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SEQUENT CALCULI FOR NORMAL MODAL PROPOSITIONAL LOGICS

Heinrich Wansing

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Sequent calculi for normal modal propositional logics

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Gentzen's proof-theoretical methods
have not yet been properly applied to modal logic.
Serebriannikov [1982, p. 79]

Abstract. In this paper we present a systematic sequent-style proof theory for many important systems of normal modal propositional logic based on classical propositional logic CPL. After discussing philosophical, methodological, and computational aspects of the problem of Gentzenizing modal logic, we introduce a variant of Belnap's Display Logic and show that within this proof theory the modal axiom schemes D, T, 4, 5, and B (and some others) can be captured by characteristic structural inference rules. We show that for all sequent systems under consideration (i) Cut is admissible, (ii) the subformula property holds, and (iii) all connectives are uniquely characterized. Eventually we briefly deal with modal systems based on substructural subsystems of CPL.

The aim of this work is to present a systematic and perspicious sequent-style proof theory for many axiomatic systems of normal modal propositional logic based on classical propositional logic CPL. As Segerberg [Bull & Segerberg 1984] has observed, "Gentzen methods have never really flourished in modal logic, but some work has been done, mostly on sequent formulations". We first briefly review several known sequent-style proof systems for various normal modal propositional logics. Subsequently, in Section 2 we lay down certain philosophical, methodological, and computational requirements which the proof theory to be presented should fulfill. It will turn out that each of the earlier sequent systems falls short of satisfying some of these conditions. In Section 3, an informal outline is given of Belnap's [1982, 1990] Display Logic **DL**. The proof theory of the present paper is a modification of **DL** and is developed at length in Section 4. Finally, extensions to modal systems on a substructural base are considered in Section 5.

The Gentzen-style proof theory to be developed is only a slight and simple modification of Belnap's **DL**. Nevertheless it seems worth a separate presentation and discussion, since

1. It allows an extension to a broader class of substructural base logics than Belnap's **DL**.

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2. It associates the *unary* modal operators with a *unary* structural connective and not with a binary structural operation as in [Belnap 1982].

3. Whereas Belnap considers modal logic as comprising connectives from *two different families*, we regard the language $L_{\{\mathbf{1},\mathbf{0},\neg,\wedge,\vee,\supset,\Box,\Diamond\}}$ of modal logic as a *one-family* language.

4. Belnap's application of **DL** to modal propositional logics in [Belnap 1982] is somewhat sketchy.

5. There seems to be a need for a more systematic methodological discussion of Gentzenizing modal propositional logic.

Moreover, **DL** does not yet seem to have attracted the attention it deserves.

1 Some known sequent systems for normal modal logics

1.1 Standard systems

Sequent systems for the axiomatic calculi $S4$ ($= KT4$) and $S5$ ($KT5$) are known for a long time. The following schematic sequent rules for $\Box$ and $\Diamond$ go back to Curry and Feyes, and have been studied by Ohnishi and Matsumoto [1957]:

$$\begin{array}{ll}
(\rightarrow \Box)_0 & \Box X \rightarrow \Box Y, A \vdash \Box X \rightarrow \Box Y, \Box A \\
(\Box \rightarrow)_0 & X, A \rightarrow Y \vdash X, \Box A \rightarrow Y \\
(\rightarrow \Diamond)_0 & X \rightarrow Y, A \vdash X \rightarrow Y, \Diamond A \\
(\Diamond \rightarrow)_0 & \Diamond Y, A \rightarrow \Diamond X \vdash \Diamond Y, \Diamond A \rightarrow \Diamond X.
\end{array}$$

Here X and Y range over finite sets of formulas and $\Box X$ ($\Diamond X$) denotes $\{\Box A \mid A \in X\}$ ($\{\Diamond A \mid A \in X\}$). If either the rules $(\rightarrow \Box)_0$ and $(\Box \rightarrow)_0$ or the rules $(\rightarrow \Diamond)_0$ and $(\Diamond \rightarrow)_0$ are added to (an appropriate version of) the standard sequent system $LCPL$ for CPL, then the result is a sequent calculus $LS5$ for $S5$. Various other modal propositional logics can be obtained by modifying $(\rightarrow \Box)_0$ resp. $(\Diamond \rightarrow)_0$. If Y is empty in $(\rightarrow \Box)_0$ resp. in $(\Diamond \rightarrow)_0$, this yields a sequent calculus $LS4$ for $S4$. Ohnishi and Matsumoto also show that if $(\rightarrow \Box)_0$ resp. $(\Diamond \rightarrow)_0$ is replaced by

$$\begin{array}{lll}
(\rightarrow \Box)_1 & X \rightarrow A \vdash \Box X \rightarrow \Box A & \text{resp.} \\
(\Diamond \rightarrow)_1 & A \rightarrow Y \vdash \Diamond A \rightarrow \Diamond Y, &
\end{array}$$

one obtains a Gentzen-system for T ($= KT$).[1] If one just adds $(\rightarrow \Box)_1$ to $LCPL$, this results in a sequent calculus LK for the minimal normal modal propositional logic K (see e.g. [Leivant 1981], [Sambin & Valentini 1982], [Mints 1990]). A sequent calculus $LK4$ for $K4$ can for instance be obtained by adding to $LCPL$ the rule

[1] It has been observed by Routley [1975] that the equivalences between $\Box A$ and $\neg\Diamond\neg A$, and $\Diamond A$ and $\neg\Box\neg A$ cannot be proved by means of Ohnishi's and Matsumoto's rules.

$$(\to \Box)_2 \quad X, \Box X \to A \vdash \Box X \to \Box A$$

(see [Sambin & Valentini 1982]). As shown in [Goble 1974], the pair of modal sequent rules $(\to \Box)_1$ and

$$(\Box \to)_1 \quad X, A \to \emptyset \vdash \Box X, \Box A \to \emptyset,$$

yields a sequent system for KD, and, if $(\to \Box)_1$ is then modified into the rule

$$(\to \Box)_3 \quad X' \to A \vdash \Box X \to \Box A,$$

where X' results from X by prefixing zero or more formulas in X by $\Box$, one obtains a sequent calculus for $KDT4$. Shvarts [1989] gives a sequent calculus formulation of $K45$ by means of supplementing $LCPL$ with the following rule for $\Box$:

$$[\Box] \quad \Box X_1, X_2 \to \Box Y_1, Y_2 \vdash \Box X_1, \Box X_2 \to \Box Y_1, \Box Y_2,$$

where Y_2 contains at most one formula. If in addition Y_1 and Y_2 are required to be non-empty, this results in a sequent system for $KD45$. Avron [1984] (see also [Shimura 1991]) presents a sequent calculus $LS4Grz$ for $S4Grz$ (= $KGrz$) by replacing the rule $(\to \Box)_0$ in Ohnishi's and Matumoto's sequent calculus for $S4$ by the rule

$$(\to \Box)_4 \quad \Box(A \supset \Box A), \Box X \to A \vdash \Box X \to \Box A.$$

