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THE COMPLETENESS OF THE LAMBEK CALCULUS WITH RESPECT TO RELATIONAL SEMANTICS

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THE COMPLETENESS OF THE LAMBEK CALCULUS WITH RESPECT TO RELATIONAL SEMANTICS

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Abstract. The problem whether Lambek Calculus is complete w.r.t. Relational Semantics was raised several times, cf. [vB88] and [vB91]. In this paper, we show that the answer is in the affirmative. More precisely, we will prove that that version of the Lambek Calculus which does not use the empty sequence is strongly complete w.r.t. those relational Kripke models where the set of possible worlds, W, is a transitive binary relation, while that version of the Lambek Calculus where we admit the empty sequence as the antecedent of a sequent is strongly complete w.r.t. those relational models where $W = U \times U$ for some set U. We will also look into extendability of this completeness result to various fragments of Girard's Linear Logic as suggested in [vB91, p.235], and investigate the connection between the Lambek Calculus and language models.

In [vB88], Johan van Benthem introduces *Relational Semantics* (RelSem for short), and states Soundness Theorem for *Lambek Calculus* (LC) w.r.t. RelSem. After doing this, he writes: "it would be very interesting to have the converse too", i.e., to have Completeness Theorem. The same question is in [vB91, p.235]. In the following, we give proofs for Strong Completeness Theorems for different versions of LC.

* * *

First of all, let us define the language of LC. Given a denumerable set P of *primitive symbols*, we let the set of *formulae* $\mathrm{Form}_{\mathrm{LC}}$ be the smallest set containing every primitive symbol and closed under '\', '/', and '•', i.e., if $A, B \in \mathrm{Form}_{\mathrm{LC}}$, then $A\backslash B, A/B, A \bullet B \in \mathrm{Form}_{\mathrm{LC}}$. The set of *sequents* is the set of all expressions of the form $A_1, \ldots, A_n \Rightarrow A_0$ where n is a positive integer and $A_i \in \mathrm{Form}_{\mathrm{LC}}$ for each $i \leq n$.

LC is given by the following axiom and rules of inference, where A, B, C stand for formulae and x, y, z stand for finite sequences of formulae including the empty sequence $\ominus$ unless the contrary is asserted.

Axiom:

$$(0) \quad A \Rightarrow A.$$

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Rules of inference:

$$(1\backslash)\quad \frac{x \Rightarrow A \quad y, B, z \Rightarrow C}{y, x, A\backslash B, z \Rightarrow C}\ x \text{ non-empty} \qquad (2\backslash)\quad \frac{A, x \Rightarrow B}{x \Rightarrow A\backslash B}\ x \text{ non-empty}$$

$$(1/)\quad \frac{x \Rightarrow A \quad y, B, z \Rightarrow C}{y, B/A, x, z \Rightarrow C}\ x \text{ non-empty} \qquad (2/)\quad \frac{x, A \Rightarrow B}{x \Rightarrow B/A}\ x \text{ non-empty}$$

$$(3)\quad \frac{x \Rightarrow A \quad y \Rightarrow B}{x, y \Rightarrow A \bullet B}\ x, y \text{ non-empty} \qquad (4)\quad \frac{x, A, B, y \Rightarrow C}{x, A \bullet B, y \Rightarrow C}$$

$$(5)\quad \frac{x \Rightarrow A \quad A \Rightarrow B}{x \Rightarrow B}\ x \text{ non-empty.}$$

A *theorem* of LC is a sequent deducible in LC ($\vdash_{\mathrm{LC}}$), i.e., by the usual recursive definition, a sequent is a theorem iff it is an instance of (0), or it is given by some rule of inference from some theorem(s). More generally, let Γ be a set of sequents and φ be a sequent. We say that φ is LC-*deducible from* Γ iff

(i) $\varphi \in \Gamma$ or
(ii) φ is an instance of (0) or
(iii) there is a set of sequents Δ each of its elements is LC-deducible from Γ and there is an inference rule such that $\frac{\Delta}{\varphi}$ is an instance of this rule.

REMARK: If the set of primitive symbols is the set of basic types, then the formulae are types and, roughly speaking, '$\Rightarrow$' of LC corresponds to the derivability relation of *Categorial Grammar*. On the other hand, if P is considered as a set of propositional variables, then LC is a Gentzen-type inference system, and hence it is a fragment of *Linear Logic*.

We give a Kripke-style semantics for LC, where we restrict the class of ordinary Kripke models with ternary accessibility relation to the class of models where the set of possible worlds consists of ordered pairs.

DEFINITION OF RELATIONAL SEMANTICS: By a *relational (Kripke) model* for LC we mean an ordered triple $\langle W, C, v\rangle$ which is a Kripke model in the usual sense (i.e., W is a set of possible worlds, C is a (ternary) accessibility relation, and v is an evaluation of expressions) and for which the following hold. W is a transitive binary relation on some set U, and $C \subseteq W \times W \times W$ such that, for every $x, y, z \in W$, $Cxyz$ holds iff $\{z\} = \{x\} \circ \{y\}$, i.e., iff there are $a, b, c \in U$ such that $x = \langle a, b\rangle$, $y = \langle b, c\rangle$ and $z = \langle a, c\rangle$. Moreover, let

$$Exp \stackrel{\text{def}}{=} \{A_1, \ldots, A_n : A_i \in \mathrm{Form}_{\mathrm{LC}}, 1 \leq i \leq n, \text{ for some } n\} \cup \{\varphi : \varphi \text{ is a sequent}\}$$

and let $v : Exp \to \mathcal{P}(W)$ be such that, for every $A, B \in \mathrm{Form}_{\mathrm{LC}}$ and sequence x of formulae, $v(x, A) = v(x \bullet A)$ and

$$v(A \bullet B) \stackrel{\text{def}}{=} \{z \in W : (\exists x \in v(A))(\exists y \in v(B))Cxyz\}$$
$$v(A\backslash B) \stackrel{\text{def}}{=} \{y \in W : (\forall x \in v(A))\forall z(Cxyz \to z \in v(B))\}$$
$$v(B/A) \stackrel{\text{def}}{=} \{x \in W : (\forall y \in v(A))\forall z(Cxyz \to z \in v(B))\}$$
$$v(x \Rightarrow A) \stackrel{\text{def}}{=} (W \setminus v(x)) \cup v(A).$$

We say that a sequent φ of LC is *true in a model* $\langle W, C, v\rangle$, in symbols $\langle W, C, v\rangle \models \varphi$, iff $v(\varphi) = W$, or, equivalently, the sequent $A_1, \ldots, A_n \Rightarrow A_0$ is true in the model above iff

$$v(A_1) \circ \cdots \circ v(A_n) \subseteq v(A_0).$$

A formula is *valid with respect to* RelSem iff it is true in every model. We denote this by $\models_{\mathsf{R}} \varphi$. We say that φ is a (RelSem) *consequence of* Γ, in symbols $\Gamma \models_{\mathsf{R}} \varphi$, iff, for every model $\langle \mathfrak{A}, v\rangle$, if $\langle \mathfrak{A}, v\rangle \models \Gamma$, then $\langle \mathfrak{A}, v\rangle \models \varphi$, where $\langle \mathfrak{A}, v\rangle \models \Gamma$ abbreviates that, for every $\psi \in \Gamma$, $\langle \mathfrak{A}, v\rangle \models \psi$.

REMARK: In the above definition, *Exp* can be thought of as the set of formulae of a modal logic with three binary modalities: '$\bullet$', '$\backslash$', '$/$'. Then C is the accessibility relation corresponding to the possibility-type modality '$\bullet$', and the residuations are, in a certain sense, dual modalities of '$\bullet$'. (Indeed, the modality '$\backslash$' is related to the modality '$\bullet$' in a similar fashion as the temporal modality '*Always in the past*', denoted as '$[P]$', is related to '*Sometime in the future*', denoted as '$\langle F\rangle$', in, e.g., [ANS91] and [Go87]. In [ANS91], '$\langle P\rangle$' is called the *conjugate* of '$\langle F\rangle$' and '$[P]$' is the *dual* of '$\langle P\rangle$'. So '$\backslash$' is a dual of a conjugate of '$\bullet$' $(-(A\backslash -B) = A^{-1} \bullet B)$. It is instructive to meditate over the two steps leading to '$\backslash$' from '$\bullet$'. The obvious dual of '$\bullet$' is given by

$$v(A^{-1}\backslash B) = \{z : (\forall x \in v(A))(\exists y \in v(B))Cxyz\}.$$

A conjugate of this is defined as

$$v(A\backslash B) = \{z : (\forall x \in v(A))(\exists y \in v(B))Cxzy\}.$$

Another conjugate of the same modality is defined by

$$v(B/A) = \{z : (\forall x \in v(A))(\exists y \in v(B))Czxy\}.)$$

Further, '$\Rightarrow$' is considered as classical implication. We can extend the relation $\langle W, C, v\rangle \models \psi$ for $\psi \in Exp$ in the usual way, i.e., it holds iff $v(\psi) = W$.

Now, we can formulate the main theorem (which was first presented in [Mi91]) of this section.

THEOREM 0. (Strong Completeness Theorem for LC w.r.t. RelSem) *For any set Γ of sequents, and for any sequent φ,*

$$\Gamma \vdash_{\mathrm{LC}} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathsf{R}} \varphi.$$

REMARK: In the case of $\Gamma = \emptyset$, we have Weak Completeness Theorem w.r.t. RelSem.

COROLLARY 0. (Compactness Theorem) *For any set Γ of sequents and sequent φ, if $\Gamma \models_{\mathsf{R}} \varphi$, then there is a finite $\Delta \subseteq \Gamma$ such that $\Delta \models_{\mathsf{R}} \varphi$.*

PROOF OF COROLLARY 0: By Theorem 0 it is enough to show that if $\Gamma \vdash_{\mathrm{LC}} \varphi$, then $\Delta \vdash_{\mathrm{LC}} \varphi$ for some finite subset Δ of Γ. And this is straightforward by the definition of LC-deduction. ∎

We will prove Theorem 0 using Theorems 1 and 2 below, but we need some lemmas and definitions before formulating them.

LEMMA 0. *For any set Γ of sequents (including the empty set $\emptyset$),*

$$\Gamma \vdash_{\mathrm{LC}} A, (A\backslash B) \Rightarrow B \qquad \textit{and} \qquad \Gamma \vdash_{\mathrm{LC}} (B/A), A \Rightarrow B.$$

PROOF: Let $x = A$, $C = B$ and $y = z = \ominus$ the empty sequence. Then apply $(1\backslash)$ and $(1/)$, respectively. ∎

LEMMA 1. *For any set* Γ *of sequents, if* $\Gamma \vdash_{\mathrm{LC}} A \Rightarrow C$, *then* $\Gamma \vdash_{\mathrm{LC}} C\backslash B \Rightarrow A\backslash B$ *and* $\Gamma \vdash_{\mathrm{LC}} B/C \Rightarrow B/A$.

