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Binary Quantifiers and Relational Semantics

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Abstract

The present paper is an attempt to extend the treatment of unary generalized quantifiers proposed in [van Lambalgen 1991] and [van Benthem & Alechina 1993] to the binary case. A binary quantifier $\Pi x(\varphi, \psi)$ can be understood as "typically, φ's are ψ's". After investigating some formal properties of these quantifiers, I define an embedding of predicate conditional logic into a logic with binary quantifiers and discuss possible applications of binary quantifiers in formalizing defeasible reasoning.

1 Introduction

This paper consists of three parts, which - hopefully - have some connection to each other.

On the one hand, I am going to give one more suggestion concerning a formal way to treat statements like "φ's are (normally, typically) ψ's": common sense generalizations which can be used as premises in defeasible reasoning. The common feature of those statements is that they can be true while some exceptions (φ's which are not ψ's) are present. In the paper they are formalized with the use of a binary generalized quantifier Π, with $\Pi x(\varphi(x), \psi(x))$ informally meaning "for all typical φ's ψ is true". A related work in this respect is, first of all, [Badaloni & Zanardo, 1990,1991,1992]. (A more detailed comparison is given in section 6.) Another obvious connection is the way such statements are formalized in the theory of circumscription: if $\varphi(x)$ and *x is not abnormal*, then $\psi(x)$. In section 6 I discuss possible applications of the logic with generalized quantifiers to defeasible reasoning.

On the other hand, in its formal approach, this paper departs from a very different work ([van Lambalgen 1991, 1992], [van Benthem & Alechina 1993]) concerning alternative semantics for generalized quantifiers. This semantics and some properties of the corresponding logic (completeness theorems, facts concerning definability) are the subject of the first three sections.

Finally, in section 5 an analogy between relational binary generalized quantifiers and conditionals, suggested by J. van Benthem, is investigated. I show that first-order conditional logic can be embedded in a logic with binary quantifiers. After that two possible readings of commonsense generalizations: "normally, φ implies ψ" and "all typical φ's are ψ's" and the corresponding intensional and extensional semantics are compared.

2 Unary relational quantifiers

In the paper [van Lambalgen 1991] some well-known generalized quantifiers (like "for almost all", "for uncountably many", "for many") were translated into the language of the first-order logic enriched with a relational symbol R of indefinite arity (or a family of relational symbols, for every arity n) in the following way:

$$(Qx\varphi(x,\bar{y}))^* = \forall x(R(x,\bar{y}) \to \varphi(x,\bar{y})),$$

where Q is a universal generalized quantifier (say, "almost all" or the dual of "(uncountably) many"), and the $\bar{y}$ are *all* the free variables of $Qx\varphi$. It was proved that if a formula is provable in an axiom system for a given generalized quantifier, than its translation is provable in first-order logic with some additional axioms for R, and conversely. In the case of the quantifiers studied in [van Lambalgen 1991], R can be intuitively understood as an independence relation (with properties resembling those of a linear independence relation).

It is a natural move to consider new *models* for the generalized quantifiers, with a relation R of indefinite arity on the domain and a truth definition

$$M \models^{\alpha} Qx\varphi(x,\bar{y}) \Leftrightarrow \forall d(R(d,\bar{d}) \Rightarrow M \models \varphi[d,\bar{d}]),$$

where $\bar{d}$ is a sequence of elements assigned by the variable assignment α to $\bar{y}$. An analogy with modal logic becomes quite transparent now. Those (to be called *relational*) generalized quantifiers were studied in the paper [van Benthem & Alechina 1993]. In general, not every relational quantifier also has a standard semantics. (In standard semantics Q is interpreted as a set of subsets of the domain, $\mathbf{Q}$. $Qx\varphi(x)$ is true if $\{d : \varphi[d]\} \in \mathbf{Q}$.) For example, the minimal logic based on the above truth definition lacks extensionality: if $\{d : \varphi[x/d]\} = \{d : \psi[x/d]\}$, but if φ and ψ have different free variables, under some variable assignment $Qx\varphi(x)$ can be satisfied and $Qx\psi(x)$ not; this means that a standard semantics is not possible.

The quantifiers, studied by Michiel van Lambalgen, have both standard and relational interpretations. Although these quantifiers are not first-order definable, the relational interpretation provides a translation into first-order logic: quantifier properties correspond to first-order conditions on R. This is used in [van Lambalgen 1991] to construct a natural deduction system for these quantifiers, where variable restrictions correspond to R-conditions.

3 Weak and strong binary relational quantifiers

Consider a first-order language $L_{\forall\Pi}$ with a binary generalized quantifier Π. A well-formed formula is defined as usual; if φ and ψ are well-formed formulas, so is $\Pi x(\varphi,\psi)$. A standard semantics for this language would interpret Π as a binary relation on the powerset of the domain. We shall now look at a relational semantics (Π is interpreted by means of some relations between *the elements* of the domain). The intended interepretation of Π is described in the introduction to this paper: it means something like "for all typical".

For every formula $\varphi(x)$ in the language, let $R_{\varphi(x)}$ be a relation on the domain. The intuitive understanding of $R_{\varphi(x)}(d,\bar{e})$ is: d is a typical $\varphi(x)$ with respect to $\bar{e}$. [1] I would like to mention here that "typicality relative to other elements" is a different notion from the one which involves considering typical tuples of elements (and which corresponds to a polyadic quantifier). For example, the fact that a is a typical brother of b does not imply that a pair $< a, b >$ belongs to the set of typical brothers, and vice versa. I will not consider quantification over (typical) tuples of objects in this paper, although it can be very useful in formalizing generics about polyadic properties (cf. section 6.5).

[1] Another option (suggested by Johan van Benthem) is to have a *property* $R_{\varphi(x,\bar{e})}(d)$. This would amount to the same result given that the order of argument places in every formula is somehow fixed ($\bar{e}$ is a sequence).

We shall look at the following two readings of $\Pi x(\varphi, \psi)$. Let $\bar{y}$ be the free variables of φ (other than x), and $\bar{z}$ - the free variables of ψ (other than x).

- "all φ-typical (with respect to $\bar{y}$ *and* $\bar{z}$) x's satisfy $\psi(x)$", and
- "all φ-typical (with respect to $\bar{y}$) x's satisfy $\psi(x)$"

We shall consider both of them, and call the quantifiers "sensitive" to the free variables of the first and second arguments "weak quantifiers" and those which take into account only the variables of the first argument "strong quantifiers". The second reading is more or less clear: cf. the example about brothers. The first one is less intuitively appealing: what can "typicality with respect to $\bar{z}$" mean, if those parameters are not in the formula? I must confess that I don't have a good answer to this question; only that weak quantifiers provide a natural generalization of unary quantifiers with relational semantics.

The first example below is meant to show that the free variables are important in reasoning about typical objects.

Example 1 Consider the sentence "Some people are liked by their colleagues". It can be formalized as $\exists x \Pi y(Colleague(y, x), Likes(y, x))$, and the informal understanding suggested above is that there exists some x such that for all typical colleagues *of* x it is true that those colleagues like x. We could not quantify just over "typical colleagues": for example, this set can be empty, if it is defined as an intersection of all sets of typical colleagues of anybody, or typical colleagues of x can be extremely untypical as colleagues, and so on.

The second example shows the difference between weak and strong quantifiers:

Example 2 The sentence "Usually, cats and dogs don't like each other" can be formalized by means of $L_{\forall\Pi}$ (of course, a good formalization requires a polyadic quantifier) in two different ways:

$$\Pi x(Cat(x), \Pi y(Dog(y), \neg Likes(x, y) \wedge \neg Likes(y, x)))$$

and

$$\Pi y(Dog(y), \Pi x(Cat(x), \neg Likes(x, y) \wedge \neg Likes(y, x)))$$

For the weak quantifiers they have different truth conditions: informally, the first one is true if for all typical cats and for all relatively typical (with respect to the chosen cat) dogs the statement is true; the second one is true if it holds for all typical dogs and for all relatively typical cats. For the strong quantifiers the two sentences are equivalent: one quantifies over typical cats and typical dogs, and the order does not matter.

Formally, the semantics for weak and strong quantifiers looks as follows.

A model for strong generalized quantifiers is a triple $< D, R_{\varphi(x)}, I >$, where D is a domain, I an interpretation function, and $R_{\varphi(x)}$ - for every w.f.f. φ with n free variables and for every variable x, free in φ, is a n-ary relation between an element from D and a sequence of elements. [2]). If x is not free in φ, there are several options to define $R_{\varphi(x)}$; I choose for the minimal logic $R_{\varphi(x)}$ to be an arbitrary $n+1$-ary relation. Of course, a reasonable option is to postulate $R_{\varphi(x)}$ to be a trivial relation, if x is not free in φ. This condition corresponds to an additional axiom (see Appendix 8.2, Proposition 7).

The truth definition for the strong generalized quantifier reads as follows:

[2]The order of the elements in the sequence is fixed. It is important because d_1 can be typical with respect to d_2 d_3, but not with respect to d_3 d_2. Example: let $P(x, y, z)$ denote the property "x is a citizen of y willing to emigrate to z". Obviously, $\{d : R_{P(x)}(d, \ Russia, \ USA)\}$ is a set different from $\{d : R_{P(x)}(d, \ USA, \ Russia)\}$.

Strong $M \models^\alpha \Pi x(\varphi,\psi) \Leftrightarrow \forall d(R_{\varphi(x)}(d,\bar{d}) \wedge M \models \varphi[d,\bar{d}] \Rightarrow M \models \psi[x/d])$, where α assigns $\bar{d}$ to the free variables of φ other than x.

A model for the weak generalized quantifiers is a triple $< D, R_{\varphi(x)}, I >$, where D is a domain, I an interpretation function, and $R_{\varphi(x)}$ - for every w.f.f. φ of the language - is a relation (of indefinite arity) between an element from D and a sequence of elements. The truth definition reads as follows:

Weak $M \models^\alpha \Pi x(\varphi,\psi) \Leftrightarrow \forall d(R_{\varphi(x)}(d,\bar{d}\bar{e}) \wedge M \models \varphi[d,\bar{d}] \Rightarrow M \models \psi[d,\bar{e}])$, where α assigns $\bar{d}$ to the free variables of φ and $\bar{e}$ to the free variables of ψ.

In both models a semantic condition, corresponding to the axiom of Alphabetic Variants below, holds: if $\varphi(y)$ is a result of correct substitution of y instead of x in $\varphi(x)$, then $R_{\varphi(x)} = R_{\varphi(y)}$. (Which means, basically, that only the argument place in φ, and not the variable x itself, is important in determining $R_{\varphi(x)}$.) The semantics described above is not extensional: if two properties $\varphi(x)$ and $\psi(x)$ denote the same sets of objects, it is still not necessarily true that typical φ's and typical ψ's constitute the same sets. This seems rather natural, because typicality is an intensional property. Below I shall also consider logics with full or restricted extensionality for the first argument.

