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Serge Lapierre

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A FUNCTIONAL PARTIAL SEMANTICS FOR INTENSIONAL LOGIC

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Abstract In this paper a partial semantics for the higher-order modal language of Intensional Logic is suggested. Partial semantic values of functional types are defined as monotonic functions on partially ordered sets; it is showed that this characterization is materially adequate for representing partial values and that it overcomes the difficulties which arise in the attempt to introduce one-place partial functions in the hierarchy of types. Partial semantic values of any type are related to classical semantic values of the same type by the mean of a relation of approximation. This allows us to compare partial models with classical models. Classical semantics then appears to be a part of the partial semantics to the extent that there exists a bijective mapping from classical models onto totally defined partial models. Also, this allows us to define, relatively to the partial semantics, a notion of entailment which is coextensive with the classical notion.

Content:

A. Introduction.

Though a very large amount of works has been done in partial semantics for modal or non-modal propositional or quantified first-order languages, very few has been said about partial semantics for modal or non-modal higher-order languages. According to our knowledge the first attempt to introduce partiality in the theory of types goes back to Pavel Tichý [1982]. There is also the nice work of Reinhard Muskens [1989], which suggestes a partialised version of Montague's Semantics. Musken's Semantics is relational, in the sense that partial semantical objects are defined as partial relations and not as partial functions. This strategy is in large part justified by the apparent impossibility, discussed in this paper, to code partial relations or partial many-places functions by one-place functions, as Shönfinkel's theorem could suggest. Notice that the same problem has motivated Tichý's own strategy, which consists to consider only many-places partial functions. On our side, we don't claim that Tichý's and Musken's strategies are not adequate or that they are not interesting. We just claim that it is possible to construct a partial semantics for the higher-order modal language of Intensional Logic which uses only one-place functions.

Our strategy, which will be explained in detail in Section C.1 of this paper, is in fact inspired by another interesting attempt to introduced partiality in the semantics of Intensional Logic, that is the one of François Lepage [1984] (see also F.Lepage [1987] and [1989a,b]). However, according to us many notions used by Lepage were not precise enough to be used as foundations of a partial semantics for Intensional Logic (notice that the purpose of Lepage was not to suggest a foundational theory of partiality in the theory of types, but rather to suggest a formal analysis of sentences of belief and knowing by using the notion of incomplete knowledge of semantic values - of course, these two enterprises aren't exclusive). In particular, his notion of a *good representation of a classical semantic value*, or equivalently, of an *approximation of a classical semantic* is total behaves exactly as one and only one classical value. Consequently, the notion of a *partial model which is total* was not precise enough neither. In this paper we shall fill these gaps by providing a unifying view on partial semantics and classical semantics. In particular, we shall give a precise account of the idea that classical models are limit cases of partial models. Though this seems trivial at first glance, in fact it is not, for the class of classical models is disjoint from the class of partial models.

Let us first present our formal framework: the language of Intensional Logic, and its standard classical semantics.

B. The language of Intensional Logic and its standard classical semantics.

The formal system described in this Section is very close to the one described by Daniel Gallin [1975], chap.1, as a version of Montague's Intensional Logic (Richard Montague [1970]).

B.1. The modal hierarchy of types.

Let e, t and s be three distinct objects. *The modal hierarchy of types* is the smallest set T such that:

(i) $e, t \in T$;

(ii) if $\alpha \in T$ and $\beta \in T$, then $(\alpha,\beta) \in T$;

(iii) if $\alpha \in T$, then $(s,\alpha) \in T$.

When no confusion arises, (α,β) and (s,α) are abbreviated by $\alpha\beta$ and $s\alpha$ respectively. We use the Greek letters α, β and σ as variables of types.

B.2. The formal language of Intensional Logic.

The language of *Intensional Logic* (henceforth, *the language of IL*) has the following resources:

(1) The following improper expressions: [,], $\equiv$, $\wedge$, $^\wedge$, $^\vee$, λ.

(2) For each $\alpha \in T$, a countable set Con_α of *constants* of type α and a countable set Var_α of *variables* of type α.

(3) For each $\alpha \in T$, a set of *terms of type* α, recursively defined as the smallest set Trm_α such that:

(i) $Con_\alpha \cup Var_\alpha \subseteq Trm_\alpha$;

(ii) if $A \in Trm_{\alpha\beta}$ and $B \in Trm_\alpha$, then $[AB] \in Trm_\beta$;

(iii) if $A \in Trm_\alpha$, then $^\wedge A \in Trm_{s\alpha}$;

(iv) if $A \in Trm_{s\alpha}$, then $^\vee A \in Trm_\alpha$;

(v) if $x \in Var_\alpha$ and $A \in Trm_\beta$, then $\lambda x A \in Trm_{\alpha\beta}$;

(vi) if $A, B \in Trm_\alpha$, then $[A \equiv B] \in Trm_t$;

(vii) if $A, B \in Trm_t$, then $[A \wedge B] \in Trm_t$.

Notice that $\equiv$ and $\wedge$ intuitively mean identity and conjunction respectively. Actually however expression $\wedge$ and rule (vii) are redundant, since $\wedge$ can be defined in terms of $\equiv$ and λ. The reason of this redundancy, which appears to be untimely here, will be explained latter.

Henceforth, for any type α, we use the symbols A_α, B_α, C_α, etc., as schemata of terms of type α. We use more particulary the symbols x_α, y_α, z_α, etc., as schemata of variables of type α and the symbols c_α, d_α, e_α, etc., as shemata of constants of type α. When no confusion arises, subscripts are dropped. We introduce by definitions some terms of the language of **IL**.

$T := [\lambda x_t x \equiv \lambda x_t x]$

$F := [\lambda x_t x \equiv \lambda x_t T]$

$\neg A_t := [A \equiv F]$

$[A_t \vee B_t] := \neg[\neg A \wedge \neg B]$

$[A_t \supset B_t] := [\neg A \vee B]$

$\forall x_\alpha A_t := [\lambda x A \equiv \lambda x T]$

$\exists x_\alpha A_t := \neg \forall x \neg A$

$[A_\alpha \equiv B_\alpha] := [^\wedge A \equiv {}^\wedge B]$

$\Box A_t := [A \equiv T]$

$\Diamond A_t := \neg \Box \neg A.$

B.3. The standard classical semantics for the language of IL.

Let E and I be two non-empty and disjoint sets. Intuitively, one considers E as a set of individuals and I as a set of possible worlds. *The standard system of classical objects based on E and I* is the indexed family $\{M_\alpha\}_{\alpha \in T}$ of sets, such that:

(i) $M_e = E$

(ii) $M_t = \{0,1\}$

(iii) $M_{\alpha\beta} = M_\beta^{M\alpha}$

(iv) $M_{s\alpha} = M_\alpha^{I}$

A standard classical model based on E and I is an ordered pair:

$$M = \langle \{M_\alpha\}_{\alpha\in T}, m \rangle$$

where: $\{M_\alpha\}_{\alpha\in T}$ is the standard system of classical objects based on E and I; m is a function:

$$m: \bigsqcup_{\alpha\in T} Con_\alpha \rightarrow \bigsqcup_{\alpha\in T} M_{s\alpha}$$

such that $m(c_\alpha) \in M_{s\alpha}$ for any $c_\alpha \in Con_\alpha$. We denote by As(M) *the set of assignments over M*, that is the set of all functions:

$$\mathbf{a}: \bigsqcup_{\alpha\in T} Var_\alpha \rightarrow \bigsqcup_{\alpha\in T} M_\alpha$$

such that $\mathbf{a}(x_\alpha) \in M_\alpha$ for any $x_\alpha \in Var_\alpha$. For each $\mathbf{a} \in$ As(M), each $x_\alpha \in Var_\alpha$ and each $y \in M_\alpha$, we denote by $\mathbf{a}(x_\alpha/y)$ the assignment in As(M) which differs at most from $\mathbf{a}$ by assigning y to x_α. We define recursively *the classical value* $[\![A_\alpha]\!]^M_{\mathbf{a},i}$ *in* M of a term A_α *according* to an assignment $\mathbf{a} \in$ As(M) and an $i \in I$ as follows:

(i) $[\![c_\alpha]\!]^M_{\mathbf{a},i} = (m(c_\alpha))(i)$;

(ii) $[\![x_\alpha]\!]^M_{\mathbf{a},i} = \mathbf{a}(x_\alpha)$;

(iii) $[\![A_{\alpha\beta}B_\alpha]\!]^M_{\mathbf{a},i} = [\![A_{\alpha\beta}]\!]^M_{\mathbf{a},i}([\![B_\alpha]\!]^M_{\mathbf{a},i})$;

(iv) $[\![{}^\wedge A_\alpha]\!]^M_{\mathbf{a},i}$ = the function f from I such that for every $j \in I$, $f(j) = [\![A_\alpha]\!]^M_{\mathbf{a},j}$;

(v) $[\![{}^\vee A_{s\alpha}]\!]^M_{\mathbf{a},i} = [\![A_{s\alpha}]\!]^M_{\mathbf{a},i}(i)$;

(vi) $[\![\lambda x_\alpha A_\beta]\!]^M_{\mathbf{a},i}$ = the function f from M_α such that for every $y \in M_\alpha$, $f(y) = [\![A_\beta]\!]^M_{\mathbf{a}',i}$, where $\mathbf{a}' = \mathbf{a}(x_\alpha/y)$;

(vii) $[\![[A_\alpha \equiv B_\alpha]]\!]^M_{\mathbf{a},i} = 1$ if $[\![A_\alpha]\!]^M_{\mathbf{a},i} = [\![B_\alpha]\!]^M_{\mathbf{a},i}$, and 0 otherwise;

(viii) $[\![[A_t \wedge B_t]]\!]^M_{\mathbf{a},i} = 1$ if $[\![A_t]\!]^M_{\mathbf{a},i} = [\![B_t]\!]^M_{\mathbf{a},i} = 1$, and 0 otherwise.

We can easily check that for each $\alpha \in T$ and each $A_\alpha \in Trm_\alpha$, $[\![A_\alpha]\!]^M_{\mathbf{a},i} \in M_\alpha$.

B.4. The classical notions of entailment and validity.

A *formula* of the language of **IL** is a term of type t. Let A be a formula, M be a standard classical model, $\mathbf{a} \in$ As(M) and $i \in I$. A is *satisfied in* M *according to* **a** and i, formally: $\models_{M,\mathbf{a},i} A$, iff $[\![A]\!]^M_{\mathbf{a},i} = 1$. A *is not satisfied in* M *according to* **a** and i, formally: $\not\models_{M,\mathbf{a},i} A$, iff $[\![A]\!]^M_{\mathbf{a},i} = 0$. If Γ is a set of formulas, then Γ is *satisfied in* M *according to* **a** and i, formally: $\models_{M,\mathbf{a},i} \Gamma$, iff $\models_{M,\mathbf{a},i} A$ for every $A \in \Gamma$. If A is a formula, then A is *true in* M iff $\models_{M,\mathbf{a},i} A$ for every $\mathbf{a} \in As(M)$ and every $i \in I$. A set Γ of formulas *classically entails* a formula A, formally: $\Gamma \models A$, iff for every standard classical model M, every $\mathbf{a} \in As(M)$ and every $i \in I$, *if* $\models_{M,\mathbf{a},i} \Gamma$, then $\models_{M,\mathbf{a},i} A$. At last, A is *classically valid*, formally: $\models A$, iff $\emptyset \models A$, that is, iff A is true in every standard classical model.

C. The standard partial semantics for the language of IL.

C.1 Introduction.

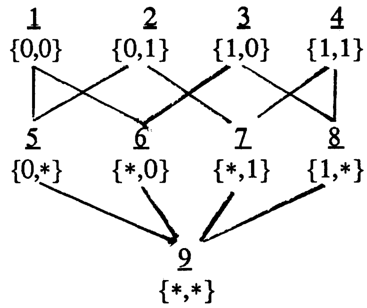
Our only basic intuition for the construction of a partial semantics for the language of **IL** is the following: partial semantic values must be *approximations* of classical semantic values. How to define formally the partial values in order to meet this intuitive requirement? Since in the classical semantics of the language of **IL**, the values of any functional type (type $\alpha\beta$ or $s\alpha$) are *total functions*, then it seems appropriate to define the partial values of any given functional type as *partial functions* approximating total functions of this type. This raises some questions however.