1.2 Higher-level systems

Došen [1985] has developed certain non-standard sequent systems for $S4$ and $S5$. In these Gentzen-style systems one is dealing with sequents of arbitrary finite level. Sequents of level 1 are like ordinary sequents, whereas sequents of level $n+1$ ($n > 0$) have finite sets of sequents of level n on both sides of the sequent arrow. Moreover, the main sequent arrow in a sequent of level n carries the superscript ^{n}, and $\emptyset$ is regarded as a set of any finite level. The rules for logical operations are presented as *double-line* rules. A double-line rule

$$\frac{s_1, \ldots, s_n}{s_0}$$

involving sequents $s_0, \ldots, s_n$, denotes the rules

$$\frac{s_1, \ldots, s_n}{s_0}, \quad \frac{s_0}{s_1}, \ldots, \frac{s_0}{s_n}.$$

Došen gives the following double-line sequent rules for $\Box$ and $\Diamond$:

$$\frac{X + \{\emptyset \to^1 \{A\}\} \to^2 X_2 + \{X_3 \to^1 X_4\}}{X_1 \to^2 X_2 + \{X_3 + \{\Box A\} \to^1 X_4\}} \qquad \frac{X_1 + \{\{A\} \to^1 \emptyset\} \to^2 X_2 + \{X_3 \to^1 X_4\}}{X_1 \to^2 X_2 + \{X_3 \to^1 X_4 + \{\Diamond A\}\}},$$

where $+$ refers to the union of disjoint sets. If these rules together with the axiomatic sequents

$$\{X_1 + \{\emptyset \rightarrow^1 \{A\}\} \rightarrow^2 X_2 + \{X_3 \rightarrow^1 X_4\}\} \rightarrow^3 \{X_1 \rightarrow^2 X_2 + \{X_3 + \{\Box A\} \rightarrow^1 X_4\}\}$$
$$\{X_1 \rightarrow^2 X_2 + \{X_3 + \{\Box A\} \rightarrow^1 X_4\}\} \rightarrow^3 \{X_1 + \{\emptyset \rightarrow^1 \{A\}\} \rightarrow^2 X_2 + \{X_3 \rightarrow^1 X_4\}\}$$
$$\{X_1 + \{\{A\} \rightarrow^1 \emptyset\} \rightarrow^2 X_2 + \{X_3 \rightarrow^1 X_4\}\} \rightarrow^3 \{X_1 \rightarrow^2 X_2 + \{X_3 \rightarrow^1 X_3 + \{\Diamond A\}\}\}$$
$$\{X_1 \rightarrow^2 X_2 + \{X_3 \rightarrow^1 X_3 + \{\Diamond A\}\}\} \rightarrow^3 \{X_1 + \{\{A\} \rightarrow^1 \emptyset\} \rightarrow^2 X_2 + \{X_3 \rightarrow^1 X_4\}\}$$

are added to Došen's higher-level sequent calculus Cp/D for CPL, this results in the sequent system S5p/D for $S5$. The sequent calculus S4p/D for $S4$ is then obtained by imposing a structural restriction on the monotonicity (or thinning) rule of level 2:

$$\frac{X \rightarrow^2 Y}{X \cup Z_1 \rightarrow^2 Y \cup Z_2}.$$

The restriction is this: if $Y = \emptyset$, then Z_2 must be a singleton or empty; if $Y \neq \emptyset$, then Z_2 must be empty. If the same restriction is applied to monotonicity of level 1 in Cp/D, then this gives one a higher-level sequent system for intuitionistic propositional logic IPL.

2 Desiderata

2.1 Rules as meaning assignments

Gentzen-style proof theory is usually associated with a certain philosophy of meaning. The idea is that the schematic introduction rules for an n-ary connective $\sharp$, together with a set of structural assumptions, specify the meaning of $\sharp$, which has certain consequences for the format of rules. In the first place, the meaning assignment should not make the meaning of $\sharp$ dependent on the meaning of other connectives. That is to say, the sequent rules for $\sharp$ should give a purely structural account of $\sharp$'s meaning, in the sense that they should not exhibit any connective other than $\sharp$. This property may be called *separation.*[2] Moreover, the rules for $\sharp$ should be *weakly symmetric*; every rule should either belong to a set of rules ($\rightarrow \sharp$) which introduce $\sharp$ into premises (i.e. on the left side of $\rightarrow$ in the conclusion sequent) or to a set of rules ($\sharp \rightarrow$) which introduce $\sharp$ into conclusions (i.e. on the right side of $\rightarrow$ in the conclusion sequent). The sequent rules for $\sharp$ can then be called *symmetric*, if they are weakly symmetric and both ($\rightarrow \sharp$) and ($\sharp \rightarrow$) are non-empty. The sequent rules for $\sharp$ will be called *weakly explicit*, if the rules ($\rightarrow \sharp$) and ($\sharp \rightarrow$) exhibit $\sharp$ in their conclusion sequents only, and they will be called *explicit*, if in addition to being weakly explicit, the rules in ($\rightarrow \sharp$) resp. ($\sharp \rightarrow$) exhibit only one occurrence of $\sharp$ on the right resp. the left side of $\rightarrow$. Separation, symmetry, and explicitness of the rules imply that in a sequent calculus for a given logic $\mathcal{L}$, every connective that is explicitly definable in $\mathcal{L}$ also has separated, symmetric, and explicit introduction rules.[3] Therefore we would like to have rules for $\Box$ and $\Diamond$ as primitives. With CPL as our base logic, these rules should allow us to prove $\Box A \rightarrow \neg\Diamond\neg A$, $\neg\Diamond\neg A \rightarrow \Box A$, $\Diamond A \rightarrow \neg\Box\neg A$, and $\neg\Box\neg A \rightarrow \Diamond A$. Let us call this property (*inter* $\Box\Diamond$).

[2] Cf. [Zucker & Tragesser 1978].

[3] These rules can be found by decomposition of the defined connective. We have to assume that $\mathcal{L}$ has a compositional semantics, or, in syntactic terms, that the deductive role of $\sharp(A_1, \ldots, A_n)$ depends on the deductive relationships between $A_1, \ldots, A_n$ only.

It can easily be verified that each of the standard rule systems presented in the previous section fails to satisfy some of the philosophical requirements mentioned so far:[4]

	sep.	weak sym.	sym.	weak expl.	expl.	(*inter*□◇)
LK	√	√	-	√	-	-
LT	√	√	√	√	-	-
$LK4$	√	√	-	-	-	-
$LK45$	√	√	-	-	-	-
LKD	√	√	√	√	-	-
$LKDT4$	√	√	√	-	-	-
$LKD45$	√	√	-	-	-	-
$LS4$	√	√	√	-	-	-
$LS4Grz$	-	√	√	-	-	-
$LS5$	√	√	√	-	-	-

Note that Došen's [1985] higher-level sequent rules for □ and ◇ being *double-line rules* do not satisfy weak symmetry and weak explicitness. However, (*inter*□◇) holds for these rules.

