j,j)$ (6.19)

$\Rightarrow$ $\Phi(\psi \circ (\varphi\wedge h(\varphi_n)) \in \Lambda'(i,j)$ (16.1, 12.3)

$\Rightarrow$ $\Phi(h(\varphi_n)\wedge\psi\circ\varphi) \in \Lambda'(i,j)$ (def. of $\ominus$, $h(\varphi_n)$)

$\Rightarrow$ $\psi\circ\varphi \in \Lambda'(i,n)$ (def. of $\Lambda'(i,n)$)

The triangle property for other triples involving n can be proved likewise, or by using lemma 12.3 and 12.4.
The third condition for coherency follows immediately from the fact that λ, the matrix we started with, already satisfied $\delta \in \Lambda'(i,j) \Leftrightarrow i=j$. □

□Lemma 13.

Lemma 14.

If Γ is an L-MCS, $\varphi\circ\psi \in \Gamma$ and $p \notin L$, then $\Gamma \cup \{(\varphi\wedge h(p))\circ\psi\}$ is consistent.

Proof.

By lemma 12.6 there are L-MCSs Π and Σ such that Γ, Π and Σ form a triangle and $\varphi \in \Pi$, $\psi \in \Sigma$. $\Pi \cup \{h(p)\}$ is consistent, for otherwise there would be a $\pi \in \Pi$ such that $\vdash h(p) \rightarrow \neg\pi$. But as $p \notin L$, this would mean by derivation rule 3 that $\vdash \neg\pi$, quod non.

By the previous lemma it easily follows that there is an MCS Γ' extending Γ such that $\varphi\wedge h(p))\circ\psi$ is in Γ'. This means that $\Gamma \cup \{(\varphi\wedge h(p))\circ\psi\}$ is consistent.

□Lemma 14.

Remark: This is the only place in the completeness proof where we need the derivation rule R3.

The following lemma says that every defect of an md matrix λ can be 'repaired' in an md matrix extending λ, both in language and in size.

Lemma 15.

If λ is a maximally distinguishing L-matrix of size n with a certain defect and $L'=L\cup\{p\}$, where $p\notin L$, then there is a maximally distinguishing L'-matrix $\lambda'\supseteq\lambda$ of size n+1, lacking this defect.

Proof.

Assume the elements of $\mathcal{I} = \{0,...,n-1\}$ are numbered in such a way that the defect has the form $(0,n-1,\varphi,\psi)$.
By lemma 14 $\Lambda(0,n-1) \cup \{(\varphi\wedge h(p))\circ\psi\}$ is consistent, so by lemma 13 we can extend λ' into an L'-matrix $(\mathcal{I},\Lambda')$ such that $\varphi\wedge h(p))\circ\psi$ is in $\Lambda'(0,n-1)$. We now define
$\Lambda'(i,n) = \{\varphi|\ \Diamond(\varphi\wedge h(p)) \in \Lambda'(i,j)\}$ and $\Lambda'(n,i) = \otimes\Lambda'(i,n)$,
$\Lambda'(n,n) = \Phi\Lambda'(n,j)$.
It then easily follows that $\varphi \in \Lambda'(0,n)$, $\psi \in \Lambda'(n,n-1)$. The proof of the coherency runs just like the proof of the second claim in lemma 18, except for condition 3; we want to show that $\delta\in\Lambda'(i,n)$ implies i=n.
So suppose $\delta\in\Lambda'(i,n)$, $i\neq n$. Then $\Lambda'(i,n) = \Phi\Lambda'(i,n) = \Lambda'(i,i)$ by 9.6.
So $\Lambda'(0,n) = \Lambda'(i,n) = \Lambda'(i,i)$. This implies $h(\varphi_i) \in \Lambda'(0,n)$, so with lemma 7.7 we have $\Lambda'(0,n) = \Lambda'(0,i)$. Likewise we prove $\Lambda'(n,n-1) = \Lambda'(i,n-1)$. But then $\varphi\in\Lambda'(0,i)$ and $\psi\in\Lambda'(i,n-1)$, so $\varphi\in\Lambda(0,i)$ and $\psi\in\Lambda(i,n-1)$. This contradicts the fact that $(0,n-1,\varphi,\psi)$ was a defect of λ.
To prove that λ' is maximally distinguishing it suffices to show that $h(p)$ distinguishes the n+1-th row and $v(\otimes p)$ the n+1-th column.

□Lemma 15.

Theorem 16. (COMPLETENESS).

ACC is complete with respect to $\mathfrak{C}2$.

Proof.

Suppose Σ is consistent. We're going to construct a model in which Σ is satisfiable.

Define $L_0=\{p|\ p$ is a propositional constant in $\varphi\}$.

Fix an L_0-MCS Σ_0 with $\Sigma_0 \supseteq \Sigma$. Define $\lambda_0 = (\ell_0, \Lambda_0)$ as follows:

-if $\delta \in \Sigma$: $\ell_0=1$, $\Lambda_0(0,0)=\Sigma$.

-if $\delta \notin \Sigma$: $\ell_0=2$, $\Lambda_0(0,1)=\Sigma$, $\Lambda_0(0,0)=\oplus\Sigma$, $\Lambda_0(1,0)=\otimes\Sigma$, $\Lambda_0(1,1)=\ominus\Sigma$.

For the sake of notational simplicity we now assume $\delta \in \Sigma$, so λ_0 is a coherent matrix of size 1.

Suppose for all $n \in \omega$, p_{n+1} is a constant not in L_n and define, for all $n \in \omega$, $L_{n+1}=L_n \cup \{p_{n+1}\}$.

Then $\Sigma \cup \{h(p_1), v(\otimes p_1)\}$ is consistent (cf. the proof of lemma 14), so it can be extended to an L_1-MCS Σ_1.

Fix an enumeration of all quadruples (m,n,φ,ψ) with m,n natural numbers and φ,ψ L_0-formulas.

Now, an iterative application of lemma 20 yields the existence of a chain of matrices $\lambda_1 \subseteq \lambda_2 \subseteq \ldots$ such that

(1) $\lambda_1 = (1, \{(0,0,\Sigma_1)\})$

(2) Every λ_n is a maximally distinguishing L_n-matrix, and if $\lambda_{n+1} \neq \lambda_n$, $\ell_{n+1} = n+1 = \{0,1,\ldots,n\}$.

(3) If λ_n has no defect then $\lambda_{n+1}=\lambda_n$. Otherwise, in λ_{n+1} the first defect of λ_n (i.e. the one appearing first in the fixed enumeration of quadruples (m,n,φ,ψ)) is removed in λ_{n+1}.

One can easily show that, if λ_n has a certain defect, this will eventually be repaired, i.e. there is an $m > n$ such that λ_m does not have this defect.

Now let Λ be the "union of the L_0-part of the Λ_n's", i.e. Λ is a map assigning L_0-MCSs to elements of $\ell \times \ell$, where $\ell = \bigcup_{n\in\omega} \ell_n$.

By the following definition of V we give a model $M = (\ell, V)$:

$V(p) = \{(s,t)|\ p \in \Lambda(s,t)\}$

Truth lemma: For every L_0-formula φ: $M,s,t \models \varphi$ iff $\varphi \in \Lambda(s,t)$.

Proof: by formula-induction:

(1) for atomic formulas the assertion is clear by definition of V.

(2) for $\varphi \equiv \neg\psi$ or $\varphi \equiv (\psi \wedge \chi)$, the proof is a routine check.

(3) the case $\varphi \equiv \otimes\psi$ is treated by observing that $\otimes\Lambda(s,t)=\Lambda(t,s)$.

So consider the case where $\varphi \equiv \psi \circ \chi$.

First, suppose $M,s,t \models \varphi$. Then there is a u in ℓ with $M,s,u \models \psi$ and $M,u,t \models \chi$. By induction hypothesis $\psi \in \Lambda(s,u)$, $\chi \in \Lambda(u,t)$. Let $n = 1 + \max(s,t,u)$. Then $s,t,u \in \ell_n$ and by definition of Λ, ψ

$\in \Lambda_n(s,u)$ and $\chi \in \Lambda_n(u,t)$; as Λ_n is coherent, this implies $\psi \circ \chi \in \Lambda_n(s,t) \subseteq \Lambda(s,t)$.
For the other direction, suppose $\psi \circ \chi \in \Lambda(s,t)$; let n = 1+ max(s,t), then s,t $\in \ell_n$ and $\psi \circ \chi \in \Lambda_n(s,t)$. If there is no u in ℓ_n with $\psi \in \Lambda_n(s,u)$ and $\chi \in \Lambda_n(u,t)$ then this is a defect of λ_n. In that case there must be an $\lambda_m \supseteq \lambda_n$ in which this defect occurs no longer. This means there is a v in ℓ_m with $\psi \in \Lambda_m(s,v)$ and $\chi \in \Lambda_m(v,t)$. As $\Lambda_m \subseteq \Lambda$ this yields $M,s,t \models \psi \circ \chi$ by the induction hypothesis.

□Truth lemma.

As $\Sigma \subseteq \Lambda(0,0)$, we have indeed found a model in which Σ is satisfiable.

QED.

1.5. CC and the 'difference operator'.

Lately two related extensions of modal logic have been developed which increase the expressive power of the language. Blackburn ([Bl]) proposed to add so-called 'nominals' to the language, i.e. propositional constants which are to be true in exactly one world. In [Ko] Koymans suggests adding a unary operator [D] (and its dual, $\langle D\rangle\varphi \equiv \neg[D]\neg\varphi$) to the language of modal (and temporal) logic. This modal operator is intended to have inequality as its accessibility relation:
(*) $M,V \Vdash [D]\varphi\,[w]$ iff $M,V \Vdash \varphi\,[w']$ for all w' with $w \neq w'$.
This extension of the language makes the logic a lot more expressive. E.g., one now can express the fact that a formula φ is true

somewhere in the frame:	$\langle D\rangle\varphi \vee \varphi$	(short: $\langle A\rangle\varphi$)
at a unique point:	$\langle A\rangle(\varphi \wedge [D]\neg\varphi)$	(short: $U\varphi$).

Furthermore, on the frame level there are new classes of frames which can be defined, e.g.

the irreflexive frames by	$F\varphi \rightarrow \langle D\rangle\varphi$,	or
the connected frames by	$\langle A\rangle\varphi \rightarrow (P\varphi \vee \varphi \vee F\varphi)$	

in the case of temporal logic.
Koymans gives the following sound and complete axiomatisation for pure D-logic:
Besides the propositional tautologies, we have as axioms
(D1) $[D](\varphi \rightarrow \psi) \rightarrow ([D]\varphi \rightarrow [D]\psi)$

(D2) $\varphi \rightarrow [D]\langle D\rangle\varphi$
(D3) $\langle D\rangle\langle D\rangle\varphi \rightarrow (\varphi \vee \langle D\rangle\varphi)$,
and as derivation rules we have Modus Ponens and [D]-Necessitation.

Here we consider the addition of the D-operator to the system CC, from the point of view of both expressibility and completeness.

Definition 1.
Let CCD be the language CC extended with the D-operator. Arrowframes with valuations form the models for CCD, where the semantics for the D-operator are as defined in (*).

The following theorem expresses the fact that in the language CCD we can distinguish the two-dimensional frames (modulo isomorphy):

Theorem 2.
Let φ_{RA} be defined as in 1.3.2 and let φ_{2D} be the conjunction of the following two CCD-formulas:
(EX) $\langle A\rangle\varphi \wedge \langle A\rangle\psi \rightarrow \langle A\rangle((\varphi\circ\underline{tt})\circ\psi)$
(UN) $(U\varphi\circ\underline{tt})\circ U\psi \rightarrow U(U(\varphi\circ\underline{tt})\circ U\psi)$
Then an arrowframe F is isomorphic to a two-dimensional frame iff $F \models \varphi_{RA} \wedge \varphi_{2D}$.

Proof.
As the other direction is straightforward, suppose F is an arrowframe validating φ_{RA} and φ_{2D}. Then F is in FRA, so by 1.3.6 we only need prove that F ⊨ (10). Now one easily verifies that (EX) and (UN) correspond to the existence and uniqueness part of the formula (10). □Theorem 1.5.2.

Now consider the following formula which may be seen as a CC-definition of the D-operator:

$$\langle D\rangle\varphi \leftrightarrow (\neg\delta\circ\varphi \vee \varphi\circ\neg\delta \vee \neg\delta\circ\varphi\circ\neg\delta) \qquad (*)$$

It is easy to show that for frames in ArRRA, this formula holds iff F is two-dimensional. We now have an easy axiomatisation of CCD-logic:

Definition 3.
Let ACCD be the CCD-logic given by the following axioms:
(1) all D-axioms
(2) all ACC-axioms
(3) $\langle D\rangle\varphi \leftrightarrow (\neg\delta\circ\varphi \vee \varphi\circ\neg\delta \vee \neg\delta\circ\varphi\circ\neg\delta)$
and derivation rules:

(1) Modus Ponens
(2) D-Necessitation
(3) CC-Necessitation
(4) To infer $\vdash\varphi$ from $\vdash(p \wedge [D]\neg p) \rightarrow \varphi$, provided p does not occur in φ.

Theorem 4.
ACCD is sound and complete with respect to the class of two-dimensional frames: $\varphi \in$ ACCD iff $\mathfrak{C}_2 \models \varphi$.

Proof.
As the proof is essentially the same as the completeness proof for CC-logic, the details won't be spelled out here: for an MCS Σ in ACCD, replace all occurrences of the D-operator in all formulas by its definition (*). The obtained set Σ^* then can be showed to be consistent in ACC, so there is a proper model M such that $M,x,y \Vdash \Sigma^*$ for a world (x,y) in M. It is then straightforward to verify that $M,x,y \Vdash \Sigma$ as well. □

One might wonder whether adding the D-operator to the language might enable us to get rid of the odd derivation rule IR and give a finite axiomatization with only regular derivation rules. That this is not possible, neither for for the D-operator, nor for any other extension of the language with finitely many 'logical' operators, was shown by Biro in [Bi]. (Here we call an operator *logical* if it has a first-order truth definition on proper models.) However, by adding a non-logical operator to the algebraic language of Relation Algebras, one can get a finite axiomatizaton of the identities holding in RRA, as Maddux shows in [M4].

1.6. Applications for relation algebras.

In this section we show how two of the results of this chapter can be applied in the theory of Relation Algebras. First we will show that the completeness proof for CC gives an analogous recursive enumeration of all identities in RRA, and second, we will give a simple version of the Sahlqvist theorem for Relation Algebras. Of course these results could be both generalized and placed in a more abstract framework (of Boolean Algebras with operators, cf. [JT]), but for the sake of clarity we prefer our presentation.

Definition 1.

Let $P = \{p_1, p_2, \ldots\}$ be a set of propositional constants for CC and $X = \{x_1, x_2, \ldots\}$ a set of individual variables. Define a mapping τ of CC-formulas in P into RA-terms in X: for the atomic formulas we set $\tau(\bot) = 0$, $\tau(\top) = 1$, $\tau(\delta) = 1'$, $\tau(p_i) = x_i$, and τ is a homomorphism with respect to the operators, i.e. $\tau(\varphi \circ \psi) = \tau(\varphi);\tau(\psi)$, etc. It is easily seen that τ is a bijection; let σ be the inverse of τ.