What is a partial function? Under which conditions a partial function is an approximation of a given total function? According to the set-theoretical definition of the notion of function, if X and Y are two non-empty sets, then a *function from* X (starting set) *into* Y (target set) is a relation $f \subset X \times Y$ such that for any $(x,y), (x',y') \in f$, if $x = x'$, then $y = y'$. Let f be a function from X into Y; the *domain* of f is the set $D(f) \subseteq X$ such that for every $x \in X$, $x \in D(f)$ iff there is $y \in Y$ such that $(x,y) \in f$. We say that f is *total* if $D(f) = X$, *non-total* if $D(f) \subset X$, and *partial* if $D(f) \subseteq X$. If f is a partial function and $x \in D(f)$, then we write "$f(x) = y$", instead of "$(x,y) \in f$"; if $x \notin D(f)$, then we write "f(x) is undefined" (or "f is not defined for x"). If f is a total function, then we write "$f:X \rightarrow Y$" (or "$f \in Y^X$"), in order to indicate that f is a function having X as domain and $Y' \subseteq Y$ as range. Let f and f' be two partial functions from X into Y; we say that f is *an approximation of* f' (or that f' *is at least as well defined than* f), and we write: $f \leq f'$, iff $D(f) \subseteq D(f')$ and for every $x \in D(f)$, $f(x) = f'(x)$. Let $P(Y^X)$ be the set of all partial functions from X into Y; one checks easily that the relation of approximation defined on $P(Y^X)$ is a partial order; in fact, $(P(Y^X), \leq)$ is a meet-semilattice, whose smallest element is the less defined function (that is, the function f such that $D(f) = \emptyset$). For instance, if we represent each $f \in P(\{0,1\}^{\{0,1\}})$ by the image of $\{0,1\}$ under f:

$$f = \{f(0), f(1)\}$$

and use the star * to indicate that f is undefined for the given argument, then we can represent the whole semilattice $P(\{0,1\}^{\{0,1\}})$ by the following figure (underlined numerals are just names):

FIGURE 1 The approximating semilattice $P(\{0,1\}^{\{0,1\}})$



Are the notions we have just presented rich anough to build on them an adequate theory of partial functions in the theory of types? More specifically, can we restrict ourselves to these notions and use them to characterize the partial semantic values of a given functional type as partial functions which approximate classical values of the same type? Unfortunately, no; three major difficulties appear.

Difficulty 1. Of course, it would be possible to use these notions in order to define the partial values of types (t,t), (e,e), (t,e), (e,t), (s,t) and (s,e), and then to compare, in terms of approximation, the partial values of each of these types with the classical values of the same type. But for higher-order types, things would become much more complicated, for partial functions of these types can take partial functions as arguments. For instance, the domain of a partial function of type ((t,t)(t,t)), if non-empty, may contain total or non-total functions in $P(\{0,1\}^{\{0,1\}})$ and the values of such function can also be total or non-total functions in $P(\{0,1\}^{\{0,1\}})$. Then we already make out the problem to compare, in terms of approximation, the partial functions of a given higher-order type with the classical functions of the same type: given the fact that the starting and target sets of the partial functions will be different from the starting and target sets of the classical functions, the conditions of inclusion of domains and of identity of values will not be directly applicable.

Difficulty 2. This problem concerns the reiteration of functional applications and raises correlatively the question about the status of the indefiness in the theory of types. By the way of example, let f be a classical function of type (t,t), that is to say, let $f \in \{0,1\}^{\{0,1\}}$. Obviously, for each $x \in \{0,1\}$, f(x) is of type t, that is, $f(x) \in \{0,1\}$; therefore, f(f(x)) is also of type t. We believe that such reiteration of functional applications should be possible in the universe of partial functions. But consider a partial function g of type (t,t), that is to say, a function $g \in P(\{0,1\}^{\{0,1\}})$, such that g(1) is *undefined.* Try then to apply g to g(1). Strictly speaking, this application has no sense. But on the other hand, if g is the value of an expression A_{tt} of type (t,t) and 1 is the value of an expression B_t of type t, then by the principle of compositionality ,g(g(1)) must be the value of the expression [A[AB]] of type t. But again, one cannot see what could be this g(g(1)).

Difficulty 3. It is Pavel Tichý [1982] who has pointed out this difficulty. The famous Schönfinkel's theorem:

$$X^{Y\times Z} \approx (X^Y)^Z$$

which is fundamental in lambda calculus, appears to be not valid in the universe of partial functions. The counterexample of Tichý is the following. Consider the partial function f from $\{0,1\} \times \{0,1\}$ into $\{0,1\}$ such that:

(1)
$$f(x,y) = \begin{cases} y \text{ if } x = 0 \\ \text{undefined, otherwise} \end{cases}$$

There are *two* distinct partial functions from {0,1}which correspond to f: one assigns to 0 the identity function and is undefined for 1; the other assigns to 0 the identity function and assigns to 1 the function which is undefined both for 0 and 1. The idea here is that wathever is, between these two functions, the one which *really* corresponds to f, the other function must also correspond to f, if it corresponds to something.

We believe that these difficulties can be simultaneously solved by the application of the following three measures: (i) to give at the level of types e and t the status of object to undefiness, and for each functional type, to identify the undefiness with the less defined function; (ii) to define on each domain of partial objects a partial order relation, interpreted as a relation of approximation; (iii) to restrict the function spaces to monotonic functions. Let us see how the application of these measures can solve our problems.

In the sequel we shall adopt the following notational convention. If X and Y are two non-empty sets (not necessarily distinct) and O is an operation such that when it is applied to an element $x \in X$ gives one element $O(x) \in Y$, then we shall denote by "$\lambda x \in X.O(x)$" the function $f{:}X \rightarrow Y$ such that $f(x) = O(x)$ for every $x \in X$. When no confusion can arise, we shall simply write "$\lambda x.O(x)$".

Consider in first place Tichý's difficulty. Tichý's counterexample reveals the fact that :

$$P(X^{Y\times Z}) \not\approx P(P(X^Y)^Z).$$

Indeed, writting $|X|$ for the cardinality of a set X, it is the case that for any non-empty sets X, Y and Z:

$$|P(X^{Y\times Z})| = (|X|+1)^{|Y\times Z|} = ((|X|+1)^{|Y|})^{|Z|} < (((|X|+1)^{|Y|})+1)^{|Z|} = |P(P(X^Y)^Z)|.$$

However, since $(|X|+1)^{|Y\times Z|} = ((|X|+1)^{|Y|})^{|Z|} = |P(X^Y)|^{|Z|}$, then:

$$P(X^{Y\times Z}) \approx (P(X^Y)^Z).$$

This means that we can go toward a solution of the problem if we identify, in a domain of partial functions, the undefiness with the less defined function. Accordingly, to the function f defined by (1) corresponds an unique function, that is the function f' from $\{0,1\}$ into $P(\{0,1\}^{\{0,1\}})$ such that:

$$(2) \qquad f'(x) = \begin{cases} \lambda y.y \text{ if } x = 0 \\ \lambda y.\text{undefined, otherwise.} \end{cases}$$

Of course, this raises another problem, which is reminiscent of the second of our difficulties: the application of f' so defined to (f'(1))(1) has no sense.

However, the application of f' to (f'(1))(1) would have a sense if the undefiness were admitted among the possible arguments of f'. The only intuitive constraint we would like to see be respected is that the result of the application of f' to the undefiness be the undefiness.

Let us represent the undefiness of type t by φ; define *the set of partial objects of type t* as the set $PM_t := \{0,1,\varphi\}$. As usual 0 and 1 can be considered as the truth values *false* and *true* respectively; therefore $\varphi \in PM_t$ can be considered as a truth value which is not defined.

It seems natural to define on PM_t a relation of approximation in this way: for $x,y \in PM_t$

$$x \leq y \text{ iff } x = \varphi \text{ or } x = y.$$

Under this relation, PM_t is a (flat) meet-semilattice which can be pictured as follows:

$$\begin{matrix} 0 & & 1 \\ & \varphi & \end{matrix}$$

(Notice that PM_t is reminiscent of the approximating lattice BOOL of Dana Scott, minus the top). We can define in a similar manner the set PM_e of partial objects of type e. Given a non-empty set E of individuals, let $PM_e := E \cup \{\varphi\}$ (we can of course distinguish between $\varphi \in PM_e$ and $\varphi \in PM_t$ by using PM_e and PM_t as subscripts of φ). Then a relation of approximation can be defined on PM_e in the same way than on PM_t.

It is obviously possible to identify formally each function $f \in PM_t^{PMt}$ with the function $p(f) \in P(\{0,1\}^{\{0,1\}})$ by the surjection $p:PM_t^{PMt} \rightarrow P(\{0,1\}^{\{0,1\}})$ defined as follows:

$$(3) \qquad p(f) = \lambda x \in \{0,1\}.\begin{cases} f(x) \text{ if } f(x) \neq \varphi \\ \text{undefined, otherwise.} \end{cases}$$

From this point of view, a function $f \in PM_t^{PMt}$ is *total* if $f(x) \neq \varphi$ for every $x \neq \varphi$; then it is *non-total* if $f(x) = \varphi$ for at least one $x \neq \varphi$. So it seems that only elements in $\{0,1\}$are relevant arguments for the functions in PM_t^{PMt}. Consequently, we may think that only *strict functions* in PM_t^{PMt} represent suitably partial functions from $\{0,1\}$ into

$\{0,1\}$, where by a strict function in $PM_t^{PM_t}$ we mean any function f in this set such that $f(\varphi) = \varphi$ (that is, any function in this set having φ as fixed point). But we must be carefull, for it is not excluded that a total function be defined for an undefined argument of the right type.

For instance, consider the function $f = \lambda x.0 \in \{0,1\}^{\{0,1\}}$ and let y = 1 or y = 0. So f is of type (t,t) and then the object y, being of type t, is of the good type to be a possible argument of f, but it is not defined precisely: it is either 0 or 1. But can we "infer" the value of f for y? Obviously yes: f(y) = 0. From this point of view, the function $f' = \lambda x.0 \in PM_t^{PM_t}$ represents suitably the function f, since $f'(\varphi) = 0$. On the other hand, given $g \in \{0,1\}^{\{0,1\}}$ such that g(0) = 1 and g(1) = 0, it is impossible to infer the value of g for y; then we must let g(y) be undefined. So the *strict* function $g' \in PM_t^{PM_t}$ such that g'(1) = 0 and g'(0) = 1 represents suitably g.

To conclude, we must remove from the set $PM_t^{PM_t}$ all non-total functions in $PM_t^{PM_t}$ which are non-strict, and also all non-strict and total functions $f \in PM_t^{PM_t}$ such that $f(x) \neq f(\varphi)$ for at least one $x \neq \varphi$. All remainders represent suitably partial functions of type (t,t).

The set PM_{tt} of all partial functions of type (t,t) can be defined precisely with the help of the monotonicity constraint. Let X (= (X, ≤)) and Y (= (Y, ≤)) be two partially ordered sets; we say that a function $f \in Y^X$ is *monotonic* when for all $x, x' \in X$, if $x \leq x'$, then $f(x) \leq f(x')$. Let X and Y be two partially ordered sets; define:

$$(X \rightarrow Y) := \{f \in Y^X \mid f \text{ is monotonic}\}.$$

The set $(X \rightarrow Y)$ can be naturally ordered pointwise: for all $f, g \in (X \rightarrow Y)$, $f \leq g$ iff for every $x \in X$, $f(x) \leq g(x)$.

Now, define:

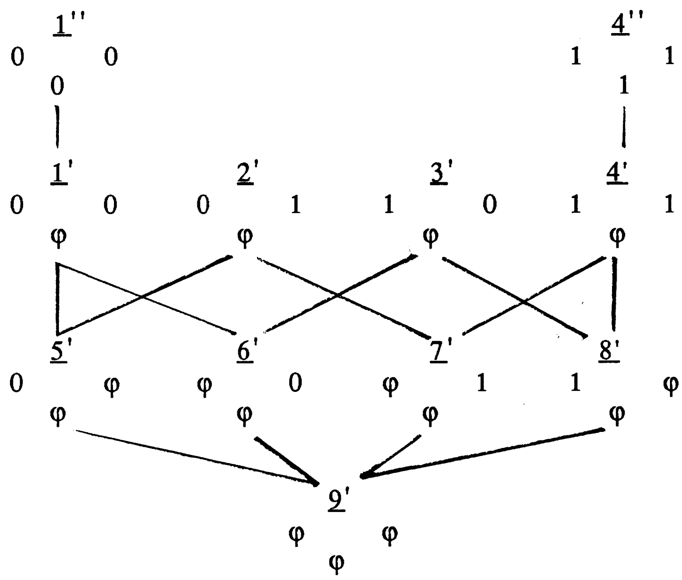
(4) $$PM_{tt} := (PM_t \rightarrow PM_t)$$

Again $(PM_{tt}, \leq)$ is a meet-semilattice. If we represent each $f \in PM_{tt}$ by the image of PM_t under f:

$$f = \begin{matrix} f(0) & & f(1) \\ & f(\varphi) & \end{matrix}$$

then we may represent PM_{tt} by the following figure.