PROOF: Assume $\Gamma \vdash_{\mathrm{LC}} A \Rightarrow C$. Then, by (0) and (3), $\Gamma \vdash_{\mathrm{LC}} A, (C\backslash B) \Rightarrow C \bullet (C\backslash B)$. Applying (4) to the result of Lemma 0 we have $\Gamma \vdash_{\mathrm{LC}} C \bullet (C\backslash B) \Rightarrow B$. Then, by (5), $\Gamma \vdash_{\mathrm{LC}} A, (C\backslash B) \Rightarrow B$, so, by (2\), $\Gamma \vdash_{\mathrm{LC}} C\backslash B \Rightarrow A\backslash B$.

Using (2/) instead of (2\) we get $\Gamma \vdash_{\mathrm{LC}} B/C \Rightarrow B/A$ too. ∎

LEMMA 2. *For any set* Γ *of sequents, if* $\Gamma \vdash_{\mathrm{LC}} B \Rightarrow D$, *then* $\Gamma \vdash_{\mathrm{LC}} A\backslash B \Rightarrow A\backslash D$ *and* $\Gamma \vdash_{\mathrm{LC}} B/A \Rightarrow D/A$.

PROOF: By Lemma 0 we have $\Gamma \vdash_{\mathrm{LC}} A, (A\backslash B) \Rightarrow B$, so, by the assumption and by (5), $\Gamma \vdash_{\mathrm{LC}} A, (A\backslash B) \Rightarrow D$. Thus, by (2\), $\Gamma \vdash_{\mathrm{LC}} A\backslash B \Rightarrow A\backslash D$.

Again, use (2/) for the other case. ∎

LEMMA 3. *For any set* Γ *of sequents,*

$$\Gamma \vdash_{\mathrm{LC}} C \Rightarrow A\backslash B \quad \textit{iff} \quad \Gamma \vdash_{\mathrm{LC}} A, C \Rightarrow B \qquad \textit{and}$$
$$\Gamma \vdash_{\mathrm{LC}} C \Rightarrow B/A \quad \textit{iff} \quad \Gamma \vdash_{\mathrm{LC}} C, A \Rightarrow B.$$

PROOF: The 'if' direction is given by (2\) and (2/), respectively.

So assume $\Gamma \vdash_{\mathrm{LC}} C \Rightarrow A\backslash B$. Then, by (3), $\Gamma \vdash_{\mathrm{LC}} A, C \Rightarrow A \bullet (A\backslash B)$. By Lemma 0 and (4) $\Gamma \vdash_{\mathrm{LC}} A \bullet (A\backslash B) \Rightarrow B$, so, by (5), $\Gamma \vdash_{\mathrm{LC}} A, C \Rightarrow B$.

The proof is similar for '/'. ∎

Now we define the relations '$\leq_\Gamma$' and '$\equiv_\Gamma$' on $\mathrm{Form}_{\mathrm{LC}}$, for any set Γ of sequents. We let, for every $A, B \in \mathrm{Form}_{\mathrm{LC}}$,

$$A \leq_\Gamma B \quad \text{iff} \quad \Gamma \vdash_{\mathrm{LC}} A \Rightarrow B \qquad \text{and}$$
$$A \equiv_\Gamma B \quad \text{iff} \quad (A \leq_\Gamma B \quad \text{and} \quad B \leq_\Gamma A).$$

Let $\mathfrak{T}$ be the *formula algebra* of LC, i.e.,

$$\mathfrak{T} \stackrel{\mathrm{def}}{=} \langle \mathrm{Form}_{\mathrm{LC}}, \backslash, /, \bullet \rangle$$

where '\', '/' and '•' denote the obvious operations on $\mathrm{Form}_{\mathrm{LC}}$.

LEMMA 4. *For any set* Γ *of sequents, '*$\equiv_\Gamma$*' is a congruence relation on* $\mathfrak{T}$ *and, for any* A, B, C, D *such that* $A \equiv_\Gamma B$ *and* $C \equiv_\Gamma D$, *we have*

$$A \leq_\Gamma C \quad \textit{iff} \quad B \leq_\Gamma D.$$

PROOF: By (0) and (5) it is an equivalence relation.

Assume that $A \equiv_\Gamma B$ and $C \equiv_\Gamma D$. Then (3) and (4) imply $A \bullet C \equiv_\Gamma B \bullet D$. Using Lemmas 1, 2 one easily shows that $A\backslash C \equiv_\Gamma B\backslash D$ and $A/C \equiv_\Gamma B/D$. Now, we assume that $A \leq_\Gamma C$ holds too, i.e., we have $\Gamma \vdash_{\mathrm{LC}} A \Rightarrow C$ as well. Then, since $\Gamma \vdash_{\mathrm{LC}} B \Rightarrow A$, we have, by (5), $\Gamma \vdash_{\mathrm{LC}} B \Rightarrow C$. Since $\Gamma \vdash_{\mathrm{LC}} C \Rightarrow D$, using (5) again, we get $\Gamma \vdash_{\mathrm{LC}} B \Rightarrow D$. The proof for the other direction is similar. ∎

Let $\mathfrak{L}_\Gamma$ be the *factor structure* of $\mathfrak{T}$ by '$\equiv_\Gamma$', i.e.,

$$\mathfrak{L}_\Gamma \stackrel{\mathrm{def}}{=} \langle L, \backslash, /, \bullet, \leq_\Gamma \rangle$$

where L is the set of equivalence classes, i.e., $L = \{\overline{A} : A \in \mathrm{Form}_{\mathrm{LC}}\}$ where $\overline{A} = \{B : A \equiv_\Gamma B\}$, and $\overline{A}\backslash\overline{B} = \overline{A\backslash B}$, $\overline{A}/\overline{B} = \overline{A/B}$, $\overline{A} \bullet \overline{B} = \overline{A \bullet B}$ and $\overline{A} \leq_\Gamma \overline{B}$ iff $A \leq_\Gamma B$. Note that this last definition is correct by Lemma 4.

Definition of Relational Structure (RS):

$$\mathfrak{A} \in \mathsf{RS} \qquad \text{iff} \qquad \mathfrak{A} = \langle A, \backslash, /, \bullet, \leq \rangle$$

where A is a non-empty set, '$\backslash$', '$/$' and '$\bullet$' are arbitrary binary operations on A, and '$\leq$' is a binary relation on A.

Clearly, $\mathfrak{L}_\Gamma \in \mathsf{RS}$ for arbitrary set Γ of sequents.

Let Σ be the following set of formulae (in the first-order language with equality of RS) where x, y, z, u are variables:

$$
\begin{array}{rlrl}
(A1) & x \leq x & & \\
(A2) & (x \bullet y) \bullet z \leq x \bullet (y \bullet z) & (A3) & x \bullet (y \bullet z) \leq (x \bullet y) \bullet z \\
(A4) & x \bullet (x\backslash y) \leq y & (A5) & (y/x) \bullet x \leq y \\
(A6) & x \bullet y \leq z \rightarrow y \leq x\backslash z & (A7) & x \bullet y \leq z \rightarrow x \leq z/y \\
(A8) & x \leq y \wedge y \leq z \rightarrow x \leq z & & \\
(A9) & x \leq y \wedge z \leq u \rightarrow x \bullet z \leq y \bullet u & & \\
(A10) & x \leq y \wedge y \leq x \rightarrow x = y. & &
\end{array}
$$

These axioms say that an $\mathfrak{A} \in \mathsf{RS}$ satisfying them is an ordered semigroup, where '$\leq$' is a partial ordering, '$\bullet$' is the semigroup operation, which is monotonic w.r.t. '$\leq$', and $x\backslash y$ is the greatest element such that $x \bullet (x\backslash y) \leq y$ and similarly for y/x.

Now we are ready to formulate Theorem 1, which we will prove a little later.

Theorem 1. *For any set Γ of sequents,*

$$\mathfrak{L}_\Gamma \models \Sigma$$

where $\mathfrak{L}_\Gamma$ is the factor structure and Σ is the set of formulae above.

Definition of Representable Relational Structure (RRS):

$$\mathfrak{A} \in \mathsf{fullRRS} \qquad \text{iff} \qquad \mathfrak{A} = \langle A, \backslash, /, \circ, \subseteq \rangle$$

where $A = \mathcal{P}(W)$, the power set of W, for some fixed transitive binary relation W and the operations of $\mathfrak{A}$ are left and right residuations relativized to W and relational composition, respectively, i.e., for any binary relations on U,

$$
\begin{aligned}
a\backslash b &\stackrel{\text{def}}{=} \{\langle x, y\rangle \in W : \forall z(\langle z, x\rangle \in a \rightarrow \langle z, y\rangle \in b)\} \\
b/a &\stackrel{\text{def}}{=} \{\langle x, y\rangle \in W : \forall z(\langle y, z\rangle \in a \rightarrow \langle x, z\rangle \in b)\} \\
a \circ b &\stackrel{\text{def}}{=} \{\langle x, y\rangle \in W : \exists z(\langle x, z\rangle \in a \wedge \langle z, y\rangle \in b)\}.
\end{aligned}
$$

Let $\mathsf{RRS} = \mathsf{S}\mathsf{fullRRS}$, i.e., RRS consists of the substructures of every fullRRS.

We say that a sequent $A_1, \ldots, A_n \Rightarrow A_0$ is *true* in a RRS $\mathfrak{A}$ under the valuation v, in symbols $\langle \mathfrak{A}, v \rangle \models A_1, \ldots, A_n \Rightarrow A_0$, iff

$$v(A_1) \circ \cdots \circ v(A_n) \subseteq v(A_0)$$

where $v(A_i)$ $(i \leq n)$ is given by the natural extension of v from P to Form_{LC}, i.e., for any formulae A, B,

$$v(A \backslash B) = v(A) \backslash v(B) \quad v(B/A) = v(B)/v(A) \quad v(A \bullet B) = v(A) \circ v(B).$$

In other words, v is a homomorphism from the formula algebra given above into an $\mathfrak{A} \in$ RRS (here we disregard '$\subseteq$', of course).

The main step in the proof of Theorem 0 is the following representation theorem, where **I**RRS denotes the collection of isomorphic copies of all elements of RRS.

THEOREM 2. *For every* $\mathfrak{A} \in$ RS,

$$\mathfrak{A} \models \Sigma \qquad \textit{iff} \qquad \mathfrak{A} \in \mathbf{I}\text{RRS}.$$

Now we can prove Theorem 0 applying Theorems 1 and 2.

PROOF OF THEOREM 0: Soundness is easy to check.

For the other direction we need the following.