If the notion of an element d being $\varphi(x)$-typical taken to presuppose $\varphi[d]$, then the truth definitions become (for weak and for strong quantifiers, respectively)

$$M \models^\alpha \Pi x(\varphi,\psi) \Leftrightarrow \forall d(R_{\varphi(x)}(d,\bar{d}\bar{e}) \Rightarrow M \models \psi[d,\bar{e}])$$

and

$$M \models^\alpha \Pi x(\varphi,\psi) \Leftrightarrow \forall d(R_{\varphi(x)}(d,\bar{d}) \Rightarrow M \models \psi[d,\bar{e}]).$$

In the sequel it will become clear that the former two truth definitions and the latter two truth definitions plus the semantic conditions

$$R_{\varphi(x)}(d,\bar{d}) \Rightarrow M \models \varphi[d,\bar{d}] \quad (for\ strong\ quantifiers)$$

and

$$R_{\varphi(x)}(d,\bar{d}) \Rightarrow M \models \varphi[x/d] \quad (for\ weak\ quantifiers),$$

in the latter case given that all parameters of φ are among $\bar{d}$, give rise to the same logics. (See Appendix 8.1 and 8.2.) But until that time we shall apply the former ones (because they make the relation to unary quantifiers more transparent).

The notions of truth, validity and semantic consequence are standard.

Now, let us consider the logical systems corresponding to the described semantics. In the sequel, given a formula $\Pi x(\varphi,\psi)$, I shall use notation $FV(\varphi)$ (resp., $FV(\psi)$) to denote the free variables of φ (ψ) *other than* x.

Definition 1 *The* minimal logic Min_1 *for weak quantifiers is the least set of formulas closed under first-order derivability and*

Reflexivity $\vdash \Pi x(\varphi,\varphi)$

Restricted Distribution for the Second Argument

$$\vdash \Pi x(\varphi,\psi_1) \wedge \Pi x(\varphi,\psi_2) \rightarrow \Pi x(\varphi,\psi_1 \wedge \psi_2),$$

given that $FV(\varphi) \cup FV(\psi_2) = FV(\varphi) \cup FV(\psi_1)$;

Restricted Monotonicity for the Second Argument

$$\vdash \forall x(\varphi(x) \to \psi(x)) \wedge \Pi x(\chi, \varphi) \to \Pi x(\chi, \psi)$$

given that $FV(\varphi) \cup FV(\chi) = FV(\chi) \cup FV(\psi)$;

Tautology $\vdash \Pi x(\varphi, \psi \to \psi)$

Alphabetic Variants $\vdash \Pi x(\varphi, \psi) \leftrightarrow \Pi y(\varphi, \psi)$, *where* y *is free for* x *in* $\Pi x(\varphi, \psi)$.

Theorem 1 Min_1 *is complete.*

Proof See Appendix 8.1.

The properties of strong binary quantifiers are somewhat different from the properties of unary quantifiers (see [van Benthem & Alechina 1993]). For example, binary quantifiers thus defined are monotone without any restrictions. Indeed, if

$$\forall x(\psi \to \psi'),$$

then $\psi(d, \bar{e}) \to \psi'(d, \bar{e'})$,

$$(R_{\varphi(x)}(d, \bar{d}) \to \psi(d, \bar{e})) \to (R_{\varphi(x)}(d, \bar{d}) \to \psi'(d, \bar{e'}))$$

and

$$\Pi x(\varphi, \psi) \to \Pi x(\varphi, \psi').$$

For the same reason extensionality (and the rule of substitution) for the second argument is unrestricted.

Definition 2 *The* minimal logic Min_2 *for strong quantifiers is the least set of formulas closed under first-order derivability and*

Reflexivity $\vdash \Pi x(\varphi, \varphi)$;

Distribution for the Second Argument $\vdash \Pi x(\varphi, \psi_1) \wedge \Pi x(\varphi, \psi_2) \to \Pi x(\varphi, \psi_1 \wedge \psi_2)$

Monotonicity for the Second Argument $\vdash \forall x(\varphi(x) \to \psi(x)) \wedge \Pi x(\chi, \varphi) \to \Pi x(\chi, \psi)$;

Exchange $\vdash \forall y \Pi x(\varphi, \psi) \to \Pi x(\varphi, \forall y \psi)$, *given that* y *is not free in* φ;

Alphabetic Variants $\vdash \Pi x(\varphi(x), \psi(x)) \leftrightarrow \Pi y(\varphi(y), \psi(y))$, *given that* y *is free for* x *in* $\Pi x(\varphi(x), \psi(x))$.

Theorem 2 Min_2 *is complete.*

Proof See Appendix 8.2.

Since in both logics the only inference rules are Modus Ponens and Generalization, the deduction theorem has the same conditions (and proof) as in classical predicate logic.

One look at the axiom systems is enough to see that Min_1 is a subsystem of Min_2. Min_1 is much closer to the unary generalized quantifiers. From this point of vieuw Min_2 can be considered to be actually too strong.

If a binary quantifier $\Pi x(\varphi, \psi)$ is understood as $Qx(\varphi \rightarrow \psi)$, the property of Exchange corresponds to a very strong (actually, rather counterintuitive) property

$$\forall y Q x \phi \leftrightarrow Q x \forall y \phi$$

It makes the connection between strong quantifiers and unary quantifiers such as "for almost all" rather doubtful, because for this quantifier that property is not valid. On an uncountable domain D, if for every $d \in D$ the set $\{e : \phi[e, d]\}$ has measure 1, it does not follow that the (uncountable) intersection of those sets has measure 1 (namely, the set $\{e : \forall y \phi[e, y]\}$).

An intuitive counterexample against the Exchange axiom can be stated as follows. For every *specific* location (for example, Rome), the generic "Generally, birds live outside Rome" seems valid, that is, (after quantifying over locations)

$$\forall y \Pi x(Bird(x), Lives - outside(y, x))$$

can be accepted. But $\Pi x(Bird(x), \forall y Lives - outside(y, x))$ is obviously not true.

Now we shall consider some extensions of minimal logics. The first semantic property one can introduce is Restricted Extensionality:

Restricted Extensionality $M \models \forall x(\varphi(x, \bar{d}) \leftrightarrow \psi(x, \bar{d})) \Rightarrow \forall e(R_{\varphi(x)}(e, \bar{d}) \Leftrightarrow R_{\psi(x)}(e, \bar{d}))$.

The logic corresponding to the strong quantifier with this property is Bin': Min_2 plus

Restricted Substitution on the Left $\vdash \forall x(\varphi(x) \leftrightarrow \psi(x)) \wedge \Pi x(\varphi, \chi) \rightarrow \Pi x(\psi, \chi)$, given that φ and ψ have the same free variables.

Proof can be found in the Appendix 8.2, Proposition 5.

Full extensionality can be also allowed. The axiom

Substitution on the Left $\vdash \forall x(\varphi \leftrightarrow \psi) \wedge \Pi x(\varphi, \chi) \rightarrow \Pi x(\psi, \chi)$

corresponds to the condition of

Extensionality $M \models \forall x(\varphi(x, \bar{d}) \leftrightarrow \psi(x, \bar{e})) \Rightarrow \forall e(R_{\varphi(x)}(e, \bar{d}) \Leftrightarrow R_{\psi(x)}(e, \bar{e}))$

The proof is very straightforward (it is given in the Appendix 8.2, Proposition 4).

Another possible condition is that for every non-empty property there exists a typical element with this property:

Existence $\exists d(M \models \varphi[x/d, \bar{d}]) \Rightarrow \exists d(R_{\varphi(x)}(d, \bar{d}))$

For the semantics with this property Min_2 plus

Existence $\vdash \exists x \varphi(x) \rightarrow \neg \Pi x(\varphi, \neg \varphi)$

is complete. The proof is also given in Appendix 8.2 (Proposition 6).

If in the models for strong quantifiers $R_{\varphi(x)}$ is equal to D^{n+1} in case if x is not free in φ, the following property becomes valid:

Trivialization $\Pi x(\varphi, \psi) \rightarrow \forall x(\varphi \rightarrow \psi)$, where x is not free in φ, and, conversely, the logic Min_2 with this axiom added is complete for the above class of models. The proof is given in the Appendix 8.2, Proposition 7. (This property is not good, however, if $\Pi x(\varphi,\psi)$ is to be understood as $Qx(\varphi \to \psi)$.)

Let us now turn back to the Example 2. To make

$$\Pi x(Cat(x), \Pi y(Dog(y), \neg Likes(x,y) \wedge \neg Likes(y,x)))$$

and

$$\Pi y(Dog(y), \Pi x(Cat(x), \neg Likes(x,y) \wedge \neg Likes(y,x)))$$

equivalent in Min_1, or, in other words, to be able to quantify over pairs, the following semantical property is needed:

Permutation $R_{\varphi(x)}(x,\bar{z}) \wedge R_{\psi(y)}(y,x\bar{z}) \Rightarrow R_{\psi(y)}(y,\bar{z}) \wedge R_{\varphi(x)}(x,y\bar{z})$

The corresponding axiom is

Permutation $\vdash \Pi x(\varphi, \Pi y(\psi,\chi)) \to \Pi y(\psi, \Pi x(\varphi,\chi))$, x is not free in ψ.

(The completeness proof is given in Appendix 8.1, Proposition 3.)

In Min_2 this formula is derivable (see Appendix 8.2, Proposition 8).

4 Definability

In this section we shall consider the matters of definability of unary quantifiers via binary and binary quantifiers via unary. A usual question asked in case of standard generalized quantifiers is whether a binary quantifier is definable via a unary one semantically; that is, whether the interpretation of a unary quantifier in a model determines the interpretation of a given binary one. The way to disprove it is to show two models isomorphic with respect to the unary quantifier and non-isomorphic with respect to the binary one. In our case, for relational quantifiers, the corresponding result can be obtained for free and is, unfortunately, not very informative.

Proposition 1 *Not all (weak or strong) binary quantifiers are semantically definable via unary quantifiers.*

Proof Consider two models of the language $L_{\forall Q\Pi}$, $M_1 =< D_1, R_1, \{R_\varphi\}_1, V_1 >$ and $M_2 =< D_2, R_2, \{R_\varphi\}_2, V_2 >$. It is clear that if $< D_1, R_1, V_1 > \cong < D_2, R_2, V_2 >$, then for every formula θ without binary quantifiers

$$(*) \quad M_1 \models^{\alpha_1} \theta \;\Leftrightarrow\; M_2 \models^{\alpha_2} \theta,$$

where $\alpha_2(x) = h(\alpha_1(x))$, and h is the isomorhism. If the binary quantifier is definable via the unary one, then $(*)$ holds for every formula (because every formula has an equivalent one which does not contain binary quantifiers). But it is obviously wrong: consider, for example, the case when R_1 and R_2 are universal relations, in the first model $R_{\varphi(x)}$ is empty for every formula $\varphi(x)$, and in the second one $R_{\varphi(x)}(d,\bar{d})$ iff $\varphi[d,\bar{d}]$. Assume $M_{1,2} \models \exists x P(x)$. Then $M_1 \models \Pi x(P(x), \neg P(x))$ and $M_2 \not\models \Pi x(P(x), \neg P(x))$.
□

The same argument shows that neither weak, nor strong binary quantifiers in the sense of Min_1 and Min_2, respectively, are semantically definable via a unary quantifier which allows for models where R is a universal relation (for example, "for almost all").