FIGURE 2 The approximating semilattice PM_{tt}



If we compare this figure with figure 1, we immediately see that each non-total function n in $P(\{0,1\}^{\{0,1\}})$ can be identified with the non-total function n' in PM_{tt}. Moreover, the total functions 2 and 3 in $P(\{0,1\}^{\{0,1\}})$ can naturally be identified with the total functions 2' and 3' in PM_{tt} respectively. Concerning the total and constant functions 1 and 4 in $P(\{0,1\}^{\{0,1\}})$, considerations above lead us to identify them with the functions 1'' and 4'' in PM_{tt} respectively.

Consider again the function f' defined by (2). By looking the figure 1, we see that the functions f'(0) and f'(1) are respectively the functions 2 and 9 in $P(\{0,1\}^{\{0,1\}})$. By looking the figure 2, we see that to these functions correspond respectively the functions 2' and 9' in PM_{tt} . Hence the function f' can as such be identified with the function $g':PM_t \rightarrow PM_{tt}$ such that:

(5) $$g'(x) = \begin{cases} \lambda y.y \text{ if } x = 0 \\ \lambda y.\varphi, \text{ otherwise} \end{cases}$$

and then we see that unlike the application of f' to (f'(1))(1), the application of g' to (g'(1))(1) is meaningfull. Indeed, $g'((g'(1))(1)) = g'(\lambda y.\varphi(1)) = g'(\varphi) = \lambda y.\varphi$. Notice that according to our convention, the function $\lambda y.\varphi$ stands for the undefiness of type (t,t); hence from this point of view g'((g'(1))(1)) is undefined. Such identification is without doubt artificial, but it is nevertheless adequate for our purpose. Moreover, it is easy to check that the function g' is monotonic relatively to the elements of PM_t, so it belongs to the set:

$$PM_{t(tt)} := (PM_t \rightarrow (PM_t \rightarrow PM_t)).$$

Now if we define a partial order on the product $PM_t \times PM_t$ in the standard way, that is to say, for all (x,y), (x',y') $\in PM_t \times PM_t$:

$$(x,y) \leq (x',y') \text{ iff } x \leq x' \text{ and } y \leq y',$$

then we can easily check that $(PM_t \times PM_t \rightarrow PM_t)$ is isomorphic to $PM_{t(tt)}$. This is in general true for all spaces of monotonic functions. Hence not only monotonicity allows us to define suitably partial functions, also the sets of functions restricted to monotonic ones do satisfy Shönfinkel's theorem.

Consider again the figure 2. According to the terminology we shall adopte in the next Sections, the functions $\underline{1}'$, $\underline{1}''$ $\underline{2}'$, $\underline{3}'$, $\underline{4}'$ and $\underline{4}''$ in PM_{tt} are *(partial) total objects* . However, only the functions $\underline{1}''$ $\underline{2}'$, $\underline{3}'$ and $\underline{4}''$ are *maximal approximations* of classical objects in $\{0,1\}^{\{0,1\}}$. By the determination of the maximal approximation of each classical object, we can compare, in terms of approximation, partials objects with classical objects, and vice versa. For instance, the function $\underline{1}''$ in PM_{tt} maximally approximates the function $\lambda x.0 \in \{0,1\}^{\{0,1\}}$. Hence every function $f \in PM_{tt}$ such that $f \leq \underline{1}''$ can naturally be seen as an approximation of $\lambda x.0 \in \{0,1\}^{\{0,1\}}$.

We are now able to apply our approache to the modal theory of types.

C.2. The domains of partial objects.

1. Notational convention We shall use the letters a, b, c, ..., x, y, z, ..., x', y', z', ... as metavariables of semantical objects. We shall use more particulary the letters f, g, h, ..., f', g', h', ... as metavariables of semantical objects of functional types.

2. Definition Let T be the modal hierarchy of types defined in **B.1** and let E, I be two non-empty disjoint sets. *The standard system of partial objects based on E and I* is the indexed family $\{PM_\alpha\}_{\alpha \in T}$ of partially ordered sets such that:

(i) $PM_e = E \cup \{\varphi\}$, where for all $x,y \in PM_e$: $x \leq y$ iff $x = \varphi$ or $x = y$

(ii) $PM_t = \{0,1,\varphi\}$, where for all $x,y \in PM_t$: $x \leq y$ iff $x = \varphi$ or $x = y$

(iii) $PM_{\alpha\beta} = (PM_\alpha \rightarrow PM_\beta)$, where for all $f,g \in PM_{\alpha\beta}$:

$$f \leq g \text{ iff for every } x \in PM_\alpha,\ f(x) \leq g(x)$$

(iv) $PM_{s\alpha} = PM_\alpha{}^I$, where for all $f,g \in PM_{s\alpha}$:

$$f \leq g \text{ iff for every } i \in I,\ f(i) \leq g(i).$$

3. Remark It is useless to define a partial order relation on the set I, since there is no function having I as target set. Each domain PM_α stands for the set of Lepage's *good representations* of classical objects in M_α (see F. Lepage [1984]), provided that the systems $\{M_\alpha\}_{\alpha\in T}$ and $\{PM_\alpha\}_{\alpha\in T}$ are both based on the same sets E and I.

The next proposition is standard and can be proved easily.

4. Proposition For each $\sigma \in T$, PM_σ is an inf-semi-lattice, where for all $x,y \in PM_\sigma$, the meet of x and y, denoted by $x \wedge y$, is inductively given as follows:

(i) for $\sigma = e$ or t: $x \wedge y = x$ if $x = y$, otherwise $x \wedge y = \varphi$;
(ii) for $\sigma = \alpha\beta$: $f \wedge g = \lambda x \in PM_\alpha.f(x) \wedge g(x)$;
(iii) for $\sigma = s\alpha$: $f \wedge g = \lambda i \in I.f(i) \wedge g(i)$.

Moreover, if the joint of x and y, denoted by $x \vee y$, exists, then:

(i) for $\sigma = e$ or t: $x \vee y = y$ if $x = \varphi$,
$= x$ if $y = \varphi$ or $x = y$;
(ii) for $\sigma = \alpha\beta$: $f \vee g = \lambda x \in PM_\alpha.f(x) \vee g(x)$;
(iii) for $\sigma = s\alpha$: $f \vee g = \lambda i \in I.f(i) \vee g(i)$.

The notion defined below is very important. We borrow it from F.Lepage [1987].

5. Definition For each $\sigma \in T$, we define the *strong difference* between objects in PM_σ (formally $x \neq_S y$) as follows:
(i) for $\sigma = e$ or t: $x \neq_S y$ iff $x \neq \varphi$ and $y \neq \varphi$ and $x \neq y$;
(ii) for $\sigma = \alpha\beta$: $f \neq_S g$ iff there is $x \in PM_\alpha$ such that $f(x) \neq_S g(x)$;
(iii) for $\sigma = s\alpha$: $f \neq_S g$ iff there is $i \in I$ such that $f(i) \neq_S g(i)$.

Intuitively, strong difference means *incompatibility* : two distinct partial objects aren't necessarily incompatible. Hence, the next proposition says that incompatibility is transmitting "high up".

6. Proposition for each $\sigma \in T$ and all $x,y,x',y' \in PM_\sigma$:

$$\text{if } x \neq_S y,\ x \leq x' \text{ and } y \leq y', \text{then } x' \neq_S y'.$$

Proof We proceed by induction on the basis of types e and t.

(i) $\sigma = e$ or t. If $x \neq_S y$, then by definition $x \neq \varphi$, $y \neq \varphi$ and $x \neq y$. So by definition again, if $x \leq x'$ and $y \leq y'$, then $x = x'$ and $y = y'$. Therefore $x' \neq_S y'$.

(ii) $\sigma = \alpha\beta$. If $f \neq_S g$, then by definition there is $x \in PM_\alpha$ such that $f(x) \neq_S g(x)$. Let $a \in PM_\alpha$ be an object of this sort. So if $f \leq f'$ and $g \leq g'$, then $f(a) \leq f'(a)$ and $g(a) \leq g'(a)$. Then by the induction hypothesis, $f'(a) \neq_S g'(a)$. Therefore $f' \neq_S g'$.

(iii) $\sigma = s\alpha$. As in (ii).□

The next proposition says that compatible objects are dominated by an object which sum up them, and no more.

7. Proposition For each $\sigma \in T$, let C_σ be the set of all non-empty subsets of PM_σ which contain only objects which aren't pairwise strongly different; formally (we write $\neg(x \neq_S y)$ to express that x and y aren't strongly different):

$$C_\sigma := \{X \subseteq PM_\sigma \mid X \neq \emptyset \ \& \ \forall x,y \in PM_\sigma: x,y \in X \Rightarrow \neg(x \neq_S y)\};$$

then for every $X \in C_\sigma$, the supremum of X, denoted by $\bigvee X$, exists in PM_σ.

Proof We proceed by induction on the basis of types e and t.

(i) σ = e or t. It is easily checked that C_σ is the set:

$$\{X \subseteq PM_\sigma \mid \exists x \in PM_\sigma: X = \{x\} \text{ or } X = \{\varphi, x\}\}.$$

It is clear that for each element X of this set, $\bigvee X$ exists in PM_σ.

(ii) $\sigma = \alpha\beta$. Let $F \in C_\sigma$; so for each $x \in PM_\alpha$, $\{f(x) \mid f \in F\} \in C_\beta$. Indeed, suppose the contrary. Then there is $x \in PM_\alpha$ and there are $f, g \in F$ such that $f(x) \neq_S g(x)$. But this implies that $f \neq_S g$, which contredicts the supposition that $f, g \in F$.

Since for every $x \in PM_\alpha$, $\{f(x) \mid f \in F\} \in C_\beta$, then by the induction hypothesis, for every $x \in PM_\alpha$, $\bigvee\{f(x) \mid f \in F\}$ exists in PM_β. Therefore, the function:

$$\lambda x \in PM_\alpha . \bigvee\{f(x) \mid f \in F\}$$

is well defined, exists in PM_σ and is exactly $\bigvee F$.

(iii) $\sigma = s\alpha$. As in (ii).□

Now we are able to begin to compare partial objects with classical objects.

C.3 Comparing partial and classical objects.

Henceforth, given two non-empty sets E and I, we shall consider $\{M_\alpha\}_{\alpha\in T}$ and $\{PM_\alpha\}_{\alpha\in T}$ respectively as the standard systems of classical objects and partial objects based on E and I . No confusion should arise.

At first glance, comparaison in terms of approximation between partials and classical objects can be done easily. Consider the following definition, suggested by F. Lepage [1989a]. First, define inductively for each $\sigma \in T$ a relation $\leq_a \sqsubseteq PM_\sigma \times M_\sigma$ as follows ($x \leq_a y$ means "x is an approximation of y"):

(i) for $\sigma = e$ or t: $x \leq_a y$ iff $x = \varphi$ or $x = y$;

(ii) for $\sigma = \alpha\beta$: $f \leq_a g$ iff for all $x \in PM_\alpha$ and $y \in M_\alpha$ such that $x \leq_a y$, $f(x) \leq_a g(y)$;

(iii) for $\sigma = s\alpha$: $f \leq_a g$ iff for every $i \in I$, $f(i) \leq_a g(i)$.

Surely this definition is intuitively suitable. Moreover, each classical object $y \in M_\sigma$ can naturally be identified with the partial object $x \in PM_\sigma$ which approximates maximally y, x being simply $\bigvee\{x' \in PM_\sigma \mid x' \leq_a y\}$. Unfortunately, it appears that the demonstration of the existence of such object is far from be obvious; in fact, provided that there is a proof of this, it would requier a lot of lemmas. But we think that there is a more simple, general and elegant way for reaching the same result.

To begin with, we shall define, for each $\sigma \in T$, the property of *to be a total object in* PM_σ. Intuitively, to be a total object in PM_σ is to be a very good approximation of one classical object in M_σ.

8. Definition For each $\sigma \in T$, we define inductively the set Q_σ ($\sqsubseteq PM_\sigma$) of *total objects* in PM_σ as follows:

(i) for $\sigma = e$ or t: $Q_\sigma := \{x \in PM_\sigma \mid x \neq \varphi\}$;

(ii) for $\sigma = \alpha\beta$: $Q_\sigma := \{f \in PM_\sigma \mid \forall x \in Q_\alpha: f(x) \in Q_\beta\}$;

(iii) for $\sigma = s\alpha$: $Q_\sigma := \{f \in PM_\sigma \mid \forall i \in I: f(i) \in Q_\alpha\}$.