LEMMA 5. *Let $\mathfrak{A}$ be a* RRS *with universe A and v be a valuation. Then there is a relational (Kripke) model $\langle W, C, v \rangle$ with $W = \bigcup A$ such that*

$$\langle \mathfrak{A}, v \rangle \models \varphi \qquad \textit{iff} \qquad \langle W, C, v \rangle \models \varphi$$

for each sequent φ. The other direction holds as well.

PROOF OF LEMMA 5: Easy by the definitions. ∎

Now, assume $\Gamma \nvdash_{\text{LC}} A_1, \ldots, A_n \Rightarrow B$. We will show that $\Gamma \not\models_{\mathsf{R}} A_1, \ldots, A_n \Rightarrow B$. Let $A = A_1 \bullet \cdots \bullet A_n$. By (0) and (3) we have $\Gamma \vdash_{\text{LC}} A_1, \ldots, A_n \Rightarrow A_1 \bullet \cdots \bullet A_n$, so, by (5), $\Gamma \nvdash_{\text{LC}} A_1, \ldots, A_n \Rightarrow B$ implies $\Gamma \nvdash_{\text{LC}} A \Rightarrow B$, i.e., $\overline{A} \nleq_\Gamma \overline{B}$ in $\mathfrak{L}_\Gamma$. By Theorems 1 and 2 $\mathfrak{L}_\Gamma$ is isomorphic to a RRS $\mathfrak{L}'_\Gamma$, so let $h : \mathfrak{L}_\Gamma \to \mathfrak{L}'_\Gamma$ be an isomorphism. Let v be a map such that $v(p) = \overline{p}$ for every $p \in P$. Then it is easy to see that v can be extended such that $v(C) = \overline{C}$ for every $C \in \text{Form}_{\text{LC}}$. Then $\langle \mathfrak{L}_\Gamma, v \rangle$ is isomorphic, in the obvious sense, to $\langle \mathfrak{L}'_\Gamma, v' \rangle$, where $v'(D) = h(v(D))$ for every $D \in \text{Form}_{\text{LC}}$. By $\overline{A} \nleq_\Gamma \overline{B}$ we have $\langle \mathfrak{L}'_\Gamma, v' \rangle \not\models A_1, \ldots, A_n \Rightarrow B$, so, by Lemma 5, $\langle W, C, v' \rangle \not\models A_1, \ldots, A_n \Rightarrow B$ for a (Kripke) model $\langle W, C, v' \rangle$. Since Γ is true in $\langle \mathfrak{L}'_\Gamma, v' \rangle$, so is it in $\langle W, C, v' \rangle$. So we have $\Gamma \not\models_{\mathsf{R}} A_1, \ldots, A_n \Rightarrow B$. ∎

To make the proof above complete we prove Theorems 1 and 2.

PROOF OF THEOREM 1: (A1) is given by (0). (A2) and (A3) are true in $\mathfrak{L}_\Gamma$ because ',' and so '$\bullet$' are associative. (A4) and (A5) are guaranteed by Lemma 0. (A6) and (A7) are true by (2\) and (2/), respectively. (A8) holds by (5), and (3) and (4) imply (A9). Finally, (A10) is true because of the following. If $\overline{A} \leq_\Gamma \overline{B}$, then $\Gamma \vdash_{\text{LC}} A \Rightarrow B$ and if $\overline{B} \leq_\Gamma \overline{A}$, then $\Gamma \vdash_{\text{LC}} B \Rightarrow A$. So $A \equiv_\Gamma B$, i.e., $\overline{A} = \overline{B}$, the two equivalence classes are the same. ∎

The next proof will be similar to the proof of Lemma 3 in [An91].

PROOF OF THEOREM 2: It is easy to check that Σ is valid in every RRS.

For the other direction, let us assume that $\mathfrak{A} \in \mathsf{RS}$ and $\mathfrak{A} \models \Sigma$. Step by step we will build a directed graph $G = \langle U, E, \ell \rangle$ the edges (E) of which will be labelled (ℓ) by the elements of our structure $\mathfrak{A}$. We will use this graph to define a representation function 'rep', which will be an isomorphism from $\mathfrak{A}$ to a structure of binary relations on U.

In each step α, we will define a directed graph $G_\alpha = \langle U_\alpha, E_\alpha, \ell_\alpha \rangle$, where U_α is the set of nodes, $E_\alpha \subseteq U_\alpha \times U_\alpha$ is the set of edges, $\ell_\alpha : E_\alpha \to A$ is the labelling function (A is the universe of $\mathfrak{A}$) such that

(I) E_α is irreflexive and transitive

(II) $\langle x, y \rangle, \langle y, z \rangle \in E_\alpha$ imply $\ell_\alpha \langle x, z \rangle \leq \ell_\alpha \langle x, y \rangle \bullet \ell_\alpha \langle y, z \rangle$.

Choose an infinite cardinal κ such that $|A| \leq \kappa$. Let V be a set of cardinality κ, and let $\sigma : \kappa \to {}^3A \times {}^2V \times 3$ be such that

$$(\forall \langle a, b, c, x, y, i \rangle \in {}^3A \times {}^2V \times 3)(\forall \lambda < \kappa)(\exists \nu < \kappa)\lambda \leq \nu \wedge \sigma(\nu + 1) = \langle a, b, c, x, y, i \rangle .$$

To see that there is such a function σ, let $f : \kappa \to {}^3A \times {}^2V \times 3 \times \kappa$ be a bijection. If we fix a, b, c, x, y, i, then, for κ many ordinals γ, $f(\gamma) = \langle a, b, c, x, y, i, \delta \rangle$ for some $\delta < \kappa$. So, for each $\lambda < \kappa$, there is $\nu \geq \lambda$ such that $f(\nu) = \langle a, b, c, x, y, i, \delta' \rangle$ for some $\delta' < \kappa$. Let $g : {}^3A \times {}^2V \times 3 \times \kappa \to {}^3A \times {}^2V \times 3$ with $g\langle a, b, c, x, y, i, \lambda \rangle = \langle a, b, c, x, y, i \rangle$ for each $\lambda < \kappa$. If we define $\sigma(\nu + 1) = g(f(\nu))$ and $\sigma(\nu)$ arbitrary for limit $\nu < \kappa$, then σ meets the requirements.

0th step. For each element c of A, we choose two different elements from V, say u_c and v_c. Let $U_0 = \{u_c, v_c : c \in A\}$. We can assume that $|V \setminus U_0| = \kappa$. Let $E_0 = \{\langle u_c, v_c \rangle : c \in A\}$ and $\ell_0 \langle u_c, v_c \rangle = c$. Clearly, (I) and (II) hold.

$\alpha + 1$st step. Let $\sigma(\alpha + 1) = \langle c, a, b, x, y, i \rangle$. If $\ell_\alpha \langle x, y \rangle \neq c$, then go to α + 2nd step. Otherwise we have three subcases according to the value of i.

$i = 0$. Choose an element from $V \setminus U_\alpha$, say, u. Let

$$\begin{aligned} U_{\alpha+1} &= U_\alpha \cup \{u\} \\ E_{\alpha+1} &= E_\alpha \cup \{\langle u, p \rangle : \langle x, p \rangle \in E_\alpha\} \cup \{\langle u, x \rangle\} \\ \ell_{\alpha+1} &= \ell_\alpha \cup \{\langle \langle u, p \rangle, a \bullet \ell_\alpha \langle x, p \rangle \rangle : \langle x, p \rangle \in E_\alpha\} \cup \{\langle \langle u, x \rangle, a \rangle\}. \end{aligned}$$

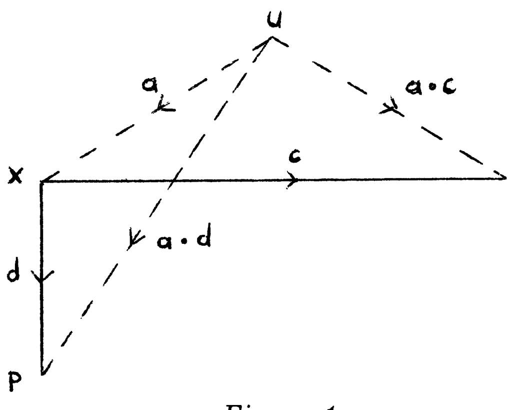

Figure 1.

$i = 1$. Choose an element from $V \setminus U_\alpha$, say, v. Let

$$\begin{aligned} U_{\alpha+1} &= U_\alpha \cup \{v\} \\ E_{\alpha+1} &= E_\alpha \cup \{\langle q, v \rangle : \langle q, y \rangle \in E_\alpha\} \cup \{\langle y, v \rangle\} \\ \ell_{\alpha+1} &= \ell_\alpha \cup \{\langle \langle q, v \rangle, \ell_\alpha \langle q, y \rangle \bullet a \rangle : \langle q, y \rangle \in E_\alpha\} \cup \{\langle \langle y, v \rangle, a \rangle\}. \end{aligned}$$

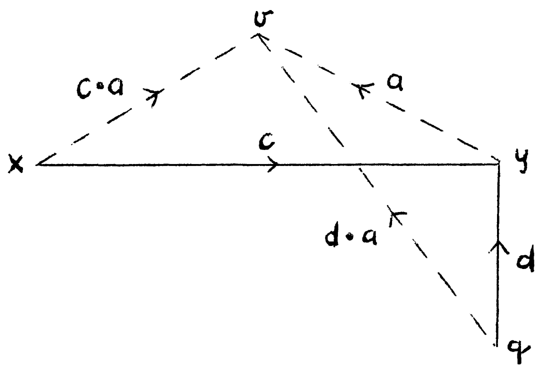

Figure 2.

$i = 2$. If $c \nleq a \bullet b$, then go to next step. Otherwise let $z \in V \setminus U_\alpha$ and

$$\begin{aligned} U_{\alpha+1} &= U_\alpha \cup \{z\} \\ E_{\alpha+1} &= E_\alpha \cup \{\langle r, z\rangle : \langle r, x\rangle \in E_\alpha\} \cup \{\langle z, s\rangle : \langle y, s\rangle \in E_\alpha\} \cup \{\langle x, z\rangle, \langle z, y\rangle\} \\ \ell_{\alpha+1} &= \ell_\alpha \cup \{\langle\langle x, z\rangle, a\rangle, \langle\langle z, y\rangle, b\rangle\} \cup \\ &\quad \{\langle\langle r, z\rangle, \ell_\alpha\langle r, x\rangle \bullet a\rangle : \langle r, x\rangle \in E_\alpha\} \cup \{\langle\langle z, s\rangle, b \bullet \ell_\alpha\langle y, s\rangle\rangle : \langle y, s\rangle \in E_\alpha\}. \end{aligned}$$

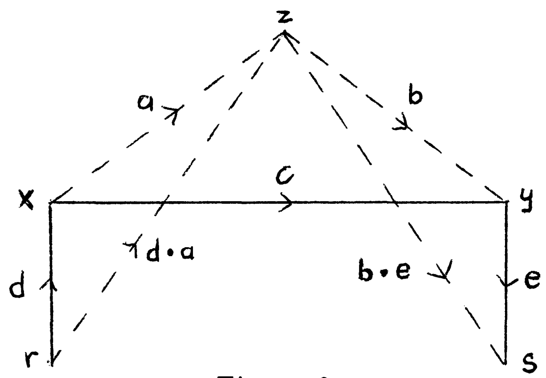

Figure 3.

