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DYNAMIC SEMANTICS AND CIRCULAR PROPOSITIONS

Willem Groeneveld

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1. Introduction

If there is a problem about the Liar paradox, it is not so much the puzzle it presents, but the vast number of solutions that have been proposed for it. In view of the extensive literature on the Liar paradox[1], the problem is not to *solve* it, but *how* to solve it. The large number of purported solutions might even lead to the contention that there is no real solution.[2] Of course, between a particular philosophical puzzle and the opinion that there must be a unique right solution there is a considerable amount of speculation, especially if a completely satisfactory solution has not yet been found. And that seems to be the case for the Liar: there are a number of reasonable proposals, but none of them is obviously the right one.

The present paper is an attempt to add another reasonable proposal to the list. I will not argue that the present proposal is *the* right one. I will of course make a case for it. The theory of this paper is an extension of Barwise and Etchemendy's Austinian account of truth and circular propositions (Barwise and Etchemendy [1987]). The main ingredient of the present proposal is a form of dynamic semantics.[3] The idea that paradoxical sentences have a certain 'context change potential' also derives from Barwise and Etchemendy. They take the view that one of the lessons of the Austinian account is that the Liar shows that there is a "contextual parameter, one corresponding to Austin's described situation, a parameter whose value necessarily *changes* [my italics] with the utterance of, or reasoning about, a sentence like the Liar." (BE 175)[4] We will use update semantics for formalizing this idea (Veltman [1991]). Its main ideas are explained in section 2 below.

The theory of this paper is an extension of Barwise and Etchemendy's Austinian account in a very literal sense: we take over the ontology and the formal language, and devise a dynamic semantics for that language. I will assume the reader to be familiar with the Austinian account, as well as the theory of non-well-founded sets that is used to develop it. Nevertheless I will give a short summary of the Austinian account, in the hope that this will give the uninitiated reader a rough idea of its main aspects.[5]

The ontology of the Austinian account comprises four classes of entities: a class SOA of states of affairs, a class SIT of situations, a class TYPE of types, and a class PROP of propositions. States of affairs are of the form $\langle H,a,c;i\rangle$ (where H is a set theoretic atom, a is Max or Claire, c is one of the standard cards, and $i \in \{0,1\}$), or of the form $\langle Tr,p;i\rangle$ for some proposition $p \in PROP$ (where Tr is a set theoretic atom, $i \in \{0,1\}$). The latter states of affairs are called semantical facts. Situations are sets of states of affairs. Types are of the form $[\sigma]$ for some state of affairs σ, or of the form $[\wedge X]$ or $[\vee X]$ for some set of types X. Propositions are of the form {s;T} for some situation s and some type T.[6]

These objects are constructed in Aczel's theory of non-well-founded sets. As a consequence there are propositions that are constituents of themselves. For example, there exist a proposition p satisfying the identity

$$p=\{s;[Tr,p;1]\}$$

But what is specifically 'Austinian' of these propositions is that they have two main constituents, the situation the proposition is about and the type. Moreover, the situation the proposition is about can also be a constituent of the type of the proposition, as in a proposition q satisfying the identity

$$q=\{s;[Tr,\{s;[Tr,q;1]\};1]\}$$

Notice that in Aczel's set theory p and q are actually identical.

The class of true propositions is defined as follows: a proposition of the form $\{s;[\sigma]\}$ is true iff $\sigma\in s$; $\{s;[\wedge X]\}$ is true iff $\{s;T\}$ is true for all $T\in X$; and $\{s;[\vee X]\}$ is true iff $\{s;T\}$ is true for some $T\in X$.

Next a class of situations of special interest is singled out: a possible situation is a situation that is coherent (that is, if a state of affairs σ is in s then the dual of σ is not in s) and respects its semantical facts (i.e., if $\langle Tr,p;1\rangle\in s$ then p is true, and if $\langle Tr,p;0\rangle\in s$ then p is not true).

The formal language has the following structure. The basic formulas are the form (<u>a</u> Has <u>c</u>), where a is Max or Claire, c is one of the standard cards; or of the form True(<u>this</u>), where <u>this</u> is the primitive symbol called propositional reflexive; or of the form True(<u>that$_i$</u>), where $i<\omega$ and <u>that$_i$</u> is a primitive symbol called a propositional demonstrative. If φ, ψ are formulas then so are Trueφ, $\neg\varphi$, $(\varphi\wedge\psi)$, $(\varphi\vee\psi)$ and $\downarrow\varphi$. An occurrence of <u>this</u> is loose in φ if it is not in the scope of the symbol "$\downarrow$". A sentence is a formula without loose occurrences of <u>this</u>.

The semantics for this language is developed in two steps. For each φ a parametric proposition Val(φ) is defined. Such a parametric proposition contains the situation indeterminate **s**, and may contain the propositional indeterminates **p** and $\mathbf{q}_i$ ($i<\omega$).

(i) Val(<u>a</u> Has <u>c</u>)={**s**;[H,a,c;1]}

(ii) Val(True(<u>that$_i$</u>))={**s**;[Tr,$\mathbf{q}_i$;1]}

(iii) Val(True(<u>this</u>))={**s**;[Tr,**p**;1]}

(iv) Val(True ψ)={**s**;[Tr,Val(ψ);1]}

(v) Val($\neg\psi$)={**s**;Type(Val(ψ))*}

(vi) $Val(\psi \wedge \chi)=\{s;[\wedge\{Type(Val(\psi)), Type(Val(\chi))\}]\}$

(vii) $Val(\psi \vee \chi)=\{s;[\vee\{Type(Val(\psi)), Type(Val(\chi))\}]\}$

(viii) $Val(\downarrow\psi)=p$, where the parametric proposition p is the unique solution to the equation $\mathbf{p}=Val(\psi)(\mathbf{p},\mathbf{q_1},...)$.

Here * is a negation operation on parametric types defined by: $[\sigma]^*=[\sigma']$, where σ' is the dual of σ; $[\wedge X]^*=[\vee\{T^*|T\in X\}]$; $[\vee X]^*=[\wedge\{T^*|T\in X\}]$. Type(Val($\varphi$)) is the type constituent of the parametric proposition Val(φ). Aczel's set theory includes a socalled Solution Lemma, which guarantees that equations as in clause (viii) do indeed have unique solutions.

Finally, the parameters in the parametric proposition Val(φ) are filled in by the context. On the Austinian account, propositions are the semantic counterparts of statements. A statement is a triple $<\varphi,s,c>$, where φ is a formula, s a situation, and c an assignment (a function of demonstratives to propositions). The proposition expressed by φ in context $<s,c>$, notation Exp(φ,s,c), is defined as $Val(\varphi)(s/s, \mathbf{q_1}/c(\underline{that_1}),...,\mathbf{q_i}/c(\underline{that_i}),...)$.[7]

For example, the Liar is rendered as the sentence $\downarrow\neg$True(<u>this</u>), where the scope symbol "$\downarrow$" indicates that, in any occasion of use of the sentence, the occurrence of the propositional reflexive <u>this</u> refers to the same proposition as the whole sentence. On the Austinian account, if this sentence is used to make a statement about a situation s, it expresses a circular proposition f_s, which has the following form:

(1) $f_s = \{s; [Tr,f_s;0]\}$

So Exp($\downarrow\neg$True(<u>this</u>),s)=f_s (we will usually not mention the assignment c when discussing formulas that do not contain demonstratives). The Austinian proposition f_s is true if the semantical fact $<Tr,f_s;0>$ is a member of s. Since by definition a possible situation respects its semantical facts, i.e. it only contains correct semantical information, the proposition f_s is not true if s is a possible situation. So suppose s is indeed a possible situation, and consider the situation s':

(2) $s'=s\cup\{<Tr,f_s;0>\}$

Then s' will also be a possible situation, because the additional semantical fact is correct (i.e. f_s is not true). Moreover, $s'\neq s$ since $<Tr,f_s;0>\notin s$ (again, because f_s is not true). So possible situations are incomplete in the following sense: although their Liar proposition will not be true, the information that this is so cannot be reflected in the situation itself. But it can be reflected in a larger situation: the situation s' is possible.

The procedure can be repeated ad infinitum: since s' is a possible situation, the Liar proposition that is about s', i.e.

(3) $f_{s'} = \{s'; [Tr,f_{s'};0]\}$

is not true, so the situation s" given by

(4) $s''=s'\cup\{<Tr,f_{s'};0>\}$

is a possible situation. And so on.

The analysis is attractive, because without contradiction the fact that a Liar proposition is not true can be actual, although it cannot be a fact of the situation the proposition is about. And the problem with the Liar always seemed to be that once you accept its not being true as a fact, you wind up contradicting yourself. On the other hand, equations (1)-(4) above only report some connections between some objects in Barwise and Etchemendy's ontology. Although the Austinian semantics assigns the proposition in (1) to the Liar sentence if it is used to make a statement about s, and the proposition in (3) if it is used to make a statement about s', the 'context-shifts' in (2) and (4) are not in any way triggered by the semantics. If it is really so important that the Liar brings about a change, then this 'context change potential' deserves to be regarded as an aspect of the *meaning* of the Liar. Moreover, saying that an utterance changes the described situation comes down to classifying the utterance as a performative speech act. But in the case of the Liar that seems to be wrong. I would rather say that an utterance of the Liar changes the information state of someone, than say that an utterance changes the described situation. If it changes any situation at all, it changes the *discourse* situation, or, more precisely, it changes the information of the participants of the discourse.

In this paper we will show that these objections can be met quite easily. The objections point in the direction of a dynamic, information oriented semantics. We will extend Barwise and Etchemendy's semantics with a form of update semantics. Update semantics is precisely what we need, since its central conception is that the meaning of a sentence is a relation between information states. A theoretical pay-off of the extended semantics will be a semantics for discourses with circular cross-references.

2. Basic ideas

In dynamic semantics, the meaning of a sentence is given by update conditions rather than by truth conditions. Veltman uses the following slogan: "You know the meaning of a sentence if you know the change it brings about in the information state of anyone who wants to incorporate the piece of news conveyed by it" (Veltman [1991]). One way of explicating this is by taking the meaning of a sentence to be a relation between information states.

What are information states? Intuitively, an information state models the information that a cognitive agent has of a real situation. This can be described by the set of situations that the agent cannot distinguish from the actual situation. So a proper information state can be seen as a set of situations. Under this perspective, there are two ways in which there can be lack of information. First, a set of situations that is not a singleton is in some sense 'disjunctive', since for all the agent knows, the actual situation could be one of many she thinks possible. Second, situations are partial, they do not settle all issues. Getting better informed can thus be seen as a combination of two things: elimination of options and filling in more detail of other options.[8]

In concreto, consider the sentence (<u>Max</u> Has <u>♠A</u>), and let σ be a set of possible situations. The update of σ with (<u>Max</u> Has <u>♠A</u>) can now be explained thus:

(5) [[<u>Max</u> Has <u>♠A</u>]] $(\sigma) = \{t \mid \exists s \in \sigma\colon t = s \cup \{\langle H, Max, ♠A; 1\rangle\}$ and t is possible$\}$

Those situations in σ that cannot be consistently extended with the fact that Max has the ace of spades will be eliminated, and the remaining ones are extended with this fact. In general, the meaning of a sentence in this set-up will be a function from sets of possible situations to sets of possible situations.

Although this is the basic picture, in the implementation below we will follow a different line. We will not define updates as functions on sets of situations, but as relations between situations. It is clear that any binary relation R between situations determines a unique function on the higher level, given by $F_R(\sigma)=\{t \mid \exists s\in\sigma\colon sRt\}$. Conversely, if F is a function on sets of situations that distributes over arbitrary unions, there is a unique binary relation R on situations such that $F=F_R$, namely $R=\{<s,t> \mid t\in F(\{s\})\}$. So the two approaches are interchangeable as long as the functions on the higher level are distributive. But for the fairly simple language we will devise a dynamic semantics for, this is the case.[9]

What kind of relations are we after? The slogan we started with gives the following clue: two situations s and t stand in the update relation $[\varphi]$ of a sentence φ only if t contains the information already in s and additionally covers the information presented by φ. From a semantical point of view, this will be the only respect in which t may differ from s: t is an option that is minimal (w.r.t. $\subseteq$) in the set of all options that are stronger than s and cover the information of φ. These considerations give two global constraints on update relations:

(6) for all s and t, if s $[\varphi]$ t then $s \subseteq t$ (Update)

(7) for all s and t, if s $[\varphi]$ t then for no t', $t' \subset t$ and s $[\varphi]$ t' (Minimality) [10]

It may happen that for some particular option s an update doesn't change anything, that is, s $[\varphi]$ s. Apparently, s already covered the information of φ. In this case we say that s *supports* φ.[11] On the face of it, then, another reasonable constraint on update relations is *success*. That is, if you are in state s and φ brings you to state t, then t supports φ.