Another question one can ask is whether a quantifier is *syntactically* definable: that is, if there is a translation from one language to another, preserving satisfiability (provability). Below are some answers to this question.

Proposition 2 *Let Π be a (weak or strong) binary quantifier. A corresponding unary quantifier can be always syntactically defined as*

$$Qx\varphi(x,\bar{y}) =_{df} \Pi x(\top(x,\bar{y}), \varphi(x,\bar{y})),$$

where $\top$ is any fixed tautology (for example, $x = x, y_1 = y_1, \ldots, y_n = y_n$).

Proof We show that any model for unary quantifier can be changed into a model for binary quantifier and vice versa, so that $Qx\varphi(x,\bar{y})$ is satisfiable if and only if $\Pi x(\top(x,\bar{y}), \varphi(x,\bar{y}))$ is. Let $R(d,\bar{d}) := R_{\top(x)}(d,\bar{d})$. Note that $R_{\top(x)} = R_{\top(z)}$ if $\top(z)$ is the result of substituting z instead of x in $\top(x)$ (z not free in $\top(x)$).

$$M \models Qx\varphi(x,\bar{d}) \Leftrightarrow \forall d(R(d,\bar{d}) \Rightarrow M \models \varphi[d,\bar{d}]) \Leftrightarrow$$

$$\Leftrightarrow \forall d(R_{\top(x)}(d,\bar{d}) \wedge \top(d,\bar{d}) \Rightarrow M \models \varphi[d,\bar{d}]) \Leftrightarrow M \models \Pi x(\top(x,\bar{d}), \varphi(x,\bar{d}))$$

□

If Π is a strong quantifier, $\top$ can be only as described above; if Π is weak, $\top$ can be any (fixed) tautology with its free variables among $\{x,\bar{y}\}$ - for example, $x = x$.

A sufficient condition to make weak binary quantifiers syntactically definable is adding to Min_1 as an axiom a formula (let us call it **D** for "definability")

$$\Pi x(\varphi,\psi) \leftrightarrow \Pi x(\top, \varphi \rightarrow \psi)$$

where $\top$ is a tautology such that $FV(\top) = FV(\varphi) \cup FV(\psi)$. Then $\Pi x(\varphi,\psi)$ can be defined as $Qx(\varphi \rightarrow \psi)$, where Q is a unary quantifier corresponding to Π ($Qx\theta := \Pi x(\top,\theta)$).

Theorem 3 *The logic with the truth definition*

$$M \models \Pi x(\varphi(x,\bar{d}), \psi(x,\bar{e})) \Leftrightarrow \forall d(R(d,\bar{d}\bar{e}) \Rightarrow (M \models \varphi[d,\bar{d}] \Rightarrow M \models \psi[d,\bar{e}]))$$

is axiomatized by the axioms for Min_1 and

D $\Pi x(\varphi,\psi) \leftrightarrow \Pi x(\top, \varphi \rightarrow \psi)$, *where $FV(\top) = FV(\varphi) \cup FV(\psi)$.*

Proof is given in Appendix 8.3.

The theorem shows that a sentence $\Pi x(\varphi,\psi)$ of $Min_1 + D$ is provable if and only if $Qx(\varphi \rightarrow \psi)$ is valid (provable) in the minimal logic for unary quantifiers.

5 Binary quantifiers and conditionals

There is a certain similarity in the behaviour of generalized quantifiers and conditionals. In [van Benthem 1986] and [Lapierre 1991] conditional propositional statements are analyzed in the generalized quantifier perspective. A conditional connective is understood as a relation between two sets (of possible worlds), that is, a generalized quantifier. Some reasonable constraints on conditional quantifiers yield conditional logics corresponding to those quantifiers.

Here another (syntactic) perspective is taken: I am going to take an existing system of conditional logic and to define an embedding of it into the language with generalized quantifiers. On the one hand, this approach allows to deal with *iterations* of conditionals as well as with single conditionals (contrary to the "horizontal" approach mentioned above). The motivation behind defining such an embedding is not, however, the ability to consider iterations per se, but rather to investigate the difference between extensional and intensional approaches to conditional statements/generics [3]. More precisely, I want to investigate, what changes if we move from the interpretation

(1) "for all elements a and for all worlds w which are normal with respect to $\varphi(a)$ (in which holds everything what is normally the case when $\varphi(a)$ is true), $\psi(a)$ holds in w", to

(2) "for all typical objects having a property φ, ψ holds". [4]

First of all we need a definition of first-order conditional logic. Let $L_{\forall>}$ denote the language of first-order conditional logic (the language of the first-order predicate logic plus binary modal operator $>$). M is *a model* for $L_{\forall>}$ if $M = < D, W, S, I >$, where D is a non-empty universe (the same for all possible worlds), W is a non-empty set of possible worlds, $S : W \times \mathcal{P}(W) \to \mathcal{P}(W)$ is a selection function, and I is an interpretation function. The truth definition for conditionals is as follows:

$$M, w \models^{\alpha} \varphi > \psi \Leftrightarrow S(w, [\varphi]_{M,\alpha}) \subseteq [\psi]_{M,\alpha},$$

where $[\varphi]_{M,\alpha} = \{w' : M, w' \models^{\alpha} \varphi\}$. The selection function satisfies the following constraint:

ID $S(w, [\varphi]_{M,\alpha}) \subseteq [\varphi]_{M,\alpha}$

Definition 3 *Cond is the minimal set of formulas derivable from the axioms and rules below:*

Pred *First-order predicate logic;*

CI $\varphi > \varphi$;

CC $(\varphi > \psi_1) \wedge (\varphi > \psi_2) \to (\varphi > \psi_1 \wedge \psi_2)$;

E $\forall x(\varphi > \psi) \to (\varphi > \forall x \psi)$, *if x is not free in φ;*

RCEA

$$\frac{\vdash \varphi \leftrightarrow \psi}{\vdash (\varphi > \chi) \leftrightarrow (\psi > \chi)}$$

RCM

$$\frac{\vdash \varphi \to \psi}{\vdash (\chi > \varphi) \to (\chi > \psi)}$$

[3] In this intention I was motivated by discussions with Michael Morreau.

[4] I assume that "normal" and "typical" have more or less the same meaning: a normal (typical) φ is an object which has all the properties one would expect from a φ-object without having any more specific information about this very object. In particular, I am not making a distinction between "normal" as "average" and "typical" as "having all the specific features of the kind in the most condensed form".

Cond is complete (see [Morreau 1992]).

I am going to show that *Cond* can be faithfully embedded into a logic with binary generalized quantifiers which is in fact a slightly extended Min_2. However, the language of this logic is two-sorted, and in addition to the axioms of Min_2 it satisfies a version of Substitution on the Left axiom.

Logic SBin. The language of *SBin* is the same as that of Min_2, but has two sorts of individual variables: SORT 1 (objects): $x_1, x_2, \ldots$ and SORT 2 (worlds): w_1, $\mathrm{w}_2, \ldots$. The definition of w.f.f. is standard: if u is a sort 1 or sort 2 variable, and φ and ψ are w.f.f.'s, so are $\forall u\varphi$ and $\Pi u(\varphi, \psi)$.

A model for *SBin* $M =< O \cup W, \{R\varphi(u)\}, V >$, where $O \cap W = \emptyset$, O and W are nonempty, O is a domain for the first sort, and W is a domain for the second sort; $R_{\varphi(u)}$ - for every w.f.f. $\varphi(u)$ - is as before, and V is a valuation. Semantic constraints on R:

AV if u_1 and u_2 are variables of the same sort, and $\varphi(u_2)$ is a result of correct substitution of u_2 instead of u_1 in $\varphi(u_1)$, then $R_{\varphi(u_1)} = R_{\varphi(u_2)}$;

S if $M \models^\alpha \forall u(\varphi(u) \leftrightarrow \psi(u))$ and $FV(\varphi) \cap SORT\ 2 = FV(\psi) \cap SORT\ 2$, then $\{d : R_{\varphi(u)}(d, \bar{w}\bar{d})\} = \{d : R_{\psi(u)}(d, \bar{w}\bar{e})\}$, where $\bar{w}\bar{d}$ is a sequence of elements assigned by α to the free variables of φ (other than u), and $\bar{w}\bar{e}$ - to $FV(\psi)$.

Definition 4 *SBin is the minimal set of formulas derivable from the axioms and rules of* Min_2 *(for both sorts) plus*

S $\forall u(\varphi \leftrightarrow \psi) \wedge \Pi u(\varphi, \chi) \rightarrow \Pi u(\psi, \chi)$, *given that* $FV(\varphi) \cap SORT\ 2 = FV(\psi) \cap SORT\ 2$, *and* u *is a variable of any sort.*

Theorem 4 *SBin is complete for the semantics described above.*

Proof The easiest way to prove the completeness theorem is to translate the language of *SBin* into the language of Min_2 with an additional predicate O (for object sort) in an obvious way: every atomic formula $P(u_1, \ldots, u_n)$ of *SBin* gets translated as a conjunction of $P(u_1, \ldots, u_n)$ and, for every u_i $(1 \leq i \leq n)$, either $O(u_i)$ or $\neg O(u_i)$, depending on whether u_i was a sort 1 or a sort 2 variable. The translation commutes with logical connectives. It is easy to check that this translation gives rise to an equivalent system. We construct a canonical model for this system in the usual way. The proof that R-conditions **S** and **AV** hold is very much the same as the analogous proofs for the systems *Bin* and *Bin'* given in Appendix 8.2.
□

Now I define a translation function * from $L_{\forall >}$ to $L_{\forall\Pi}$ (two- sorted), such that

$$Cond \vdash \varphi \Leftrightarrow SBin \vdash \varphi^*.$$

Definition 5 * *:* $L\forall > \rightarrow L_{\forall\Pi}$ *is the following function:*

- *for every* n*-place predicate symbol* P *of* $L_{\forall >}$, *let* P^{*i} *be an* $n{+}i{+}1$*-place predicate symbol of* $L_{\forall\Pi}$*;* $(P(x_1, \ldots, x_n))^{*0} = P(x_1, \ldots, x_n, \mathrm{w}_0)$*;* $(P(x_1, \ldots, x_n))^{*i} = P^{*i}(x_1, \ldots, x_n, \mathrm{w}_0, \ldots, \mathrm{w}_i)$*;*
- ^{*i} *commutes with Boolean connectives and ordinary quantifiers;*

- $(\varphi > \psi)^{*i} = \Pi \mathrm{w}_{i+1}((\varphi)^{*i+1}, (\psi)^{*i+1})$.

- $^* = ^{*0}$.

The trick of adding an extra variable (for a possible world) when translating a modal language into a classical one is quite common. Here we need to keep count of the worlds which occurred in the path to the given world; the deeper a subformula sits, the more variables it accumulates. (Basically, not to make the quantification vacuous, i.e. not to bind variables not occurring free in the formula.)
Example:

$$(P(x) > (Q(y) > P(x)))^* = \Pi \mathrm{w}_1(P(x)^{*1}, (Q(y) > P(x))^{*1}) =$$
$$= \Pi \mathrm{w}_1(P^{*1}(x, \mathrm{w}_0, \mathrm{w}_1), \Pi \mathrm{w}_2(Q^{*2}(y, \mathrm{w}_0, \mathrm{w}_1, \mathrm{w}_2), P^{*2}(x, \mathrm{w}_0, \mathrm{w}_1, \mathrm{w}_2)))$$

This example shows already a little bit how the translation works to avoid translating unprovable formulas of *Cond* by provable formulas of *SBin*. (In this case, the similar looking formula $\Pi u(\varphi(u), \Pi y(\psi(y), \varphi(u)))$ is provable.)

