Institute for Language, Logic and Information

THE MODAL LOGIC OF INEQUALITY

Maarten de Rijke

ITLI Prepublication Series
for Logic, Semantics and Philosophy of Language LP-90-15

University of Amsterdam

The ITLI Prepublication Series

1986

86-01 The Institute of Language, Logic and Information
86-02 Peter van Emde Boas A Semantical Model for Integration and Modularization of Rules
86-03 Johan van Benthem Categorial Grammar and Lambda Calculus
86-04 Reinhard Muskens A Relational Formulation of the Theory of Types
86-05 Kenneth A. Bowen, Dick de Jongh Some Complete Logics for Branched Time, Part I Well-founded Time, Forward looking Operators
86-06 Johan van Benthem Logical Syntax

1987

87-01 Jeroen Groenendijk, Martin Stokhof Type shifting Rules and the Semantics of Interrogatives
87-02 Renate Bartsch Frame Representations and Discourse Representations
87-03 Jan Willem Klop, Roel de Vrijer Unique Normal Forms for Lambda Calculus with Surjective Pairing
87-04 Johan van Benthem Polyadic quantifiers
87-05 Víctor Sánchez Valencia Traditional Logicians and de Morgan's Example
87-06 Eleonore Oversteegen Temporal Adverbials in the Two Track Theory of Time
87-07 Johan van Benthem Categorial Grammar and Type Theory
87-08 Renate Bartsch The Construction of Properties under Perspectives
87-09 Herman Hendriks Type Change in Semantics: The Scope of Quantification and Coordination

1988

Logic, Semantics and Philosophy of Language:

LP-88-01 Michiel van Lambalgen Algorithmic Information Theory
LP-88-02 Yde Venema Expressiveness and Completeness of an Interval Tense Logic
LP-88-03 Year Report 1987
LP-88-04 Reinhard Muskens Going partial in Montague Grammar
LP-88-05 Johan van Benthem Logical Constants across Varying Types
LP-88-06 Johan van Benthem Semantic Parallels in Natural Language and Computation
LP-88-07 Renate Bartsch Tenses, Aspects, and their Scopes in Discourse
LP-88-08 Jeroen Groenendijk, Martin Stokhof Context and Information in Dynamic Semantics
LP-88-09 Theo M.V. Janssen A mathematical model for the CAT framework of Eurotra
LP-88-10 Anneke Kleppe A Blissymbolics Translation Program

Mathematical Logic and Foundations:

ML-88-01 Jaap van Oosten Lifschitz' Realizabiility
ML-88-02 M.D.G. Swaen The Arithmetical Fragment of Martin Löf's Type Theories with weak Σ-elimination
ML-88-03 Dick de Jongh, Frank Veltman Provability Logics for Relative Interpretability
ML-88-04 A.S. Troelstra On the Early History of Intuitionistic Logic
ML-88-05 A.S. Troelstra Remarks on Intuitionism and the Philosophy of Mathematics

Computation and Complexity Theory:

CT-88-01 Ming Li, Paul M.B.Vitanyi Two Decades of Applied Kolmogorov Complexity
CT-88-02 Michiel H.M. Smid General Lower Bounds for the Partitioning of Range Trees
CT-88-03 Michiel H.M. Smid, Mark H. Overmars, Leen Torenvliet, Peter van Emde Boas Maintaining Multiple Representations of Dynamic Data Structures
CT-88-04 Dick de Jongh, Lex Hendriks, Gerard R. Renardel de Lavalette Computations in Fragments of Intuitionistic Propositional Logic
CT-88-05 Peter van Emde Boas Machine Models and Simulations (revised version)
CT-88-06 Michiel H.M. Smid A Data Structure for the Union-find Problem having good Single-Operation Complexity
CT-88-07 Johan van Benthem Time, Logic and Computation
CT-88-08 Michiel H.M. Smid, Mark H. Overmars, Leen Torenvliet, Peter van Emde Boas Multiple Representations of Dynamic Data Structures
CT-88-09 Theo M.V. Janssen Towards a Universal Parsing Algorithm for Functional Grammar
CT-88-10 Edith Spaan, Leen Torenvliet, Peter van Emde Boas Nondeterminism, Fairness and a Fundamental Analogy
CT-88-11 Sieger van Denneheuvel, Peter van Emde Boas Towards implementing RL

Other prepublications:

X-88-01 Marc Jumelet On Solovay's Completeness Theorem

1989

Logic, Semantics and Philosophy of Language:

LP-89-01 Johan van Benthem The Fine-Structure of Categorial Semantics
LP-89-02 Jeroen Groenendijk, Martin Stokhof Dynamic Predicate Logic, towards a compositional, non-representational semantics of discourse
LP-89-03 Yde Venema Two-dimensional Modal Logics for Relation Algebras and Temporal Logic of Intervals
LP-89-04 Johan van Benthem Language in Action
LP-89-05 Johan van Benthem Modal Logic as a Theory of Information
LP-89-06 Andreja Prijatelj Intensional Lambek Calculi: Theory and Application
LP-89-07 Heinrich Wansing The Adequacy Problem for Sequential Propositional Logic
LP-89-08 Víctor Sánchez Valencia Peirce's Propositional Logic: From Algebra to Graphs
LP-89-09 Zhisheng Huang Dependency of Belief in Distributed Systems

Mathematical Logic and Foundations:

ML-89-01 Dick de Jongh, Albert Visser Explicit Fixed Points for Interpretability Logic
ML-89-02 Roel de Vrijer Extending the Lambda Calculus with Surjective Pairing is conservative
ML-89-03 Dick de Jongh, Franco Montagna Rosser Orderings and Free Variables
ML-89-04 Dick de Jongh, Marc Jumelet, Franco Montagna On the Proof of Solovay's Theorem
ML-89-05 Rineke Verbrugge Σ-completeness and Bounded Arithmetic
ML-89-06 Michiel van Lambalgen The Axiomatization of Randomness
ML-89-07 Dirk Roorda Elementary Inductive Definitions in HA: from Strictly Positive towards Monotone
ML-89-08 Dirk Roorda Investigations into Classical Linear Logic
ML-89-09 Alessandra Carbone Provable Fixed points in $I\Delta_0+\Omega_1$

Computation and Complexity Theory:

CT-89-01 Michiel H.M. Smid Dynamic Deferred Data Structures
CT-89-02 Peter van Emde Boas Machine Models and Simulations
CT-89-03 Ming Li, Herman Neuféglise, Leen Torenvliet, Peter van Emde Boas On Space Efficient Simulations
CT-89-04 Harry Buhrman, Leen Torenvliet A Comparison of Reductions on Nondeterministic Space
CT-89-05 Pieter H. Hartel, Michiel H.M. Smid, Leen Torenvliet, Willem G. Vree A Parallel Functional Implementation of Range Queries
CT-89-06 H.W. Lenstra, Jr. Finding Isomorphisms between Finite Fields
CT-89-07 Ming Li, Paul M.B. Vitanyi A Theory of Learning Simple Concepts under Simple Distributions and Average Case Complexity for the Universal Distribution (Prel. Version)
CT-89-08 Harry Buhrman, Steven Homer, Leen Torenvliet Honest Reductions, Completeness and Nondeterminstic Complexity Classes
CT-89-09 Harry Buhrman, Edith Spaan, Leen Torenvliet On Adaptive Resource Bounded Computations
CT-89-10 Sieger van Denneheuvel The Rule Language RL/1
CT-89-11 Zhisheng Huang, Sieger van Denneheuvel, Peter van Emde Boas Towards Functional Classification of Recursive Query Processing

Other Prepublications:

X-89-01 Marianne Kalsbeek An Orey Sentence for Predicative Arithmetic
X-89-02 G. Wagemakers New Foundations: a Survey of Quine's Set Theory
X-89-03 A.S. Troelstra Index of the Heyting Nachlass
X-89-04 Jeroen Groenendijk, Martin Stokhof Dynamic Montague Grammar, a first sketch
X-89-05 Maarten de Rijke The Modal Theory of Inequality
X-89-06 Peter van Emde Boas Een Relationele Semantiek voor Conceptueel Modelleren: Het RL-project

1990 SEE INSIDE BACK COVER

Instituut voor Taal, Logica en Informatie
Institute for Language, Logic and Information

Faculteit der Wiskunde en Informatica
(Department of Mathematics and Computer Science)
Plantage Muidergracht 24
1018TV Amsterdam

Faculteit der Wijsbegeerte
(Department of Philosophy)
Nieuwe Doelenstraat 15
1012CP Amsterdam

THE MODAL LOGIC OF INEQUALITY

Maarten de Rijke
Department of Mathematics and Computer Science
University of Amsterdam

ITLI Prepublications
for Logic, Semantics and Philosophy of Language
ISSN 0924-2082

Received November 1990

Research upported by the Netherlands Organization for Scientific Research (NWO).
This paper is a revised version of X-89-05

The Modal Logic of Inequality

Maarten de Rijke*

Abstract

We consider some modal languages with a modal operator D whose semantics is based on the relation of inequality. Basic logical properties such as definability, expressive power and completeness are studied. Also, some connections with a number of other recent proposals to extend the standard modal language are pointed at.

1 Introduction

As is well-known standard (propositional) modal and temporal logic cannot define all the natural assumptions one would like to make on the accessibility relation. One obvious move to try and overcome this lack of expressive power, is to extend the languages of modal and temporal logic with new operators. One particular such extension consists in adding an operator D whose semantics is based on the relation of inequality. The proposal to consider the D-operator is due to several people independently, including Koymans [15], and Gargov, Passy and Tinchev [10]. This particular extension of the standard modal language is of interest for a number of reasons. First of all, it shows that some of the most striking deficiencies in expressive power may be removed with relatively simple means. Secondly, several recent proposals to enhance the expressive power of the standard language naturally give rise to considering the D-operator; thus the language with the operators $\Diamond$ and D appears as a kind of fixed point amongst the wide range of recently introduced extensions of the standard language. And thirdly, many of the interesting logical phenomena that one encounters in the study of enriched modal languages are illustrated by this particular extension.

Applications of the D-operator can be found in [9], where it has been used in the study of various enriched modal languages, and in [15], where it is applied in the specification of message passing and time-critical systems.

The main subject of this paper is the modal language $\mathcal{L}(\Diamond, D)$ whose operators are $\Diamond$ and D. The remainder of §1 introduces the basic notions, and examines which of the (anti-) preservation results known from standard modal logic remain valid in the extended formalism. Next, §2 compares the expressive powers of modal languages that contain the D-operator with a number of other modal languages. In §3 we present the basic logics in some languages with the D-operator, and we give complete axiomatizations for several special structures; in it we also prove an analogue of the Sahlqvist Theorem for $\mathcal{L}(\Diamond, D)$. §4 then deals with definability—both of classes of frames and of classes of models.

*Research supported by the Netherlands Organization for Scientific Research (NWO). This paper is based on my Master's Thesis written at the University of Amsterdam, Department of Philosophy, under the supervision of Johan van Benthem; I am grateful to him for his questions and suggestions.

1.1 Basics

The (multi-) modal languages we consider have an infinite supply of proposition letters (p, q, r, ...), propositional constants $\bot$, $\top$ and the usual Boolean connectives. Furthermore, they contain one or more unary modal operators. The standard language $\mathcal{L}(\Diamond)$ has operators $\Diamond$ and $\Box$—$\Diamond$ being regarded as primitive, and $\Box$ being defined as $\neg\Diamond\neg$. (In general, $\mathcal{L}(O_1, \ldots, O_n)$ denotes the (multi-) modal language with operators $O_1, \ldots, O_n$.) We use φ, ψ, $\chi, \ldots$ to denote (multi-) modal formulas. The semantic structures are *frames*, i.e. ordered pairs $\langle W, R\rangle$ consisting of a non-empty set W with a binary relation R on W. To save words, we assume that $\mathcal{F}$ denotes the frame $\langle W, R\rangle$. In addition to these frames, structures called *models* will be used, consisting of a frame $\mathcal{F}$ together with a *valuation V on $\mathcal{F}$* assigning subsets of W to proposition letters. We assume that $\mathcal{M}$ denotes the model $\langle \mathcal{F}, V\rangle$.