9. Remark Obviously, $Q_e = M_e = E$ and $Q_t = M_t = \{0,1\}$. However, for any type σ of the form $\alpha\beta$, it is not the case that $M_\sigma \approx Q_\sigma$. For instance, M_{tt} contains exactly four objects, while Q_{tt} contains six objects - if one takes a look on the figure 2 of the Section **C.1**, one sees that $Q_{tt} = \{\underline{1}', \underline{2}', \underline{3}', \underline{4}', \underline{1}'', \underline{4}''\}$. But an equivalence relation can be defined on a set of total objects. For instance, the functions $\underline{1}'$ and $\underline{1}''$ in Q_{tt}, *restricted* to Q_t, are identical; so we can consider that both are very good approximations of the function $\lambda x.0$ in M_{tt}. Similarly, the functions $\underline{4}'$ and $\underline{4}''$ in Q_{tt} can both be seen as very good approximations of the function $\lambda x.1$ in M_{tt}. From this point of view, we can consider $\underline{1}'$ and $\underline{1}''$ (and $\underline{4}'$ and $\underline{4}''$) as equivalent total objects. The next definition generalizes this view to all types.

10. Definition For each $\sigma \in$ T, we define inductively an equivalence relation between objects in Q_σ (formally, $x<>y$) as follows:

(i) for σ = e or t and $x,y \in Q_\sigma$: $x<>y$ iff $x = y$

(ii) for $\sigma = \alpha\beta$ and $f,g \in Q_\sigma$: $f<>g$ iff for every $x \in Q_\alpha$: $f(x)<>g(x)$

(iii) for $\sigma = s\alpha$ et $f,g \in Q_\sigma$: $f<>g$ iff for every $i \in I$: $f(i)<>g(i)$.

The next proposition says that a partial object, y, which is as well defined than a total object x, is total and equivalent to x.

11. Proposition For each $\sigma \in$ T, every $x \in Q_\sigma$ and every $y \in PM_\sigma$:

$$\text{if } x \leq y, \text{ then } y \in Q_\sigma \text{ and } x<>y.$$

Proof We proceed by induction on the basis of types e and t.

(i) σ = e or t. If $x \in Q_\sigma$, then by definition $x \neq \varphi$. Hence if $x \leq y$, then by definition again, $x = y$. Therefore $y \in Q_\sigma$ and $x<>y$.

(ii) $\sigma = \alpha\beta$. If $f \in Q_\sigma$, then by definition $f(x) \in Q_\beta$ for every $x \in Q_\alpha$. So if $f \leq g$, then $f(x) \leq g(x)$ for every $x \in PM_\alpha$ and hence, $f(x) \leq g(x)$ for every $x \in Q_\alpha$. Therefore, by the induction hypothesis, $g(x) \in Q_\beta$ and $f(x)<>g(x)$ for every $x \in Q_\alpha$. Hence by definition, $g \in Q_\sigma$ and $f<>g$.

(iii) $\sigma = s\alpha$. As in (ii).□

The next proposition is not only obvious by itself but it will be useful in the proofs of many future propositions. It says that each non-total object is an approximation of a total one.

12. Proposition For each $\sigma \in$ T and every $x \in PM_\sigma$:

$$\text{if } x \notin Q_\sigma, \text{ then there exists } y \in Q_\sigma \text{ such that } x \leq y.$$

Proof We proceed by induction on the basis of types e and t.

(i) σ = e or t. If $x \notin Q_\sigma$, then by definition $x = \varphi$. By definition again, every $y \in Q_\sigma$ is such that $\varphi \leq y$.

(ii) $\sigma = \alpha\beta$. If $f \notin Q_\sigma$, then by definition there is $x \in Q_\alpha$ such that $f(x) \notin Q_\beta$. But by the induction hypothesis, for any $x \in Q_\alpha$ such that $f(x) \notin Q_\beta$, there is $y \in Q_\beta$ such that $f(x) \leq y$. This assures that for *each* $x \in Q_\alpha$, the set:

$$S_{f(x)} := \{y \in Q_\beta \mid f(x) \leq y\},$$

is not empty. Define:

$$U_f := \{X \subseteq Q_\beta \mid \exists x \in Q_\alpha : X = S_{f(x)}\},$$

et let χ be any function from U_f into Q_β such that:

(i) for every $X \in U_f$, $\chi(X) \in X$;
(ii) for all $X, X' \in U_f$, if $X \subseteq X'$, then $\chi(X') \leq \chi(X)$.

We can check that there are functions of this sort. Now consider $g{:}PM_\alpha \rightarrow PM_\beta$ such that:

$$g(x) = \begin{cases} f(x) \text{ if } x \notin Q_\alpha \\ \chi(S_{f(x)}) \text{ otherwise.} \end{cases}$$

One easily verifies that if g is monotonic, then $g \in Q_\sigma$ and $f \leq g$. So let us check that g is monotonic.

Let $x, x' \in PM_\alpha$ such that $x \leq x'$. A priori, there are four possible cases for x,x':

(**1**) $x, x' \notin Q_\alpha$;

(**2**) $x \notin Q_\alpha$ and $x' \in Q_\alpha$;

(**3**) $x \in Q_\alpha$ and $x' \notin Q_\alpha$;

(**4**) $x, x' \in Q_\alpha$.

The case (**3**) is excluded by proposition 11. Let us see the other cases.

In case (**1**), we have $g(x) = f(x) \leq f(x') = g(x')$. Hence $g(x) \leq g(x')$.

In case (**2**), we have $g(x) = f(x) \leq f(x') \leq \chi(S_{f(x')}) = g(x')$. So $g(x) \leq g(x')$.

In case (**4**), it is easily checked, by the monotonicity of f, that $S_{f(x')} \subseteq S_{f(x)}$. Therefore we have $g(x) = \chi(S_{f(x)}) \leq \chi(S_{f(x')}) = g(x')$. Hence $g(x) \leq g(x')$. This means that g is monotonic.

(iii) $\sigma = s\alpha$. As in (ii), but much more simple. □

The following proposition requires, in one direction (the direction $\Rightarrow$), the monotonicity of functions of type $\alpha\beta$ (for all total objects x and y, we write $\neg(x<>y)$ to express the fact that x and y aren't equivalent).

13. Proposition For each $\sigma \in T$ and for all $x,y \in Q_\sigma$:

$$x \neq_S y \text{ iff } \neg(x<>y).$$

Proof ($\Rightarrow$) We proceed by induction on the basis of types e and t.

(i) $\sigma = e$ or t. If $x \neq_S y$, then $x \neq y$ by definition. So by definition again, $\neg(x<>y)$.

(ii) $\sigma = \alpha\beta$. If $f \neq_S g$, then by definition there is $x \in PM_\alpha$ such that $f(x) \neq_S g(x)$. Let $a \in PM_\alpha$ be an object of this sort. If $a \in Q_\alpha$, then by the induction hypothesis, $\neg(f(a)<>g(a))$ and therefore by definition, $\neg(f<>g)$. If $a \notin Q_\alpha$, then by proposition 12, there is $y \in Q_\alpha$ such that $a \leq y$. Let $b \in Q_\alpha$ be an object of this sort. Since f and g are monotonic, then $f(a) \leq f(b)$ and $g(a) \leq g(b)$. But $f(a) \neq_S g(a)$. Therefore $f(b) \neq_S g(b)$, by proposition 6. So by the induction hypothesis, $\neg(f(b)<>g(b))$ and so $\neg(f<>g)$ by definition.

(iii) $\sigma = s\alpha$. As in (ii), but much more simple.

($\Leftarrow$) Again we proceed by induction on the basis of types e and t.

(i) $\sigma = e$ or t. If $\neg(x<>y)$, the by definition $x \neq y$ and hence, by definition again, $x \neq_S y$.

(ii) $\sigma = \alpha\beta$. If $\neg(f<>g)$, then by definition there is $x \in Q_\alpha$ such that $\neg(f(x)<>g(x))$ and hence, by the induction hypothesis, such that $f(x) \neq_S f(x)$. So $f \neq_S g$ by definition.

(iii) $\sigma = s\alpha$. As in (ii).□

The next proposition is analogous to the Leibniz's law. Its proof requires the monotonicity of functions of type $\alpha\beta$.

14. Proposition For every $\sigma \in T$ of the form $\alpha\beta$, and for $f \in Q_\sigma$ and $x, y \in Q_\alpha$:

$$\text{if } x<>y, \text{ then } f(x)<>f(y).$$

Proof If $x<>y$, then $\neg(x \neq_S y)$ by proposition 13. Hence by proposition 7, there is $z \in PM_\alpha$ such that $z = x \vee y$ and so such that $x \leq z$ and $y \leq z$. Therefore, by the monotonicity of f, $f(x) \leq f(z)$ and $f(y) \leq f(z)$. But $f(x), f(y) \in Q_\beta$; so by proposition 11, $f(z) \in Q_\beta$, $f(x)<>f(z)$ and $f(y)<>f(z)$. Therefore $f(x)<>f(y)$.□

15. Proposition Let, for each $\sigma \in T$, Π_σ be the partition of Q_σ generated by $<>$; formally:

$$\Pi_\sigma := \{X \subseteq Q_\sigma \mid \exists x \in Q_\sigma : X = \{y \in Q_\sigma \mid x<>y\}\};$$

then for each $\sigma \in T$ and for every $X \in \Pi_\sigma$, $\bigvee X$ exists in PM_σ and belongs to X.

Proof By the definition of Π_σ and proposition 13, each $X \in \Pi_\sigma$ belongs to C_σ. Hence by proposition 7, $\bigvee X$ exists in PM_σ. Moreover, it is obviously the case that $x \leq \bigvee X$ for every $x \in X$. Therefore, $x<>\bigvee X$ for every $x \in X$, according to proposition 11. So by definition, $\bigvee X \in X$.□

16. Proposition (i) Let $F \in \Pi_{\alpha\beta}$, $X \in \Pi_\alpha$ and $Y \in \Pi_\beta$ such that $\bigvee F(\bigvee X) \in Y$; then $\bigvee F(\bigvee X) = \bigvee Y$. (ii) Let $F \in \Pi_{s\alpha}$, $i \in I$ and $Y \in \Pi_\alpha$ such that $\bigvee F(i) \in Y$; then $\bigvee F(i) = \bigvee Y$.

Proof (i) Since for every $x \in X$ and every $f \in F$, $x \leq \bigvee X$ and $f \leq \bigvee F$, then for every $x \in X$ and every $f \in F$, $f(x) \leq \bigvee F(\bigvee X)$. Similarly, for every $y \in Y$, $y \leq \bigvee Y$; therefore, since by hypothesis, $\bigvee F(\bigvee X) \in Y$, then $\bigvee F(\bigvee X) \leq \bigvee Y$ and consequently:

(1) for every $x \in X$ and every $f \in F$: $f(x) \leq \bigvee F(\bigvee X) \leq \bigvee Y$.

Now by proposition 15, $\bigvee X \in X$ and $\bigvee F \in F$; so this plus **(1)** imply:

(2) for every $x \in X$ and every $f \in F$, if $\bigvee F(\bigvee X) \leq f(x) \leq \bigvee Y$, then $f(x) = \bigvee F(\bigvee X)$.

Suppose that $\bigvee F(\bigvee X) \neq \bigvee Y$. Then there is $y \in Y$ such that $\bigvee F(\bigvee X) \neq y$ and $\bigvee F(\bigvee X) \leq y \leq \bigvee Y$. Let $a \in Y$ be an object of this sort and consider the function $g:PM_\alpha \rightarrow PM_\beta$ such that:

$$g(x) = \begin{cases} a \text{ if } x = \bigvee X \\ \bigvee F(x) \text{ otherwise.} \end{cases}$$

One can easily check that if g exists, then $g \in PM_{\alpha\beta}$, $g \neq \bigvee F$ but $\bigvee F \leq g$. By proposition 11 this implies that $g \in F$, contradicting **(2)** above. Therefore, $\bigvee F(\bigvee X) = \bigvee Y$. (iii) As in (ii), but more simple.□

We would like to associate to each classical object $x \in M_\sigma$, and this for each $\sigma \in T$, the equivalence class $X \in \Pi_\sigma$ of objects in Q_σ which are very good approximations of x. Inversely, we would like to associate to each equivalence class $X \in \Pi_\sigma$ the classical object $x \in M_\sigma$ which is approximated by the objects in X. This is the purpose of the next definition.

17. Definition We define inductively two functions:

$$\Phi: \bigsqcup_{\sigma \in T} M_\sigma \to \bigsqcup_{\sigma \in T} \Pi_\sigma$$

$$\Psi: \bigsqcup_{\sigma \in T} \Pi_\sigma \to \bigsqcup_{\sigma \in T} M_\sigma$$

as follows.

(i) for $\sigma = e$ or t: $\Phi(x) = \{x\}$ and $\Psi(\{x\}) = x$;

(ii) for $\sigma = \alpha\beta$:

$$\Phi(f) = \{f' \in Q_\sigma \mid \forall x \in M_\alpha: \forall y \in \Phi(x): \ f'(y) \in \ \Phi(f(x))\};$$

$$\Psi(F) = \lambda x \in M_\alpha.\Psi(\{z \in Q_\beta \mid \forall g \in F: \forall y \in \Phi(x): \ z <> g(y)\});$$

(iii) for $\sigma = s\alpha$:

$$\Phi(f) = \{f' \in Q_\sigma \mid \forall i \in I: \ f'(i) \in \ \Phi(f(i))\}$$

$$\Psi(F) = \lambda i \in I.\Psi(\{z \in Q_\alpha \mid \forall g \in F: \ z <> g(i)\}).$$

18. Proposition Φ is bijective and $\Phi^{-1} = \Psi$.

Proof It is sufficient to show that $\Psi(\Phi(x)) = x$ and $\Phi(\Psi(X)) = X$. One proceeds by induction on types e and t.