(8) for all s and t, if s $[\varphi]$ t then t $[\varphi]$ t (Success)

As a consequence of Barwise and Etchemendy's 'dynamic' analysis of the Liar, it turns out that success cannot hold in general. That is, we take it that their analysis shows that it *is* possible to 'incorporate the piece of news' of a statement with the Liar. If you are in state s, then the piece of news of $\downarrow\neg$True(this), if taken to be about s, is given by the semantical information that the circular proposition about s that claims of itself that it is not true, is in fact not true. If we equate this semantical information with the semantical fact $\langle Tr, f_s; 0\rangle$, the update relation of the Liar will satisfy:

(9) s $[\downarrow\neg$True(this)$]$ t iff $t = s \cup \{\langle Tr, f_s; 0\rangle\}$

If we demand that semantic information must be correct, the range of s and t in (9) must be restricted to possible situations. Now the following rephrasal of Barwise and Etchemendy's analysis results: *you can always consistently extend your information with the message of the Liar, but you can never wind up in an information state that supports the Liar*. In fact, in the semantics developed below the typical property of $\downarrow\neg$True(this) is expressed by

(10) for all s and t, if s $[\varphi]$ t then $s \neq t$ (Anti-success)

The Liar could be called a Zeno tortoise: every attempt to catch up with it will fail, although the attempt does change your position.[12]

For normal descriptive sentences success will still hold. However, consider the following property:

(11) for all s and t, if s $[\varphi]$ t then $s = t$ (Pre-conditional success)

Pre-conditional success implies Success.[13] But it is an awkward form of success, since what (11) expresses is roughly "you can accept the information presented by φ only if you already have this information". This is a rather strange property, since it expresses something very close to question begging. As we will see below, pre-conditional success is a typical property of the Truthteller $\downarrow$True(this) ("This proposition is true"). In terms of our earlier metaphor, a sentence with this property is a *dead* tortoise: you can only catch up with it if you're already sitting on it.

We will now proceed to develop these semi-formal considerations in detail.

3 The dynamic semantics

Our informal discussion of update relations in section 2 gave us the following picture: if t is an update of s with φ, then t is a state stronger than s that covers the information of φ; moreover there is no state weaker than t with this property. One strategy for the implementation would then run as follows: first we explain what we mean by "covering the information of φ", and then we define "t is an update of s with φ" as "t is minimal with respect to $\subseteq$ in the class $s+\varphi=\{v|\ s\subseteq v$ and v covers $\varphi\}$". This strategy is followed in Groeneveld [1989]. In general, the class $s+\varphi$ will be proper, so some work has to be done in order to show that these classes have minimal elements. The strategy we follow here avoids this extra work, by using Barwise and Etchemendy's static semantics to estimate in advance the possible updates of a given situation.

1 Definition Let s be a situation, and c be an assignment of propositions to propositional demonstratives. R(s,c), the set of relevant facts for s under c, is given by

$$R(s,c)=\{\sigma\in SOA|\ [\sigma]=Type(Exp(\varphi,s,c)) \text{ for some simple formula } \varphi\}$$

Here Type(Exp(φ,s,c)) is the type constituent of the proposition Exp(φ,s,c). A formula is simple if it has one of the following forms: (a̲ Has c̲), $\neg$(a̲ Has c̲), True(this̲),$\neg$True(this̲), Trueφ or $\neg$Trueφ.
The set of possible updates of s for c, is:

$$U(s,c) =_{def} \{t|\ s\subseteq t\subseteq s\cup R(s,c)\}$$

2 Definition If P is a set of situations then $\mu P =_{def} \{\ s\in P|\ \neg\exists t\in P: t\subset s\}$

3 Definition By simultaneous recursion we define, for any formula φ, and any assignment c, the positive update relation of φ and the negative update relation of φ. In the following, s ranges over situations, t over parametric situations. **p** is the indeterminate that is used in the static semantics to fix the reference of the propositional reflexive this̲. **t** is an additional situation indeterminate.

(i) $s\,[\underline{a}\ \mathrm{Has}\ \underline{c}]^+_c\,t$ iff $t= s \cup \{\langle H,a,c;1\rangle\}$

$s\,[\underline{a}\ \mathrm{Has}\ \underline{c}]^-_c\,t$ iff $t= s \cup \{\langle H,a,c;0\rangle)\}$

(ii) $s\,[\mathrm{True}(\underline{\mathrm{this}})]^+_c\,t$ iff $t=s\cup\{\langle Tr,\mathbf{p};1\rangle\}$

$s\,[\mathrm{True}(\underline{\mathrm{this}})]^-_c\,t$ iff $t=s\cup\{\langle Tr,\mathbf{p};0\rangle\}$

(iii) $s\,[\mathrm{True}(\underline{\mathrm{that}_i})]^+_c\,t$ iff $t=s\cup\{\langle Tr,c(\underline{\mathrm{that}_i});1\rangle\}$

$s\,[\mathrm{True}(\underline{\mathrm{that}_i})]^-_c\,t$ iff $t=s\cup\{\langle Tr,c(\underline{\mathrm{that}_i}),0\rangle\}$

(iv) $s\,[\mathrm{True}\varphi]^+_c\,t$ iff $t= s\cup\{\langle Tr,Exp(\varphi,s,c);1\rangle\}$

$s\,[\mathrm{True}\varphi]^-_c\,t$ iff $t=s\cup\{\langle Tr,Exp(\varphi,s,c);0\rangle\}$

(v) $s\,[\neg\varphi]^+_c\,t$ iff $s\,[\varphi]^-_c\,t$

$s\,[\neg\varphi]^-_c\,t$ iff $s\,[\varphi]^+_c\,t$

(vi) $s\,[\varphi\wedge\psi]^+_c\,t$ iff $t\in \mu\{u\in U(s,c)\mid\ \exists v\sqsubseteq u\text{: } s\,[\varphi]^+_c v \text{ and } \exists v\sqsubseteq u\text{: } s\,[\psi]^+_c v\}$

$s\,[\varphi\wedge\psi]^-_c\,t$ iff $t\in \mu\{u\in U(s,c)\mid\ \exists v\sqsubseteq u\text{: } s\,[\varphi]^-_c v \text{ or } \exists v\sqsubseteq u\text{: } s\,[\psi]^-_c v\}$

(vii) $s\,[\varphi\vee\psi]^+_c\,t$ iff $t\in \mu\{u\in U(s,c)\mid\ \exists v\sqsubseteq u\text{: } s\,[\varphi]^+_c v \text{ or } \exists v\sqsubseteq u\text{: } s\,[\psi]^+_c v\}$

$s\,[\varphi\vee\psi]^-_c\,t$ iff $t\in \mu\{u\in U(s,c)\mid\ \exists v\sqsubseteq u\text{: } s\,[\varphi]^-_c v \text{ and } \exists v\sqsubseteq u\text{: } s\,[\psi]^-_c v\}$

(viii) $s\,[\downarrow\varphi]^+_c\,t$ iff there is a t' such that $s\,[\varphi]^+_c\,t'$, and $t=F(\mathbf{t})$ for the unique solution F of the system of equations:

$$\mathbf{t} = t'$$
$$\mathbf{p} = Exp(\varphi,s,c)$$

$s\,[\downarrow\varphi]^-_c\,t$ iff there is a t' such that $s\,[\varphi]^-_c\,t'$, and $t=F(\mathbf{t})$ for the unique solution F of the system of equations:

$$\mathbf{t} = t'$$
$$\mathbf{p} = Exp(\varphi,s,c)$$

Basically, the definition is a dynamic version of the double recursion that has become usual in partial logic. We are mainly interested in the positive update relations, and the negative relations are a technical device that is needed for negation. However, the negative relations do have an intuitive meaning that resembles the meaning of the positive relations: roughly, $s[\varphi]^+_c t$ can be read as "t is the weakest extension of s that covers the information of φ", and $s[\varphi]^-_c t$ can be read as "t is the weakest extension of s that rejects the information of φ".

Before we turn to a detailed discussion of the Liar and other sentences, we discuss some general properties of the update relations. The constraints of Update and Minimality we discussed in the previous section are satisfied:

4 Lemma Let φ be a sentence, s a situation, and c an assignment for φ. Then:

(i) $s\,[\varphi]_c^+\,t \Rightarrow s \sqsubseteq t$ (Update)

(ii) $s\,[\varphi]_c^+\,t \Rightarrow \neg\exists u(s \sqsubseteq u \sqsubset t \wedge s\,[\varphi]_c^+ u)$ (Minimality)

This can be shown with formula induction, where the the corresponding versions of Update and Minimality for the negative relations $[\varphi]_c^-$ have to be proven simultaneously. We will omit proofs as simple as this.

The most important non-properties are:

$s\,[\varphi]_c^+\,t \Rightarrow t\,[\varphi]_c^+\,t$ (Success)

$s\,[\varphi]_c^+\,s \wedge s \sqsubseteq t \Rightarrow t\,[\varphi]_c^+\,t$ (Persistence)

These properties fail for sentences that are context dependent in the following way. In an update of s with Trueφ, a semantical fact with s as a constituent will be added to s. But t $[\text{True}\varphi]_c^+$ t will only hold if the corresponding semantical fact with t instead of s is a member of t. In fact, this form of context dependency can be seen as an instance of the following connection with Barwise and Etchemendy's static set-up:

$s\,[\varphi]_c^+\,t \Rightarrow \{t;\ \text{Type}(\text{Exp}(\varphi,s,c))\}$ is true (Weak success)

So an output t of an update of s with φ will cover the information of φ about s, but not necessarily the information of φ about t. For sentences like ($\underline{a}$ Has $\underline{c}$) everything is 'normal', that is, they are successful and persistent.

Besides Weak success, there are some other important connections between the static and the dynamic semantics. These are given by the next lemmata. The simple proofs are omitted again.

5 Lemma Let φ be a sentence, s a situation, and c an assignment for φ. Then:

(i) $s\,[\varphi]_c^+\,s \Leftrightarrow \text{Exp}(\varphi,s,c)$ is true (Support)

(ii) $s\,[\varphi]_c^-\,s \Leftrightarrow \text{Exp}(\neg\varphi,s,c)$ is true (Refutation)

6 Proposition Statically indiscernible sentences have the same update relation. More precisely, let φ and ψ be sentences, s and t situations, and c an assignment defined for both φ and ψ. Moreover, suppose $\text{Exp}(\varphi,s,c) = \text{Exp}(\psi,s,c)$. Then $s\,[\varphi]_c^+\,t$ iff $s\,[\psi]_c^+\,t$, and $s\,[\varphi]_c^-\,t$ iff $s\,[\psi]_c^-\,t$.

4 Dynamic notions of paradoxality

In this section we will investigate the dynamic behaviour of paradoxical sentences. The dynamic semantics of the previous section will only be interesting if we restrict the update relations to possible situations. We want information to be 'consistent' information, and we are interested in those updates that bring us to information states that are 'consistent' too. Let's introduce some terminology.

7 Definition A sentence is *tangible* if there are possible situations s and t and an assignment c such that $s[\varphi]^{+}_{c}t$; *acceptable* if there is a possible situation s and an assignment c such that $s[\varphi]^{+}_{c}s$.

By lemma 5(i), acceptability comes down to static consistency, whereas tangibility can be seen as dynamic consistency.

Since the paradoxical sentences we deal with in this paper all involve truth, we start our discussion by looking at sentences of the form Trueφ . Let s and t be possible situations, c an assignment, and suppose $s[\mathrm{True}\varphi]^{+}_{c}t$. By definition 3, $t=s\cup\{<Tr,Exp(\varphi,s,c);1>\}$. This means that the information of s is extended with the information that the proposition expressed by φ about s under c is true. Since by assumption t is a possible situation, its semantical information must be correct, that is to say that Exp(φ,s,c) is true. Conversely, suppose s is possible and that Exp(φ,s,c) is true. Then $s\cup\{<Tr,Exp(\varphi,s,c);1>\}$ is a possible situation since s is possible and the additional semantical information given by $<Tr,Exp(\varphi,s,c);1>$ is correct. Summarizing, we see that the dynamic treatment of truth is connected with the static treatment in the following way:

8 Proposition For all sentences φ, possible situations s and assignments c for φ: there is a possible situation t such that $s[\mathrm{True}\varphi]^{+}_{c}t$ if and only if Exp(φ,s,c) is true.

By lemma 5(i) this implies that Trueφ is tangible if and only iff φ is acceptable.

On possible situations the update relation of Trueφ combines 'forward' and 'backward' aspects: the forward aspect is the addition of semantical information, the backward aspect is the test for truth of φ in the antecedent of the update; these are combined in the sense that the forward action can only be carried out if the backward test has a positive outcome. Hence we can call φ a pre-condition of Trueφ.

In the semantics we do not treat "True" in Trueφ very differently from the relation symbol "Has" in ($\underline{\text{Max}}$ Has $\underline{\spadesuit\text{A}}$): in an update of s with ($\underline{\text{Max}}$ Has $\underline{\spadesuit\text{A}}$) we simply add the information given by the state of affairs <H,Max,$\spadesuit$A;1> to s, and in an update of s with Trueφ we add the semantic information <Tr,Exp(φ,s,c);1> to s. The connection between φ and Trueφ is not a direct consequence of the connection between sentences and update relations, but of the coherence conditions that must be observed by possible situations. In definition 3, we could have taken a clause like

$$s\ [\mathrm{True}\varphi]^{+}_{c}\ t \ \text{ iff } \ t = s\cup\{\langle \mathrm{Tr},\mathrm{Exp}(\varphi,s,c);1\rangle\}\ \textit{and}\ \mathrm{Exp}(\varphi,s,c) \text{ is true}$$

as definition of the positive update relation of φ, but in view of proposition 8 above, this gives the same result for possible situations.