Σ is the smallest set of identities in L_{RA} containing Id_{RA} which is closed under deduction and under the following rule R:

if $1;-y \vee 1;y;0' \vee t(\underline{x}) = 1$ is in Σ,

then so is $t(\underline{x}) = 1$, provided y does not occur in $\underline{x}$.

For our purpose it is convenient to assume that all identities in Id_{RRA} have the form $t(\underline{x})=1$. This assumption causes no loss of genreality, as in every Boolean Algebra every identity $s(\underline{x})=t(\underline{x})$ is equivalent to $(s(\underline{x}) \wedge t(\underline{x})) \vee (-s(\underline{x}) \wedge -t(\underline{x})) = 1$.

Lemma 2.

For all RA-terms t: If ACC $\vdash \sigma(t)$ then $t = 1$ is in Σ.

Proof.

For any ACC-provable formula φ we show $\tau(\varphi) = 1 \in \Sigma$ by induction on the length of the (shortest) proof for φ.

For ACC-axioms the claim follows from the observation that they are the translations of the RA-axioms, so suppose φ is derived by applying a rule. If this rule is Modus Ponens or Necessitation, the claim follows more or less directly, so suppose ACC $\vdash \varphi$ by ACC $\vdash h(p_i) \rightarrow \varphi$, where p_i does not occur in φ. By the induction hypothesis we get, after rewriting $\tau(h(p_i)\rightarrow\varphi)$, $1;-x_i \vee 1;x_i;0' \vee \tau(\varphi) = 1 \in \Sigma$, where x_i does not occur in $\tau(\varphi)$. As Σ is closed under the rule R, this gives $\tau(\varphi) \in \Sigma$.

□Lemma 1.6.2.

Theorem 4.3. $\Sigma = Id_{RRA}$.

Proof.

$\subseteq$: (Soundness).

As Id_{RRA} is an equational theory, it suffices to show that Id_{RRA} is closed under R. Suppose $t(\underline{x})=1 \notin Id_{RRA}$, then there is a representable relation algebra A with elements $\underline{a}$ of A such that $A \models t(\underline{a}) \neq 1$. As $A \in PS(FRA)$ (cf. section 1.1), we may assume that A is a subalgebra of a full relation algebra B with $B \models t(\underline{a}) \neq 1$. So there is an element (p,q) of VxV such that $(p,q) \notin t(\underline{a})$. Now define b as $\{(r,s) \in V \times V \mid s=q\}$, then $(p,q) \notin 1;-b \vee 1;b;0'$. So $B \not\models 1;-y \vee 1;y;0' \vee t(\underline{x}) = 1$, whence the latter identity is not in Id_{RRA}.

$\supseteq$: (Completeness)

We have to show that if $t=1 \notin \Sigma$, then there is a RRA **A** such that $\mathbf{A} \not\models t=1$.

Now if $t=1 \notin \Sigma$, then $\sigma(-t)$ is ACC-consistent by lemma 4.1, whence it has a model $M=(W,V)$. Let **A** be the (representable!) relation algebra generated by $\{V(p_i) \mid x_i$ occurs in $\sigma(-t)\}$. It is then straightforward to verify that $\mathbf{A} \models t(V(p_i)) \neq 1$.

□Theorem 4.2.

The second and last example we treat deals with some varieties K of relation-type algebras for which ArK has a first order characterization. We might give a straightforward analogon of 1.3.5, but we only give a simple case. A *negationless* identity is an identity in which the complement symbol "-" does not occur.

Lemma 4.

Let F be an arrowframe, so φ and ψ are CC-formulas. Then we have $F \models \varphi \rightarrow \psi \Rightarrow \mathrm{CmF} \models \tau(\varphi) \leqslant \tau(\psi)$.

Proof.

Straightforward. □

Theorem 5.

If K is a variety of relation-type algebras, defined by negationless identities, then ArK has a first order characterization.

Proof.

Let Σ_K be the set of negationless identities characterizing K, then for any arrowframe F we have by the previous lemma

$$F \in \mathrm{ArK} \iff F \models \{\sigma(s) \leftrightarrow \sigma(t) \mid s=t \in \Sigma_K\}.$$

But as every CC-formula $\sigma(s) \rightarrow \sigma(t)$ meets the constraints of theorem 1.3.5, the result is immediate. □Theorem 1.6.5.

2. INTERVAL TEMPORAL LOGIC AS TWODIMENSIONAL MODAL LOGIC.

2.1. Representation of time in intervals.

When thinking of time, we usually have an ontology in mind in which time*points* are the basic entities. One might argue, however, that from a philosophical or psychological standpoint, it is more natural to start with *periods* of time, and this century has seen several logical studies of a period-based represenntation of time. Especially in the field of artificial intelligence and natural language processing there is an increasing tendency towards modelling time as a set of periods with appropriate relations. Reading the literature (e.g. [vB1], [A], [L]), one finds a wide variety of approaches, differences lying in the fields of both language and ontology, eg.:
Should linguistically inspired *events* be basic, or should periods be *intervals*, i.e. uninterrupted stretches of times; should time be linear or branching, dense, discrete or continuous; should point-like periods of no duration exist or not; what are to be the basic relations between intervals; should we define periods as sets of points after all. Should we use (first order) predicate calculus, a modal system or perhaps an algebraic language to talk about periods of time.

Concerning the latter choice, we should mention here that our primary interest is in modal logics of intervals. Though, contrasting the point-based approach of time, classical logic has been dominant in the field, some research has been devoted to modal systems, cf. [vB1],[HMM],[H1],[R],[HS],[V].
In the ontology we have in mind, time is considered to be linear: we will use the term 'interval' for an uninterrupted stretch of time, informally visualized by a horizontal line segment: ____________i .
In general, two intervals i,j can have one of thirteen relative positions: one of the seven as showed in the figure below, or one of the converse relatons of these (where equals is its own converse).
In [LM] this is the starting point for Ladkin and Madux to develop a relation algebraic theory of intervals in the ordering of the rationals: the thirteen posible relations are seen as atoms in a finite relation algebra.

i equals j:

i meets j:

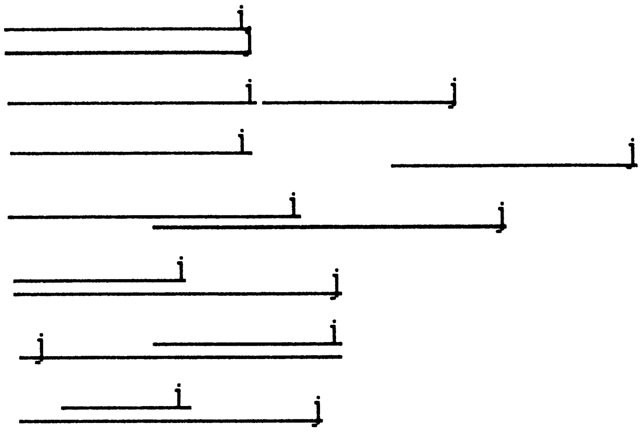
i precedes j:

i overlaps j:

i starts j:

i ends j:

i during j:



In a modal logic of intervals, each of these relations (or their unions) may be taken as an accessibility relation for a modal operator. In [HS] a logic is defined with operators F, P and $\Diamond$, corresponding to resp. the following relations: precedes$^{\vee}$, precedes, and the union of during$^{\vee}$, starts$^{\vee}$ and ends$^{\vee}$. In [HS] one has operators corresponding to the meets, starts and ends relations and their converses, e.g. $\langle B\rangle\varphi$ holding at an interval i if it has an interval j such that j starts i and φ holds at j. In the same article one can find some results concerning the complexity of the validity problem for several classes of frames for this logic. In [V] sound and complete axiom systems are given for several classes of frames.
Here we will present our results concerning a more expressive logic called CDT in which we have binary operators having a ternary accessibility relation A. This relation A may be defined as follows:

$Aijk \equiv$ i starts k $\wedge$ i meets j $\wedge$ j ends k, viz.

2.2. CDT, a modal logic chopping intervals.

1. Syntax.

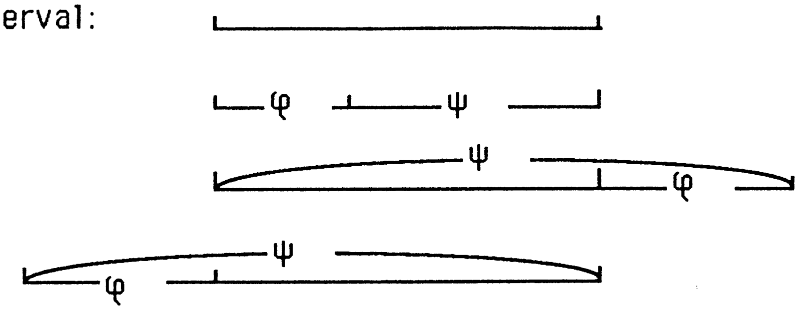
Besides the usual Boolean connectives, the system CDT has three binary modal operators: C, D and T, and a propositional constant, π. The intuitive picture is given by the following figure:

current interval:

$\varphi C\psi$

$\varphi T\psi$

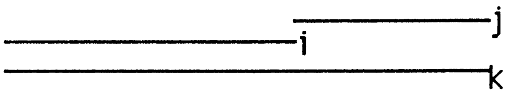
$\varphi D\psi$



i.e. $\varphi C\psi$ holds in an interval if it can be chopped into two pieces, at the first of which φ is to hold, and at the second ψ. The constant π will hold for "point-intervals", i.e. intervals of zero duration.

<u>Semantics.</u>
We have a choice between an ontology in which intervals are the primary objects and one in which they are defined as sets of points in an underlying linear order. We present both approaches, the most general one first:

<u>Definition 2.</u>
An *i-frame* is a triple $J = (I,A,P)$ with $A \subseteq {}^3I$ and $P \subseteq I$. Intuitively, $Aijk$ represents the following situation:



i.e. k is the "sum" of two adjacent intervals i and j.
An *i-model* is a pair $M = (J,V)$ with J an i-frame and V a valuation, i.e. a map assigning subsets of I to atomic propositions. We can define a *truth forcing relation* $\Vdash$ for i-models in the following way:

$M,i \Vdash p$ if $i \in V(p)$
$M,i \Vdash \pi$ if $i \in P$
$M,i \Vdash \varphi \wedge \psi$ if $M,i \Vdash \varphi$ and $M,i \Vdash \psi$,
$M,i \Vdash \varphi C\psi$ if there are j,k in I with $Ajki$ and $j \Vdash \varphi$ and $k \Vdash \psi$,
$M,i \Vdash \varphi T\psi$ if there are j,k in I with $Aijk$ and $j \Vdash \varphi$ and $k \Vdash \psi$,
$M,i \Vdash \varphi D\psi$ if there are j,k in I with $Ajik$ and $j \Vdash \varphi$ and $k \Vdash \psi$.

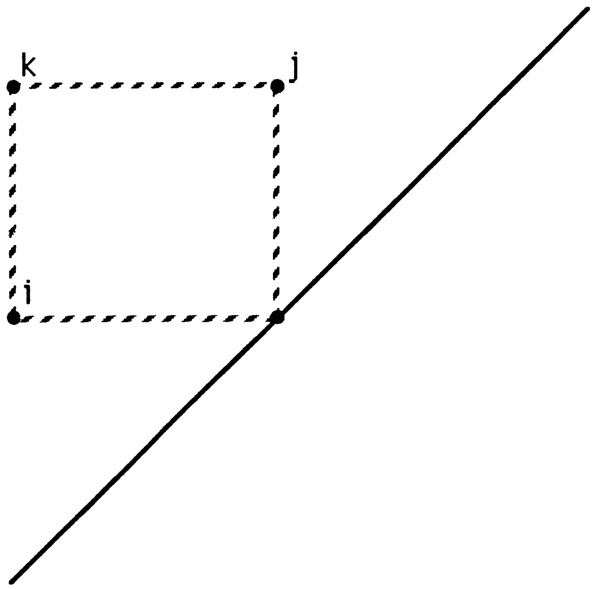
A *frame* is a pair $F = (T,<)$ with $<$ a linear ordering on T. For a frame F we define the *set of F-intervals* $INT(F) = \{[s,t] \mid s,t \in F, s \leqslant t\}$ and the *i-frame on F*, $I(F) = (INT(F),A,P)$ with
$([s,t],[u,v],[w,x]) \in A$ if $s=w$, $t=u$ and $v=x$,
$[s,t] \in P$ iff $s=t$.

A *model* is an i-model (J,V) with J based on a frame F. In such a case we denote the model by (F,V). Note that for models we have F,V,[s,t] $\Vdash$ φCψ iff there is a u with $s \leqslant t \leqslant u$, [s,u]$\Vdash$φ and [u,t]$\Vdash$ψ.

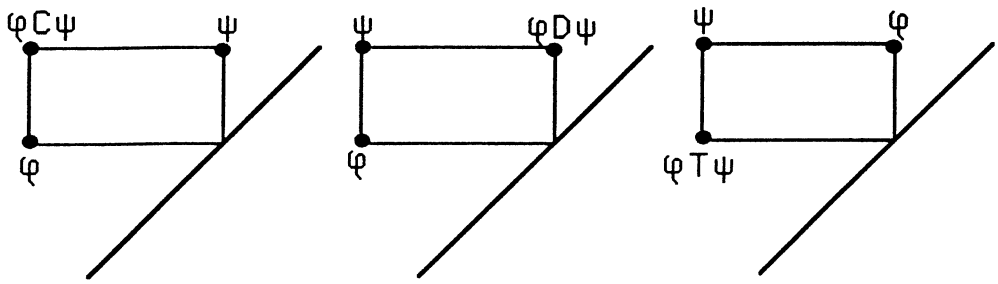
For both kind of models, the notions of *satisfiability* and *validity* are defined in the usual way.