(i) $\sigma = e$ or t. $\Psi(\Phi(x)) = \Psi(\{x\}) = x$ and $\Phi(\Psi(\{x\})) = \Phi(x) = \{x\}$.

(ii) $\sigma = \alpha\beta$. Suppose that the property is satisfied by the objects of types α and β. Let $f \in M_\sigma$ and $F = \Phi(f)$. So $\Psi(\Phi(f)) = \Psi(F) =$

(1) $$\lambda x \in M_\alpha.\Psi(\{z \in Q_\beta \mid \forall g \in F: \forall y \in \Phi(x): \ z <> g(y)\});$$

By definition, $g(y) \in \Phi(f(x))$ for every $g \in F$, every $x \in M_\alpha$ and every $y \in \Phi(x)$; therefore (**1**) is equal to:

$$\lambda x \in M_\alpha.\Psi(\{z \in Q_\beta \mid z \in \Phi(f(x))\})$$

and this is of course equal to:

$$\text{(2)} \qquad \lambda x \in M_\alpha.\Psi(\Phi(f(x))).$$

Therefore (**2**) is equal to $\lambda x \in M_\alpha.f(x)$, that is f.

On the other hand, let $F \in \Pi_\sigma$ and $f = \Psi(F)$. So $\Phi(\Psi(F)) = \Phi(f) =$

$$\text{(1')} \qquad \{f' \in Q_\sigma \mid \forall x \in M_\alpha: \forall y \in \Phi(x): f'(y) \in \Phi(f(x))\}.$$

But by definition, for every $x \in M_\alpha$:

$$f(x) = \Psi(\{z \in Q_\beta \mid \forall g \in F: \forall y \in \Phi(x): z <> g(y)\}).$$

Therefore (**1'**) is the set:

$$\{f' \in Q_\sigma \mid \forall x \in M_\alpha: \forall y \in \Phi(x): f'(y) \in \{z \in Q_\beta \mid \forall g \in F: \forall y \in \Phi(x): z <> g(y)\}\},$$

which is equal to the set:

$$F' = \{f' \in Q_\sigma \mid \forall x \in M_\alpha: \forall y \in \Phi(x): \forall g \in F: \forall z \in \Phi(x): f'(y) <> g(z)\}$$

Now $F \subseteq F'$. Indeed, suppose there is $f \in F$ such that $f \notin F'$. Therefore:

$$\text{(2')} \qquad \exists x \in M_\alpha: \exists y \in \Phi(x): \exists g \in F: \exists z \in \Phi(x): \neg(f(y) <> g(z)).$$

But for every $x \in M_\alpha$ and for all $y,z \in \Phi(x)$, $y,z \in Q_\alpha$ and $y <> z$. Hence (**2'**) implies:

$$\text{(3')} \qquad \exists g \in F: \exists y,z \in Q_\alpha: y <> z \ \ \& \ \neg(f(y) <> g(z)).$$

But since $f \in F$, then for every $g \in F$, $f(x) <> g(x)$ for every $x \in Q_\alpha$. This and proposition 14 imply that for all $y,z \in Q_\alpha$ such that $y <> z$, $f(y) <> f(z) <> g(z)$, so $f(y) <> g(z)$, and this clearly contradicts (**3'**).

On the other hand, $F' \subseteq F$. Indeed, suppose there is $f' \in F'$ such that $f' \notin F$. Therefore, it is obviously the case that for any $g \in F$, $\neg(f' <> g)$ and so:

(4') $\exists z \in Q_\alpha: \neg(f'(z)<>g(z))$.

But since $f' \in F'$:

(5') $\forall x \in M_\alpha: \forall y \in \Phi(x): f'(y)<>g(y)$.

So **(4')**, **(5')** and proposition 14 imply:

(6') $\exists z \in Q_\alpha: \forall x \in M_\alpha: \forall y \in \Phi(x): \neg(z<>y)$.

But $Q_\alpha = \{y \in \Phi(x) \mid x \in M_\alpha\}$. Therefore **(6')** is equivalent to:

$$\exists z \in Q_\alpha: \forall y \in Q_\alpha: \neg(z<>y),$$

and this implies that there is $z \in Q_\alpha$ such that $\neg(z<>z)$, which is absurd.

(iii) $\sigma = s\alpha$. Assuming that the property is satisfied by the objects of type α, the proof is as in (ii), but much more simple.□

19. Remark Now we know that for any $\sigma \in T$, $M_\sigma \approx \Pi_\sigma$. Moreover, proposition 15 assures that for every classical object $x \in M_\sigma$, the object $\bigvee \Phi(x) \in Q_\sigma$ is the unique partial total object which approximates maximally x. Therefore we can identify each classical object $x \in M_\sigma$ with its maximal approximation $\bigvee \Phi(x)$, and then consider any partial object $y \in PM_\sigma$ such that $y \leq \bigvee \Phi(x)$, as an approximation of x.

20. Notational convention For any $\sigma \in T$ and any $x \in M_\sigma$, we denote $\bigvee \Phi(x)$ (the maximal approximation of x) by *ma(x)* .

21. Proposition (i) Let $f \in M_{\alpha\beta}$ and $x \in M_\alpha$; then $(ma(f))(ma(x)) = ma(f(x))$. (ii) Let $f \in M_{s\alpha}$; then for every $i \in I$, $(ma(f))(i) = ma(f(i))$.

Proof (i) By definition 17-(ii) and proposition 15, $(ma(f))(ma(x)) \in \Phi(f(x))$. Therefore, by proposition 16-(i), $(ma(f))(ma(x)) = ma(f(x))$. (ii) By definition 17-(iii) and proposition 15, $(ma(f))(i) \in \Phi(f(i))$. Therefore, by proposition 16-(ii), $(ma(f))(i) = ma(f(i))$.□

22. Remark The partial functions of type (t,t) or (t,(t,t)) of our system which correspond to the truth functions of Kleene's strong three-valued logic (KSL) are exactly the maximal approximations of the classical truth functions. Indeed, to the function of negation according to KSL corresponds the function $\underline{3}'$ in PM_{tt} (see figure 2, Section **C.1**) and this function is obvioully the maximal approximation of the classical truth function of negation. Moreover, to each binary truth function of KSL corresponds a function f in $PM_{t(tt)}$ which can be represented by the image of PM_t under f, as follows:

<u>Conjunction</u>

```
0  0    0  1
 0        φ

   0  φ
     φ
```

<u>Disjunction</u>

```
0  1    1  1
 φ        1

   φ  1
     φ
```

<u>Conditional</u>

```
1  1    0  1
 1        φ

   φ  1
     φ
```

<u>Biconditional</u>

```
1  0    0  1
 φ        φ

   φ  φ
     φ
```

It is then easy to check that each of these functions belongs to $Q_{t(tt)}$ and that it is not dominated by another function and so that it is the maximal approximation of its classical analogue.

This ends the description of the domains of partials objects. We are now able to define the notion of standard partial model for the language of **IL**.

C.4. Definition of the notion of standard partial model.

Let E and I be two non-empty disjoint sets. *A standard partial model based on E and I* is an ordered pair:

$$PM = <\{PM_\alpha\}_{\alpha\in T}, pm>$$

where: $\{PM_\alpha\}_{\alpha\in T}$ is the standard system of partial objects based on E and I; pm is a function:

$$pm: \bigcup_{\alpha\in T} Con_\alpha \rightarrow \bigcup_{\alpha\in T} PM_{s\alpha}$$

such that $pm(c_\alpha) \in PM_{s\alpha}$ for each $c_\alpha \in Con_\alpha$. We denote by As(PM) *the set of assignments over PM*, that is the set of all functions:

$$\mathbf{pa}: \bigcup_{\alpha\in T} Var_\alpha \rightarrow \bigcup_{\alpha\in T} PM_\alpha$$

such that $\mathbf{pa}(x_\alpha) \in PM_\alpha$ for any $x_\alpha \in Var_\alpha$. For any $\mathbf{pa} \in As(PM)$, any $x_\alpha \in Var_\alpha$ and any $y \in M_\alpha$, we denote by $\mathbf{pa}(x_\alpha/y)$ the assignment in As(PM) which differs at most from **pa** by assigning y to x_α. We define recursively *the partial value* $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},i}$ *in* PM of a term A_α *according to* an assignment $\mathbf{pa} \in As(PM)$ and an $i \in I$ as follows:

(i) $[\![c_\alpha]\!]^{PM}_{\mathbf{pa},i} = (pm(c_\alpha))(i)$;

(ii) $[\![x_\alpha]\!]^{PM}_{\mathbf{pa},i} = \mathbf{pa}(v_\alpha)$;

(iii) $[\![A_{\alpha\beta}B_\alpha]\!]^{PM}_{\mathbf{pa},i} = [\![A_{\alpha\beta}]\!]^{PM}_{\mathbf{pa},i}([\![B_\alpha]\!]^{PM}_{\mathbf{pa},i})$;

(iv) $[\![{}^\wedge A_\alpha]\!]^{PM}_{\mathbf{pa},i}$ = the function f from I such that for any $j \in I$, $f(j) = [\![A_\alpha]\!]^{PM}_{\mathbf{pa},j}$;

(v) $[\![{}^\vee A_{s\alpha}]\!]^{PM}_{\mathbf{pa},i} = [\![A_{s\alpha}]\!]^{PM}_{\mathbf{pa},i}(i)$;

(vi) $[\![\lambda x_\alpha A_\beta]\!]^{PM}_{\mathbf{pa},i}$= the function f from PM_α such that for any $y \in PM_\alpha$, $f(y) = [\![A_\beta]\!]^{PM}_{\mathbf{pa}',i}$ where $\mathbf{pa}' = \mathbf{pa}(x_\alpha/y)$;

$$\text{(vii) } [\![[A_\alpha \equiv B_\alpha]]\!]^{PM}_{\mathbf{pa},i} = \begin{cases} 1 \text{ if } [\![A_\alpha]\!]^{PM}_{\mathbf{pa},i}, [\![B_\alpha]\!]^{PM}_{\mathbf{pa},i} \in Q_\alpha \text{ and } [\![A_\alpha]\!]^{PM}_{\mathbf{pa},i} <> [\![B_\alpha]\!]^{PM}_{\mathbf{pa},i} \\ 0 \text{ if } [\![A_\alpha]\!]^{PM}_{\mathbf{pa},i} \neq_S [\![B_\alpha]\!]^{PM}_{\mathbf{pa},i} \\ \varphi \text{ otherwise.} \end{cases}$$

$$\text{(viii) } [\![[A_t \wedge B_t]]\!]^{PM}_{\mathbf{pa},i} = \begin{cases} 1 \text{ if } [\![A_t]\!]^{PM}_{\mathbf{pa},i} = [\![B_t]\!]^{PM}_{\mathbf{pa},i} = 1 \\ 0 \text{ if } [\![A_t]\!]^{PM}_{\mathbf{pa},i} = 0 \text{ or } [\![B_t]\!]^{PM}_{\mathbf{pa},i} = 0 \\ \varphi \text{ otherwise.} \end{cases}$$

23. Remark The rule (vii) of identity makes indiscernable, in the object language, all total equivalent objects, and by proposition 13, it makes discernable all non-equivalent total objects. So identity between total objects has the same behaviour than standard identity between classical objects. This expresses the idea the if the *known denotations* of two expressions are total objects, then one can safely determine if the two expressions denote *in fact* the same thing or not. On the other hand, it is sufficient that the known denotations of two expressions be incompatible to be able to infer that the denotations of the expressions aren't in fact the same. However, if the known denotations of two expressions aren't total but compatible, one cannot determine if the two expressions denote in fact the same thing or not; this is the case even if the known denotations of the two expressions are both equally undefined.

24. Proposition For any $\alpha \in T$, any $A_\alpha \in Trm_\alpha$, any $\mathbf{pa} \in As(PM)$ and any $i \in I$, $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},i} \in PM_\alpha$.