By some simple considerations similar to those above, it can be seen that sentences of the form $\neg$Trueφ behave as follows:

9 Proposition For all φ, possible situations s and assignments c for φ: there is a possible situation t such that $s[\neg\mathrm{True}\varphi]^{+}_{c}t$ if and only if Exp(φ,s,c) is not true.

We will now discuss some concrete examples.

The Truthteller: $\downarrow$True($\underline{\text{this}}$). Let s and t be situations. Definition 3 leads to the following calculations. By 3(viii) we must find a t' such that $s\ [\mathrm{True}(\underline{\mathrm{this}})]^{+}_{c}\ t'$ and t'=F(**t**) for the unique solution F of the equations

$$\mathbf{t} = t'$$
$$\mathbf{p} = \mathrm{Exp}(\mathrm{True}(\underline{\mathrm{this}}),s,c)$$

By 3(iv), t'=s$\cup$\{<Tr,**p**;1>\}, and by the static semantics, Exp(True($\underline{\text{this}}$),s,c)= \{s;[Tr,**p**;1]\}. So we have to solve the system of equations given by:

$$\mathbf{t} = s\cup\{\langle \mathrm{Tr},\mathbf{p};1\rangle\}$$
$$\mathbf{p} = \{s;[\mathrm{Tr},\mathbf{p};1]\}$$

If F is the unique solution of these equations, then F(**p**) will be the Truthteller proposition t_S about s, i.e.

$$t_S=\{s;[\mathrm{Tr},t_S;1]\}$$

By the static semantics t_S=Exp($\downarrow\neg$True($\underline{\text{this}}$),s,c). So the t we are after is given by

$$t=F(\mathbf{t})= s\cup\{\langle \mathrm{Tr},\mathrm{Exp}(\downarrow\mathrm{True}(\underline{\mathrm{this}}),s,c);1\rangle\}=s\cup\{\langle \mathrm{Tr},t_S;1\rangle\}$$

If s and t are possible situations something peculiar happens. Since t is a possible situation,

$\mathrm{Exp}(\downarrow\mathrm{True}(\underline{\mathrm{this}}),s,c)$ is true, in which case $<\mathrm{Tr},\mathrm{Exp}(\downarrow\mathrm{True}(\underline{\mathrm{this}}),s,c);1> \in s$. But then t=s!

10 Definition A sentence φ has the property of *pre-conditional success* if for all c and all possible s,t: $s[\varphi]^{\dagger}_{c}t$ implies $s[\varphi]^{\dagger}_{c}s$.

Other examples of sentences with this property are all sentences of the form $\downarrow(\varphi\wedge\mathrm{True}(\underline{\mathrm{this}}))$. It turns out that pre-conditional success has a simple characterization.

11 Proposition A sentence φ has pre-conditional success if and only if $\varphi\wedge\neg\mathrm{True}\varphi$ is intangible.

Proof: straightforward.□

The Liar: $\downarrow\neg\mathrm{True}(\underline{\mathrm{this}})$. Calculations similar to those for the Truthteller give us

(*) $s[\downarrow\neg\mathrm{True}(\underline{\mathrm{this}})]^{\dagger}_{c}t$ iff $t = s \cup \{<\mathrm{Tr},\mathrm{Exp}(\downarrow\neg\mathrm{True}(\underline{\mathrm{this}}),s,c);0>\}$

But can there be possible situations that stand in this update relation? The answer is a definite YES. If s is a possible situation, then we know from the static semantics that the proposition $\mathrm{Exp}(\downarrow\neg\mathrm{True}(\underline{\mathrm{this}}),s,c)$ is not true. But then the situation $s\cup\{<\mathrm{Tr},\mathrm{Exp}(\downarrow\neg\mathrm{True}(\underline{\mathrm{this}}),s,c);0>\}$ is a possible situation, since s is possible and the additional semantic information is correct. Since s was arbitrary, this means that *every possible situation can be updated with the Liar*. Moreover, such an update must be insuccessful in the sense that it cannot bring you in a state t such that $t[\downarrow\neg\mathrm{True}(\underline{\mathrm{this}})]^{\dagger}_{c}t$, for this would imply by lemma 5(i) that $\mathrm{Exp}(\downarrow\neg\mathrm{True}(\underline{\mathrm{this}}),t,c)$ is true, which cannot be if t is a possible situation. So *no update with the Liar can bring you to a state in which it is accepted.*

12 Definition A sentence φ is *anti-succesful* if $s[\varphi]^{\dagger}_{c}t$ implies not $t[\varphi]^{\dagger}_{c}t$, for all possible situations s,t and assignments c.

More in line with the terminology of definition 10 we can also call this property *post-conditional failure*. Besides the Liar, all instances of the schemata $\varphi\wedge\neg\mathrm{True}\varphi$ and $\downarrow(\varphi\wedge\neg\mathrm{True}(\underline{\mathrm{this}}))$ are anti-successful. There is a simple characterization of anti-success in terms of static consistency.

13 Proposition φ is anti-succesful if and only if φ is unacceptable.

Proof: straightforward.□

The intrinsic sentence: next consider the sentence $\downarrow$(True(<u>this</u>)$\vee\neg$True(<u>this</u>)), the double wide scope reading of

This proposition is true or this proposition is not true

Abbreviate the formula by ι. Barwise and Etchemendy's axiom 4 tells us that ι is indiscernible from True $\iota \vee\neg$True ι.[14] The next proposition tells it all.

14 Proposition Every possible situation can be updated with every sentence of the form (True$\varphi\vee\neg$Trueφ). However, depending on φ, there need not be a situation in which (True$\varphi\vee\neg$Trueφ) is accepted.

Proof: The first claim follows from propositions **8** and **9**. The last claim follows directly from the fact that, by the static semantics, both the Liar and its negation are necessarily false; moreover (writing λ for $\downarrow\neg$True(<u>this</u>)) λ is indiscernible from $\neg$Trueλ, while $\neg\lambda$ is indiscernible from Trueλ.□

So we can always have an update with the intrinsic sentence. Also, by the static semantics, there are possible situations which accept it. In fact, for possible situations s, Exp(ι,s) is true if and only if Exp(True ι,s) is true, which, by the observed indiscernibility of ι and (True $\iota \vee\neg$True ι), implies that $\neg$True ι is necessarily false. There is no such preference for the first disjunct of (True $\iota\vee\neg$True ι) in the dynamic case: in any situation that contains no semantical information, ι is not true, hence an update with $\neg$True ι is possible. The choice is not open, however, and is fully determined by the input of the update: if ι is true of s then the update of s with ι is the update of s with True ι, and if ι is not true of s the update will 'choose' $\neg$True ι. As far as the negation $\neg\iota$ is concerned, it is both unacceptable and intangible, which follows from the fact that $\neg\iota$ is indiscernible from True $\iota \wedge\neg$True ι.

Contingent paradoxes: Suppose φ is a closed sentence. Then the sentence of the form $\downarrow(\varphi\wedge\neg$True(<u>this</u>)) behaves much like the Liar, since it is anti-successful. It is tangible if and only if φ is tangible and φ does not have the property of preconditional success. The difference with the Liar is that, depending on φ, it need not be the case that every possible situation has an update, for if $\neg\varphi$ is acceptable, a state that accepts $\neg\varphi$ cannot have an update with $\downarrow(\varphi\wedge\neg$True(<u>this</u>)).

Sentences of the form $\downarrow(\varphi\vee\neg$True(<u>this</u>)) can also have Liar-like effects. Let α be $\downarrow(\varphi\vee\neg$True(<u>this</u>)). By the static semantics, α is indiscernible form $\varphi\vee\neg$Trueα. If φ is intangible, then both φ and α are unacceptable. From this it easily follows that α is anti-successful, and that every possible situation has an update with α (which will be an update with the 'disjunct' $\neg$Trueα). If φ is tangible the behaviour is as follows. If φ is true in s, then s[φ]s, in which case s[α]s and by update and minimality s will be the only output. If $\neg\varphi$ is true in s, the only output for an update of s with α is $s \cup \{<Tr, Exp(\alpha,s); 0>\}$. If neither φ nor $\neg\varphi$ is true in s, then an update of s with α can 'choose' any of the disjuncts.

We conclude this section with some logical issues. Technical details and proofs can be found in the appendix. In what follows, we consider the language without demonstratives, so assignments are irrelevant. Barwise and Etchemendy give a deductive characterization of the following relation of indiscernibility:

(I) for all s, Exp(φ,s) = Exp(ψ,s)

They construct a deductive system that is sound and complete with respect to indiscernibility; moreover indiscernibility is shown to be a decidable relation.[15] Notice that indiscernibility can be seen as an 'intensional' relation, since sentences with the same truth conditions need not express the same proposition. So, another interesting relation is the relation of static equivalence given by

(S) for all possible situations s, Exp(φ,s) is true iff Exp(ψ,s) is true

Moreover, the dynamic semantics naturally gives rise to the following relation of dynamic equivalence:

(D) for all possible situations s,t, $s[\varphi]^+t$ iff $s[\psi]^+t$

It is possible to give sound and complete proof theoretic characterizations of both (S) and (D). Moreover, both relations are decidable (see appendix). The three notions of equivalence are interrelated as follows:

$$I \subseteq D \subseteq S$$

All inclusions are proper: $\varphi \wedge \varphi$ and φ are not indiscernible, but dynamically equivalent; the Liar and its negation are statically equivalent, but not dynamically. The decidability of static and dynamic equivalence has some interesting corollaries, for it turns out that many semantic properties we have discussed above are decidable. That is, the following properties of sentences are all decidable:

φ is tangible	φ is anti-successful
φ is acceptable	φ has pre-conditional success

This can be seen as follows. Let $\perp$ be short for (Max Has ♠A)$\wedge\neg$(Max Has ♠A). Then φ is tangible if and only if φ is not dynamically equivalent with $\perp$; and the latter is decidable. φ is acceptable if and only if φ is not statically equivalent with $\perp$. Decidability of anti-success now follows from proposition **13** and the decidability of acceptability. Decidability of pre-conditional success follows from proposition **11** and the decidability of tangibility. Whether success is also a decidable property is still open (the conjecture being: decidable).

5 Discourses

One of the main achievements of dynamic semantics, if not its 'raison d'etre', has been its ability to account for the semantic structure of discourses, in particular anaphoric structure. In this section we develop a semantics for texts with 'propositional anaphors'. We conceive of a discourse as a sequence of sentences, and develop a conception of a reading of a discourse, in such a way that the propositional demonstratives $\underline{\text{that}_i}$ of our formal language get the force of "the proposition expressed by the i-th sentence of this sequence". Our main objective for doing so is to be able to give a description of texts with *circular* cross-reference.

15 Definition A discourse is a finite sequence of sentences. Notation: $D=\varphi_1;...;\varphi_n$

Our notion of a reading of a discourse is governed by the following idea. Reflecting on what happens if someone reads a story, we can say that the result of reading is a sequence of pictures. The next-picture relation corresponds with the effect of processing a sentence of the text. So we can concieve of a reading of a discourse as a sequence of situations produced by a sequence of updates.

We will exploit the fact that the formal language we are working with contains propositional demonstratives by allowing these demonstratives to be linked to sentences in the discourse. This is for the same reason why Barwise and Etchemendy included them in the language, namely in order to be able to analyze those semantical paradoxes that consist of several sentences that refer to each other.

16 Definition A discourse $D=\varphi_1;...;\varphi_n$ is closed if $i\leq n$ for all $\underline{\text{that}_i}$ occurring in D.

17 Definition A reading of a closed discourse $D=\varphi_1;...;\varphi_n$ consists of a sequence of contexts $<s_1,c_1>;...;<s_{n+1},c_{n+1}>$ such that:

(i) all s_i are possible situations, for $1 \le i \le n$

(ii) $s_i\ [\varphi_i]_{c_i}\ s_{i+1}$, for $1 \le i \le n$

(iii) $c_i(\underline{that}_j)=Exp(\varphi_j,s_i,c_i)$, for all $1 \le i,j \le n$

Clauses (ii) reflects the sequential nature of a reading of a discourse. Clause (iii) fixes the interpretation of propositional demonstratives in a discourse. The idea behind the clause is that the interpretation of demonstratives is governed by 'paging' through the text. Suppose you are in state s_8 and and you are about to read φ_8, but φ_8 turns out to have an occurrence of $\underline{that}_{13}$. Clause (iii) tells you that you that you must interpret $\underline{that}_{13}$ as φ_{13} *in your current state.* This proposal seems to give a correct account of what happens when you read a text and hit upon an expression of the form "the 13-th sentence of this text".[16]

One of the consequences of this procedure is that the interpretation of a demonstrative in a discourse will not be uniform, since different occurrences of the demonstrative may refer to different propositions. But in this respect demonstratives do not behave differently from sentences, because multiple occurrences of context-dependent sentences will also express different propositions.