3. Geometrical representation.

If we represent intervals [s,t] of INT(F) as points (s,t) in the 'North-Western Halplane' of F^2, we get the following interpretation for (i,j,k) ∈ A:



For the interpretation of the modal operators, we then obtain the following picture:



4. Compass-operators.

We will use the following abbreviations:

σ for ¬π,

$\Diamond^{\rightarrow}$φ for σCφ, $\Diamond^{\downarrow}$φ for φCσ

$\Diamond^{\leftarrow}$φ for σDφ, $\Diamond^{\uparrow}$φ for σTφ,

$\Diamond_{EW}\varphi$ for $(\Diamond_E\varphi \vee \varphi \vee \Diamond_W\varphi)$, $\Diamond_{NS}\varphi$ for $\Diamond_S\varphi \vee \varphi \vee \Diamond_N\varphi)$,
$\Box\varphi$ for $\neg\Diamond_{NS}\neg\varphi$, etc.
In the light of the previous paragraph it will be clear how we can give these operators a 'compass-interpretation' in point-based frames, e.g. $\Diamond_S\varphi$ holds at a two-dimensional point X iff there is a φ-point right *south* of X. In terms of intervals, the meaning of the compass operators is given by the following scheme:
$\Diamond_S\varphi$ holds at an interval if it has a *starting* interval where φ holds.
$\Diamond_E\varphi$ holds at an interval if it has an *ending* interval where φ holds.
$\Diamond_N\varphi$ holds at an interval if it *starts* a φ-interval,
$\Diamond_W\varphi$ holds at an interval if it *ends* a φ-interval.
Here the relations 'starts' and 'ends' are supposed to be irreflexive: an interval does not start or end itself. We use the name *HS* for the sublogic of CDT in which we only have the compass operators. This name is after Halpern and Shoham, who defined this system in [HS]. In [V] it is shown that, using the compass operators, one can define, for each possible relative position P of two intervals, an operator $\langle P\rangle$ such that $\langle P\rangle\varphi$ holds at i iff φ holds at an interval j which has position P with respect to i.

2.3. Some CDT-theory.

When a new modal logic is defined, it is natural to apply the familiar theory of modalities to this logic and to ask old questions about the new subject. We will do this here, but for lack of space (and surprising results) we will only touch a small number of topics, and only lightly.
To start with, we can look at the capacity of CDT to characterize classes of frames and develop its correspondence theory. Analogous to the choice in semantics for CDT, we have a choice in the first-order language which is to be the target of the correspondence translation (cf. 1.3.3): we may choose a language with a triadic predicate symbol A and a monadic P, or a language with only a dyadic symbol $<$. We won't give the formal definitions, but treat some examples instead. First we treat some formulas which will come back as axioms for the class of p-frames later on. In the propositions 1-6 we consider correspondences for i-frames J. The fact that the involved formulas have first order correspondents is a direct consequence of their Sahlqvist form (cf. 1.3.6).

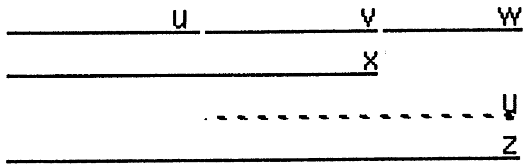
Proposition 1.

$$J \models (\varphi C\psi)C\xi \rightarrow \varphi C(\psi C\xi) \qquad (1)$$

$$\Leftrightarrow \quad J \models \forall uvwz \;\; [\; \exists x\,(Auvx \wedge Axwz) \rightarrow \exists y\,(Auyz \wedge Avwy) \;] \qquad (*1)$$

Proof.

First, the following picture should make clear what is going on:



Suppose $J \models (*1)$ and there are V,z with $z \Vdash (\varphi C\psi)C\xi$. Then there are u,v,w and x with Axwz and $w \Vdash \varphi C\psi$, Auvx, $u \Vdash \varphi$ and $v \models \psi$. By (*1) there is a y with $Auyz \wedge Avwy$. This means $y \Vdash \psi C\xi$, and then z $\Vdash \varphi C(\psi C\xi)$.

For the other direction, suppose $J \not\models (*1)$. Then there are u,v,w,z and x with Auvx and Axwz, but for no y we have Auyz and Avwy. If V is a valuation with V(p) = {u}, V(q) = {v} and V(r) = {w}, then it is straightforward to show that $J,V,z \Vdash (pCq)Cr \wedge \neg(pC(qCr))$. □

In the same way one can show that (*1) is characterized by the following CDT-formulas as well:

$$\varphi T(\psi T\xi) \rightarrow (\varphi T\psi)T\xi \qquad (2)$$

$$\psi D(\varphi T\xi) \rightarrow \varphi T(\psi D\xi) \qquad (3)$$

Note that (*1) implies transitivity of the starting relation of intervals: if u starts x and x starts z then u starts z. This transitivity is reflected in (1) if we take both φ and ψ to be $\underline{tt}$. (1) then reads as $\Diamond\Diamond\xi \rightarrow (\underline{tt}C\underline{tt})T\xi$, and as $\underline{tt}C\underline{tt}$ implies $\underline{tt}$, we get $\Diamond\Diamond\xi \rightarrow \Diamond\xi$, a familiar correspondent of transitivity.

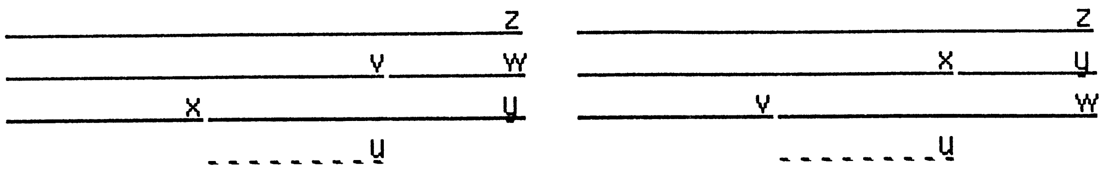
Proposition 2.

$$J \models \varphi T(\psi C\xi) \rightarrow [\; \psi C(\varphi T\xi) \vee (\xi T\varphi)T\psi \;] \qquad (4)$$

$$\Leftrightarrow \quad J \models \forall vwxy \;\; [\; \exists z \;(\, Avwz \wedge Axyz)$$
$$\rightarrow \exists u \;[\; (Axuv \wedge Auwy) \vee (Avux \wedge Auyw) \;]\,] \qquad (*4)$$

Proof.

The proof runs in the same line as the previous one. The following picture may be of use; here v should be seen as the current interval and φ, ψ and ξ as belonging to w, x and y.



□

Note that in (*4) some kind of 'internal linearity is involved: if both meetingpoints, of v and w and of x and y, are in the interval z, then one of them is before the other, or they are identical.
In some respect the following pair of formulas can be seen as the converses of the previous ones:

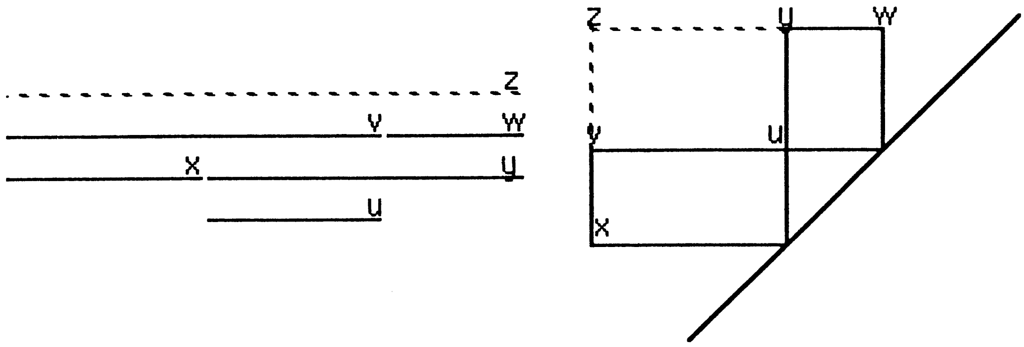
<u>Proposition 3.</u>

$$J \models \psi C(\varphi T\xi) \to \varphi T(\psi C\xi) \qquad (5)$$

$$\Leftrightarrow\ J \models \forall vwxy\ [\ \exists u\,(Axuv \wedge Auwy) \to \exists z\,(Axyz \wedge Avwz)\ ] \qquad (*5)$$

<u>Proof.</u>

Again we only give a picture, and here its geometrical representation too:



Again, v is the current interval and in the proof one should consider a valuation V with $V(\varphi)=\{w\}$, $V(\psi)=\{x\}$ and $V(\xi)=\{y\}$. □

Note that (*5) implies what is called in [V] 'North-Western directedness': if v and y are resp. west and north of u, then there is a z north of v and west of y. One can easily show that this property is characterized by the HS-formula $\Diamond\Diamond\xi \to \Diamond\Diamond\xi$, the same one we obrtain by taking both φ and ψ to be <u>tt</u> in (3).

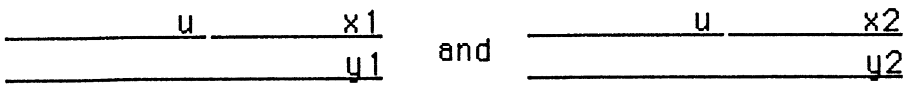
<u>Proposition 4.</u>

$$J \models \varphi C(\xi D\psi) \vee (\psi D\xi)T\varphi \to (\xi T\varphi)C\psi \vee (\varphi T\xi)D\psi \qquad (6)$$

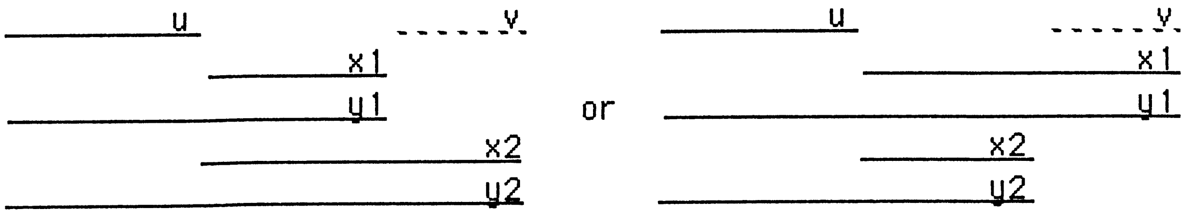
$$\Leftrightarrow\ J \models \forall x1x2y1y2\ [\ \exists u\,(Aux1y1 \wedge Aux2y2) \to \exists v\ [\,(Ax1vx2 \wedge Ay1vy2) \vee (Ax2vx1 \wedge Ay2vy1)\,]\ ] \qquad (*6)$$

Proof.

(*6) expresses that if we have both



then there is an interval v connecting the x1,y1- and the x2,y2-endpoints; either



If we keep this in mind, the proof is straighforward. □

Here we may consider (*6) as a sort of generalization of 'external linearity': if x1 and x2 have the same beginning point then one of them starts the other, or they are identical.
We conclude this this part with mentioning two correspondences involving the π-constant:

Proposition 5.

$J \models \pi C\varphi \leftrightarrow \varphi$ (7)

$\Leftrightarrow$ $J \models \forall vw\ (\exists u\,(Auvw \wedge Pu) \leftrightarrow v=w)$ (*7)

Proposition 6.

$J \models \neg\pi Ctt \rightarrow \neg\pi$ (8)

$J \models \forall w\ (\exists uv\,(Auvw \wedge \neg Pu) \rightarrow \neg Pw)$. (*8)

Next, we have to consider the problem how to characterize the class of two-dimensional frames among the i-frames. As in the case of CC-frames, this cannot be done in CDT. Again we can apply a version of the wellknown theorem of preservation of modal validity under zigzagmorphisms:

Definition 8.

Let J = (I,A,P) and J' = (I',A',P') be two i-frames. A function f: I↦I' is a *zigzagmorphism from J onto J'* if

(1) f is surjective

(2) f is a homomorphism

(3) Assume A'u'v'w'.

If $f(u)=u'$ then there are v,w with $Auvw$, $f(v)=v'$ and $f(w)=w'$.
If $f(v)=v'$ then there are u,w with $Auvw$, $f(u)=u'$ and $f(w)=w'$.
If $f(w)=w'$ then there are u,v with $Auvw$, $f(u)=u'$ and $f(w)=w'$.

Proposition 9.

If f is a zigzagmorphism from F onto F' then for all CDT-formulas: $F \models \varphi \Rightarrow F' \models \varphi$.

Proposition 10.

There is no (set of) CDT-formulas characterizing the two-dimensional frames.

Proof.

Consider the i-frames $F = (INT(\mathbb{Q},<),A,P)$ and $F'=(I,A,P)$ with $I = \{q \in \mathbb{Q} \mid q \geqslant 0\}$, $A = \{\langle p,q,r \rangle \mid p+q=r\}$ and $P = \{0\}$. Let f be the function from $INT(\mathbb{Q},\leqslant)$ onto I mapping intervals on their length, i.e. $f([p,q]) = q-p$. It is straightforward to verify that f is a zigzagmorphism, so if Φ were a set of CDT-formulas characterizing the two-dimensional i-frames, by the previous proposition we would have $F' \models \varphi$ for all φ in Φ whence F' would be isomorphic to a p-frame, which cannot be the case. □

In the first order language with predicates A and P we don't have any problems in defining twodimensional frames. We might do this by defining the meets relation $\|$ ($u \| v \equiv \exists w\, Auvw$) and then proceeding like Allen and Hayes do in [AH] (cf. [L], Ch. 5), where points are defined as equivalence classes of meeting pairs of intervals. In this way we wouldn't even need the P-predicate (except for some defining formulas, as our i-frames need an interpretation for P), but for lack of space we will give an analogous formula to the one in theorem 1.3.6 which characterizes the twodimensional arrowframes.

Definition 11.

Let POINT-BASED be the conjunction of the following formulas:
(*1), (*1m),(*4),(*5),(*6),(*6m),(*7),(*7m),(*8),
(*9) $\forall puvq\ (Pp \wedge Pq \wedge Apuu \wedge Apvv \wedge Auqu \wedge Avqu \rightarrow u=v)$
(*10) $\forall pq\ (\ Pp \wedge Pq \rightarrow [\ p=q \veebar \exists u\,(\neg Pu \wedge Apuu \wedge Auqu) \veebar \exists u\,(\neg Pu \wedge Aquu \wedge Aupu)\ ]\ )$
where (*1m),... are the mirror-images (obvious definiton) of (*1),... and $\veebar$ stands for the exclusive or, i.e. $\varphi \veebar \psi \equiv (\varphi \vee \psi) \wedge \neg(\varphi \wedge \psi)$.

Theorem 12.

Let J = (I,A,P) be an i-frame. Then

J ⊨ POINT-BASED ⟺ J is isomophic to a p-frame.

Proof.

This proof runs along the same lines as the one for arrowframes (1.3.6): for each interval a unique 'startpoint-interval' s(u) and 'endpoint-interval' e(u) are defined and we show that the map g: u↦[s(u),e(u)] is an isomorphism between J and the frame on the point-intervals P (suitably ordered). □

Considering correspondence from the viewpoint where points are the primary entities, we only give two examples: call a frame dense if it satisfies $\forall st\,(s<t \rightarrow \exists u(s<u<t))$, discrete when $F \models \forall st\,(s<t \rightarrow [\exists u\,(s<u \wedge \neg\exists v\,(s<v<u)) \wedge \exists u\,(u<t \wedge \neg\exists v\,(u<v<t))]$.
Let length1 be the CDT-formula $\sigma \wedge \neg(\sigma C \sigma)$

Proposition 13.

Let F be a p-frame.

(1) $F \models \pi \vee (\sigma C \sigma) \Longleftrightarrow$ F is dense

(2) $F \models \sigma \rightarrow [\,(\text{lengtht1}C\underline{tt}) \wedge (\underline{tt}C\text{length1})\,] \Longleftrightarrow$ F is discrete.

Proof.