It is easy to check that property (I) is preserved in the $\alpha + 1$st step.

We also have to prove that the new transitive triangles constructed in the $\alpha+1$st step have property (II). We have three cases according to the value of i above.

$i = 0$. We have to show that $a_5 \leq a_4 \bullet a_3$. By induction we have $a_2 \leq a_1 \bullet a_3$, so, using (A1), (A3) and (A9), $a_5 = a \bullet a_2 \leq a \bullet a_1 \bullet a_3 = a_4 \bullet a_3$, hence, by (A1) and (A8), $a_5 \leq a_4 \bullet a_3$.

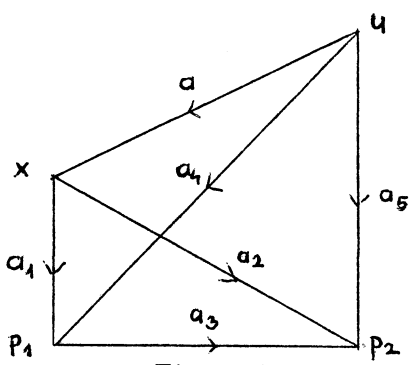

Figure 4.

$i = 1$. We want $a_5 \leq a_3 \bullet a_4$. By induction hypothesis $a_2 \leq a_3 \bullet a_1$, so $a_5 = a_2 \bullet a \leq a_3 \bullet a_1 \bullet a = a_3 \bullet a_4$.

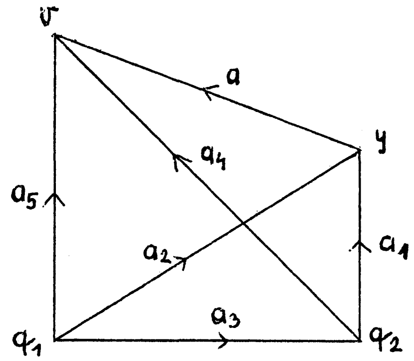

Figure 5.

$i = 2$. We show that $a_3 \leq a_5 \bullet a_6$. By induction hypothesis $a_3 \leq a_4 \bullet a_2$ and $a_4 \leq a_1 \bullet c$. So $a_3 \leq a_4 \bullet a_2 \leq a_1 \bullet c \bullet a_2 \leq a_1 \bullet a \bullet b \bullet a_2 = a_5 \bullet a_6$.

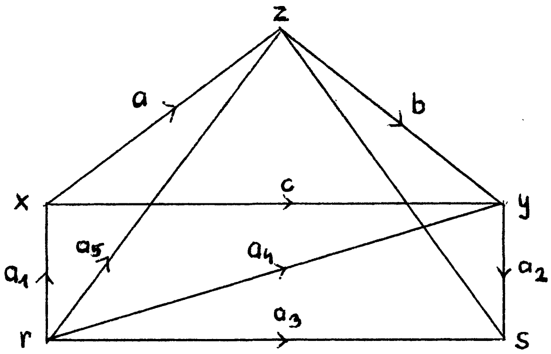

Figure 6.

Thus $G_{\alpha+1}$ satisfies (II) as well.

Limit step. If α is a limit ordinal, then let $U_\alpha = \bigcup_{\beta<\alpha} U_\beta$, $E_\alpha = \bigcup_{\beta<\alpha} E_\beta$ and $\ell_\alpha = \bigcup_{\beta<\alpha} \ell_\beta$.

Let $G = G_\kappa$, i.e.,

$$U = \bigcup_{\alpha<\kappa} U_\alpha, \qquad E = \bigcup_{\alpha<\kappa} E_\alpha \qquad \text{and} \qquad \ell = \bigcup_{\alpha<\kappa} \ell_\alpha.$$

Clearly, G satisfies (I) and (II).

Now, we are ready to define the representation function 'rep'. For every $c \in A$, let

$$\text{rep}(c) = \{\langle u, v\rangle : \ell\langle u, v\rangle \leq c\}.$$

We have to show that 'rep' is an isomorphism from $\mathfrak{A}$ to a structure whose elements are binary relations on the set of nodes of our graph.

We prove that 'rep' preserves '$\leq$', i.e.,

$$\text{if} \qquad a \leq b, \qquad \text{then} \qquad \text{rep}(a) \subseteq \text{rep}(b).$$

Indeed, if $\ell\langle u, v\rangle \leq a$, then, by (A8), $\ell\langle u, v\rangle \leq b$, so $\langle u, v\rangle \in \text{rep}(a)$ implies $\langle u, v\rangle \in \text{rep}(b)$.

Now we show that 'rep' is one-one, i.e.,

$$\text{if} \qquad a \neq b, \qquad \text{then} \qquad \text{rep}(a) \neq \text{rep}(b).$$

Indeed, if $\text{rep}(a) \subseteq \text{rep}(b)$, then for every $\langle u, v\rangle \in E$, if $\ell\langle u, v\rangle \leq a$, then $\ell\langle u, v\rangle \leq b$. Since $\ell\langle u_a, v_a\rangle = a$ (see the 0th step), we have $a \leq b$. By symmetry $b \leq a$ whenever $\text{rep}(b) \subseteq \text{rep}(a)$. Thus, by (A10), $a = b$.

We check that 'rep' preserves the operations too.

$$\begin{aligned}\text{rep}(a \bullet b) &= \{\langle u, v\rangle : \ell\langle u, v\rangle \leq a \bullet b\} \overset{\text{(i)}}{=} \{\langle u, v\rangle : \exists z(\ell\langle u, z\rangle \leq a \wedge \ell\langle z, v\rangle \leq b)\} = \\ &= \{\langle u, z\rangle : \ell\langle u, z\rangle \leq a\} \circ \{\langle z, v\rangle : \ell\langle z, v\rangle \leq b\} = \\ &= \text{rep}(a) \circ \text{rep}(b).\end{aligned}$$

(i) ($\subseteq$): Let $c = \ell\langle u, v\rangle$. Then, for some $\alpha + 1$, $\sigma(\alpha + 1) = \langle c, a, b, u, v, 2\rangle$. So in the $\alpha + 1$st step we put a z into the graph such that $\ell\langle u, z\rangle = a$ and $\ell\langle z, v\rangle = b$.

($\supseteq$): By properties (I) and (II), and by the transitivity of '$\leq$'.

$$\begin{aligned}\text{rep}(a\backslash b) &= \{\langle u, v\rangle : \ell\langle u, v\rangle \leq a\backslash b\} \overset{\text{(ii)}}{=} \{\langle u, v\rangle : a \bullet \ell\langle u, v\rangle \leq b\} \overset{\text{(iii)}}{=} \\ &= \{\langle u, v\rangle : \forall z(\ell\langle z, u\rangle \leq a \rightarrow \ell\langle z, u\rangle \bullet \ell\langle u, v\rangle \leq b)\} \overset{\text{(iv)}}{=} \\ &= \{\langle u, v\rangle : \forall z(\ell\langle z, u\rangle \leq a \rightarrow \ell\langle z, v\rangle \leq b)\} = \\ &= \text{rep}(a)\backslash\text{rep}(b).\end{aligned}$$

(ii) $c \leq a\backslash b$ iff $a \bullet c \leq b$. The proof is essentially the same as that of Lemma 3.

(iii) ($\subseteq$): By monotonicity of '$\bullet$'.

($\supseteq$): The following triangle is in the graph.

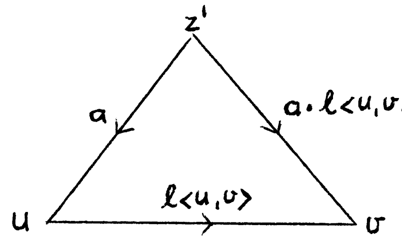

Figure 7.

(iv) ($\subseteq$): By properties (I) and (II).

($\supseteq$): The figure above is in G, so if $\ell\langle z, u\rangle \leq a$, then $\ell\langle z, u\rangle \bullet \ell\langle u, v\rangle \leq a \bullet \ell\langle u, v\rangle = \ell\langle z', u\rangle \bullet \ell\langle u, v\rangle = \ell\langle z', v\rangle \leq b$.

$$\begin{aligned}\text{rep}(b/a) &= \{\langle u, v\rangle : \ell\langle u, v\rangle \leq b/a\} \overset{\text{(v)}}{=} \{\langle u, v\rangle : \ell\langle u, v\rangle \bullet a \leq b\} \overset{\text{(vi)}}{=} \\ &= \{\langle u, v\rangle : \forall z(\ell\langle v, z\rangle \leq a \rightarrow \ell\langle u, v\rangle \bullet \ell\langle v, z\rangle \leq b)\} \overset{\text{(vii)}}{=} \\ &= \{\langle u, v\rangle : \forall z(\ell\langle v, z\rangle \leq a \rightarrow \ell\langle u, z\rangle \leq b)\} = \\ &= \text{rep}(b)/\text{rep}(a).\end{aligned}$$

(v) By the "translation" of Lemma 3.

(vi) ($\subseteq$): By monotonicity of '$\bullet$'.
($\supseteq$): The following picture is in G.

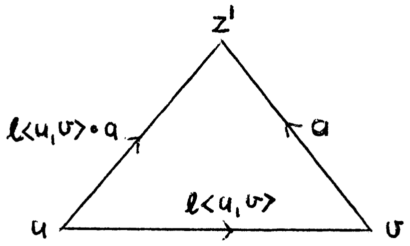

Figure 8.

(vii) ($\subseteq$): By properties (I) and (II).
($\supseteq$): By the picture above $\ell\langle u,v\rangle \bullet a \leq b$.
Thus 'rep' is the desired isomorphism, since the image of $\mathfrak{A}$, $\langle\{\text{rep}(a) : a \in A\}, \backslash, /, \circ, \subseteq\rangle$, is in RRS. So Theorem 2 is proved. ∎

REMARK: If, in the definition of Relational Semantics, we require that $W = U \times U$ for some set U, then (Weak) Completeness Theorem fails. Indeed, the sequent $y \Rightarrow y \bullet (y\backslash y)$ is valid in those RRS's in which W has the form $U \times U$, because $v(y\backslash y) \supseteq \text{Id} \cap (U \times U)$, i.e., the value of $y\backslash y$ contains the identity relation. On the other hand, let $W = \{\langle 0,0\rangle, \langle 0,1\rangle\}$ and consider $\mathcal{P}(W) \in$ RRS. Let $a = \{\langle 0,1\rangle\}$. Then $a\backslash a = \{\langle 0,0\rangle, \langle 0,1\rangle\}$ and $a \circ (a\backslash a) = \emptyset$, so $a \not\leq a \circ (a\backslash a)$. Thus the sequent $y \Rightarrow y \bullet (y\backslash y)$ is not valid in RRS and, therefore, it is not derivable in LC. See also [Do90].