Of course the condition of clause (iii) is circular, but, as always in Aczel's set theory, the Solution Lemma comes to the rescue.

18 Lemma If $D=\varphi_1;...;\varphi_n$ is a closed discourse and s a situation, then there is an assignment c such that $c(\underline{that}_i)=Exp(\varphi_i,s,c)$ for all $1 \le i \le n$. This assignment is unique in its values relevant for demonstratives in D, that is: if $d(\underline{that}_i)=Exp(\varphi_i,s,d)$ for all $1 \le i \le n$, then $c(\underline{that}_i)=d(\underline{that}_i)$ for all $1 \le i \le n$.

Proof: Use the solution lemma to obtain the unique solution of the following system of equations in the indeterminates $\mathbf{q_1},...,\mathbf{q_n}$:

$$\mathbf{q_1}=Val(\varphi_1)(s,\mathbf{q_1},...,\mathbf{q_n})$$

$$...$$

$$\mathbf{q_n}=Val(\varphi_n)(s,\mathbf{q_1},...,\mathbf{q_n})$$

Let F be the solution, and define c by $c(\underline{that}_i)=F(\mathbf{q_i})$ for $1 \le i \le n$, and undefined otherwise. Then $c(\underline{that}_i) = F(\mathbf{q_i}) = Val(\varphi_i)(s,F(\mathbf{q_1}),...,F(\mathbf{q_n})) = Val(\varphi_i)(s,c(\underline{that}_1),...,c(\underline{that}_n)) = Exp(\varphi_i,s,c)$, where the last identity follows from the fact that φ_i can contain no other demonstratives than the ones shown, since D is closed. Now any assignment d satisfying

$d(\underline{that_i})=Exp(\varphi_i,s,d)$ for all $1 \leq i \leq n$ determines a solution of the above system of equations. By the solution lemma, solutions are unique, so for the relevant values we must have $c(\underline{that_i})=d(\underline{that_i})$.$\square$

What happens if some φ_j in D contains $\underline{that_j}$, for example in the case that φ_j *is* $\neg True(\underline{that_j})$? Then for any i, $c_i(\underline{that_j}) = Exp(\neg True(\underline{that_j}),s_i,c_i) = \{s_i;[Tr,c_i(\underline{that_j});0]\}$, i.e. $c_i(\underline{that_j})$ is the Liar proposition f_{s_j} about s_j! So in this case we could substitute $\downarrow\neg True(\underline{this})$ for φ_j without changing the reading of the discourse.

What does the semantics of discourses have to say about Liar cycles, for example the discourse

(LC1) $True(\underline{that_2}); \neg True(\underline{that_1})$

Does it have any readings? If a sequence of contexts $<s_1,c_1>;<s_2,c_2>;<s_3,c_3>$ is a reading, then by definition 17, the following conditions must obtain:

(i) $s_2=s_1 \cup \{<Tr,c_1(\underline{that_2});1>\}$

(ii) $s_3=s_2 \cup \{<Tr,c_2(\underline{that_1});0>\}$

(iii) $c_1(\underline{that_1}) = Exp(True(\underline{that_2}),s_1,c_1) = \{s_1;[Tr,c_1(\underline{that_2});1]\}$

(iv) $c_1(\underline{that_2}) = Exp(\neg True(\underline{that_1}),s_1,c_1) = \{s_1;[Tr,c_1(\underline{that_1});0]\}$

(v) $c_2(\underline{that_1}) = Exp(True(\underline{that_2}),s_2,c_2) = \{s_2;[Tr,c_2(\underline{that_2});1]\}$

(vi) $c_2(\underline{that_2}) = Exp(\neg True(\underline{that_1}),s_2,c_2) = \{s_2;[Tr,c_2(\underline{that_1});0]\}$

Moreover, s_1, s_2 and s_3 all have to be possible situations. So we must have:

(vii)	$<Tr,c_1(\underline{that_1});0> \in s_1$	by (i) and (iv)
(viii)	$<Tr,c_1(\underline{that_2});1> \notin s_1$	by (vii) and (iii)
(ix)	$<Tr,c_2(\underline{that_2});1> \notin s_2$	by (ii) and (v)

With the help of the Solution Lemma it is not hard to construct possible situations that meet these conditions. So the discourse has readings.

But what do the conditions mean? Condition (vii) means that in any successful reading of the cycle, your initial information s_1 must already contain the semantic information that the proposition expressed by the first sentence is not true. So suppose s_1 is a possible situation satisfying (vii). By combining (iii) and (iv) we see that the first sentence expresses that it is *true* that the first sentence expresses a proposition that is not true:

$$c_1(\underline{that_1}) = \{s_1;[Tr,\{s_1;[Tr,c_1(\underline{that_1});0]\};1]\}$$

But since you already believe $c_1(\underline{that_1})$ (i.e. the proposition expressed by the first sentence about s_1) not to be true, you can consistently add this additional semantic information. Hence

s_2 (as in (i)) is also a possible situation. Now the second sentence claims that in your current state (i.e. s_2) the first sentence still expresses a proposition that is not true. Intuitively, this is correct, since this is what you initially believed and has been acknowledged by the first sentence. Formally, there is an itch: it might be so that in s_1 you believe that the first sentence expresses a false proposition of s_1, but you also believe that once you will be in s_2 the first sentence will be true. This predicament is described by

(x) $\langle Tr,Exp(True(\underline{that_2}),s_2,c_2);1\rangle \in s_1$

But given that s_1 and s_2 are possible situations, you cannot have this information, since by (v) and (vi), (x) implies that $Exp(True(\underline{that_2}),s_2,c_2)$ is not true.

Summarizing, we see that a possible situation s_1 is the initial state of a reading of the Liar Cycle if and only if it contains the semantical information that the proposition expressed by the first sentence of the cycle is not true. So although the cycle has readings, it has no unbiased reading, since the initial state of a reading must contain the 'semantic prejudice' that the first sentence is not true.

In Barwise and Etchemendy's treatment of the Liar Cycle the formulas $True(\underline{that_2})$ and $\neg True(\underline{that_1})$ are used to make statements about the same situation s, and $\underline{that_2}$ is taken to refer to $Exp(\neg True(\underline{that_1}),s)$ and vice versa. The results are: $Exp(True(\underline{that_2}),s)$ is not true if s is a possible situation; there are possible situations s such that $Exp(\neg True(\underline{that_1}),s)$ is true, but if s is T-closed for expressible propositions, then $Exp(\neg True(\underline{that_1}),s)$ is not true.[17] The important difference with our analysis of the Liar cycle is not so much the result as the fact that we treat it as a sequence. The problem described by Barwise and Etchemendy involves two speakers who both make a claim about the same situation; the problem described here involves a text and the changes of information it induces upon the reader. These are different problems, so a comparison of the outcomes seems rather senseless (but see below).

As a second example we take a contingent Liar cycle:

$\underline{Max}$ Has $\underline{♠A}$; $True(\underline{that_3})$; $\neg True(\underline{that_1}) \vee \neg True(\underline{that_2})$

This discourse has readings. We will not spell out all details, but give the main steps. Suppose you are in state s_1, and (a) you do not have the information that Max does not have the ace of spades ($\langle H,Max,♠A;0\rangle \notin s_1$); moreover (b) you believe in s_1 that once you have processed the first sentence (and are in the state $s_2=s_1\cup\{\langle H,Max,♠A;1\rangle\}$), the proposition expressed by the second sentence about s_2 is not true (i.e. $\langle Tr,Exp(True(\underline{that_3}),s_2,c_2);0\rangle \in s_1$).

Step 1: By (a) it is possible to carry out the update of s_1 with (Max Has ♠A) and arrive in s_2.

Step 2: So suppose you are in s_2. Now the second sentence asserts that the third sentence expresses a true proposition about s_2. The first disjunct of the third sentence is out, for it expresses the proposition that the first sentence is not true of s_2, but $\langle H,Max,♠A;1\rangle \in s_2$. On the other hand, the second disjunct of the third sentence expresses a proposition about s_2 that is true: $Exp(\neg True(\underline{that_2}),s_2,c_2)$ is true since $\underline{that_2}$ refers to $Exp(True(\underline{that_3}),s_2,c_2)$ and by (b), $\langle Tr,Exp(True(\underline{that_3}),s_2,c_2);0\rangle \in s_2$. Hence we can add the semantical information that the third sentence is true to s_2. So we arrive in $s_3=s_2\cup\{\langle Tr,Exp(\neg True(\underline{that_1}) \vee \neg True(\underline{that_2})),s_2,c_2);1\rangle\}$.

Step 3: As before the first disjunct of the third sentence is still out, so the question is whether or not $s_4=s_3\cup\{\langle Tr,Exp(True(\underline{that_3}),s_3,c_3);0\rangle\}$ is a possible situation. Suppose not; then $Exp(True(\underline{that_3}),s_3,c_3)$ is true, so $\langle Tr,c_3(\underline{that_3});1\rangle \in s_3$. Then $c_3(\underline{that_3})$ is true, but $c_3(\underline{that_3})$ is $Exp(\neg True(\underline{that_1}) \vee \neg True(\underline{that_2}),s_3,c_3)$; since the first disjunct is out it follows that $Exp(\neg True(\underline{that_2}),s_3,c_3)$ is true, so $\langle Tr,c_3(\underline{that_2});0\rangle \in s_3$. So $c_3(\underline{that_2})$ is not true, but $c_3(\underline{that_2})$ is $Exp(True(\underline{that_3}),s_3,c_3)$, so $\langle Tr,c_3(\underline{that_3});1\rangle \notin s_3$, hence $Exp(True(\underline{that_3}),s_3,c_3)$ is not true which contradicts our initial assumption. So $s_3\cup\{\langle Tr,Exp(True(\underline{that_3}),s_3,c_3);0\rangle\}$ is a possible situation.

Both examples have no unbiased readings, that is, the initial state of a reading cannot be the empty situation. In this respect, readings of discourses are comparable to updates of single sentences, since the input of an update of Trueφ cannot be the empty situation either. There are more similarities. For example, we can call a discourse acceptable if it has a reading in which the first and the last (hence all) situations are the same. For example, the Liar cycle is unacceptable. This follows immediately from Barwise and Etchemendy's static analysis of the cycle. In effect, a discourse is acceptable if and only if it is consistent in the static analysis. So in a sense the static versions are special cases of the dynamic versions.

The final part of this section deals with manipulations of demonstratives in a discourse. In the informal discussion on the concept of a reading of a discourse, we decided to treat a demonstrative $\underline{that_i}$ in a discourse as having the force of "the i-th sentence of this text". We show that our formalization is correct in this respect: substitution of the i-th sentence of a discourse for some occurrence of $\underline{that_i}$ doesn't change the descriptive content of the discourse.

19 Definition If $<s_1,c_1>;...;<s_{n+1},c_{n+1}>$ is a reading of the discourse $\varphi_1;...;\varphi_n$, then $s_1;...;s_{n+1}$ is a trace of $\varphi_1,...,\varphi_n$. Two discourses are strongly equivalent if they have the same traces.

So we abstract from the contribution of the demonstratives, and focus on the descriptive content of a discourse, which can be seen as a labeled graph in logical space. Several weaker notions of equivalence are of interest; for example, we could also abstract from the 'stylistic' features of the discourse and only consider the input-output behaviour of the discourse as a whole. But we will not pursue this here.

20 Citation principle Let $D=\varphi_1;...;\varphi_n$ be a discourse in which some φ_i has an occurrence of $\underline{\text{that}}_j$, where $j\leq n$. Let the discourse E be the result of substituting $\varphi_i(\varphi_j/\underline{\text{that}}_j)$ for φ_i in D, where $\varphi_i(\varphi_j/\underline{\text{that}}_j)$ is the result of substituting φ_j for one or more occurrences of $\underline{\text{that}}_j$ in φ_i. Then D and E are strongly equivalent.

Proof: let $<s_1,c_1>;...;<s_{n+1},c_{n+1}>$ be a reading of D. Use the fact that $c_i(\underline{\text{that}}_j)=\text{Exp}(\varphi_j,s_i,c_i)$ to prove with induction on the complexity of φ_i that $\text{Exp}(\varphi_i,s_i,c_i)=\text{Exp}(\varphi_i(\varphi_j/\underline{\text{that}}_j),s_i,c_i)$. Conclude that $<s_1,c_1>;...;<s_{n+1},c_{n+1}>$ is also a reading of E. Use the same argument for the converse. □

We can even do better: in some cases, we can eliminate demonstratives altogether.

21 Definition A discourse $D=\varphi_1;...;\varphi_n$ is well-founded if its referential structure is well-founded (that is, the relation R_D defined as
$\{<i,j>|\ i,j\leq n \text{ and } \underline{\text{that}}_j \text{ occurs in } \varphi_i\}$ is conversely well-founded).