Straighforward. □

Note that the CDT-formulas characterizing density and discreteness are variable-free. This will turn out to be of great use in section 2.5; we denote the CDT-formulas of 7.1 and 7.2 by DENSE resp. DISCRETE.
In CDT we can also characterize the classes of e.g. the Dedekind-complete frames or the 'iso-choppable' ones (i.e. F consisting of a frame F' with an isomorphic copy of F' glued behind it), but as this has already been shown in [HS], resp [V], for the subsysten HS of CDT, we omit it here.

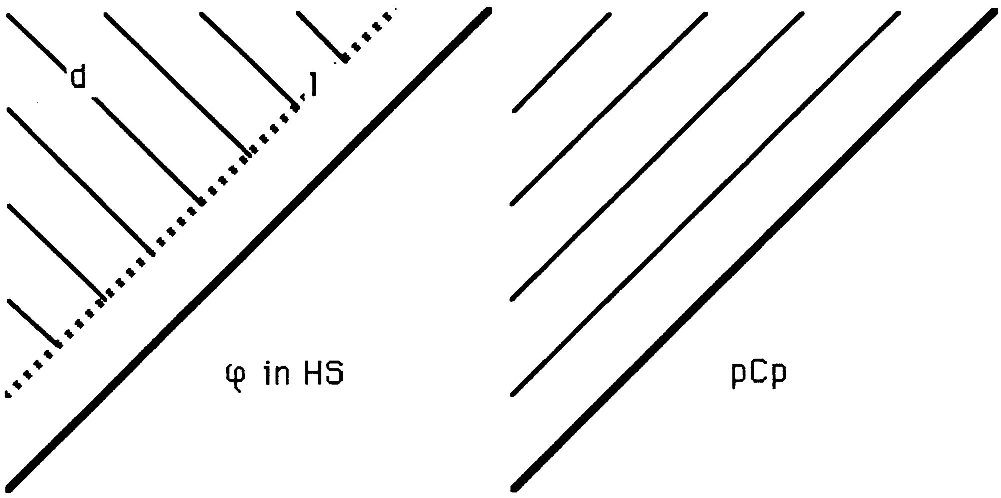
We finish this section with a result concerning the expressive power of CDT and HS. In [V] it is proved that on the class of dense frames, CDT is more expressive: the formula pCp has no equivalent in HS. Here we show that the same holds if we consider the ordering of the integer numbers.

Theorem 14.

There is no HS-formula φ such that for all valuations V, we have $\mathbb{Z},<,V,[s,t] \Vdash pCp$ iff $\mathbb{Z},<,V,[s,t] \Vdash \varphi$.

Proof.

Consider the model $(\mathbb{Z},<,V)$ with p assigned to those intervals [s,t] for which s+t is even, not a four-fold. We claim that for every HS-formula φ, there is a natural number n such that if $t-s \geqslant n$ then $[s,t] \Vdash \varphi$ iff $[s-1,t+1] \Vdash \varphi$. The claim can be proved by formula-induction; its meaning is vizualized in the picture in the next figure:

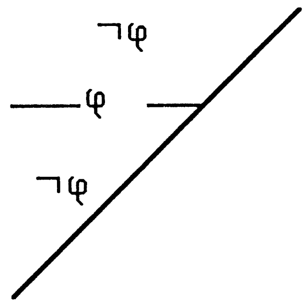


North-West of l (the line where s+t = n), each diagonal line d either satisfies $d \subseteq V(\varphi)$ or $d \subseteq V(\neg\varphi)$. As we have for the CDT-formula pCp that $[s,t] \Vdash pCp$ iff $[s-1,t+1] \nVdash pCp$, this means that pCp cannot have an equivalent in HS. □

2.4. Axiomatic completeness for CDT.

In this section we will give an axiomatic system with which we can recursively enumerate the CDT-formulas valid on the class of all point-based frames. The system is very similar to the ACC axiom system of section 1.4; it uses an inequality rule as well, for which we need the following definition:

$h(\varphi) \equiv \boxminus\varphi \wedge \square\boxminus\neg\varphi \wedge \square\boxminus\neg\varphi$, viz.



<u>Definition 1.</u>

The axiomatic system *ACDT* consists of

(1) (all substitution instances of) the following *axioms* and their mirrorimages:

A all propositional tautologies

B $(\varphi \vee \psi)C\chi \leftrightarrow \varphi C\chi \vee \psi C\chi$
$(\varphi \vee \psi)T\chi \leftrightarrow \varphi T\chi \vee \psi T\chi$
$\varphi T(\psi \vee \chi) \leftrightarrow \varphi T\psi \vee \varphi T\chi$

C 1 $\varphi C(\psi C\chi) \leftrightarrow (\varphi C\psi)C\chi$
2 $\varphi T(\psi T\chi) \leftrightarrow (\psi C(\varphi T\chi) \vee (\chi T\varphi)T\psi$
3 $[\ \varphi C(\chi D\psi \vee (\psi D\chi)T\varphi\ ] \leftrightarrow [\ (\chi T\varphi)C\psi \vee (\varphi T\chi)D\psi\ ]$

D 1 $\pi C\varphi \leftrightarrow \varphi$
2 $\pi T\varphi \leftrightarrow \varphi$
3 $\varphi T\pi \leftrightarrow \varphi$

E $\sigma C\underline{tt} \leftrightarrow \sigma$

F 1 $\neg(\varphi T\psi)C\varphi \rightarrow \neg\psi$
2 $\neg(\varphi T\psi)D\psi \rightarrow \neg\varphi$
3 $\varphi T\neg(\psi C\varphi) \rightarrow \neg\psi$

(2) the following *derivation rules* :

R1 Modus Ponens: if φ and ψ are theses, then so is ψ.

R2 Generalization: if φ is a thesis, then so are $\neg(\neg\varphi C\psi)$, $\neg(\neg\varphi T\psi)$, $\neg(\psi T\neg\varphi)$ and their mirror images.

R3 Inequality: if $h(p) \rightarrow \varphi$ is a thesis and p does not occur in φ then φ is a thesis as well.

<u>2. The meaning of the axioms.</u>

The A- and B-axioms are needed for any binary existential modal operator and the axioms under C, D and E have already been explained in section 2.3. They correspond to the axioms 5 and 6 of ACC. The E-

axioms correspond to the last RA-axioms, but there is a bit more to say of them:
In ordinary point-based temporal logic we have two modal operators $\Diamond_1$ and $\Diamond_2$. In principle these operators could have, in a Kripke semantics, *two* arbitrary accessibility relations R_1 and R_2, but we want R_2 to be the converse of R_1, thus obtaining a future operator and a past operator. This can be axiomatized by the scheme $\varphi \rightarrow \Box_1 \Diamond_2 \varphi$. (Note that the formulatoin of this scheme is independent of any other demand like transitivity we make of the accessibility relation R.
Now in the case of CDT-logic something similar is going on: as we have three binary modal operators, in general the frames for the logic will have three ternary accessibility relations R_c, R_d and R_t. Just as in the case described above, the E-axioms are expressing that R_c, R_d and R_t are no more than 'directions' of one ternary relation A.

<u>Theorem 3.</u> (SOUNDNESS AND COMPLETENESS).

A CDT-formula is a thesis iff it is valid on the class of p-frames: $\vdash \varphi \Leftrightarrow \models \varphi$.

<u>Proof.</u>

The soundness part of the proof is straightforward.
As the proof for the $\Leftarrow$-direction of this theorem is very involved, and very similar to the completeness proof for CC, it is omitted here. (The only real difference is the fact that CDT-models presuppose a linear ordering. This constraint got considerable attention in the completeness proof for HS in [V], and does not cause any real problems in the context of CDT-logic.)

$\Box$Theorem 2.4.3.

We can now proceed to define sound and complete axiom systems for the classes of the dense and the discrete frames: (cf. 2.3.7.)

<u>Definition 4.</u>

(1) $ACDT_{de}$ is the axiom system ACDT where we have added the formula DENSE as an axiom.
(2) $ACDT_{di}$ is the axiom system ACDT where we have added the formula DISCRETE as an axiom.
(3) $ACDT_Q$ is the axiom system ACDT where we have added the formulas DENSE, $\Diamond$<u>tt</u> and $\Diamond$<u>tt</u> as axioms.

Theorem 5.
(1) φ is a thesis of $ACDT_{de}$ iff φ is valid on the class of dense p-frames.
(2) φ is a thesis of $ACDT_{di}$ iff φ is valid on the class of discrete p-frames
(3) φ is a thesis of $ACDT_{Q}$ iff F is valid on $(\mathbb{Q}, <)$.

Proof.
The proofs for these propositions are more or less the same: Soundness is straightforward; for completeness, one simply copies the completeness proof for ACDT-logic. Then arriving at the truth lemma, one observes that the formula DENSE (resp. DISCRETE, resp DENSE $\wedge \Diamond \underline{tt} \wedge \Diamond \underline{tt}$) is an element of every MCS, so DENSE (resp. ...) is valid on the model, we found in the proof, and as this formula only uses the propositional constants π and $\underline{tt}$, it is valid on the underlying frame as well. It is then straighforward (cf. proposition 2.3.7.) to verify that this frame is dense (resp. discrete, resp. isomorphic to the ordering of the integers).

□Theorem 2.4.5.

6. Undecidability.
The above is about the best we can get, viz. the results in [HS] concerning the complexity of the validity problem for the compass subsystem HS of CDT. Useful *fragments* of the language may be decidable: e.g. the Sahlqvist forms (cf. def 1.3.5) have first order equivalents, so by decidability of the theory of linear orders, decidability for the Sahlqvist fragment follows immediately.

2.5. A natural deduction system for CDT.

Introduction 0.
As was mentioned in 1.1, a sound and complete natural deduction system for the identities holding in RRA was given in 197? by William Wadge ([W]). The idea behind the Wadge approach is to first translate RA-identities α into formulas $\varphi(\alpha)$ of the predicate calculus in such a way that full relation algebras where α holds correspond to standard classical models of $\varphi(\alpha)$ and vice versa, and then to give an elegant axiomatisation of the fragment of the first order language which is the codomain of the translation map. The disadvantage of the method is that it expands the formalism by adding point variables, thus violating the paradigm of the algebraic

variable-free approach towards relations. Adopting Wadges idea for our logic, we only need take special care of the fact that our semantics presupposes a linear orderering of the frames. This explains the presence of the symbols 'S' ($\simeq <$) and 'R' ($\simeq \leqslant$) in the target set of predicate formulas, and the need to provide deduction rules for these symbols.

Definition 1.

The set *AT* of *atomic terms* consists of all CDT-formulas and the symbols S, R and Ω. The set *T* of *terms* is defined as the closure of AT under Boolean operators. A *(V-)formula* is an expression of the form $x\alpha y$, where x and y are variables and α is an term. The *negation* $\neg F$ of a formula $F \equiv x\alpha y$ is defined as $\neg F \equiv x\neg\alpha y$. For a set Γ of V-formulas we denote the set of variables occurring in Γ by $Var(\Gamma)$.

Note that with this definition, the set of CDT-formulas is a (proper) subset of the set of terms.

Definition 2.

A *(V-)model* is a pair $M = (D,<,b)$, where $(D,<)$ is a linear frame and b is a function mapping variables on elements of D and terms on subsets of INT(D). Furthermore, b is required to satisfy the following properties:

(1) $b(\pi) = \{(s,s) | s \in D\}$, $b(S) = <$, $b(R) = INT(D)$, $b(\Omega) = \emptyset$.

(2) $b(\alpha \wedge \beta) = b(\alpha) \cap b(\beta)$, etc.

(3) $b(\alpha C \beta) = \{(s,t) \in INT(D) | \exists u \in D\, [(s,u) \in b(\alpha) \wedge (u,t) \in b(\beta)]\}$,
$b(\alpha T \beta) = \{(s,t) \in INT(D) | \exists u \in D\, [(s,u) \in b(\beta) \wedge (t,u) \in b(\alpha)]\}$,
$b(\alpha D \beta) = \{(s,t) \in INT(D) | \exists u \in D\, [(u,s) \in b(\alpha) \wedge (u,t) \in b(\beta)]\}$

Definition 3.

Let M be a model, F a formula, say $x\alpha y$, and Γ a set of formulas. *F is true in M, M satisfies F,* or *M is a model for F,* notation $M \models F$, if $(b(x),b(y))$ is in $b(\alpha)$. *Γ is true in M* if all its formulas are, *F (Γ) is valid*, notation $\models F$ ($\models \Gamma$), if it is true in all models. *F follows semantically from Γ,* or *Γ implies F* if F is true in all the models for Γ.

Lemma 4.

For CDT-formulas φ, φ is valid on the class of linear models iff $x(\neg R \vee \varphi)y$ is valid.

Proof.

By a straighforward induction on the complexity of φ. □

Thus we may obtain a recursive enumeration of all valid CDT-formulas by doing so for the valid entailments $\models F$.

<u>Definition 5.</u>

We give a *natural deduction system* for V-formulas.by defining a notion "$\vdash$" of *deducibility* between sets of formulas and formulas. Formally, $\vdash$ is defined as the smallest relation for which $\{\varphi\}\vdash\varphi$ holds and which is closed under the following *rules of inference* :

In these rules x,y and z are arbitrary variables, φ is an arbitrary CDT-formula, α and β are arbitrary terms, F and G are arbitrary formulas and Γ is an arbitrary set of formulas, with the exception that the rules C^+, D^+, and T^+ and may only be applied when $z \notin Var(\Gamma)$.

(F^+) $$\frac{\Gamma \vdash F}{\Gamma, G \vdash F}$$

(F^-) $$\frac{\Gamma \vdash G \quad \Gamma, G \vdash F}{\Gamma \vdash F}$$

($\wedge^+$) $$\frac{x\alpha y \quad x\beta y}{x\alpha\wedge\beta y}$$

($\wedge^-$) $$\frac{x\alpha\wedge\beta y}{x\alpha y} \qquad \frac{x\alpha\wedge\beta y}{x\beta y}$$

($\vee^+$) $$\frac{x\alpha y}{x\alpha\vee\beta y} \qquad \frac{x\beta y}{x\alpha\vee\beta y}$$

($\wedge^-$) $$\frac{\Gamma, x\alpha y \vdash F \quad \Gamma, x\beta y \vdash F \quad \Gamma \vdash x\alpha\vee\beta y}{\Gamma \vdash F}$$

($\neg^+$) $$\frac{\Gamma, x\alpha y \vdash z\Omega z}{\Gamma \vdash x\neg\alpha y}$$

($\neg^-$) $$\frac{\Gamma, x\alpha y \vdash F \quad \Gamma, x\neg\alpha y \vdash F}{\Gamma \vdash F}$$

(Ω^+) $$\frac{x\alpha y \quad x\neg\alpha y}{z\Omega z}$$

(Ω^-) $$\frac{x\Omega x}{F}$$

(C^+) $$\frac{x\alpha z,\ z\beta y}{x\alpha C\beta\varphi y}$$

(C^-) $$\frac{\Gamma, x\alpha z, z\beta y \vdash F \quad \Gamma \vdash x\alpha C\beta y}{\Gamma \vdash F}$$

(T^+) $$\frac{xRy,\ x\alpha z,\ y\beta z}{x\alpha C\beta y}$$

(T^-) $$\frac{\Gamma, xRy, y\alpha z, x\beta z \vdash F \quad \Gamma \vdash x\alpha T\beta y}{\Gamma \vdash F}$$

(D^+) $$\frac{xRy,\ z\alpha x,\ z\beta y}{x\alpha D\beta y}$$

(D^-) $$\frac{\Gamma, xRy, z\alpha x, z\beta y \vdash F \quad \Gamma \vdash x\alpha D\beta y}{\Gamma \vdash F}$$

(R^+) $$\frac{xS\vee\pi y}{xRy}$$

(R^+) $$\frac{xRy}{xS\vee\pi y}$$

(S_1)	$\dfrac{xSy \quad ySz}{xSz}$	(π_1)	$\dfrac{}{x\pi x}$
(S_2)	$\dfrac{x\neg Sy \quad x\neg \pi y}{ySx}$	(π_2)	$\dfrac{x\pi y \quad y\pi z}{x\pi z}$
(S_3)	$\dfrac{xSy \quad ySx}{x\Omega x}$	(π_3)	$\dfrac{x\pi y}{y\pi x}$
(S_4)	$\dfrac{x\varphi y}{xRy}$	(π_4)	$\dfrac{x\alpha y \quad y\pi z}{x\alpha z}$

Definition 6.

