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ITERATED QUANTIFIERS

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ITERATED QUANTIFIERS

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Abstract. We study the logic of polyadic quantifiers definable by generalized quantifier prefixes, called iterations. Besides being of general logical interest, the study is also motivated by the fact that iterations provide a perspicuous way of displaying scope dependencies in formalizations of many quantified sentences in natural language. First, two results by Keenan on quantifier prefixes are presented in a generalized and global form, and some techniques used in their proofs are made explicit. Then, these techniques are applied to logical definability issues for quantifiers, more precisely to questions as to when certain kinds of polyadic quantifiers are iterations. Among other things, necessary and sufficient conditions are given for resumption quantifiers, branching quantifiers, and cumulative quantifiers, respectively, to be iterations on finite models.

1. Introduction

This paper deals with a special kind of generalized quantifiers, called *iterations*. As expected, iterations are obtained by *iterating* quantifiers, of certain types. Equivalently, they are definable by (generalized) *quantifier prefixes*. This generalizes the notions of a quantifier prefix, and of prenex form, familiar from elementary logic, to logic with generalized quantifiers. Another motive for studying iterations is linguistic. A wide range of sentences in natural languages have truth conditions representable by means of iterations. When this is possible, the *scope relations* between noun phrases in the sentences are directly reflected in the corresponding prefix, by the left-right order. Scope *ambiguities* are accounted for by permutations of that order. Furthermore, there are other sentences, seemingly similar to the ones using iterations, whose truth conditions can be represented by other kinds of generalized quantifiers, but, on a closer look, *not* by iterations. Thus, it becomes of interest to know just when these other kinds of quantifiers are iterations, and when they are not. Several such questions will be addressed in this paper.

The first significant results on generalized quantifier prefixes were obtained by Edward Keenan. In fact, Part I of the present paper is my way of understanding his two main results in this field, the 'Reducibility Equivalence Theorem' in Keenan 1991a and the 'Generalized Linear Prefix Theorem' in Keenan 1991b.[1] Keenan usually writes with particular linguistic applications in mind, but these theorems also have a purely logical interest. I will reformulate them in a setting more familiar to logicians, generalize them slightly, and bring out certain techniques which are implicit in their proofs.

[1] In the case of Keenan 1991b, I have only read an early draft of the paper, not the final version.

That this is not only in the interest of exegesis and clarity is shown in Part II, which contains a number of applications of these techniques to questions of definability of generalized quantifiers.

In more detail, the paper is organized as follows. In section 1, the iteration operation is defined for a suitable class of quantifiers, and motivated by a number of linguistic examples. Section 2 presents some useful properties of iterations, and section 3 contains (generalizations of) the two results by Keenan mentioned above. In section 4, the *convertible* iterations are characterized, i.e., those which are 'closed under converses', and as a corollary we also obtain necessary and sufficient conditions for a *resumption* (an ordinary monadic quantifier applied to n-tuples instead of individuals) to be an iteration. The main result of Section 5 gives a similar characterization for *branching* quantifiers, and section 6 one for *cumulations* (quantifiers rendering the so-called cumulative readings of certain sentences). Section 7 takes up the issue (raised in van Benthem 1989) of when a quantifier is a Boolean combination of iterations, and we prove, among other things, that the resumption of the quantifier ***most*** to pairs instead of individuals is not such a Boolean combination. Section 8, finally, lists some problems for further study.[2]

I

ITERATIONS AND THEIR PROPERTIES

2. Motivation and definitions

As usual, a (generalized) *quantifier of type* $<k_1,...,k_n>$ $(k_i \geq 1)$ is a functional Q which to each non-empty set M assigns a *quantifier* Q_M *of type* $<k_1,...,k_n>$ *on* M, that is, an n-ary relation between subsets of $M^{k_1},...,M^{k_n}$, respectively. Q is *monadic* if all $k_i = 1$, *polyadic* otherwise. Q is *simple* if $n = 1$.

To Q corresponds a *quantifier symbol* Q (of the same type), which acts as a variable-binding operator according to the following formation rule: if $\phi_1,...,\phi_n$ are formulas and $x_{11},...,x_{nk_n}$ are distinct variables, then

$$Qx_{11}...x_{1k_1}, \; ... \; ,x_{n1}...x_{nk_n}(\phi_1,...,\phi_n)$$

is a formula. By adding Q to elementary logic, with this formation rule and a corresponding additional clause in the definition of satisfaction, one obtains the *logic* $L(Q)$, and similarly $L(Q_1,...,Q_k)$. The formation rules and satisfaction clauses for the usual type <1> quantifiers $\forall$ and $\exists$ can be seen as instances of this.

Call the quantifier symbol followed by appropriate variables, $Qx_{11}...x_{1k_1}, \; ... \; ,x_{n1}...x_{nk_n}$, a *quantifier expression.* Quantifier expressions with simple quantifier symbols, i.e., those applying to just one formula, can be iterated: put one more in front of a formula and you get a new formula. A

[2] The main results of this paper were announced, in weaker forms and without proofs, in Westerståhl 1992.

(generalized quantifier) *prefix* is a finite string of simple quantifier expressions, with all variables distinct. If $Q_1,...,Q_k$ are simple, a formula of $L(Q_1,...,Q_k)$ is in *prenex form* if it has the form of a prefix (which may contain $\forall$ and $\exists$) followed by a quantifier-free formula.

Iterating quantifier expressions is one thing, iterating quantifiers is another, though of course related, thing. To see which kind of quantifiers we want to iterate, let us look at a few examples from natural language.

The canonical quantified English sentence has quantified subject and object noun phrases and a transitive verb, as in

(1) Most critics reviewed two books.

This can be formalized as a quantifier Q applied to three arguments, the set of critics (A), the set of books (B), and the relation 'reviewed' (R); Q is thus of type <1,1,2>. But clearly it is more informative to represent the truth condition of (1) by means of the two familiar type <1,1> quantifiers ***most*** and ***two***. Indeed (suppressing the universe M),

$$QAB,R \;\Leftrightarrow\; \textit{\textbf{most}}\,A\{a\colon \textit{\textbf{two}}\;B\;\{b\colon Rab\}\}.$$

We will call Q the *iteration* of ***most*** and ***two***, and formalize (1) as

$$\textit{\textbf{most}}\cdot\textit{\textbf{two}}\;AB,R.$$

One advantage of this is that the other *reading* of (1), that there were two books such that most critics reviewed both of them, can now be represented as another iteration $\textit{\textbf{two}}\;B\{b\colon \textit{\textbf{most}}\;A\;\{a\colon Rab\}\}$, i.e.,

$$\textit{\textbf{two}}\cdot\textit{\textbf{most}}\;BA,R^{-1}$$

(note that we always take the first set argument to be linked to the first argument of the relation, and the second set argument to the second relation argument; hence the appearance of R^{-1} above).

There are more complex iterations. Consider

(2) Two boys gave more dahlias than roses to three girls.

Here three quantifiers are iterated, the first and the third of type <1,1>, but the second is the type <1,1,1> quantifier ***more-than*** (defined by $\textit{\textbf{more-than}}ABC \Leftrightarrow |A\cap C| > |B\cap C|$), and the resulting quantifier has type <1,1,1,1,3>. We should have

$$\textit{\textbf{two}}\cdot\textit{\textbf{more-than}}\cdot\textit{\textbf{three}}\;ABCD,R \;\Leftrightarrow\; \textit{\textbf{two}}\;A\{a\colon \textit{\textbf{more-than}}\;BC\{b\colon \textit{\textbf{three}}\;D\{c\colon Rabc\}\}\};$$

this gives *one* reading of (2).[3]

It thus seems clear that we should be able, in principle, to iterate arbitrary monadic quantifiers. In fact, we will define iteration for an even larger class, which includes (certain) polyadic quantifiers as

[3] It is instructive at this point to work out just what this reading says, and which the other five readings are. Some of these may strike the reader as more natural for (2), whereas some will seem *very* unnatural. But if the latter are ruled out, this is mainly due to contingent facts about the relation of giving, and not, it seems to me, to any principled impossibility of these readings.

well, and which is *closed* under iteration. It is not surprising that this turns out to simplify the definition; after all, the iteration of two monadic quantifiers is polyadic. More interesting is the fact that this move also has a linguistic motivation. Keenan 1991a gives several examples involving 'unreducible' polyadic quantifiers, among them

(3) Every student criticized himself

(4) Every boy likes a different girl.

One reading of (4) uses the type <1,1,2> quantifier ***ED***, defined by

$$\textbf{\textit{ED}}AB,R \quad \Leftrightarrow \quad \forall a,b\in A(a \neq b \Rightarrow \exists c\in B(Rac \;\&\; \neg Rbc)),$$

and (3) uses the type <1,2> quantifier $\textbf{\textit{EH}}A,R \Leftrightarrow \forall a\in A\; Raa$. Now, although none of these are iterations, as Keenan shows, they can themselves be iterated with other quantifiers:

(5) Every student introduced himself to two professors

(6) Every boy gave different flowers to two girls.

For example, one reading of (6) should be rendered

$$\textbf{\textit{ED}}\cdot\textbf{\textit{two}}\; ABC,R \quad \Leftrightarrow \quad \textbf{\textit{ED}}\; AB,\{(a,b)\colon \textbf{\textit{two}}\; C\{c\colon Rabc\}\},$$

and the other reading is obtained by permuting the two quantifiers as before.

We are now ready to define iteration. Here is the relevant class of quantifiers.

1.1. Definition. CIT is the class of quantifiers of types <1,...,1,*k*> with $m+1$ arguments, $m \geq 0$, $k \geq 1$. For obvious reasons, the first m arguments are called the *noun arguments*, and the last argument the *verb argument*. Thus, simple quantifiers in CIT have only verb arguments. For every $Q \in$ CIT and all sets $A_1,\ldots,A_m$, define the simple quantifier

(7) $$(Q^{A_1,\ldots,A_m})_M R \quad \Leftrightarrow \quad Q_M A_1\ldots A_m,R$$

(if some A_j is not included in M, $(Q^{A_1,\ldots,A_m})_M R$ is false).

Now, the idea is to first define iteration for simple quantifiers, and then extend the definition to all quantifiers in CIT via (7). We need the following

1.2. Notation. If R is an n-ary relation on M, $k < n$, and $a_1,\ldots,a_k \in M$, let $R_{a_1\ldots a_k}$ be the $(n-k)$-ary relation defined by

$$R_{a_1\ldots a_k} \quad = \quad \{(a_{k+1},\ldots,a_n)\in M^{n-k}\colon\; Ra_1\ldots a_n\}.$$

Note that

$$(R_{a_1\ldots a_k})_{b_1\ldots b_m} \quad = \quad R_{a_1\ldots a_k b_1\ldots b_m}.$$

Here is how to iterate two simple quantifiers.

1.3. Definition. If Q_1 is of type $<k>$, Q_2 of type $<m>$, define $Q_1 \cdot Q_2$ of type $<k+m>$ as follows:

$$Q_1 \cdot Q_2 R \quad \Leftrightarrow \quad Q_1\{(a_1,...,a_k) : Q_2 R_{a_1...a_k}\}$$

(the universe M is suppressed as usual).

We will often omit the '$\cdot$' and write just Q_1Q_2. It is easily verified that the iteration operation is associative:

$$(Q_1Q_2)Q_3 \quad = \quad Q_1(Q_2Q_3).$$

Thus, $Q_1Q_2Q_3$, and in general

$$Q_1...Q_k,$$

is well-defined.

We have defined iteration of simple quantifiers in a purely set-theoretic way. Of course, we could have gone via prefixes instead:

1.4. Fact. *If Q_i is of type $<p_i>$, $Q_1...Q_k$ is the quantifier defined by the sentence*

$$Q_1x_{11}...x_{1p_1} \ ... \ Q_kx_{k1}...x_{kp_k}Rx_{11}...x_{kp_k}.$$

Finally, we extend the notion of iteration to arbitrary quantifiers in CIT.