22 Elimination principle Every closed and well-founded discourse is strongly equivalent with a discourse that doesn't contain demonstratives.

Proof: since the referential structure R_D is conversely well-founded, there is a pair $<i,j>\in R_D$ such that for no k, $<j,k>\in R_D$. What this means is that $\underline{\text{that}}_j$ occurs in φ_i, and that φ_j doesn't contain demonstratives. We now substitute φ_j for every occurrence of $\underline{\text{that}}_j$ in the discourse. By the citation principle we obtain a strongly equivalent discourse D'. Moreover, D' has no occurrences of $\underline{\text{that}}_j$ anymore, and it is still a well-founded and closed discourse. By repeating this procedure we can get rid of all demonstratives.□

Well-foundedness is not a necessary condition. In some cases we can replace circular reference via propositional demonstratives with circular reference via a propositional reflexive. The following proposition is a generalization of the remarks immediately following lemma 18.

23 Non-well-founded elimination Suppose $D=\varphi_1;\ldots;\varphi_n$ is a discourse, and for some i ($1\leq i\leq n$), (a) all occurrences of $\underline{\text{that}_i}$ in φ_i are not within the scope of any $\downarrow$, or (b) φ_i is of the form $\downarrow\psi$ and ψ has no occurrences of $\downarrow$. Define $\varphi_i'=\downarrow\varphi_i(\underline{\text{this}}/\underline{\text{that}_i})$ if (a) holds, $\varphi_i'=\downarrow\psi(\underline{\text{this}}/\underline{\text{that}_i})$ if (b) holds. The discourse obtained by substitution of φ_i' for φ_i in D is strongly equivalent with D.

Proof: omitted.□ [18]

In general, citation and elimination will increase the average length of the sentences in the discourse. This can be illustrated by the Contingent Liar cycle:

$$\underline{\text{Max}}\text{ Has }\underline{\spadesuit\text{A}};\ \text{True}(\underline{\text{that}_3});\ \neg\text{True}(\underline{\text{that}_1}) \vee \neg\text{True}(\underline{\text{that}_2})$$

Two applications of the citation principle, followed by one application of non-well-founded elimination and one more citation, give the strongly equivalent discourse: [19]

$$\underline{\text{Max}}\text{ Has }\underline{\spadesuit\text{A}};\ \text{True}(\downarrow(\neg\text{True}(\underline{\text{Max}}\text{ Has }\underline{\spadesuit\text{A}}) \vee \neg\text{True}(\text{True}(\underline{\text{this}}))));$$
$$\downarrow(\neg\text{True}(\underline{\text{Max}}\text{ Has }\underline{\spadesuit\text{A}}) \vee \neg\text{True}(\text{True}(\underline{\text{this}})))$$

The moral to be drawn is a platitude: cross-references in a text allow for a more concise presentation. The effect of citation and elimination is that global computational procedures ('paging' back and forth in order to link demonstratives to sentences) are replaced by longer local procedures (increase of sentence length).

6 Problems and prospects

As it stands, the approach developed in this paper is an extension of Barwise and Etchemendy's Austinian framework. The dynamic semantics formalizes the idea that propositions expressed by sentences like the Liar bring about a change of information. A theoretical pay-off was that we were able to construct a semantics for texts with circular cross-reference. Of course, the Austinian semantics deserves credit for providing the background in which the dynamic semantics could be developed. On the other hand the dependence has two disadvantages. First, it makes the dynamic approach vulnerable to criticism that might be launched against the Austinian set-up. Second, it is not clear to which extent the results depend on the static semantics and the ontology, rather than on the dynamic semantics that is built on top of it; thus it is hard to estimate the value of the dynamic approach. Here I will not attempt to treat these two issues in full detail, but instead I will briefly discuss some important questions and problems.

(1) The logic is too weak. One argument against Barwise and Etchemendy's implementation of the Austinian approach is that the logic is too weak. The reason for this is the fact that the Austinian propositions have too much syntactical structure. For example, a sentence of the form $(\alpha\wedge\alpha)$ has the same truth conditions as α but expresses a different proposition. Hence True$(\alpha\wedge\alpha)$ and Trueα do not have the same truth conditions. By contrast, $\varphi\wedge\psi$ and $\neg(\neg\varphi\vee\neg\psi)$ do express the same proposition and thus have the same truth conditions. I see no good reason why de Morgan's Law is more fundamental than idempotency of conjunction. The distinction in behaviour is a consequence of an accidental choice of modeling techniques (conjunction and disjunction signs become 'constituents' of the types of the propositions, but negation is a defined operation). The price to be paid is that the following representation principle does not hold:

(RP) *if φ and ψ have the same truth conditions and the same falsity conditions, then so do Trueφ and Trueψ*

(where the falsity conditions of a formula are the truth conditions of its negation). Given the analytic task of providing an account of truth and self-reference, this is too high a price. It is to be hoped that it is possible to equip the Austinian account with a different notion of proposition in such a way that (RP) holds. Notice that I do not claim that propositions should be exhausted by truth conditions. The familiar substitutivity puzzles in intensional contexts suggest they should not. I do claim that non-truth-conditional distinctions between two sentences φ and ψ cannot induce truth-conditional distinctions between Trueφ and Trueψ. But this is not to deny that there are sentential operators (such as "John believes that ...") which behave differently in this respect.

(2) Non-well-foundedness. Those readers who feel uneasy about non-well-founded sets should notice the following. Working in ZFC$^-$ (i.e. ZFC without the axiom of foundation) Aczel constructs an inner model of ZFC$^-$ + the axiom of anti-foundation (Aczel[1988], chapter 3). Trivially, every model of ZFC is a model of ZFC$^-$. This means that we can translate talk of non-well-founded sets into talk of well-founded sets (of course, the 'translation' of the non-well-founded $\in$ will not be the well-founded $\in$). So in principle it is possible to do without

non-well-founded sets. It is to be expected, however, that the set-theoretical coding would rather obscure than clarify matters. The attractive aspect of the theory of non-well-founded sets is that it provides an elegant mathematical tool for modeling self-reference. Compared to Gödel numbering this modeling is more direct and less artificial.

So the important question is not whether Barwise and Etchemendy really need non-well-founded sets, but whether or not the sort of circularity they model by them really exists. There are at least two basic claims of their theory of truth that imply non-well-foundedness: (1) circular propositions exist; (2) situations can support semantical information about any situation whatsoever, in particular about itself, or even about a larger situation. Gupta has argued that (1) is not essential for the Austinian semantics.[20] But (2) is crucial, especially in connection with the idea that the truth of a statement depends on the situation the statement is about. Much of the results of the dynamic semantics developed in this paper depend on (2) and the use of 'context-dependent' semantical facts for representing truth. For example, in a dynamic version of Barwise and Etchemendy's Russellian semantics the Liar is simply inconsistent (i.e., it has an empty update relation).

(3) Semantical facts. The representation of truth by semantical facts also distinguishes Barwise and Etchemendy's approach from other theories of truth and self-reference, e.g. those of Kripke, Gupta and Herzberger.[21] A sentence Trueφ is true in a situation s iff the semantical fact $\langle Tr, Exp(\varphi, s); 1\rangle$ is a member of s. This semantical fact is of the form $\langle Tr, \{s; X\}; 1\rangle$. What is represented is not the truth of a sentence but the truth of a statement consisting of a sentence and a described situation. But in Kripke's theory the evaluation clause for sentences of the form Trueφ is basically: $\langle D,T,F\rangle \models \text{True}\varphi$ iff $\varphi \in T$ (where T is the extension and F the anti-extension of the truth predicate). What is represented by Kripke's scheme is truth of a sentence.

I favour Barwise and Etchemendy's scheme of representation. The reason is that I think that partiality is a central feature of most occasions on which we use natural language. In partial semantics the basic notion of truth will be "truth on the basis of the available evidence" (Veltman [1985], p.155). So truth is a relation between available evidence s and a sentence φ. If we want to reflect this relation internally, then both relata will be significant for the representation. This does not force us to adopt all details of Barwise and Etchemendy's Austinian framework. But it does point in the direction of a representation of truth that employs constructions that look a lot like Barwise and Etchemendy's semantical facts.

(4) Revisions and updates. The dynamic semantics of this paper describes a process, and so do the theories of Kripke, Gupta and Herzberger. But the processes are of a different kind.

An update s[φ]t is intended to describe the change of information of someone who has initial information s, accepts the message of φ, and so arrives at a new information state t. The process of the KGH approaches describes a 'revision' of the extension and anti-extension of the truth predicate, and, moreover, in a jump from a model to the successor model in the sequence *all* sentences are considered. The basic goal of the process is that at some stage a model is obtained with a fairly strong representation of truth.

Granted the different nature of the processes, it might seem that we can mimic the revision process by updates. The important observation is that whenever s is a possible situation, Exp(φ,s) is true if and only if there is a unique possible situation t such that s[Trueφ]t, namely t=s∪{<Tr,Exp(φ,s);1>}; likewise, if Exp(φ,s) is not true, the situation t=s∪{<Tr,Exp(φ,s);0>} is the unique possible situation t such that s[¬Trueφ]t. So maybe we could simulate a revision jump by two parallel simultaneous updates:

$$t\,(s) = \cup\{t \mid t \text{ is possible and } \exists\varphi\text{: } s[\text{True}\varphi]t\}$$

$$n\,(s) = \cup\{t \mid t \text{ is possible and } \exists\varphi\text{: } s[\neg\text{True}\varphi]t\}$$

$$\phi\,(s) = t\,(s) \cup n\,(s)$$

But this proposal fails to achieve its goal. Though the revision ϕ (s) is a possible situation whenever s is, it does not satisfy:

(T) Exp(φ,s) is true $\Rightarrow$ Exp(Trueφ, ϕ (s)) is true

(N) Exp(φ,s) is not true $\Rightarrow$ Exp(¬Trueφ, ϕ (s)) is true

It is possible to construct an operation that satisfies the first clause (not so for the second) but the connection with updates would be lost.

In conclusion, I do not think that the form of the dynamic semantics of this paper is final. It was motivated by some remarks of Barwise and Etchemendy and initially I thought that I was just formalizing these ideas, thus obtaining an extension of the Austinian semantics. However, I've come to think that the dynamic approach can stand on its own, and should not be made all-dependent on the peculiarities of Barwise and Etchemendy's Austinian set-up. Under this perspective an improvement of the theory could as well be a revision of the Austinian set-up as a choice for a different framework. But I'm not yet sure what is the right way to go.

Appendix: Logical issues

The following notational conventions are useful: "$s\models\varphi$" abbreviates "Exp(φ,s) is true"; "$\forall_p$st" abbreviates "for all possible situations s and t". Other symbols used in the meta language have their usual meaning.

What we are after then, is a characterization of the following two notions of equivalence:

(D) $\forall_p st(\ s[\varphi]^+t \leftrightarrow s[\psi]^+t)$

(S) $\forall_p st(\ s\models\varphi \leftrightarrow s\models\varphi)$

Here we take φ to range over sentences that don't contain demonstratives. As usual, we do not characterize these equivalence relations directly, but instead characterize the corresponding notions of implication. For static equivalence (S) this gives:

(s) $\forall_p st(\ s\models\varphi \rightarrow s\models\varphi)$

For dynamic equivalence (D) this gives

(*) $\forall_p st(\ s[\varphi]t \rightarrow s[\psi]t)$

(We will drop the superscript $^+$ from now on). But this notion is a bit hard to handle. The reason is that (*) requires an exact match between the information change induced by φ and the change induced by ψ. For example, a rule for conjunction like $\varphi\wedge\psi\Rightarrow\varphi$ fails for (*), since in general an update with φ will require a smaller expansion than an update of $\varphi\wedge\psi$. We could use the rule $\varphi\wedge\psi\Rightarrow(\varphi\wedge\psi)\wedge\varphi$, which is correct for (*). In effect, all rules for (*) would have a copy of the antecedent in the consequent. This can be avoided by taking the following observation at face value: the gist of $\varphi\wedge\psi\Rightarrow(\varphi\wedge\psi)\wedge\varphi$ is that "every $\varphi\wedge\psi$-jump covers a φ-jump". This gives as notion of implication:

(d) $\forall_p st(\ s[\varphi]t \rightarrow \exists t'\sqsubseteq t\text{: } s[\psi]t'\ )$

The formal justification of this notion is that the equivalence corresponding to (d) is precisely (D). We omit the simple proof of this fact (hint: use minimality of updates).

A.1 Definition Let Γ and Δ be non-empty sets of sentences.