The following theorem guarantees that the method works in general.

Theorem 5 $SBin \vdash \varphi^{*0} \Rightarrow Cond \vdash \varphi$

Proof We shall show that there is a transformation mapping an $L_{\forall>}$-model M, a world w_0 and a variable assignment α to an $L_{\forall\Pi}$ model M^* and a variable assignment α^* such that

$$M, w_0 \models^{\alpha} \varphi \ iff \ M^* \models^{\alpha^*} (\varphi)^{*0}$$

(it will show that if something is consistent with *Cond*, then its translation is consistent with *SBin*).

Given an $L_>$ model $M =< D, W, S, I >$, α and $w_0 \in W$, construct M^* and α^* as follows:

- $O^* = D$;

- $W^* = W$;

- $I^* :< d_1, \ldots, d_n, w_0, \ldots, w_m > \in I^*(P^{*m}) \ iff \ < d_1, \ldots, d_n > \in I_{w_m}(P)$;

- α^* agrees with α on O^* and assigns w_0 to w_0;

- $R_{\varphi^{*i}(\mathrm{w}_i)}(w', w, \ldots, w_0, \bar{d}) \ iff \ w' \in S(w, [\varphi[\bar{d}]])$, where w is standing on the argument place of w_{i-1}.

For other formulas $\psi(x)$ $R_{\psi(x)}$ will be defined after the proof of the following lemma (note that the only case when the definition of R is used in that proof is for the formulas which are translations of conditional formulas and only for one of their free variables, and for this case R is uniquely defined).

Lemma 1 *Let α, α^* be as above. For all i, for any formula φ of Cond, for any world w and for any variable assignment α', which possibly differs from α^* in its assignment of values to* $\mathrm{w}_1, \ldots, \mathrm{w}_{i-1}$,

$$M, w \models^{\alpha} \varphi \Leftrightarrow M^* \models^{\alpha'} (\varphi)^{*i}[\mathrm{w}_i/w]$$

The choice of α' is due to the fact that at depth i a subformula is in the scope of i restricted universal quantifiers, which corresponds to any possible chain of possible worlds in M (starting from w_0). The proof goes by induction on the complexity of φ.

(i) $M, w \models^{\alpha} P(x_1, \ldots, x_n) \Leftrightarrow < \alpha(x_1), \ldots, \alpha(x_n) > \in I_w(P) \Leftrightarrow$

$$\Leftrightarrow < \alpha(x_1), \ldots, \alpha(x_n), w_0, \alpha'(\mathrm{w}_1), \ldots, \alpha'(\mathrm{w}_{i-1}), w > \in I^*(P^{*i}) \Leftrightarrow M^* \models^{\alpha'} P^{*i}[w_i/w]$$

(ii) - (iii) $\neg, \wedge, \forall$: easy (note that in φ and in φ^{*i} ordinary quantifiers bind only sort 1 variables);

(iv) $M, w \models^{\alpha} \chi > \psi \Leftrightarrow \forall w'(w' \in S(w, [\chi]_{M,\alpha}) \rightarrow M, w' \models^{\alpha} \psi) \Leftrightarrow$

$$(from\ ID: \ S(w, [\varphi]) \subseteq [\varphi])$$

$$\Leftrightarrow \forall w'(w' \in S(w, [\chi]_{M,\alpha}) \wedge M, w' \models^{\alpha} \chi \rightarrow M, w' \models^{\alpha} \psi) \Leftrightarrow$$

$$\Leftrightarrow \forall w'(R_{\chi^{*i+1}(\mathrm{w}_{i+1})}(w', w, \ldots, w_0, \alpha(FV(\chi))) \ \wedge \ M^* \models^{\alpha'} \chi^{*i+1}[\mathrm{w}_i/w, \mathrm{w}_{i+1}/w'] \rightarrow$$

$$\rightarrow M^* \models^{\alpha'} \psi^{*i+1}[\mathrm{w}_i/w, \mathrm{w}_{i+1}/w']) \Leftrightarrow$$

$$\Leftrightarrow M^* \models^{\alpha'} \Pi \mathrm{w}_{i+1}(\chi^{*i+1}[\mathrm{w}_i/w], \psi^{*i+1}[\mathrm{w}_i/w]) \Leftrightarrow M^* \models^{\alpha'} (\chi > \psi)^{*i}[\mathrm{w}_i/w]$$

□

Now I must show that in this model **S** holds: if $M \models \forall u(\varphi(u, \bar{w}\bar{d}) \leftrightarrow \psi(u, \bar{w}\bar{e}))$ and $FV(\varphi) \cap SORT\ 2 = FV(\psi) \cap SORT\ 2$, then $\{d : R_{\varphi(u)}(d, \bar{w}\bar{d})\} = \{d : R_{\psi(u)}(d, \bar{w}\bar{e})\}$. But first I am going to specify the relation $R_{\psi(u)}$ for the formulas ψ which are not translations of conditional formulas (or u is not a sort 2 variable). Since the lemma above is proved for all possible extensions of $R_{\psi(u)}$, it will still hold for one specific definition of $R_{\psi(u)}$. But to show that this definition is correct, I need the lemma first to prove the following step: if

$$\{w : M^* \models \varphi^{*i}[w_i/w, \bar{w}\bar{d}]\} = \{w : M^* \models \psi^{*i}[w_i/w, \bar{w}\bar{e}]\},$$

then (by the lemma above),

$$\{w : M, w \models \varphi[\bar{d}]\} = \{w : M, w \models \psi[\bar{e}]\}$$

and thus, for every w, w',

$$w \in S(w', [\varphi[\bar{d}]]) \Leftrightarrow w \in S(w', [\psi[\bar{e}]])$$

that is,

$$\{w : R_{\varphi^{*i}(\mathrm{w}_i)}(w, w', \ldots, w_0, \bar{d})\} = \{w : R_{\psi^{*i}(\mathrm{w}_i)}(w, w', \ldots, w_0, \bar{e})\}$$

It shows that the condition **S** holds for the case when φ and ψ are translations of the same depth i and $u = \mathrm{w}_i$.

Now I define $R_{\psi(u)}$ for the case $\psi \neq \theta^{*j}$ or $u \neq \mathrm{w}_j$ as follows:

- if $\psi(u) \neq \theta^{*j}(\mathrm{w}_j)$ for any formula $\theta^{*j}(\mathrm{w}_j)$, but

  $$\{w : M \models \psi[u/w]\} = \{w : M \models \theta^{*j}[\mathrm{w}_j/w]\}$$

  for some θ^{*j} which depends on the same sort 2 variables as $\psi(u)$, then $R_{\psi(u)} = R_{\theta^{*j}(\mathrm{w}_j)}$. It guarantees not only the restricted extensionality, but also the condition that if $\theta^{*j}[\mathrm{w}_j/\mathrm{w}]$ is a result of a correct substitution of w on the argument place of w_j, $R_{\theta^{*j}(\mathrm{w})} = R_{\theta^{*j}(\mathrm{w}_j)}$ **(AV)**.

  Note that, for every formula ψ, it can have the same free variables only with a translation of one fixed depth: ψ can be satisfied by the same set of elements as θ^{*i} and θ^{*j}, $i \neq j$, but it either has the same sort 2 variables as θ^{*i} does, or as θ^{*j} does. This guarantees (together with the fact that extensionality holds for translations of the same depth) that R for all formulas is well defined.

- For all other formulas ψ and variables u $R_{\psi(u)}$ is empty.

It follows that M^* is a model for *SBin*; and by lemma 1, $M^* \models^{\alpha^*} \varphi^{*0}$. □

Theorem 6 $Cond \vdash \varphi \Rightarrow SBin \vdash (\varphi)^*$.

Proof

The proof of the theorem goes by induction on the length of the derivation of φ. The case when φ is an axiom is trivial (it is immediate to see that translations of the axioms of *Cond* are axioms of *SBin*). Slightly more complicated is the inductive step (for every inference rule of *Cond*, if the translation of the premise is provable in *SBin*, then so is the translation of the conclusion).

First we need the following fact:

$$SBin \vdash \varphi^{*i} \Leftrightarrow SBin \vdash \varphi^{*i+1}$$

From the definition of ^{*i} we know that every predicate letter in φ^{*i} has $\mathrm{w}_0,\ldots,\mathrm{w}_i$ as its free sort 2 variables, that ordinary quantifiers bind only the variables of φ (not the new variables) and that the generalized quantifiers bind w_{i+1}, ..., w_k (if φ has k nested conditionals). Assume that φ^{*i} is an axiom of *SBin*. If we rename the variables bound by the generalized quantifiers so that w_{i+1} becomes w_{i+2} etc., it will remain an axiom. If we substitute for every predicate letter P^{*j} $(i \leq j \leq k)$ a predicate letter P^{*j+1}, no free variable becomes bound and the resulting formula will be the instance of the same axiom schema. It means that φ^{*i+1} is also an axiom. Backwards: substitute P^{*j-1} for every predicate letter P^{*j} $(i+1 \leq j \leq k+1)$ in φ^{*i}. The result will remain an axiom. Rename the bound variables so that j becomes $j-1$. We have obtained φ^{*i}, which is again a substitution instance of the same axiom schema.

The only inference rules we have in *SBin* are MP and Generalization: $\vdash \varphi \Rightarrow \vdash \forall x \varphi$. We assume that for the premises θ the proof of θ^{*i} can be transformed in the proof of θ^{*i+1} and backwards, and show that the application of the rule does not destroy the proof. For MP it is obvious:

$$\psi^{*i}, \psi^{*i} \rightarrow \chi^{*i}$$

imply

$$\chi^{*i}$$

as well as

$$\psi^{*i+1}, \psi^{*i+1} \rightarrow \chi^{*i+1}$$

imply

$$\chi^{*i+1}.$$

For Generalization (and for subsequent use of the deduction theorem) adding or deleting one extra free variable which never becomes bound by $\forall$ also makes no change.

Now we continue the proof of

$$Cond \vdash \varphi \Rightarrow SBin \vdash (\varphi)^*$$

RCEA Let $\vdash (\varphi)^{*i} \leftrightarrow (\psi)^{*i}$. By the above fact, $\vdash (\varphi)^{*i+1} \leftrightarrow (\psi)^{*i+1}$, and thus $\vdash \forall w_{i+1}(\varphi^{*i+1} \leftrightarrow \psi^{*i+1})$. Also, φ^{*i+1} and ψ^{*i+1} have the same sort 2 variables. Axiom **S** gives

$$\vdash \Pi w_{i+1}((\varphi)^{*i+1}, (\chi)^{*i+1}) \leftrightarrow \Pi w_{i+1}((\psi)^{*i+1}, (\chi)^{*i+1}),$$

that is,

$$\vdash (\varphi > \chi)^{*i} \leftrightarrow (\psi > \chi)^{*i}$$

RCM (analogously).