A *theory* is a set formulas which is closed under $\vdash$, i.e. $\Gamma \vdash F$ implies $F \in \Gamma$. A theory Γ is *consistent* if for no x, $x\Omega x$ is deducible from Γ, it is *complete* if for every CDT-term α and variables x,y, one of $x\alpha y$, $y\alpha x$, $x\neg\alpha y$ or $y\neg\alpha x$ belongs to Γ.

A theory Γ' is a *saturation* of Γ, if it contains Γ and satisfies the following condition for the operator C:

whenever $x\alpha C\beta y$ is in Γ, there is a z with $x\alpha z$ and $z\alpha y$ in Γ',

and the analogons for the operators D and T. (Note that in such a case Γ' might need more variables than Γ.)

A theory Γ is *saturated* if it is a saturation of itself.

Theorem 7: Soundness.

If $\Gamma \vdash F$ then $\Gamma \models F$.

Proof.

Straightforward. □Theorem 7.

We now proceed to prove completeness, i.e. we want to show that $\Gamma \models \varphi$ implies $\Gamma \vdash \varphi$. As usual this is done by contraposition: given a Γ and an F for which $\Gamma \not\vdash F$, we will construct a model for Γ in which F is not true. It easily follows from the inference rules for negation that this can be done by showing that every consistent theory has a model. We need the following lemmas:

Lemma 8.

Every consistent theory has a complete extension in the same variables.

Proof.

By the standard Lindenbaum construction. □Lemma 8.

Lemma 9.

Every consistent theory has a consistent saturation.

Proof.

Suppose Γ is a consistent theory, $x\alpha C\beta y \in \Gamma$ but there is no z in Γ with $x\alpha z$ and $z\beta y$ in Γ, Let z be a variable not occurring in Γ. We claim that the set $\Gamma \cup \{x\alpha z, z\beta y\}$ is consistent. For, suppose this is not the case, then $\Gamma, x\alpha z, z\beta y \vdash a\Omega a$. As $x\alpha C\beta y \in \Gamma$ we have $\Gamma \vdash x\alpha C\beta y$. Applying the rule C^- this gives $\Gamma \vdash a\Omega a$, contradicting the consistency of Γ.

So, using a similar procedure for every quintuple like the above (x,y,C,α,β), we can find extensions $\Gamma_1 \subseteq \Gamma_2 \subseteq \Gamma_3$... of Γ such that the union of these extensions wil be a consistent saturation of Γ. (By introducing a *new* variable z_i for every Γ_i we can ensure that no added formulas in different Γ_i's will conflict.) □Lemma 9.

Lemma 10.

Every consistent theory has a complete, saturated extension.

Proof.

Using the previous two lemmas, we can construct for any consistent theory Γ, a sequence of consistent theories $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \Gamma_3$ ... satisfying

(1) every Γ_{2i+1} is a complete extension of Γ_{2i}.

(2) every Γ_{2i+2} is a saturation of Γ_{2i+1}.

The union of these theories will be the complete, consistent and saturated extension of Γ we were looking for. □Lemma 10.

Theorem 11. Completeness.

Every consistent theory has a model.

Proof.

Let Γ be a consistent theory. By the previous lemma we may assume for this proof that Γ is complete and saturated.

Define the following relation ≡ on Var(Γ): $x \equiv y$ if $x\pi y$ is in Γ. Using the rules (π_1), (π_2) and (π_3) one can easily verify that ≡ is an equivalence relation. Let [x] be the equivalence class of the variable x and define $D = Var(\Gamma)/\equiv\ = \{[x] \mid x \in Var(\Gamma)\}$.

We want D to be the domain of our model; we set b(x) = [x] for variables x, $b(\alpha) = \{ ([x],[y]) \mid x\alpha y \in \Gamma\}$. (Rule π_4 ensures that this is a correct definition). The ordering $<$ is defined as b(S). It is easily proved that $<$ is a strict linear ordering, the rules S1, S2 and S3 giving that $<$ is resp. transitive, linear and asymmetric. That $b(\alpha) \subseteq \leqslant$ for any α, follows from S4. Condition (1) of definition 2 is satisfied by definition of D and b, and by the rules R^+ and R^-, (2) is obtained by using the inference rules for the Boolean connectives. Of condition (3) we only prove the part for C:

$b(\alpha C\beta) = \{(s,t) \in INT(D) | \exists u \in D\, [(s,u) \in b(\alpha) \wedge (u,t) \in b(\beta)]\}$.

This statement is equivalent to the following ones:

- $([x],[y]) \in b(\alpha C\beta)$ iff there is a z with $([x],[z]) \in b(\alpha)$ and $([z],[y]) \in b(\beta)$
- $x\alpha C\beta y \in \Gamma$ iff there is a z with $x\alpha z$ and $z\beta y$ in Γ.

Now of the above equivalence the direction from right to left is immediate, as by definition theories are closed under deduction and $x\alpha z$, $z\beta y \vdash x\alpha C\beta y$ by rule C^+. The other direction is just the statement that Γ is saturated.

It then only remains to be shown that $M = (D,<,b)$ is a model for Γ. But by definition of D,$<$ and b it is almost immediate that $M \models F$ for all formulas F in Γ.

□Theorem 11.

3. CONNECTIONS.

3.1. CCλ and CDT.

The main line of this report is the similarity between interval temporal logic and relation algebras, although up till now this line has hardly been made explicit. In this section we will say something about the connection between the two two-dimensional logics, but as the matter soon gets very complex (yet deep nor difficult), we won't give many details, and only consider proper models here.
To start with, we want to stress the differences with respect to these semantics:
First, CDT-logic presupposes a linear ordering $<$ of the set of timepoints, wheras CC does not. But the existence of a linear ordering is very easy expressible in the language of relation algebras:

Proposition 1.
A binary relation Λ on a set V is a linear ordering iff it satisfies:

(1)	$\Lambda \mid \Lambda \subseteq \Lambda$	(transitivity)
(2)	$\Lambda \cap Id = \emptyset$	(irreflexivity)
(3)	$\Lambda \cup \Lambda^{-1} \cup Id = U$.	(linearity)

Proof.
Straightforward. □

One might proceed here in an abstract direction by defining the concept of a relation-algebra-with-a-linear-ordering (being an element λ satisfying $\lambda;\lambda \leqslant \lambda$, $\lambda \wedge 1' = 0$ and $\lambda \vee \lambda^{\smile} \vee 1' = 1$) and looking at the connection of these algebras with the associated algebras of CDT-logic. (In fact, this was the way how the ACDT-axioms were obtained.) Here we will do something like that for the modal logic side: we add an atomic constant λ to the language and demand that $V(\lambda)$ is a linear ordering of the domain of the model.
But we have to take the second difference between CC and CDT into account as well: the fact that for two timepoints s,t, CDT-formulas can only be evaluated at either [s,t] or [t,s], whereas CC-formulas can always be evaluated at both pairs. This gap can be bridged by only taking possible worlds in CC-models into account where λ holds.

So now we can define an adapted version CCλ of CC and show that CDT can be embedded as a subsystem of it:

Definition 2.

CCλ is a set of formulas defined just as CC, with the one difference that there is a new logical constant λ; we abbreviate $\underline{\lambda} \equiv \lambda \vee \delta$.

A model for CCλ is a pair $M = (W,V)$ which is a model for CC and satisfies the constraints that $V(\lambda)$ is a linear ordering of W and that $V(p) = (V(p))^{-1}$ for all propositional variables p.

The forcing relation $\Vdash_{cc\lambda}$, validity $\vDash_{cc\lambda}$, etc. are defined as usual.

Next we define a translation from CDT-formulas into CCλ-formulas. Before giving the formal definition, we treat the case of the operator T as an example:

Suppose $\varphi T \psi$ holds at an interval $[s,t]$, then $s \leqslant t$ and there is a u with $t \leqslant u$, $[s,u] \Vdash_{cdt} \psi$ and $[t,u] \Vdash \varphi$. As the translation $\theta(\varphi T \psi)$ we will get, in principle, $\theta(\psi) \circ \theta(\varphi)$, but to capture the relative position of s and u, resp. t and u, we 'turn' $\theta(\psi)$ 'into' $\theta(\psi) \wedge \underline{\lambda}$ (as $s \leqslant u$) and $\theta(\varphi)$ 'into' $\otimes(\theta(\varphi) \wedge \underline{\lambda})$ (as $u \geqslant t$ and the definition for $s,t \Vdash_{cc\lambda} \varrho \circ \xi$ is 'there is a u with $s,u \Vdash_{cc\lambda} \varrho$ and $\underline{u,t} \Vdash_{cc\lambda} \xi$'). So we get the following

Definition 3.

θ is the following translation from CDT into CCλ:

(1) $\theta(\pi) = \delta$
$\theta(p) = p \wedge \underline{\lambda}$, for other atomic formulas

(2) $\theta(\varphi \wedge \psi) = \theta(\varphi) \wedge \theta(\psi)$,
$\theta(\neg \varphi) = \neg \theta(\varphi) \wedge \underline{\lambda}$,

(3) $\theta(\varphi C \psi) = (\theta(\varphi) \wedge \underline{\lambda}) \circ (\theta(\psi) \wedge \underline{\lambda})$
$\theta(\varphi D \psi) = \otimes(\theta(\varphi) \wedge \underline{\lambda}) \circ (\theta(\psi) \wedge \underline{\lambda})$
$\theta(\varphi T \psi) = (\theta(\psi) \wedge \underline{\lambda}) \circ \otimes(\theta(\varphi) \wedge \underline{\lambda})$

Definition 4.

(1) Let $M = (W,V)$ be a CCλ-model. The *associated CDT-model* of M is defined as $M^{cdt} = (T,<,U)$ where $T = W$, $< = V(\lambda)$ and the valuation U is given by $U(\pi) = V(\delta)$ and $U(p) = V(p) \cap \leqslant$ for any other propositional constant p.

(2) Let $M = (T,<,U)$ be a CDT-model. The *associated CCλ-model* of M is defined as $M^{cc\lambda} = (W,V)$ where $W = T$ and the valuation V is given by $V(\delta) = U(\pi)$, $V(\lambda) = <$ and $V(p) = U(p) \cup \{(s,t) | \, t \leqslant s$ and $[t,s] \in U(p)\}$.

Without proof we mention the following

Proposition 5.

(1) For any CCλ-model M, $M \simeq (M^{cdt})^{cc\lambda}$.

(2) For any CDT-model M, $M \simeq (M^{cc\lambda})^{cdt}$.

We are now able to give the embedding theorem:

Theorem 6.

(1) For any CDT-formula φ, CDT-model M and interval [s,t] in M:
$M,[s,t] \Vdash_{cdt} \varphi \iff M^{cc\lambda},s,t \Vdash_{cc\lambda} \Delta \wedge \theta(\varphi)$.

(2) For any CDT-formula φ, $\vDash_{cdt} \varphi \iff \vDash_{cc\lambda} \Delta \rightarrow \theta(\varphi)$.

Proof.

(1)This proposition is proved by induction on φ:
For atomic CDT-formulas, the proposition is by definition and the induction step for the Boolean operators is standard, so we only need consider the interval temporal operators, and of those we only treat T:

$M,[s,t] \Vdash_{cdt} \varphi T \psi$

$\iff$ there is a $u \geqslant t$ with $[s,u] \Vdash_{cdt} \psi$ and $[t,u] \Vdash_{cdt} \varphi$

$\iff_{\text{Induction}}$ $(s,u) \Vdash_{cc\lambda} \Delta \wedge \theta(\psi)$ and $(t,u) \Vdash_{cc\lambda} \Delta \wedge \theta(\varphi)$

$\iff$ $(s,u) \Vdash_{cc\lambda} \Delta \wedge \theta(\psi)$ and $(u,t) \Vdash_{cc\lambda} \otimes(\Delta \wedge \theta(\varphi))$

$\iff M^{cc\lambda},[s,t] \Vdash \Delta \wedge \theta(\varphi T \psi)$.

(2)Immediate by (1) and the previous fact. □Theorem 6.

By theorem 6.(2) we have yet another recursive enumeration of all valid CDT-formulas if we axiomatise CCλ, and this can easily be done:

Definition 7.

ACCλ is the axiom system ACC with the following formulas as added axioms:

(Λ1) $\lambda \circ \lambda \rightarrow \lambda$

(Λ2) $\lambda \rightarrow \neg \delta$

(Λ3) $\lambda \vee \otimes\lambda \vee \delta$.

For the *derivability* relation we use the symbol $\vdash_{cc\lambda}$

Theorem 8. (SOUNDNESS and COMPLETENESS).

$$\vdash_{cc\lambda} \varphi \iff \vDash_{cc\lambda} \varphi.$$

Proof.

Soundness is straightforward.

For completeness, let Γ be a maximal ACCλ-consistent set. We can now just copy the completeness proof for ACC, obtaining a CC-model M = (W,V) for F. By the construction, the set $\{\varphi | (s,t) \Vdash_{cc} \varphi\}$ is a maximal ACCλ-consistent set, for each pair (s,t) of M. So we have M $\vDash_{cc}$ Λ1 ∧ Λ2 ∧ Λ3, as Λ1, Λ2 and Λ3 are *formulas* and axioms of ACCλ, and therefor elements of every maximal ACCλ-consistent set. Now it is straightforward to prove that in that case V(λ) is a linear ordering of W, whence M is a CCλ-model as well. □Theorem 8.

One can also show that conversely, CCλ can be embedded in CDT. However, the fact that in CDT-models we only have one of the pairs [s,t] and [t,s] at our disposal causes technical problems: in the translation one must give *two* images for each propositional variable. For lack of space, this matter will not be pursued here.