1.5. Definition. If Q_i is of type $<1,...,1,p_i>$ with m_i+1 arguments, define the quantifier $Q_1...Q_k$ of type $<1,...,1, \ ... \ ,1,...,1, p_1+...+p_k>$ (with $m_1+...+m_k+1$ arguments) by

$$Q_1...Q_kA_{11}...A_{km_k},R \quad \Leftrightarrow \quad (Q_1^{A_{11},...,A_{1m_1}}...Q_k^{A_{k1},...,A_{km_k}})R \ .$$

Thus, the class CIT is Closed under ITeration. The reader can check that Definition 1.5 indeed gives the truth conditions we wanted in the examples above. To account for ambiguities we can introduce permutations of iterations:

1.6. Definition. For simple $Q_1,...,Q_k$, a permutation $i_1,...,i_k$ of $1,...,k$ induces a *permutation* $(Q_1...Q_k)^{(i_1,...,i_k)}$ of $Q_1...Q_k$ as follows: $(Q_1...Q_k)^{(i_1,...,i_k)}$ is the quantifier defined by the sentence

$$Q_{i_1}x_{i_1 1}...x_{i_1 p_{i_1}} \; ... \; Q_{i_k}x_{i_k 1}...x_{i_k p_{i_k}} Rx_{11}...x_{kp_k}.$$

This can be extended to arbitrary $Q_1,...,Q_k \in$ **CIT** as usual:

$$(Q_1...Q_k)^{(i_1,...,i_k)}A_{11}...A_{km_k},R \quad \Leftrightarrow \quad (Q_1^{A_{11},...,A_{1m_1}}...Q_k^{A_{k1},...,A_{km_k}})^{(i_1,...,i_k)}R \;.$$ [4]

2 Basic properties of iterations

The familiar properties of type <1,1> quantifiers,

CONSERV $\quad Q_M AB \;\Leftrightarrow\; Q_M A\, A{\cap}B$

EXT $\quad$ If $A,B \subseteq M,M'$ then $Q_M AB \;\Leftrightarrow\; Q_{M'} AB$

ISOM $\quad$ If $(M,A,B) \cong (M',A',B')$ then $Q_M AB \;\Leftrightarrow\; Q_{M'} A'B'$,

can be generalized to quantifiers in **CIT**. This is immediate for EXT and ISOM. For CONSERV, let Q be of type <1,...,1,k> with m+1 arguments, and assume $m > 0$ to avoid trivialities. For example, Q could be an iteration $Q_1...Q_r$, and then we know precisely which noun arguments are *linked* to which arguments of the k-ary relation. This information is required for CONSERV to make sense: CONSERV says that each relation argument can be restricted to the *union* of those sets which are linked to it:

2.1. Definition. Let Q be as above. For each $m_1,...,m_k > 0$ with $m_1+...+m_k = m$, we say that Q is $(m_1,...,m_k)$*-conservative* if the following holds:

CONSERV $\quad QA_{11}...A_{1m_1},...,A_{k1}...A_{km_k},R \quad \Leftrightarrow$

$$QA_{11}...A_{1m_1},...,A_{k1}...A_{km_k},\; ((A_{11}\cup...\cup A_{1m_1})\times...\times(A_{k1}\cup...\cup A_{km_k})) \cap R.$$

When $k = m = 1$ we have the old notion of CONSERV for type <1,1> quantifiers. When $k = 1$, the above definition coincides with the notion of conservativity for monadic quantifiers proposed in the literature. Quantifiers like ***ED*** above of type <1,1,2> are (1,1)-conservative. In most cases a specific linking of noun arguments to the verb argument is understood; we then drop the prefix and talk about plain conservativity.

2.2. Fact. *Iteration preserves CONSERV, EXT, and ISOM. More specifically, for CONSERV: if Q_1 is $(m_1,...,m_k)$-conservative and Q_2 is $(p_1,...,p_n)$-conservative, then Q_1Q_2 is $(m_1,...,m_k,p_1,...,p_n)$-conservative.*

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[4] It would have been aesthetically more satisfactory, though slightly longer, to write down the $L(Q_1...Q_k)$-sentence defining the permutation also in the general case. On the other hand, a purely set-theoretic formulation of Definition 1.6 is of course possible but considerably more involved.

Proof. Straightforward calculation. —|

A quantifier Q is said to be *trivial on M*, if Q_M is either the empty or the universal relation between relations on M (of the appropriate type); otherwise Q is *non-trivial on M*. This is a *local* notion of non-triviality. We also need a *global* notion — one not confined to a particular universe:

2.3. Definition. Let Q and $m_1,...,m_k$ be as in Definition 2.1. Q is $(m_1,...,m_k)$*-non-trivial* — but '$(m_1,...,m_k)$' is usually left out — if there are $n_1,...,n_k \geq 0$ such that whenever $A_{11},...,A_{km_k} \subseteq M$ and $|A_{i1}\cup...\cup A_{im_i}| \geq n_i$, $1 \leq i \leq k$, $Q^{A_{11},...,A_{km_k}}$ is non-trivial on M. If Q is simple we require instead that there be an $n \geq 0$ such that Q is non-trivial on M whenever $|M| \geq n$. $n_1,...,n_k$ (n) are called the *triviality bounds* of Q. If the condition is not satisfied, Q is *trivial.*

For example, the type <1,1> quantifier ***at least 5*** is non-trivial, with a triviality bound of 5, but the quantifier

$$QAB \Leftrightarrow |A| \text{ is even and } |A\cap B| \geq 5$$

is trivial. Note that this quantifier is (globally) trivial even though it is (locally) non-trivial *on* every universe with at least 5 elements. This is because of the special role of the noun arguments in Definition 2.3: to be non-trivial, Q has to be non-trivial on all large enough noun arguments, as it were, not just on all large enough universes.

2.4. Triviality Lemma. (i) *For* $Q_1,...,Q_k \in$ CIT: $Q_1...Q_k$ *is trivial* $\Leftrightarrow$ *some* Q_i *is trivial.*
(ii) (Keenan) *For simple* $Q_1,...,Q_k$: $Q_1...Q_k$ *is trivial on M* $\Leftrightarrow$ *some* Q_i *is trivial on M.*

$Q_1,...,Q_k$ can of course also be simple in (i), but the restriction to simple quantifiers is necessary in the local version (ii). To see this, consider the iteration ***every·0***, where ***0*** is the empty quantifier of type <1>. We have ***every·0*** $A,R \Leftrightarrow A = \varnothing$, so ***every·0*** is in fact non-trivial on every M, although one of its components is trivial on every M.

Proof of Lemma 2.4. We first give Keenan's proof of (ii), and then derive (i) from (ii).
(ii): An immediate induction shows that it is sufficient to consider the case $k = 2$. Let Q_1 be of type <m> and Q_2 of type <n>. Thus,

$$Q_1Q_2R \Leftrightarrow Q_1\{(a_1,...,a_m) : Q_2R_{a_1...a_m}\}.$$

Now, if either Q_1 or Q_2 is trivial on M, it is straightforward to calculate that so is Q_1Q_2. So suppose Q_1 and Q_2 are both non-trivial on M. Hence there are $R_1,R_2 \subseteq M^m$ and $S_1,S_2 \subseteq M^n$ such that Q_1R_1, $\neg Q_1R_2$, Q_2S_1, and $\neg Q_2S_2$ (on M). It follows that the following claim establishes the result, i.e., that Q_1Q_2 is non-trivial on M:

CLAIM: $\forall R \subseteq M^m \, \exists R' \subseteq M^{m+n} \, (R = \{(a_1,...,a_m) : Q_2R'_{a_1...a_m}\})$.

The Claim is proved by taking

$$R' = \{(a_1,...,a_m,b_1,...,b_n): (Ra_1...a_m \ \& \ S_1b_1...b_n) \vee (\neg Ra_1...a_m \ \& \ S_2b_1...b_n)\}.$$

Then $Ra_1...a_m$ implies that $R'_{a_1...a_m} = S_1$, and hence $Q_2R'_{a_1...a_m}$. Similarly, $\neg Ra_1...a_m$ implies $\neg Q_2R'_{a_1...a_m}$.

(i): We leave it as an exercise to check that if all of $Q_1,...,Q_k$ are non-trivial, (ii) can be used to verify that $Q_1...Q_k$ too is non-trivial (with triviality bounds given by those for $Q_1,...,Q_k$). For the other direction, suppose some Q_i is trivial. More precisely, suppose it is $(m_1,...,m_p)$-trivial, and hence is of type $<1,...,1,p>$ with $m_1+...+m_p+1$ arguments. Now choose any putative triviality bounds $n_1,...,n_r$ for $Q_1...Q_k$. Let $n = max(n_1,...,n_r)$. By the triviality of Q_i, we can find M and $A_{11},...,A_{pm_p} \subseteq M$ such that $|A_{j1} \cup ... \cup A_{jm_j}| \geq n$ for $1 \leq j \leq p$, and $Q_i^{A_{11},...,A_{pm_p}}$ is trivial on M. Now, from (ii) and Definition 1.5 it follows that for *any* choice of the remaining noun arguments for $Q_1...Q_k$ — let us indicate such a choice by $\underline{C}$ — $(Q_1...Q_k)^{\underline{C},A_{11},...,A_{pm_p}}$ is trivial on M. Moreover, by the choice of n, we can take $\underline{C}$ such that all the sizes of the relevant unions of sets are $> n_1,...,n_r$, and all noun arguments are still subsets of M. So we have shown that however these bounds are chosen, we can find an M including noun arguments 'above' the respective bounds such that the corresponding simple 'instance' of $Q_1...Q_k$ is trivial on M. In other words, $Q_1...Q_k$ is trivial. —|

Next, let us look at iteration and negation. For $Q \in$ CIT, the *inner negation* of Q is defined by $(Q\neg)_M A_1...A_m,R \Leftrightarrow Q_M A_1...A_m,M^p—R$, and the *dual* is $Q^d = \neg(Q\neg) = (\neg Q)\neg$. The following lemma is simple but useful.

2.5. Negation Lemma. *For* $Q_1,...,Q_k \in$ CIT:

(i) $Q_1...Q_k = Q_1...Q_{i-1} \cdot Q_i\neg \cdot \neg Q_{i+1} \cdot Q_{i+2}...Q_k$

(ii) $\neg(Q_1...Q_k) = \neg Q_1 \cdot Q_2...Q_k$

(iii) $(Q_1...Q_k)\neg = Q_1...Q_{k-1} \cdot Q_k\neg$

(iv) $(Q_1...Q_k)^d = Q_1^d...Q_k^d$

Proof. Almost immediate, using Fact 1.4, and the fact that $(Q\neg)x_1...x_p\phi \leftrightarrow Qx_1...x_p\neg\phi$. —|

Call a quantifier Q *positive* (on M) if $\neg Q\varnothing$ (on M). One frequent use of the Negation Lemma is that when Q is an iteration, we can always assume that Q is of the form $Q_1...Q_k$ with $Q_2...Q_k$ positive (on a particular M, or on every M).

Our last two lemmas, which are more or less implicit in Keenan 1991a, concern the characteristic behaviour of iterations on Cartesian products. In particular the first of these lemmas turns out to be very useful.

2.6. Product Decomposition Lemma. *Suppose Q_1 is of type <k>, Q_2 of type <m>, and that Q_2 is positive on M. Then, for all $R \subseteq M^k$ and all $S \subseteq M^m$,*

$$Q_1Q_2R{\times}S \quad \Leftrightarrow \quad (Q_1R \;\&\; Q_2S) \lor (Q_1\varnothing \;\&\; \neg Q_2S).$$

Proof. This is almost immediate once you understand the mechanism of iteration. The argument goes like this. Since

(a) $\quad Ra_1...a_k \;\Rightarrow\; (R{\times}S)_{a_1...a_k} = S$

(b) $\quad \neg Ra_1...a_k \;\Rightarrow\; (R{\times}S)_{a_1...a_k} = \varnothing,$

it follows from the positivity of Q_2 that

(c) $\quad Q_2S \;\Rightarrow\; \{(a_1,...,a_k)\colon Q_2(R{\times}S)_{a_1...a_k}\} = R$

(d) $\quad \neg Q_2S \;\Rightarrow\; \{(a_1,...,a_k)\colon Q_2(R{\times}S)_{a_1...a_k}\} = \varnothing.$

And since $Q_1Q_2R{\times}S \Leftrightarrow Q_1\{(a_1,...,a_k)\colon Q_2(R{\times}S)_{a_1...a_k}\}$, the desired result follows readily from (c) and (d). —|

2.7. Product Lemma. *Suppose that, on M, $Q = Q_1...Q_k$, where Q_i is of type <p_i>, and let $m = p_1+...+p_k$. Then, for every $R \subseteq M^m$, there is a product $P = R_1{\times}...{\times}R_k$, with $R_i \subseteq M^{p_i}$, such that $Q_MR \Leftrightarrow Q_MP$.*

Proof. By induction on k, the case $k = 1$ being trivial. Suppose the result holds for k, and consider $Q_0...Q_k$, where we may assume that $Q_1...Q_k$ is positive. Take any $R \subseteq M^{p_0+...+p_k}$. Let

$$R_0 \;=\; \{(a_1,...,a_{p_0}) : Q_1...Q_kR_{a_1...a_{p_0}}\}.$$

Thus, $Q_0...Q_kR \Leftrightarrow Q_0R_0$. If $R_0 = \varnothing$, we can take $P = \varnothing$. So suppose $R_0 \neq \varnothing$. Since $Q_1...Q_kR_{b_1...b_{p_0}}$ for some $(b_1,...,b_{p_0})$, there is by induction hypothesis a product $P' = R_1{\times}...{\times}R_k$ such that $Q_1...Q_kP'$. Set $P = R_0{\times}R_1{\times}...{\times}R_k$. Essentially the same argument as in the previous proof now gives the result.[5] —|

[5] It may be pointed out that the Product Lemma is very weak. Call a simple quantifier Q *non-trivial on products on M* if there is a product on M (of the suitable form) for which Q holds, and another for which it does not hold. Then Q trivially has the property stated in the Product Lemma. The Product Lemma just gives a way (for iterations) of finding a corresponding product regardless of such a non-triviality assumption.