Theorems 9 and 10 investigate this $U \times U$-type semantics and state strong completeness of certain versions of LC w.r.t. it.

Let LCD be LC plus the following two rules:

$$(6)\quad \frac{x \Rightarrow A \quad x \Rightarrow B}{x \Rightarrow A \sqcap B} \qquad (7)\quad \frac{x \Rightarrow A \sqcap B}{x \Rightarrow A \quad x \Rightarrow B}.$$

Let Θ be Σ plus the following two formulae:

$$(A10)\quad z \leq x \wedge z \leq y \rightarrow z \leq x \sqcap y \qquad (A11)\quad z \leq x \sqcap y \rightarrow z \leq x \wedge z \leq y.$$

THEOREM 3. *For each sequent φ of the language of* LCD *and set Γ of sequents,*

$$\Gamma \vdash_{\mathsf{LCD}} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathsf{R}'} \varphi$$

where in the definition of the consequence relation '$\models_{\mathsf{R}'}$' we further require that $v(A \sqcap B) = v(A) \cap v(B)$.

PROOF: Soundness is easy to check.

For the other direction, since the other modifications are obvious, we show that the above representation function remains isomorphism. Indeed, by (A10) and (A11),

$$\text{rep}(a \sqcap b) = \{\langle u,v\rangle : \ell\langle u,v\rangle \leq a \sqcap b\} = \{\langle u,v\rangle : \ell\langle u,v\rangle \leq a\} \cap \{\langle u,v\rangle : \ell\langle u,v\rangle \leq b\} = \\ = \text{rep}(a) \cap \text{rep}(b).$$

∎

COROLLARY 1. *If we add the axiom*

$$A, B \Rightarrow B \bullet A$$

to LC *(or to* LCD*), then we have Strong Completeness Theorem w.r.t.* Symmetric RelSem, *i.e., we consider only those models* $\langle W, C, v\rangle$ *for which* $v(A \bullet B) = v(B \bullet A)$ *for all formulae* A, B. *Moreover, we have Compactness Theorem too.*

PROOF: We add the formula

$$a \bullet b \leq b \bullet a$$

to Σ (or to Θ). Then

$$\text{rep}(a) \circ \text{rep}(b) = \text{rep}(a \bullet b) = \text{rep}(b \bullet a) = \text{rep}(b) \circ \text{rep}(a),$$

so 'rep' is an isomorphism again. Compactness can be proved as in Corollary 0. ∎

REMARK: As in the case of Theorem 0, we have also Weak Completeness Theorem, i.e., a sequent $A_1, \ldots, A_n \Rightarrow B$ is derivable in the above version of LC iff $v(A_1) \circ \cdots \circ v(A_n) \subseteq v(B)$ for every symmetric representable relational structure and valuation v.

Now we prove that strong completeness fails if we add '$\sqcup$' to the set of operations of LC.

DEFINITION: Let $\{\cup, \cap, \circ\} \subseteq M \subseteq \{\cup, \cap, \circ, -, {}^{-1}, \emptyset, \text{Id}, \backslash, /\}$. Then $\text{R}(M)$ is the class of all algebras (isomorphic to ones) whose elements are binary relations and whose operations are the members of M.

THEOREM 4. $\text{R}(M)$ *is a quasi-variety which is not finitely axiomatizable.*

PROOF: $\text{R}(M)$ is a quasi-variety because it is closed under **I**, **P**, **Pu** and **S** and contains the trivial algebra, since the class of all representable relational algebras, RRA, has these properties.

In the proof of Theorem 4, in [An91], Hajnal Andréka defines algebras A_n whose operations are the members of a fixed M' which satisfies $\{\sqcap, \sqcup, \bullet\} \subseteq M' \subseteq \{\sqcap, \sqcup, \bullet, -, \breve{}, 0, 1'\}$, and whose '$\sqcap, \sqcup, \bullet$'-reducts are not representable while their ultraproduct is representable.

If we add '$\backslash$' and '$/$' to the operations of A_n, then the '$\sqcap, \sqcup, \bullet$'-reducts are still not representable, and the ultraproduct of these algebras is representable, since '$\backslash$' and '$/$' are term definable. Indeed, $a\backslash b = -(\breve{a} \bullet (-b))$ and $b/a = -((-b) \bullet \breve{a})$. Thus the above mentioned proof of Andréka works in our case as well. ∎

Let $\text{Qe}(\text{R}(M))$ denote the class of all quasi-equations that hold in $\text{R}(M)$.

COROLLARY 2. $\text{Qe}(\text{R}(M))$ *is not axiomatizable by finitely many quasi-equations.*

PROOF: Since $\text{R}(M)$ is a quasi-variety, it is axiomatized by the quasi-equations that hold in it. So if $\Sigma \models \text{Qe}(\text{R}(M))$ (and Σ is valid in $\text{R}(M)$), then $\text{Mod}(\Sigma) = \text{R}(M)$. Thus Σ cannot be finite. ∎

THEOREM 5. *The Relational Semantics with a set of connectives M has no strongly complete and sound inference system.*

PROOF: For the sake of simplicity, we will prove the $M = \{\cap, \cup, \circ\}$ case.

Assume that there is a sound and strongly complete inference system, say L. Let 'tr' be a function translating the sequent schemes and rules to equations and quasi-equations of the language of R(M), respectively, and satisfying the following. We assume that there are so many formula variables (X_i) as algebraic variables (x_i) and they are enumerated. Let

$$\mathrm{tr}(X_i) = x_i \qquad \mathrm{tr}(A \sqcap B) = \mathrm{tr}(A) \cap \mathrm{tr}(B) \qquad \mathrm{tr}(A \sqcup B) = \mathrm{tr}(A) \cup \mathrm{tr}(B)$$
$$\mathrm{tr}(A, B) = \mathrm{tr}(A \bullet B) = \mathrm{tr}(A) \circ \mathrm{tr}(B) \qquad \mathrm{tr}(A \Rightarrow B) = (\mathrm{tr}(A) \subseteq \mathrm{tr}(B))$$

for formula schemes A, B. Moreover, let

$$\mathrm{tr}(\frac{\varphi_1 \cdots \varphi_n}{\varphi_0}) = (\mathrm{tr}(\varphi_1) \wedge \cdots \wedge \mathrm{tr}(\varphi_n)) \rightarrow \mathrm{tr}(\varphi_0)$$

for sequent schemes $\varphi_0, \ldots, \varphi_n$.

Let Ξ be the translation of L. Then, since R(M) cannot be finitely axiomatized, there is a quasi-equality e such that $\mathrm{R}(M) \models e$ and $\Xi \not\models e$. We can assume that e and the elements of Ξ are closed formulae, so there is an algebra $\mathfrak{B}$ such that $\mathfrak{B} \models \Xi$ and $\mathfrak{B} \not\models e$. Let $e = ((e_1 \wedge \cdots \wedge e_n) \rightarrow e_0)$.

Now, we extend the translation function, 'tr', to the set of sequents in the following way. For any sequent $A_1, \ldots, A_n \Rightarrow A_0$, let

$$\mathrm{tr}(A_1, \ldots, A_n \Rightarrow A_0) = (\mathrm{tr}(A_1(p_j : X_j)) \circ \cdots \circ \mathrm{tr}(A_n(p_j : X_j)) \subseteq \mathrm{tr}(A_0(p_j : X_j)))$$

where $A_i(p_j : X_j)$ denotes the formula scheme given by substituting the jth formula variable (X_j) for the jth element (p_j) of the set of basic symbols P.

LEMMA 6. *Let, for* $1 \leq i \leq n$, $\varphi_i^1 = (\tau_i \Rightarrow \sigma_i)$, $\varphi_i^2 = (\sigma_i \Rightarrow \tau_i)$, *and* $(\mathrm{tr}(\tau_i) = \mathrm{tr}(\sigma_i)) = e_i$. *Assume that*

$$\{\varphi_1^1,\ \varphi_1^2,\ \ldots,\ \varphi_n^1,\ \varphi_n^2\} \vdash_{\mathrm{L}} \varphi_0^1.$$

Then, for every $\mathfrak{B}$,

$$\textit{if} \quad \mathfrak{B} \models \Xi, \quad \textit{then} \quad \mathfrak{B} \models (e_1 \wedge \cdots \wedge e_n) \rightarrow \mathrm{tr}(\tau_0 \Rightarrow \sigma_0).$$

PROOF: Assume that $\{\varphi_1^1,\ \varphi_1^2,\ \ldots,\ \varphi_n^1,\ \varphi_n^2\} \vdash_{\mathrm{L}} \varphi_0^1$ and $\mathfrak{B}$ is such that $\mathfrak{B} \models \Xi$. We use induction on the derivation of φ_0^1.

Base step. $\varphi_0^1 = \chi(X_j : \theta_j)$ is an axiom, i.e., it is given by substituting formulae (θ_j) for variables (X_j) in an axiom scheme (χ). Then $\mathrm{tr}(\chi) \in \Xi$.

Now, let $k : \{x_j : j \in J\} \rightarrow B$ be arbitrary, and let $k'(x_j) = \mathrm{tr}(X_j)[k'] = (\mathrm{tr}(\theta_j))[k]$. Since χ does not contain any p_i (so $\chi(X_j : (\theta_j(p_i : X_i))) = ((\chi(X_j : \theta_j))(p_i : X_i)))$ and $(\mathrm{tr}(\theta_j))[k] = (\mathrm{tr}(\theta_j(p_i : X_i)))[k]$, we have

$$(\mathrm{tr}(\varphi_0^1))[k] = (\mathrm{tr}(\varphi_0^1(p_i : X_i)))[k] = (\mathrm{tr}((\chi(X_j : \theta_j))(p_i : X_i)))[k] =$$
$$= (\mathrm{tr}(\chi(X_j : \theta_j(p_i : X_i))))[k] = (\mathrm{tr}(\chi))[k'].$$

Since $\mathfrak{B} \models \mathrm{tr}(\chi)$, we also have $\mathfrak{B} \models \mathrm{tr}(\varphi_0^1)$.

Induction step. There are $\psi_1, \ldots, \psi_m$ such that, for each $1 \leq i \leq m$,

$$\{\varphi_1^1,\ \varphi_1^2,\ \ldots,\ \varphi_n^1,\ \varphi_n^2\} \vdash_{\mathrm{L}} \psi_i$$

and $\frac{\psi_1 \ldots \psi_m}{\varphi_0^1}$ is a rule instance, i.e., $\psi_i = \chi_i(X_j : \theta_j)$ and $\varphi_0^1 = \chi(X_j : \theta_j)$, and $\frac{\chi_1 \ldots \chi_m}{\chi}$ is a rule of L.