2.2 Uniqueness

Suppose that $\mathcal{L}$ is a logical system with a syntactic presentation S in which the connective $\sharp$ occurs. Let S^* be the result of rewriting $\sharp$ everywhere in S as $\sharp^*$, and let $\mathcal{LL}^*$ be the system presented by the union of S and S^* in the language with both $\sharp$ and $\sharp^*$. Let $A_\sharp$ denote a formula (in this language) that contains a certain occurrence of $\sharp$, and let $A_{\sharp^*}$ denote the result of replacing this occurrence of $\sharp$ in A by $\sharp^*$. The connectives $\sharp$ and $\sharp^*$ are said to be *uniquely characterized* in $\mathcal{LL}^*$ iff for every formula A in the language of $\mathcal{LL}^*$, $A_\sharp$ is provable in SS^* iff $A_{\sharp^*}$ is provable in SS^*. Unique characterization of the logical operations of a system $\mathcal{L}$ can be considered as a desirable property of a syntactic presentation of $\mathcal{L}$. Došen [1985] has proved that unique characterization is a non-trivial property and that the connectives in his higher-level systems S4p/D and S5p/D are uniquely characterized.

2.3 The range of systems: axioms versus rules

Compared with the multitude of not only existing but also interesting axiomatically presentable normal modal propositional logics, the number of systems for which sequent calculus presentations (of some sort) are known is disappointingly small. In contrast to the axiomatic approach, the standard sequent-style proof theory for normal modal logic fails to be 'modular', and the very mechanism behind the small range of known possible variations is not very clear. Also in Došen's higher-level framework it is not clear how restrictions similar to the one used to obtain S4p/D from S5p/D would allow to capture further axiomatic systems of normal modal propositional logic.[5] One might be inclined to agree with Segerberg's [Bull & Segerberg 1984, p. 30] remark (in connection with natural deduction systems for modal logic) that "only exceptional systems ... seem to be

[4] The rule [□] of $LK45$ and $LKD45$ can be considered either as a left or a right rule for □.

[5] This point of view is shared by Cerrato [1990, p. 1].

characterizable in terms of reasonably simple rules". Applying Segerberg's methodology, on the basis of the above sequent system LK for K "different logics would have to be characterized by special axioms. This means giving up the idea of finding characteristic rules for those systems." Apart from the absence of symmetric and explicit introduction rules for $\Box$ and $\Diamond$, the problem is that it is simply not clear which parameters could be *systematically* modifed so as to obtain characteristic sequent rules. Structural constraints like those mentioned in the previous section simply do not seem to give enough systematic flexibility. What one would need, it seems, is an extension of the usual Gentzen format that (i) conforms to the usual philosophy of meaning present in studies inspired by Gentzen, and (ii) offers sufficient degrees of structural freedom. The proof theory we are in search of should exemplify a principle that has most emphatically been advocated by Došen [1985, 1988] and that may therefore be called *Došen's Principle*:

> "[T]he rules for the logical operations are never changed: all changes are made in the structural rules" [Došen 1988, p. 352].

We shall take as our basic modal system the minimal normal modal propositional logic K, that is to say the introduction rules for $\Box$ and $\Diamond$ should be such that, together with appropriate structural assumptions, their addition to a suitable sequent calculus for CPL results in a sequent system LK for K. Sequent calculus presentations of certain extensions of K by modal axiom schemes should then be obtainable by adding suitable structural inference rules to LK. It would be nice, if quite a few important systems of normal modal logic will turn out to be 'Gentzenizable'. Of interest would then be a *classification* of the systems which can be Gentzenized and those which cannot.

2.4 Subformula property and Cut-freedom

Certainly, "Hilbert systems are not suited for the purpose of actual deductions" [Bull & Segerberg 1984, p. 28]. In order to be computationally attractive, a sequent calculus presentation $L\mathcal{L}$ of an axiomatic system $\mathcal{L}$ should, however, enjoy certain well-known properties. According to Sambin and Valentini [1982, p. 316], it "is usually not difficult to choose suitable [sequent] rules for each modal logic if one is content with completeness of rules. The real problem however is to find a set of rules also satisfying the subformula-property". Moreover, if in $\mathcal{L}$, $\vdash A$, then it would be nice, if $\emptyset \rightarrow A$ has a Cut-free proof in $L\mathcal{L}$. In sequent systems whose rules are separate, symmetric, and weakly explicit the redundancy of Cut implies the subformula-property, if these systems do not comprise silly structural rules like $X, Y \rightarrow Z \vdash X \rightarrow Z$. In principle, it is desirable to have Cut as an *admissible* rule. Ohnishi's and Matsumoto's $LS5$ does not allow Cut-elimination (see [Ohnishi & Matsumoto 1959] for an example of an $S5$-theorem without any Cut-free proof in $LS5$). Apart from this system the sequent calculi presented in Section 1.1 allow Cut-elimination. The sequent calculi for $S5$ in [Mints 1970] and [Sato 1980], although admitting of Cut-elimination, do not have the subformula-property. In Došen's higher-level calculi Cut of all levels is not eliminable [1985, Lemma 1].

2.5 Digressions

2.5.1 The constructive approach of Benevides and Maibaum

Recently, Benevides and Maibaum [1992] have suggested to present $\Box$ as a constructive connective in a generalized system of natural deduction. In this system, a modal propositional logic is presented as a denumerable series of sets of modal formulas Σ_0, Σ_1, Σ_2 .... In every Σ_i, the formulas are indexed by $\vdash_{\Sigma_i}$, and Σ_i is supposed to be closed under the usual natural deduction rules for CPL. In addition, for every Σ_i, there are introduction rules for conjunctions and disjunctions prefixed by $\Box$ and elimination rules for conjunctions and implications prefixed by $\Box$. This latter set of rules consists of the modal distribution principles valid in K and is called R_2. Moreover, there is a set R_3 of introduction and elimination rules for completely unspecified formulas A in the scope of one occurrence of $\Box$, which are based on an interaction between sets Σ_i, Σ_{i+1}:

$\Box$**-I** $\dfrac{\vdash_{i+1} A}{\vdash_i \Box A}$ where the proof of $\vdash_{i+1} A$ does not depend on an undischarged assumption in Σ_{i+1}

$\Box$**-E** $\dfrac{\vdash_i \Box A}{\vdash_{i+1} A}$.

Benevides and Maibaum [1992, p. 45] state a theorem to the effect that, if no further rules are assumed, $\Sigma_0 = K$, where Σ_0 may be any (initial) set of modal formulas. Proof systems for certain extensions of K are obtained by adding further proof rules which operate within the sets Σ_i, e.g. the rule associated with T is

$$\frac{\vdash_i A}{\vdash_i \Box A}.$$

Obviously, these additional rules violate Došen's Principle. Moreover, no rules are given for $\Diamond$ as a primitive operation.