3. Keenan's Prefix Theorems

The two theorems by Keenan mentioned in the Introduction provide answers to the following questions:

1. To what extent does an iteration determine its components, or, equivalently, the prefix that defines it?
2. Are iterations uniquely determined by their behaviour on Cartesian products?

Versions of these answers are given in this section. To distinguish them from Keenan's original theorems, I will call them the Prefix Theorem and the Product Theorem, respectively. They generalize Keenan's results in the following ways:

(i) The theorems are global, not (only) local.

(ii) They apply to iterations of arbitrary quantifiers in CIT whereas Keenan deals with iterations of type <1> quantifiers.

(iii) The relation between the two results is exhibited: the Prefix Theorem can be seen as a corollary to the (proof of the) Product Theorem.

Starting with the first question, it is clear by associativity that an iteration by itself determines neither the number of its components nor their types. In other words, the notion of a component is not yet precise enough.

3.1. Definition. If $\tau_1,...,\tau_k$ are types of quantifiers in CIT, $\sigma = <\tau_1,...,\tau_k>$ is called a (k-ary) *iteration form*. σ is an iteration form *of* Q if there are Q_i of type τ_i such that $Q = Q_1...Q_k$. $Q_1,...,Q_k$ are called σ-*components* of Q.

A first version of the first question is then: Does an iteration Q of form σ determine its σ-components? The answer is NO. First, if Q is trivial we can, by the Triviality Lemma, get no useful information about the σ-components (except that at least one of them must also be trivial). Second, the Negation Lemma shows that there are 2^{k-1} different ways to distribute inner and outer negations in $Q_1...Q_k$ without changing the resulting quantifier or the iteration form.

Thus, we need to disregard trivial quantifiers and provide some information about how negations are distributed. Moreover, we wish to do this globally, not just on a particular universe. A global notion of non-triviality was introduced in Definition 2.3. As to negations, it turns out that it suffices to know, for each choice of the noun arguments, the behaviour of the σ-components when the verb argument is the empty or the universal relation. We could extend iteration forms to, say, *weighted* iteration forms by encoding this information as well. The Prefix Theorem then says that a non-trivial iteration together with a weighted iteration form does determine the components uniquely. But such a formulation would be cumbersome, and instead we proceed as follows.

3.2. Definition. Let $Q,Q' \in$ CIT be non-trivial quantifiers of the same type. Q and Q' are *balanced*, if for all large enough M and all $A_1,...,A_m \subseteq M$ (i.e., with the cardinality of the relevant unions above the maxima of the respective triviality bounds), $QA_1..A_m,\varnothing \Leftrightarrow Q'A_1..A_m,\varnothing$, and similarly for $Q\neg$ and $Q'\neg$. The *sequences* $(Q_1,...,Q_k)$ and $(Q_1',...,Q_k')$ are *balanced* if Q_i and Q_i' are balanced for each *i*. The corresponding local notion of *balance on M* is obtained by restricting attention to a particular universe M and leaving out the non-triviality requirements.

For example, if the Q_i and Q_i' are non-trivial and MON↑ (upward monotone in the verb argument), then $(Q_1,...,Q_k)$ and $(Q_1',...,Q_k')$ are balanced. We leave the proof of the next lemma as an easy exercise.

3.3. Lemma. *If (Q_1,Q_2) and (Q_1',Q_2') are balanced on M, then so are $Q_1 \cdot Q_2$ and $Q_1' \cdot Q_2'$.*

3.4. The Prefix Theorem. *If $Q_1...Q_k = Q_1'...Q_k'$, where $(Q_1,...,Q_k)$ and $(Q_1',...,Q_k')$ are non-trivial and balanced, then for each i, Q_i is eventually equal to Q_i' (they are equal above the triviality bounds of Q_i). For the local version we must assume that the quantifiers involved are simple; then, if $Q_1...Q_k = Q_1'...Q_k'$ on M, where $(Q_1,...,Q_k)$ and $(Q_1',...,Q_k')$ are non-trivial and balanced on M, we have for each i, $Q_i = Q_i'$ on M.*

Remark. Keenan's Generalized Linear Prefix Theorem is essentially the local version of this for k = 2 without the assumption of balance. The conclusion then becomes that *either* $Q_1 = Q_1'$ and $Q_2 = Q_2'$ on M, *or* $Q_1 = Q_1'\neg$ and $Q_2 = \neg Q_2'$ on M. Balance reduces the options to one, and hence allows generalization to any k.

The answer to the question whether iterations are determined by their product behaviour is YES, once we make clear what 'product behaviour' means.

3.5. Definition. Two quantifiers Q and Q' in CIT of the same iteration form $\sigma = <\tau_1,...,\tau_k>$, where $\tau_i = <1,...,1,p_i>$ with m_i+1 arguments, are said to be *equal on products on M w.r.t.* σ, if for all $A_{11},...,A_{km_k} \subseteq M$ and all $R_i \subseteq M^{p_i}$, $QA_{11}...A_{km_k},R_1\times...\times R_k \Leftrightarrow Q'A_{11}...A_{km_k},R_1\times...\times R_k$ on M. They are *equal on products w.r.t.* σ if this holds for all M.

3.6. The Product Theorem. *If two iterations in* CIT *are equal on products (on M) w.r.t. the same iteration form, then they are equal (on M).*

Proof. First, it is clearly enough to prove the local version. Second, it suffices to prove the result for simple quantifiers. For then, if Q and Q' are arbitrary iterations in CIT which are equal on products w.r.t. σ on M, choose noun arguments $A_{11},...,A_{km_k} \subseteq M$. By definition 1.5, $Q^{A_{11},...,A_{km_k}}$ and

$Q'^{A_{11},\ldots,A_{km_k}}$ are simple iterations, equal on products on M w.r.t. the simple iteration form σ' corresponding to σ. Hence $Q^{A_{11},\ldots,A_{km_k}} = Q'^{A_{11},\ldots,A_{km_k}}$ on M, and since $A_{11},\ldots,A_{km_k}$ were arbitrary, $Q = Q'$ on M.

Next, we dispose of the case where one of the iterations is trivial on M. So suppose $Q_1\ldots Q_k$ and $Q_1'\ldots Q_k'$ are equal on products on M w.r.t. the iteration form $\langle\langle p_1\rangle,\ldots,\langle p_k\rangle\rangle$, and that, say, $Q_1\ldots Q_k$ is trivial on M (the other case is symmetric). Let $p = p_1+\ldots+p_k$. Suppose

$$\forall R \subseteq M^p Q_1\ldots Q_k R.$$

Then, we claim, the same holds for $Q_1'\ldots Q_k'$. For, it follows from our assumption that $Q_1'\ldots Q_k'P$ for any product $P = R_1\times\ldots\times R_k \subseteq M^p$. But then, by the Product Lemma, $Q_1'\ldots Q_k'R$ holds for all $R \subseteq M^p$. A similar argument applies if $\forall R \subseteq M^p \neg Q_1\ldots Q_k R$. Hence, $Q_1\ldots Q_k = Q_1'\ldots Q_k'$.

To prove the theorem for simple iterations which are non-trivial on M we use induction on the length k of the iteration form. The result is trivial for $k = 1$, so suppose it holds for k, and let $Q_0\ldots Q_k$ and $Q_0'\ldots Q_k'$ be equal on products on M w.r.t. $\langle\langle p_0\rangle,\ldots,\langle p_k\rangle\rangle$. As noted before, we can assume that $Q_1\ldots Q_k$ and $Q_1'\ldots Q_k'$ are positive on M. Thus, by product decomposition:

(*) For all $R \subseteq M^{p_0}$ and all $S \subseteq M^p$ ($p = p_1+\ldots+p_k$),

$$(Q_0R \;\&\; Q_1\ldots Q_kS) \vee (Q_0\varnothing \;\&\; \neg Q_1\ldots Q_kS) \;\Leftrightarrow\; (Q_0'R \;\&\; Q_1'\ldots Q_k'S) \vee (Q_0'\varnothing \;\&\; \neg Q_1'\ldots Q_k'S).$$

Then,

(i) $Q_0\varnothing \;\Leftrightarrow\; Q_0'\varnothing$

(we suppress mention of M here and below). To see this, suppose, say, that $\neg Q_0\varnothing$ but $Q_0'\varnothing$. But then (*) is false for $R = \varnothing$.

(ii) $Q_1\ldots Q_k$ and $Q_1'\ldots Q_k'$ are equal on products.

This is proved as follows. Suppose first that $\neg Q_0\varnothing$, and so $\neg Q_0'\varnothing$ by (i). Take any product $P = R_1\times\ldots\times R_k$. Fix R such that Q_0R (non-triviality of Q_0). Then

$Q_1\ldots Q_kP$	$\Rightarrow$	$Q_0R \;\&\; Q_1\ldots Q_kP$	
	$\Rightarrow$	$Q_0\ldots Q_kR\times P$	(product decomposition)
	$\Rightarrow$	$Q_0'\ldots Q_k'R\times P$	(assumption)
	$\Rightarrow$	$Q_1'\ldots Q_k'P$	(product decomposition).

Similarly, $Q_1'\ldots Q_k'P \Rightarrow Q_1\ldots Q_kP$. If instead $Q_0\varnothing$, and hence $Q_0'\varnothing$, apply the above argument to $\neg Q_0\ldots Q_k$ and $\neg Q_0'\ldots Q_k'$. This proves (ii).

By (ii) and the induction hypothesis,

(iii) $Q_1\ldots Q_k = Q_1'\ldots Q_k'$.

Now take S such that $Q_1\ldots Q_kS$. It follows immediately from (iii) and (*) that

(iv) $Q_0 = Q_0'$.

This concludes the proof. —|

The proof of the Prefix Theorem follows the same pattern. This is, in fact, not surprising. First, the hypothesis that the two iterations are equal could be replaced by the assumption that they are equal on products, as we have just seen. Second, the Prefix Theorem makes the additional assumptions of non-triviality and balance. Thus, it is only natural that if we carry out roughly the same proof under these stronger assumptions, the conclusion becomes stronger too: not only the iterations are equal, but also their respective components.

Proof of the Prefix Theorem. We first show that the local version implies the global one. So suppose $Q_1,...,Q_k$ and $Q_1',...,Q_k'$ satisfy the assumptions of the global result. Take large enough M and $A_{11},...,A_{km_k} \subseteq M$ (so that the cardinality of the relevant unions are above the respective triviality bounds for $Q_1,...,Q_k$). Then $Q_1^{A_{11},...,A_{1m_1}}$, ... ,$Q_k^{A_{k1},...,A_{km_k}}$ and $Q_1'^{A_{11},...,A_{1m_1}}$, ..., $Q_k'^{A_{k1},...,A_{km_k}}$ satisfy the assumptions of the local version, relative to M. In particular, they are non-trivial on M by definition. So the local result gives us that $Q_i^{A_{i1},...,A_{im_i}} = Q_i'^{A_{i1},...,A_{im_i}}$ on M, for each i. Hence $Q_i = Q_i'$, for large enough arguments.

We now prove the local result by an induction similar to the one in the previous proof. Suppose the result holds for k, and let $Q_0,...,Q_k$ and $Q_0',...,Q_k'$ satisfy the assumptions of the theorem, relative to M. This time we cannot simply assume that $\neg Q_1...Q_k\varnothing$. $Q_0...Q_k$ can always be *written* so that this holds, but that may change some of the components, and here we need to show that the *given* components are pairwise equal. So we distinguish two cases.

Case A: $\neg Q_1...Q_k\varnothing$.

By balance, and Lemma 3.3, $\neg Q_1'...Q_k'\varnothing$. But then we can argue just as in the last part of the preceding proof (because of the assumption of non-triviality on M), concluding that $Q_0 = Q_0'$ and $Q_1...Q_k = Q_1'...Q_k'$ on M. But $(Q_1,...,Q_k)$ and $(Q_1',...,Q_k')$ are balanced and non-trivial on M, so the induction hypothesis applies, and we are done: $Q_i = Q_i'$ on M for each i.

Case B: $Q_1...Q_k\varnothing$.

As before, $Q_1'...Q_k'\varnothing$. But the assumptions of the theorem also hold, by the Negation Lemma, for $Q_0\neg,\neg Q_1,...,Q_k$ and $Q_0'\neg,\neg Q_1',...,Q_k'$. So Case A applies (this is where we need the balance assumption that also $Q_0\neg$ and $Q_0'\neg$ behave the same on $\varnothing$), and we conclude that $Q_0\neg = Q_0'\neg$ and $\neg Q_1...Q_k = \neg Q_1'...Q_k'$ on M. Hence, $Q_0 = Q_0'$ and $Q_1...Q_k = Q_1'...Q_k'$ on M. The proof is complete. —|

The following corollary is easily obtained by inspecting the previous proofs.