□

Backward translation

It is easy to show that a very natural translation of $\Pi x(\varphi, \psi)$ as $\forall x(\varphi > \psi)$ *does not* work. A counterexample (not involving ordinary quantifiers) is:

$$SBin \vdash \Pi x(Q(x), \Pi y(P(y), Q(x)))$$

but

$$Cond \nvdash \forall x(Q(x) > \forall y(P(y) > Q(x)))$$

(by a straightforward semantic argument). If we somehow manage to translate the above formula into propositional conditional logic, still

$$Cond \nvdash q > (p > q).$$

(This shows also that an $\forall$-free fragment of Min_2, where $\Pi x(Q(x), \Pi y(P(y), Q(x)))$ is true, can not be axiomatized by using just "conditional" axioms and rules.)

Embedding propositional conditional logic. It is evident that one does not need two-sorted language to obtain an embedding of propositional conditional logic into a logic with binary quantifiers. If $\Pi w(\varphi, \psi)$ is a translation of a conditional formula, then φ and ψ will have the same free variables; this fact can be used to show (an easy check of the embedding proof for the predicate case) that conditional propositional logic can be embedded in the minimal logic for either weak or strong quantifiers enriched with the axiom of Restricted Substitution.

5.1 Some comparisons

In [Morreau 1992] it is repeatedly stressed that the author understands generics as quantifying over normal (typical) individuals. However, representing the sentence "Normally, φ's are ψ's" by $\forall x(\varphi(x) > \psi(x))$ involves quantifying both over individuals and worlds and it is not obvious that this can be reduced to quantifying only over individuals.

First I shall compare representing generic sentences by $\forall x(\varphi > \psi)$ and representing them by $\Pi x(\varphi, \psi)$ in a one-sorted language. The first approach can be called intensional, and the second one - extensional. Then I shall show that the two-sorted language not only allows to express everything about conditionals which is expressible in conditional logic (this follows from the embedding theorem), but also the statements which are true, but not expressible in the conditional language.

Consider a one-sorted language with generalized quantifiers. Then the difference between extensional and intensional semantics becomes clear in the following fact:

$$\forall x(\chi > \varphi) \land \forall x(\varphi \to \psi) \to \forall x(\chi > \psi)$$

is not valid in *Cond*, while

$$\forall x(\varphi \to \psi) \land \Pi x(\chi, \varphi) \to \Pi x(\chi, \psi),$$

is valid in Min_2.

One more symptom of the same difference between extensional and intensional is the fact that $\forall x(\varphi \to \psi) \to \Pi x(\varphi, \psi)$ is obviously true in Min_2, whereas $\forall x(\varphi \to \psi) \to \forall x(\varphi > \psi)$ false for conditionals.

A special case of

$$\forall x(\varphi \to \psi) \to \Pi x(\varphi, \psi),$$

the statement that if there are no φ's, then anything is plausible with respect to something being a typical φ:

$$\neg\exists x\varphi \to \Pi x(\varphi, \psi)$$

caused the strongest objections towards the extensional approach in [Morreau 1992]. I shall borrow his example here: imagine a club where in principle elderly members drink on the house, $\Pi x(Elderly - member(x), Drinks - on - the - house(x))$. But at the moment there are no elderly members. So the following becomes trivially true: $\Pi x(EM(x), \neg DH(x))$. It does not look nice indeed that a "trivial" second generic seems to have the same status with the "non-trivial" first one, and that together they imply $\Pi x(EM(x), \bot(x))$. But if one is mostly interested in using generics as premises in the defeasible reasoning, that is, from $\Pi x(\varphi(x), \psi(x))$ and $\varphi(a)$ to $\psi(a)$, the problem does not seem to be that grave. The contradiction is not derivable even defeasibly because there are no elderly members! And as soon as one appears, the second sentence is not true any more. I shall discuss some other problems related to defeasible reasoning in the following section.

Let us turn to the two-sorted language.

$$\forall w(\varphi \to \psi) \to \Pi w(\varphi, \psi)$$

is now not an "extensional principle", but a true statement, not expressible in the language of conditional logic. A possible way to make this statement expressible is to add to the conditional language an absolute modality $\Box$: $M, w \models \Box\varphi \Leftrightarrow \forall w' M, w' \models \varphi$. Then this principle would become

$$\Box(\varphi \to \psi) \to (\varphi > \psi).$$

This example shows that the two-sorted language with binary quantifiers, as it was introduced in the previous section, is more expressive than the conditional language.

6 Applications to defeasible reasoning

In this section I want to give a short review of related approaches to formalizing defeasible reasoning. By defeasible reasoning I mean using commonsense generalizations (laws with exceptions) in order to arrive at conclusions referring to particular objects.

Whether a logic for generalizations provides a good tool for making defeasible inferences, depends on two things: first, on how well it captures the properties of generalizations themselves (i.e., whether one is satisfied with logically derivable properties of generalizations, such as, for example, the properties which were discussed in the previous section), and, second, on how well it works as a mechanism for defeasible inference.

In what follows, I shall comment on both aspects of the problem.

The following principles of defeasible reasoning are widely accepted as the desirable ones:

Defeasible modus ponens (the name comes from [Morreau 1992])

Normally, A's are B's
<u>*a is A*</u>
(It is plausible that) *a is B*

Penguin principle *Normally, A's are B's*
Normally, C's are not B's
All C's are A's
a is A
a is C
(It is plausible that) *a is not B*

Irrelevance *Normally, A's are B's*
a is A
a is C
(It is plausible that) *a is B* (if C is irrelevant or positive with respect to B).

Of course, if C is not irrelevant (for example, $C = \neg B$), the inference should not go through.

Those principles can be used as criteria of adequacy for the systems of defeasible reasoning.

6.1 The approach of Badaloni and Zanardo

Both the logic of relational quantifiers and conditional logic, using explicit reference to the set of typical objects or normal worlds, accept some facts about generalizations as valid while their plausibility is doubted in other approaches. Those facts are the axioms CC (conjunction of consequents) and Exchange. They can be shown to be not always intuitively appealing. (Cf. the example with "Birds live outside ..." from section 3).

In the papers [Badaloni & Zanardo 1991, 1993] a different approach was taken. Badaloni and Zanardo give the following semantics to their high plausibility quantifier Π ($\Pi x(\varphi, \psi)$ to be read as "generally, φ's are ψ's", or "φ's are highly plausible to be ψ's"). It corresponds to a relation π on the powerset of the domain (to be understood intuitively as "$\pi(X,Y)$ if a relevant part of X is in Y") with the following properties:

P1 $\pi(X, X)$;

P2 $\pi(X,Y) \Rightarrow \pi(X, Y \cap X)$;

P3 $\pi(X,Y)$ and $Y \subseteq Y' \Rightarrow \pi(X, Y')$;

P4 $\pi(X, Y_i)$ for all $i \in I \Rightarrow \cap_{i \in I} Y_i \neq \emptyset$.

So, instead of assuming that the set of sets of X-typical objects is closed under intersections, they assume only that their intersection is non-empty (P4): that *there are* some typical objects, but they don't have to be definable as an intersection of "typical" properties. However, this assumption has rather unpleasant consequences. Due to P4, not for all elements $x \in X$ $\pi(X, X \backslash \{x\})$. It is a strange property if one accepts that "for all but one" is much stronger than "for all typical".

From the fact that π is not closed under intersections follows that Π is not a relational quantifier, so that we do not have an explicit treatment of typical objects.

Badaloni and Zanardo formalize defeasible reasoning using a three-valued logic. In their system constants refer not to single objects but to sets of objects (possible denotations of the constant). $A(a)$ is neither true nor false if some of the objects referred to by a have the property A, and some do not. To express the fact that "*all* we know about a is A", one can say that an object is a possible denotation of a if and only if it has the property A:

$$\forall x (x \stackrel{\diamond}{=} a \leftrightarrow A(x))$$

The following valid inference corresponds in their system to defeasible modus ponens:

$$\frac{\Pi x(A(x), B(x)) \qquad \forall x(x \stackrel{\diamond}{=} a \leftrightarrow A(x))}{\Pi x(x \stackrel{\diamond}{=} a, B(x))}$$

The advantage of their approach as compared to circumscription, for example, is that it allows in principle to make all reasoning inside the system without referring to semantical arguments (considering all minimal models, or all consistent extensions etc.). It is well known that in general non-monotonic reasoning in the sense of the circumscription theory is highly complex: it is not first-order axiomatizable. The logic of Badaloni and Zanardo has a complete axiomatization. At the same time, the theory of circumscription or conditional logic with the notion of minimal entailment provide better treatment of the last issue in our list of patterns: they are not sensitive to irrelevant information, while the logic of Badaloni and Zanardo is. (From "It is plausible that a bird can fly" and "Tweety is a yellow bird" they cannot draw the conclusion "It is plausible that Tweety can fly". This conclusion is possible only if *all* what is known about Tweety is that it is a bird.) A solution proposed in [Badaloni & Zanardo 1991] is to add a new property of π (and a new axiom for the quantifier) which would enable one to infer "It is plausible that a yellow bird can fly". One suggestion is

P6 $\pi(X,Y) \& X \cap Y \cap Z \neq \emptyset \Rightarrow \pi(X \cap Z, Y)$

which is extremely strong. It forbids any subsets of X to be envolved in some rules with exceptions: let Z be a subset of X, and *normally* Z's are not-Y's, but one or two of them happen to be Y's. Then two contradictory (due to P4) π-statements are derivable:

$$\pi(Z, \neg Y)$$

and, by P6,

$$\pi(Z, Y)$$

The same holds not only for rules with $\neg Y$ in the consequent, but for any property, disjoint with Y (a contradiction is derivable with the use of P4). One more complication here is that in order to infer from "It is plausible that a bird can fly" and "Tweety is a yellow bird" the conclusion "It is plausible that Tweety can fly", one must first prove $\exists x(Bird(x) \wedge Yellow(x) \wedge CanFly(x))$. Another option is to accept the weaker property

P6* $\pi(X,Y) \& \neg\pi(X \cap Z, \neg Y) \Rightarrow \pi(X \cap Z, Y)$

This is intuitively more acceptable. But it seems that in order to reason with irrelevant information, one has first to make an explicit list of all irrelevant properties, respectively non-rules (like $\neg\Pi x(Bird(x) \wedge Yellow(x), \neg CanFly(x))$). Perhaps this problem can be solved better using some techniques developed in the theory of conditionals and non-monotonic reasoning.