2.5.2 Cerrato's framework

An extended Gentzen-style proof-theoretic framework for normal modal propositional logics has also been suggested by Cerrato [1990]. In this sequent calculus framework a formula A may be signed as $\langle A\rangle$ or $[A]$ indicating that A occurs "in a modal (possible or necessary) way" [1990, p. 1]. There are four types of sequent rules: (i) structural rules including a reflexivity rule and Cut for arbitrary expressions, (ii) 'logical rules' for the classical connectives, (iii) modal rules for the axiom schemes K, D, T, 4, 5, B, and (iv) 'duality rules':

$$\begin{array}{ll}
[A], X \rightarrow Y \vdash \Box A, X \rightarrow Y & X \rightarrow Y, [A] \vdash X \rightarrow Y, \Box A \\
\langle A\rangle, X \rightarrow Y \vdash \Diamond A, X \rightarrow Y & X \rightarrow Y, \langle A\rangle \vdash X \rightarrow Y, \Diamond A \\
X \rightarrow Y, [A] \vdash \langle\neg A\rangle, X \rightarrow Y & [A], X \rightarrow Y \vdash X \rightarrow Y, \langle\neg A\rangle \\
X \rightarrow Y, \langle A\rangle \vdash [\neg A], X \rightarrow Y & \langle A\rangle, X \rightarrow Y \vdash X \rightarrow Y, [\neg A].
\end{array}$$

These duality rules make sure that ($inter\Box\Diamond$) holds and that one can prove $[A] \rightarrow \Box A$, $\Box A \rightarrow [A]$, $\langle A\rangle \rightarrow \Diamond A$, and $\Diamond A \rightarrow \langle A\rangle$. Although copying $\Box A$ and $\Diamond A$ as structural elements $[A]$ resp. $\langle A\rangle$ introduces a certain amount of flexibility, Došen's principle fails to be satisfied: some of Cerrato's modal rules exhibit $\Box$ or $\Diamond$, like for instance the rules corresponding with 5:

$$[A], X \to Y \vdash \langle \Box A \rangle, X \to Y \quad X \to Y, \langle A \rangle \vdash X \to Y, [\Box A].$$

Moreover, Cerrato proves Cut-elimination only for his sequent system for K.

3 The solution: some general remarks

The proof theory to be presented is a modification of Belnap's [1982, 1990] Display Logic **DL** which is a very elegant and general proof-theoretic framework. At the moment we are not interested in **DL**'s full generality, but rather in a case study application of its modified version to normal modal propositional logics based on CPL. Therefore our presentation of modified **DL** will to a large extent be restricted to this limited purpose.[6] **DL** is rooted in two fundamental and ingenious ideas. (i) In **DL** the usual inventory of structural connectives in sequent systems is slightly extended. This extension is already inherent in the standard presentation of $LCPL$. Usually, sequent calculi dispose of only one structural connective, viz. the comma ',', which can take an arbitrary finite number of arguments. This connective is to be interpreted as Boolean conjunction in the antecedent and as Boolean disjunction in the succedent of a sequent. Instead of working with the polyvalent ',', Belnap uses the *binary* structure connective '$\circ$' and also introduces the structural constant **I** to denote the empty sequence. In antecedent position **I** is to be understood as the truth constant **1** and in succedent position **I** is to be understood as the falsum constant **0**. Moreover, there is a unary structural 'shift connective' * (to use terminology from [Gabbay 1991]) that shuffles structures from the right side to the left side of $\to$, and conversely, and that is used to introduce negation as a 'declaratively identical' operation in the logical language. While a polyvalent connective similar to $\circ$ is already present in ordinary Gentzen-systems, **I** and * are 'copied' from the logical language into the language describing the structure of premises and conclusions. (ii) Some systems comprise connectives of different sorts, or from different *families*. Belnap e.g. thinks of modal systems based on CPL as combinations of Boolean connectives with intensional connectives. Drawing on work of Dunn and Mints in relevance logic, he associates with each family of logical connectives its own set of structural connectives. That is to say, the basic structural connectives can be indexed as belonging to certain families of connectives, say, $\mathbf{I}_b$, $\circ_b$, $*^b$, and $\mathbf{I}_m$, $\circ_m$, $*^m$, where b stands for 'Boolean' and m for some system of modal propositional logic. The combination of both ideas turns out to be flexible enough to provide sequent calculi for certain 'hybrid' systems from relevance logic. In contrast to Belnap, however, we shall *not* consider the language of modal logic as hybrid. Whereas for Belnap this language combines the Boolean family with the modal one, we think of dealing with only one family, viz. the Boolean-modal family. (Other families e.g. are the intuitionistic-modal and the relevant-modal ones.) Accordingly, our application of **DL** to modal logic will differ from Belnap's [1982]. Instead of working with two sets of structural connectives, we shall introduce an additional unary structural connective $\bullet$ that will be used to formulate introduction rules for the modal operators.

[6] See, however, the remarks in Section 5 on substructural base logics.

Belnap's approach. Let us briefly describe Belnap's [1982] application of **DL** to modal logic. Belnap assumes structural rules to the effect that $\mathbf{I}_m = \mathbf{I}_b$ and ${}^{*m} = {}^{*b}$ and uses the binary modal structural connective $\circ_m$ to give introduction rules for $\Box$ and $\Diamond$:

$$
\begin{array}{ll}
(\Box) & X \circ_m \mathbf{I}_m \to A \vdash X \to \Box A \\
 & A \to X \vdash \Box A \to \mathbf{I}_m^{*m} \circ_m X \\
(\Diamond) & X \to A \vdash (X^{*m} \circ_m \mathbf{I}_m)^{*m} \to \Diamond A \\
 & X^{*m} \circ_m A \to \mathbf{I}_m^{*m} \vdash \Diamond A \to X.
\end{array}
$$

Now, in order to prove equivalences with axiomatic systems, one would like to translate sequents into formulas of $L_{\{\mathbf{1},\mathbf{0},\neg,\wedge,\vee,\supset,\Box,\Diamond\}}$. In the absence of the B-axiom-scheme $A \supset \Box\Diamond A$, however, $(X \circ_m Y)$ cannot always be translated into $L_{\{\mathbf{1},\mathbf{0},\neg,\wedge,\vee,\supset,\Box,\Diamond\}}$. The reason is that in antecedent position, $(X \circ_m Y)$ is considered to hold at a world y in a Kripke frame $\langle W, R\rangle$ just in case there is a world $x \in W$, xRy, the translation of X holds at x, and the translation of Y holds at y. (In succedent position $(X \circ_m Y)$ is considered to hold at a world y just in case for every world $x \in W$, if xRy, then the translation of X holds at y or the translation of Y holds at y.) In the standard vocabulary such a 'backwards-looking' operation cannot be defined on non-symmetric frames. Therefore one has to consider conservative extensions by a logical connective corresponding to $\circ_m$ in antecedent position. One can then show that, in the presence of suitable structural inference rules, Belnap's connective rules $(\Box)$ and $(\Diamond)$ give rise to a Cut-free sequent system LK for K with separated, symmetric, and explicit introduction rules. Moreover, sequent calculus presentations of various axiomatic extensions of K can be obtained by adding suitable further structural rules to LK, that is to say Došen's Principle is satisfied.