3.7. Corollary. *If $Q_1...Q_k$ and $Q_1'...Q_k'$ are equal on products on M w.r.t. $<<p_1>,...,<p_k>>$, and all Q_i and Q_i' are positive on M, then $Q_i = Q_i'$ on M, for $i = 1,...,k$.*

II

APPLICATIONS TO DEFINABILITY

4. Convertibility and resumptions

Keenan's motive for studying generalized quantifier prefixes was partly to obtain methods for showing that certain polyadic quantifiers are *not* iterations. One such method is this: Show that a particular quantifier is equal to some iteration on products, but not on all relations. Then it follows by the Product Theorem that the quantifier in question cannot be an iteration. This is quite efficient in many cases; for example, it can be used to show that the quantifiers *ED* and *EH* from section 1 are not iterations.[6]

In sections 4 - 6 we apply the results in Part I not to particular quantifiers, but to some natural classes of quantifiers, and give necessary and sufficient conditions for a quantifier in such a class to be an iteration (these characterizations are in fact also quite useful for showing particular quantifiers not to be iterations). Although the characterizations are not simple applications of the previous results but require extra work, the basic facts of Part I are used repeatedly, and the present results would have been much less feasible without them.

We now make a few assumptions that will hold, unless otherwise stated, for the rest of the paper. *First*, we restrict attention to *simple* quantifiers. We have already seen that this is no real restriction – think of these quantifiers as *noun phrase denotations*, obtained from quantifiers in CIT by fixing the noun arguments. *Second*, we consider for simplicity only iteration forms <<1>,...,<1>>. So the information about a quantifier that it is a k-ary iteration uniquely determines its iteration form (and its type, i.e., <*k*>). *Third*, we restrict attention to *finite* universes. *Fourth* and last, we assume ISOM of all quantifiers.

In contrast with the first three assumptions, the last one may seem completely unrealistic from a linguistic point of view, since a noun phrase denotation of the form $Q^{A_{11},\ldots}$ practically *never* satisfies ISOM (A_{11},... are *fixed* sets)! However, this simplifying assumption does have an adequate motivation: our results in fact have stronger versions which only rely on the (quite realistic) assumption that Q, not $Q^{A_{11},\ldots}$, satisfies (CONSERV and) ISOM.

The present section is supposed to deal with convertible quantifiers and with resumptions, so we had better define these notions. First, however, another piece of

4.1. Notation. If R is a k-ary relation on M and $i_1,\ldots,i_k$ a permutation of $1,\ldots,k$, let $R^{(i_1,\ldots,i_k)}$ be the relation on M defined by the following condition:

$$R^{(i_1,\ldots,i_k)}a_{i_1}\ldots a_{i_k} \quad\Leftrightarrow\quad Ra_1\ldots a_k.$$

[6] More examples can be found in Keenan 1991a. Ben-Shalom 1992 further extends Keenan's methods.

Thus, if R is binary, $R^{(2,1)}$ is the *converse* of R, or R^{-1}. This notation is related to the one we introduced for permutations of iterations in Definition 1.6 by the following easily verified

4.2. Fact. *For type <1> quantifiers,* $(Q_1...Q_k)^{(i_1,...,i_k)}R \;\Leftrightarrow\; Q_{i_1}...Q_{i_k}R^{(i_1,...,i_k)}$.

4.3. Definition. A quantifier Q of type <k> is *convertible* (on M) if for every permutation $i_1,...,i_k$ of $1,...,k$ and every k-ary relation R (on M), $QR \Rightarrow QR^{(i_1,...,i_k)}$ (on M).

4.4. Fact. (i) *If Q is convertible, so are $\neg Q$, $Q\neg$, and (hence) Q^{d}.*
(ii) *If Q is of type <k> and closed under permutations of k-tuples, then Q is convertible.*

Proof. For (i), note that $M^k - R^{(i_1,...,i_k)} = (M^k - R)^{(i_1,...,i_k)}$. (ii) follows from the fact that if Q is closed under permutations of k-tuples, then only the *cardinality* of R matters for whether QR holds or not, so Q is clearly convertible. —|

There are lots of convertible quantifiers, but for our main example we need one more

4.5. Definition. If Q is of type <1,...,1> with n arguments and $k \geq 1$, the *k-ary resumption $Q^{(k)}$ of Q* is defined as follows: for all M and all $R_1,...,R_n \subseteq M^k$,

$$Q^{(k)}{}_M R_1...R_n \;\Leftrightarrow\; Q_{M^k}R_1...R_n.$$

For $k = 0$ we let $Q^{(0)}{}_M = \mathrm{T}_M$, the trivially true quantifier on M (of the type of Q).

For example, ***most***$^{(2)}$ is the type <2,2> quantifier defined by

$$\textit{most}^{(2)}RS \;\Leftrightarrow\; |R\cap S| > |R - S|.$$

The use of resumption (quantification over pairs) in natural language is proposed in May 1989; cf. van Benthem 1989 and Westerståhl 1989 for further discussion. Note that ***most***$^{(2)}$ is not in CIT. To remain within CIT we will only consider resumptions of type <1> quantifiers here. In particular, instead of ***most***$^{(2)}$ we consider the type <2> quantifier

$$(Q^{\mathrm{R}})^{(2)}{}_M R \;\Leftrightarrow\; |R| > |M^2 - R|,$$

i.e., the resumption of the 'type <1> counterpart' Q^{R} of ***most***.

Here are some familiar examples of convertible quantifiers:

- All resumptions $Q^{(k)}$ (since Q is assumed to satisfy ISOM, $Q^{(k)}$ is closed under permutations of k-tuples).
- $Q^{\mathrm{E}}{}_n R \Leftrightarrow R$ is an equivalence relation with at least n equivalence classes.
- $\textit{Tot}R \Leftrightarrow R$ is a total ordering of the universe.

- $Q^k R \Leftrightarrow$ there is an infinite set A such that $A\times\ldots\times A \subseteq R$ (a 'Ramsey quantifier'; only interesting on infinite universes, of course).

Which iterations are convertible? Well, clearly $\exists\ldots\exists$ and $\forall\ldots\forall$, but it is not so easy to find other examples. Here, however, is one. Let $Q_{\text{odd}}A \Leftrightarrow |A|$ is odd.

4.6. Fact. $Q_{\text{odd}}^{(k)} = Q_{\text{odd}}\cdot\ldots\cdot Q_{\text{odd}}$ *(k components). Hence, the iteration* $Q_{\text{odd}}\cdot\ldots\cdot Q_{\text{odd}}$ *is convertible.*

Proof. Induction on k. $k = 1$ is trivial. Suppose the result is true for k. Write $It^{(k)}(Q_{\text{odd}})$ for $Q_{\text{odd}}\cdot\ldots\cdot Q_{\text{odd}}$ with k components.

$$\begin{array}{lll} It^{(k+1)}(Q_{\text{odd}})R & \Leftrightarrow & Q_{\text{odd}}\{a\colon It^{(k)}(Q_{\text{odd}})R_a\} \text{ (by definition)} \\ & \Leftrightarrow & Q_{\text{odd}}\{a\colon |R_a| \text{ is odd}\} \quad \text{(by induction hypothesis).} \end{array}$$

But clearly, for $M = \{a_1,\ldots,a_m\}$,

$$|R| = |R_{a_1}| + \ldots + |R_{a_m}|.$$

Now it is easy to verify that $|R|$ is odd iff there is an odd number of a_i such that $|R_{a_i}|$ is odd. —|

The main result of this section says, essentially, that these three iterations are the *only* convertible ones. For the proof we shall need the following

4.7. Lemma. *If Q_0 is non-trivial on M and $Q_0\ldots Q_k$ is convertible on M, then $Q_1\ldots Q_k$ is also convertible on M.*

Proof. Case 1: $\neg Q_1\ldots Q_k\varnothing$.

Subcase 1A: $\neg Q_0\varnothing$.

Take A such that Q_0A (non-triviality). By product decomposition, for every $R \subseteq M^k$,

$$Q_0\ldots Q_kA\times R \Leftrightarrow Q_0A \;\&\; Q_1\ldots Q_kR \Leftrightarrow Q_1\ldots Q_kR,$$

and similarly,

$$Q_0\ldots Q_kA\times R^{(i_1,\ldots,i_k)} \Leftrightarrow Q_1\ldots Q_kR^{(i_1,\ldots,i_k)}.$$

But $A\times R^{(i_1,\ldots,i_k)}$ is a 'converse' of $A\times R$, and so, by the convertibility of $Q_0\ldots Q_k$, $Q_1\ldots Q_kR \Leftrightarrow Q_1\ldots Q_kR^{(i_1,\ldots,i_k)}$.

Subcase 1B: $Q_0\varnothing$.

This time, taking A such that $\neg Q_0A$, product decomposition gives

$$\begin{array}{lll} Q_0\ldots Q_kA\times R & \Leftrightarrow & \neg Q_1\ldots Q_kR \\ Q_0\ldots Q_kA\times R^{(i_1,\ldots,i_k)} & \Leftrightarrow & \neg Q_1\ldots Q_kR^{(i_1,\ldots,i_k)}, \end{array}$$

etc.

Case 2: $Q_1...Q_k\emptyset$.

Consider $Q_0\neg\cdot\neg(Q_1...Q_k)$ and repeat Case 1. —|

4.8. Theorem. *$Q_1...Q_k$ is convertible iff, on each M where $Q_1...Q_k$ is non-trivial, $Q_1...Q_k$ is either $\exists...\exists$ or $\forall...\forall$ or $Q_{odd}\cdot...\cdot Q_{odd}$ (k components), or one of their negations ($k \geq 2$).*

Proof. 'If': This follows from Fact 4.6, and the fact that if Q is trivial on M, it is convertible on M.

'Only if': We shall prove by induction on k that if $Q_1...Q_k$ is convertible, and nontrivial on M, and if both Q_1 and $Q_2...Q_k$ are positive on M, then $Q_1...Q_k$ is either $\exists...\exists$ or $\forall...\forall$ or $Q_{odd}\cdot...\cdot Q_{odd}$ on M. This is sufficient, for we can always assume that $Q_2...Q_k$ is positive on M, and if Q_1 is not positive on M, we apply the result to $\neg Q_1...Q_k$, and conclude that $Q_1...Q_k$ is the negation of one the three alternatives.

INDUCTION BASE, $k = 2$: In this inductive proof the basis case requires more work than the induction step. Suppose Q_1Q_2 is convertible and non-trivial on M, and that Q_1 and Q_2 are positive (from now on in the proof, the phrase "on M" will usually be omitted). We start with the

CLAIM: $\quad Q_1 = Q_2$

To prove this, we argue as follows. For all $A,B \subseteq M$,

$$\begin{array}{llll} Q_1Q_2A\times B & \Leftrightarrow & Q_1Q_2B\times A & \text{(by convertibility)} \\ & \Leftrightarrow & Q_1B \ \& \ Q_2A & \text{(by product decomposition, since } Q_1,Q_2 \text{ are positive)} \\ & \Leftrightarrow & Q_2Q_1A\times B & \text{(by product decomposition again).} \end{array}$$

This shows that Q_1Q_2 and Q_2Q_1 are equal on products. Hence, by Corollary 3.7, $Q_1 = Q_2$.

Let $M = \{a_1,...,a_m\}$. Because of ISOM (and the fact that M is finite), we can regard Q_1 simply as a subset of $\{0,...,m\}$. Thus, we often write $k \in Q_1$ instead of "there is $A \subseteq M$ with $|A| = k$ such that Q_1A". We now consider two cases.

Case 1: $1 \notin Q_1$.

Since $0,1 \notin Q_1$, we have $m > 1$ by the non-triviality of Q_1. Thus, Q_1 cannot be $\exists$ or Q_{odd}. We show that it must be $\forall$. Let n be the smallest number in Q_1; n exists by non-triviality, and is > 1 by assumption. It suffices to show that $n = m$. Suppose, for contradiction, that $n < m$. Choose a binary relation R such that a_i has exactly n (R-)successors for $1 \leq i \leq n$, but such that *not* all of these a_i have the *same* successors, and that the remaining elements of M have no successors. Note that this choice of R is possible because $1 < n < m$. Since $n \in Q_1$ but $0 \notin Q_1$, $|\{a : Q_1R_a\}| = n$, and so Q_1Q_1R. But, by the construction of R, the number of elements in M with exactly n *predecessors* must be smaller than n (all predecessors are among $a_1,...,a_n$). So this number is *not* in Q_1, and it follows that $\neg Q_1 Q_1 R^{-1}$. This contradicts the convertibility of $Q_1 Q_1$, and we have shown that $Q_1 = \forall = \{m\}$.

Case 2: $1 \in Q_1$.

Subcase 2A: $2 \in Q_1$.