Assume we know the lemma for $\psi_1, \ldots, \psi_m$. Again, let $k : \{x_j : j \in J\} \to B$ be arbitrary, and let $k'(x_j) = (\mathrm{tr}(\theta_j))[k]$. Since $\mathrm{tr}(\frac{\chi_1 \cdots \chi_m}{\chi}) \in \Xi$, we have

$$\mathfrak{B} \models (\mathrm{tr}(\chi_1) \wedge \cdots \wedge \mathrm{tr}(\chi_m) \to \mathrm{tr}(\chi))[k'].$$

Assume

$$\mathfrak{B} \models (e_1 \wedge \cdots \wedge e_n)[k],$$

i.e., $\mathfrak{B} \models (\mathrm{tr}(\varphi_j^1))[k]$ and $\mathfrak{B} \models (\mathrm{tr}(\varphi_j^2))[k]$ for $1 \leq j \leq n$. Then, by hypothesis, for $1 \leq i \leq m$, $\mathfrak{B} \models (\mathrm{tr}(\psi_i))[k]$, i.e., $\mathfrak{B} \models (\mathrm{tr}(\chi_i))[k']$. Then $\mathfrak{B} \models (\mathrm{tr}(\chi))[k']$, i.e., $\mathfrak{B} \models (\mathrm{tr}(\varphi_0^1))[k]$ as desired. ∎

Then, since, for the above $\mathfrak{B}$ and e, $\mathfrak{B} \not\models (e_1 \wedge \cdots \wedge e_n) \to e_0$, i.e., $\mathfrak{B} \not\models (e_1 \wedge \cdots \wedge e_n) \to \mathrm{tr}(\tau_0 \Rightarrow \sigma_0)$ or $\mathfrak{B} \not\models (e_1 \wedge \cdots \wedge e_n) \to \mathrm{tr}(\sigma_0 \Rightarrow \tau_0)$, by Lemma 6, either

$$\{\varphi_1^1,\ \varphi_1^2,\ \ldots,\ \varphi_n^1,\ \varphi_n^2\} \nvdash_{\mathrm{L}} \varphi_0^1 \qquad \text{or} \qquad \{\varphi_1^1,\ \varphi_1^2,\ \ldots,\ \varphi_n^1,\ \varphi_n^2\} \nvdash_{\mathrm{L}} \varphi_0^2.$$

On the other hand, $\mathrm{R}(M) \models e$ means that, for every $\mathfrak{A} \in \mathrm{R}(M)$, $\mathfrak{A} \models e$. Thus, for every valuation k, if $\mathfrak{A} \models e_i[k]$ for $1 \leq i \leq n$, then $\mathfrak{A} \models e_0[k]$, i.e.,

$$\{\varphi_1^1,\ \varphi_1^2,\ \ldots,\ \varphi_n^1,\ \varphi_n^2\} \models_{\mathrm{R}(M)} \varphi_0^k$$

for $k = 1, 2$, where '$\models_{\mathrm{R}(M)}$' is defined in the obvious way. Contradiction. ∎

Now we prove that LC is not (weakly) complete w.r.t. language models (LM) and that there is no extension of LC which is sound w.r.t. $U \times U$-type Relational Semantics and is strongly complete w.r.t. LM. First, we recall the definition of language models from [vB91, p.189].

Definition of Language Model: A *family of languages* is a set $\{L_i : i \in I\}$, where L_i is a set of finite sequences (words) over a finite alphabet.

A *language model* is a family of languages enriched with the following operations.

$$L_a \bullet L_b \stackrel{\text{def}}{=} \{xy : x \in L_a,\ y \in L_b\}$$
$$L_a \backslash L_b \stackrel{\text{def}}{=} \{x : (\forall y \in L_a) yx \in L_b\}$$
$$L_b / L_a \stackrel{\text{def}}{=} \{x : (\forall y \in L_a) xy \in L_b\}.$$

A sequent $A_1, \ldots, A_n \Rightarrow A_0$ is *true* in a language model if

$$v(A_1) \bullet \cdots \bullet v(A_n) \subseteq v(A_0)$$

where v is the *valuation* function defined in the obvious way. The consequence relation '$\models_{\mathrm{LM}}$' is the usual as well.

Theorem 6. LC *is not (weakly) complete w.r.t. language models.*

Proof: By the definition of '$\backslash$' the empty sequence ($\ominus$) is in $L \backslash L$ for every language L. Thus $x \Rightarrow x \bullet (x \backslash x)$ is valid in every language model.

On the other hand, if $W = a = \{\langle 0, 1\rangle\}$, then $a \not\subseteq a \circ (a \backslash a)$ in the RRS with universe W, so, by Theorem 0, $x \Rightarrow x \bullet (x \backslash x)$ is not deducible in LC. (See also the Remark after the proof of Theorem 2.) ∎

Let '$\models_{\mathsf{R}^+}$' be that semantic consequence relation which is determined by those relational Kripke models where $W = U \times U$.

THEOREM 7. *There is no calculus containing* LC *which is sound w.r.t.* $U \times U$*-type Relational Semantics and strongly complete w.r.t. language models.*

PROOF: We will show that there are a set Γ of sequents and a sequent φ such that $\Gamma \not\models_{\mathsf{R}+} \varphi$ but $\Gamma \models_{\mathrm{LM}} \varphi$.

Let $u = v$ abbreviate that $u \Rightarrow v$ and $v \Rightarrow u$. It is easy to check that $\{x = x \bullet x,\ y = y \bullet y,\ x = y \bullet x,\ y = x \bullet y\} \not\models_{\mathsf{R}+} y = x$ (let $\mathrm{rep}(x) = \{\langle 1,0\rangle, \langle 0,0\rangle\}$ and $\mathrm{rep}(y) = \{\langle 0,1\rangle, \langle 1,1\rangle\}$).

On the other hand, $\{x = x \bullet x,\ y = y \bullet y,\ x = y \bullet x,\ y = x \bullet y\} \models_{\mathrm{LM}} y = x$ because of the following. Let 'rep' be a function representing x and y as languages. If $\mathrm{rep}(x) = \emptyset$, then $\mathrm{rep}(y) = \emptyset \bullet \mathrm{rep}(y) = \emptyset$. So we can assume that $\mathrm{rep}(x) \neq \emptyset$. Let $a \in \mathrm{rep}(x)$. Then, since $\mathrm{rep}(x) = \mathrm{rep}(x) \bullet \mathrm{rep}(x)$, there are $x_1, y_1 \in \mathrm{rep}(x)$ such that $a = x_1 y_1$. By the same argument, for each number i, there are $x_{i+1}, y_{i+1} \in \mathrm{rep}(x)$ with $x_i = x_{i+1} y_{i+1}$. Sooner or later, since a is finite, either x_i or y_i is the empty sequence ($\ominus$). So $\ominus \in \mathrm{rep}(x)$. Then, since $\mathrm{rep}(x) = \mathrm{rep}(y) \bullet \mathrm{rep}(x)$, $b = b\ominus \in \mathrm{rep}(x)$ for every $b \in \mathrm{rep}(y)$, i.e., $\mathrm{rep}(y) \subseteq \mathrm{rep}(x)$. Mutatis mutandis, $\mathrm{rep}(x) \subseteq \mathrm{rep}(y)$, hence $\mathrm{rep}(x) = \mathrm{rep}(y)$. ∎

COROLLARY 3. LC^0, *that version of the Lambek Calculus where we admit sequents with empty antecedent, is not strongly complete w.r.t.* LM.

PROOF: LC^0 contains LC. ∎

* * *

Now we turn to investigating the connection between $U \times U$-type Relational Semantics and (various versions of) the Lambek Calculus. The main results are Theorems 9 and 10, but, as before, the piths of the proofs are two representation theorems (Theorems 8 and 11).

Let Σ^+ be Σ plus the following four formulae.

$$x \leq y \rightarrow z \leq z \bullet (x \backslash y) \qquad x \leq y \rightarrow z \leq (x \backslash y) \bullet z$$
$$x \leq y \rightarrow z \leq z \bullet (y / x) \qquad x \leq y \rightarrow z \leq (y / x) \bullet z$$

Let RRS^+ be the class of those $\mathfrak{A} \in \mathsf{RRS}$, where there is a $\mathfrak{B} \in \mathsf{fullRRS}$ such that $B = \mathcal{P}(U \times U)$ for some set U, and $\mathfrak{A}$ is a substructure of $\mathfrak{B}$.

THEOREM 8. (Andréka–Mikulás) *For every* $\mathfrak{A} \in \mathsf{RS}$,

$$\mathfrak{A} \models \Sigma^+ \qquad \textit{iff} \qquad \mathfrak{A} \in \mathsf{IRRS}^+.$$

PROOF: The 'if' part is easy and omitted.

Assume that $\mathfrak{A} \models \Sigma^+$. Then we will construct, as in the case of Theorem 2, a directed and labelled graph, and we will define the representation function using this graph.

Let $G = \langle V, E, \ell \rangle$, where V is the set of nodes, $E = V \times V$ is the set of edges and $\ell : E \rightarrow \mathcal{P}(A)$ is the labelling function. G will have the following five propeties.

(I) $(\forall u, v, w \in V)(\forall a, b)(a \in \ell\langle u, w\rangle \wedge b \in \ell\langle w, v\rangle \rightarrow \exists c(c \leq a \bullet b \wedge c \in \ell\langle u, v\rangle))$
(II) $(\forall u, v \in V)(\forall a, b, c \in A)(a \leq b \bullet c \wedge a \in \ell\langle u, v\rangle \rightarrow (\exists w \in V) b \in \ell\langle u, w\rangle \wedge c \in \ell\langle w, v\rangle)$
(III) $(\forall u \in V)(\forall a \in A)\exists w(a \in \ell\langle w, u\rangle \wedge (\forall v \in V) u \neq v \rightarrow \ell\langle w, v\rangle = \{a \bullet h : h \in \ell\langle u, v\rangle\})$
(IV) $(\forall v \in V)(\forall a \in A)\exists w(a \in \ell\langle v, w\rangle \wedge (\forall u \in V) u \neq v \rightarrow \ell\langle u, w\rangle = \{h \bullet a : h \in \ell\langle u, v\rangle\})$
(V) $(\forall u \in V)\ell\langle u, u\rangle \supseteq I$,

where $I = \{a\backslash b : a \leq b\} \cup \{b/a : a \leq b\}$.

We will define G by recursion. Let κ and σ be as in the proof of Theorem 2. We will use the following notation. If $X, Y \subseteq A$, then let $X \bullet Y = \{x \bullet y : x \in X,\ y \in Y\}$.