3.2. CC and extended modal/dynamic logics.

Dynamic logic ([H]) is a system devised for reasoning about program behaviour. In the syntax programs and formulas are defined by simultaneous induction. Starting with sets of atomic programs and atomic formulas, one has clauses saying that if α, β are programs and φ, τ are formulas, then φ∧ψ, ¬φ and <α>φ are formulas and α;b, α∪β, α* and φ? are programs. These entities have the following intended meaning:

- α;β sequential execution of first α, then β,
- α∪β non-deterministic choice: execution of either α or β
- α* sequential execution of α, a finite number of times
- φ? execution of the empty command if φ holds; otherwise abortion
- <α>φ after performing α, φ holds.

In the Kripke semantics for dynamic logic, every program α has its own accessibility relation $R_\alpha \subseteq WxW$, where W is the set of possible worlds called 'states'. The induction clauses for the definition of R_α are: $R_{\alpha;\beta} = R_\alpha \mid R_\beta$, $R_{\alpha\cup\beta} = R_\alpha \cup R_\beta$, $R_{\alpha*} = (R_\alpha)*$ and $R_{\varphi?} = \{(w,w) | \varphi$ holds at w}. (For the forcing relation the obvious definitions hold.) Extensions of dynamic logic have been proposed in which the set of operators on programs is augmented. An example is [], where, among other things, the converse operator (ˇ) of "undoing programs" is added (with $R_{\alpha^\vee} = (R_\alpha)^{-1}$).

There are also examples ([H1],[Go][GP]) of systems of extended modal logic, in which there are operators $\wedge$, $\neg$ giving $\alpha\wedge\beta$, $\neg\alpha$, connected with the intersection $R_\alpha \cap R_\beta$ of the accessibility relations R_α, R_β, resp. the complement $(R_\alpha)^c$ of R_α (the so-called inaccessibility relation).

Orlowska ([O]) gives a system in which there are 'operator-constituents' for every Relation Algebra-operator. We show that her system can be seen as a subsystem of our logic CC, and henceforth the same holds for every system mentioned above (forgetting the *-operator for the moment). First a formal definition:

Definition 1.

Let PV be a set of propositional variables, RV a set of relational variables, Δ a subset of the relational operators $\{\neg, \wedge, \vee, ;, ^\vee\}$. By simultaneous induction we define $P\Delta$ and $R\Delta$, the set of Δ-modal formulas and relational expressions:

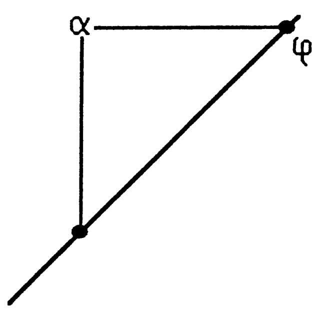
$PV \subseteq P\Delta$, $RV \subseteq R\Delta$,

if $\varphi,\psi \in P\Delta$ and $\alpha,\beta \in R\Delta$, then $\neg\varphi$, $\varphi\vee\psi$, $\langle\alpha\rangle\varphi$ are in $P\Delta$ and $\neg\alpha$, $\alpha\wedge\beta$, $\alpha\vee\beta$, $\alpha;\beta$ and $\alpha^\vee$ are in $R\Delta$, of course with the restriction that we only use Δ-operators.

The semantics for Δ-logics will be defined in the obvious fashion and also concepts like validity ($\models_\Delta$) are defined as usual.

It should be noted here that where the Δ-logics defined above have a two-sorted syntax (there being formulas as well as relations/ programs), CC only has formulas. So the embedding should be given by a translation of both Δ-formulas and Δ-relational expressions onto CC-formulas. The main idea of the embedding is that Δ-formulas are mapped on CC-formulas 'on the diagonal', cf. the translation of the formula $\langle\alpha\rangle\varphi$:

$$\theta(\langle\alpha\rangle\varphi) = \delta \wedge \Diamond(\ \theta(\alpha) \wedge \ominus\theta(\varphi)\ ),\text{ viz.}$$



Definition 2.

Let $\Delta \subseteq \{\neg, \cup, \cap, ;, ^{\smile}\}$. θ is a function mapping $P\Delta \cup R\Delta$ onto CC-formulas in the language $PV \cup RV$, as follows:

$\theta(p) = \delta \wedge p$, for $p \in PV$,
$\theta(a) = a$, for $a \in RV$,
$\theta(\varphi \wedge \psi) = \theta(\varphi) \wedge \theta(\psi)$,
$\theta(\neg\varphi) = \delta \wedge \neg\theta(\varphi)$,
$\theta(\langle\alpha\rangle\varphi) = \delta \wedge \Phi(\ \theta(\alpha) \wedge \ominus\theta(\varphi)\ )$,
and θ is a homomorphism with respect to relational expressions, e.g. $\theta(\alpha;\beta) = \theta(\alpha) \circ \theta(\beta)$, etc

By induction on the complexity of Δ-formulas one easily prooves the following

Proposition 3.

For any Δ-formula φ one has $\models_\Delta \varphi \iff \mathfrak{C}2 \models_{cc} \delta \rightarrow \theta(\varphi)$.

It might be interesting to try and find a similar result for Δ-logics in which the Kleene star-operator $*$ is in Δ as well. This would mean to define a logic CC* in which the language of CC is augmented with a unary operator $*$, $*\varphi$ having the following truth definition on proper models:

$M,x,y \models *\varphi$ if there are $n \in \omega$, $z_0,\ldots,z_n$ with $M,z_i,z_{i+1} \Vdash \varphi$ for all $i \in \{0,\ldots,n-1\}$ and $z_0=x$, $z_n=y$.

Our conjecture is that the set of valid CC*-formulas can be recursively axiomatised by adding the following axioms (analogons of the Segerberg axioms in dynamic logic) to the system ACC:

(*1) $*\varphi \leftrightarrow \delta \vee \varphi \circ *\varphi$.
(*2) $[\ \delta \wedge \Phi(*\varphi \wedge \ominus\psi)\ ] \rightarrow [\ \psi \vee \Phi(\ *\varphi \wedge \ominus(\neg\psi \wedge \Phi(\varphi \wedge \ominus\psi)))\ ]$

3.3. A modal logic for cylindric algebras.

Cylindric algebras form an algebraic approach towards the predicate calculus of relations, in the same manner as relation algebras do towards the calculus of binary relations. This also means that cylindric algebras are similar to relation algebras, whence often results holding for the one apply to the other. Here we will show that the D-rule can be used here as well to give a recursive enumeration of the identities holding in RCA_α. (For definitions of concepts concerning cylindric algebras, we refer to [HMT]. We should mention here that we only treat cylindric algebras of finite dimensions and that our approach is similar to that of Kuhn in [K1].

Definition 1. Syntax.

Let $\alpha < \omega$. *CML_α, the set of cylindric formulas of dimension α,* is defined by induction, the atomic propositions being $\top,\bot,p_0,p_1,\ldots$ and δ_{ij} for $0 \leqslant i,j < \alpha$; as operators we have the Booleans and a unary operator $\langle i\rangle$ for every $0 < i < \alpha$.

Definition 2. Semantics.

An α-frame is a pair $M = (W,V)$ with W a set called the *domain* of the model. Elements of W^α are called *possible worlds,* notation $\underline{x} = (x_0,\ldots,x_n)$. V is an α-valuation, i.e. a set assigning a subset of W^α to each atomic formula, such that $V(\top) = W^\alpha$, $V(\bot) = \emptyset$ and $V(\delta_{ij}) = \{\underline{x} \mid x_i = x_j\}$. A *forcing relation* $\Vdash_\alpha$ is defined by induction; we give the clause for the operators $\langle i\rangle$:

$M,\underline{x} \Vdash \langle i\rangle\varphi$ if there is a $\underline{z} \in W^\alpha$ with $M,\underline{z} \Vdash \varphi$ and $z_j = x_j$ for $j \neq i$.

Notions like *validity* ($\vDash_\alpha$) are defined as usual.

So possible worlds in this semantics are n-tuples of elements of the domain W, δ_{ij} holds at the i-j-diagonal and $\langle i\rangle\varphi$ holds at a world if we can reach a φ-world from it by varying the i-th coordinate.
Just like in the case of CC we might proceed by defining abstract frames for CML_α-logic and consider expressiveness matters first. For lack of space we only give an analogon of the completeness result; here too our axiom system consists of a set of axioms which can be seen as translations of the algebraic identities governing cylindric algebras, and of three types of derivation rules: Modus Ponens, modal necessitation and an inequality rule.
To formulate this rule, we first need, analogous to the CC-case, a formula expressing that φ holds in a certain hyperplane of W^α while $\neg\varphi$ holds in all parallel hyperplanes. We define some abbreviations: first an operator $\langle c(i)\rangle$, $\langle c(i)\rangle\varphi$ expressing that 'φ holds in a world with the same i-coordinate'

$\langle d(i)\rangle\varphi \equiv \langle 0\rangle\varphi \vee \langle 1\rangle\varphi \vee \ldots \vee \langle i-1\rangle\varphi \vee \langle i+1\rangle\varphi \vee \ldots \vee \langle\alpha-1\rangle\varphi$.

$\langle c(i)\rangle\varphi \equiv \langle d(i)\rangle\ldots\langle d(i)\rangle\varphi$ (α times).

Then we could use an irreflexive version of the $\langle i\rangle$-operators. We believe that such an operator cannot be defined in the formalism, unless the current point is on a diagonal, say δ_{ij}. Then we can use

$\langle r(i,j)\rangle\varphi \equiv \delta_{ij} \wedge \langle i\rangle(\neg\delta_{ij} \wedge \varphi)$

for the purpose. So, setting

$h(i,\varphi) \equiv [c(i)]\varphi \wedge \bigwedge_{j\neq i} \langle c(i)\rangle (\delta_{ij} \wedge [r(i,j)][c(i)]\neg\varphi)$,

we have $M,\underline{x} \Vdash_\alpha h(i,\varphi) \Leftrightarrow (M,\underline{z} \Vdash_\alpha \varphi$ iff $z_i = x_i$.)

Definition 3.

(1) Consider the following axiomatic system $ACML_\alpha$:

A. Axioms.

(A1) all propositional tautologies.

(A2) $\varphi \rightarrow \langle i \rangle \varphi$

(A3) $\varphi \rightarrow [i]\langle i \rangle \varphi$

(A4) $\langle i \rangle\langle i \rangle \varphi \rightarrow \langle i \rangle \varphi$

(A5) $\langle i \rangle\langle j \rangle \varphi \rightarrow \langle j \rangle\langle i \rangle \varphi$

(A6) δ_{ii}

(A7) $\delta_{ik} \rightarrow \langle j \rangle(\delta_{ij} \wedge \delta_{jk})$

(A8) $\langle i \rangle(\delta_{ij} \wedge \varphi) \rightarrow \neg\langle i \rangle(\delta_{ij} \wedge \neg\varphi)$

B. Rules of inference.

(R1) Modus Ponens: $\vdash\varphi$ and $\vdash\varphi\rightarrow\psi$ give $\vdash\psi$.

(R2) i-Necessitation: $\vdash\varphi$ gives $\vdash[i]\varphi$

(R3) Inequality: $\vdash h(i,p) \rightarrow \varphi$ implies $\vdash\varphi$, provided p does not occur in φ.

(2)The notions of *deduction*, etc. are defined as in definition 1.4.1.

We now state soundness and completeness of this system:

Theorem 4. (SOUNDNESS)

Every thesis of $ACML_\alpha$ is α-valid: $\vdash_\alpha\varphi \Rightarrow \models_\alpha \varphi$.

Proof.

Straightforward. □Theorem 3.3.4.

Theorem 5. (COMPLETENESS).

Every α-valid CML_α-formula is an α-thesis: $\models_\alpha \varphi \Rightarrow \vdash_\alpha \varphi$.

Proof sketch.

The completeness proof proceeds in the same fashion as for ACC: the notion of a maximal consistent set is defined, and the aim of the proof is to show that every MCS has a model. A relation $\equiv_i$ is defined satisfying $\Gamma\equiv_i\Delta \Leftrightarrow \langle i \rangle\varphi \in \Gamma$ iff $\langle i \rangle\varphi \in \Delta$. Then α-dimensional lattices are defined, being finite approximations of models. Every shortcoming of such an approximation with respect to the final model we are after, is shown to be eliminable in a larger lattice, so that for any particular MCS Γ we can build a chain of lattices such that the union of these lattices is the perfect model for Γ we are looking for. □Theorem 3.3.5.

Just as in the case of CC and relation algebras we may use the previous theorem to give a recursive enumeration of all identities holding in RCA_α; for lack of space we will not do this here.

3.4. Two simple two-dimensional logics.

We consider two simple subsystems of CC, one of which was defined by Segerberg in [Se], the other one being a natural extension of a logic given by Kuhn in [Ku2]. We show that for both logics we can prove completeness applying the matrix method of section 1.4 (without using a D-rule!). We should mention that Segerberg gives a D-rule-free axiom system himself and proves completeness in a Henkin-style using a 'twodimensional bulldozing' technique, where Kuhn does use a D-rule for his system. Furthermore, Segerbergs logic is very similar both to the two-dimensional cylindric logic (cf. section 3.3) and to a system treated by Davis in [Da].

<u>Definition 1.</u>
Segerbergs system Seg is a modal logic with unary operators ◫, ⊟, ⊖, ⦶ and ⊗. The interpretation of these operators is given in 1.2.2 and 1.2.3, e.g.

M,x,y ⊩ ⊟φ if for all z, M,x,z ⊩ φ,
M,x,y ⊩ ⦶φ if M,x,x ⊩ φ.

Validity for Seg-formulas, notation ⊨φ, is defined as validity on the class of two-dimensional models.

<u>Definition 2.</u>
ASeg is the set of formulas obtained by closing the set of *axioms* (all instances of the following schemata):

(A1)	all propositional tautologies		
(A2)	o(φ→ψ) → (oφ→oψ) for o ∈ {◫,⊟,⊖,⦶,⊗}		
(A3)	◫φ → φ	⊟φ → φ.	
(A4)	φ → ◫◇φ	φ → ⊟⟠φ.	
(A5)	⊖φ → ¬⊖¬φ	⦶φ → ¬⦶¬φ	⊗φ → ¬⊗¬φ.
(A6)	◫φ → ◫◫φ	⊟φ → ⊟⊟φ	
	⊖φ → ⊖⊖φ	⦶φ → ⦶⦶φ.	
(A7)	φ → ⊗⊗φ.		
(A8)	◫⊟φ → ⊟◫φ	⊟◫φ → ◫⊟φ	
(A9)	◫φ → ⦶φ	⊟φ → ⊖φ	◫⊟φ → ⊗φ.
(A10)	◫φ → ⦶◫φ	⊟φ → ⊖⊟φ.	