We show that $Q_1 = \exists = \{1,...,m\}$. This follows from the next

CLAIM: If $k \in Q_1$, then $k+1 \in Q_1$ $(k < m)$.

If $m = 2$ we are done, so assume $m > 2$. Assume $k \in Q_1$ but $k+1 \notin Q_1$. Then define R as follows:

$$R_{a_1} = \{a_1,...,a_{k+1}\}$$
$$R_{a_2} = \{a_1,a_2\}$$
$$R_{a_i} = \{a_i\}, \text{ for } 3 \leq i \leq k+1, \text{ and } R_a = \varnothing \text{ otherwise.}$$

Then $\{a : Q_1 R_a\} = \{a_2,...,a_{k+1}\}$, so $Q_1 Q_1 R$. But $\{a : Q_1 {R^{-1}}_a\} = \{a_1,...,a_{k+1}\}$, and hence $\neg Q_1 Q_1 R^{-1}$, contradicting our hypothesis, and the Claim is proved.

Subcase 2B: $2 \notin Q_1$.

That $Q_1 = Q_{\text{odd}}$ now follows from the

CLAIM: $k \notin Q_1$ iff $k+1 \in Q_1$ $(k < m)$.

To prove this we define R as follows:

$$R_{a_1} = \{a_1,...,a_k\}$$
$$R_{a_2} = \{a_{k+1}\}, \text{ and } R_a = \varnothing \text{ otherwise.}$$

Suppose first that $k \notin Q_1$ and $k+1 \notin Q_1$. Then $\{a : Q_1 R_a\} = \{a_2\}$, so $Q_1 Q_1 R$. But $\{a : Q_1 {R^{-1}}_a\} = \{a_1,...,a_{k+1}\}$, and hence $\neg Q_1 Q_1 R^{-1}$, contradiction. Next, suppose instead $k \in Q_1$ and $k+1 \in Q_1$. This time $\{a : Q_1 R_a\} = \{a_1,a_2\}$, and so $\neg Q_1 Q_1 R$. But $\{a : Q_1 {R^{-1}}_a\} = \{a_1,...,a_{k+1}\}$ as before, and it follows that $Q_1 Q_1 R^{-1}$, again a contradiction. This proves the Claim, and thereby concludes the proof of the induction basis.

INDUCTION STEP: Suppose the result if true for k, and let $Q_0...Q_k$ be convertible, nontrivial on M, and such that both Q_0 and $Q_1...Q_k$ are positive. Since Q_0 is non-trivial (by the Triviality Lemma), it follows from Lemma 4.7 and the induction hypothesis that $Q_1...Q_k$ is either $\exists...\exists$ or $\forall...\forall$ or $Q_{\text{odd}} \cdot ... \cdot Q_{\text{odd}}$.

Case 1: $Q_1...Q_k = \exists...\exists$.

Thus, $Q_0...Q_k R \Leftrightarrow Q_0\{a : R_a \neq \varnothing\}$. Since Q_0 is positive and non-trivial, there is $A \neq \varnothing$ such that $Q_0 A$. Fix a non-empty $S \subseteq M^{k-1}$, and let B be any non-empty subset of M. Then

$$Q_0...Q_k A{\times}B{\times}S \Leftrightarrow Q_0 A$$
$$Q_0...Q_k B{\times}A{\times}S \Leftrightarrow Q_0 B.$$

Thus, by the convertibility of $Q_0...Q_k$, Q_0B. We have shown that Q_0B for any non-empty subset B of M, i.e., $Q_0 = \exists$ on M.

Case 2: $Q_1...Q_k = \forall...\forall$.

By the Negation Lemma, $(Q_1...Q_k)^d = \exists...\exists$. But $(Q_0...Q_k)^d = Q_0{}^d \cdot Q_1{}^d ... Q_k{}^d$ is convertible (Fact 4.4), and $Q_0{}^d$ and $(Q_1{}^d...Q_k{}^d)$ are positive, so by Case 1 we get $Q_0{}^d = \exists$, i.e., $Q_0 = \forall$.

Case 3: $Q_1...Q_k = Q_{odd} \cdot ... \cdot Q_{odd}$.

This time, by Fact 4.6, $Q_0...Q_kR \Leftrightarrow Q_0\{a : |R_a| \text{ is odd}\}$. Fix $S \subseteq M^{k-1}$ such that $|S|$ is odd. Then, for all $A,B \subseteq M$:

(i) If $|B|$ is odd, then $Q_0...Q_kA{\times}B{\times}S \Leftrightarrow Q_0A$ (since $|B{\times}S|$ is odd).
(ii) If $|A|$ is odd, then $Q_0...Q_kB{\times}A{\times}S \Leftrightarrow Q_0B$.
(iii) If $|A|$ is even, then $\neg Q_0...Q_kB{\times}A{\times}S$ (since $|A{\times}S|$ is even and $\neg Q_0\varnothing$).
(iv) If $|A|$ is even, then $\neg Q_0A$ (by convertibility from (iii) and (i)).
(v) If $|B|$ is odd, then Q_0B.

To see that (v) holds, suppose $|B|$ is odd, and take (by non-triviality) an A such that Q_0A. By (iv), $|A|$ is odd. Thus, by (i), (ii), and convertibility, Q_0B.

(iv) and (v) show that $Q_0 = Q_{odd}$ on M. This concludes the proof of the induction step, and thereby of the theorem. —|

As a bonus, we get the following characterization for free:

4.9. Corollary. *$Q^{(k)}$ is a k-ary iteration iff, on each M where $Q^{(k)}$ is non-trivial, $Q^{(k)}$ is either $\exists^{(k)}$ or $\forall^{(k)}$ or $Q_{odd}{}^{(k)}$, or one of their negations ($k \geq 2$).*

Proof. 'Only if': If $Q^{(k)} = Q_1...Q_k$, then $Q_1...Q_k$ is convertible, so the result is immediate from the theorem (and Fact 4.6).

'If': Suppose the right hand side holds. We must find type <1> quantifiers $Q_1,...,Q_k$ such that $Q^{(k)} = Q_1...Q_k$. So we must define each Q_i on every universe M. If $Q^{(k)}$ is trivial on M, we can clearly find $Q_1,...,Q_k$ on M such that $Q^{(k)} = Q_1...Q_k$ on M. If $Q^{(k)}$ is non-trivial on M, it is either $\exists^{(k)}$ or $\forall^{(k)}$ or $Q_{odd}{}^{(k)}$, or one of their negations, on M. So again $Q_1,...,Q_k$ can be defined on M in the desired way. (For example, if $Q^{(k)} = \neg\exists^{(k)}$ on M, we let $(Q_1)_M = \neg\exists_M$, and $(Q_2)_M = ... = (Q_k)_M = \exists_M$. Of course, we cannot guarantee that $Q_1,...,Q_k$ are defined in the *same* way on every M where $Q^{(k)}$ is non-trivial, but that is not required.) —|

Remark: In view of the fact (4.4) that not only $\neg Q$, but also $Q\neg$ and Q^d are convertible if Q is, did we not forget a few cases in Theorem 4.8? Well, $(\exists...\exists)\neg$ is convertible, but $(\exists...\exists)\neg$ =

$\neg(\forall\ldots\forall)$, so this is covered by the theorem. But what about $(Q_{odd}\cdot\ldots\cdot Q_{odd})\neg$? To see that this too is covered, note first that

(1) If $|M|$ is odd then $Q_{odd}{}^{d} = Q_{odd}$ on M, and if $|M|$ is even, $Q_{odd}\neg = Q_{odd}$ on M.

Thus, if $|M|$ is odd, $(Q_{odd}\cdot\ldots\cdot Q_{odd})\neg = \neg(Q_{odd}{}^{d}\cdot\ldots\cdot Q_{odd}{}^{d}) = \neg(Q_{odd}\cdot\ldots\cdot Q_{odd})$ as above, and if $|M|$ is even, $(Q_{odd}\cdot\ldots\cdot Q_{odd})\neg = Q_{odd}\cdot\ldots\cdot Q_{odd}\cdot Q_{odd}\neg = Q_{odd}\cdot\ldots\cdot Q_{odd}$. —|

It should be noted that both the assumptions of ISOM and of finiteness of the universe are used essentially in these results. As to ISOM, consider typical non-ISOM quantifiers like *proper names*:

$$JohnA \iff John \in A.$$

Then $\textbf{\textit{John}}\cdot\textbf{\textit{John}}R \iff (John,John) \in R$, so $\textbf{\textit{John}}\cdot\textbf{\textit{John}}$ is convertible. Similarly, it can be seen that the results in the next two sections fail for proper names.

However, we said earlier that the requirement of ISOM for the type <1> quantifiers $Q_1,\ldots,Q_k$ in Theorem 4.8 can be weakened to a linguistically more realistic assumption. The next proposition points the way to the correct formulation of this result.

4.10. Proposition. *$some^A\cdot some^B$ is convertible iff $A = B$ or one of A,B is $\varnothing$. Similarly for $all^A\cdot all^B$ and $odd^A\cdot odd^B$.*

Proof. If $A = \varnothing$ or $B = \varnothing$, $some^A\cdot some^B R$ is always false, so convertibility holds. And clearly $some^A\cdot some^A$ is convertible. For the other direction, suppose $\varnothing \neq A,B$, and take $a \in A$. Then take $b \in B$ and let $R = \{(a,b)\}$. Thus $some^A\cdot some^B R$, so by convertibility, $some^A\cdot some^B R^{-1}$, i.e., there is $c \in A$ such that $some^B(R^{-1})_c$. Hence, $c = b$, $(R^{-1})_b = \{a\}$, and $B\cap(R^{-1})_b \neq \varnothing$, so $a \in B$. We have shown that $A \subseteq B$, and by symmetry, that $B \subseteq A$. This takes care of the case of $some^A\cdot some^B$. For $all^A\cdot all^B$ the result follows by taking duals. For $odd^A\cdot odd^B$ essentially the same argument as for $some^A\cdot some^B$ works. —|

4.11. Lemma. *Suppose that $Q^A = some^B$ on M, where Q^A and $some^B$ are non-trivial on M, Q is ISOM and CONSERV, and $\neg Q^A\varnothing$ on M. Then $A = B$. The same conclusion holds if $Q^A = all^B$, or $Q^A = odd^B$.*

Proof. Note that the assumptions imply that $\varnothing \neq A,B \subseteq M$. (in particular, if $A = \varnothing$, then by CONSERV, $Q^A C \iff Q^A\varnothing$ for all $C \subseteq M$, which makes Q^A trivial on M). Suppose first $a \in B$. Then $some^B\{a\}$, so $Q^A\{a\}$, and $Q^A A\cap\{a\}$ by CONSERV. Hence, $A\cap\{a\} \neq \varnothing$, i.e., $a \in A$. So $B \subseteq A$. Next, suppose $a \in A$. Take any $b \in B$. Since $some^B\{b\}$, we have $Q^A\{b\}$. But $a,b \in A$, and hence it follows from ISOM that $Q^A\{a\}$. Hence, $some^B\{a\}$, so $a \in B$. This shows that $A = B$ when $Q^A = some^B$. If instead $Q^A = all^B$, we get the same result by taking duals, and if $Q^A = odd^B$, the same proof as for *some* works. —|

We can now give the following strengthening of Theorem 4.8.

4.12. Theorem. *Suppose that the type <1,1> quantifiers $Q_1,...,Q_k$ are CONSERV and ISOM, that $A_1,...,A_k \neq \emptyset$, and that $Q_1^{A_1}...Q_k^{A_k}$ is non-trivial on some universe. Then, $Q_1^{A_1}...Q_k^{A_k}$ is convertible iff $A_1 = ... = A_k = A$, and, on every M where $Q_1^A...Q_k^A$ is non-trivial, $Q_1^A...Q_k^A$ is $some^A...some^A$ or $all^A...all^A$ or $odd^A...odd^A$(k components), or a negation of these.*

Note that Theorem 4.8 follows from this theorem: if $Q_1,...,Q_k$ are of type <1> and ISOM, then $Q_1',...,Q_k'$ defined by $Q_i'BC \Leftrightarrow Q_iB\cap C$ are CONSERV and ISOM, and $Q_i = Q_i'^M$ on every M, so the theorem applies.

Proof of Theorem 4.12 (outline). 'If': This is just as before.
'Only if': We consider only the case $k = 2$. Suppose $Q_1^AQ_2^B$ is convertible, and non-trivial on M; such an M exists by hypothesis. As before we may assume that Q_1^A and Q_2^B are both positive on M. Now the first claim in the proof of Theorem 4.8 used only the Product Decomposition Lemma, not ISOM, so exactly the same argument gives

$$Q_1^A = Q_2^B$$

(on M). Thus, $Q_1^AQ_1^A$ is convertible. Since Q_1 is CONSERV, only the behaviour of Q_1^A on subsets of A need be considered. Since Q_1 is ISOM (and A is finite), only the size of these subsets matters. Thus, Q_1^A can be considered as a subset of $\{0,...,|A|\}$, where $k \in Q_1^A$ means that there is $C \subseteq A$ such that $|C| = k$ and Q_1^AC. But then, the same arguments as for the case $k = 2$ in the proof of Theorem 4.8 show that Q_1^A is either $some^A$ or all^A or odd^A on M. It also follows by Lemma 4.11 that $A = B$, and we are done. —|

The results in the next two sections also have stronger versions, where the type <1> quantifiers involved are 'instances' of CONSERV and ISOM type <1,1> quantifiers, but I will not state these versions explicitly.