0th step. Let $V_0 = \{u_a, v_a : a \in A\}$, $E_0 = V_0 \times V_0$ and $W = \{\langle u_a, v_a\rangle, \langle u_a, u_a\rangle, \langle v_a, v_a\rangle : a \in A\}$. Moreover, let $\ell_0 \langle u_a, v_a\rangle = \{a\}$ and $\ell_0 \langle u_a, u_a\rangle = \ell_0 \langle v_a, v_a\rangle = I$, and let $\ell_0 \langle u, v\rangle = \emptyset$ if $\langle u, v\rangle \in V_0 \times V_0 \smallsetminus W$.

(I) holds because of the new formulae in Σ^+, and (V) is satisfied as well.

$\alpha + 1$st step. Let $\sigma(\alpha + 1) = \langle a, b, c, x, y, i\rangle$. We have three subcases according to the value of i.

$i = 0$. Let z be a new point ($z \notin V_\alpha$), and let

$$\begin{aligned} V_{\alpha+1} &= V_\alpha \cup \{z\} \\ E_{\alpha+1} &= V_{\alpha+1} \times V_{\alpha+1} \\ \ell_{\alpha+1} &= \ell_\alpha \cup \{\langle\langle z, z\rangle, I\rangle,\ \langle\langle z, x\rangle, \{a\} \bullet \ell_\alpha \langle x, x\rangle \cup \{a\}\rangle\} \cup \\ &\quad \{\langle\langle z, p\rangle, \{a\} \bullet \ell_\alpha \langle x, p\rangle\rangle : p \in V_\alpha \wedge p \neq x\} \cup \{\langle\langle p, z\rangle, \emptyset\rangle : p \in V_\alpha\}. \end{aligned}$$

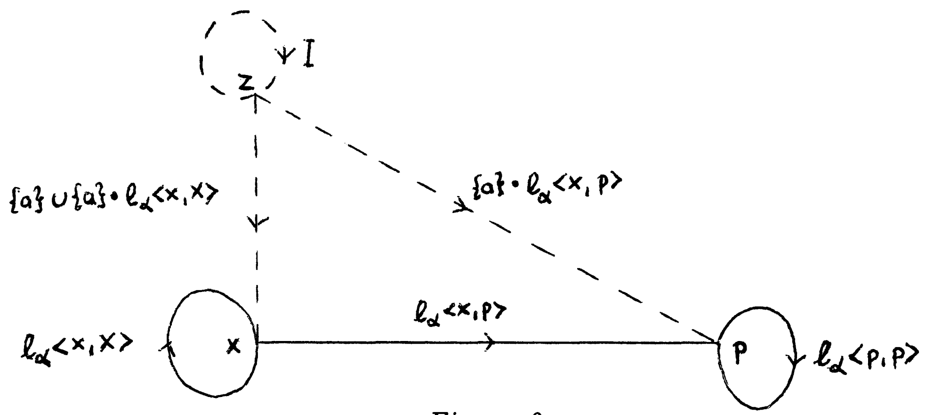

Figure 9.

$i = 1$. Let z be a new point, and let

$$\begin{aligned} V_{\alpha+1} &= V_\alpha \cup \{z\} \\ E_{\alpha+1} &= V_{\alpha+1} \times V_{\alpha+1} \\ \ell_{\alpha+1} &= \ell_\alpha \cup \{\langle\langle z, z\rangle, I\rangle,\ \langle\langle y, z\rangle, \ell_\alpha \langle y, y\rangle \bullet \{a\} \cup \{a\}\rangle\} \cup \\ &\quad \{\langle\langle q, z\rangle, \ell_\alpha \langle q, y\rangle \bullet \{a\}\rangle : q \in V_\alpha \wedge q \neq y\} \cup \{\langle\langle z, q\rangle, \emptyset\rangle : q \in V_\alpha\}. \end{aligned}$$

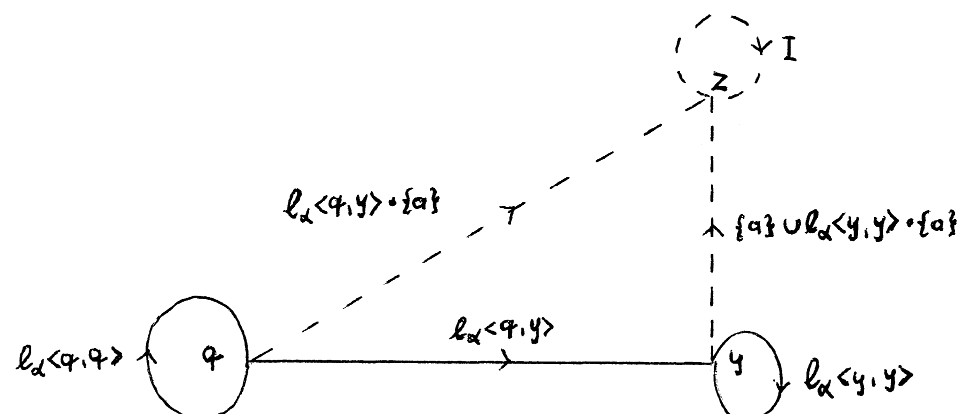

Figure 10.

$i = 2$. If $a \not\leq b \bullet c$, or $a \notin \ell_\alpha \langle x, y \rangle$, then go to $\alpha + 2$nd step. Otherwise let z be a new point, and let

$$\begin{aligned} V_{\alpha+1} &= V_\alpha \cup \{z\} \\ E_{\alpha+1} &= V_{\alpha+1} \times V_{\alpha+1} \\ \ell_{\alpha+1} &= \ell_\alpha \cup \{\langle \langle z, z \rangle, \{c\} \bullet \ell_\alpha \langle y, x \rangle \bullet \{b\} \cup I \rangle\} \cup \\ &\quad \{\langle \langle x, z \rangle, \ell_\alpha \langle x, x \rangle \bullet \{b\} \cup \{b\} \rangle\} \cup \{\langle \langle r, z \rangle, \ell_\alpha \langle r, x \rangle \bullet \{b\} \rangle : r \in V_\alpha \wedge r \neq x\} \cup \\ &\quad \{\langle \langle z, y \rangle, \{c\} \bullet \ell_\alpha \langle y, y \rangle \cup \{c\} \rangle\} \cup \{\langle \langle z, s \rangle, \{c\} \bullet \ell_\alpha \langle y, s \rangle \rangle : s \in V_\alpha \wedge s \neq y\}. \end{aligned}$$

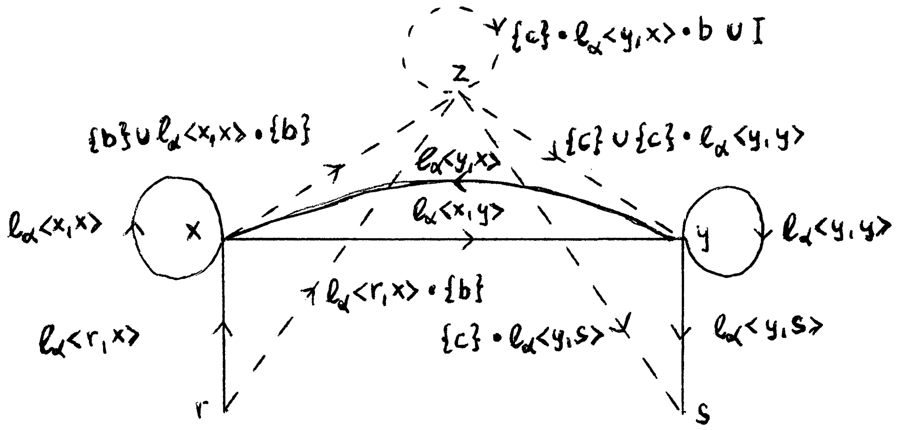

Figure 11.

Limit step. If α is a limit ordinal, then let

$$V_\alpha = \bigcup_{\beta<\alpha} V_\beta, \qquad E_\alpha = \bigcup_{\beta<\alpha} E_\beta, \qquad \ell_\alpha = \bigcup_{\beta<\alpha} \ell_\beta.$$

Let $G = G_\kappa$. G satisfies (I) and (II), since in each step these properties were preserved (it is a mechanical and tiresome calculation). Moreover, (II), (III) and (IV) are realized by the construction.

Let, for every $a \in A$,

$$\text{rep}(a) = \{\langle u, v \rangle : (\exists h \in \ell \langle u, v \rangle) h \leq a\}.$$

Then 'rep' clearly preserves '$\leq$', and is one-one because of the 0th step in the construction.

Now we show that 'rep' is a homomorphism. First we show that

$$\text{rep}(a) \circ \text{rep}(b) = \text{rep}(a \bullet b).$$

Indeed, if $\langle u, v \rangle \in \text{rep}(a) \circ \text{rep}(b)$, then

$$\exists w((\exists h_a \in \ell \langle u, w \rangle) h_a \leq a \wedge (\exists h_b \in \ell \langle w, v \rangle) h_b \leq b)$$

and, by (I),

$$\exists w(\exists h \in \ell \langle u, v \rangle)(\exists h_a \in \ell \langle u, w \rangle)(\exists h_b \in \ell \langle w, v \rangle) h \leq h_a \bullet h_b \leq a \bullet b,$$

i.e., $\langle u, v \rangle \in \text{rep}(a \bullet b)$. The other direction is a straightforward consequence of (II).

We also have

$$\text{rep}(a\backslash b) \subseteq \text{rep}(a)\backslash\text{rep}(b),$$

since if $\langle u, v\rangle \in \text{rep}(a\backslash b)$, then $(\exists h \in \ell\langle u,v\rangle)h \leq a\backslash b$, so, by (I),

$$\forall w((\exists h_a \in \ell\langle w,u\rangle)h_a \leq a \rightarrow (\exists h' \in \ell\langle w,v\rangle)h' \leq a \bullet (a\backslash b) \leq b),$$

i.e., $\forall w\,\langle w,u\rangle \in \text{rep}(a) \rightarrow \langle w,v\rangle \in \text{rep}(b)$ whence $\langle u,v\rangle \in \text{rep}(a)\backslash\text{rep}(b)$.

To show that

$$\text{rep}(a)\backslash\text{rep}(b) \subseteq \text{rep}(a\backslash b)$$

we have to distinguish two cases. In the first case, we assume that $u \neq v$ and $\langle u,v\rangle \in \text{rep}(a)\backslash\text{rep}(b)$. Then

$$\forall w(\langle w,u\rangle \in \text{rep}(a) \rightarrow \langle w,v\rangle \in \text{rep}(b)),$$

i.e.,

$$\forall w((\exists h_a \in \ell\langle w,u\rangle)h_a \leq a \rightarrow (\exists h_b \in \ell\langle w,v\rangle)h_b \leq b),$$

so, by (III),

$$\exists w((\exists h \in \ell\langle u,v\rangle)(\exists h_b \in \ell\langle w,v\rangle)a \bullet h = h_b \leq b).$$

Thus $(\exists h \in \ell\langle u,v\rangle)a \bullet h \leq b$, so $(\exists h \in \ell\langle u,v\rangle)h \leq a\backslash b$, i.e., $\langle u,v\rangle \in \text{rep}(a\backslash b)$.