(A11)	$⦶\varphi \rightarrow ◫⦶\varphi$	$\ominus\varphi \rightarrow \boxminus\ominus\varphi$.
(A12)	$⦶\varphi \rightarrow ⦶\ominus\varphi$	$\ominus\varphi \rightarrow \ominus⦶\varphi$.
(A13)	$⦶\varphi \rightarrow ⦶\otimes\varphi$	$\ominus\varphi \rightarrow \ominus\otimes\varphi$.
(A14)	$⦶\varphi \rightarrow \otimes\ominus\varphi$	$\ominus\varphi \rightarrow \otimes⦶\varphi$
(A15)	$\otimes◫\varphi \rightarrow \boxminus\otimes\varphi$	$\otimes\boxminus\varphi \rightarrow ◫\otimes\varphi$

under the following *rules of inference*:
(MP) From φ and ψ, infer $\varphi \rightarrow \psi$.
(Nec) From φ, infer $◫\varphi$, $\boxminus\varphi$, $\ominus\varphi$, $⦶\varphi$ and $\otimes\varphi$.

A *deduction* in ASeg is a finite string of formulas each of which is either an axiom or follows from earlier formulas by a rule of inference.
A formula φ is a *thesis* of ASeg (notation: ASeg $\vdash \varphi$ or $\vdash \varphi$ if no confusion arises) if it appears as the last item of a deduction.
A formula φ is a *consequence* of a set Γ of formulas, notation $\Gamma \vdash \varphi$, if there are formulas $\gamma_1,\ldots,\gamma_n$ in Γ such that $\vdash(\gamma_1 \wedge \ldots \wedge \gamma_n) \rightarrow \varphi$.
A set of formulas Γ is *consistent* if <u>ff</u> is not a consequence of Γ; a *maximal consistent* set (short: *MCS*) is a consistent set which has no consistent extension.

With Segerberg we "hasten to remark that this axiom system is not independent". Soundness of the system is straighforward:

<u>Theorem 3. (SOUNDNESS of ASeg)</u>
$\vdash \varphi \Rightarrow \models \varphi$.

For completeness ($\models \varphi \Rightarrow \vdash \varphi$), we show that for any consistent set Σ of formulas we can build a model $M = (W,V)$ with $x,y \in W$ such that $M,x,y \Vdash \sigma$ for all $\sigma \in \Sigma$. We need the following definitions and lemmas:

<u>Definition 4.</u>
For an MCS Σ we define
$⦶\Sigma = \{\varphi \mid ⦶\varphi \in \Sigma\}$,
$\ominus\Sigma = \{\varphi \mid \ominus\varphi \in \Sigma\}$,
$\otimes\Sigma = \{\varphi \mid \otimes\varphi \in \Sigma\} \cup \{\otimes\varphi \mid \varphi \in \Sigma\}$.
An MCS Δ is called *on the diagonal* if $\Delta = \ominus\Delta$.

<u>Lemma 5.</u>
(1) Every consistent set has a maximal consistent extension.
(2) If Σ is an MCS, then so are $⦶\Sigma$, $\ominus\Sigma$ and $\otimes\Sigma$.

Proof.

(1) By a standard Lindenbaum construction.

(2) By the axioms A2, A5 and A7. $\square$

Definition 6.

Γ *is on a row with* Δ, notation $\Gamma = \Delta$, if $\{\varphi \mid \boxminus\varphi \in \Gamma\} \subseteq \Delta$.

Likewise we define: Γ *is in a column with* Δ, and write $\Gamma \| \Delta$.

Lemma 7.

(0) $\Gamma = \Delta \iff$ for all $\varphi \in \Delta$, $\diamondminus\varphi \in \Gamma$.

(1) If $\diamondminus\varphi$ is in Γ, then there is a Σ with $\varphi \in \Sigma$ and $\Sigma = \Gamma$.

(2) $=$ and $\|$ are equivalence relations.

(3) If $\Gamma(0,0) = \Gamma(1,0) \| \Gamma(1,1)$ then there is a $\Gamma(0,1)$ with $\Gamma(1,1) = \Gamma(0,1) \| \Gamma(0,0)$.

(4) If $\Gamma = \Sigma$ then $\ominus\Gamma = \ominus\Sigma$.

(5) $\ominus\Gamma = \Gamma$.

(6) $\ominus\Gamma$ is on the diagonal.

(7) $\otimes\Gamma \| \ominus\Gamma$

(8) $\Gamma = \Delta \Rightarrow \otimes\Gamma \| \otimes\Delta$.

(9) $\otimes\otimes\Gamma = \Gamma$.

Proof.

(0),(1): Standard.

(2) Standard, as $\boxminus$ and $\boxbar$ are S5-modalities.

(3) Straightforward, by axiom A8.

(4),(5): By the axioms A9, A10 and A11.

(6) By A6.

(7) By A13 and A14.

(8) By A15.

(9) By A7. $\square$Lemma 7.

We will prove completeness by constructing for a given MCS Γ, a model M in which Γ is satisfied. In every finite stage of the construction we will be dealing with a finite approximation of M called a matrix:

Definition 8.

For n a natural number we define a *matrix* of *size* n to be a pair $\lambda = (n,\Lambda)$, where Λ is a map assigning an MCS to each pair (u,v) with $u,v < n$, such that for all $u,v,w < n$:

(1) $\Lambda(u,v) = \Lambda(w,v)$, $\Lambda(u,v) \| \Lambda(u,w)$

(2) $\Lambda(v,u) = \otimes\Lambda(u,v)$.
(3) $\Lambda(u,u)$ is on the diagonal.

If λ,λ' are matrices of sizes n, n', then λ' is said to *extend* λ (notation: $\lambda\subseteq\lambda'$) if $n\leqslant n'$, and for all u,v in n: $\Lambda(u,v) = \Lambda'(u,v)$. The size of a matrix λ is denoted by $|\lambda|$.

A matrix is only an approximation of a model; it will not be perfect:

<u>Definition 9.</u>

An *defect* of a matrix $\lambda = (n,\Lambda)$ is a quadruple (X,φ,u,v) with $v,w<n$, φ a formula and one of the following holds:
(1) $X=\ominus$ and $\ominus\varphi\in\Lambda(u,v)$, but there is no $w<n$ with $\varphi\in\Lambda(w,v)$
(2) $X=\oplus$ and $\oplus\varphi\in\Lambda(u,v)$, but there is no $w<n$ with $\varphi\in\Lambda(u,w)$.

We will prove in the next lemma that we can repair any defect of a matrix λ.

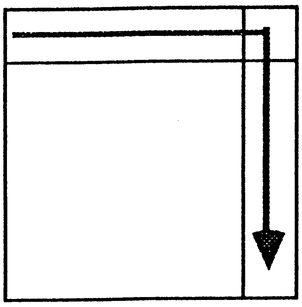
<u>Lemma 10.</u>

If (X,φ,u,v) is a defect of λ, then there is a $\lambda' = (|\lambda|+1,\Lambda')$ extending λ and lacking this defect.

<u>Proof.</u>

Without loss of generality we may assume $X = \oplus$, $u=0$, $v=n-1$. (viz. the figure). We need to define MCSs $\Lambda'(u,v)$ with $u=n$ or $v=n$ in such a way that the extended map meets the above constraints.

n	$\Lambda'(0,n)$	$\Lambda'(1,n)$	–	$\Lambda'(i,n)$	–	$\Lambda'(n-1,n)$	$\Lambda'(n,n)$
n-1	$\Lambda(0,n-1)$	$\Lambda(1,n-1)$	–	$\Lambda(i,n-1)$	–	$\Lambda(n-1,n-1)$	$\Lambda'(n,n-1)$
	|	|		|		|	|
i	$\Lambda(0,i)$	$\Lambda(1,i)$	–	$\Lambda(i,i)$	–	$\Lambda(n-1,i)$	$\Lambda'(n,i)$
	|	|		|		|	|
1	$\Lambda(0,1)$	$\Lambda(1,1)$	–	$\Lambda(i,1)$	–	$\Lambda(n-1,1)$	$\Lambda'(n,1)$
0	$\Lambda(0,0)$	$\Lambda(1,0)$	–	$\Lambda(i,0)$	–	$\Lambda(n-1,0)$	$\Lambda'(n,0)$
	0	1		i		n-1	n



The order in which we will construct the MCSs is indicated by the arrow.

By lemma 7.1 there is a $\Pi(0,n)$ such that $\Pi(0,n) \parallel \Lambda(0,n-1)$ and $\varphi \in \Pi(0,n)$.
Then by lemma 7.2 $\Pi(0,n) \parallel \Lambda(0,i)$ for all $0 \leqslant i \leqslant n$.
By lemma 7.3, there are $\Pi(i,n)$ for $0 < i < n$ such that $\Pi(0,n) \doteq \Pi(i,n) \parallel \Lambda(i,n-1)$, and by lemma 7.2 again, we have $\Pi(i,n) \doteq \Pi(j,n)$ and $\Lambda(i,k) \parallel \Pi(i,n)$ for $0 \leqslant i,j,k < n$.
Setting $\Pi(n,n) = \ominus\Pi(0,n)$, we easily prove by lemma 7.4 and 7.5 that $\Pi(n,n) = \ominus\Pi(i,n)$ and $\Pi(n,n) \doteq \Pi(i,n)$ for $0 \leqslant i \leqslant n$; while 7.6 gives that $\Pi(n,n)$ is on the diagonal.
Putting $\Pi(n,i) = \otimes\Pi(i,n)$ for $0 \leqslant i < n$ we prove by lemma 7.7 that $\Pi(n,n) \parallel \Pi(n,i)$, by lemma 7.2 that $\Pi(n,i) \parallel \Pi(n,j)$ and $\Pi(n,i) \doteq \Pi(n,i)$, by lemma 7.8 that $\Pi(n,i) \doteq \Lambda(k,i)$ and by lemma 7.9 that $\otimes\Pi(n,i) = \Pi(i,n)$, all with $0 \leqslant i,j \leqslant n$, $0 \leqslant k < n$.
It is now straightforward to verify that $\lambda' = (n+1,\Lambda)$ defined by

$$\Lambda'(u,v) = \begin{cases} \Pi(u,v) & \text{if } u=n \text{ or } v=n \\ \Lambda(u,v) & \text{otherwise,} \end{cases}$$

is a matrix extending λ. As $\varphi \in \Lambda'(0,n)$ we have repaired the defect $(\Diamond\!\!\!\mid,\varphi,0,n)$ of λ.

□Lemma 10.

<u>Theorem 11.</u> (COMPLETENESS of ASeg).

$\models\varphi \Rightarrow \vdash\varphi$.

<u>Proof.</u>

Suppose Σ is consistent. We will construct a model in which Σ is satisfiable. Without loss of generality we may assume that Σ is maximal. Define $\lambda_2 = (2,\Lambda_2)$ as follows: $\Lambda_2(0,1)=\Sigma$, $\Lambda_2(0,0)=\Phi\Sigma$, $\Lambda_2(1,0)=\otimes\Sigma$, $\Lambda_2(1,1)=\ominus\Sigma$. It is straighforward to verify that λ_2 is a matrix.
Fix an enumeration of all quadruples (X,φ,m,n) with m,n natural numbers, $X \in \{\Diamond\!\!\!\mid, \Diamond\!\!\!-\}$ and φ a Seg-formula.
Now an iterative application of lemma 10 yields the existence of a chain of matrices $\lambda_2 \subseteq \lambda_3 \subseteq \ldots$ such that if λ_n has no defect then

$\lambda_{n+1} = \lambda_n$. Otherwise, in λ_{n+1} the first defect of λ_n (i.e. the one appearing first in the fixed enumeration of quadruples (m,n,φ,ψ)) is removed in λ_{n+1}.
One can easily show that, if λ_n has a certain defect, this will eventually be repaired, i.e. there is an $m > n$ such that λ_m does not have this defect.
Now let Λ be the union of the Λ_n's, i.e. Λ is a map assigning MCSs to elements of $\mathbb{N}\mathrm{x}\mathbb{N}$ where $\mathbb{N}$ is the union of the domains of the Λ_n.
By the following definition of V we give a model $M = (\mathbb{N},V)$:

$V(p) = \{(u,v) | p \in \Lambda(u,v)\}$

<u>Truth lemma</u>: For every Seg-formula φ: $M,u,v \Vdash \varphi$ iff $\varphi \in \Lambda(u,v)$.
<u>Proof</u>: by formula-induction:
(1) for atomic formulas the assertion is clear by definition of V.
(2) for $\varphi \equiv \neg\psi$ or $\varphi \equiv (\psi \wedge \chi)$, the proof is a routine check.
(3) the case $\varphi \equiv \otimes\psi$ is treated by observing that $\otimes\Lambda(u,v) = \Lambda(v,u)$.
For $\varphi \equiv \ominus\psi$, by property (1) of matrices we have $\Lambda(v,v) = \Lambda(u,v)$; so lemma 7.5 gives $\ominus\Lambda(v,v) = \ominus(\Lambda(u,v)$; as $\Lambda(v,v)$ is on the diagonal we get $\Lambda(v,v) = \ominus\Lambda(u,v)$. So $M,u,v \Vdash \ominus\psi \Leftrightarrow M,v,v \Vdash \psi \Leftrightarrow_{IH} \psi \in \Lambda(v,v) \Leftrightarrow \ominus\varphi \in \Lambda(u,v)$.
Now consider the case where $\varphi \equiv \diamond\psi$.
First, suppose $M,u,v \Vdash \varphi$. Then there is a w in $\mathbb{N}$ with $M,u,w \Vdash \psi$. By the induction hypothesis $\psi \in \Lambda(u,w)$. Let $n = 1+\max(s,t,u)$. Then $u,v,w < n$ and by definition of Λ, $\psi \in \Lambda_n(u,w)$; then by definition 8.1 and lemma 7.0 we have $\diamond\psi \in \Lambda_n(u,w) = \Lambda(u,w)$.
For the other direction, suppose $\diamond\psi \in \Lambda(u,v)$; let $n = 1+\max(s,t)$, then $u,v < n$ and $\diamond\psi \in \Lambda_n(u,v)$. If there is no $w < n$ with $\psi \in \Lambda_n(u,w)$ then this is a defect of λ_n. In that case there must be an $\lambda_m \supseteq \lambda_n$ in which this defect occurs no longer. This means there is a $w < n$ with $\psi \in \Lambda_m(u,w)$. As $\Lambda_m \subseteq \Lambda$ this yields $M,u,v \Vdash \diamond\psi$ by the induction hypothesis.

□Truth lemma.

As $\Sigma = \Lambda(0,1)$, we have indeed found a model in which Σ is satisfiable.

□Theorem 3.4.11.

Kuhn's system is even simpler then the previous one: it has only one operator □, □φ being true at (x,y) if φ is true at all (y,z). (it will be clear why he calls the corresponding accessibility relation the *domino* relation). Kuhn gives a sound and complete axiom system for this logic with infinitely many axioms and a D-rule. Here we will show that we can reduce the axioms to a finite set and get rid of this odd rule if we add the 'converse operator' of □ to the language:

<u>Definition 12.</u>

Ku is the set of formulas we obtain in a language with propositional constants T,⊥,p_0,p_1,..., the usual Boolean connectives and two modal operators ⬖, ⬗.