$\mathcal{M} \models \varphi[w]$ is defined as usual—the important case being: $\mathcal{M} \models \Diamond\varphi[w]$ iff for some $v \in W$, Rwv and $\mathcal{M} \models \varphi[v]$. For temporal logic the clause for $\Diamond$ is replaced by two clauses for F and P: $\mathcal{M} \models F\varphi[w]$ iff for some $v \in W$, Rwv and $\mathcal{M} \models \varphi[v]$; $\mathcal{M} \models P\varphi[w]$ iff for some $v \in W$, Rvw and $\mathcal{M} \models \varphi[w]$. The semantics of the D-operator is given by $\mathcal{M} \models D\varphi[w]$ iff for some $v \neq w$, $\mathcal{M} \models \varphi[v]$. From this, notions like $\mathcal{M} \models \varphi$, $\mathcal{F} \models \varphi[w]$, and $\mathcal{F} \models \varphi$ are defined as usual.

G and H are short for $\neg F \neg$ and $\neg P \neg$, respectively. Ds dual $\neg D \neg$ is denoted by $\overline{D}$. Using the D-operator some useful abbreviations can be defined: $E\varphi := \varphi \vee D\varphi$ (there exists a point at which φ holds); $A\varphi := \varphi \wedge \overline{D}\varphi$ (φ holds at all points); $U\varphi := E(\varphi \wedge \neg D\varphi)$ (φ holds at a unique point).

The fact that some notions are sensitive to the language we are working with, is reflected in our notation: e.g. we write $\mathcal{F} \equiv_{\Diamond,D} \mathcal{G}$ for $\mathcal{F}$ and $\mathcal{G}$ validate the same $\varphi \in \mathcal{L}(\Diamond, D)$, and $\mathrm{Th}_{\Diamond,D}(\mathcal{F})$ for the set of formulas in $\mathcal{L}(\Diamond, D)$ that are valid on $\mathcal{F}$.

We will sometimes refer to the first-order languages $\mathcal{L}_0$ and $\mathcal{L}_1$: $\mathcal{L}_0$ has one binary predicate symbol R as well identity; $\mathcal{L}_1$ extends $\mathcal{L}_0$ with unary predicate symbols $P_1, P_2, \ldots, P$, $Q, \ldots$ corresponding to the proposition letters of the (multi-) modal language. First-order formulas will be denoted by α, β, $\gamma, \ldots$. α is called *locally definable* in $\mathcal{L}(O_1, \ldots, O_n)$ if for some $\varphi \in \mathcal{L}(O_1, \ldots, O_n)$, for all $\mathcal{F}$, and all $w \in W$, $\mathcal{F} \models \alpha[w]$ iff $\mathcal{F} \models \varphi[w]$; it is called *(globally) definable* in $\mathcal{L}(O_1, \ldots, O_n)$ if for some $\varphi \in \mathcal{L}(O_1, \ldots, O_n)$, for all $\mathcal{F}$, $\mathcal{F} \models \alpha$ iff $\mathcal{F} \models \varphi$.

1.2 (Anti-) preservation and filtrations

It is well-known that standard modal formulas are preserved under surjective p-morphisms, disjoint unions and generated subframes:

Definition 1.1 1. A surjective function f from a frame $\mathcal{F}_1$ to a frame $\mathcal{F}_2$ is called a *p-morphism* if (i) for all w, $v \in W_1$, if R_1wv then $R_2f(w)f(v)$; and (ii) for all $w \in W_1$, $v \in W_2$, if $R_2f(w)v$ then there is a $u \in W_1$ such that R_1wu and $f(u) = v$.

2. $\mathcal{F}_1$ is called a *generated subframe* of a frame $\mathcal{F}_2$ if (i) $W_1 \subset W_2$; (ii) $R_1 = R_2 \cap (W_2 \times W_2)$; and (iii) for all $w \in W_1$, $v \in W_2$, if R_2wv then $v \in W_1$.

3. Let $\mathcal{F}_i$ ($i \in I$) be a collection of disjoint frames. Then the *disjoint union* $\biguplus_{i\in I} \mathcal{F}_i$ is the frame $\langle \cup\{ W_i : i \in I \}, \cup\{ R_i : i \in I \}\rangle$.

Here are some examples showing that adding the D-operator to $\mathcal{L}(\Diamond)$ gives an increase in expressive power:

1. $\Diamond p \to Dp$ defines $\forall x\, \neg Rxx$;

2. $Dp \to \Diamond p$ defines $R = W^2$;

3. $\Diamond\top \vee D\Diamond\top$ defines $R \neq \emptyset$.

Using the above preservation results it is easily verified that none of these three conditions is definable in $\mathcal{L}(\Diamond)$. And conversely, the fact that they are definable in $\mathcal{L}(\Diamond, D)$ implies that we no longer have these preservation results in $\mathcal{L}(\Diamond, D)$. Moreover, they can be restored only at the cost of trivializing the constructions concerned.

A fourth important construction in standard modal logic is the following:

Definition 1.2 Let $\mathcal{F}$ be a frame, and $X \subseteq W$. Then $M_R(X) = \{ w \in W : \forall v \in W\, (Rwv \to v \in X) \}$. The *ultrafilter extension* $ue(\mathcal{F})$ is the frame $\langle W_{\mathcal{F}}, R_{\mathcal{F}} \rangle$, where $W_{\mathcal{F}}$ is the set of ultrafilters on W, and $R_{\mathcal{F}} U_1 U_2$ holds if for all $X \subseteq W$, $\mathcal{M}_R(X) \in U_1$ implies $X \in U_2$.

Standard modal formulas are *anti-preserved* under ultrafilter extension, i.e. if $ue(\mathcal{F}) \models \varphi$ then $\mathcal{F} \models \varphi$. (Cf. [4, Lemma 2.25].) Perhaps surprisingly, for formulas $\varphi \in \mathcal{L}(\Diamond, D)$ this results still holds good—as one easily deduces from the following result.

Proposition 1.3 *Let V be a valuation on $\mathcal{F}$. Define the valuation $V_{\mathcal{F}}$ on $ue(\mathcal{F})$ by putting $V_{\mathcal{F}}(p) = \{ U : V(p) \in U \}$. Then, for all ultrafilters U on W, and all formulas $\varphi \in \mathcal{L}(\Diamond, D)$ we have $\langle ue(\mathcal{F}), V_{\mathcal{F}} \rangle \models \varphi[U]$ iff $V(\varphi) \in U$.*

Proof. This is by induction on φ. The cases $\varphi \equiv p$, $\neg\psi$, $\psi \wedge \chi$, $\Diamond\psi$ are proved in [4, Lemma 2.25]. The only new case is $\varphi \equiv D\psi$. Suppose $V(D\psi) = \{ w : \exists v \neq w\, (v \in V(\psi)) \} \in U$. We must find an ultrafilter $U_1 \neq U$ such that $\langle ue(\mathcal{F}), V_{\mathcal{F}} \rangle \models \psi[U_1]$. First assume that U contains a singleton—say, $U = \{ X \subseteq W : X \supseteq \{ w_0 \} \}$. Then $w_0 \in V(D\psi)$, so there exists a $v \neq w_0$ with $v \in V(\psi)$. Since $v \neq w_0$, we must have $\{ v \} \notin U$. Let U_1 be the ultrafilter generated by $\{ v \}$; then $U \neq U_1$. Furthermore, $v \in V(\psi)$ implies $V(\psi) \in U_1$, and hence $\langle ue(\mathcal{F}), V_{\mathcal{F}} \rangle \models \psi[U_1]$, by the induction hypothesis. It follows that $\langle ue(\mathcal{F}), V_{\mathcal{F}} \rangle \models D\psi[U]$. Next, suppose that U does not contain a singleton. Since $V(D\psi) \in U$, we find some $w_0 \in V(D\psi)$. Let v be a point such that $v \neq w_0$ and $v \in V(\psi)$. Then $\{ v \} \notin U$—and we can proceed as in the previous case.

Conversely, assume that $V(D\psi) \notin U$. We have to show that $\langle ue(\mathcal{F}), V_{\mathcal{F}} \rangle \not\models D\psi[U]$. Since $V(D\psi) \notin U$, we have that $X = \{ w : \forall v (v \neq w \to v \notin V(\psi)) \} \in U$, and hence $X \neq \emptyset$. Let $w_0 \in X$. Clearly, if $w_0 \notin V(\psi)$, then $X = W$ and $V(\psi) = \emptyset$. Consequently, for all ultrafilters $U_1 \neq U$ we have $V(\psi) \notin U_1$. So, by the induction hypothesis, $\langle ue(\mathcal{F}), V_{\mathcal{F}} \rangle \not\models \psi[U_1]$, and hence $\langle ue(\mathcal{F}), V_{\mathcal{F}} \rangle \not\models D\psi[U]$—as required. If, on the other hand $w_0 \in V(\psi)$, then $X = \{ w_0 \} = V(\psi)$, and U is generated by X. It follows that for any ultrafilter $U_1 \neq U$, $X = V(\psi) \notin U_1$. So by the induction hypothesis $\langle ue(\mathcal{F}), V_{\mathcal{F}} \rangle \not\models \psi[U_1]$, for such U_1. This implies $\langle ue(\mathcal{F}), V_{\mathcal{F}} \rangle \not\models D\psi[U]$. QED.

Corollary 1.4 *For any frame $\mathcal{F}$ and all $\varphi \in \mathcal{L}(\Diamond, D)$, if $ue(\mathcal{F}) \models \varphi$ then $\mathcal{F} \models \varphi$.*

Corollary 1.5 *$\exists x\, Rxx$ is not definable in $\mathcal{L}(\Diamond, D)$.*

Proof. Evidently, $\mathcal{F} = \langle \mathbb{N}, < \rangle \not\models \exists x\, Rxx$. Some elementary reasoning shows that for any principal ultrafilter U on $\mathbb{N}$, $R_{\mathcal{F}} UU$. Hence, $ue(\mathcal{F}) \models \exists x\, Rxx$. Now apply 1.4. QED.

Another important notion from standard modal logic is that of a *filtration*. It has a straightforward adaptation to $\mathcal{L}(\Diamond, D)$:

Definition 1.6 Let $\mathcal{M}_1$, $\mathcal{M}_2$ be models, and let Σ be a set of formulas $\varphi \in \mathcal{L}(\Diamond, D)$ closed under subformulas. A surjective function $g : \mathcal{M}_1 \to \mathcal{M}_2$ is an *extended filtration with respect to* Σ, if

1. for all $w, v \in W_1$, if R_1wv then $R_2g(w)g(v)$,
2. for all $w \in W_1$, and all proposition letters p in Σ, $w \in V_1(p)$ iff $g(w) \in V_2(p)$,
3. for all $w \in W_1$, and all $\triangle\varphi \in \Sigma$, if $\mathcal{M}_1 \models \triangle\varphi[w]$ then $\mathcal{M}_2 \models \triangle\varphi[g(w)]$, where $\triangle \in \{\Diamond, D\}$.

Proposition 1.7 *If g is an extended filtration w.r.t. Σ from $\mathcal{M}_1$ to $\mathcal{M}_2$, then for all $w \in W_1$, and all $\varphi \in \Sigma$, $\mathcal{M}_1 \models \varphi[w]$ iff $\mathcal{M}_2 \models \varphi[g(w)]$.*

Recall that the standard example of a filtration in ordinary modal logic is the *modal collapse*: given a model $\mathcal{M}$ and a set Σ that is closed under subformulas, it is defined as the model $\mathcal{M}'$, where for $g(w) = \{\varphi \in \Sigma : \mathcal{M} \models \varphi[w]\}$, $W' = g[W]$, $R'g(w)g(v)$ holds iff for all $\Box\varphi \in \Sigma$, $\Box\varphi \in g(w)$ implies $\varphi \in g(v)$, and $V'(p) = \{g(w) : p \in g(w)\}$. To obtain an analogue of the modal collapse for $\mathcal{L}(\Diamond, D)$, take the ordinary modal collapse and double points that correspond to more than one point in the original model. A simple inductive proof then shows that corresponding (doubled) points verify the same formulas.