(i) $\Gamma\models_S\Delta$ iff there are sentences $\varphi_1,\ldots,\varphi_n\in\Gamma$ and $\psi_1,\ldots,\psi_m\in\Delta$ such that
$\forall_p s(\ s\models\varphi_1\wedge\ldots\wedge\varphi_n \Rightarrow\ s\models\psi_1\vee\ldots\vee\psi_m)$

(ii) $\Gamma\models_D\Delta$ iff there are sentences $\varphi_1,\ldots,\varphi_n\in\Gamma$ and $\psi_1,\ldots,\psi_m\in\Delta$ such that
$\forall_p st(\ s[\varphi_1\wedge\ldots\wedge\varphi_n]t \Rightarrow \exists t'\sqsubseteq t\text{: } s[\psi_1\vee\ldots\vee\psi_m]t')$

Barwise and Etchemendy give a synatactic characterization of indiscernibility, the equivalence relation given by

(I) for all s, $Exp(\varphi,s)=Exp(\psi,s)$

The system has has the following axioms and rules:

(A1) $\varphi\Leftrightarrow\neg\neg\varphi$

(A2) $\neg(\varphi\wedge\psi)\Leftrightarrow(\neg\varphi\vee\neg\psi)$

(A3) $\neg(\varphi\vee\psi)\Leftrightarrow(\neg\varphi\wedge\neg\psi)$

(A4) $\downarrow\varphi\Leftrightarrow\varphi(\underline{this}/\downarrow\varphi)$

(R1) if $\varphi\Leftrightarrow\psi$ then $\psi\Leftrightarrow\varphi$

(R2) if $\varphi\Leftrightarrow\psi$ and $\psi\Leftrightarrow\chi$ then $\varphi\Leftrightarrow\chi$

(R3) Substitution: if $\varphi\Leftrightarrow\psi$ then $\chi \Leftrightarrow \chi\ [\varphi/\psi]$

(R4) Identity of Indiscernibles

where in (A4) $\varphi(\underline{this}/\downarrow\varphi)$ is the result of substituting $\downarrow\varphi$ for all loose occurrences of <u>this</u> in φ; and in (R3) $\chi\ [\varphi/\psi]$ is the result of substituting ψ for one or more occurrences of φ in χ. It would take up too much space to explain the rule Identity of Indiscernibles (vide BE 111-112). Barwise and Etchemendy also show that indiscernibility is decidable. We will be lazy here and simply make one rule which inputs all equivalences $\varphi\Leftrightarrow\psi$ that are correct for (I). We use "$\mathrm{I}\vdash \varphi\Leftrightarrow\psi$" as an abbreviation for "$\varphi\Leftrightarrow\psi$ is provable in Barwise and Etchemendy's deductive system".

We present the deductive systems in the form of a sequent calculus.

A.2 Deduction rules

In the following list, Γ and Δ are finite sets of sentences. If every occurrence of Γ in a rule is accompanied by a side formula, then $\Gamma=\varnothing$ is allowed in that rule, otherwise it isn't. Similarly for Δ. $\Gamma\cup\{\varphi\}$ is written as Γ,φ. $\varphi[\alpha/\beta]$ is the result of substituting the sentence β for one or more occurrences of the sentence α in φ.

Identity: $$\frac{}{\varphi\Rightarrow\varphi}$$

Weakening: $$\frac{\Gamma\Rightarrow\Delta}{\Gamma,\varphi\Rightarrow\Delta} \qquad \frac{\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\varphi,\Delta}$$

Cut: $$\frac{\Gamma,\varphi\Rightarrow\Delta \quad \Gamma\Rightarrow\varphi,\Delta}{\Gamma\Rightarrow\Delta}$$

Substitution of indiscernibles: $$\frac{I\vdash\alpha\Leftrightarrow\beta \quad I\vdash\gamma\Leftrightarrow\delta}{\varphi[\alpha/\beta]\Rightarrow\varphi[\gamma/\delta]}$$

$\wedge$L: $$\frac{\Gamma,\varphi\Rightarrow\Delta}{\Gamma,\varphi\wedge\psi\Rightarrow\Delta} \qquad \frac{\Gamma,\psi\Rightarrow\Delta}{\Gamma,\varphi\wedge\psi\Rightarrow\Delta}$$

$\wedge$R: $$\frac{\Gamma\Rightarrow\varphi,\Delta \quad \Gamma\Rightarrow\psi,\Delta}{\Gamma\Rightarrow\varphi\wedge\psi,\Delta}$$

$\vee$R: $$\frac{\Gamma\Rightarrow\varphi,\Delta}{\Gamma\Rightarrow\varphi\vee\psi,\Delta} \qquad \frac{\Gamma\Rightarrow\psi,\Delta}{\Gamma\Rightarrow\varphi\vee\psi,\Delta}$$

$\vee$L: $$\frac{\Gamma,\varphi\Rightarrow\Delta \quad \Gamma,\psi\Rightarrow\Delta}{\Gamma,\varphi\vee\psi\Rightarrow\Delta}$$

Ex falso: $$\frac{}{\Gamma,\varphi,\neg\varphi\Rightarrow\Delta}$$

T ex falso: $$\frac{}{\Gamma,\varphi,\neg\mathrm{True}\varphi\Rightarrow\Delta}$$

TL: $$\frac{\Gamma,\varphi\Rightarrow\Delta}{\Gamma,\mathrm{True}\varphi\Rightarrow\Delta}$$

TR: $$\frac{\Gamma\Rightarrow\mathrm{True}\varphi,\Delta}{\Gamma\Rightarrow\varphi,\Delta}$$

$\neg$T ($k\geq 0$) $$\frac{\varphi 1,\ldots,\varphi k\Rightarrow\psi 1,\ldots,\psi m,\Delta}{\mathrm{True}\varphi 1,\ldots,\mathrm{True}\varphi_k,\neg\mathrm{True}\psi 1,\ldots,\neg\mathrm{True}\psi_m\Rightarrow\Delta}$$

The notions of proof and of provability of a sequent are as usual in sequent calculus.[22]

A.3 Definition An s-proof is a proof in which any of the above rules may be used. A d-proof is an s-proof in which the rule T ex falso is not used.

Notice that in the full system the rule $\neg$T can be derived, but only with the help of the T ex falso rule.

A.4 Definition Let Γ and Δ be non-empty sets of sentences.

(i) $\Gamma \vdash_s \Delta$ if and only if there are finite sets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ for which there is an s-proof of $\Gamma' \Rightarrow \Delta'$.

(ii) $\Gamma \vdash_d \Delta$ if and only if there are finite sets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ for which there is a d-proof of $\Gamma' \Rightarrow \Delta'$.

Notice that the derivability relations are thus compact by definition.

We will prove soundness, completeness and decidability for $\vdash_d$ and $\vdash_s$. We give details only for the dynamic notion $\vdash_d$ (simplified versions of the techniques below also work for $\vdash_s$). First some preliminaries.

A.5 Definition Consider the following conditions:

(i) Every occurrence of $\neg$ in φ is immediately in front of a formula of the form ($\underline{a}$ Has $\underline{c}$), True($\underline{\text{this}}$) or Trueψ.

(ii) Every occurrence of $\downarrow$ in φ is in the scope of some occurrence of True.

If a formula φ satisfies clause (i) then φ is in *negation normal form.* If φ satisfies both (i) and (ii) then it is in *normal form.*

Observe that a formula φ is in normal form if and only if one of the following four conditions obtains:

(i) φ is ($\underline{a}$ Has $\underline{c}$) or $\neg$($\underline{a}$ Has $\underline{c}$);

(ii) φ is True($\underline{\text{this}}$) or $\neg$True($\underline{\text{this}}$);

(iii) φ is Trueψ or $\neg$Trueψ, where ψ is a formula in negation normal form;

(iv) φ is of the form $\psi \wedge \chi$ or $\psi \vee \chi$, where both ψ and χ are normal form formulas.

A *sentence* φ is in normal form if and only if one of the following three conditions obtains:

(i)' φ is ($\underline{a}$ Has $\underline{c}$) or $\neg$($\underline{a}$ Has $\underline{c}$);

(iii)' φ is Trueψ or $\neg$Trueψ, where ψ is a *sentence* in negation normal form;

(iv)' φ is of the form $\psi \wedge \chi$ or $\psi \vee \chi$, where both ψ and χ are normal form *sentences.*

This will enable us to prove properties of normal form sentences with induction on the complexity of these sentences (the base clauses being both (i)' and (iii)').

Next we present a slight strengthening of Barwise and Etchemendy's normal form lemma (BE 110). The proof is a explicitation of Barwise and Etchemendy's proof.

A.6 Normal form lemma There is an effective operation $.^{NF}$ such that for every formula φ, φ is indiscernible from φ^{NF} and φ^{NF} is in normal form. Moreover, if φ is in normal form, then $\varphi^{NF}=\varphi$.

Proof: as a preliminary, we define an operation * on normal form formulas that has the property that

> for every normal form formula φ, φ^* is indiscernible from $\neg\varphi$, and φ^* is in normal form

Define * as follows

$(\underline{a}\text{ Has }\underline{c})^* = \neg(\underline{a}\text{ Has }\underline{c})$ $\qquad (\text{True}\psi)^* = \neg\text{True}\psi$

$(\neg(\underline{a}\text{ Has }\underline{c}))^* = (\underline{a}\text{ Has }\underline{c})$ $\qquad (\neg\text{True}\psi)^* = \text{True}\psi$

$\text{True}(\underline{\text{this}})^* = \neg\text{True}(\underline{\text{this}})$ $\qquad (\varphi\wedge\psi)^* = \varphi^*\vee\psi^*$

$(\neg\text{True}(\underline{\text{this}}))^* = \text{True}(\underline{\text{this}})$ $\qquad (\varphi\vee\psi)^* = \varphi^*\wedge\psi^*$

Next we define $.^{NF}$ by the following recursion:

$$(\underline{a}\text{ Has }\underline{c})^{NF} = (\underline{a}\text{ Has }\underline{c})$$

$$\text{True}(\underline{\text{this}})^{NF} = \text{True}(\underline{\text{this}})$$

$$(\text{True}\psi)^{NF} = \begin{cases}\text{True}\psi \text{ if } \psi \text{ is in negation normal form}\\ \text{True}(\psi^{NF}) \text{ otherwise}\end{cases}$$

$$(\neg\varphi)^{NF} = (\varphi^{NF})^*$$

$$(\varphi\wedge\psi)^{NF} = \varphi^{NF}\wedge\psi^{NF}$$

$$(\varphi\vee\psi)^{NF} = \varphi^{NF}\vee\psi^{NF}$$

$$(\downarrow\varphi)^{NF} = \varphi^{NF}(\underline{\text{this}}/\downarrow\varphi^{NF})$$

The check that this works is left to the reader. □

The completeness proof uses techniques familiar from completeness proofs for intuitionistic logic (Aczel [1968], Thomason [1968]). The important difference with completeness proofs for static notions of consequence is that in the dynamic case the model construction has to provide two models (situations) instead of just one, since update relations are binary relations between situations. This is reflected in the notion of *update theory* below. Typically, an update with Trueφ requires the 'pre-condition' φ to be true in the antecedent of the update; therefore, an update theory consist of two sets of sentences P and U, where P is to keep track of the pre-conditions of the sentences in U (for which an update has to be constructed).

A.7 Definition Let Δ, Γ, P and U be sets of sentences.

(i) Δ is *d-consistent* iff there is some φ such that $\Delta \nvdash_d \varphi$

(ii) Δ is *saturated* iff for every φ and ψ, if $(\varphi \vee \psi) \in \Delta$ then $\varphi \in \Delta$ or $\psi \in \Delta$

(iii) Δ is a *d-theory within* Γ iff for every $\varphi \in \Gamma$, if $\Delta \vdash_d \varphi$ then $\varphi \in \Delta$

(iv) <P,U> is an *update theory within* Γ iff the following conditions obtain:

  (a) both P and U are consistent saturated d-theories within Γ

  (b) $P \subseteq U \subseteq \Gamma$

  (c) if $\text{True}\varphi \in U$ then $\varphi \in P$

  (d) if $\neg\text{True}\varphi \in U$ then $\varphi \notin P$

(v) A set of formulas Δ is *rich* iff the following conditions hold:

  (a) Δ is closed under subformulas

  (b) if $\varphi \in \Delta$ then $\varphi^{NF} \in \Delta$

A.8 Lemma

(a) For every set of sentences Δ there is a rich set of sentences Γ such that $\Delta \subseteq \Gamma$.

(b) There is an effective operation γ such that for each *finite* set of sentences Δ, $\gamma(\Delta)$ is a rich and finite set with $\Delta \subseteq \gamma(\Delta)$.