As abbreviations we use: ⊖φ ≡ ⬗⬖φ, ⦶φ ≡ ⬖⬗φ, and the duals ◧,◨,⊟ and ◫ of ⬖,⬗,⊖ and ⦶.

The *mirror image* $\mu(\varphi)$ of a formula φ is obtained by simultaneously replacing all occurrences of ⬗ by ⬖ and vice versa.

In models M= (W,V) the *interpretation* of the operators is given by

M,x,y ⊩ ⬖φ if there is a z with M,z,x ⊩ φ

M,x,y ⊩ ⬗φ if there is a z with M,y,z ⊩ φ.

(Note that in this way ⊖ and ⦶ get their standard interpretation).

<u>Definition 13.</u>

AKu is the set of axioms obtained by closing the set of *axioms* (all instances of the following schemata and their mirror images):

(B1) all propositional tautologies

(B2) ◨(φ→ψ) → (◨φ → ◨ψ)

(B3) φ → ⬗⬖φ

(B4) ⬗⬖φ → ◨⬖φ

(B5) ⬗φ → ⬗⬗φ

(B6) ⬗⬗φ → ⬖⬖φ

under the following *rules of inference* :

(MP) From φ and φ→ψ, infer ψ.

(Nec) From φ, infer ◨φ and ◧φ.

The notions of deduction etc. are as defined in definition 2.

<u>Theorem 14.</u> (SOUNDNESS of AKu))

$\vdash\varphi \Rightarrow \models\varphi$.

Completeness is proved in the same manner as for ASeg. We need some derived theses:

Proposition 15.

The following formulas are derived theses of AKu:

(T0) ⬗T.
(T1) φ → ◨⬖φ
(T2) φ → ⟠φ
(T3) ◨φ → ⬗φ
(T4) φ → ⊟⟠φ
(T5) ⬗⬖⬗φ → ⬗φ
(T6) ⟠⟠φ → ⟠φ
(T7) ⟠◇⃒φ → ◇⃒⟠φ
(T8) (φ∧⬗ψ) → ⬗(⬖φ∧⬗ψ)
(T9) (⬗φ∧⬗φ) ↔ ⬗(◇⃒φ∧◇⃒φ).

Proof.

In the following derivatons, PL and ML stand for propositional and modal logic, resp.

(T0) by B3.
(T1) By B3 and B4.
(T2) B3.
(T3) By T0.
(T4)
1. ⬖φ → ⬖⬗⬖φ (B3)
2. ⬖⬗⬖φ → ◧⬗⬖φ (B4)
3. ⬖φ → ◧⬗⬖φ (1,2,PL)
4. ◨⬖φ → ◨◧⬗⬖φ (3,Nec)
5. ⬗⬖φ → ◨⬖φ (B4)
6. ⟠φ → ⟠⟠φ. (4,5,PL)

(T5)
1. ⬖⬗φ → ◧⬗φ (B4)
2. ◨⬖⬗φ → ◨◧⬗φ (1,Nec)
3. ◨◧⬗φ → ⬗φ (B3)
4. ◨⬖⬗φ → ⬗φ (2,3,PL)
5. ⬗⬖⬗φ → ◨⬖⬗φ (B4)
6. ⬗⬖⬗φ → ⬗φ (4,5,PL)

(T6) By T5 with ⬖φ

(T7)
1. ⬗⬖⬖⬗φ → ⬗⬗⬗⬗φ (B6,ML)
2. ⬗⬗⬗⬗φ → ⬖⬖⬖⬖φ (B6,ML)
3. ⬖⬖⬖⬖φ → ⬖⬗⬗⬖φ (B6,ML)
4. ⟠◇⃒φ → ◇⃒⟠φ (1,2,3,ML)

(T8)
1. ⬗ψ → ⬗⬗ψ (B5)
2. φ → ◨⬖φ (T1)
3. (φ∧⬗ψ) → (◨⬖φ ∧ ⬗⬗φ) (1,2,PL)
4. (φ∧⬗ψ) → ⬗(⬖φ ∧ ⬗ψ) (3,ML)

(T9)
1. ⬗ψ → ◨◇⃒ψ (T1)
2. (⬗φ∧⬗ψ) → ⬗(φ∧◇⃒ψ) (1,ML)

3. φ → ◇⃒φ (T2)
4. (⬗φ∧⬗φ) → ⬗(◇⃒φ∧◇⃒φ). (2,3,ML)
5. ⬗⬖⬗φ → ◨◧⬗φ (1,ML)
6. ◨◧⬗φ → ⬗φ (B3)
7. ⬗◇⃒φ → ⬗φ (5,6,ML)
8. (⬗φ∧⬗φ) ↔ ⬗(◇⃒φ∧◇⃒φ) (4,7,PL).

□

Definition 16.

For MCSs Γ and Δ we have the following notions:
Γ is *on a row with* Δ, notation ΓhhΔ, if {φ|⊟φ ∈ Γ} ⊆ Δ. Likewise we define Γ is *in a column with* Δ, and write ΓvvΔ. Γ *meets* Δ if {φ| ◨φ ∈ Γ} ⊆ Δ. Δ is *on the diagonal* if Δ meets Δ.

Lemma 17.

(1) ΓhhΔ ⟺ for all φ∈Δ, ⟠φ∈Γ.
(2) Γ meets Δ ⟺ for all φ∈Δ, ⬗φ∈Γ
for all ◧φ∈Δ, φ∈Γ
for all φ∈Δ, ⬖φ∈Γ.
(3) If ⬗φ∈Γ then there is a Δ with φ∈Δ and Γ meets Δ.

Proof.

Standard □

Lemma 18.

Let Γ, Δ and Σ be MCSs.
(1) hh and vv are equivalence relations.
(2) If Γ(0,0)hhΓ(1,0)vvΓ(1,1) then there is a Γ(0,1) with Γ(1,1)hhΓ(0,1)vvΓ(0,0).
(3) If Γ meets Δ and Σ meets Δ, then ΓhhΣ.
(4) If Δ is on the diagonal, then ΔhhΔ and ΔvvΔ.
(5) If Δ is on the diagonal, then Δ meets Σ ⟺ ΔvvΣ.
(6) If Δ is on the diagonal, ΓhhΔ and Γ meets Σ, then ΔvvΣ.
(7) If Δ is on the diagonal and ΓhhΔvvΣ, then Γ meets Σ.
(8) For every Γ there is a Δ on the diagonal with ΓhhΔ.

Proof.

(1) Standard, as ⊟ and ◫ are S5-modalities (T2,T4 and T6).
(2) Standard by T7.
(3) Immediate by the definition of ⊟ and lemma 17.2.
(4) Immediate by (1).

(5) From left to right: φ∈Σ ⇒ ⬗φ∈Δ ⇒ by 17.2, ⬖⬗φ ∈ Δ as Δ meets Δ. From right to left: φ∈Σ ⇒ ⟠φ∈Δ by 17.1 ⇒ ◧⬗φ ∈ Δ by B4 ⇒ by 17.2, ⬗φ ∈ Δ as Δ meets Δ.

(6) By 18.6, ΓhhΔ implies Γ meets Δ, so 18.3 gives ΔvvΣ.

(7) Let φ∈Σ, then ⟠φ ≡ ⬖⬗φ in Δ, so ⬗⬖⬗φ in Δ, as Δ meets Δ. Then by B4, ◨◧⬗φ ≡ ⊟⬗φ in Δ ⇒ φ ∈ Γ.

(8) We first prove the following

<u>Claim:</u> {⬗φ|⬗φ∈Γ} ∪ {⬖φ|φ∈Γ} is consistent.

<u>Proof.:</u> Suppose otherwise, then there are $\varphi_1,\dots,\varphi_n,\psi_1,\dots,\psi_m$ with all ⬗φ_i,ψ_j in Γ and ⊢($\bigwedge_i$⬗φ_i ∧ $\bigwedge_j \psi_j$) → ⊥ (*).

Defining φ ≡ $\bigwedge_i$⟠φ_i and ψ ≡ $\wedge_j \psi_j$, we have ⊢ ⬖ψ → $\bigwedge_j$⬖ψ_j by plain modal logic and ⊢ ⬗φ ↔ $\bigwedge_i$⬗φ_i by (T9). With (*) this gives

⊢ (⬗φ ∧ ⬖ψ) → ⊥, whence by B2 and T0

⊢ ⬗(⬗φ ∧ ⬖ψ) → ⊥, so by the thesis T9 we get

⊢ (ψ ∧ ⬗φ) → ⊥, which gives a contradiction as both ψ and ⬗φ are in Γ. □

Now let Δ be a maximal extension of the above set, then Γ meets Δ by 17.2. For an arbitrary φ in Δ, ⬗φ is in Γ, as Γ meets Δ, whence ⬗φ is in Δ: Δ meets itself, so it is on the diagonal. Then by 18.6 we have ΓhhΔ.

□Lemma 3.4.18.

<u>Definition 19.</u>

For n a natural number we define a *(Ku-)matrix* of *size* n to be a pair λ = (n,Λ), where Λ is a map assigning an MCS to each pair (u,v) < n such that for all u,v,w < n, Λ(u,v) meets Λ(v,w).

For λ,λ' matrices of sizes n,n', λ' is said to *extend* λ if n ⩽ n' and for all u,v ⩽ n: Λ(u,v) = Λ'(u,v).

The size of a matrix λ is denoted by |λ|.

<u>Lemma 20.</u>

Let Λ be a map assigning an MCS to every pair (u,v) with u,v < n.

λ = (n,Λ) is a matrix

iff Λ satisfies (1) for all u,v,w < n: Λ(u,w)vvΛ(u,v)hhΛ(w,v)

and (2) for all u < n: Λ(u,u) is on the diagonal.

<u>Proof.</u>

⇒: As both Λ(u,v) and Λ(w,v) are meets-predecessors of Λ(u,v), they are on a row by 18.3.

⇐: Λ(v,v) being on a row with Λ(u,v) and in a column with Λ(v,w), by lemma 18.7 Λ(u,v) must meet Λ(v,w), as Λ(v,v) is on the diagonal. □

Definition 21.

An *defect* of a matrix $\lambda = (n,\Lambda)$ is a quadruple (X,φ,u,v) with $v,w < n$, φ a formula and one of the following holds:

(1) $X=R$ and ⬗$\varphi \in \Lambda(u,v)$, but there is no $w<n$ with $\varphi \in \Lambda(v,w)$

(2) $X=L$ and ⬖$\varphi \in \Lambda(u,v)$, but there is no $w<n$ with $\varphi \in \Lambda(w,u)$.

Lemma 22.

If (X,φ,u,v) is a defect of a matrix λ, then there is a matrix $\lambda' = (|\lambda|+1,\Lambda')$ extending λ and lacking this defect.

Proof.

Without loss of generality we assume that $X=R$, $u=n-1$ and $v=0$.
By lemma 17.3 there is a $\Pi(0,n)$ with $\varphi \in \Pi(0,n)$ and $\Lambda(n-1,0)$ meets $\Pi(0,n)$. Lemma 18.6 gives $\Lambda(0,0)vv\Pi(0,n)$ as $\Pi(0,0)$ is on the diagonal and $\Lambda(n-1,0)$ is on a row with $\Lambda(0,0)$.
Then by lemma 18.1 $\Pi(0,n)vv\Lambda(0,i)$ for all $0 \leqslant i \leqslant n$.
Lemma 18.2 yields MCSs $\Pi(i,n)$ for $o<i<n$ such that $\Pi(0,n)hh\Pi(i,n)vv\Lambda(i,n-1)$ and lemma 18.1 again ensures that everything defined up till now is coherent.
Using lemma 18.8 we obtain an MCS $\Pi(n,n)$ on the diagonal, and on a row with $\Pi(0,n)$. By 18.1 then $\Pi(i,n)hh\Pi(n,n)$.
Finally, using 18.2 again, we may define MCSs $\Pi(n,j)$ such that, setting $\Lambda'(u,v) =$ $\Pi(u,v)$ if $u=n$ or $v=n$
$\Lambda(u,v)$ otherwise,
we can prove that, for all $u,v,w<n$, $\Lambda'(u,u)$ is on the diagonal and $\Lambda'(u,v)$ is on a row with $\Lambda'(w,v)$ and in a column with $\Lambda'(u,w)$.
By lemma 20 then $\lambda' = (n+1,\Lambda')$ is a matrix. Moreover, λ' extends λ and it lacks the defect $(R,\varphi,n-1,0)$ as $\varphi \in \Lambda'(0,n)$.

□Lemma 3.4.22.

Theorem 23. (COMPLETENESS of AKu)

$\models \varphi \Rightarrow \vdash \varphi$.

Proof.

Just as for ASeg (Theorem 11) □Theorem 3.4.23.

4. SUMMARY.

The central idea of this paper was to view both relation algebras and temporal logics of intervals as twodimensional modal logics. Therefor we have defined two systems of generalized modal logics, CC and CDT, generalized in the sense that not all modal operators are unary. For both systems two kinds of semantics could be defined: in the first one frames and models are two-dimensional, i.e. possible worlds are formed by pairs of objects. The other semantics are in a sense more general and abstract; for CC these frames can be seen as atomic structures of relation algebras, for CDT they fit in the approach to representing time where intervals are the basic entities instead of timepoints.

To define these abstract frames we had to generalize the notion of accessibility relation known in Kripke semantics: an n-ary modal operator has an n+1-placed accessibility relation. Having these we can speak in two languages about abstract frames: the modal one and a first order language with a predicate for every accessibility relation. We have developed the correspondence theory of these pairs of languages and showed how to apply the Sahlqvist theorem. For both CC and CDT, we have treated the problem how to characterize the (isomorphic copies of) the twodimensional frames amongst the abstract frames: it turned out that only the first order language is powerfull enough to do the job.

Concerning the modal formulas valid on the class of twodimensional frames, we have given for both CC and CDT a finite axiomatization, though in both cases we used an odd derivation rule. From the fact that for CC no ordinary finite axiomatization exists, it follows that the rule really adds new theorems. We have derived two corollaries of our results for the theory of relation algebras, one giving a new enumeration of the identities holding in the variety of representable relation algebras. Using a method known for relation algebras, we have given a sound and complete deduction system for the "twodimensional CDT-theorems".

In the last chapter we have made clear some connections of CC, CDT and other logics. First we embedded CDT in CCλ, a small extension of CC; then we showed how to embed in CC several systems of "extended modal logic" (i.e. multimodal logics with operations on the accessibility relations and corresponding operations on the modal operators). We also briefly treated a generalized modal logic for cylindric algebras (like CC is for relation algebras), leading to a recursive enumeration of all identities holding in the variety of cylindric algebras of dimension α. Finally we gave two completeness theorems (without any odd rule) for twodimensional subsystems of CC, developed by Segerberg resp. related to a system given by Kuhn.

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