Using the extended collapse one may show in a standard way that formulas $\varphi \in \mathcal{L}(\Diamond, D)$ satisfy the finite model property, and that the validities in $\mathcal{L}(\Diamond, D)$ form a recursive set.

2 Some comparisons

In this section we compare modal languages with the D-operator to some languages without it. It is not our aim to give a complete description of all the aspects in which languages of the former kind differ from, or are the same as, languages of the latter kind, but merely to highlight some of the features of the former languages.

2.1 The language $\mathcal{L}(D)$

Proposition 2.1 *All formulas $\varphi \in \mathcal{L}(D)$ define first-order conditions.*

Proof. Using the ST-translation as defined in §4.2, such formulas can be translated into equivalent second-order formulas containing only monadic predicate variables. By a result in [1, Chapter IV] these formulas are in turn equivalent to first-order ones. QED.

Proposition 2.1 marks a considerable difference with $\mathcal{L}(\Diamond)$: as is well-known, not all $\mathcal{L}(\Diamond)$-formulas correspond to first-order conditions. In the opposite direction, there are also some natural conditions undefinable in $\mathcal{L}(\Diamond)$ that are definable in $\mathcal{L}(D)$. For example, using the preservation of standard modal formulas under generated subframes and disjoint unions, it is easily verified that no finite cardinality is definable in $\mathcal{L}(\Diamond)$; on the other hand, although 2.1 implies that 'infinity' is not definable in $\mathcal{L}(D)$, we do have

Proposition 2.2 (Koymans) *All finite cardinalities are definable in $\mathcal{L}(D)$.*

Proof. For $n \in \mathbb{N}$, $|W| \leq n$ is defined by $\bigwedge_{1\leq i\leq n+1} Up_i \longrightarrow \bigvee_{1\leq i<j\leq n+1} E(p_i \wedge p_j)$, while $|W| > n$ is defined by $A(\bigvee_{1\leq i\leq n} p_i) \longrightarrow E\bigvee_{1\leq i\leq n}(p_i \wedge Dp_i)$. QED.

Theorem 2.3 (Functional Completeness) *On frames $\mathcal{L}(D)$ is equivalent with the language of first-order logic over $=$.*

Proof. All first-order formulas over identity can be defined as a Boolean combination of formulas expressing the existence of at least a certain number of elements. By 2.2 these formulas are definable in $\mathcal{L}(D)$. The converse follows from 2.1. QED.

2.2 The languages $\mathcal{L}(\Diamond, D)$ and $\mathcal{L}(\Diamond)$

One way to compare the expressive powers of two languages is to examine their ability to discriminate between special (read: well-known) structures. For example, in contrast to $\mathcal{L}(\Diamond)$, $\mathcal{L}(\Diamond, D)$ is able to distinguish $\mathbb{Z}$ from $\mathbb{N}$: $\langle \mathbb{N}, < \rangle \not\equiv_{\Diamond,D} \langle \mathbb{Z}, < \rangle$. This follows from the fact that the existence of a (different) predecessor is expressible in $\mathcal{L}(\Diamond, D)$ by means of the formula $p \to D\Diamond p$.

So $\forall x \exists y\, (x \neq y \wedge Ryx)$ is an $\mathcal{L}_0$-condition on frames which is definable in $\mathcal{L}(\Diamond, D)$, but not in $\mathcal{L}(\Diamond)$. Other well-known $\mathcal{L}_0$-conditions undefinable in $\mathcal{L}(\Diamond)$ are irreflexivity and anti-symmetry. By the next result, these conditions do have an $\mathcal{L}(\Diamond, D)$-equivalent:

Proposition 2.4 *All Π^1_1-sentences in R, $=$ of the purely universal form*

$$\forall P_1 \ldots \forall P_m \forall x_1 \ldots \forall x_n\, BOOL(P_i x_j, Rx_i x_j, x_i = x_j)$$

are definable in $\mathcal{L}(\Diamond, D)$.

Proof. Let $p_1, \ldots, p_m$, $q_1, \ldots, q_n$ be proposition letters such that each of $p_1, \ldots, p_m$ is different from each of $q_1, \ldots, q_n$. Now take $Uq_1 \wedge \ldots \wedge Uq_n \to BOOL(E(q_i \wedge p_j), E(q_i \wedge \Diamond q_j), E(q_i \wedge q_j))$. QED.

It is well-known that two finite, rooted frames that validate the same formulas $\varphi \in \mathcal{L}(\Diamond)$, are isomorphic. This is improved upon in $\mathcal{L}(\Diamond, D)$:

Corollary 2.5 *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are finite frames, then $\mathcal{F}_1 \equiv_{\Diamond,D} \mathcal{F}_2$ iff $\mathcal{F}_1 \cong \mathcal{F}_2$.*

Proof. Finite frames are isomorphic iff they have the same universal first-order theory. So from 2.4 the result follows. Alternatively, one may give, for each finite frame $\mathcal{F}$, a 'characteristic formula' $\chi_{\mathcal{F}}$ such that for all $\mathcal{G}$, $\mathcal{G} \not\models \neg\chi_{\mathcal{F}}$ iff $G \cong \mathcal{F}$ (cf. §4.1). QED.

Let us call a set T of (multi-) modal formulas *(frame) categorical* if, up to isomorphism, there is only one frame validating T; T is called *λ-categorical* if, up to isomorphism, T has only one frame of power λ validating it. (λ-) categoricity is an important notion in first-order logic that is meaningless in standard modal languages: by some elementary manipulations one easily establishes that if $\mathcal{F} \models T$, where T is a theory in either $\mathcal{L}(\Diamond)$ or $\mathcal{L}(F, P)$, and if I is a set of indices, then for each $i \in I$ there is a frame $\mathcal{F}_i \models T$ such that $\mathcal{F}_i \not\cong F_j$ if $i \neq j$. In contrast, for any *finite* frame $\mathcal{F}$ the complete $\Diamond, D$-theory $\mathrm{Th}_{\Diamond,D}(\mathcal{F})$ is easily seen to be categorical by 2.4.

The classical example of an ω-categorical theory in first-order logic is the complete theory of the rationals. By standard techniques one can show that $\mathrm{Th}_{\Diamond}(\mathbb{Q})$ is not ω-categorical; but $\mathrm{Th}_{\Diamond,D}(\mathbb{Q})$ *is* ω-categorical:

Proposition 2.6 *The complete $\Diamond, D$-theory of $\mathbb{Q}$ is ω-categorical.*

Proof. It suffices to give formulas $\varphi \in \mathcal{L}(\Diamond, D)$ which are equivalent to the axioms for the theory of dense linear order without endpoints:

$\forall xyz\,(x < y \wedge y < z \rightarrow x < z)$	$\Box\Box p \rightarrow \Box p$
$\forall xyz\,(x < y \wedge y < x \rightarrow x = y)$	$Up \wedge Uq \rightarrow E(p \wedge q)$
$\forall x\,\neg(x < x)$	$\Diamond p \rightarrow Dp$
$\forall xy\,(x = y \vee x < y \vee y < x)$	$Up \wedge Uq \rightarrow E(p \wedge q) \vee E(p \wedge \Diamond q) \vee E(q \wedge \Diamond p)$
$\forall xy \exists z\,(x < y \rightarrow x < z \wedge z < y)$	$\Box p \rightarrow \Box\Box p$
$\exists xy\,(x \neq y)$	$D\top$
$\forall x \exists y\,(x < y)$	$\Diamond\top$
$\forall x \exists y\,(y < x)$	$Up \rightarrow D\Diamond p$. QED.

Recall that a *modal sequent* is a pair $\sigma = \langle \Gamma_0, \Theta_0 \rangle$ where Γ_0 and Θ_0 are finite sets of (multi-) modal formulas; $\mathcal{F} \models \sigma$ if for every V, if $\langle \mathcal{F}, V \rangle \models \Gamma_0$ then there is a $\theta \in \Theta_0$ with $\langle \mathcal{F}, V \rangle \models \theta$. A class K of frames is *sequentially definable* if there is a set L of modal sequents such that $K = \{\, \mathcal{F} : \forall \sigma \in L\,(\mathcal{F} \models \sigma) \,\}$. Kapron [14] shows that in $\mathcal{L}(\Diamond)$ sequential definability is strictly stronger than ordinary definability. By our remarks in §1.2 and the fact that validity of sequents is preserved under p-morphisms (cf. [14]), it follows that definability in $\mathcal{L}(\Diamond, D)$ is still stronger. Furthermore, in $\mathcal{L}(\Diamond, D)$ the notions of ordinary and sequential definability coincide; as is pointed out in [13] this is due to the fact that we can define the 'universal modality' A in $\mathcal{L}(\Diamond, D)$:

Proposition 2.7 *Let K be a class of frames. K is sequentially definable in $\mathcal{L}(\Diamond, D)$ iff it is definable in $\mathcal{L}(\Diamond, D)$.*

Proof. One direction is clear. To prove the other one, assume that K is defined by a set L of sequents. For each $\sigma = \langle \{\, \varphi_0, \ldots, \varphi_n \,\}, \{\, \psi_0, \ldots, \psi_m \,\} \rangle \in L$ put $\sigma^* := \bigwedge_{0 \leq i \leq n} A\varphi_i \rightarrow \bigvee_{0 \leq i \leq m} A\psi_i$. Then K is defined by $\{\sigma^* : \sigma \in L\,\}$. QED.

It should be clear by now that adding the D-operator to $\mathcal{L}(\Diamond)$ greatly increases the expressive power. Limitations are easily found, however. As we have seen, $\exists x\, Rxx$ is still not definable in $\mathcal{L}(\Diamond, D)$. And just as with the standard modal language we find that on well-orders a sort of 'stabilization of discriminatory power' occurs at a relatively early stage (cf. [5] for a proof of this result for the standard modal language). To prove this we recall that the *clusters* of a transitive frame $\mathcal{F}$ are the equivalence classes of W under the relation $x \sim y$ iff $(Rxy \wedge Ryx) \vee x = y$. Clusters are divided into three kinds: proper, with at least two elements, all reflexive; simple, with one reflexive element; and degenerate, with one irreflexive element.

Theorem 2.8 *If $\varphi \in \mathcal{L}(\Diamond, D)$, and $\mathcal{F}$ is a well-ordered frame with $\mathcal{F} \not\models \varphi$, then there is a well-ordered frame $\mathcal{G}$ such that $\mathcal{G} < \omega^2$ and $\mathcal{G} \not\models \varphi$.*

Proof. Suppose that for some V, $w \in W$, $\mathcal{M} = \langle \mathcal{F}, V \rangle \models \neg\varphi[w]$. Let Σ^- be the set of subformulas of $\neg\varphi$, and define $\Sigma := \Sigma^- \cup \{\, \Diamond\psi : D\psi \in \Sigma^- \,\}$. Let $\mathcal{M}_1$ be the (extended) collapse of $\mathcal{M}$ w.r.t. Σ. Then $\mathcal{M}_1$ is transitive and linear. Consequently, $\mathcal{M}_1$ may be viewed as a finite linear sequence of clusters.