A moment's reflection about the use of $\mathbf{I}_m = \mathbf{I}$ in Belnap's rules $(\Box)$ and $(\Diamond)$ shows that it would be enough to use a *unary* structural connective instead of the binary $\circ_m$, and that this unary connective is just the tense logical operation P ("sometimes in the past"). In other words, sequents can be translated into the simple (modal) tense logical language $L_{\{\mathbf{1},\mathbf{0},\neg,\wedge,\vee,\supset,\Box,\Diamond,P\}}$. (If $\mathcal{L}$ is a normal modal propositional logic that is sound and complete for a certain class F of Kripke frames, then let $\mathcal{L}P$ denote the logic of F in $L_{\{\mathbf{1},\mathbf{0},\neg,\wedge,\vee,\supset,\Box,\Diamond,P\}}$. Clearly, $\mathcal{L}P$ is a conservative extension of $\mathcal{L}$.) In antecedent position our unary operation $\bullet$ will be nothing but P at the structural level. In succedent position $\bullet$ is to be understood as $\Box$. Note that P is a natural companion to $\Box$; it is the converse of the dual of $\Box$. With this modification we shall still be able to benefit from Belnap's general development of **DL**, in particular from the Display Theorem and his admirably general Cut-elimination theorem. Moreover, we can give more general basic rules governing the structural connectives.

4 The proof systems

In **DL** there are first of all certain basic structural rules which are postulated for *every* system of **DL**. These basic structural rules describe the 'geometry' of structures. Moreover, there are logical rules and connective rules, which also remain the same for every display logic. And there are additional structural rules which may successively be added to the basic structural rules, the logical rules, and the connective rules. It is well known that in the presence of certain structural rules certain connectives can become interreplaceable. Since in what follows we shall consider modal systems based on CPL, many structural rules will already be present, and we need not consider **DL**'s full inventory of non-modal formula connectives here, but may stick to the language $L_{\{\mathbf{1},\mathbf{0},\neg,\wedge,\vee,\supset,\Box,\Diamond\}}$. Formulas (structures) are built up from propositional variables and formula connectives (formulas and structure connectives) (of any family) in the obvious way. A substructure of a structure is positive (negative), if it is in the scope of an even (uneven) number of $*$'s. We shall use p, p_1, p_2 ... to denote propositional variables, A, B, C, A_1, A_2 ... to denote formulas, and X, Y, Z, X_1, X_2, ... to denote structures. An expression $X \rightarrow Y$ is called a sequent; the structure X (Y) is the antecedent (succedent) of $X \rightarrow Y$. An antecedent (succedent) part of a sequent $X \rightarrow Y$ is a positive substructure of X or a negative substructure of Y (a positive substructure of Y or a negative substructure of X). We shall use $\mathbf{I}$, $\circ$, $*$, and $\bullet$ to denote the Boolean-modal structural connectives; the basic structural rules are, however, postulated for every family of structural connectives.

Logical rules:

$$\text{(Id)} \quad \vdash p \rightarrow p \qquad \text{(Cut)} \quad X \rightarrow A \quad A \rightarrow Y \vdash X \rightarrow Y.$$

Basic structural rules:

$$\begin{array}{ll}
(1) & X \circ Y \rightarrow Z \dashv\vdash X \rightarrow Z \circ Y^* \dashv\vdash Y \rightarrow X^* \circ Z \\
(2) & X \rightarrow Y \circ Z \dashv\vdash X \circ Z^* \rightarrow Y \dashv\vdash Y^* \circ X \rightarrow Z \\
(3) & X \rightarrow Y \dashv\vdash Y^* \rightarrow X^* \dashv\vdash X \rightarrow Y^{**} \\
(4) & X \rightarrow \bullet Y \dashv\vdash \bullet X \rightarrow Y.
\end{array}$$

where $X_1 \rightarrow Y_1 \dashv\vdash X_2 \rightarrow Y_2$ abbreviates $X_1 \rightarrow Y_1 \vdash X_2 \rightarrow Y_2$ and $X_2 \rightarrow Y_2 \vdash X_1 \rightarrow Y_1$. If two sequents are interderivable by means of (1) - (4), then these sequents are said to be *structurally equivalent*. It can easily be verified that the following pairs of sequents are structurally equivalent by means of the rules (1) - (3):

$$\begin{array}{ll}
X \circ Y \rightarrow Z & Z^* \rightarrow Y^* \circ X^* \\
X \rightarrow Y \circ Z & Z^* \circ Y^* \rightarrow X^* \\
X \rightarrow Y & X^{**} \rightarrow Y \\
X^* \rightarrow Y & Y^* \rightarrow X \\
X \rightarrow Y^* & Y \rightarrow X^*.
\end{array}$$

Moreover (see [Belnap 1982, p. 381]) one can prove

Theorem 1 (Display Theorem) For every sequent s and every antecedent (succedent) part X of s there exists a sequent s' structurally equivalent with s, such that X is the antecedent (succedent) of s'.

Operational rules:

$$\begin{array}{ll}
(\rightarrow \mathbf{1}) & \vdash \mathbf{I} \rightarrow \mathbf{1} \\
(\mathbf{1} \rightarrow) & \mathbf{I} \rightarrow X \vdash \mathbf{1} \rightarrow X \\
 & \\
(\rightarrow \mathbf{0}) & X \rightarrow \mathbf{I} \vdash X \rightarrow \mathbf{0} \\
(\mathbf{0} \rightarrow) & \vdash \mathbf{0} \rightarrow \mathbf{I} \\
 & \\
(\rightarrow \neg) & X \rightarrow A^* \vdash X \rightarrow \neg A \\
(\neg \rightarrow) & A^* \rightarrow X \vdash \neg A \rightarrow X
\end{array}$$

$$
\begin{array}{ll}
(\rightarrow\wedge) & X \rightarrow A \quad Y \rightarrow B \vdash X \circ Y \rightarrow A \wedge B \\
(\wedge\rightarrow) & A \circ B \rightarrow X \vdash A \wedge B \rightarrow X \\
(\rightarrow\vee) & X \rightarrow A \circ B \vdash X \rightarrow A \vee B \\
(\vee\rightarrow) & A \rightarrow X \quad B \rightarrow Y \vdash A \vee B \rightarrow X \circ Y \\
(\rightarrow\supset) & X \circ A \rightarrow B \vdash X \rightarrow A \supset B \\
(\supset\rightarrow) & X \rightarrow A \quad B \rightarrow Y \vdash A \supset B \rightarrow X^* \circ Y \\
(\rightarrow\Box) & \bullet X \rightarrow A \vdash X \rightarrow \Box A \\
(\Box\rightarrow) & A \rightarrow Y \vdash \Box A \rightarrow \bullet Y \\
(\rightarrow\Diamond) & X \rightarrow A \vdash (\bullet(X^*))^* \rightarrow \Diamond A \\
(\Diamond\rightarrow) & Y^* \rightarrow \bullet(A^*) \vdash \Diamond A \rightarrow Y.
\end{array}
$$