6.2 Circumscription

The way commonsense generalizations are formalized in circumscription, is intuitively rather close to the way it is done in this paper: "for all but exceptional A's, B holds". But an individual can be exceptional in different aspects, which implies that the property of being exceptional is binary: $\forall x(A(x) \wedge \neg ab(aspect, x) \rightarrow B(x))$. Again, here from "Normally, A's are B's" and "Normally, A's are C's" does not follow "Normally, A's are B's and C's": from

$$\forall x(A(x) \wedge \neg ab(aspect_1, x) \rightarrow B(x))$$

and

$$\forall x(A(x) \wedge \neg ab(aspect_2, x) \rightarrow C(x))$$

does not follow

$$\forall x(A(x) \wedge \neg ab(aspect_3, x) \rightarrow B(x) \wedge C(x))$$

A sentence follows from a set of premises by circumscribing *ab* if it is true in all minimal models for the premises, that is, the models where the extension of *ab* is minimal. Circumscription does not have difficulties in dealing with irrelevant information and with some other patterns of the commonsense defeasible reasoning it is supposed to follow. Most criticism (rather philosophically than practically motivated) of the theory of circumscription concerns the following points (see, for example, [Morreau 1992]). The theory is not applied to reason about nested generalizations or about generalizations themselves (although in principle this is not impossible); it includes a lot of auxiliary devices (stipulating which aspects have to be circumscribed first, for example) intended to make the inference mechanism work really as it must do. Badaloni and Zanardo add to this criticism that the whole process of making defeasible inferences is semantical and involves considering *models*, while in their logic they do it purely syntactically. Indeed, the notion of minimal entailment is not in general first-order characterizable, although some important special cases are (see [Morreau 1992] for more detailed analysis).

6.3 Predicate conditional logic with the notion of minimal entailment

Predicate conditional logic was first used to formalize default reasoning in [Delgrande 1988]. A different system of conditional logic together with a notion of minimal entailment was introduced in [Morreau 1993]. The latter notion is based on the same intuition as in the theory of circumscription: as many individuals as possible (i.e., so that the premises remain true) are assumed to be normal, and if the conclusion is true, then it follows defeasibly from the premises. Here are some formal definitions:

Definition 6 *Let M be a model and $w \in W_M$, and φ is a formula with n free variables. The anomalous φ's (at w, in M), written $An_\varphi(M, w)$, are the following subset of D^n_M:*

$$An_\varphi(M, w) = \{< d_1, \ldots, d_n > \in D^n_M : \exists\psi\ M, w \models \forall x_1 \ldots \forall x_n(\varphi > \psi) \wedge \varphi[\bar{x}/\bar{d}] \wedge \neg\psi[\bar{x}/\bar{d}]\}$$

Definition 7 *Let M and N be models, $w \in W_M$ and $v \in W_N$. $M, w \leq N, v$ just in case for all φ $An_\varphi(M, w) \subseteq An_\varphi(N, v)$.*

Definition 8 *Let Γ be a set of sentences. $v \in W_N$* minimally satisfies Γ *(in N), in symbols: $N, v \models_{min} \Gamma$, just in case*

1. $N, v \models \Gamma$;

2. for any M, w, $M, w \models \Gamma$ and $M, w \leq N, v$ only if $N, v \leq M, w$.

Definition 9 *φ is a minimal consequence of Γ, written $\Gamma \models_{min} \varphi$, just in case for each M and w, if $M, w \models_{min} \Gamma$, then $M, w \models \varphi$.*

As in circumscription, the attention can be restricted to some set of properties with respect to which individuals can be anomalous. Let V be such a set; then the definition 7 will become

Definition 10 *Let M and N be models, $w \in W_M$ and $v \in W_N$. $M, w \leq^V N, v$ just in case for all $\varphi \in V$ $An_\varphi(M, w) \subseteq An_\varphi(N, v)$.*

The definitions of $M, w \models_{V\ min} \Gamma$ and $\Gamma \models_{V\ min} \varphi$ use $\leq^V$ instead of $\leq$.

The study of examples in [Morreau 1993] shows that this notion of minimal entailment captures well the natural language intuitions about defeasible reasoning.

There are, however, some problems with defining minimal entailment this way. One of those problems was pointed out by Frank Veltman. If defeasible inferences about a are made by assuming (if possible) that a is typical, then one exceptional property which a has will destroy all other possible conclusions which could have been made about this object.

To me, this seems to be a serious limitation. One way to solve this problem is to use aspects, as in circumscription; another possibility is to make the notion of minimality depend also on the number of generalizations violated: i.e., if M_1 has one abnormal element which is abnormal because of violating one generalization, and M_2 has also only one abnormal element, but it violates more nonequivalent generalizations, then M_2 is not a minimal model).

Here is one more problem:

$$Normally,\ birds\ fly$$

$$\exists x(Bird(x) \wedge \neg Flies(x))$$

$$\underline{Bird(Tweety)}$$

$$Flies(Tweety$$

This is not a valid inference either in the sense of circumscription, or in Morreau's minimal entailment, although the information about the existence of exceptions seems to be insufficient to block the general rule. There are minimal models with domains consisting only of Tweety, and there it has to be the nonflying bird. This can be repaired by making the notion of a suitable minimal model more sophisticated; however, it is somewhat disappointing that the initial notion does not work right from the very beginning.

Finally, giving syntactic characterization for defeasible inferences in this semantics is problematic.

6.4 Defaults in update semantics

In Frank Veltman's approach [Veltman 1991], default rules are formalized by means of a binary modality ("Normally...") and plausible conclusions by means of a unary modality ("Presumably..."). From "Normally, p implies q" and p *logically* follows "Presumably, q". The dynamic semantics for this logic works perfectly well with respect to all paradigmatic examples; however, it was designed so far only for propositional logic.

6.5 Relational binary quantifiers

Unlike the logic constructed by Badaloni and Zanardo, the logic of relational quantifiers does not give means to express the statements "all we know about a is A" and "it is plausible that a is B". The first problem can be solved by making it explicit which information was used to arrive at the conclusion: not just "plausible $B(a)$", but "*given that* normally A's are B's, and $A(a)$, it is plausible that $B(a)$".

The second problem is more tricky. I shall use a rather complicated way to solve it; probably, there are better ways to do it.

Consider a finite set of formulas Γ, where Γ contains generalizations (that is, expressions of the form $\Pi x(\chi, \psi)$ and $\forall x(\chi, \psi)$) and atomic sentences which contain a constant a. Let us denote the conjunction of generalizations from Γ by θ and the conjunction of atomic sentences by $\gamma(a)$.

Definition 11 *Let* Γ, $\gamma(a)$ *and* θ *be as before. Replace everywhere in* γ a *by a new variable* y. Γ makes a formula $\varphi(a)$ plausible, *if*

$$\Gamma \vdash \Pi y(\gamma(y/a), \theta \rightarrow \varphi(y/a))$$

Then *defeasible modus ponens* is derivable in both Min_1 and Min_2:

$$\vdash \Pi y(A(y), \Pi x(A(x), B(x)) \rightarrow B(y))$$

(by Iteration, see Appendix 8.1).

Assume that Γ contains one more sentence, $C(a)$. Then $B(a)$ is not any more plausible with respect to Γ:

$$\Pi y(A(y) \wedge C(y), \Pi x(A(x), B(x)) \rightarrow B(y))$$

is not derivable. In this respect I have the same properties of plausible inference as in [Badaloni & Zanardo 1991]: adding more information blocks the inference (even if it is highly irrelevant information).

The *penguin principle* requires a stronger system (Bin in the general case). We must derive from Γ

$$\Pi y(A(y) \wedge C(y), \forall x(C(x) \rightarrow A(x)) \wedge \Pi x(A(x), B(x)) \wedge \Pi x(C(x), \neg B(x)) \rightarrow \neg B(y))$$

The following is derivable by Iteration:

$$\Pi y(C(y), \Pi x(C(x), \neg B(x)) \rightarrow \neg B(y))$$

By Monotonicity,

$$\Pi y(C(y), \forall x(C(x) \rightarrow A(x)) \wedge \Pi x(A(x), B(x)) \wedge \Pi x(C(x), \neg B(x)) \rightarrow \neg B(y)).$$

From $\forall x(C(x) \rightarrow A(x))$ is derivable $\forall x(C(x) \leftrightarrow A(x) \wedge C(x))$. Then in Bin (in Bin', if A and C have the same free variables) from Γ follows

$$\Pi y(A(y) \wedge C(y), \forall x(C(x) \rightarrow A(x) \wedge \Pi x(A(x), B(x)) \wedge \Pi x(C(x), \neg B(x)) \rightarrow \neg B(y)).$$

If the fragment of Min_1 or Min_2 without ordinary quantifiers is decidable (this is still an open question), then the problem whether a formula follows from a finite set of premises (of a special form) by defeasible modus ponens is also decidable. The penguin principle requires first-order quantifiers.

There is one more example, related to the same patterns of reasoning, but with polyadic properties, which shows, to my mind, that the way generalizations about those properties are formalized in Michael Morreau's approach is not satisfactory. Consider the following set Γ:

Brothers like each other
Brothers of Jozeph don't like him
Ruben is a brother of Joseph

In conditional logic, Michael proposes to formalize it as follows:

$$\forall x \forall y (B(x,y) > Like(x,y) \wedge Like(y,x))$$

$$\forall x (B(x,j) > \neg Like(x,j))$$

$$B(r,j)$$

From the first sentence follows

$$\forall x (B(x,j) > Like(x,j))$$

It makes it impossible to derive a plausible conclusion about the attitude of Ruben to Jozeph. At the same time, this example is in certain sense a polyadic variant of the penguin principle, and the conclusion should be: it is plausible that Ruben does not like Jozeph.

This example can be formalized in the language of relational quantifiers (as well as in conditional logic) in several different ways, so that the desired result can be achieved, for example:

$$\Pi x (Person(x), \Pi y (Brother(x,y), Like(x,y) \wedge Like(y,x))$$

$$\forall x (Person(x) > \forall y (Brother(x,y) > Like(x,y) \wedge Like(y,x))$$

but to make it a really good formalization one needs a polyadic quantifier (over pairs):

$$\Pi^2 < x, y > (B(x,y), Like(x,y) \wedge Like(y,x)).$$

This is one of the directions I am going to work on in the future.

7 Conclusions

The above comparison shows that the ideal system for making defeasible inferences has not been devised yet. I would think that the most difficult problem here is an efficient treatment of irrelevant information (which consists not just in adding to the given default theory all statements about irrelevance which are compatible with it, in the spirit of [Badaloni & Zanardo 1991] or [Delgrande 1988]). That such more efficient treatment is in principle possible, suggests the result achieved in [Veltman 1991] (for the propositional case).

I believe that relational generalized quantifiers can be used in formalizing defeasible reasoning quite successfully. However, the proposal given in this paper is only a first attempt in this direction. I plan to consider the matters of decidability (of a weak system with only defeasible modus ponens in it), dealing with irrelevance, and using polyadic quantifiers in future work.

8 Appendix

8.1 Completeness for Min_1

First of all we need the following derivabilities:

Contraction $\vdash \Pi x(\varphi, \varphi \to \psi) \to \Pi x(\varphi, \psi)$

Proof

1. $\Pi x(\varphi, \varphi \to \psi)$ assumption
2. $\Pi x(\varphi, \varphi)$ Reflexivity
3. $\Pi x(\varphi, \varphi \wedge (\varphi \to \psi))$ from 1, 2, Restricted Distribution $(FV(\varphi) \subseteq FV(\varphi) \cup FV(\psi))$;
4. $\Pi x(\varphi, \psi)$ from 3, Restricted Monotonicity $(FV(\varphi) \cup FV(\varphi \wedge (\varphi \to \psi)) = FV(\varphi) \cup FV(\psi))$.
5. $\Pi x(\varphi, \varphi \to \psi) \to \Pi x(\varphi, \psi)$ 1,4 Deduction theorem for classical predicate logic.