Proof: The first part is trivial. For (b) we first define an effective operation that assigns to each formula φ in negation normal form a set of formulas that is finite and rich and contains φ:

$$f(\underline{a} \text{ Has } \underline{c}) = \{(\underline{a} \text{ Has } \underline{c})\}$$
$$f(\text{True}(\underline{\text{this}})) = \{\text{True}(\underline{\text{this}})\}$$
$$f(\text{True}\varphi) = \{\text{True}\varphi\} \cup f(\varphi)$$
$$f(\neg(\underline{a} \text{ Has } \underline{c})) = \{(\underline{a} \text{ Has } \underline{c}), \neg(\underline{a} \text{ Has } \underline{c})\}$$
$$f(\neg\text{True}(\underline{\text{this}})) = \{\text{True}(\underline{\text{this}}), \neg\text{True}(\underline{\text{this}})\}$$
$$f(\neg\text{True}\varphi) = \{\text{True}\varphi, \neg\text{True}\varphi\} \cup f(\varphi)$$
$$f(\varphi\wedge\psi) = \{\varphi\wedge\psi, (\varphi\wedge\psi)^{NF}\} \cup f(\varphi) \cup f(\psi)$$
$$f(\varphi\vee\psi) = \{\varphi\vee\psi, (\varphi\vee\psi)^{NF}\} \cup f(\varphi) \cup f(\psi)$$
$$f(\downarrow\varphi) = \{\downarrow\varphi\} \cup \{\psi(\underline{\text{this}}/\downarrow\varphi) \mid \psi \in f(\varphi)\}$$

That for each negation normal form formula φ, $f(\varphi)$ is finite and rich and contains φ, is checked by induction. The only problematic case is closure under $.^{NF}$ of $f(\downarrow\varphi)$, where one needs a (long) sub-induction to show that

$$\text{if } \psi \in f(\varphi) \text{ then } (\psi(\underline{\text{this}}/\downarrow\varphi))^{NF} = (\psi^{NF})(\underline{\text{this}}/\downarrow\varphi)$$

where it is important that $\downarrow\varphi$ is in negation normal form.

Next, define two operations on finite sets of formulas as follows:

$$c\Delta = \{\psi \mid \psi \text{ is a subformula of some } \varphi \in \Delta\}$$
$$n\Delta = \{\psi^{NF} \mid \psi \in \Delta\}$$

and define γ by

$$\gamma(\Delta) = c\Delta \cup nc\Delta \cup cnc\Delta \cup \bigcup\{f(\psi) \mid \psi \in cnc\Delta \ \&\ \psi^{NF} \notin cnc\Delta\}$$

Observe that this is well-defined: if we decompose a normal form formula and hit upon a formula not in normal form this can only be because we have a normal form formula Trueψ and ψ is not in normal form; but then ψ (and its subformulas) will be in negation normal form. It is easy to check that $\gamma(\Delta)$ has the desired properties. □

A.9 Lemma If Δ is a d-theory within Γ, and Γ is rich, then $\varphi^{NF} \in \Delta \cap \Gamma$ for every $\varphi \in \Delta \cap \Gamma$.
Proof: If $\varphi \in \Delta \cap \Gamma$, then $\varphi \in \Gamma$ and $\Delta \vdash_d \varphi$. So $\Delta \vdash_d \varphi^{NF}$ by the rule of Substitution of Indiscernibles, and $\varphi^{NF} \in \Gamma$ since Γ is rich. So $\varphi^{NF} \in \Delta$ since Δ is a d-theory within Γ, so $\varphi^{NF} \in \Delta \cap \Gamma$.□

A.10 Saturation lemma Suppose $\Delta \nvdash_d \varphi$, Γ is rich and $\Delta \cup \{\varphi\} \subseteq \Gamma$. Then there is an update theory <P,U> within Γ such that $\Delta \subseteq U$ and $\varphi \notin U$.
Proof: Suppose $\Delta \nvdash_d \varphi$ and $\Delta \cup \{\varphi\} \subseteq \Gamma$. Let $\varphi_0, \ldots, \varphi_k, \ldots$ be an enumeration of all sentences in Γ in which each sentence of Γ occurs countably many times. Define an increasing sequence of sets of sentences in the following way:

- $U_0 = \Delta$
- - if $U_n \nvdash_d \varphi_n$ then $U_{n+1} = U_n$
  - if $U_n \vdash_d \varphi_n$ and φ_n is not a disjunction then $U_{n+1} = U_n \cup \{\varphi_n\}$
  - if $U_n \vdash_d \varphi_n$ and $\varphi_n = \psi \vee \chi$ then if $U_n, \psi \nvdash_d \varphi$ then $U_{n+1} = U_n \cup \{\varphi_n, \psi\}$
  else $U_{n+1} = U_n \cup \{\varphi_n, \chi\}$

Now put $U = \cup_n U_n$. Notice that $U \subseteq \Gamma$, since Γ is rich and $\Delta \subseteq \Gamma$.
We use the same technique to construct P, except that:

- we have an enumeration $\psi_0, \ldots, \psi_k, \ldots$ of U in which each sentence occurs countably many times.
- we now decide disjunctions with $\nvdash_d \Phi$, where Φ is given by:
  $\Phi = \{\psi \in \Gamma \mid \neg\text{True}\psi \in U\} \cup \{\psi \in \Gamma \mid \psi \notin U\}$.
- we start with $P_0 = \{\psi \in \Gamma \mid \text{True}\psi \in U\}$

Now proceed as before:

- if $P_n \nvdash_d \psi_n$ then $P_{n+1}=P_n$
  - if $P_n \vdash_d \psi_n$ and ψ_n is not a disjunction then $P_{n+1}=P_n \cup \{\psi_n\}$
  - if $P_n \vdash_d \psi_n$ and $\psi_n=\psi \vee \chi$ then if $P_n, \psi \nvdash_d \Phi$ then $P_{n+1}=P_n \cup \{\psi_n, \psi\}$

  else $P_{n+1}=P_n \cup \{\psi_n, \chi\}$

Put $P=\cup_n P_n$.

We must now check that P and U have the desired properties.

(1) U is a d-theory within Γ: suppose $\psi \in \Gamma$ and $U \vdash_d \psi$. By compactness, some finite subset of U proves ψ, so by the way we have chosen the enumeration there must be some n such that $\psi=\varphi_n$ and $U_n \vdash_d \varphi_n$. But then $\psi \in U_{n+1}$.

(2) U is d-consistent: show that $U \nvdash_d \varphi$.

(3) U is saturated: suppose $\psi \vee \chi \in U$. Then for some k, $\varphi_k=\psi \vee \chi$ (since $U \subseteq \Gamma$) and $U_k \vdash_d \varphi_k$. But then $\psi \in U_{k+1}$ or $\chi \in U_{k+1}$.

(4) $P \subseteq U$: left to the reader.

(5) P is a d-theory within Γ: suppose $\psi \in \Gamma$ and $P \vdash_d \psi$. Since $P \subseteq U$, $U \vdash_d \psi$, so $\psi \in U$ by (1). By compactness, the construction of the enumeration and the construction of P, there is an n such that $\psi=\psi_n$ and $P_n \vdash_d \psi_n$. So $\psi \in P_{n+1}$.

(6) P is d-consistent: we show that $P \nvdash_d \Phi$. Suppose $P_0 \vdash_d \Phi$, i.e.

$\{\psi \in \Gamma \mid \mathrm{True}\psi \in U\} \vdash_d \{\psi \in \Gamma \mid \neg\mathrm{True}\psi \in U\} \cup \{\psi \in \Gamma \mid \psi \notin U\}$

Then by rule $\neg$T

$\{\mathrm{True}\psi \mid \psi \in \Gamma, \mathrm{True}\psi \in U\} \cup \{\neg\mathrm{True}\psi \mid \psi \in \Gamma, \neg\mathrm{True}\psi \in U\} \vdash_d \{\psi \in \Gamma \mid \psi \notin U\}$

so $U \vdash_d \{\psi \in \Gamma \mid \psi \notin U\}$, which can't be, since U is a d-theory within Γ.

For the induction step use the Cut rule and the induction hypothesis.

(7) P is saturated: suppose $\psi \vee \chi \in P$. Then for some k, $\psi_k=\psi \vee \chi$ (since $P \subseteq U$) and $P_k \vdash_d \psi_k$. But then $\psi \in P_{k+1}$ or $\chi \in P_{k+1}$.

(8) if $\mathrm{True}\psi \in U$ then $\psi \in P$: in fact $\psi \in P_0$.

(9) if $\neg\mathrm{True}\psi \in U$ then $\psi \notin P$: suppose $\neg\mathrm{True}\psi \in U$ but $\psi \in P$. Then $\psi \in \Phi$, so $P \vdash_d \Phi$, which we showed to be impossible under (6). □

The idea is to use update theories for constructing counter examples. In fact, at this point we could go directly to the update construction lemma (**A.13**). Instead we make a small detour which will facilitate the proof of decidability below. It turns out to be more convenient to construct updates from sets of sentences that are defined without mentioning derivability.

A.11 Definition Let P, U, Γ be sets of sentences. Then <P,U> is a *syntactic update* within Γ if the following conditions obtain:

(i) $P \subseteq U \subseteq \Gamma$, and Γ is rich

(ii) if $\varphi \in \Gamma$, then $\varphi \in P$ iff $\varphi^{NF} \in P$; if $\varphi \in \Gamma$, then $\varphi \in U$ iff $\varphi^{NF} \in U$

(iii) if $\psi \wedge \chi \in \Gamma$ then $\psi \wedge \chi \in U$ iff $\psi \in U$ and $\chi \in U$

(iv) if $\psi \vee \chi \in \Gamma$ then $\psi \vee \chi \in U$ iff $\psi \in U$ or $\chi \in U$

(v) if $\psi \wedge \chi \in \Gamma$ then $\psi \wedge \chi \in P$ iff $\psi \in P$ and $\chi \in P$

(vi) if $\psi \vee \chi \in \Gamma$ then $\psi \vee \chi \in P$ iff $\psi \in P$ or $\chi \in P$

(vii) if True$\varphi \in U$ then $\varphi \in P$

(viii) if $\neg$True$\varphi \in U$ then $\varphi \notin P$

(ix) there are no $\varphi, \psi \in U$ such that φ is indiscernible from $\neg\psi$

Remark. If Γ is finite and rich, then the set of syntactic updates within Γ is recursive: clearly, $\{<P,U> \mid P \subseteq U \subseteq \Gamma\}$ is finite in this case, and the remaining clauses in **A.11** express decidable properties if P, U, Γ are finite (for (ii) it is crucial that $.^{NF}$ is recursive; for (ix), recall that indiscernibility is decidable).

A.12 Lemma If <P,U> is an update theory within Γ, then <P,U> is a syntactic update within Γ.

A.13 Update construction lemma Let <P,U> be a syntactic update within Γ. Then there are possible situations s and t such that:

(1) $\psi \in \Gamma \Rightarrow$ (Exp(ψ,s) is true $\Leftrightarrow$ $\psi \in P$)

(2) $\psi \in \Gamma \Rightarrow$ ($\exists t' \subseteq t$: $s[\psi]t'$ $\Leftrightarrow$ $\psi \in U$)

Proof: Consider the equation:

$$\mathbf{s} = \{\sigma \mid [\sigma]=\text{Type}(\text{Val}(\psi)) \text{ for some simple sentence } \psi \in P\}$$

Let s=F(**s**), where F is the unique solution of the equation. It is essential here that the indeterminate **s** is the situation indeterminate that Barwise and Etchemendy use in the static semantics. Next define t by:

$$t = s \cup \{\sigma \mid [\sigma]=\text{Type}(\text{Exp}(\psi,s)) \text{ for some simple sentence } \psi \in U\}$$

We now show that

(1) $\psi \in \Gamma \Rightarrow$ (Exp(ψ,s) is true $\Leftrightarrow$ $\psi \in P$)

First, we show this for normal form sentences ψ. For simple sentences (1) is immediate from the construction of s; and for the disjunction and conjunction cases (1) follows from the induction hypotheses and clauses (v) and (vi) in definition **A.11**. Secondly, we show (1) in general: if $\psi \in \Gamma$ and Exp(ψ,s) is true, then Exp(ψ^{NF},s) is true, so $\psi^{NF} \in P$ (by (1)), so $\psi \in P$ (by **A.11**(ii)). Conversely, if $\psi \in \Gamma$ and $\psi \in P$, then $\psi^{NF} \in P \cap \Gamma$, so Exp($\psi^{NF}$,s) is true, so Exp($\psi$,s) is true.

A similar argument shows that:

(2) $\psi \in \Gamma \Rightarrow (\exists t' \subseteq t\text{: } s[\psi]t' \Leftrightarrow \psi \in U)$

What remains is to show that s and t are possible situations. Since s⊆t, it is sufficient to show that t is possible:

- suppose for some state of affairs σ both it and its dual σ^* are in t. By construction of s and t and the fact that P⊆U, there are simple sentences ψ and χ in U such that ψ and $\neg\chi$ are indiscernible. But this contradicts **A.11**(ix).
- suppose <Tr,p,1>$\in$ t. By construction of t, p=Exp(ψ,s) for some sentence True$\psi \in$ U. Hence $\psi \in P$ by **A.11**(vii), and so by (1) above, Exp(ψ,s) is true.
- suppose <Tr,p,0>$\in$ t. By construction of t, p=Exp(ψ,s) for some sentence $\neg$True$\psi \in$ U. So $\psi \notin P$ by **A.11**(viii). Moreover $\psi \in \Gamma$, since $\neg$True$\psi \in$ U, Γ is rich and U$\subseteq$ Γ. So by (1), Exp(ψ,s) is not true.□

A.14 Dynamic completeness theorem For all set of sentences Δ and all sentences φ, $\Delta \models_d \varphi$ if and only if $\Delta \vdash_d \varphi$.