Proof This is immediately verified if A_α is a constant or a variable - this follows from the definitions of pm and **pa**. This is also verified by induction when A_α is of the form [BC], λxB, $^\vee$B or $^\wedge$B. Indeed, we note that to these four sorts of terms, correspond the functional application and its inverse, the functional abstraction, two operations which always preserve partial order and monotonicity. Remain identity and conjunction. Though it is obviously the case that for all A_α and B_α, $[\![[A_\alpha \equiv B_\alpha]]\!]^{PM}_{\mathbf{pa},i} \in PM_t$, we have to check that identity can only generate, by functional abstraction, monotonic functions; that is, we must check that it is effectively the case that:

$$[\![\lambda x_\alpha \lambda y_\alpha [x_\alpha \equiv y_\alpha]]\!]^{PM}_{pa,i} \in PM_{\alpha(\alpha t)}.$$

We easily check that $[\![\lambda x_\alpha \lambda y_\alpha [x_\alpha \equiv y_\alpha]]\!]^{PM}_{pa,i}$ is the function, say *id*, from PM_α into $PM_t^{PM\alpha}$ such that for all x,y $\in PM_\alpha$:

$$(id\,(x))(y) = \begin{cases} 1 \text{ if } x,y \in Q_\alpha \text{ and } x<>y \\ 0 \text{ if } x \neq_S y \\ \varphi \text{ otherwise.} \end{cases}$$

It is then sufficient to show that *id* is monotonic. This is shown by proving that for every x $\in PM_\alpha$, the one-place functions $\lambda y.(id\,(x))(y)$ and $\lambda y.(id\,(y))(x)$ are monotonic. We shall show this by restricting our proof to the case of the first function; the proof for the other function is similar, anyway.

Let x,y,z $\in PM_\alpha$ and $y \leq z$. Four cases are possible for x, y:

(1) x,y $\in Q_\alpha$;

(2) x $\in Q_\alpha$ and y $\notin Q_\alpha$;

(3) x $\notin Q_\alpha$ and y $\in Q_\alpha$;

(4) x,y $\notin Q_\alpha$.

In case (1), if x<>y, then $(id\,(x))(y) = 1$. But by proposition 11, z $\in Q_\alpha$ and y<>z, and this implies that x<>z, so $(id\,(x))(z) = 1$. Therefore, $(id\,(x))(y) \leq (id\,(x))(z)$. If, on the other hand, $\neg$(x<>y), then by proposition 13, $x \neq_S y$ and hence $(id\,(x))(y) = 0$. But by proposition 6, $x \neq_S z$ and so $(id\,(x))(z) = 0$. Therefore, $(id\,(x))(y) \leq (id\,(x))(z)$.

In case (2) proposition 11 implies $\neg(x \leq y)$. Therefore, either $y \leq x$, or either $x \neq_S y$, or either $\neg(y \leq x)$ and $\neg(x \neq_S y)$. First, if $y \leq x$, then of course $\neg(x \neq_S y)$, so we have $(id\,(x))(y) \neq 1$ and $(id\,(x))(y) \neq 0$. Therefore $(id\,(x))(y) = \varphi$ and hence $(id\,(x))(y) \leq (id\,(x))(z)$. Secondly, if $x \neq_S y$, then $(id\,(x))(y) = 0$. But by proposition 6, $x \neq_S z$ and hence $(id\,(x))(z) = 0$. So $(id\,(x))(y) \leq (id\,(x))(z)$. At last, if $\neg(y \leq x)$ and $\neg(x \neq_S y)$, then again $(id\,(x))(y) \neq 1$, $(id\,(x))(y) \neq 0$ and hence $(id\,(x))(y) = \varphi$. Therefore, $(id\,(x))(y) \leq (id\,(x))(z)$.

The case (3) is similar to the previous one. The case (4) is just the sum of the two previous ones.

It is shown in a similar manner that the conjunction is monotonic. □

25. Proposition Let PM = <$\{PM_\alpha\}_{\alpha\in T}$, pm> and PM' = <$\{PM_\alpha\}_{\alpha\in T}$, pm'> be two standard partial models and let **pa**, **pa'** ∈ As(PM) = As(PM'); then if for every $\alpha \in T$, every $c_\alpha \in Con_\alpha$ and every $x_\alpha \in Var_\alpha$, $pm(c_\alpha) \leq pm'(c_\alpha)$ and $\mathbf{pa}(x_\alpha) \leq \mathbf{pa'}(x_\alpha)$, then for every $\alpha \in T$, every $A_\alpha \in Trm_\alpha$ and every $i \in I$, $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},i} \leq [\![A_\alpha]\!]^{PM'}_{\mathbf{pa'},i}$.

Proof This is immediately verified for constants and variables. Let us see the other cases.

• Consider a term $[A_{\alpha\beta}B_\alpha]$. By the induction hypothesis:

$$[\![A_{\alpha\beta}]\!]^{PM}_{\mathbf{pa},i} \leq [\![A_{\alpha\beta}]\!]^{PM'}_{\mathbf{pa'},i} \text{ and } [\![B_\alpha]\!]^{PM}_{\mathbf{pa},i} \leq [\![B_\alpha]\!]^{PM'}_{\mathbf{pa'},i}.$$

By the definition of the partial order on $PM_{\alpha\beta}$ and by the monotonicity of the objects in $PM_{\alpha\beta}$, this implies:

$$[\![A_{\alpha\beta}]\!]^{PM}_{\mathbf{pa},i}([\![B_\alpha]\!]^{PM}_{\mathbf{pa},i}) \leq [\![A_{\alpha\beta}]\!]^{PM'}_{\mathbf{pa'},i}([\![B_\alpha]\!]^{PM'}_{\mathbf{pa'},i}).$$

This means that $[\![[A_{\alpha\beta}B_\alpha]]\!]^{PM}_{\mathbf{pa},i} \leq [\![[A_{\alpha\beta}B_\alpha]]\!]^{PM'}_{\mathbf{pa'},i}$.

• Consider a term $^\wedge A_\alpha$. By definition, for any $j \in I$, $[\![^\wedge A_\alpha]\!]^{PM}_{\mathbf{pa},i}(j) = [\![A_\alpha]\!]^{PM}_{\mathbf{pa},j}$ and $[\![^\wedge A_\alpha]\!]^{PM'}_{\mathbf{pa'},i}(j) = [\![^\wedge A_\alpha]\!]^{PM'}_{\mathbf{pa'},j}$. By the induction hypothesis, $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},j} \leq [\![A_\alpha]\!]^{PM'}_{\mathbf{pa'},j}$ for any $j \in I$. By the definition of the partial order on $PM_{s\alpha}$, this means that $[\![^\wedge A_\alpha]\!]^{PM}_{\mathbf{pa},i} \leq [\![^\wedge A_\alpha]\!]^{PM'}_{\mathbf{pa'},i}$.

• Consider a term $^\vee A_{s\alpha}$. By definition, for any $i \in I$, $[\![^\vee A_{s\alpha}]\!]^{PM}_{\mathbf{pa},i} = [\![A_{s\alpha}]\!]^{PM}_{\mathbf{pa},i}(i)$ and $[\![^\vee A_{s\alpha}]\!]^{PM'}_{\mathbf{pa'},i} = [\![A_{s\alpha}]\!]^{PM'}_{\mathbf{pa'},i}(i)$. By the induction hypothesis, $[\![A_{s\alpha}]\!]^{PM}_{\mathbf{pa},i} \leq [\![A_{s\alpha}]\!]^{PM'}_{\mathbf{pa'},i}$ for any $i \in I$. Therefore $[\![A_{s\alpha}]\!]^{PM}_{\mathbf{pa},i}(i) \leq [\![A_{s\alpha}]\!]^{PM'}_{\mathbf{pa'},i}(i)$. This means that $[\![^\vee A_{s\alpha}]\!]^{PM}_{\mathbf{pa},i} \leq [\![^\vee A_{s\alpha}]\!]^{PM'}_{\mathbf{pa'},i}$.

• Consider a term $\lambda x_\alpha A_\beta$. By definition, for any $y \in PM_\alpha$, $[\![\lambda x_\alpha A_\beta]\!]^{PM}_{\mathbf{pa},i}(y) = [\![A_\beta]\!]^{PM}_{\mathbf{pb},i}$ where $\mathbf{pb} = \mathbf{pa}(x_\alpha/y)$, and $[\![\lambda x_\alpha A_\beta]\!]^{PM'}_{\mathbf{pa'},i}(y) = [\![A_\beta]\!]^{PM'}_{\mathbf{pb'},i}$ where $\mathbf{pb'} = \mathbf{pa'}(x_\alpha/y)$. By the induction hypothesis, $[\![A_\beta]\!]^{PM}_{\mathbf{pa},i} \leq [\![A_\beta]\!]^{PM'}_{\mathbf{pa'},i}$ and of course, $\mathbf{pa}(x_\alpha/y)(x_\alpha) \leq \mathbf{pa'}(x_\alpha/y)(x_\alpha)$ for every $y \in PM_\alpha$. This means that $[\![\lambda x_\alpha A_\beta]\!]^{PM}_{\mathbf{pa},i} \leq [\![\lambda x_\alpha A_\beta]\!]^{PM'}_{\mathbf{pa'},i}$.

• Consider a term $[A_\alpha \equiv B_\alpha]$. By the induction hypothesis, $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},i} \leq [\![A_\alpha]\!]^{PM'}_{\mathbf{pa'},i}$ and $[\![B_\alpha]\!]^{PM}_{\mathbf{pa},i} \leq [\![B_\alpha]\!]^{PM'}_{\mathbf{pa'},i}$. Since we have just shown that identity is monotonic, this implies that $[\![[A_\alpha \equiv B_\alpha]]\!]^{PM}_{\mathbf{pa},i} \leq [\![[A_\alpha \equiv B_\alpha]]\!]^{PM'}_{\mathbf{pa'},i}$.

• At last, for a term $[A_t \wedge B_t]$, the reasoning is similar to the previous one.□

26. Définition Let PM = $<\{PM_\alpha\}_{\alpha\in T}, pm>$ be a standard partial model and let **pa** ∈ As(PM); *an extension of PM* is a standard partial model PM' = $<\{PM_\alpha\}_{\alpha\in T}, pm'>$ such that for every $\alpha \in T$ and every $c_\alpha \in Con_\alpha$, $pm(c_\alpha) \leq pm'(c_\alpha)$; *an extension of* ***pa*** is an assignment **pa'** ∈ As(PM) such that for every $\alpha \in T$ and every $x_\alpha \in Var_\alpha$, $\mathbf{pa}(x_\alpha) \leq \mathbf{pa'}(x_\alpha)$; *a maximal extension of PM* is an extension PM' = $<\{PM_\alpha\}_{\alpha\in T}, pm'>$ of PM such that for every $\alpha \in T$ and every $c_\alpha \in Con_\alpha$, $pm'(c_\alpha) = \bigvee X$ for an $X \in \Pi_{s\alpha}$; *a maximal extension of* ***pa*** is an extension **pa'** of **pa** such that for every $\alpha \in T$ and every $x_\alpha \in Var_\alpha$, $\mathbf{pa'}(x_\alpha) = \bigvee X$ for an $X \in \Pi_\alpha$. A standard partial model is *total* if it is a maximal extension of some standard partial model.

27. Proposition Let PM be a standard partial model and let **pa** ∈ As(PM); then: (i) PM has a maximal extension (not necessarily unique); (ii) **pa** has a maximal extension (not necessarily unique).