Next, $\mathcal{M}_1$ will be blown up into a well-ordered model $\mathcal{M}_2$ by replacing each cluster with an appropriate well-order. If $C = \{\, w \,\}$ is a degenerate cluster, then C is itself a well-order, and we do nothing. Non-degenerated clusters $\{\, w_1, \ldots, w_k \,\}$ are replaced with a copy of ω; the valuation is adapted by verifying p in a newly added n iff $n = i \bmod k$ and $w_i \in V_1(p)$. The resulting model is a well-order, and since $\mathcal{M}_1$ is finite it will have order type $< \omega^2$.

If $w \in W_1$, we write $\overline{w}$ for (a) point(s) corresponding to w in $\mathcal{M}_2$. Then, for all $\psi \in \Sigma$, and $w \in W_1$, $\mathcal{M}_1 \models \psi[w]$ iff $\mathcal{M}_2 \models \psi[\overline{w}]$. This equivalence is proved by induction on ψ. The only non-trivial case is when $\psi \equiv D\chi$, and $\mathcal{M}_2 \models D\chi[\overline{w}]$. In that case one uses the fact that $D\chi \in \Sigma$ implies $\Diamond\chi \in \Sigma$. QED.

From 2.8 and [5, Theorem 5.2] it follows that the well-orders of type $< \omega \cdot k + n$ ($k \leq \omega$, $n < \omega$) all have distinct $\Diamond, D$-theories, while for $k \geq \omega$, $\omega \cdot k + n \equiv_{\Diamond,D} \omega \cdot \omega + n$.

2.3 The languages $\mathcal{L}(\Diamond, D)$ and $\mathcal{L}(F, P)$

On strict linear orders the D-operator becomes definable in $\mathcal{L}(F, P)$: on such frames we have $\mathcal{F} \models (P\varphi \vee F\varphi) \leftrightarrow D\varphi$. In fact, this may be generalized somewhat; call a frame $\mathcal{F}$ *n-connected* ($n > 0$) if for any w, $v \in W$ with $w \neq v$, there exists a sequence $w_1, \ldots, w_k$ such that $w_1 = w$, $w_k = v$ and for each j ($1 \leq j < k$) either Rw_jw_{j+1} or $Rw_{j+1}w_j$. Then, using a suitable translation, one may show that on irreflexive, n-connected frames every $\Diamond, D$-formula is equivalent to one in $\mathcal{L}(F, P)$. This shows that new results about standard modal languages may be obtained by studying extended ones: for it follows from 2.4 that on the class of irreflexive, n-connected frames every purely universal Π^1_1-sentence in $R, =$ is definable in $\mathcal{L}(F, P)$.

By the next result there is no converse to our previous remarks: P is not definable in $\mathcal{L}(\Diamond, D)$—not even on strict linear orders.

Theorem 2.9 *1.* $\langle \mathbb{Q}, < \rangle \not\equiv_{F,P} \langle \mathbb{R}, < \rangle$,
2. $\langle \mathbb{Q}, < \rangle \equiv_{\Diamond,D} \langle \mathbb{R}, < \rangle$.

Proof. The first part is well-known. To prove the second part, assume first that for some $\varphi \in \mathcal{L}(\Diamond, D)$ and valuation V, $\langle \mathbb{R}, <, V \rangle \not\models \varphi$. Using the ST-translation as defined in §4.2 we find that $\langle \mathbb{R}, <, V \rangle \models \exists x\, ST(\neg\varphi)$. Hence by the Löwenheim-Skolem Theorem, $\langle \mathbb{Q}, <, V' \rangle \models \exists x\, ST(\neg\varphi)$, where $V'(p) = V(p) \upharpoonright \mathbb{Q}$, for all proposition letters p. It follows that $\langle \mathbb{Q}, < \rangle \not\models \varphi$.

Conversely, assume that for some $\varphi \in \mathcal{L}(\Diamond, D)$ and a valuation V, $\langle \mathbb{Q}, <, V \rangle \not\models \varphi$. Define Σ and $\mathcal{M}_1$ as in the proof of 2.8. Then $\mathcal{M}_1$ is transitive, linear and successive—both to the right and to the left. A model $\mathcal{M}_2$ may then be constructed by replacing each cluster with an ordering of type λ if it is the left-most cluster, and otherwise, if it is degenerated it and its *non-degenerated* successor (by [18, Lemma 1.1] $\mathcal{M}_1$ does not contain adjacent degenerated clusters) are replaced in one go with an ordering of type $1 + \lambda$; after that, the remaining non-degenerated clusters are also replaced by $1 + \lambda$. The valuation may then be extended to newly added points in such a way that an induction similar to the one in the proof of 2.8 yields $\mathcal{M}_2 \not\models \varphi$. QED.

2.4 The language $\mathcal{L}(\Diamond, D)$ and some other enriched languages

In [6] a simple method of incorporating reference into modal logic is presented by introducing a new sort of atomic symbols—*nominals*—to the modal language. These new symbols combine with other symbols of the language in the usual way to form formulas. Their only non-standard feature is that they are true at exactly *one* point in a model. Let $\mathcal{L}_n(\Diamond)$ denote the language $\mathcal{L}(\Diamond)$ with nominals added to it. From [6] we know that $\mathcal{L}_n(\Diamond)$ is much more expressive than $\mathcal{L}(\Diamond)$: important classes of frames undefinable in $\mathcal{L}(\Diamond)$ become definable in $\mathcal{L}_n(\Diamond)$. But it turns out that $\mathcal{L}(\Diamond, D)$ is even more expressive than $\mathcal{L}_n(\Diamond)$. To see this, let $n_0, n_1, n_2, \ldots$ range over nominals; let $p_0, p_1, p_2, \ldots$ denote the proposition letters in $\mathcal{L}_n(\Diamond)$ and $\mathcal{L}(\Diamond, D)$, and define $\tau : \mathcal{L}_n(\Diamond) \to \mathcal{L}(\Diamond, D)$ by putting $\tau(p_i) = p_{2i}$ and $\tau(n_i) = p_{2i+1}$, and by letting τ commute with the connectives and operators. Given a formula $\varphi \in \mathcal{L}_n(\Diamond)$, let $n_1, \ldots, n_k$ be the nominals occurring in φ, and define $(\varphi)^* \in \mathcal{L}(\Diamond, D)$ to be $U\tau(n_1) \wedge \ldots \wedge U\tau(n_k) \to \tau(\varphi)$.

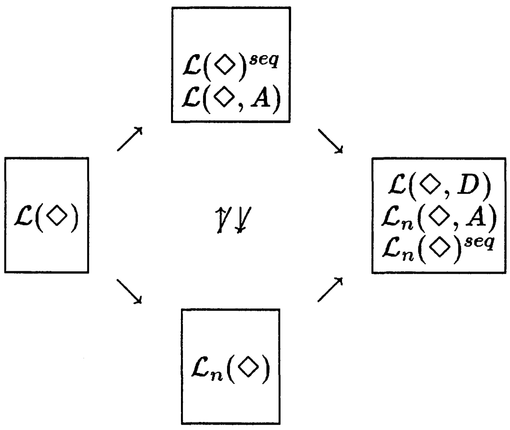
Proposition 2.10 *Every class of frames that is definable in $\mathcal{L}_n(\Diamond)$ is definable in $\mathcal{L}(\Diamond, D)$, but not conversely.*

Proof. The first part follows from the observation that for any formula $\varphi \in \mathcal{L}_n(\Diamond)$, and any model $\langle W, R, V\rangle$, $\langle W, R, V\rangle \models \varphi[w]$ iff $\langle W, R, V^*\rangle \models \varphi^*[w]$, where $V^*(p) = V(\tau^{-1}(p))$. The second part follows from 2.2 and the fact that 1 is the only cardinality definable in $\mathcal{L}_n(\Diamond)$ (cf. [6]). QED.

In [6] and [9] the extension $\mathcal{L}_n(\Diamond, A)$ of $\mathcal{L}_n(\Diamond)$ is studied—here A is the operator defined in §1.1, whose semantics is given by $\mathcal{M} \models A\varphi[w]$ iff for all $v \in W$, $\mathcal{M} \models \varphi[v]$; it is sometimes called the *shifter* (in [6]), or the *universal modality* (in [9]). By the above observations $\mathcal{L}_n(\Diamond, A)$ is no more expressive than $\mathcal{L}(\Diamond, D)$. Moreover, by a nice result in [9] the converse holds as well:

Theorem 2.11 (Gargov and Goranko) *A class of frames is definable in $\mathcal{L}_n(\Diamond, A)$ iff it is definable in $\mathcal{L}(\Diamond, D)$.*

Combining results from this section and earlier ones together with results from [9] and [13], we arrive at the following picture:



(Here, $\mathcal{L}(\Diamond)^{seq}$ is $\mathcal{L}(\Diamond)$ with sequential definability; each box contains languages that are equivalent w.r.t. definability of frames, and arrows point to more expressive languages.)

3 Axiomatics

Starting from the basic logic K in $\mathcal{L}(\Diamond)$ some obvious questions concerning its extensions in $\mathcal{L}(\Diamond, D)$ may be asked. The following such questions will be considered in this section:

What is the basic logic in $\mathcal{L}(\Diamond, D)$? What are the logics (in $\mathcal{L}(\Diamond, D)$) of structures like $\mathbb{N}$ or $\mathbb{Z}$? Is there a general completeness theorem in $\mathcal{L}(\Diamond, D)$ for a wide class of extensions of the basic logic—like the Sahlqvist Theorem for $\mathcal{L}(\Diamond)$? And, given a logic $K + \varphi$ in $\mathcal{L}(\Diamond)$ with property P, does its minimal extension in $\mathcal{L}(\Diamond, D)$ have P?

3.1 The basic logic

Definition 3.1 DL^- is propositional logic plus the following schemata:

(A1) $\overline{D}(p \to q) \to (\overline{D}p \to \overline{D}q)$,
(A2) $p \to \overline{D}Dp$ (symmetry),
(A3) $DDp \to (p \vee Dp)$ (pseudo-transitivity).

As rules of inference it has Modus Ponens, Substitution, and a 'Necessitation Rule' for $\overline{D}$: $\vdash \varphi \Rightarrow \vdash \overline{D}\varphi$.

Theorem 3.2 (Koymans) *Let* $\Sigma \cup \{\varphi\} \subseteq \mathcal{L}(D)$. *Then* $\Sigma \vdash_{DL^-} \varphi$ *iff* $\Sigma \models \varphi$.

Proof. Soundness is immediate. To prove completeness, assume $\Sigma \not\vdash_{DL^-} \varphi$, and let $\Delta \supseteq \Sigma \cup \{\neg\varphi\}$ be a maximal DL^--consistent set. Consider $W_\Delta := \{\Gamma : \exists n\, (R_D)^n \Delta\Gamma\}$, where Γ ranges over maximal DL^--consistent sets and R_D is the canonical relation defined by: $R_D\Gamma_1\Gamma_2$ iff for all $\overline{D}\psi \in \Gamma_1$, $\psi \in \Gamma_2$. Then $\forall xy\, (R_Dxy \to R_Dyx)$ and $\forall xyz\, (R_Dzy \wedge R_Dyz \to R_Dxz \vee x = z)$. If there are any R_D-reflexive points, let c be such a point; replace it with *two* points c_1, c_2, and adapt R_D by putting $R_Dc_1c_2$, and conversely, and by putting R_Dc_iw (R_Dwc_i) if R_Dcw (R_Dwc) ($i = 1, 2$). In the resulting structure R_D is real inequality, and φ is refuted somewhere. QED.