Now we assume that $u = v$, i.e., $\langle u,u\rangle \in \text{rep}(a)\backslash\text{rep}(b)$. By the construction $\exists w(\ell\langle w,u\rangle = \{a\} \bullet \ell\langle u,u\rangle \cup \{a\})$, so we conclude that

$$\exists w(a \leq b \vee (\exists h \in \ell\langle u,u\rangle)(\exists h_b \in \ell\langle w,u\rangle)a \bullet h = h_b \leq b).$$

Then, by (V), and because $\ell\langle u,u\rangle \subseteq I$, $(\exists h \in \ell\langle u,u\rangle)h \leq a\backslash b$, i.e., $\langle u,u\rangle \in \text{rep}(a\backslash b)$.

Similar argument, using (IV), shows that

$$\text{rep}(a/b) = \text{rep}(a)/\text{rep}(b).$$

Since the 'rep'-image of $\mathfrak{A}$ is a representable structure, we are done. ∎

Let LC^+ be LC plus the following four rules.

$$\frac{A \Rightarrow B}{C \Rightarrow C \bullet (A\backslash B)} \qquad \frac{A \Rightarrow B}{C \Rightarrow (A\backslash B) \bullet C}$$

$$\frac{A \Rightarrow B}{C \Rightarrow C \bullet (B/A)} \qquad \frac{A \Rightarrow B}{C \Rightarrow (B/A) \bullet C}$$

As before, let '$\models_{\mathsf{R}+}$' be that semantic consequence relation which is determined by those relational Kripke models where $W = U \times U$.

THEOREM 9. (Andréka–Mikulás) (Strong Completeness Theorem for LC^+ w.r.t. RelSem) *For any set Γ of sequents and sequent φ,*

$$\Gamma \vdash_{\text{LC}+} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathsf{R}+} \varphi.$$

PROOF: As before, one can prove that the factor structure of the formula algebra of LC^+ (by the congruence relation '$\equiv_\Gamma$', defined as before) satisfies Σ^+. Then Theorem 8 gives the proof, as Theorem 2 did in the case of Theorem 0. ∎

Now, we turn to showing that if we allow empty sequence to be antecedent of sequents in the Lambek Calculus, then it will become strongly complete w.r.t. $U \times U$-type Relational Semantics.

Let the language of LC^0 be defined as that of LC except that we do not exclude the sequents $A_1, \ldots, A_n \Rightarrow A_0$ where $n = 0$, i.e., we allow sequents with empty sequence as antecedent. These sequents will be denoted as $\Rightarrow A_0$ or $\ominus \Rightarrow A_0$.

Let LC^0 be given by the axiom (0) and rules $(1\backslash), \ldots, (5)$ without any restriction, i.e., any sequence of sequents (denoted as x, y or z) can be empty. Let '$\models_{\mathsf{R}+}$' be as above. Then the following theorem holds.

THEOREM 10. (Andréka–Mikulás) (Strong Completeness Theorem for LC^0 w.r.t. RelSem) *Let $\Gamma \cup \{\varphi\}$ be a set of sequents in the language of* LC^0. *Then*

$$\Gamma \vdash_{\mathrm{LC}^0} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathsf{R}+} \varphi.$$

Before proving Theorem 10, we will prove a representation theorem. To do this we need the following definitions.

DEFINITION OF RRS^0:

$$\mathfrak{A} \in \mathsf{fullRRS}^0 \qquad \text{iff} \qquad \mathfrak{A} = \langle A, \backslash, /, \circ, \subseteq, \mathrm{Id}_U, \emptyset \rangle$$

where $A = \mathcal{P}(U \times U)$ for some set U, and $\langle A, \backslash, /, \circ, \subseteq \rangle \in \mathsf{RRS}$. Further, $\mathrm{Id}_U = \{\langle u, u \rangle : u \in U\}$, and the empty set, $\emptyset$, is considered as a binary relation.

Let $\mathsf{RRS}^0 = \mathsf{SfullRRS}^0$.

DEFINITION OF RS^0:

$$\mathfrak{A} \in \mathsf{RS}^0 \qquad \text{iff} \qquad \mathfrak{A} = \langle A, \backslash, /, \bullet, \leq, e, 0 \rangle$$

where $\langle A, \backslash, /, \bullet, \leq \rangle \in \mathsf{RS}$ and $e, 0 \in A$.

Let Σ^0 be Σ plus the following formulae

$$e \bullet x = x \bullet e = x \qquad 0 \bullet x = x \bullet 0 = 0 \qquad 0 \leq x.$$

Let Δ be the set of the following formulae

$$x \bullet y = 0 \longleftrightarrow (x = 0 \vee y = 0)$$
$$x \bullet y \leq e \longleftrightarrow (x = 0 \vee y = 0 \vee x = y = e).$$

Note that Δ is not valid in RRS^0 (while Σ^0 is). That is why, in the following theorem, only one direction is stated.

THEOREM 11. (Andréka–Mikulás) *For every* $\mathfrak{A} \in \mathsf{RS}^0$,

$$\textit{if} \qquad \mathfrak{A} \models \Sigma^0 \cup \Delta, \qquad \textit{then} \qquad \mathfrak{A} \in \mathsf{IRRS}^0.$$

PROOF: We make essentially the same construction as in the proof of Theorem 8 with some modifications.

We will construct a directed and labelled graph, $G = \langle V, E, \ell \rangle$, satisfying the following six properties. Properties (I), (II) and (V) will be the same as in the proof of Theorem 8. We require properties (III) and (IV) only for $a \in A \setminus \{e, 0\}$. The graph will have this feature too:

(VI) $(\forall \langle u, v \rangle \in E) 0 \notin \ell \langle u, v \rangle \wedge (e \in \ell \langle u, v \rangle \rightarrow u = v)$.

Let σ and κ be as before. We define the graph by recursion using the original construction in the proof of Theorem 8.

0th step. This is the same as before, we just choose v_a and u_a for $a \in A \setminus \{e, 0\}$ only.

$\alpha + 1$st step. Let $\sigma(\alpha + 1) = \langle a, b, c, x, y, i \rangle$. We have three subcases according to the value of i again.

$i = 0$ or $i = 1$. Do the original construction.

$i = 2$. If $a \not\leq b \bullet c$, or $a \notin \ell_\alpha \langle x, y \rangle$, then go to next step. Otherwise, by property (VI), we have that $0 \notin \{b, c\}$. If $b = e$ or $c = e$, then go to next step. Otherwise, by Δ, $b \not\leq e$ and $c \not\leq e$. In this case, do the original construction.

Limit step. Take the union as before.

Let $G = G_\kappa$. Then properties (I)–(V) can be checked as before. (VI) is clearly preserved in each step.

Let

$$\text{rep}(a) = \{\langle u, v \rangle : (\exists h \in \ell\langle u, v \rangle) h \leq a\}.$$

One can check, in the usual way, that 'rep' is an isomorphism. ∎

Now we are ready to prove the completeness theorem.

PROOF OF THEOREM 10: Let Γ be an arbitrary set of sequents of LC^0. We begin the proof by defining an analogue, $\mathfrak{L}^0_\Gamma$, of the factor structure of LC, $\mathfrak{L}_\Gamma$.

Let $e, 0, 1$ be three new elements not in $\mathrm{Form}_{\mathrm{LC}}$, and let $T = \mathrm{Form}_{\mathrm{LC}} \cup \{e, 0, 1\}$. Let

$$\mathfrak{T} = \langle T, \backslash, /, \bullet, \leq_\Gamma, e, 0 \rangle$$

where the definitions of the operations and the relation '$\leq_\Gamma$' go as follows. On $\mathrm{Form}_{\mathrm{LC}}$ these are defined as before. For every $x \in T$ and $A \in \mathrm{Form}_{\mathrm{LC}}$, let $0 \leq_\Gamma x \leq_\Gamma 1$ and $e \leq_\Gamma e$, and let $e \leq_\Gamma A$ iff $\Gamma \vdash_{\mathrm{LC}^0} \ominus \Rightarrow A$. Let $0 \bullet x = x \bullet 0 = 0$ and $e \bullet x = x \bullet e = x$, and if $x \neq 0$, then let $1 \bullet x = x \bullet 1 = 1$. Let $0 \backslash x = 1$ and $e \backslash x = x$, and if $x \neq 0$, then let $x \backslash 0 = 0$. Moreover, if $x \notin \{e, 0\}$, then let $x \backslash e = 0$ and $x \backslash 1 = 1$. Finally, if $x \neq 1$, then let $1 \backslash x = 0$. The other slash, '/', can be defined in a similar way. Let

$$x \equiv_\Gamma y \qquad \text{iff} \qquad (x \leq_\Gamma y \qquad \text{and} \qquad y \leq_\Gamma x),$$

and let $\mathfrak{L}^0_\Gamma = (\mathfrak{T}/{\equiv_\Gamma})$.

LEMMA 7.

$$\mathfrak{L}^0_\Gamma \models \Sigma^+ \cup \Delta.$$

PROOF: It is easy to check, using the definition above, that '$\leq_\Gamma$' is a partial ordering, and that '$\bullet$' is an associative operation which is monotonic w.r.t. '$\leq_\Gamma$'. It is not difficult to show, by case distinction, that if $a \leq_\Gamma b$ and $c \leq_\Gamma d$, then $a \bullet c \leq_\Gamma b \bullet d$. The rest of Σ^+ is easy, by the definitions of the slashes, and Δ holds, by the definition of '$\bullet$', as well. ∎

By the lemma above and Theorem 11 $\mathfrak{L}^0_\Gamma \in \mathsf{IRRS}^0$.

From now on, the proof of Theorem 10 proceeds as the proof of Theorem 0. In more detail, let φ be a sequent of LC^0 and assume that $\Gamma \not\vdash_{\mathrm{LC}^0} \varphi$. Let φ be $A_1, \ldots, A_n \Rightarrow B$. Now, assume that $n = 0$, i.e., φ is $\ominus \Rightarrow B$. Then, as in the proof Lemma 4, it can be verified, by the definition of '$\leq_\Gamma$', that $\overline{e} \not\leq_\Gamma \overline{B}$ in $\mathfrak{L}_\Gamma$. Then we have a relational Kripke model (with a set of possible worlds of the form $U \times U$) which falsifies φ and satisfies Γ. That is, $\Gamma \not\models_{\mathsf{R}+} \varphi$. The case $n > 0$ is analogous to the previous proof. Thus Theorem 10 is proved. ∎

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