Note that $\vdash A \rightarrow A$. This can be shown by induction on the complexity of A. Also (*inter* $\Box\Diamond$) can easily be established, e.g.:

$$
\begin{array}{ll}
\underline{A \rightarrow A} & (3) \\
\underline{A^* \rightarrow A^*} & (\neg\rightarrow) \\
\underline{\neg A \rightarrow A^*} & (1\text{–}3) \\
\underline{A \rightarrow (\neg A)^*} & (\Box\rightarrow) \\
\underline{\Box A \rightarrow \bullet((\neg A)^*)} & (1\text{–}3) \\
\underline{(\Box A)^{**} \rightarrow \bullet((\neg A)^*)} & (\Diamond\rightarrow) \\
\underline{\Diamond\neg A \rightarrow (\Box A)^*} & (1\text{–}3) \\
\underline{\Box A \rightarrow (\Diamond\neg A)^*} & (\rightarrow\neg) \\
\Box A \rightarrow \neg\Diamond\neg A &
\end{array}
\qquad
\begin{array}{ll}
\underline{A \rightarrow A} & (3) \\
\underline{A^* \rightarrow A^*} & (\rightarrow\neg) \\
\underline{A^* \rightarrow \neg A} & (\rightarrow\Diamond) \\
\underline{(\bullet(A^{**}))^* \rightarrow \Diamond\neg A} & (1\text{–}3) \\
\underline{(\Diamond\neg A)^* \rightarrow \bullet(A^{**})} & (4) \\
\underline{\bullet((\Diamond\neg A)^*) \rightarrow A^{**}} & (1\text{–}3) \\
\underline{\bullet((\Diamond\neg A)^*) \rightarrow A} & (\rightarrow\Box) \\
\underline{(\Diamond\neg A)^* \rightarrow \Box A} & (\neg\rightarrow) \\
\neg\Diamond\neg A \rightarrow \Box A. &
\end{array}
$$

In order to obtain K as our modal base logic we assume the following additional structural rules:

$$
\begin{array}{ll}
(\mathbf{I}+) & X \rightarrow Z \vdash \mathbf{I} \circ X \rightarrow Z \\
 & X \rightarrow Z \vdash X \circ \mathbf{I} \rightarrow Z \\
(\mathbf{I}-) & \mathbf{I} \circ X \rightarrow Z \vdash X \rightarrow Z \\
 & X \circ \mathbf{I} \rightarrow Z \vdash X \rightarrow Z \\
(ex\ \mathbf{0}) & X \rightarrow \mathbf{I} \vdash X \rightarrow Z \\
(\mathbf{A}) & X_1 \circ (X_2 \circ X_3) \rightarrow Z \dashv\vdash (X_1 \circ X_2) \circ X_3 \rightarrow Z \\
(\mathbf{P}) & X_1 \circ X_2 \rightarrow Z \vdash X_2 \circ X_1 \rightarrow Z \\
(\mathbf{C}) & X \circ X \rightarrow Z \vdash X \rightarrow Z \\
(\mathbf{M}) & X_1 \rightarrow Z \vdash X_1 \circ X_2 \rightarrow Z \\
 & X_1 \rightarrow Z \vdash X_2 \circ X_1 \rightarrow Z \\
(MN) & \mathbf{I} \rightarrow A \vdash \bullet\mathbf{I} \rightarrow A.
\end{array}
$$

Here we have opted for mnemonically easy names, eg: (**A**) ((**P**), (**C**), (**M**)) stands for *association* (*permutation, contraction, monotonicity*). Let us now call the above collection of logical, structural, and operational rules LK. Consider the following translation τ of sequents into (tense logical) formulas:

$$\tau(X \rightarrow Y) = \tau_1(X) \supset \tau_2(Y),$$

where τ_i $(i = 1, 2)$ is defined as follows:

$$
\begin{array}{lcl}
\tau_i(A) & = & A \\
\tau_1(\mathbf{I}) & = & \mathbf{1} \\
\tau_2(\mathbf{I}) & = & \mathbf{0} \\
\tau_1(X^*) & = & \neg\tau_2(X) \\
\tau_2(X^*) & = & \neg\tau_1(X) \\
\tau_1(X \circ Y) & = & \tau_1(X) \wedge \tau_1(Y) \\
\tau_2(X \circ Y) & = & \tau_2(X) \vee \tau_2(Y) \\
\tau_1(\bullet X) & = & P\tau_1(X) \\
\tau_2(\bullet X) & = & \Box\tau_2(X).
\end{array}
$$

Theorem 2 (i) If $\vdash A$ in K, then $\vdash \mathbf{I} \rightarrow A$ in LK.
(ii) If $\vdash X \rightarrow Y$ in LK, then $\vdash \tau(X \rightarrow Y)$ in KP[7].

PROOF By induction on the complexity of proofs in K resp. LK. In order to further acquaint the reader with LK we here verify (i) wrt K based on Łukasiewicz's axiomatisation of CPL. We first consider Łukasiewicz's three axiom schemes and modus ponens:

$$
\begin{array}{ll}
A \rightarrow A & \\ \hline
A^* \rightarrow A^* & (\mathbf{M}) \\ \hline
A^* \circ B^* \rightarrow A^* & \\ \hline
A^* \rightarrow A^* \circ B & \\ \hline
\neg A \rightarrow A^* \circ B & \\ \hline
A \circ \neg A \rightarrow B & \\ \hline
A \rightarrow \neg A \supset B & \\ \hline
\mathbf{I} \circ A \rightarrow \neg A \supset B & \\ \hline
\mathbf{I} \rightarrow A \supset (\neg A \supset B) &
\end{array}
$$

$$
\begin{array}{ll}
A \rightarrow A & \\ \hline
A^* \rightarrow A^* & \\ \hline
A^* \rightarrow \neg A \quad A \rightarrow A & \\ \hline
\neg A \supset A \rightarrow A^{**} \circ A & \\ \hline
A^* \circ (\neg A \supset A) \rightarrow A & \\ \hline
A^* \rightarrow A \circ (\neg A \supset A)^* & \\ \hline
A^* \circ A^* \rightarrow (\neg A \supset A)^* & (\mathbf{C}) \\ \hline
A^* \rightarrow (\neg A \supset A)^* & \\ \hline
\neg A \supset A \rightarrow A & \\ \hline
\mathbf{I} \circ (\neg A \supset A) \rightarrow A & \\ \hline
\mathbf{I} \rightarrow (\neg A \supset A) \supset A &
\end{array}
$$