Proof: Suppose $\Delta \nvdash_d \varphi$. Let Γ be some rich set of sentences for which $\Delta \cup \{\varphi\} \subseteq \Gamma$. Then for some sets P and U, <P,U> is an update theory within Γ such that $\Delta \subseteq U$ and $\varphi \notin U$, by the saturation lemma. By lemmas **A.12** and **A.13**, there are possible situations s and t such that

(1) $\psi \in \Gamma \Rightarrow (\exists t' \subseteq t\text{: } s[\psi]t' \Leftrightarrow \psi \in U)$

Since $\varphi \in \Gamma$ but $\varphi \notin U$, and $\Delta \subseteq U$, we see that (1) implies that it cannot be the case that $\Delta \models_d \varphi$.This establishes completeness. We leave soundness to the reader.□

A.15 Theorem $\{\langle\Gamma,\Delta\rangle \mid \Gamma \vdash_d \Delta$ and Γ and Δ are finite$\}$ is decidable.

Proof: Since interpunction on the left is interchangeable with conjunction, and on the right with disjunction, it is sufficient to show that $\{\langle\varphi,\psi\rangle \mid \varphi \vdash_d \psi\}$ is decidable. Let φ and ψ be sentences, and let γ be the effective operation of lemma **A.8**. Then :

$\varphi \nvdash_d \psi$ iff there is a syntactic update <P,U> within $\gamma(\{\varphi,\psi\})$ such that $\varphi \in U$ and $\psi \notin U$

The effectiveness of γ and the remark immediately after lemma **A.11** guarantee that the right hand side is recursive. So the non-derivability relation between sentences is recursive. Hence the derivability relation between sentences is also recursive. □

We conclude with an observation on anti-success, the typical property of the Liar.

A.16 Theorem φ is anti-successful if and only if there is a finite set of sentences $\{\varphi_1,\ldots,\varphi_n\}$ such that $\varphi \vdash_d \varphi_1\wedge\neg\mathrm{True}\varphi_1,\ldots, \varphi_n\wedge\neg\mathrm{True}\varphi_n$

Proof: If $\varphi\vdash_d \vee_{i\leq m}(\psi_i\wedge\neg\mathrm{True}\psi_i)$ then $\varphi\vdash_s \vee_{i\leq m}(\psi_i\wedge\neg\mathrm{True}\psi_i)$, but by the T ex falso rule this implies that φ is unacceptable. So for every possible situation t, not $t[\varphi]t$, so a fortiori $\forall_p st(\ s[\varphi]t \rightarrow \text{not } t[\varphi]t\ )$.

For the converse we argue by contraposition. First an abbreviation: we say that a set of sentences Γ has (property) K if and only if

(K) there is no finite set of sentences $\{\varphi_1,\ldots,\varphi_n\}$ such that

$$\Gamma \vdash_d \varphi_1\wedge\neg\mathrm{True}\varphi_1,\ldots, \varphi_n\wedge\neg\mathrm{True}\varphi_n$$

Notice that if Γ has K, then Γ is d-consistent.

Now suppose that

(1) φ has K

We will construct a possible situation t such that $t[\varphi]t$, which is sufficient because φ cannot be anti-successful in that case. We use the construction of the static completeness theorem.

Let $\varphi_0,\ldots,\varphi_n,\ldots$ be an enumeration of all sentences in which each sentence occurs countably many times. Define:

(0) $\Delta_0=\{\varphi\}$

(n+1) - if $\Delta_{3n}\nvdash\varphi_n$ then $\Delta_{3n+3}=\Delta_{3n+2}=\Delta_{3n+1}=\Delta_{3n}$

- if $\Delta_{3n}\vdash\varphi_n$ then
  - $\Delta_{3n+1}=\Delta_{3n}\cup\{\varphi_n\}$ and

    if φ_n is of the form $\psi\vee\chi$ then
    - $\Delta_{3n+2}=\Delta_{3n+1}\cup\{\psi\}$ if $\Delta_{3n+1}\cup\{\psi\}$ has K, else $\Delta_{3n+2}=\Delta_{3n+1}$
    - $\Delta_{3n+3}=\Delta_{3n+2}\cup\{\chi\}$ if $\Delta_{3n+2}\cup\{\chi\}$ has K, else $\Delta_{3n+3}=\Delta_{3n+2}$

    else $\Delta_{3n+3}=\Delta_{3n+2}=\Delta_{3n+1}$

Now take $\Delta = \cup_n \Delta_n$. Then:

(1) Δ is a d-theory, i.e. $\Delta\vdash_d\psi$ iff $\psi\in\Delta$

(2) Δ is d-consistent, in particular Δ has property K

(3) Δ is saturated, that is: if $\psi\vee\chi\in\Delta$ then $\psi\in\Delta$ or $\chi\in\Delta$

To see the last, suppose $\psi\vee\chi\in\Delta$ but $\psi\notin\Delta$ and $\chi\notin\Delta$. Then there must be a natural number n such that $\Delta 3_n \vdash_d \psi \vee \chi$ while both $\Delta 3_n \cup \{\psi \vee \chi, \psi\}$ and $\Delta 3_n \cup \{\psi \vee \chi, \chi\}$ do not have property K. But this means that there are sentences $\psi_1, \ldots, \psi_m$ and $\chi_1, \ldots, \chi_n$ such that

- $\Delta 3_n \cup \{\psi \vee \chi, \psi\} \vdash_d \psi_1 \wedge \neg \mathrm{True}\psi_1, \ldots, \psi_m \wedge \neg \mathrm{True}\psi_m$
- $\Delta 3_n \cup \{\psi \vee \chi, \chi\} \vdash_d \chi_1 \wedge \neg \mathrm{True}\chi_1, \ldots, \chi_m \wedge \neg \mathrm{True}\chi_m$

which by Cut and the assumption $\Delta 3_n \vdash_d \psi \vee \chi$ imply that Δ doesn't have property K, but this contradicts (2).

Now consider the equation

(4) $\quad s = \{\sigma \mid [\sigma] = \mathrm{Type}(\mathrm{Val}(\psi))$ for some simple sentence $\psi \in \Delta\}$.

and define t by $t =_{df} F(\mathbf{s})$ where F solves (4).

Then show that

(5) $\quad t[\psi]t$ iff $\psi \in \Delta$

which is simple (it is again sufficient to consider normal forms only). What remains to show is that t is a possible situation. t must be coherent, by the construction of t and the fact that Δ is d-consistent; moreover, if $\mathrm{True}\psi \in \Delta$ then $\psi \in \Delta$ by (1), which together with (5) suffices to show that t respects its positive semantical facts. For the negative semantical facts, suppose that $\neg\mathrm{True}\psi \in \Delta$; if $\psi \in \Delta$, then $\Delta \vdash_d \psi \wedge \neg \mathrm{True}\psi$, which contradicts (2), so $\psi \notin \Delta$ which by (5) implies that $\mathrm{Exp}(\psi, t)$ is not true.□

The theorem shows that our earlier observation that $\downarrow\neg$True(this) and all instances of $\varphi \wedge \neg\mathrm{True}\varphi$ and $\downarrow(\varphi \wedge \neg$True(this)) are anti-successful was not a coincidence.[23]

Notes

1 See the references in Martin [1970], Martin [1984] and Visser [1989].

2 The feeling that there is no real solution to the semantic paradoxes (yet) is also expressed by Kripke and Visser, who write, respectively: "I do not regard any proposal, including the one to be advanced here, as definitive in the sense that it gives *the* interpretation of the ordinary use of 'true', or *the* solution to the semantic paradoxes." (Kripke [1975], in Martin [1984], p. 63); "But perhaps there is no true solution, maybe we should be content with a number of ways to block the paradox, the choice among which is to be governed by local considerations of utility and simplicity." (Visser [1989], p. 624).

3 Dynamic semantics gained a lot of momentum with the development of Discourse Representation Theory by Kamp and Heim (Kamp [1984], Heim [1982]), though the basic ideas where already formulated in Stalnaker [1972] and Seuren [1976]. For more recent developments see Barwise [1986], Groenendijk and Stokhof [1990], [1991], van Benthem [1990], Veltman [1991].

4 Throughout this paper, references to Barwise and Etchemendy [1987] will have the form (BE pagenumber).

5 The Austinian account comprises chapters 8 to 13 of Barwise and Etchemendy [1987], and chapter 3 provides a good introduction to Peter Aczel's theory of non-well-founded sets. More technical information about this set theory can be found in Aczel [1988].

6 So, propositions are modeled as sets of the form {s;T}. Barwise and Etchemendy leave it open what kind of construction is meant, as long as {s;T} is some set having s and T as components. I think of {s;T} as non-standard notation for the ordered pair <s,T>. Similar remarks apply to other sequences, such as $[Tr,f_s;0]$ and $[\wedge X]$.

7 In this summary of the Austinian account, we have deleted states of affairs of the form <Bel,a,p;i> as well as formulas of the form (a Believes this), (a Believes that_i) and (a Believes φ), because they are irrelevant for the theory below. Moreover, as Barwise and Etchemendy indicate themselves, the treatment of belief sentences in the Austinian semantics is unsatisfactory (Open Problem 2, BE 146-147).

8 This does not imply, however, that the number of options in the output state cannot exceed the number of options in the input state. An update with a *disjunction* may extend an input option into two or more output options.

9 Probably modalities are not distributive. Veltman treats *might* as follows: an update with a sentence of the form *might*φ leaves everything as it is if at least one of the options is a φ-option, but otherwise the output of the update will be the inconsistent state (see Veltman[1991]). Clearly, this defines a non-distributive function.

10 We use "$\subseteq$" for "is a subset of" and "$\subset$ " for "is a proper subset of".

11 Notice that Update and Minimality imply that, if s supports φ, then s is the unique update of s with φ.

12 Notice that anti-success is equivalent with: if s[φ]t then *not* t[φ]t. And the latter is equivalent with: [φ] is irreflexive.

13 Pre-conditional success and Anti-success are in some sense contrary properties: no consistent sentence can have both. Observe that if Update and Minimality hold, then Pre-conditional success is equivalent to: if s[φ]t then s[φ]s. This doesn't imply reflexivity (compare anti-success), but rather a restriction of the reflexive relation (either s[φ]s or for no t, s[φ]t).

14 Axiom 4 amounts to the indiscernibility of ↓φ and φ(this/↓φ), where φ(this/↓φ) is the result of substituting ↓φ for all loose occurrences of this in φ.

15 For the Austinian completeness theorem see BE 152; however, the real work is done in chapter 7, BE 107-115.

16 In fact clause (iii) is a bit stronger than the informal description suggests, for (iii) implies that the assignments in a reading have to be defined for any demonstrative with an index smaller than the number of sentences in the discourse, even if such a demonstrative doesn't occur in the discourse at all. The choice for (iii) was made for simplicity.

17 BE 148, proposition 16.

18 Barwise and Etchemendy observe that in their version of the Liar Cycle, Exp(True($\underline{\text{that}}_2$),s)=Exp(↓True(¬True(this)),s) and Exp(¬True($\underline{\text{that}}_1$),s)=Exp(↓¬True(True(this)),s) (vide BE 149). For the sequential version of the Liar Cycle, the discourse True($\underline{\text{that}}_2$);¬True($\underline{\text{that}}_1$), we find the same correspondence. By non-well-founded elimination the discourse is strongly equivalent to ↓True(¬True(this)) ; ¬True(↓True(¬True(this))). The last formula of this sequence is statically indiscernible from ↓¬True(True(this)), hence both formulas also have the same update relation. The indiscernibility can be shown with the rule Identity of Indiscernibles; intuitively, they express the same proposition since the 'unfolding' of the formulas is the same, namely an infinite repetition of ¬TrueTrue.

19 Use the copy and paste options of your favorite word processor to see how this works.

20 Gupta writes: "However, I should note that the idea of circular propositions is not at all central to their diagnosis of the Liar. A philosopher who eschews propositions and takes objects of truth to be sentences can put forward a diagnosis essentially similar to the one they offer. Barwise and Etchemendy's attitude towards the role of propositions is somewhat vacillating. At one point in their book (p.138) they suggest that the ambiguities they claim to find in the Liar cannot be accounted for by the sentential theorist. At another point (p. 175) they allow that such a theorist may take truth to depend on a contextual parameter, one that reflects the situation the statement is about." (Gupta [1989], p.708)

21 Kripke [1975], Gupta [1982], Herzberger[1982].

22 I do not know whether or not the present systems have Cut Elimination. The absence of "symmetric" rules

for the T and $\neg$T rules make this rather implausible.

[23] Since clearly $\varphi\wedge\neg\mathrm{True}\varphi \vdash_d \varphi\wedge\neg\mathrm{True}\varphi$, and for $\downarrow(\varphi\wedge\neg\mathrm{True}(\underline{this}))$ it is not hard to show that $\downarrow(\varphi\wedge\neg\mathrm{True}(\underline{this})) \vdash_d \downarrow(\varphi\wedge\neg\mathrm{True}(\underline{this}))\wedge\neg\mathrm{True}(\downarrow(\varphi\wedge\neg\mathrm{True}(\underline{this})))$.

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