5. Branching

Barwise 1979 introduced branching generalized quantifiers in connection with natural language semantics. Here we shall only consider branching of MON↑ type <1> quantifiers, defined (by Barwise) as follows:

5.1. Definition. For MON↑ $Q_1,...,Q_k$, define the quantifier $B(Q_1,...,Q_k)$ of type <k> by

$$B(Q_1,...,Q_k)_M R \quad \Leftrightarrow \quad \exists X_1,...,X_k \subseteq M\ [Q_{1M}X_1\ \&\ ...\ \&\ Q_{kM}X_k\ \&\ X_1\times...\times X_k \subseteq R].$$

We call the corresponding syntactic expression '$B(Q_1,...,Q_k)$' a *branching prefix* (for typographical reasons; a vertical alignment of $Q_1,...,Q_k$ would have been better).

Note that $B(Q_1,...,Q_k)$, as defined above, is always MON↑, regardless of the monotonicity behaviour of $Q_1,...,Q_k$. However, in what follows we presuppose that $Q_1,...,Q_k$ are MON↑, whenever '$B(Q_1,...,Q_k)$' occurs.

5.2. Lemma. *If each of $Q_1,...,Q_k$ is MON↑, then so is $Q_1...Q_k$.*

Proof. Straightforward calculation. —|

Here are some further useful facts about branching.

5.3. Lemma. *$B(Q_1,...,Q_k)$ is non-trivial on M iff each Q_i is non-trivial on M.*

Proof. 'Only if': Suppose Q_i is trivial on M. If $Q_i = \emptyset$ then $B(Q_1,...,Q_k)R$ is always false. If $Q_i = P(M)$, then $Q_i\emptyset$, and so, by the definition of branching,

$$B(Q_1,...,Q_k)R \quad \Leftrightarrow \quad \exists X_1,...,X_k\ [Q_1X_1\ \&\ ...\ \&\ Q_kX_k],$$

and the right hand side is independent of R.
'If': Suppose that each Q_i is non-trivial on M. Then there are X_i such that $Q_i X_i$, and hence $B(Q_1,...,Q_k)X_1\times...\times X_k$. It remains to show that $\neg\forall R \subseteq M\ B(Q_1,...,Q_k)R$. Suppose this is not so. Then $B(Q_1,...,Q_k)\emptyset$, so it follows that for some i, $Q_i\emptyset$. But then Q_i is trivial on M by MON↑, contrary to assumption. —|

5.4. Lemma. *If $B(Q_1,...,Q_k)$ is non-trivial on M, then $B(Q_1,...,Q_k)$ and $Q_1...Q_k$ are equal on products on M.*

Proof. By the previous lemma, each Q_i is non-trivial on M. We have

$B(Q_1,...,Q_k)A_1\times...\times A_k$
$\Leftrightarrow\ \exists X_1,...,X_k\ [Q_1X_1\ \&\ ...\ \&\ Q_kX_k\ \&\ X_1\times...\times X_k \subseteq A_1\times...\times A_k]$
$\Leftrightarrow\ \exists X_1[Q_1X_1\ \&\ X_1 \subseteq A_1]\ \&\ ...\ \&\ \exists X_k[Q_kX_k\ \&\ X_k \subseteq A_k]$ ($X_i \neq \emptyset$, by MON↑ + non-triv.)
$\Leftrightarrow\ Q_1A_1\ \&\ ...\ \&\ Q_kA_k$ (by MON↑)
$\Leftrightarrow\ Q_1...Q_kA_1\times...\times A_k$ (by product decomposition).

—|

The next lemma shows that the branching of $Q_1,...,Q_k$ is stronger than the iteration of $Q_1,...,Q_k$, in any order.

5.5. Lemma. *For any permutation* $i_1,...,i_k$ *of* $1,...,k$ *and any k-ary R,*

$$B(Q_1,...,Q_k)R \quad \Rightarrow \quad (Q_1...Q_k)^{(i_1,...,i_k)}R.$$

Proof. If there are $X_1,...,X_k$ such that Q_jX_j for each j and $X_1 \times ... \times X_k \subseteq R$, it follows that

(*) $$X_{i_1} \times ... \times X_{i_k} \subseteq R^{(i_1,...,i_k)}.$$

Suppose first that $Q_j\varnothing$ for some j. Since Q_j is MON↑, $\{a : Q_jA\} = M$ for every set A. But, by MON↑ and our assumption, Q_iM for *each i*. From this it follows that $(Q_1...Q_k)^{(i_1,...,i_k)}R$.

Next, suppose $\neg Q_j\varnothing$ for each i. Then, by the Product Decomposition Lemma,

$$Q_{i_1}...Q_{i_k}(X_{i_1} \times ... \times X_{i_k}).$$

Thus, by (*) and Lemma 5.2, $Q_{i_1}...Q_{i_k}R^{(i_1,...,i_k)}$, i.e., $(Q_1...Q_k)^{(i_1,...,i_k)}R$. —|

This is a good time to note the following fact, which is immediate from the definition of branching:

5.6. Fact. $B(Q_1,...,Q_k)R \quad \Leftrightarrow \quad B(Q_{i_1},...,Q_{i_k})R^{(i_1,...,i_k)}$.

We may express this by saying that branching prefixes are *order-independent*, in contrast with (most) linear prefixes. Note carefully that order-independence, i.e., invariance under permutations of quantifier expressions in a prefix, is a property of syntactic prefixes, but *not* of the quantifiers defined by these prefixes. For example, the quantifiers $\exists\exists$ and $\exists\neg\cdot\neg\exists$ are identical, but whereas the prefix $\exists x \exists y$ is order-independent, the prefix $(\exists\neg)x(\neg\exists)y$ is not!

Also note the difference between order-independence of a prefix and convertibility of the corresponding quantifier, i.e., invariance under permutations of the arguments of the relation (this *is* a property of quantifiers). Branching quantifiers are not in general convertible, not even when they are equal to iterations, as we shall see. However, for both linear and branching prefixes, the two properties coincide in the case of iterations of the *same* quantifier expression (compare the notion of self-commutativity in van Benthem 1989), so, for example, $B(Q,...,Q)$ is convertible. Is this the *only* case when a branching quantifier is convertible? The positive answer to this question (which was posed by Jaap van der Does) turns out to be a simple application of the methods developed here:

5.7. Proposition. $B(Q_1,...,Q_k)$ *is convertible iff, on any M where* $B(Q_1,...,Q_k)$ *is non-trivial,* $Q_1 = ... = Q_k$.

Proof. Suppose $B(Q_1,...,Q_k)$ is convertible and non-trivial on M. Let $i_1,...,i_k$ be any permutation of $1,...,k$, and let $j_1,...,j_k$ be its inverse permutation. Then

$$
\begin{aligned}
B(Q_{i_1},...,Q_{i_k})R &\Leftrightarrow B(Q_1,...,Q_k)R^{(j_1,...,j_k)} && \text{(by Fact 5.6)} \\
&\Leftrightarrow B(Q_1,...,Q_k)R && \text{(by convertibility).}
\end{aligned}
$$

Thus, $B(Q_1,...,Q_k) = B(Q_{i_1},...,Q_{i_k})$. But then, by Lemmas 5.3 and 5.4, $Q_1...Q_k$ and $Q_{i_1}...Q_{i_k}$ are equal on products (on M). Since each Q_i is positive on M, it follows as usual from Corollary 3.7 that $Q_r = Q_{i_r}$, for $r = 1,...,k$. But since $i_1,...,i_k$ was an arbitrary permutation, this can only hold if $Q_1 = ... = Q_k$ on M. —|

For linear prefixes, the properties of order-independence and convertibility (of the corresponding quantifier) don't seem to have much to do with each other. However, it is shown in Westerståhl 1986 that at least in the MON↑ case, order-independence too is a rare phenomenon (under ISOM, and for finite universes). In fact, the results there together with Theorem 4.8 show that the following *local* result is true.

5.8. Proposition. If *$Q_1,...,Q_k$ are MON↑ and non-trivial on M, then*

$Q_1...Q_k$ is convertible on M $\Leftrightarrow$ *the prefix $Q_1x_1...Q_kx_k$ is order-independent on M.*

The main result of this section says that branchings are iterations only in very few cases.

5.9. Theorem. *$B(Q_1,...,Q_k)$ is a k-ary iteration iff, on each M where $B(Q_1,...,Q_k)$ is non-trivial, for some n with $0 \le n < k$, $B(Q_1,...,Q_k) = \exists^{(n)}\cdot Q_{n+1}\cdot\forall^{(k-n-1)}$ on M; in fact, $Q_1 = ... = Q_n = \exists$, and $Q_{n+2} = ... = Q_k = \forall$, on M ($k \ge 2$).*

Proof. 'If': We have to find $Q_1',...,Q_k'$ such that $B(Q_1,...,Q_k) = Q_1'...Q_k'$. If M is such that $B(Q_1,...,Q_k)$ is trivial, clearly this is possible. Otherwise, we take $Q_i' = Q_i$, where $Q_1 = ... = Q_n = \exists$, and $Q_{n+2} = ... = Q_k = \forall$, on M. From Lemma 5.5, we know that $B(Q_1,...,Q_k)R \Rightarrow Q_1...Q_kR$. For the other direction, suppose that $Q_1...Q_kR$, i.e., that $\exists^{(n)}\cdot Q_{n+1}\cdot\forall^{(k-n-1)}R$. Then

$$\{(a_1,...,a_n) : Q_{n+1}\cdot\forall^{(k-n-1)}R_{a_1...a_n}\} = \{(a_1,...,a_n) : Q_{n+1}\{b : R_{a_1...a_nb} = M^{k-n-1}\}\} \neq \emptyset.$$

Take $(a_1,...,a_n)$ in this set. It follows that

$$\{a_1\} \times ... \times \{a_n\} \times \{b : R_{a_1...a_nb} = M^{k-n-1}\} \times M \times ... \times M \subseteq R.$$

Thus, $B(Q_1,...,Q_k)R$.

'Only if': Suppose $B(Q_1,...,Q_k) = Q_1'...Q_k'$, and that $B(Q_1,...,Q_k)$ is non-trivial on M. By Lemma 5.3, each Q_i is non-trivial on M (mention of M will be omitted in what follows). By Lemma 5.4 and the assumption, $Q_1...Q_k$ and $Q_1'...Q_k'$ are equal on products. Hence, by the Product Theorem, $Q_1 ... Q_k = Q_1' ... Q_k'$. That is, $B(Q_1,...,Q_k) = Q_1 ... Q_k$. Now we are in a position to show that the quantifiers $Q_1,...,Q_k$ have the required form, by induction on k.

INDUCTION BASE, $k = 2$: Suppose $B(Q_1,Q_2) = Q_1Q_2$ on $M = \{a_1,...,a_m\}$. Let n_i be the smallest number in Q_i, $i = 1,2$. $n_1,n_2 > 0$, by non-triviality. If $n_1 = 1$, then $Q_1 = \exists$, and we are done. Assume $n_1 > 1$. We must show that $n_2 = m$ (so $Q_2 = \forall$). Suppose instead that $n_2 < m$. Let

$$R \quad = \quad (\{a_1,...,a_{n_1-1}\}\times\{a_1,...,a_{n_2}\}) \cup (\{a_{n_1}\}\times\{a_2,...,a_{n_2},a_m\}).$$

It follows that if $X\times Y \subseteq R$, then $|X| < n_1$. Thus, $\neg B(Q_1,Q_2)R$. On the other hand, $|\{a: |R_a| \geq n_2\}| = n_1$, so Q_1Q_2R. This contradicts our assumption.

INDUCTION STEP: Suppose the result is true for k, and $B(Q_0,...,Q_k) = Q_0 ... Q_k$. We use the following

5.10. Lemma. *If* $B(Q_0,...,Q_k) = Q_0 ... Q_k$ *and* $B(Q_0,...,Q_k)$ *is non-trivial on* M, $B(Q_1,...,Q_k) = Q_1 ... Q_k$ *and* $B(Q_0,...,Q_{k-1}) = Q_0 ... Q_{k-1}$.