$$
\begin{array}{lr}
 & B \rightarrow B \quad C \rightarrow C \\ \hline
 & B \supset C \rightarrow B^* \circ C \\ \hline
 & B \circ (B \supset C) \rightarrow C \\ \hline
 & A \rightarrow A \quad B \rightarrow C \circ (B \supset C)^* \\ \hline
 & A \supset B \rightarrow A^* \circ (C \circ (B \supset C)^*) \\ \hline
 & A \circ (A \supset B) \rightarrow C \circ (B \supset C)^* \\ \hline
(\mathbf{A}) & (A \circ (A \supset B)) \circ (B \supset C) \rightarrow C \\ \hline
(\mathbf{P}) & A \circ ((A \supset B) \circ (B \supset C)) \rightarrow C \\ \hline
 & ((A \supset B) \circ (B \supset C)) \circ A \rightarrow C \\ \hline
 & (A \supset B) \circ (B \supset C) \rightarrow A \supset C \\ \hline
 & (A \supset B) \rightarrow (B \supset C) \supset (A \supset C) \\ \hline
 & \mathbf{I} \circ (A \supset B) \rightarrow (B \supset C) \supset (A \supset C) \\ \hline
 & \mathbf{I} \rightarrow (A \supset B) \supset ((B \supset C) \supset (A \supset C))
\end{array}
$$

$$
\begin{array}{cc}
 & A \rightarrow A \quad B \rightarrow B \\ \hline
\mathbf{I} \rightarrow A \quad \mathbf{I} \rightarrow A \supset B & A \supset B \rightarrow A^* \circ B \\ \hline
\mathbf{I} \circ \mathbf{I} \rightarrow A \wedge (A \supset B) & A \circ (A \supset B) \rightarrow B \\ \hline
\mathbf{I} \rightarrow A \wedge (A \supset B) & A \wedge (A \supset B) \rightarrow B \\ \hline
\multicolumn{2}{r}{\mathbf{I} \rightarrow B.}
\end{array}
$$

For the purpose of readability we shall split up the proof of the K-axiom-scheme into a

[7] This minimal tense logical system is called $\mathbf{L}_0$ in [Burgess 1984].

number of steps. In a first step we prove $\Box((A \supset B) \wedge A) \rightarrow \bullet B$. This step is very easy and left to the reader. Next we prove $\Box(A \supset B) \wedge \Box A \rightarrow \Box((A \supset B) \wedge A)$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \supset B \rightarrow A \supset B}{\Box(A \supset B) \rightarrow \bullet(A \supset B)}}{\Box(A \supset B) \circ \Box A \rightarrow \bullet(A \supset B)}}{\Box(A \supset B) \wedge \Box A \rightarrow \bullet(A \supset B)}}{\bullet(\Box(A \supset B) \wedge \Box A) \rightarrow A \supset B}
\quad
\dfrac{\dfrac{\dfrac{\dfrac{A \rightarrow A}{\Box A \rightarrow \bullet A}}{\Box(A \supset B) \circ \Box A \rightarrow \bullet A}}{\Box(A \supset B) \wedge \Box A \rightarrow \bullet A}}{\bullet(\Box(A \supset B) \wedge \Box A) \rightarrow A}}
{\dfrac{\dfrac{\bullet(\Box(A \supset B) \wedge \Box A) \circ \bullet(\Box(A \supset B) \wedge \Box A) \rightarrow (A \supset B) \wedge A}{\bullet(\Box(A \supset B) \wedge \Box A) \rightarrow (A \supset B) \wedge A}}{\Box(A \supset B) \wedge \Box A \rightarrow \Box((A \supset B) \wedge A).}}
$$

Applying Cut we obtain $\Box(A \supset B) \wedge \Box A \rightarrow \bullet B$ and can then continue as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box(A \supset B) \rightarrow \Box(A \supset B) \quad \Box A \rightarrow \Box A}{\Box(A \supset B) \circ \Box A \rightarrow \Box(A \supset B) \wedge \Box A} \quad \Box(A \supset B) \wedge \Box A \rightarrow \bullet B}{\Box(A \supset B) \circ \Box A \rightarrow \bullet B}}{\bullet(\Box(A \supset B) \circ \Box A) \rightarrow B}}{\Box(A \supset B) \circ \Box A \rightarrow \Box B}}{\Box(A \supset B) \rightarrow \Box A \supset \Box B}}{\mathbf{I} \circ \Box(A \supset B) \rightarrow \Box A \supset \Box B}}{\mathbf{I} \rightarrow \Box(A \supset B) \supset (\Box A \supset \Box B).}
$$

The necessitation rule is captured by (MN).

Corollary 1 In LK, $\vdash \mathbf{I} \rightarrow A$ iff $\vdash A$ in K.

In addition to the K-axiom-scheme we shall also consider the following modal axiom schemes:

D	$\Box A \supset \Diamond A$
T	$\Box A \supset A$
4	$\Box A \supset \Box\Box A$
5	$\Diamond A \supset \Box\Diamond A$
B	$A \supset \Box\Diamond A$
Tr	$(\Box A \supset A) \wedge (A \supset \Box A)$
V	$\Box A$
$Alt1$	$\Diamond A \supset \Box A$.

If R is any of these axiom schemes, we shall associate with R one or two structural rules R':

R	R'
D	$\bullet A \circ \bullet B \rightarrow \mathbf{I}^* \vdash A \rightarrow B^*$
T	$X \rightarrow \bullet Y \vdash X \rightarrow Y$
4	$X \rightarrow \bullet Y \vdash X \rightarrow \bullet\bullet Y$
5	$(\bullet(X^*))^* \rightarrow Y \vdash \bullet((\bullet(X^*))^*) \rightarrow Y$
B	$(\bullet(X^*))^* \rightarrow Y \vdash \bullet X \rightarrow Y$
Tr	$X \rightarrow \bullet Y \vdash X \rightarrow Y \qquad X \rightarrow Y \vdash X \rightarrow \bullet Y$
V	$\vdash \bullet\mathbf{I} \rightarrow X$
$Alt1$	$X \rightarrow Y \vdash X \rightarrow (\bullet((\bullet Y)^*))^*$

Let Δ be the set of all axiom schemes R and $\Gamma \subseteq \Delta$. Then $\Gamma' = \{R' \mid R \in \Gamma\}$.

Theorem 3 In $LK \cup \Gamma'$, $\vdash \mathbf{I} \to A$ iff $\vdash A$ in $K \cup \Gamma$.