Hence, one may be inclined to think that DL^- is the basic logic in $\mathcal{L}(D)$—just like K is the basic logic in $\mathcal{L}(\Diamond)$. However, DL^- is, so to speak, not as stable as K: in $\mathcal{L}(\Diamond)$ incompleteness phenomena occur only with more exotic extensions of K (cf. [2]); in contrast, here's a very simple incomplete extension of DL^-:

Example 3.3 Consider the system $DL^- + (\varphi \to D\varphi)$. Then $DL^- + (\varphi \to D\varphi) \models \bot$, since no frame validates $DL^- + (\varphi \to D\varphi)$. On the other hand, $DL^- + (\varphi \to D\varphi) \not\vdash \bot$. To see this, recall that a *general frame* is a triple $\mathfrak{F} = \langle W, R, \mathcal{W}\rangle$, where $\mathcal{W} \subseteq P(W)$ contains $\emptyset$, and is closed under the Boolean operations as well as the operator M_R (cf. 1.2); valuations on a general frame should take their values inside $\mathcal{W}$. Now, let $\mathfrak{F} = \langle W, R, \mathcal{W}\rangle$, where $\mathcal{F} = \langle\{0,1\}, \emptyset\rangle$ and $\mathcal{W} = \{\emptyset, \{0,1\}\}$. Then $\mathfrak{F} \models DL^- + (\varphi \to D\varphi)$. Therefore, $DL^- + (\varphi \to D\varphi)$ is incomplete.

By the Sahlqvist Theorem for $\mathcal{L}(\Diamond)$ (cf. [17]) $K + \Box^m(\varphi \to \psi)$ is complete for any m and for φ, ψ that satisfy certain requirements. Any obvious adaptation of this result to $\mathcal{L}(D)$ would imply that $DL^- + (\varphi \to D\varphi)$ is complete—hence, by the above example there is no such adaptation. To avoid incompleteness phenomena as those sketched above, we follow some suggestions by Yde Venema and Valentin Goranko, and add the following rule of inference to DL^-:

(IR) if for all proposition letters p not occurring in φ, $\vdash p \wedge \overline{D}\neg p \to \varphi$ then $\vdash \varphi$.

Let DL denote DL^- plus the rule IR. Notice that, given the Substitution Rule, IR is in fact equivalent to a finitary rule: if for *some* proposition p letter not occurring in φ, $\vdash p \wedge \overline{D}\neg p \to \varphi$ then $\vdash \varphi$. Our next aim is to prove that in terms of general consequence, DL has no effects over DL^-. To this end it suffices to show that DL precisely axiomatizes the basic logic in $\mathcal{L}(D)$. In doing so we will closely follow the proof of the completeness of the basic logic in $\mathcal{L}_n(\Diamond)$ as presented in [10].

Let $L \supseteq DL$ be a logic. A set of formulas Δ is *L-closed* if it contains all theorems of L and is closed under MP and IR; $cl_L(\Delta)$ (or $cl(\Delta)$ when L is clear) denotes the smallest L-closed set containing Δ.

Theorem 3.4 *Let* $\Sigma \cup \{\varphi\} \subseteq \mathcal{L}(D)$. *Then* $\Sigma \vdash_{DL} \varphi$ *iff* $\Sigma \models \varphi$.

Proof. Soundness is immediate. To prove completeness, assume that $\Sigma \not\vdash_{DL} \varphi$. We construct, for each consistent set Δ a maximal DL-consistent set $\Delta' \supseteq \Delta$ such that Δ' is R_D-irreflexive. Let $\{\varphi_i\}_{i\in\omega}$ enumerate all D-formulas. Put $\Delta_0 = cl(\Delta)$. Assume that the consistent closed set Δ_n has been defined. If $\Delta_n \cup \{\varphi_n\}$ is consistent, put $\Delta_{n+1} := cl(\Delta_n \cup \{\varphi_n\})$. Otherwise, the rule IR yields a proposition letter p not occurring in φ_n such that $\Delta_n \cup \{\neg(p\wedge\overline{D}\neg p \to \varphi_n)\}$ is consistent; put $\Delta_{n+1} = cl(\Delta_n \cup \{\neg(p\wedge\overline{D}\neg p \to \varphi_n)\})$. Finally, put $\Delta' = \bigcup_n \Delta_n$. Since $\Delta' \not\vdash \bot$, IR yields a proposition letter p with $p \wedge \overline{D}\neg p \in \Delta'$—hence Δ' is R_D-irreflexive.

Now, let Σ' be an R_D-irreflexive maximal DL-consistent set extending $\Sigma \cup \{\neg\varphi\}$. Put $W = \{\Gamma : \exists n\,(R_D)^n\Sigma'\Gamma\}$, where Γ ranges over R_D-irreflexive maximal DL-consistent sets. Then, on W, R_D is real inequality. Define $V(p) = \{\Gamma \in W : p \in \Gamma\}$. Then $\langle W, \neq, V\rangle \models \neg\varphi[\Sigma]$. QED.

It follows from 3.2 and 3.4 that the rule IR is superfluous in the basic logic. However, it does yield new consequences in extensions of DL: $DL + (\varphi \to D\varphi)$ is inconsistent, and thus complete. (To see that it's inconsistent, notice first that for any proposition letter p, $DL+(\varphi \to D\varphi) \vdash (p\wedge\overline{D}\neg p \to \bot)$, hence by the rule IR, $DL+(\varphi \to D\varphi) \vdash \bot$.) A further justification for adding IR to DL^- may be found in §3.3, where a Sahlqvist Theorem for the basic logic in $\mathcal{L}(\Diamond, D)$ (which contains IR) is proved.

Definition 3.5 The basic logic DL_m in $\mathcal{L}(\Diamond, D)$ is $DL + K + (\Diamond p \to p \vee Dp)$; its rules of inference are those of DL plus those of K. The basic logic DL_t in $\mathcal{L}(F, P, D)$ is $DL + K_t + (Fp \to p \vee Dp) + (Pp \to p \vee Dp)$; its rules of inference are those of DL plus those of K_t.

Theorem 3.6 *1. Let* $\Sigma \cup \{\varphi\} \subseteq \mathcal{L}(\Diamond, D)$. *Then* $\Sigma \vdash_{DL_m} \varphi$ *iff* $\Sigma \models \varphi$.
2. Let $\Sigma \cup \{\varphi\} \subseteq \mathcal{L}(F, P, D)$. *Then* $\Sigma \vdash_{DL_t} \varphi$ *iff* $\Sigma \models \varphi$.

Proof. Similar to the proof of 3.4. Notice that by the additional axiom $\Diamond p \to p \vee Dp$ any set W of maximal DL_m-consistent sets that is closed under R_D, is also closed under the canonical relation $R_\Diamond$, defined by $R_\Diamond\Gamma_1\Gamma_2$ iff for all $\Box\psi \in \Gamma_1$, $\psi \in \Gamma_2$. Analogous remarks hold for DL_t and the canonical relations R_F and R_P. QED.

3.2 Some extensions of DL_m

We present axioms in $\mathcal{L}(\Diamond, D)$ for some familiar classes of frames; we also axiomatize the $\Diamond, D$-theories of some special structures. For a start, here's a list of axioms together with the corresponding conditions on frames:

$(A4)$	$\Diamond\Diamond p \to \Diamond p$	transitivity
$(A5)$	$p \to \Diamond p$	reflexivity
$(A6)$	$p \to \overline{D}p \vee \Box(\Diamond p \to p)$	anti-symmetry
$(A7)$	$\Diamond p \to Dp$	irreflexivity
$(A8)$	$p \to \Diamond q \vee \overline{D}(q \to \Diamond p)$	linearity
$(A9)$	$\Diamond\top$	successiveness to the right
$(A10)$	$p \to D\Diamond p$	successiveness to the left
$(A11)$	$\Box(\Box p \to p) \to (\Diamond\Box p \to p)$	discreteness
$(A12)$	$\Diamond p \to \Diamond\Diamond p$	denseness.

Theorem 3.7 *1.* $DL_m + A4\text{–}A6$ *is complete w.r.t. partial orders.*

2. $DL_m + A4 + A7$ *is complete w.r.t strict partial orders.*

3. $DL_m + A4 + A8$ *is complete w.r.t. linear orders.*

4. $DL_m + A4 + A7 + A8$ *is complete w.r.t. strict linear orders.*

Proof. Assume that $\Sigma \not\vdash \varphi$ in $DL_m + A4\text{–}A6$. As in the proof of 3.4 we can use an appropriate notion of a closed set to construct a canonical model consisting of R_D-irreflexive maximal consistent sets. We then take a submodel $\mathcal{M}$ of the canonical model, which is R_D-generated by some Δ extending $\Sigma \cup \{\neg\varphi\}$. Using the characteristic axioms it's a routine matter to check that $\mathcal{M}$ is a partial order. Cases 2, 3, 4 of the theorem may be proved in a similar way. QED.

Theorem 3.8 *1.* $DL_m + A4 + A7\text{–}A9 + A11$ *axiomatizes* $\mathrm{Th}_{\Diamond,D}(\mathbb{N})$.

2. $DL_m + A4 + A7\text{–}A11$ *axiomatizes* $\mathrm{Th}_{\Diamond,D}(\mathbb{Z})$.

3. $DL_m + A4 + A7\text{–}A9 + A12$ *axiomatizes* $\mathrm{Th}_{\Diamond,D}(\mathbb{Q})$ $(= \mathrm{Th}_{\Diamond,D}(\mathbb{R})$ *by 2.9).*

Proof. To prove 1, 2 and 3, start by constructing an R_D-generated submodel of the canonical model as in the proof of 3.4. In the case of 3 the resulting structure will be isomorphic to $\langle \mathbb{Q}, < \rangle$. In the case of 1 or 2 one may apply an appropriate version of the techniques of [18] to turn the model into a model based on $\mathbb{N}$ or $\mathbb{Z}$. QED.

What about decidability of the above logics? Using extended filtrations (cf. 1.7) one easily establishes that both $DL_m + A4 - A6$ and $DL_m + A4 + A8$ have the finite *frame* property (f.f.p.); from this their decidability follows in a standard way.

As for $DL_m + A4 + A7$, notice that it does not have the f.f.p.: any frame $\mathcal{F}$ with $\mathcal{F} \models DL_m + A4 + A7$ and $\mathcal{F} \not\models \neg\Box\Diamond\top$ must be infinite. However, $DL_m + A4 + A7$ does have the finite model property (f.m.p.)—thus showing that Segerberg's Theorem (which says that the f.f.p. and the f.m.p. are equivalent in $\mathcal{L}(\Diamond)$) fails in $\mathcal{L}(\Diamond, D)$. In fact, $DL_m + A4 + A7$ may be shown to be complete w.r.t. the class of finite models $\mathcal{M} = \langle \mathcal{F}, V \rangle$ which satisfy $\mathcal{F} \models DL_m + A4$, and for any $\varphi \in \mathcal{L}(\Diamond, D)$, if $\{w : Rww\} \cap V(\varphi) \neq \emptyset$ then $|V(\varphi)| \geq 2$. Soundness is immediate. The easy proof of the completeness is to lengthy to be included here, so we only mention some steps in it. By 3.7 there is a model $\mathcal{M} = \langle \mathcal{F}, V \rangle$ with $\mathcal{F} \models DL_m + A4 + A7$ and $\mathcal{M} \not\models \varphi[w]$, for some $w \in \mathcal{M}$. Let $\Sigma \ni \neg\varphi$ be some finite set of formulas that is closed under subformulas, and that satisfies $\Diamond\psi \in \Sigma \Rightarrow D\psi \in \Sigma$. We define a non-standard model $\mathcal{M}'$ as follows; let g, W', R', V' be as in our remarks following 1.7; define R_D by $R_D g(v) g(u)$ iff for all $\overline{D}\psi \in g(v)$, $\psi \in g(u)$. Then, using R_D as the interpretation of D, $\mathcal{M}' \not\models \varphi$, and moreover, R' is transitive, $R' \subseteq R_D$, R_D holds between any two different points, and $\mathcal{M}'$ is finite. Next, one may use the 'doubling-points' technique of 3.2 to obtain a model $\mathcal{M}'' \not\models \varphi$ in which R_D is real inequality, and which satisfies all our requirements.