Iteration $\vdash \Pi z(\varphi(z), \Pi x(\varphi(x), \psi(x)) \to \psi(z))$.

Proof

1. $\forall z(\psi(z) \to (\Pi x(\varphi(x), \psi(x)) \to \psi(z)))$ classical logic;
2. $\Pi z(\varphi(z), \psi(z)) \to \Pi z(\varphi(z), \Pi x(\varphi(x), \psi(x)) \to \psi(z))$ from 1, Restricted Monotonicity $(FV(\varphi) \cup FV(\psi) = FV(\varphi) \cup FV(\Pi x(\varphi, \psi) \to \psi))$;
3. $\neg \Pi z(\varphi(z), \psi(z))$ assumption
4. $\neg \Pi x(\varphi(x), \psi(x))$ 3, Alphabetic Variants;
5. $\Pi x(\varphi(x), \psi(x)) \to \psi(z)$ from 4
6. $\forall z(\Pi x(\varphi(x), \psi(x)) \to \psi(z))$ from 5
7. $\forall z((\psi \to \psi) \to (\Pi x(\varphi(x), \psi(x)) \to \psi(z)))$ from 6;
8. $\Pi z(\varphi(z), \psi \to \psi) \to \Pi z(\varphi(z), \Pi x(\varphi(x), \psi(x)) \to \psi(z))$ from 7, Restricted Monotonicity $(FV(\varphi) \cup FV(\psi \to \psi) = FV(\varphi) \cup FV(\Pi x(\varphi, \psi) \to \psi))$;
9. $\Pi z(\varphi(z), \psi \to \psi)$ Tautology;
10. $\Pi z(\varphi(z), \Pi x(\varphi(x), \psi(x)) \to \psi(z))$ from 8,9;
11. $\neg \Pi z(\varphi(z), \psi(z) \to \Pi z(\varphi(z), \Pi x(\varphi(x), \psi(x)) \to \psi(z))$ from 3,10;
12. $\Pi z(\varphi(z), \Pi x(\varphi(x), \psi(x)) \to \psi(z))$ 2,11.

Monotonicity Rule

$$\frac{\Sigma \vdash \varphi(x) \to \psi(x)}{\Sigma \vdash \Pi x(\chi, \varphi) \to \Pi x(\chi, \psi),}$$

where x is not free in Σ and $FV(\chi) \cup FV(\varphi) = FV(\chi) \cup FV(\psi)$.

Proof Immediate from the Restricted Monotonicity axiom. □

We construct a maximally consistent set (in the sense of Min_1) as usual [5], and for every formula $\neg \Pi x(\varphi, \psi)$ consistent with Σ_n add a new variable x' such that

(1) $\varphi[x/x']$

(2) $\neg\psi[x/x']$

(3) $\Pi x(\varphi, \chi) \to (\varphi[x/x'] \to \chi[x/x'])$
for all formulas χ with $FV(\chi) \cup FV(\varphi) = FV(\varphi) \cup FV(\psi)$.

The last formula is equivalent, due to (1), to $\Pi x(\varphi, \chi) \to \chi[x/x']$.

Assume that the above algorithm gives rise to inconsistencies. Then

$$\Sigma_n \vdash \bigwedge_i (\Pi x(\varphi, \chi_i) \to \chi_i(x')) \wedge \varphi(x') \to \psi(x')$$

$$\Sigma_n \vdash \bigwedge_i (\Pi x(\varphi, \chi_i) \to \chi_i(x')) \to (\varphi(x') \to \psi(x'))$$

and, by the Monotonicity Rule,

$$\Sigma_n \vdash \Pi z(\varphi(z), \bigwedge_i (\Pi x(\varphi, \chi_i) \to \chi_i(z))) \to \Pi z(\varphi(z), \varphi(z) \to \psi(z))$$

By Contraction, the consequent implies

$$\Pi z(\varphi(z), \psi(z))$$

and, by Alphabetic Variants,

$$\Pi x(\varphi(x), \psi(x)),$$

that is, if

$$\Pi z(\varphi, \bigwedge_i (\Pi x(\varphi, \chi_i) \to \chi_i(z)))$$

is derivable from Σ_n, then Σ_n is contradictory (because by assumption $\neg\Pi x(\varphi(x), \psi(x)) \in \Sigma_n$).

But

$$\Pi z(\varphi(z), \Pi x(\varphi(x), \chi_i(x)) \to \chi_i(z)))$$

is derivable by Iteration for every χ_i, and then we can apply Distribution.

The model is based on the set $\Sigma = \cup_{n<\omega} \Sigma_n$.

Now define $R_{\varphi(x)}(u, \bar{z})$ iff for all formulas χ with $FV(\chi) \cup FV(\varphi) = \bar{z}$ $\Pi x(\varphi, \chi) \in \Sigma \Rightarrow \varphi[u/x] \to \chi[u/x] \in \Sigma$.

It is easy to prove now that for any formula ϕ $\phi \in \Sigma \Leftrightarrow \Sigma \models \phi$. Let $\phi = \Pi x(\varphi, \psi)$.

1. Assume $\Pi x(\varphi, \psi) \in \Sigma$;

2. Assume $R_{\varphi(x)}(u, \bar{z})$, where $\bar{z} = FV(\varphi) \cup FV(\psi)$, and $\Sigma \models \varphi[x/u]$;

3. $\forall \chi(FV(\chi) \cup FV(\varphi) = \bar{z} \wedge \Pi x(\varphi, \chi) \in \Sigma \Rightarrow (\varphi[x/u] \to \chi[x/u]) \in \Sigma)$ from 2, definition of $R_{\varphi(x)}$;

[5] The only difference with the standard setup is that I am adding new variables to the language instead of new constants; I do it just to keep the semantic definitions simpler. But a reader can substitute the word "constants" or "parameters" everywhere instead of "free variables".

4. $\varphi[x/u] \in \Sigma$ from 2, IH;

5. $\varphi[x/u] \rightarrow \psi[x/u] \in \Sigma$ from 1,3;

6. $\psi[x/u] \in \Sigma$ from 4,5;

7. $\Sigma \models \psi[x/u]$ from 6, IH;

8. $\forall u(R_{\varphi(x)}(u,\bar{z}) \wedge \Sigma \models \varphi(u) \rightarrow \Sigma \models \psi(u)$ from 2,7;

9. $\Sigma \models \Pi x(\varphi,\psi)$ from 8.

The other direction:

1. Assume $\Pi x(\varphi,\psi) \notin \Sigma$;

2. For some x', $\varphi[x/x'] \in \Sigma$, $\psi[x/x'] \notin \Sigma$ and $R_{\varphi(x)}(x',\bar{z})$, where $\bar{z} = FV(\varphi) \cup FV(\psi)$ from 1, the construction of Σ and $R_{\varphi(x)}$;

3. For some x', $R_{\varphi(x)}(x',\bar{z})$, $\Sigma \models \varphi(x')$ and $\Sigma \not\models \psi(x')$ from 2, IH;

4. $\Sigma \not\models \Pi x(\varphi,\psi)$ from 3.

□

Let us return to the construction of the model and the definition of R. As it was mentioned already, instead of adding

$$\Pi x(\varphi,\chi) \rightarrow (\varphi[x/x'] \rightarrow \chi[x/x'])$$

for all formulas χ with $FV(\chi) \cup FV(\varphi) = FV(\varphi) \cup FV(\psi)$, we could have added just

$$\Pi x(\varphi,\chi) \rightarrow \chi[x/x']).$$

Note that $\varphi[x/x']$ is already in Σ. Then we could have defined $R_{\varphi(x)}(u,\bar{z})$ as $\varphi[x/u] \in \Sigma$ and for all θ with appropriate parameters $\Pi x(\varphi,\theta) \rightarrow \theta[x/u] \in \Sigma$. The proof will remain essentially the same, and in addition it is easy to check that the constraint of

$$R_{\varphi(x)}(u,\bar{z}) \Rightarrow M \models \varphi[x/u]$$

holds (given that the free variables of φ are among $\bar{z}$. This gives a completeness proof for Min_1 with respect to alternative semantics (with the truth definition

$$M \models \Pi x(\varphi,\psi)[\bar{d}] \Leftrightarrow \forall d(R_{\varphi(x)}(d,\bar{d}) \Rightarrow \psi[x/d])$$

and the constraint on R:

$$R_{\varphi(x)}(d,\bar{d}) \Rightarrow M \models \varphi[x/d]$$

given that the parameters of φ are among $\bar{d}$.

The construction of such alternative canonical model we shall use to prove the following fact:

Proposition 3 *Min_1 plus*

$$\Pi x(\varphi,\Pi y(\psi,\chi)) \rightarrow \Pi y(\psi,\Pi x(\varphi,\chi))$$

is complete for the class of models with the following condition in semantics:

$$R_{\varphi(x)}(x,\bar{z}) \wedge R_{\psi(y)}(y,x\bar{z}) \rightarrow R_{\psi(y)}(y,\bar{z}) \wedge R_{\varphi(x)}(x,y\bar{z}).$$

Proof Let the free variables of $\Pi x(\varphi, \Pi y(\psi, \chi))$ be $\bar{z}$. In the canonical model, assume $R_{\varphi(x)}(x, \bar{z})$ and $R_{\psi(y)}(y, x\bar{z})$. Assume $\neg R_{\psi(y)}(y, \bar{z})$. If x is free in ψ, there is no such formula θ that $FV(\psi) \cup FV(\theta) = y\bar{z}$ and, by the definition of R, $R_{\psi(y)}(y, \bar{z})$. ($\psi(y) \in \Sigma$ from the fact that $R_{\psi(y)}(y, x\bar{z})$. Assume that x is not free in ψ. Then

$$\neg(\Pi y(\psi, \theta) \to \theta) \in \Sigma$$

for some θ such that $FV(\psi) \cup FV(\theta) = y\bar{z}$. Also,

$$\neg(x = x \wedge \Pi y(\psi, \theta) \to \theta) \in \Sigma.$$

This formula has as its free variables $x, y, \bar{z}$. Assume that

$$\Pi y(\psi(y), (x = x \wedge \Pi y(\psi, \theta) \to \theta) \in \Sigma$$

Since $R_{\psi(y)}(y, x\bar{z})$, it will mean that

$$x = x \wedge \Pi y(\psi, \theta) \to \theta) \in \Sigma.$$

It is a contradiction, thus

$$\neg \Pi y(\psi(y), (x = x \wedge \Pi y(\psi, \theta) \to \theta) \in \Sigma$$

This formula has as its free variables x and $\bar{z}$. Applying the same reasoning and the fact that $R_{\varphi(x)}(x, \bar{z})$, we obtain

$$\neg \Pi x(\varphi(x), \Pi y(\psi(y), x = x \wedge \Pi y(\psi, \theta) \to \theta))$$

Permutation gives

$$\neg \Pi y(\psi(y), \Pi x(\varphi(x), x = x \wedge \Pi y(\psi, \theta) \to \theta))$$

Now it is possible to get rid of $x = x$. By Restricted Monotonicity,

$$\Pi x(\varphi(x), x = x \wedge \chi) \leftrightarrow \Pi x(\varphi(x), \chi)$$

so, applying Restricted Monotonicity once again, we obtain

$$\neg \Pi y(\psi(y), \Pi x(\varphi(x), \Pi y(\psi, \theta) \to \theta))$$

Now Permutation gives

$$\neg \Pi x(\varphi(x), \Pi y(\psi(y), \Pi y(\psi, \theta) \to \theta))$$

Since the following formula as an example of Iteration is derivable:

$$\Pi y(\psi(y), \Pi y(\psi, \theta) \to \theta),$$

$$\Pi x(\varphi(x), \Pi y(\psi(y), \Pi y(\psi, \theta) \to \theta))$$

is also derivable (from Tautology and Restricted Monotonicity). A contradiction, thus $R_{\psi(y)}(y, \bar{z})$.