Proof. Fix A such that Q_0A (non-triviality and Lemma 5.3) and take any $R \subseteq M^k$. We have

$$\begin{aligned} & B(Q_0,...,Q_k)A\times R \\ \Leftrightarrow\quad & \exists X_0,...,X_k\ [Q_0X_0\ \&\ ...\ \&\ Q_kX_k\ \&\ X_0\times...\times X_k \subseteq A\times R] \\ \Leftrightarrow\quad & \exists X_0[Q_0X_0\ \&\ X_0 \subseteq A]\ \&\ \exists X_1,...,X_k\ [Q_1X_1\ \&\ ...\ \&\ Q_kX_k\ \&\ X_1\times...\times X_k \subseteq R] \\ \Leftrightarrow\quad & Q_0A\ \&\ B(Q_1,...,Q_k)R \\ \Leftrightarrow\quad & B(Q_1,...,Q_k)R\ . \end{aligned}$$

But also,

$$Q_0 ... Q_kA\times R \quad \Leftrightarrow \quad Q_0A\ \&\ Q_1 ... Q_kR \quad \text{(product decomposition)} \quad \Leftrightarrow \quad Q_1 ... Q_kR.$$

This proves the first part of the lemma. The second part is proved by fixing B such that Q_kB and considering $R\times B$. —|

To finish the proof of Theorem 5.9, the induction hypothesis and the lemma shows that both $B(Q_1,...,Q_k)$ and $B(Q_0,...,Q_{k-1})$ have the desired 'form' on M. But then it is readily verified that the same holds for $B(Q_0,...,Q_k)$. —|

6. Cumulation

So-called *cumulative readings* (Scha 1981) of quantifiers are natural for sentences like

(1) Sixty teachers taught seventy courses at the summer school

(2) Five girls told ten stories to three boys.

(2) has a reading involving five girls, ten stories, and three boys, but not saying exactly how many stories the first girl told the second boy etc., only that each girl told some story to some boy, that each story was told by some girl to some boy, etc. This leads to the following

6.1. Definition.

$$(Q_1,\ldots,Q_k)^c x_1 \ldots x_k R x_1 \ldots x_k \;\leftrightarrow\; Q_1 x_1 \exists x_2 \ldots \exists x_k R x_1 \ldots x_k \wedge \ldots \wedge Q_k x_k \exists x_1 \ldots \exists x_{k-1} R x_1 \ldots x_k.$$

Or, equivalently,

$$(3) \qquad (Q_1,\ldots,Q_k)^c R \;\Leftrightarrow\; \bigwedge_{1 \le i \le k} Q_i \cdot \exists \ldots \exists R^{(i,1,\ldots,i-1,i+1,\ldots,k)}.$$

Thus, cumulatives are Boolean combinations of iterations, but we shall see that they are very seldom iterations themselves. We only consider cumulations of positive (but not necessarily monotone) quantifiers, and start with the following observations.

6.2. Lemma. *Let $Q_1,\ldots,Q_k$ be positive. Then $(Q_1,\ldots,Q_k)^c$ and $Q_1 \ldots Q_k$ are equal on products.*

Proof.

$$\begin{aligned}
(Q_1,\ldots,Q_k)^c A_1 \times \ldots \times A_k &\Leftrightarrow \bigwedge_{1 \le i \le k} Q_i \cdot \exists \ldots \exists (A_i \times A_1 \times \ldots A_{i-1} \times A_{i+1} \times \ldots \times A_k) && \text{(definition)} \\
&\Leftrightarrow \bigwedge_{1 \le i \le k} (Q_i A_i \;\&\; \bigwedge_{j \ne i} (A_i \ne \varnothing)) && \text{(product decomposition)} \\
&\Leftrightarrow Q_1 A_1 \;\&\; \ldots \;\&\; Q_k A_k && \text{(positivity)} \\
&\Leftrightarrow Q_1 \ldots Q_k A_1 \times \ldots \times A_k && \text{(product decomposition).}
\end{aligned}$$

⊣

6.3. Lemma. *Suppose $Q_1,\ldots,Q_k$ are positive on M. $(Q_1,\ldots,Q_k)^c$ is non-trivial on M iff each Q_i is non-trivial on M.*

Proof. If Q_j is trivial on M it must be empty on M (since it is positive), but then the corresponding conjunct in (3) is always false, so $(Q_1,\ldots,Q_k)^c$ is trivial on M. If each Q_i is non-trivial on M there are $A_i \ne \varnothing$ such that $Q_i A_i$, $1 \le i \le k$, and thus, just as in the proof of the previous lemma, $(Q_1,\ldots,Q_k)^c A_1 \times \ldots \times A_k$. We must show that $\neg \forall R \subseteq M (Q_1,\ldots,Q_k)^c R$. Otherwise, $(Q_1,\ldots,Q_k)^c \varnothing$. But then the first conjunct in (3) is $Q_1 \cdot \exists \ldots \exists \varnothing$, and hence $Q_1 \varnothing$, contradicting the positivity of Q_1. ⊣

Now we can characterize the (positive) cumulations which are iterations.

6.4. Theorem. *$(Q_1,...,Q_k)^c$ is a k-ary iteration iff, on each M where $(Q_1,...,Q_k)^c$ is non-trivial, $Q_2 = ... = Q_k = \exists$.*

Proof. 'If': As usual, we can obviously find the required iteration on universes where $(Q_1,...,Q_k)^c$ is trivial. On other universes, one easily verifies, using the positivity of Q_1, that $(Q_1,\exists,...,\exists)^c = Q_1 \cdot \exists ... \exists$.

'Only if': Suppose $(Q_1,...,Q_k)^c = Q_1'...Q_k'$, and that $(Q_1,...,Q_k)^c$ is non-trivial on M. By Lemma 6.2, $Q_1...Q_k$ and $Q_1'...Q_k'$ are equal on products, and hence equal, by the Product Theorem. Thus, $(Q_1,...,Q_k)^c = Q_1...Q_k$ on $M = \{a_1,...,a_m\}$. We shall prove the

CLAIM: If $0 < k_i < m$ then $k_i \in Q_i \Leftrightarrow k_i+1 \in Q_i$, for $i = 2,...,k$.

Since $0 \notin Q_i$ but some $p \in Q_i$, it follows from the Claim that $Q_i = \exists$, for $i = 2,...,k$.

To prove the Claim, fix such an i. For each $j \neq i$, $1 \leq j \leq k$, choose $k_j \in Q_j$. Thus, each $k_j > 0$. Now define the k-ary relation R by the following stipulations:

$$Rx_1...x_{i-1}a_1x_{i+1}...x_k \;,\; .... \;,\; Rx_1...x_{i-1}a_{k_i}x_{i+1}...x_k \;,$$
for all $x_1 \in \{a_1,...,a_{k_1-1}\}$ and $x_j \in \{a_1,...,a_{k_j}\}$, $2 \leq j \leq k, j \neq i$,

and $$Ra_{k_1}x_2...x_{i-1}a_2x_{i+1}...x_k \;,\; .... \;,\; Ra_{k_1}x_2...x_{i-1}a_{k_i+1}x_{i+1}...x_k \;,$$
for all $x_j \in \{a_1,...,a_{k_j}\}$, $2 \leq j \leq k, j \neq i$,

and no other k-tuples are in R.

Then R has the following properties.

(i) For $j \neq i$, $1 \leq j \leq k$: $\{y_j : \exists x_1...\exists x_{j-1}\exists x_{j+1}...\exists x_k Rx_1...x_{j-1}\, y_j\, x_{j+1}...x_k\} = \{a_1,...,a_{k_j}\}$, and hence, $Q_j \cdot \exists ... \exists R^{(j,1,...,j-1,j+1,...,k)}$.

(ii) $\{y_i : \exists x_1...\exists x_{i-1}\exists x_{i+1}...\exists x_k Rx_1...x_{i-1}\, y_i\, x_{i+1}...x_k\} = \{a_1,...,a_{k_i+1}\}$.

(iii) For $i < r < k$: If $x_j \in \{a_1,...,a_{k_j}\}$ for $j \neq i$, $1 \leq j < r$, and $x_i \in \{a_1,...,a_{k_i+1}\}$, then $\{y_r : Q_{r+1}...Q_kR_{x_1..x_{r-1}y_r}\} = \{a_1,...,a_{k_r}\}$. Otherwise, $\{y_r : Q_{r+1}...Q_kR_{x_1..x_{r-1}y_r}\} = \varnothing$.

(i) and (ii) are immediate from the definition of R. (iii) follows by a (downward) inductive argument on r (omitted here), using the fact that $0 \notin Q_r$ but $k_r \in Q_r$, for $i < r \leq k$. (iii) is not true for $r = i$. However, we do have

(iv) If $x_j \in \{a_1,...,a_{k_j}\}$ for $1 \leq j < i$, then $|\{y_i : Q_{i+1}...Q_kR_{x_1..x_{i-1}y_i}\}| = k_i$. Otherwise, $|\{y_i : Q_{i+1}...Q_kR_{x_1..x_{i-1}y_i}\}| = 0$.

Indeed, we see from (iii) and the definition of R that $\{y_i : Q_{i+1}...Q_kR_{x_1..x_{i-1}y_i}\}$ is either $\varnothing$ or else $\{a_1,...,a_{k_i}\}$ or $\{a_2,...,a_{k_i+1}\}$.

Now, let $0 < k_i < m$ and suppose first that $k_i \in Q_i$. Then, by (iii) and (iv), we can 'continue downward', from $r = i$ to $r = 1$, eventually obtaining

$$\{y_1 : Q_2 ... Q_k R_{y_1}\} \;=\; \{a_1,...,a_{k_1}\},$$

and hence $Q_1...Q_kR$. But then, $(Q_1,...,Q_k)^c R$, and so in particular,

$$Q_i \cdot \exists ... \exists R^{(i,1,...,i\text{-}1,i+1,...,k)}.$$

By (ii), this is equivalent to

$$Q_i\{a_1,...,a_{k_i+1}\},$$

and thus $k_i+1 \in Q_i$.

Now suppose instead $k_i \notin Q_i$. It follows that from step i-1 and downward in the above induction we always get $\emptyset$, and so $\neg Q_1...Q_kR$. Hence, $\neg(Q_1,...,Q_k)^c R$. But, by (i), all the conjuncts in the definition of $(Q_1,...,Q_k)^c R$, except the one involving Q_i, are true. It follows that

$$\neg Q_i \cdot \exists ... \exists R^{(i,1,...,i\text{-}1,i+1,...,k)},$$

which, by (ii), means that $k_i+1 \notin Q_i$. This proves the Claim, and thereby the theorem. —|

7. Unary complexes

In this section, we briefly look at a generalization of the question as to when a certain quantifier is an iteration. The linguistic interest of this question stems from the ubiquity of natural language sentences with a transitive verb and quantified subject and object noun phrases. But other means of expression are also 'natural'. For one thing, Boolean operators are clearly available. For another, sentences corresponding to iterations are often ambiguous, and hence all their readings can be used. Restricting attention to 2-ary iterations, this leads to the following definition, from van Benthem 1989 (though he uses 'unary complex' for what I here call a 'right complex').

7.1. Definition. A quantifier Q of type <2> is a *unary complex* if there is a Boolean combination Φ of iterations of the form Q_1Q_2R and inverse iterations of the form $Q_1'Q_2'R^{-1}$, such that for all R, $QR \Leftrightarrow \Phi$. Q is a *right complex* (*left complex*) if only iterations (inverse iterations) are used.

The next proposition is an example of the added expressive power of unary complexes compared to iterations. Note that, by Corollary 4.9, $(\exists_{\geq n})^{(2)}$ is not an iteration for $n \geq 2$.

7.2. Proposition. $(\exists_{\geq n})^{(2)}$ *is a right complex for all n.*

Proof. (sketch) We must express $|R| \geq n$, for binary R, as a right complex. Start with the following equivalence, which is clearly valid:

$$|R| \geq n \quad \Leftrightarrow \quad \bigvee_{1\leq k\leq n-1} (\exists_{=k}x\exists yRxy \ \wedge \ |R| \geq n) \ \vee \ \exists_{\geq n}x\exists yRxy.$$

Thus, it suffices to express each of the n - 1 first disjuncts as right complexes. Given that exactly k elements have R-successors, it is not so hard to describe the circumstances under which $|R| \geq n$. Consider the different ways, say $s_1,...,s_r$, in which n can be written as the sum of k positive integers (independent of order). Each such way corresponds to a minimal distribution of successors (over the k elements which have successors) so that $|R| \geq n$. With each s_i we will correlate a right complex ψ_i. Rather than giving precise details, we explain the idea by means of an example. There are 5 ways in which 10 (n) can be written as the sum of 6 (k) positive integers. We exhibit them below, together with the corresponding right complexes:

s_i	ψ_i
5 + 1 + 1 + 1 + 1 + 1	$\exists x\exists_{\geq 5}yRxy$
4 + 2 + 1 + 1 + 1 + 1	$\exists_{\geq 2}x\exists_{\geq 2}yRxy \ \wedge \ \exists x\exists_{\geq 4}yRxy$
3 + 3 + 1 + 1 + 1 + 1	$\exists_{\geq 2}x\exists_{\geq 3}yRxy$
3 + 2 + 2 + 1 + 1 + 1	$\exists_{\geq 3}x\exists_{\geq 2}yRxy \ \wedge \ \exists x\exists_{\geq 3}yRxy$
2 + 2 + 2 + 2 + 1 + 1	$\exists_{\geq 4}x\exists_{\geq 2}yRxy$

(note that, since $\exists_{=6}x\exists yRxy$, conjuncts corresponding to the 1's, such as $\exists_{\geq 5}x\exists yRxy$ for the first row, are not needed in ψ_i). Hopefully the idea is clear, and one may now verify that, in general,

$$\exists_{=k}x\exists yRxy \ \wedge \ |R| \geq n \quad \Leftrightarrow \quad \exists_{=k}x\exists yRxy \ \wedge \ (\psi_1 \vee ... \vee \psi_r).$$

$\dashv$

The properties of *orientation* from van Benthem 1989 provide convenient ways of showing that certain quantifiers are *not* unary (right, left) complexes. For example, in this way one easily sees that cumulations, although unary complexes, are usually not right or left complexes, and that branchings are usually not unary complexes. However, the method does not work for resumptions, since all resumptions have these orientation properties. Our final application of 'prefix techniques' in this paper shows that the resumption of Q^{R} is not a unary complex. Recall that, on finite universes, Q^{R} means "more than half of the elements of the universe". It is clear that the next result extends to any proportion quantifier "more than m/n:ths of the elements of the universe".