Proof (i) Let PM = $<\{PM_\alpha\}_{\alpha\in T}, pm>$ be a standard partial model. Obviously, for every $\alpha \in T$ and every $c_\alpha \in Con_\alpha$, either $pm(c_\alpha) \in Q_{s\alpha}$, or $pm(c_\alpha) \notin Q_{s\alpha}$; in this last case, proposition 12 assures that there is an $y \in Q_{s\alpha}$ such that $pm(c_\alpha) \leq y$. Let PM' = $<\{PM_\alpha\}_{\alpha\in T}, pm'>$ be any standard partial model such that for every $\alpha \in T$ and every $c_\alpha \in Con_\alpha$, $pm'(c_\alpha) = \bigvee\{x \in Q_{s\alpha} \mid x<>pm(c_\alpha)\}$ if $pm(c_\alpha) \in Q_{s\alpha}$; otherwise, $pm'(c_\alpha) = \bigvee\{x \in Q_{s\alpha} \mid x<>y\}$ for an $y \in Q_{s\alpha}$ such that $pm(c_\alpha) \leq y$. It is clear that PM' is an extension of PM which, moreover, is maximal. (ii) As in (i), considering **pa** ∈ As(PM) instead of pm.□

28. Remark Consider the terms defined in Section **B.2.** On the basis of the definition of identity between partial objects, it is easy to check that T denotes 1, F denotes 0 and that the definition of ¬ is equivalent to the definition of the negation according to KSL. Moreover, it is clear that the definition of ∧ is equivalent to the meaning postulate for the conjunction according to KSL; therefore it follows that the definitions of ∨ and ⊃ are respectively equivalent to the definitions of the disjonction and the conditional according to KSL. Notice that the biconditional is the sign ≡ restricted to Trm_t and it is easy to check that this sign so restricted corresponds to the biconditional according to KSL. The definition of the universal quantifier ∀ induces interesting truth conditions; for instance, one easily checks that:

$$[\![\forall x_e[A_{et}x_e]]\!]^{PM}_{\mathbf{pa},i} = \begin{cases} 1 \text{ if for every } x \in PM_e \text{ such that } x \neq \varphi,\ [\![A_{et}]\!]^{PM}_{\mathbf{pa},i}(x) = 1 \\ 0 \text{ if there is } x \in PM_e \text{ such that } [\![A_{et}]\!]^{PM}_{\mathbf{pa},i}(x) = 0 \\ \varphi \text{ otherwise.} \end{cases}$$

At first glance, this result may appear to be not consistent; indeed, one could imagine that A_{et} is such that $[\![A_{et}]\!]^{PM}_{pa,i}(\varphi) = 0$ and $[\![A_{et}]\!]^{PM}_{pa,i}(x) = 1$ for every $x \in PM_e$ such that $x \neq \varphi$. Clearly the result would be that $[\![\forall x_e[A_{et}x_e]]\!]^{PM}_{pa,i} = 1$ and 0. But the truth is that no predicate may have this behaviour, for this behaviour is not monotonic. Monotonicity implies that if there is an $x \in PM_e$ such that $[\![A_{et}]\!]^{PM}_{pa,i}(x) = 0$, then $[\![A_{et}]\!]^{PM}_{pa,i}(y) = 0$ for every $y \in PM_e$ such that $x \leq y$. One can check that the definition of the existential quantifier $\exists$ induces the following truth conditions:

$$[\![\exists x_e[A_{et}x_e]]\!]^{PM}_{pa,i} = \begin{cases} 1 \text{ if there is } x \in PM_e \text{ such that } [\![A_{et}]\!]^{PM}_{pa,i}(x) = 1 \\ 0 \text{ if for every } x \in PM_e \text{ such that } x \neq \varphi,\ [\![A_{et}]\!]^{PM}_{pa,i}(x) = 0 \\ \varphi \text{ otherwise.} \end{cases}$$

At last, one easily checks that the definitions of the modal operators induce the following truth conditions:

$$[\![\Box A_t]\!]^{PM}_{pa,i} = \begin{cases} 1 \text{ if for every } j \in I,\ [\![A_t]\!]^{PM}_{pa,j} = 1 \\ 0 \text{ if there is } j \in I \text{ such that } [\![A_t]\!]^{PM}_{pa,j} = 0 \\ \varphi \text{ otherwise.} \end{cases}$$

$$[\![\Diamond A_t]\!]^{PM}_{pa,i} = \begin{cases} 1 \text{ if there is } j \in I \text{ such that } [\![A_t]\!]^{PM}_{pa,j} = 1 \\ 0 \text{ if for every } j \in I,\ [\![A_t]\!]^{PM}_{pa,j} = 0 \\ \varphi \text{ otherwise.} \end{cases}$$

In Gallin's presentation of the language of **IL**, conjunction is introduced by the following definition:

Def.$\wedge$ $\qquad \wedge := \lambda x_t \lambda y_t[\lambda z_{tt}[[zx] \equiv y] \equiv \lambda z_{tt}[zT]]$

and the convention is to write $[A_t \wedge B_t]$ instead of $[[\wedge A_t]B_t]$. In the classical semantics, the truth conditions of $[A_t \wedge B_t]$ so defined are exactly the truth conditions given by the rule **B.3**-(viii). But in the partial semantics, the truth conditions of $[A_t \wedge B_t]$ introduced by Def.$\wedge$ do not coincide with the truth conditions given by the rule **C.4**-(viii). Indeed, we obtain:

$$[\![[A_t \wedge B_t]]\!]^{PM}_{pa,i} = \varphi \text{ if } [\![A_t]\!]^{PM}_{pa,i} = 0 \text{ and } [\![B_t]\!]^{PM}_{pa,i} = \varphi.$$

In all other cases however, the values of $[A_t \wedge B_t]$ agree with those given by the rule **C.4**-(viii). This means a non-commutative conjunction, which is of course undesirable.

C.5. The notions of entailment and validity in the partial sense.

Most of the so-called partial logics are weakened logics, lacking many fundamental laws of classical logic such the excluded middle. But a certain faintness come to us front such exclusion. What is troublesome is not the thesis that there are sentences which are neither true nor false (there are indeed many good reasons to believe that); the problem is that on this basis, one concludes that the law of excluded middle (this is just an example) is *not valid.* But this conclusion rests on a particular notion of validity: to be true in every model, or equivalently, to be true under every substitution of terms for non-logical constituents. In classical logic of course this notion is equivalent to the notion of to be not false in every model, or equivalently, to the notion of to be not false under every substitution of terms for non-logical constituents. This follows obviously from bivalence. In partial logics however, this equivalence doesn't hold anymore: the class of valid formulas according to the first notion of validity is generally smaller than the class of valid formulas according to the second. For instance, Kleene's strong three-valued propositional logic doesn't have valid formulas according to the first notion, because given any formula A, it is always possible to construct a model in which A is not true. But N. Rescher [1969] showed that for the same logic (and others like that one) the class of valid formulas according to the second notion is exactly the class of valid formulas of the classical logic. A more general result can be obtained. For the langage of propositional logic interpreted by the coherent partial situation semantics - whose meaning postutlates for the logical connectives are equivalent to those of the Kleene's strong three-valued logic - J. van Benthem [1986] defined a notion of entailment which turns out to be coextensive with the classical notion. This notion (called "weak consequence") superficially appears identical with the classical notion: a set Γ of formulas entails a formula A, if and only if there is no model in wich all fomulas in Γ are true and A is false. In the spirit of partiality however, that amounts to say that a deductively valid reasoning whose conclusion is a sentence B is a sequence $A_1, ..., A_n$, B of formulas such that, necessarily if $A_1, ..., A_n$ are *true*, then B is *not false*. According to us, this is the essential property of a valid reasoning, for, though it may appear to much weak, it is not controversial on the one hand (in the sense that nobody may seriously think of it as false), and on the other hand, it leads to a class of valid formulas which is identical with the class of valid formulas according to the classical notion. So let us apply this notion of entailment to our system.

Let A be a formula of the language of **IL**, PM be a standard partial model, **pa** $\in$ As(PM) and i $\in$ I. A is *satisfied in* PM *according to* **pa** and i, formally: $\models\!\equiv_{\mathbf{PM,pa,i}} A$, iff $[\![A]\!]^{PM}_{\mathbf{pa},i} = 1$. A is *not satisfied* (or is *unsatisfied) in* PM *according to* **pa** and i, formally: $\equiv\!|_{\mathbf{PM,pa,i}} A$, iff $[\![A]\!]^{PM}_{\mathbf{pa},i} = 0$. A *is not unsatified in* PM *according to* **pa** and i, formally: $\not\equiv\!|_{\mathbf{PM,pa,i}} A$, iff $[\![A]\!]^{PM}_{\mathbf{pa},i} = 1$ or φ. If Γ is a set of formulas of the language of **IL**, then Γ is *satisfied in* PM *according to* **pa** and i, formally: $|\!\equiv_{\mathbf{PM,pa,i}} \Gamma$, iff $|\!\equiv_{\mathbf{PM,pa,i}} A$ for every A $\in \Gamma$. If A is a formula of the language of **IL**, then A is *true or undefined in* M iff $\not\equiv\!|_{\mathbf{PM,pa,i}} A$ for every **pa** $\in$ As(M) and every i $\in$ I. A set Γ of formulas *entails* a formula A (in the partial sense), formally: $\Gamma \not\equiv\!| A$, iff for every standard partial model PM, every **pa** $\in$ As(PM) and every i $\in$ I, *if* $|\!\equiv_{\mathbf{PM,pa,i}} \Gamma$, then $\not\equiv\!|_{\mathbf{PM,pa,i}} A$. At last, a formula A is *valid* (in the partial sense), formally: $\not\equiv\!| A$, if and only if $\emptyset \not\equiv\!| A$, that is to say, iff A is true or undefined in every standard partial model.

The notion of entailment in the partial sense is equivalent to the notion of classical entailment (see proposition 33, Section **C.6**). This is due to the fact that partial models which are total can be identified with classical models. This of course presupposes that we can compare partial models with classical models.

C.6. Comparing partial and classical models.

29. Proposition Let PM be a standard partial model, **pa** ∈ As(PM), PM' be a maximal extension of PM and **pa'** ∈ As(PM) = As(PM') be a maximal extension of **pa**; then for every $\alpha \in T$, every $A_\alpha \in Trm_\alpha$ and every $i \in I$, $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},i} \leq [\![A_\alpha]\!]^{PM'}_{\mathbf{pa'},i}$.

Corrolary Let PM be a standard partial model, **pa** ∈ As(PM), PM' be a maximal extension of PM and **pa'** ∈ As(PM) = As(PM') be a maximal extension of **pa**; then for any formula A and for every i ∈ I: $|\equiv_{PM,\mathbf{pa},i} A$ implies $|\equiv_{PM',\mathbf{pa'},i} A$ and $\equiv|_{PM,\mathbf{pa},i} A$ implies $\equiv|_{PM',\mathbf{pa'},i} A$.

Proof By definition 26 and proposition 25.□

30. Definition Let $M = <\{M_\alpha\}_{\alpha\in T}, m>$ be a standard classical model and **a** ∈ As(M); *the partial replica of M* is the standard partial model $PR(M) = <\{PM_\alpha\}_{\alpha\in T}, pm>$ such that for every $\alpha \in T$ and every $c_\alpha \in Con_\alpha$, $pm(c_\alpha) = ma(m(c_\alpha))$; *the partial replica of* ***a*** is the assignment PR(**a**) = **pa** ∈ As(PR(M)) such that for every $\alpha \in T$ and every $x_\alpha \in Var_\alpha$, $\mathbf{pa}(x_\alpha) = ma(\mathbf{a}(x_\alpha))$. It is easily seen that the partial replica of a classical model is a partial model which is total.

31. Proposition Let M be a standard classical model, **a** ∈ As(M), PM = PR(M) and **pa** = PR(**a**); then for every $\alpha \in T$, every term A_α and every $i \in I$, $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},i} = ma([\![A_\alpha]\!]^{M}_{\mathbf{a},i})$.

Corrolary Let M be a standard classical model, **a** ∈ As(M), PM = PR(M) and **pa** = PR(**a**); then for any formula A and for every i ∈ I: $|\equiv_{PM,\mathbf{pa},i} A$ iff $\models_{M,\mathbf{a},i} A$ and $\equiv|_{PM,\mathbf{pa},i} A$ iff $\not\models_{M,\mathbf{a},i} A$.

Proof This is immediately verified for variables. For constants this is also straightforward: provided that $M = <\{M_\alpha\}_{\alpha\in T}, m>$, then $[\![c_\alpha]\!]^{PM}_{\mathbf{pa},i} = (ma(m(c_\alpha)))(i) = ma((m(c_\alpha))(i))$ (by proposition 21-(ii)) $= ma([\![c_\alpha]\!]^{M}_{\mathbf{a},i})$. Let us see the other cases.

• Consider a term $[A_{\alpha\beta}B_\alpha]$. By the induction hypothesis:

$$[\![A_{\alpha\beta}]\!]^{PM}_{\mathbf{pa},i} = ma([\![A_{\alpha\beta}]\!]^{M}_{\mathbf{a},i}) \text{ and } [\![B_\alpha]\!]^{PM}_{\mathbf{pa},i} = ma([\![B_\alpha]\!]^{M}_{\mathbf{a},i})$$

hence

$$[\![[A_{\alpha\beta}B_\alpha]]\!]^{PM}_{\mathbf{pa},i} = [\![A_{\alpha\beta}]\!]^{PM}_{\mathbf{pa},i}([\![B_\alpha]\!]^{PM}_{\mathbf{pa},i}) = (ma([\![A_{\alpha\beta}]\!]^{M}_{\mathbf{a},i}))(ma([\![B_\alpha]\!]^{M}_{\mathbf{a},i})).$$

But by proposition 21-(i),

$$(ma(\llbracket A_{\alpha\beta}\rrbracket^{M}_{\mathbf{a},i}))(ma(\llbracket B_{\alpha}\rrbracket^{M}_{\mathbf{a},i})) = ma(\llbracket A_{\alpha\beta}\rrbracket^{M}_{\mathbf{a},i}(\llbracket B_{\alpha}\rrbracket^{M}_{\mathbf{a},i}))$$

and by definition,

$$ma(\llbracket A_{\alpha\beta}\rrbracket^{M}_{\mathbf{a},i}(\llbracket B_{\alpha}\rrbracket^{M}_{\mathbf{a},i})) = ma(\llbracket [A_{\alpha\beta}B_{\alpha}]\rrbracket^{M}_{\mathbf{a},i}).$$

Therefore, $\llbracket [A_{\alpha\beta}B_{\alpha}]\rrbracket^{PM}_{\mathbf{pa},i} = ma(\llbracket [A_{\alpha\beta}B_{\alpha}]\rrbracket^{M}_{\mathbf{a},i})$.