Using the fact that $DL_m + A4 + A7$ has the f.m.p. one may establish the decidability of this logic. The decidability of $DL_m + A4 + A7 + A8$ and of $\mathrm{Th}_{\Diamond,D}$ may be proved in a similar fashion. To obtain decidability results for $\mathrm{Th}_{\Diamond,D}(\mathbb{N})$ and $\mathrm{Th}_{\Diamond,D}(\mathbb{Z})$ one may apply Rabin-Gabbay techniques (cf. [6, Chapter 5] for a similar move in $\mathcal{L}_n(F,P)$).

3.3 A Sahlqvist Theorem for $\mathcal{L}(\Diamond, D)$

We start with some preliminary remarks. The canonical general frame $\langle W_L, R_L, \mathcal{W}_L\rangle$ of a logic L in $\mathcal{L}(\Diamond)$ is defined as follows: W_L is the set of all maximal L-consistent sets, $R_L\Gamma\Delta$ holds if for all $\Box\varphi \in \Gamma$, $\varphi \in \Delta$, and

$$\mathcal{W}_L = \{\, X \subseteq W_L : \exists\varphi \in \mathcal{L}(\Diamond)\forall\Delta \in W_L\,(\varphi \in \Delta \leftrightarrow \Delta \in X)\,\}.$$

A canonical general frame for DL^- has as its domain a set W_L of points that correspond (possibly not uniquely) to maximal DL^--consistent sets; on W_L the canonical relation R_D holds between any two different points, and only between those (cf. the proof of 3.2); $\mathcal{W}_{DL^-}$ is defined like $\mathcal{W}_L$. The canonical general frame for a logic $L \supseteq DL_m$ in $\mathcal{L}(\Diamond, D)$ has as its domain a set of (R_D-irreflexive) maximal L-consistent sets that is R_D-generated by a single set (as in the proof of 3.6); R_L and $\mathcal{W}_L$ are defined as usual.

Next we introduce some notation. For the remainder of this section we use T $(T_0, T_1, \dots)$ as a binary relation symbol to stand for either identity, R or inequality. The set operators P_T and M_T are defined by $P_T(S) = \{\, w : \exists v\,(wTv \wedge v \in S)\,\}$, and $M_T(S) = (P_T(S^c))^c$. T may be associated with (modal) operators $\mathbf{t}$ and $\overline{\mathbf{t}}$ in the following way. If T is the identity both $\mathbf{t}$ and $\overline{\mathbf{t}}$ are the identity function; if $T = R$ then $\mathbf{t} = \Diamond$ and $\overline{\mathbf{t}} = \Box$; if T is inequality then $\mathbf{t} = D$ and $\overline{\mathbf{t}} = \overline{D}$.

In the sequel we consider propositional functions that are built up using the following basic ones:

projections: $\pi_i^n(\varphi_1, \dots \varphi_n) = \varphi_i$;
falsity: $\bot^n(\varphi_1, \dots \varphi_n) = \bot$;
truth: $\top^n(\varphi_1, \dots \varphi_n) = \top$;
$\vee, \wedge, \Diamond, \Box, D, \overline{D}, \top, \bot$.

(Cf. [17].) For each propositional function φ and frame $\langle W, R\rangle$ we define a function $F^\varphi : \mathcal{P}(W)^n \to \mathcal{P}(W)$. For non-modal φ, F^φ is the obvious Boolean set operation, while $F^{\Diamond\varphi}(S_1, \dots, S_n) = P_R(F^\varphi(S_1, \dots, S_n))$, and $F^{D\varphi}(S_1, \dots, S_n) = P_{\neq}(F^\varphi(S_1, \dots, S_n))$. The functions $F^{\Box\varphi}$ and $F^{\overline{D}\varphi}$ are defined dually.

Define a general frame $\langle W, R, \mathcal{W}\rangle$ to be *refined* if $\forall S \in \mathcal{W}\,(y \in S \to x \in P_T(S)) \to xTy$ (cf. [19]). For the proof of the original Sahlqvist Theorem it is essential that canonical general frames for logics L in $\mathcal{L}(\Diamond)$ are refined. Due to the fact that the canonical general frame for DL^- may contain doubled (hence indistinguishable) points, it need not satisfy the refinedness condition when T is the identity relation. Refinedness is restored when we add the rule IR to DL^-: the canonical general frame for any logic $L \supseteq DL_m$ is refined. To see this, let denote T the identity relation, and assume $x \neq y$; then for some $\varphi \in \mathcal{L}(\Diamond, D)$, $\varphi \in x$ but $\varphi \notin y$; so $x \in \{\,\Delta : \varphi \in \Delta\,\}$ ($\in \mathcal{W}$), but $y \notin \{\,\Delta : \varphi \in \Delta\,\}$. Next, let T denote R (the case that T is the inequality relation is similar); assume that $\neg Rxy$, then there is a $\varphi \in \mathcal{L}(\diamond, D)$ such that $\Box\varphi \in x$ with $\varphi \notin y$. So $y \notin \{\Delta : \varphi \in \Delta\}$, and $x \in M_R(\{\Delta : \varphi \in \Delta\})$.

Now that we have restored refinedness, we may proceed to give a proof of a Sahlqvist Theorem for $\mathcal{L}(\diamond, D)$; this proof is a more or less straightforward generalization of the proof of the original Sahlqvist Theorem.

Suppose that we want to prove a logic $L \supseteq DL_m$ in $\mathcal{L}(\diamond, D)$ complete. Let $\not\vdash_L \psi$. Then there is a general frame $\mathfrak{F} = \langle W, R, \mathcal{W}\rangle$ with $\mathfrak{F} \models L$, but $\mathfrak{F} \not\models \psi$. To establish completeness we want to find an ordinary frame $\mathcal{F}$ with this property. Below we indicate how one may show that under certain restrictions $\mathcal{F}$ can be taken to be the frame $\langle W, R\rangle$ underlying the above general frame $\mathfrak{F}$. First, we need to restrict our general frames to so-called simple frames which form a subclass of the refined frames; second, the logic L needs to be of the form $DL_m + \varphi$ for some so-called Sahlqvist formula φ.

Definition 3.9 A general frame $\langle W, R, \mathcal{W}\rangle$ is *simple* if for all $l, m, n_1, \ldots, n_m$, and positive $\varphi_1, \ldots, \varphi_l$ it satisfies

$$\forall S \in \mathcal{W}\Big(\bigwedge_{i=1}^{m} x_i \in M_{T_{n_i}}(\ldots(M_{T_{1_i}}(S))\ldots) \to \bigvee_{j=1}^{l} u_j \in F^{\varphi_j}(\ldots, S, \ldots)\Big)$$

$$\to \bigvee_{j=1}^{l} u_j \in F^{\varphi_j}\Big(\ldots, \bigcup_{i=1}^{m} P_{\check{T}_{1_i}}(\ldots(P_{\check{T}_{n_i}}\{x_i\})\ldots), \ldots\Big). \qquad (1)$$

To see that each simple general frame is refined, choose $m = l = n_1 = 1$, and φ_1 an appropriate projection in (1).

Theorem 3.10 *Let $L \supseteq DL_m$ be any logic in $\mathcal{L}(\diamond, D)$. Then the canonical general frame for L is simple.*

Proof. This is a generalization of [17, Theorems 16 and 18]. QED.

Definition 3.11 A formula $\varphi \in \mathcal{L}(\diamond, D)$ is said to be *positive* if it is built up using $\bot$, $\top$, proposition letters, $\vee$, $\wedge$, $\diamond$, $\Box$, D, $\overline{D}$ only. A formula $\varphi \in \mathcal{L}(\diamond, D)$ is called a *Sahlqvist formula* if it is (a conjunction) of formulas of the form $\overline{\mathfrak{t}}_1 \ldots \overline{\mathfrak{t}}_m(\psi \to \chi)$, where each $\overline{\mathfrak{t}}_i$ is either $\Box$ or $\overline{D}$, χ is a positive formula, and in ψ projections are brought outermost, negations are brought inside all other connectives ($\to$ is eliminated), and each proposition letter in ψ occurs only under sequents of connectives where no $\Box$ or $\overline{D}$ precedes any $\vee$, $\wedge$, $\diamond$ or D.

Theorem 3.12 *Let φ be a Sahlqvist formula in $\mathcal{L}(\diamond, D)$. Then φ corresponds to a first-order condition on frames, effectively obtainable from φ.*

Proof. Similar to the proof of [4, Theorem 9.10] or [17, Theorem 8]. For future reference we mention a few steps in the latter proof. Let $\varphi \equiv \overline{\mathfrak{t}}_1 \ldots \overline{\mathfrak{t}}_m(\psi \to \chi)$, and let $p_1, \ldots, p_k$ be all the proposition letters occurring in φ. Having $\langle W, R\rangle \models \varphi$ means having

$$\forall S_1 \ldots S_k, v, u\,\Big(vT_1 \circ \cdots \circ T_m u \wedge u \in F^{\psi}(S_1, \ldots, S_k) \to u \in F^{\chi}(S_1, \ldots, S_k)\Big), \qquad (2)$$

where $T_1, \ldots, T_m$ are the relations corresponding to $\overline{\mathbf{t}}_1 \ldots \overline{\mathbf{t}}_m$ respectively. This may be rewritten as a conjunction of formulas of the form

$$\forall \cdots \Big(\Phi \wedge \bigwedge_{j=1}^{k} \bigwedge_{i=1}^{m_j} x_i \in M_{T_{n_i}}(\ldots(M_{T_{1_i}}(S_j))\ldots) \rightarrow \bigvee_{j=1}^{k} u_j \in F^{\chi_j}(S_1, \ldots, S_k)\Big), \tag{3}$$

where Φ is a quantifier free formula in $\mathcal{L}_0$ ordering its variables in a certain way (each variable occurs to the right of an R or $\neq$ only once). Such formulas may in turn be rewritten as a first-order formula of the form

$$\forall \cdots \Big(\Phi \rightarrow \bigvee_{j=1}^{k} u_j \in F^{\chi_j}(\bigcup_{i=1}^{m_1} P_{\check{T}_{1_i}}(\ldots(P_{\check{T}_{n_i}}(\{x_{1_i}\}))\ldots), \ldots$$

$$\ldots, \bigcup_{i=1}^{m_k} P_{\check{T}_{1_i}}(\ldots(P_{\check{T}_{n_i}}(\{x_{k_i}\}))\ldots\Big). \text{ QED.} \tag{4}$$

Theorem 3.13 *Let φ be a Sahlqvist formula in $\mathcal{L}(\Diamond, D)$. Then $L = DL_m + \varphi$ is complete with respect to the class of frames that satisfy the first-order condition corresponding to φ.*

Proof. Soundness follows from 3.12. To prove completeness, assume that $\not\vdash_L \psi$. Let $\mathfrak{F}_L = \langle W_L, R_L, \mathcal{L}_L\rangle$ be a canonical general frame for L with $\mathfrak{F}_L \models L$ and $\mathfrak{F}_L \not\models \psi$. So $\langle W_L, R_L\rangle \not\models \psi$. By 3.10 $\mathfrak{F}_L$ is simple, and so it has the property

$$\forall S_1, \ldots S_k \in \mathcal{W}\Big(\Phi \wedge \bigwedge_{j=1}^{k} \bigwedge_{i=1}^{m_j} x_{i,j} \in M_{T_{n_{i,j}}}(\ldots(M_{T_{1_{i,j}}}(S_j))\ldots) \rightarrow \bigvee_{j=1}^{l} u_j \in F^{\varphi_j}(S_1, \ldots, S_k)\Big)$$

$$\rightarrow \Big(\Phi \rightarrow \bigvee_{j=1}^{l} u_j \in F^{\chi_j}(\bigcup_{i=1}^{m_1} P_{\check{T}_{1_{i,1}}}(\ldots(P_{\check{T}_{n_{i,k}}}(\{x_{i,1}\}))\ldots), \ldots, \bigcup_{i=1}^{m_k} P_{\check{T}_{1_{i,k}}}(\ldots(P_{\check{T}_{n_{i,k}}}(\{x_{i,k}\}))\ldots)\Big).$$

for any formula Φ not containing $S_1, \ldots, S_k$. Moreover, $\mathfrak{F}_L \models (2)$ implies $\mathfrak{F}_L \models (4)$. Hence $\mathfrak{F}_L \models (4)$. But then $\langle W_L, R_L\rangle \models (4)$, and also $\langle W_L, R_L\rangle \models L$. QED.