7.3. Theorem. *Q^{R} is not a unary complex.*

Proof. We shall prove that not even $(Q^{\mathrm{R}})^{(2)}A\times B$ can be expressed by a (fixed) unary complex. Suppose to the contrary that there is a unary complex Φ, i.e., a Boolean combination of iterations of the forms $Q'Q''A\times B$ and $Q'''Q''''B\times A$ such that for all M and all $A,B \subseteq M$,

$$|A\times B| > |M|^2/2 \quad \Leftrightarrow \quad \Phi.$$

We now perform the following operations on Φ. First, by redefining the quantifiers in Φ if necessary, make sure that the second quantifier in an iteration (or inverse iteration) is always positive. Then replace, according to the Product Decomposition Lemma, each $Q'Q''A{\times}B$ in Φ by $(Q'A \wedge Q''B) \vee (Q'\varnothing \wedge \neg Q''B)$, and similarly for the inverse iterations. Next, rewrite the result in disjunctive normal form. Finally, in each disjunct, 'pull together' the conjuncts which involve A by defining suitable new quantifiers (for example, if the conditions on A in one conjunct are $Q'A$, $Q''A$, and $\neg Q'''A$, replace $Q'A \wedge Q''A \wedge \neg Q'''A$ by QA , where $Q = Q' \wedge Q'' \wedge \neg Q'''$), and similarly for B and $\varnothing$. The result of all this is that there are type <1> quantifiers Q_i , Q_i', and Q_i'', $1 \le i \le p$, which satisfy ISOM and are such that

$$|A{\times}B| > |M|^2/2 \quad \Leftrightarrow \quad \bigvee_{1 \le i \le p} (Q_i A \wedge Q_i'B \wedge Q_i''\varnothing) \qquad (\text{on } M).$$

In other words, for all $m > 0$ there are $X_i, Y_i, Z_i \subseteq \{0,...,m\}$, $1 \le i \le p$, such that for all $k,n \le m$,

$$kn > m^2/2 \quad \Leftrightarrow \quad \bigvee_{1 \le i \le p} (k \in X_i \wedge n \in Y_i \wedge 0 \in Z_i).$$

Simplifying a little, it follows that

(*) $$\exists p\, \forall m > 0\, \exists p' \le p\, \exists X_i, Y_i \subseteq \{0,...,m\} \text{ for } 1 \le i \le p' \text{ s. t. } \forall k,n \le m,$$

$$kn > m^2/2 \quad \Leftrightarrow \quad \bigvee_{1 \le i \le p} ((k,n) \in X_i{\times}Y_i).$$

It is intuitively plausible that (*) cannot possibly be true, since p is fixed but m arbitrary. Nevertheless, here is a proof.

Suppose m is even and large enough (cf. below). For $k,n \le m$, call (k,n) *minimal* if $kn > m^2/2$, but $(k - 1)n, k(n - 1) \le m^2/2$. We shall count the number of minimal pairs.

(i) If (k,n) is minimal then (n,k) is minimal.

(ii) If (k,n) is minimal then $k,n > m/2$ and $k \ne n$.

(iii) $(m/2 + 1 , m - 1)$ is minimal.

These are all immediate or almost: that (k,k) cannot be minimal follows from the fact that if $k{\cdot}k > m^2/2$ then $k(k - 1) > m^2/2$, for large enough m ($m > 70$ suffices).

Let $k_0 = m/2 + 1$, and then $k_{i+1} = k_i + 1$ until we reach $k_l = k_{l-1} + 1 =$ the largest k such that $k{\cdot}k \le m^2/2$. Also, let n_i be the smallest n such that $k_i{\cdot}n > m^2/2$.

(iv) (k_i,n_i) is minimal, $n_i > k_i$, for $0 \le i \le l$, and $n_0 > ... > n_l$.

Proof: That $n_i > k_i$ is immediate. The rest is by induction. $(k_0,n_0) = (m/2 + 1 , m-1)$ is minimal. Suppose (k_i,n_i) is minimal. Then $k_{i+1}(n_i - 1) = (k_i + 1)(n_i - 1) = k_in_i + n_i - (k_i + 1) \ge k_in_i$ (since $n_i > k_i$) $> m^2/2$. Thus, $n_{i+1} < n_i$. To see that (k_{i+1},n_{i+1}) is minimal, it suffices to check that $(k_{i+1} - 1)n_{i+1} \le m^2/2$. But $(k_{i+1} - 1)n_{i+1} = k_{i+1}n_{i+1} - n_{i+1} < k_{i+1}n_{i+1} - k_{i+1} = k_{i+1}(n_{i+1} - 1) \le m^2/2$, by the definition of n_{i+1}.

(v) If (k,n) is minimal and $k < n$, then $(k,n) = (k_i,n_i)$, for some i.

Proof: It suffices to show that $k_0 \le k \le k_l$. Clearly $k_0 \le k$. Suppose $k > k_l$. Now $(k_l + 1)(k_l + 1) > m^2/2$, and hence $k_l(k_l + 1) > m^2/2$. But then, $(k - 1)n > m^2/2$, contradicting the minimality of (k,n).

From (i), (ii), (iv) and (v) we get

(vi) There are $2(l + 1)$ minimal pairs.

Now we can return to (*). The point of the preceding exercise is this:

(vii) Distinct minimal pairs belong to distinct $X_i{\times}Y_i$ in (*).

Proof: Suppose (k,n), (k',n') are minimal, $(k,n) \ne (k',n')$, say, $k < k'$, and $(k,n),(k',n') \in X_i{\times}Y_i$. But then $(k,n') \in X_i{\times}Y_i$, so $kn' > m^2/2$ by (*), which contradicts the minimality of (k',n').

We have shown that $p \ge p' \ge 2(l + 1)$. But this is impossible, since p is fixed and l increases with m, in fact,

(viii) $l > m/2(\sqrt{2} + 1) - 2$.

Proof: Easy calculation, from the facts that $(k_l + 1)^2 > m^2/2$, and $k_l = m/2 + l + 1$.

This concludes the proof of the theorem. —|

I have been a bit fussy about distinguishing local from global results in this paper. The usual notion of definability in logic is global, i.e., uniform over universes, and I have endeavoured to state global forms of all definability results here. We have seen that many of these results have both a global and a local version. The last result above, however, is a good illustration of the point that this is not always so. Indeed, it follows from a result in van Benthem 1989 that, *on any given universe* M, Q^R can be defined as a right complex. There is just no definition that works for all universes.[7]

8. Issues for further study

What else could said about iteration? Very briefly, here are a few suggestions.

1. *Characterizing quantifier lifts.* Can iteration be characterized in terms of their properties? That is, are there (interesting) properties such that, say, a type <2> quantifier is a 2-ary iteration iff it has these properties?[8] One necessary such property is being determined by its behaviour on products. We have seen in Part II that this is not sufficient, but combining it with other properties might give a sufficient condition.

This is part of a more general issue. Iteration, resumption, branching and cumulation can all be considered as natural *liftings* of monadic quantifiers to polyadic ones (or, more generally, liftings of

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[7] In Westerståhl 1992 I conjecture that Q^R is not even monadically definable, that is, not definable in any logic $L(Q_1,...,Q_k)$, where the Q_i are monadic.

[8] Keenan 1991b has such a characterization (the Reducibility Characterization Theorem), but the property used is too reminiscent of the definition of iteration to be of real interest here (it has other uses, mainly as a tool for showing that certain quantifiers are not iterations).

quantifiers of certain types to quantifiers of 'higher' types). These lifts have characteristic properties, and it would be interesting to know if they can be completely characterized in terms of these properties. A similar question is studied in van der Does 1992a,b in the field of collective quantification (i.e., quantification over collections, or sets, of individuals). He displays several lifts from ordinary quantification to quantification over collections, and, among other things, characterizes the lifts in terms of their respective properties, such as (versions of) conservativity, monotonicity, etc. Further, in both cases one can study which linguistic mechanisms trigger such lifts. In general (as Johan van Benthem has pointed out), polyadic quantification of the kinds studied here and collective quantification seem to have much in common.

2. *Generalizing the Prefix Theorem.* It is an immediate corollary of the Prefix Theorem that if $Q_1x_1...Q_kx_k$ and $Q_1'x_1...Q_k'x_k$ are prefixes with Q_i and Q_i' in $\{\forall,\exists\}$ such that

(*) $$\models Q_1x_1...Q_kx_kRx_1...x_k \quad \leftrightarrow \quad Q_1'x_1...Q_k'x_kRx_1...x_k ,$$

then $Q_i = Q_i'$ for each i (in view of Corollary 3.7, we could replace $Rx_1...x_k$ by $P_1x_1 \wedge ... \wedge P_kx_k$ here). This is a weak version of the Linear Prefix Theorem in Keisler and Walkoe 1973. In the original version, $Rx_1...x_k$ on the right hand side in (*) is replaced by an arbitrary quantifier-free formula ϕ (without constant or function symbols). As Keenan notes, the strong version does not hold for arbitrary type <1> quantifiers; even $\forall xPx \leftrightarrow (\neg\exists)x\neg Px$ is a counterexample. But it might still hold under some restrictions. For example, does it hold when all the quantifiers are positive?

3. *Infinite universes.* The proofs of the main theorems in Part II depend heavily on the assumption that universes are finite. Are there versions of these results for infinite models?

4. *General branching.* Barwise's branching of monotone quantifiers has been extended to other generalized quantifiers (cf. Spaan 1992 for references and a discussion of the various options here). Can the results in section 5 be extended accordingly?

5. *Iteration in other types.* The notion of iteration is not confined to quantifiers. Thinking of quantifiers as objects in type theory, one may generalize the idea to objects of (certain) other types. What is the common pattern here? Will characteristic properties of quantifier iteration, such as the Product Theorem, carry over to the general case?

Already for the case studied here, the perspective of type theory and categorial grammar, may be fruitful. Johan van Benthem remarked that our basic iteration scheme (Definition 1.3) yields a type transition

$$(e^k\to t)\to t\,,\,(e^m\to t)\to t \quad\Rightarrow\quad (e^{k+m}\to t)\to t \qquad\qquad (\text{where } e^1 = e,\ e^{n+1} = e\cdot e^n)$$

which is provable in the Lambek Calculus, and that the scheme itself is precisely the lambda term for its most straightforward derivation (cf. van Benthem 1991). Likewise, some properties of iterations, like the preservation of ISOM or CONSERV, can be predicted from a categorial analysis.

6. *Syllogistic inference.* Various kinds of syllogistic inference with non-iterated quantifiers are known from the literature. For example, consider a language which has atomic formulas of the form

QAB , where A and B are Boolean combinations of *set variables* $X_1,X_2,...$, and Q is a type <1,1> quantifier symbol selected among $Q_1,Q_2,...$, where $Q_1,Q_2,...$ are given quantifiers. The language also has the usual propositional connectives, and the obvious semantics. For simple choices of $Q_1,Q_2,...$, complete axiomatizations of validity are known; for example, van der Hoek and de Rijke 1991 axiomatize the case when Q_i = *at least i* (in this logic, quantifiers like *all* and *no* are of course definable).

This could be generalized to iterations, say, of type <1,1,2>. One then adds variables $R_1,R_2,...$ for binary relations, and in addition to the Boolean operations, one might have other operations from relational algebra, such as *converse*. New 'atomic' formulas are $QQ'AB,R$, with A, B as before, R an expression in the chosen relational algebra, and Q, Q' type <1,1> quantifier symbols. The expressive power has increased somewhat, but is still weak compared to $L(Q_1,Q_2,...)$. Do the valid sentences still have nice axiomatizations for natural choices of $Q_1,Q_2,...$?

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