PROOF It suffices to show that (i) $\vdash A$ in $K + R$ implies $\vdash \mathbf{I} \to A$ in $LK + R'$, and (ii) $\vdash X \to Y$ in $LK + R'$ implies $\vdash \tau(X \to Y)$ in $KP + R$. Here is one example concerning the verification of (i):

$$\begin{array}{l}
A \to A \\ \hline
A^* \to A^* \\ \hline
\neg A \to A^* \\ \hline
\Box\neg A \to \bullet A^* \\ \hline
\bullet\Box\neg A \to A^* \\ \hline
A \to (\bullet(\Box\neg A))^* \\ \hline
\Box A \to \bullet((\bullet(\Box\neg A))^*) \\ \hline
\bullet\Box A \to (\bullet(\Box\neg A))^* \\ \hline
\mathbf{I} \circ \bullet\Box A \to (\bullet(\Box\neg A))^* \\ \hline
\bullet\Box A \to \mathbf{I}^* \circ (\bullet(\Box\neg A))^* \\ \hline
\bullet\Box A \circ \bullet(\Box\neg A) \to \mathbf{I}^* \quad {\scriptstyle D'} \\ \hline
\Box A \to (\Box\neg A)^* \\ \hline
\Box A \to \neg\Box\neg A.
\end{array}$$

From the final sequent one can easily derive the D-axiom-scheme.[8]

Theorem 4 Applications of Cut can be eliminated from proofs in every system $LK \cup \Gamma'$.

PROOF This follows from Belnap's [1982] general Cut-elimination theorem for **DL**: all rules under consideration satisfy Belnap's conditions C2 - C8 [Belnap 1982, 389 ff.]. The only condition which cannot be verified by eye is C8 *Eliminability of matching principal constituents.* This is the situation where both premise sequents of Cut introduce the Cut-formula A. For $\Box$ and $\Diamond$ we get the following replacements by proofs with an application of Cut involving less formula connectives:

$$\dfrac{\dfrac{\bullet X \to A}{X \to \Box A} \qquad \dfrac{A \to Y}{\Box A \to \bullet Y}}{X \to \bullet Y} \quad \Longrightarrow \quad \dfrac{\dfrac{\bullet X \to A \qquad A \to Y}{\bullet X \to Y}}{X \to \bullet Y}$$

$$\dfrac{\dfrac{X \to A}{(\bullet(X^*))^* \to \Diamond A} \qquad \dfrac{Y^* \to \bullet(A^*)}{\Diamond A \to Y}}{(\bullet(X^*))^* \to Y} \quad \Longrightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{X \to A \qquad \dfrac{\dfrac{Y^* \to \bullet(A^*)}{\bullet(Y^*) \to A^*}}{A \to (\bullet(Y^*))^*}}{X \to (\bullet(Y^*))^*}}{\bullet(Y^*) \to X^*}}{Y^* \to \bullet(X^*)}}{(\bullet(X^*))^* \to Y.}$$

Corollary 2 The systems $LK \cup \Gamma'$ enjoy the subformula property.

PROOF Their operational rules are separated, symmetric and explicit.

[8] Note that the rule D' corresponds with the relational property corresponding with D, viz. seriality.

Observation The connectives of $LK \cup \Gamma'$ are uniquely characterized.

PROOF It is enough to show by induction on A that for every primitive connective $\sharp$, the following sequents are provable in $(LK \cup \Gamma')(LK \cup \Gamma')^*$: $A_\sharp \rightarrow A_{\sharp^*}$ and $A_{\sharp^*} \rightarrow A_\sharp$.

5 Substructural base logics

Giving up structural rules of inference may lead to a richer inventory of logical operations. In particular, in the absence of (**M**) we get a distinction between (i) multiplicative conjunction $\wedge$ and additive conjunction $\sqcap$, and (ii) multiplicative disjunction $\vee$ and additive disjunction $\sqcup$ (cf. [Belnap 1990], [Troelstra 1992]). In the absence of (**P**) implication $\supset$ breaks down into two directional connectives, the left-searching $\backslash$ and the right-searching $/$. We shall also add $\bot$ and $\top$ as the duals of **1** rep. **0**. The basic structural rules of LK are appropriate for formulating introduction rules for the operations from this richer stock of connectives.[9] We shall now present a modal sequent calculus based on non-commutative classical linear propositional logic without exponentials (alias, following the terminology of [Wansing 1992], classical sequential propositional logic). Let us call this system $LK\emptyset$. The rules of $LK\emptyset$ consist of all logical rules and basic structural rules of LK together with LK's rules for **1**, **0**, $\neg$, $\wedge$, $\vee$, $\Box$, and $\Diamond$ and the following operational rules:

$$
\begin{array}{ll}
(\rightarrow \top) & \vdash \mathbf{I}^* \rightarrow \top \\
(\top \rightarrow) & \mathbf{I}^* \rightarrow X \vdash \top \rightarrow X \\
(\rightarrow \bot) & X \rightarrow \mathbf{I}^* \vdash X \rightarrow \bot \\
(\bot \rightarrow) & \vdash \bot \rightarrow \mathbf{I}^* \\
(\rightarrow \sqcap) & X \rightarrow A \quad X \rightarrow B \vdash X \rightarrow A \sqcap B \\
(\sqcap \rightarrow) & A \rightarrow X \vdash A \sqcap B \rightarrow X \quad B \rightarrow X \vdash A \sqcap B \rightarrow X \\
(\rightarrow \sqcup) & X \rightarrow A \vdash X \rightarrow A \sqcup B \quad X \rightarrow B \vdash X \rightarrow A \sqcup B \\
(\sqcup \rightarrow) & A \rightarrow X \quad B \rightarrow X \vdash A \sqcup B \rightarrow X \\
(\rightarrow /) & X \circ A \rightarrow B \vdash X \rightarrow B/A \\
(/ \rightarrow) & X \rightarrow A \quad B \rightarrow Y \vdash B/A \rightarrow Y \circ X^* \\
(\rightarrow \backslash) & A \circ X \rightarrow B \vdash X \rightarrow A \backslash B \\
(\backslash \rightarrow) & X \rightarrow A \quad B \rightarrow Y \vdash A \backslash B \rightarrow X^* \circ Y.
\end{array}
$$

In $LK\emptyset$ we can prove for example $\Box(A \sqcap B) \rightarrow \Box A \sqcap \Box B$, $\Box A \sqcap \Box B \rightarrow \Box(A \sqcap B)$, and $\Box A \sqcup \Box B \rightarrow \Box(A \sqcup B)$. Clearly, ($inter\Box\Diamond$) holds for $LK\emptyset$, and we also have $\vdash A \rightarrow A$. Applications of Cut can be eliminated in $LK\emptyset$, $LK\emptyset$ enjoys the subformula property, and every connective in this system is uniquely characterized. From the display logical point of view, $LK\emptyset$ is a very natural logic. The question arises whether $LK\emptyset$ can be finitely axiomatized with modus ponens as the only rule of inference.

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[9] In Belnap's [1982] original version of **DL** the structural connective $\circ$ is assumed to be commutative in succedent position. Therefore, in succedent position $\circ$ can be interpreted as $\vee$ only in the presence of (**P**).

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