3.4 Transferring properties of logics

Let L extend K in $\mathcal{L}(\Diamond)$ with schemas $\{\varphi_i : i \in I\}$. The *minimal extension* of L in $\mathcal{L}(\Diamond, D)$ is DL_m plus the schemas φ_i read as schemas over $\mathcal{L}(\Diamond, D)$. Gargov and Goranko [9] formulate the following *Transfer Problem*: if L has property P does its minimal extension have P? Here we will consider two of the many obvious properties one may study in this context: completeness and incompleteness.

It is still open whether in general completeness is transferred. However, if φ is a Sahlqvist formula in $\mathcal{L}(\Diamond)$, then the minimal extension of $K + \varphi$ is complete by 3.13. To obtain a more general result we adopt an argument from [9]. Recall that a logic L in $\mathcal{L}(\Diamond)$ is *canonical* if its validity is preserved in passing from a descriptive general frame (cf. [4]) to its underlying full frame. From [8] we know that all complete and elementary (i.e first-order definable) logics in $\mathcal{L}(\Diamond)$ are canonical. Hence by the Sahlqvist Theorem for $\mathcal{L}(\Diamond)$, if $\varphi \in \mathcal{L}(\Diamond)$ is a Sahlqvist formula then $K + \varphi$ is canonical. (Since canonical logics need not be elementary, they form a wider class then the 'Sahlqvist logics'.)

Proposition 3.14 *If $L \subseteq \mathcal{L}(\Diamond)$ is canonical then its minimal extension L' in $\mathcal{L}(\Diamond, D)$ is complete.*

Proof. Let $\langle W, R, V\rangle$ be (an R_D-generated submodel of) the canonical model for L' (as in 3.4). Then $\mathfrak{F} = \langle W, R, \{ V(\varphi) : \varphi \in \mathcal{L}(\Diamond, D) \}\rangle$ is a descriptive general frame. Since $\mathfrak{F} \models L'$, we have $\mathfrak{F} \models L$, thus by assumption $\langle W, R\rangle \models L$, so $\langle W, R\rangle \models L'$. QED.

Incompleteness is an example of a property for which we always have transfer. By an easy argument, if L' is the minimal extension of a logic L in $\mathcal{L}(\Diamond)$, then L' is conservative over L. Therefore, if L is incomplete, so is L'. Hence all of the well-known incomplete logics in $\mathcal{L}(\Diamond)$ re-occur as incomplete systems in $\mathcal{L}(\Diamond, D)$. (As an aside, new and fairly simple incomplete logics occur as well: let X be $DL_m + (\Diamond\varphi \to D\varphi) + (\Diamond\Diamond\varphi \to \Diamond\varphi) + (\Box\Diamond\varphi \to \Diamond\Box\varphi)$. Then $X \models \bot$ since $\Diamond\varphi \to D\varphi$ defines irreflexivity of R; and given $\Diamond\Diamond\varphi \to \Diamond\varphi$, $\Box\Diamond\varphi \to \Diamond\Box\varphi$ defines $\forall x \exists y\, (Rxy \to \forall z\, (Ryz \to z = y))$. However, by a routine argument involving general frames, $X \not\models \bot$.)

4 Definability

We first make a remark or two about definability of classes of frames. After that we give a characterization of the $\mathcal{L}_0$-formulas that are equivalent to a $\Diamond, D$-formula on models, and apply this result to obtain a model-theoretic characterization of the definable classes of models.

4.1 Definability of classes of frames

The study of definability of classes of frames in $\mathcal{L}(\Diamond, D)$ in the spirit of [11] has been undertaken in [9] and [12]. For the sake of completeness we repeat the main definability result from the latter papers.

A general ultraproduct of frames $\mathcal{F}_i$ is an ultraproduct of the *full* general frames $\langle \mathcal{F}_i, 2^{W_i}\rangle$. (Cf. [4].)

Definition 4.1 $\mathcal{F}'$ is a *collapse* of the general frame $\mathfrak{F} = \langle \mathcal{F}, \mathcal{W}\rangle$ if $\mathcal{F}'$ is a subframe of $\mathcal{F}$ and if there exists a subframe $\mathfrak{G}$ of $\mathfrak{F}$ such that $(\mathcal{F}')^+ \cong (\mathfrak{G})^+$ and for each $x \in W'$, $\{ y : Rxy \} \subseteq [R'(x)]_{\mathfrak{G}^+}$, where $[X]_{\mathfrak{G}^+}$ is the least element of $(\mathfrak{G})^+$ containing X, and $(\cdot)^+$ is the mapping defined in [4, Chapter 4], that takes (general) frames to modal algebras.

Theorem 4.2 (Gargov and Goranko) *A class of frames is definable in $\mathcal{L}(\Diamond, D)$ iff it is closed under isomorphisms and collapses of general ultraproducts of frames.*

Gargov and Goranko arrive at 4.2 by using an appropriate kind of modal algebras. For an important special case a purely modal proof may be given:

Proposition 4.3 *A class K of finite frames is definable in $\mathcal{L}(\Diamond, D)$ iff it is closed under isomorphisms.*

Proof. Let $\mathcal{F}$ be a finite frame with $W = \{ w_1, \ldots, w_n \}$, and $\mathcal{F} \models \mathrm{Th}_{\Diamond,D}(K)$. Assume $p_1, \ldots, p_n$ are different proposition letters. Define $\chi_{\mathcal{F}}$ by

$$\bigwedge_{1\le i\le n} Ep_i \wedge A\Big(\bigvee_{1\le i\le n} (p_i \wedge \neg Dp_i)\Big) \wedge A\Big(\bigwedge_{1\le i\neq j\le n} (p_i \to \neg p_j)\Big) \wedge A\Big(\bigwedge_{1\le i,j\le n} (p_i \to Op_j)\Big),$$

where $O \equiv \Diamond$ if Rw_iw_j holds, and $O \equiv \neg\Diamond$ otherwise. Then for any frame $\mathcal{G}$, there is a valuation V with $\langle \mathcal{G}, V\rangle \not\models \neg\chi_{\mathcal{F}}$ iff $\mathcal{G} \cong \mathcal{F}$. In particular $\mathcal{F} \not\models \neg\chi_{\mathcal{F}}$. Hence $\neg\chi_F \notin \mathrm{Th}_{\Diamond,D}(K)$. Thus for some $\mathcal{G} \in K$, $\mathcal{G} \not\models \neg\chi_{\mathcal{F}}$. So $\mathcal{F} \in K$. QED.

4.2 Definability of classes of models

Standard modal formulas, when interpreted in models, are equivalent to a special kind of first-order formulas. Adding the D-operator does not change this.

Definition 4.4 Let x be a fixed variable. The *standard translation* $ST(\varphi)$ of a formula $\varphi \in \mathcal{L}(\Diamond, D)$ is defined as follows: it commutes with the Boolean connectives, and $ST(p) = Px$, $ST(\Diamond\psi) = \exists y\,(Rxy \wedge ST(\psi)[x := y])$, and $ST(D\psi) = \exists y\,(x \neq y \wedge ST(\psi)[x := y])$, where y is a variable not occurring in $ST(\psi)$.

Since the equivalences $\mathcal{M} \models \varphi[w]$ iff $\mathcal{M} \models ST(\varphi)[w]$, and $\mathcal{M} \models \varphi$ iff $\mathcal{M} \models \forall x\, ST(\varphi)$ hold, well-known facts about $\mathcal{L}_1$ become applicable for $\mathcal{L}(\Diamond, D)$. $\mathcal{L}_1$-formulas of the form $ST(\varphi)$ for some $\varphi \in \mathcal{L}(\Diamond, D)$ can be described independently in the following way:

Definition 4.5 The set of *MD-formulas* is the least set X of $\mathcal{L}_1$-formulas such that $Px \in X$, for unary predicate symbols P and all variables x; if $\alpha \in X$ then $\neg\alpha \in X$; if α, $\beta \in X$ have the same free variable, then $\alpha \wedge \beta \in X$; and if $\alpha \in X$, x, y are distinct variables, and y is αs free variable, then $\exists y\,(Rxy \wedge \alpha)$, $\exists y\,(x \neq y \wedge \alpha) \in X$.

The semantic characterization of MD-formulas we give generalizes a corresponding result for $\mathcal{L}(\Diamond)$ in [4]. However, whereas the proof given there uses an elementary chain construction, the proof we present uses saturated models. Clearly, the characterization will also be a characterization of the (translations of the) $\Diamond, D$-formulas in $\mathcal{L}_1$.

Definition 4.6 A binary relation Z is called a *p-relation* between two models $\mathcal{M}_1$ and $\mathcal{M}_2$ if the following holds (for $i \neq j \in \{1, 2\}$):

1. Zwv then w, v verify the same proposition letters,
2. if Zwv, $w' \in W_i$ and R_iww' then $Zw'v'$ for some $v' \in W_j$ with R_jvv',
3. if Zwv, $w' \in W_i$ and $w \neq w'$ then $Zw'v'$ for some $v' \in W_j$ with $v \neq v'$,
4. $\mathrm{dom}(Z) = W_1$, $\mathrm{ran}(Z) = W_2$.

An $\mathcal{L}_1$-formula $\alpha(x_1, \ldots, x_n)$ is *invariant for p-relations* if, for all models $\mathcal{M}_1$, $\mathcal{M}_2$, all p-relations Z between $\mathcal{M}_1$ and $\mathcal{M}_2$, and all $w_1, \ldots, w_n \in W_1$, $w'_1, \ldots, w'_n \in W_2$ such that $Zw_1w'_1, \ldots, Zw_nw'_n$, we have $\mathcal{M}_1 \models \alpha[w_1, \ldots, w_n]$ iff $\mathcal{M}_2 \models \alpha[w'_1, \ldots, w'_n]$.

Theorem 4.7 *An $\mathcal{L}_1$-formula containing exactly one free variable x is equivalent to an MD-formula iff it is invariant for p-relations.*

Proof. A simple induction proves that every MD-formula is invariant for p-relations.

Conversely, assume that the $\mathcal{L}_1$-formula α has this property, and suppose x is αs free variable. Define $MD(\alpha) := \{\beta : \beta \text{ is an MD-formula}, \alpha \models \beta, FV(\beta) \subseteq \{x\}\}$. We will prove that $MD(\alpha) \models \alpha$. Then, by compactness, there is a $\beta \in MD(\alpha)$ with $\models \alpha \leftrightarrow \beta$. Assume $\mathcal{M} \models MD(\alpha)[w]$; we have to show that $\mathcal{M} \models \alpha[w]$. Introduce a new constant $\underline{w}$

to stand for the object w, and define $\mathcal{L}^* = \mathcal{L}_1 \cup \{\underline{w}\}$. Expand $\mathcal{M}$ to an $\mathcal{L}^*$-model $\mathcal{M}^*$ by interpreting $\underline{w}$ as w. In the remainder of this proof we use the following notation: if $\beta \in \mathcal{L}_1$ then $\beta^* \equiv \beta[x := \underline{w}]$; and if T is a set of $\mathcal{L}_1$-formulas then $T^* := \{\beta^* : \beta \in T\}$.