• Consider a term $^\wedge A_{\alpha}$. By definition, for any $j \in I$:

$$\llbracket ^\wedge A_{\alpha}\rrbracket^{PM}_{\mathbf{pa},i}(j) = \llbracket A_{\alpha}\rrbracket^{PM}_{\mathbf{pa},j}$$

$$\llbracket ^\wedge A_{\alpha}\rrbracket^{M}_{\mathbf{a},i}(j) = \llbracket A_{\alpha}\rrbracket^{M}_{\mathbf{a},j},$$

and by the induction hypothesis:

$$\llbracket A_{\alpha}\rrbracket^{PM}_{\mathbf{pa},j} = ma(\llbracket A_{\alpha}\rrbracket^{M}_{\mathbf{a},j}).$$

So $\llbracket ^\wedge A_{\alpha}\rrbracket^{PM}_{\mathbf{pa},i}(j) = ma(\llbracket A_{\alpha}\rrbracket^{M}_{\mathbf{a},j}) = ma(\llbracket ^\wedge A_{\alpha}\rrbracket^{M}_{\mathbf{a},i}(j))$. But by proposition 21-(ii):

$$ma(\llbracket ^\wedge A_{\alpha}\rrbracket^{M}_{\mathbf{a},i}(j)) = (ma(\llbracket ^\wedge A_{\alpha}\rrbracket^{M}_{\mathbf{a},i}))(j).$$

Hence $\llbracket ^\wedge A_{\alpha}\rrbracket^{PM}_{\mathbf{pa},i}(j) = (ma(\llbracket ^\wedge A_{\alpha}\rrbracket^{M}_{\mathbf{a},i}))(j)$ and this implies $\llbracket ^\wedge A_{\alpha}\rrbracket^{PM}_{\mathbf{pa},i} = ma(\llbracket ^\wedge A_{\alpha}\rrbracket^{M}_{\mathbf{a},i})$.

• Consider a term $^\vee A_{s\alpha}$. By definition, for any $i \in I$:

$$\llbracket ^\vee A_{s\alpha}\rrbracket^{PM}_{\mathbf{pa},i} = \llbracket A_{s\alpha}\rrbracket^{PM}_{\mathbf{pa},i}(i)$$

$$\llbracket ^\vee A_{s\alpha}\rrbracket^{M}_{\mathbf{a},i} = \llbracket A_{s\alpha}\rrbracket^{M}_{\mathbf{a},i}(i),$$

and by the induction hypothesis, $\llbracket A_{s\alpha}\rrbracket^{PM}_{\mathbf{pa},i} = ma(\llbracket A_{s\alpha}\rrbracket^{M}_{\mathbf{a},i})$. Hence $\llbracket ^\vee A_{s\alpha}\rrbracket^{PM}_{\mathbf{pa},i} = (ma(\llbracket A_{s\alpha}\rrbracket^{M}_{\mathbf{a},i}))(i)$. But by proposition 21-(ii):

$$(ma(\llbracket A_{s\alpha}\rrbracket^{M}_{\mathbf{a},i}))(i) = ma(\llbracket A_{s\alpha}\rrbracket^{M}_{\mathbf{a},i}(i)).$$

Therefore, $\llbracket ^\vee A_{s\alpha}\rrbracket^{PM}_{\mathbf{pa},i} = ma(\llbracket A_{s\alpha}\rrbracket^{M}_{\mathbf{a},i}(i)) = ma(\llbracket ^\vee A_{s\alpha}\rrbracket^{M}_{\mathbf{a},i})$.

• Consider a term $\lambda x_{\alpha}A_{\beta}$. It is sufficient to show that for every $y \in M_{\alpha}$:

$$[\![\lambda x_\alpha A_\beta]\!]^{PM}_{\mathbf{pa},i}(ma(y)) = ma([\![\lambda x_\alpha A_\beta]\!]^{M}_{\mathbf{a},i}(y))$$

for by proposition 21-(i), $ma([\![\lambda x_\alpha A_\beta]\!]^{M}_{\mathbf{a},i}(y)) = (ma([\![\lambda x_\alpha A_\beta]\!]^{M}_{\mathbf{a},i}))(ma(y))$, and this shows effectively that $[\![\lambda x_\alpha A_\beta]\!]^{PM}_{\mathbf{pa},i} = ma([\![\lambda x_\alpha A_\beta]\!]^{M}_{\mathbf{a},i})$.

First, by definition, it is the case that for every $y \in M_\alpha$:

$$[\![\lambda x_\alpha A_\beta]\!]^{PM}_{\mathbf{pa},i}(ma(y)) = [\![A_\beta]\!]^{PM}_{\mathbf{pa'},i} \text{ where } \mathbf{pa'} = \mathbf{pa}(x_\alpha/ma(y))$$

$$[\![\lambda x_\alpha A_\beta]\!]^{M}_{\mathbf{a},i}(y) = [\![A_\beta]\!]^{M}_{\mathbf{a'},i} \text{ where } \mathbf{a'} = \mathbf{a}(x_\alpha/y).$$

Secondly, by the induction hypothesis, $[\![A_\beta]\!]^{PM}_{\mathbf{pa},i} = ma([\![A_\beta]\!]^{M}_{\mathbf{a},i})$ and it is obviously the case that $\mathbf{pa}(x_\alpha/ma(y))(x_\alpha) = ma(\mathbf{a}(x_\alpha/y)(x_\alpha))$ for every $y \in M_\alpha$. Therefore, for every $y \in M_\alpha$, $\mathbf{a'} = \mathbf{a}(x_\alpha/y)$ and $\mathbf{pa'} = \mathbf{pa}(x_\alpha/ma(y))$, $[\![\lambda x_\alpha A_\beta]\!]^{PM}_{\mathbf{pa},i}(ma(y)) = [\![A_\beta]\!]^{PM}_{\mathbf{pa'},i} = ma([\![A_\beta]\!]^{M}_{\mathbf{a'},i}) = ma([\![\lambda x_\alpha A_\beta]\!]^{M}_{\mathbf{a},i}(y))$.

• Consider a term $[A_\alpha \equiv B_\alpha]$. By the induction hypothesis, $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},i} = ma([\![A_\alpha]\!]^{M}_{\mathbf{a},i})$ and $[\![B_\alpha]\!]^{PM}_{\mathbf{pa},i} = ma([\![B_\alpha]\!]^{M}_{\mathbf{a},i})$. So $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},i}$, $[\![B_\alpha]\!]^{PM}_{\mathbf{pa},i} \in Q_\alpha$ and by proposition 15, $\neg([\![A_\alpha]\!]^{PM}_{\mathbf{pa},i} <> [\![B_\alpha]\!]^{PM}_{\mathbf{pa},i})$ iff $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},i} \neq_S [\![B_\alpha]\!]^{PM}_{\mathbf{pa},i}$. Hence we have:

$$[\![[A_\alpha \equiv B_\alpha]]\!]^{PM}_{\mathbf{pa},i} = \begin{cases} 1 \text{ if } [\![A_\alpha]\!]^{PM}_{\mathbf{pa},i} <> [\![B_\alpha]\!]^{PM}_{\mathbf{pa},i} \\ 0 \text{ otherwise} \end{cases}$$

and so obviously we have:

$$[\![[A_\alpha \equiv B_\alpha]]\!]^{PM}_{\mathbf{pa},i} = [\![[A_\alpha \equiv B_\alpha]]\!]^{M}_{\mathbf{a},i},$$

and a fortiori we have $[\![[A_\alpha \equiv B_\alpha]]\!]^{PM}_{\mathbf{pa},i} = ma([\![[A_\alpha \equiv B_\alpha]]\!]^{M}_{\mathbf{a},i})$.

• For a term $[A_t \wedge B_t]$, the reasoning is similar to the previous one.□

32. Proposition Let PM be a standard partial model, **pa** ∈ As(PM), PM' be a maximal extension of PM and **pa'** ∈ As(PM) = As(PM') be a maximal extension of **pa**; then there exists a standard classical model M and an assignment **a** ∈ As(M) such that PR(M) = PM' et PR(**a**) = **pa'**. As corrolary, to every partial model that is total corresponds one classical model.

Proof Let PM' = $<\{PM_\alpha\}_{\alpha\in T}, pm'>$ be a maximal extension of the model PM = $<\{PM_\alpha\}_{\alpha\in T}, pm>$. By definition, for every $\alpha \in T$ and every $c_\alpha \in Con_\alpha$, $pm(c_\alpha) \leq pm'(c_\alpha) = \bigvee X$ for an $X \in \Pi_{s\alpha}$. Let $M = <\{M_\alpha\}_{\alpha\in T}, m>$ be a standard classical model such that for every $\alpha \in T$ and every $c_\alpha \in Con_\alpha$, $m(c_\alpha) = \Psi(\{x \in Q_{s\alpha} \mid x<>pm'(c_\alpha)\})$. Obviously, M exists. Moreover, for every $\alpha \in T$ and every $c_\alpha \in Con_\alpha$, $pm'(c_\alpha) = \bigvee(\{x \in Q_{s\alpha} \mid x<>pm'(c_\alpha)\}) = ma(\Psi(\{x \in Q_{s\alpha} \mid x<>pm'(c_\alpha)\})) = ma(m(c_\alpha))$. Therefore PR(M) = PM'. On the other hand, let **a** $\in$ As(M) be an assignment such that for every $\alpha \in T$ and every $x_\alpha \in Var_\alpha$, $\mathbf{a}(x_\alpha) = \Psi(\{x \in Q_\alpha \mid x<>\mathbf{pa'}(x_\alpha)\})$. Again, $ma(\mathbf{a}(x_\alpha)) = \mathbf{pa'}(x_\alpha)$ for every $\alpha \in T$ and every $x_\alpha \in Var_\alpha$. Therefore PR(**a**) = **pa'**.□

33. Proposition Let Γ be a set of formulas and A be a formula of the language of **IL**; then $\Gamma \not\equiv\!| A$ iff $\Gamma \models A$.

Proof First suppose that $\Gamma \not\equiv\!| A$ but not $\Gamma \models A$. This means that there exists a standard classical model M, an assignment **a** $\in$ As(M) and an $i \in I$ such that $\models_{M,\mathbf{a},i} \Gamma$ and $\not\models_{M,\mathbf{a},i} A$. Let PM = PR(M) and **pa** = PR(**a**). So by the corrolary of proposition 31, $|\!\equiv_{PM,\mathbf{pa},i} \Gamma$ and $\equiv\!|_{PM,\mathbf{pa},i} A$. This clearly contradicts the assumption that $\Gamma \not\equiv\!| A$. Therefore $\Gamma \models A$. On the other hand, suppose that $\Gamma \models A$ but not $\Gamma \not\equiv\!| A$. This means that there exists a standard partial model PM, an assignment **pa** $\in$ As(PM) and an $i \in I$ such that $|\!\equiv_{PM,\mathbf{pa},i} \Gamma$ and $\equiv\!|_{PM,\mathbf{pa},i} A$. By proposition 27 and the corrolary of proposition 29, there exists a maximal extension PM' of PM and a maximal extension **pa'** of **pa** sucht that $|\!\equiv_{PM',\mathbf{pa'},i} \Gamma$ and $\equiv\!|_{PM',\mathbf{pa'},i} A$. But by proposition 32, there exists a standard classical model M and an assignment **a** $\in$ As(M) such that PM' = PR(M) and **pa'** = PR(**a**). Therefore, by the corrolary of proposition 31, $\models_{M,\mathbf{a},i} \Gamma$ and $\not\models_{M,\mathbf{a},i} A$. This clearly contradicts the assumption that $\Gamma \models A$. Therefore $\Gamma \not\equiv\!| A$.□

34. Remark Consider the deductive system **IL** described by D. Gallin [1975], chap. 1, § 3. Let **IL+Cnj** be the system **IL** plus the following axiom schema :

Cjn. $[[A_t \wedge B_t] \equiv [\lambda z_{tt}[[zA] \equiv B] \equiv \lambda z_{tt}[zT]]]$, if z is not free in A and B.

Of course, this schema is redundant if conjunction is introduced by Def.$\wedge$. However, **IL+Cnj** is deductively equivalent to the system **IL** and in our version of the language of **IL**, conjunction is not introduced by Def.$\wedge$. It is then easily seen that **Cjn** is not redundant given our version of the language of **IL**.

Given the fact that the system **IL** is sound in the standard and classical sense (every theorem of **IL** is valid in the standard classical semantics), the last proposition allows us to claim that **IL+Cnj** is sound in the standard and partial sense (one verifies easily that the rules of inference preserve validity in the partial sense). Moreover, we know that restricted to a certain class Σ of formulas (the class of *persistent formulas*), the system **IL**

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