Let $T := \{\beta : \mathcal{M} \models \beta[w], \beta$ is an MD-formula, $FV(\beta) \subseteq \{x\}\}$. By compactness we find an $\mathcal{L}^*$-model $\mathcal{N}^*$ with $\mathcal{N}^* \models T^* \cup \{\alpha^*\}$. By [7, Theorem 6.6.1] there are ω-saturated elementary extensions $\mathcal{M}_1^* =: \langle W_1, R_1, w_1, V_1\rangle \succ \mathcal{M}^*$ and $\mathcal{N}_1^* =: \langle W_2, R_2, w_2, V_2\rangle \succ \mathcal{N}^*$ such that both w_1 and w_2 realize T, and such that $\mathcal{N}_1^* \models \alpha^*$.

Define a relation $Z \subseteq W_1 \times W_2$ between (the $\mathcal{L}_1$-reducts of) $\mathcal{M}_1^*$ and $\mathcal{N}_1^*$ by putting Zwv iff for all $\varphi \in \mathcal{L}(\Diamond, D)$, $\langle W_1, R_1, V_1\rangle \models \varphi[w]$ iff $\langle W_2, R_2, V_2\rangle \models \varphi[v]$. We verify that Z is in fact a p-relation by checking the conditions of 4.6. Condition 1 is trivial. We only check half of condition 2: assume that R_1ww' and Zwv, with w, $w' \in W_1$ and $v \in W_2$. We have to prove $\exists v' \in W_2\,(R_2vv' \wedge Zwv')$. Define $\Psi := \{\varphi \in \mathcal{L}(\Diamond, D) : \mathcal{M}_1^* \models \varphi[w']\}$. Then $ST(\Psi) \cup \{R\underline{v}y\}$ is finitely satisfiable in $(\mathcal{N}_1^*, v)$. Hence, by saturation $(\mathcal{N}_1^*, v) \models ST(\Psi) \cup \{R\underline{v}y\}[v']$, for some $v' \in W_2$. But then we have $Zw'v'$. Condition 3 is similar to condition 2, and condition 4 is immediate from condition 3 and the fact that Zw_1w_2.

Finally, by invariance for p-relations $\mathcal{N}_1^* \models \alpha^*$ yields $\mathcal{M}_1^* \models \alpha^*$. Since $\mathcal{M}^* \prec \mathcal{M}_1^*$ it follows that $\mathcal{M}^* \models \alpha^*$, and so $\mathcal{M} \models \alpha[w]$. QED.

Next we apply 4.7 to obtain a definability result for classes of models. To this end we find it convenient to take frames $\langle \mathcal{F}, w\rangle$ with a distinguished world w (as in Kripke's original publications) as the basic notion of frame. Similarly, the basic notion of model is taken to be $\langle \mathcal{F}, w, V\rangle$.

Theorem 4.8 *Let M be a class of models. Then $M = \{\mathcal{M}\ (= \langle W, R, w, V\rangle) : \mathcal{M} \models \varphi[w]\}$, for some $\varphi \in \mathcal{L}(\Diamond, D)$ iff M is closed under p-relations and ultraproducts, while its complement is closed under ultraproducts.*

Proof. Introduce a new constant $\underline{w}$ to stand for the object w, and define $\mathcal{L}^* := \mathcal{L}_1 \cup \{\underline{w}\}$. As before we write β^* for $\beta[x := \underline{w}]$.

If $M = \{\mathcal{M}\ (= \langle W, R, w, V\rangle) : \mathcal{M} \models \varphi[w]\}$, for some $\varphi \in \mathcal{L}(\Diamond, D)$, then M is closed under p-relations and ultraproducts. The complement of M is defined by $\{\neg ST(\varphi)^*\}$, hence closed under ultraproducts.

For the other direction, suppose that $\mathcal{M}$ and its complement satisfy the stated conditions. Since $\mathcal{M}$ is closed under p-relations, it and its complement are closed under isomorphisms. So by [7, Corollary 6.1.16] there is an $\mathcal{L}^*$-sentence α^* such that for all $\mathcal{L}^*$-models $\mathcal{M}$, $\mathcal{M} \in M$ iff $\mathcal{M} \models \alpha^*$. From the fact that M is closed under p-relations one easily derives that α is closed under p-relations between 'ordinary' models. Therefore, by 4.7 α is equivalent to an MD-formula with the same free variable. Hence α is equivalent to $ST(\varphi)$ for some formula $\varphi \in \mathcal{L}(\Diamond, D)$. QED.

Remark 4.9 In [16] Piet Rodenburg uses a proof similar to the one we gave for 4.7 to characterize the definable classes of models of intuitionistic propositional logic. A reading of this characterization led to 4.8.

References

[1] W. Ackermann. *Solvable Cases of the Decision Problem*, North-Holland, Amsterdam, 1954.

[2] J.F.A.K. van Benthem. Syntactical Aspects of Modal Incompleteness Theorems. *Theoria* **45** (1979), 63–77.

[3] J.F.A.K. van Benthem. *The Logic of Time*, Reidel, Dordrecht, 1983.

[4] J.F.A.K. van Benthem. *Modal Logic and Classical Logic*, Bibliopolis, Naples, 1985.

[5] J.F.A.K. van Benthem. Notes on Modal Definability. *Notre Dame Journal of Formal Logic* **30** (1989), 20–35.

[6] P. Blackburn. *Nominal Tense Logic and Other Sorted Intensional Frameworks*, Dissertation, University of Edinburgh, 1990.

[7] C.C. Chang and H.J. Keisler. *Model Theory*, North-Holland, Amsterdam, 1973.

[8] K. Fine. Some Connections between Elementary and Modal Logic. In: S. Kanger (ed.) *Proceedings of the Third Scandinavian Logic Symposium. Uppsala 1973*, North-Holland, Amsterdam, 1975, 15–31.

[9] G. Gargov and V. Goranko. Modal Logic with Names I. Linguistic Modeling Laboratory, CICT, Bulgarian Academy of Science, and Sector of Logic, Faculty of Mathematics and Computer Science, Sofia University. Preprint, 1989.

[10] G. Gargov, S. Passy and T. Tinchev. Modal Environment for Boolean Speculations. In: D. Skordev (ed.), *Mathematical Logic and its Applications*, Plenum Press, New York, 1987, 253–263.

[11] R. Goldblatt and S.K. Thomason. Axiomatic Classes in Propositional Modal Logic. In: J. Crossley (ed.), *Algebra and Logic*, Springer LNM 450, Berlin, 1974.

[12] V. Goranko. Modal Definability in Enriched Languages. *Notre Dame Journal of Formal Logic* **31** (1990), 81–105.

[13] V. Goranko and S. Passy. Using the Universal Modality: Gains and Questions. Sector of logic, Faculty of Mathematics, Sofia University. Preprint, 1990.

[14] B.M. Kapron. Modal Sequents and Definability. *Journal of Symbolic Logic* **52** (1987), 756–762.

[15] R. Koymans. *Specifying Message Passing and Time-Critical Systems with Temporal Logic*, Dissertation, Eindhoven University of Technology, 1989.

[16] P.H. Rodenburg. *Intuitionistic Correspondence Theory*, Dissertation, University of Amsterdam, 1986.

[17] H. Sahlqvist. Completeness and Correspondence in the First and Second Order Semantics for Modal Logic. In: S. Kanger (ed.), *Proc. of the Third Scandinavian Logic Symposium Uppsala 1973*, North-Holland, Amsterdam, 1975, 110–143.

[18] K. Segerberg. Modal Logics with Linear Alternative Relations. *Theoria* **36** (1970), 301–322.

[19] S.K. Thomason. Semantic Analysis of Tense Logics. *Journal of Symbolic Logic* **37** (1972), 150–158.

The ITLI Prepublication Series

1990

Logic, Semantics and Philosophy of Language

LP-90-01 Jaap van der Does	A Generalized Quantifier Logic for Naked Infinitives
LP-90-02 Jeroen Groenendijk, Martin Stokhof	Dynamic Montague Grammar
LP-90-03 Renate Bartsch	Concept Formation and Concept Composition
LP-90-04 Aarne Ranta	Intuitionistic Categorial Grammar
LP-90-05 Patrick Blackburn	Nominal Tense Logic
LP-90-06 Gennaro Chierchia	The Variablity of Impersonal Subjects
LP-90-07 Gennaro Chierchia	Anaphora and Dynamic Logic
LP-90-08 Herman Hendriks	Flexible Montague Grammar
LP-90-09 Paul Dekker	The Scope of Negation in Discourse, towards a flexible dynamic Montague grammar
LP-90-10 Theo M.V. Janssen	Models for Discourse Markers
LP-90-11 Johan van Benthem	General Dynamics
LP-90-12 Serge Lapierre	A Functional Partial Semantics for Intensional Logic
LP-90-13 Zisheng Huang	Logics for Belief Dependence
LP-90-14 Jeroen Groenendijk, Martin Stokhof	Two Theories of Dynamic Semantics
LP-90-15 Maarten de Rijke	The Modal Logic of Inequality

Mathematical Logic and Foundations

ML-90-01 Harold Schellinx	Isomorphisms and Non-Isomorphisms of Graph Models
ML-90-02 Jaap van Oosten	A Semantical Proof of De Jongh's Theorem
ML-90-03 Yde Venema	Relational Games
ML-90-04 Maarten de Rijke	Unary Interpretability Logic
ML-90-05 Domenico Zambella	Sequences with Simple Initial Segments
ML-90-06 Jaap van Oosten	Extension of Lifschitz' Realizability to Higher Order Arithmetic, and a Solution to a Problem of F. Richman
ML-90-07 Maarten de Rijke	A Note on the Interpretability Logic of Finitely Axiomatized Theories
ML-90-08 Harold Schellinx	Some Syntactical Observations on Linear Logic
ML-90-09 Dick de Jongh, Duccio Pianigiani	Solution of a Problem of David Guaspari
ML-90-10 Michiel van Lambalgen	Randomness in Set Theory

Computation and Complexity Theory

CT-90-01 John Tromp, Peter van Emde Boas	Associative Storage Modification Machines
CT-90-02 Sieger van Denneheuvel, Gerard R. Renardel de Lavalette	A Normal Form for PCSJ Expressions
CT-90-03 Ricard Gavaldà, Leen Torenvliet, Osamu Watanabe, José L. Balcázar	Generalized Kolmogorov Complexity in Relativized Separations
CT-90-04 Harry Buhrman, Leen Torenvliet	Bounded Reductions
CT-90-05 Sieger van Denneheuvel, Karen Kwast	Efficient Normalization of Database and Constraint Expressions
CT-90-06 Michiel Smid, Peter van Emde Boas	Dynamic Data Structures on Multiple Storage Media, a Tutorial
CT-90-07 Kees Doets	Greatest Fixed Points of Logic Programs`

Other Prepublications

X-90-01 A.S. Troelstra	Remarks on Intuitionism and the Philosophy of Mathematics, Revised Version
X-90-02 Maarten de Rijke	Some Chapters on Interpretability Logic
X-90-03 L.D. Beklemishev	On the Complexity of Arithmetical Interpretations of Modal Formulae
X-90-04	Annual Report 1989
X-90-05 Valentin Shehtman	Derived Sets in Euclidean Spaces and Modal Logic
X-90-06 Valentin Goranko, Solomon Passy	Using the Universal Modality: Gains and Questions
X-90-07 V.Yu. Shavrukov	The Lindenbaum Fixed Point Algebra is Undecidable
X-90-08 L.D. Beklemishev	Provability Logics for Natural Turing Progressions of Arithmetical Theories
X-90-09 V.Yu. Shavrukov	On Rosser's Provability Predicate
X-90-10 Sieger van Denneheuvel, Peter van Emde Boas	An Overview of the Rule Language RL/1
X-90-11 Alessandra Carbone	Provable Fixed points in $I\Delta_0+\Omega_1$, revised version
X-90-12 Maarten de Rijke	Bi-Unary Interpretability Logic
X-90-13 K.N. Ignatiev	Dzhaparidze's Polymodal Logic: Arithmetical Completeness, Fixed Point Property, Craig's Property
X-90-14 L.A. Chagrova	Undecidable Problems in Correspondence Theory