Assume $\neg R_{\varphi(x)}(x, y\bar{z})$. As before, for some θ such that $FV(\varphi) \cup FV(\theta) = xy\bar{z}$,

$$\neg(\Pi x(\varphi(x), \theta) \to \theta) \in \Sigma$$

or $\varphi(x) \notin \Sigma$. But the latter contradicts $R_{\varphi(x)}(x, \bar{z})$. From $R_{\psi(y)}(y, \bar{z})$,

$$\neg \Pi y(\psi(y), \Pi x(\varphi(x), \theta) \to \theta) \in \Sigma$$

The free variables of the above formula are $\bar{z}$. From $R_{\varphi(x)}(x, \bar{z})$

$$\neg \Pi x(\varphi(x), \Pi y(\psi(y), \Pi x(\varphi(x), \theta) \to \theta)) \in \Sigma$$

By Permutation,

$$\neg \Pi y(\psi(y), \Pi x(\varphi(x), \Pi x(\varphi(x), \theta) \to \theta)) \in \Sigma$$

- a contradiction again. So, $R_{\varphi(x)}(x, y\bar{z})$.
□

8.2 Completeness for Min_2

Since Min_2 is stronger than Min_1 (the same axioms hold without restrictions; it is easy to see that the Tautology axiom of Min_1 is derivable from Reflexivity and Monotonicity in Min_2), we shall use the theorems derived in the previous section without giving their proofs in Min_2. In the sequel we shall need the following version of

Iteration(2) $\vdash \Pi z(\varphi(z), \forall y_1 \ldots \forall y_n(\Pi x(\varphi(x), \psi(x)) \to \psi(z)))$, given that $y_1, \ldots, y_n$ are not free in φ.

Proof

1. $\Pi z(\varphi(z), \Pi x(\varphi(x), \psi(x)) \to \psi(z))$ theorem;
2. $\forall y_1 \ldots \forall y_n \Pi z(\varphi(z), \Pi x(\varphi(x), \psi(x)) \to \psi(z))$ from 1;
3. $\Pi z(\varphi(z), \forall y_1 \ldots \forall y_n(\Pi x(\varphi(x), \psi(x)) \to \psi(z)))$ from 2, Exchange.

Proof Construction of the maximally consistent set: as usual, and for every formula $\neg \Pi x(\varphi, \psi)$ consistent with Σ_n add a new variable x' such that

(1) $\varphi[x/x']$:

(2) $\neg \psi[x/x']$;

(3) $\{\forall y_1 \ldots \forall y_k(\Pi x(\varphi(x), \chi(x)) \to (\varphi(x') \to \chi(x')))$:
for all formulas χ, where $y_1, \ldots, y_k$ are the free variables of χ not occurring in $\varphi\}$

The last formula is equivalent, due to (1), to $\forall y_1 \ldots \forall y_k(\Pi x(\varphi(x), \chi(x)) \to \chi(x'))$.
Assume that the above algorithm gives rise to inconsistencies. Then

$$\Sigma_n \vdash \bigwedge_i \forall y_{i1} \ldots \forall y_{ik}(\Pi x(\varphi(x), \chi_i(x)) \to \chi_i(x')) \wedge \varphi(x') \to \psi(x')$$

$$\Sigma_n \vdash \bigwedge_i \forall y_{i1} \ldots \forall y_{ik}(\Pi x(\varphi(x), \chi_i(x)) \to \chi_i(x')) \to (\varphi(x') \to \psi(x'))$$

and, by Monotonicity,

$$\Sigma_n \vdash \Pi z(\varphi(z), \bigwedge_i \forall y_{i1} \ldots \forall y_{ik}(\Pi x(\varphi(x), \chi_i(x)) \to \chi_i(z))) \to$$

$$\rightarrow \Pi z(\varphi(z), \varphi(z) \rightarrow \psi(z))$$

By Contraction, the consequent implies

$$\Pi z(\varphi(z), \psi(z))$$

and, by Alphabetic Variants,

$$\Pi x(\varphi(x), \psi(x)),$$

that is, if

$$\Pi z(\varphi(z), \bigwedge_i \forall y_{i1} \ldots \forall y_{ik}(\Pi x(\varphi(x), \chi_i(x)) \rightarrow \chi_i(z)))$$

is derivable from Σ_n, Σ_n is contradictory.

But

$$\Pi z(\varphi(z), \forall y_{i1} \ldots y_{ik}(\Pi x(\varphi(x), \chi_i(x)) \rightarrow \chi_i(z)))$$

is derivable by Iteration for every χ_i, and then we can apply Distribution.

The model is based on the set $\Sigma = \cup_{n<\omega}\Sigma_n$.

Note that if we have added just

$$\{\Pi x(\varphi, \chi) \rightarrow \chi(x/x')\},$$

we could have run into trouble, because the set of formulas χ - on every step - is increasing since the language is being expanded, and the procedure can no longer guarantee that if $\neg\Pi x(\varphi, \psi) \in \Sigma$, then for some variable x' (1), (2) and (3') hold:

(3') for all $\chi \in \Sigma$ $\Pi x(\varphi, \chi) \rightarrow \chi(x/x') \in \Sigma$

Nevertheless, the above algorithm (the idea to use (3) instead of (3') is due to [Badaloni & Zanardo 1992]) guarantees that (3') holds. Indeed, assume $\chi(x/x') \notin \Sigma$. Then $\neg\chi(x/x') \in \Sigma$ and $\exists\bar{y}(\Pi x(\varphi, \chi) \wedge \neg\chi) \in \Sigma$. But that is a contradiction.

Now define $R_{\varphi(x)}(u, \bar{z})$ iff for all formulas χ $\Pi x(\varphi(x, \bar{z}), \chi(x)) \in \Sigma \Rightarrow \varphi(u, \bar{z}) \rightarrow \chi(u) \in \Sigma$.

It is easy to prove now that for any formula ϕ $\phi \in \Sigma \Leftrightarrow \Sigma \models \phi$. Let $\phi = \Pi x(\varphi(x, \bar{d}), \psi))$.

1. Assume $\Pi x(\varphi(x, \bar{d}), \psi)) \in \Sigma$;
2. Assume $R_{\varphi(x)}(d, \bar{d})$ and $\Sigma \models \varphi(d, \bar{d})$;
3. $\forall\chi(\Pi x(\varphi(x, \bar{d}), \chi(x)) \in \Sigma \Rightarrow (\varphi(d, \bar{d}) \rightarrow \chi(d)) \in \Sigma)$ from 2, definition of $R_{\varphi(x)}$;
4. $\varphi(d, \bar{d}) \in \Sigma$ from 2, IH;
5. $\varphi(d, \bar{d}) \rightarrow \psi(d) \in \Sigma$ from 1,3;
6. $\psi(d) \in \Sigma$ from 4,5;
7. $\Sigma \models \psi(d)$ from 6, IH;
8. $\forall d(R_{\varphi(x)}(d, \bar{d}) \wedge \Sigma \models \varphi(d, \bar{d}) \rightarrow \Sigma \models \psi(d))$ from 2,7;
9. $\Sigma \models \Pi x(\varphi(x, \bar{d}), \psi(x))$ from 8.

The other direction:

1. Assume $\Pi x(\varphi(x,\bar{d}),\psi(x)) \notin \Sigma$;

2. For some d, $\varphi(d,\bar{d}) \in \Sigma$, $\psi(d) \notin \Sigma$ and $R_{\varphi(x)}(d,\bar{d})$ from 1, the construction of Σ and $R_{\varphi(x)}$;

3. For some d, $R_{\varphi(x)}(d,\bar{d})$, $\Sigma \models \varphi(d,\bar{d})$ and $\Sigma \not\models \psi(d)$ from 2, IH;

4. $\Sigma \not\models \Pi x(\varphi(x,\bar{d}),\psi(x))$ from 3.

□

The check that for the second truth definition (and defining $R_{\varphi(x)}(d,\bar{d})$ in the canonical model as $\varphi(d,\bar{d}) \in \Sigma$ and $\Pi x(\varphi(x,\bar{d}),\chi(x)) \in \Sigma \Rightarrow \chi(d) \in \Sigma$) the same procedure works, is immediate. This gives a completeness proof for Min_2 with respect to the alternative semantics: the one with the truth definition

$$M \models \Pi x(\varphi(x,\bar{d}),\psi(x)) \Leftrightarrow \forall d(R_{\varphi(x)}(d,\bar{d}) \Rightarrow M \models \varphi[d,\bar{d}])$$

and the constraint

$$R_{\varphi}(d,\bar{d}) \Rightarrow M \models \varphi[d,\bar{d}]$$

Proposition 4 *Min_2 plus the axiom of Substitution*

$$\vdash \forall x(\varphi \leftrightarrow \psi) \wedge \Pi x(\varphi,\chi) \rightarrow \Pi x(\psi,\chi)$$

(I shall call this logic Bin) is complete for the class of models with Extensionality in semantics:

$$M \models \forall x(\varphi(x,\bar{d}) \leftrightarrow \psi(x,\bar{e})) \Rightarrow (R_{\varphi(x)}(d,\bar{d}) \Leftrightarrow R_{\psi(x)}(d,\bar{e}))$$

Proof Let $\Sigma \models \forall x(\varphi(x,\bar{d}) \leftrightarrow \psi(x,\bar{e}))$. It means that $\forall x(\varphi(x,\bar{d}) \leftrightarrow \psi(x,\bar{e})) \in \Sigma$. Applying the axiom, we obtain that for every θ $\Pi x(\varphi,\theta) \rightarrow \theta$ is in Σ if and only if $\Pi x(\psi,\theta) \rightarrow \theta$ is in Σ. Also $\varphi(d,\bar{d}) \in \Sigma \Leftrightarrow \psi(d,\bar{e}) \in \Sigma$. Thus, $R_{\varphi(x)}(d,\bar{d}) \Leftrightarrow R_{\psi(x)}(d,\bar{e})$. □

Proposition 5 *Min_2 with Restricted Substitution*

$$\vdash \forall x(\varphi \leftrightarrow \psi) \wedge \Pi x(\varphi,\chi) \rightarrow \Pi x(\psi,\chi)$$

(Bin′) is complete for semantics with Restricted Extensionality:

$$M \models \forall x(\varphi(x,\bar{d}) \leftrightarrow \psi(x,\bar{d})) \Rightarrow (R_{\varphi(x)}(d,\bar{d}) \Leftrightarrow R_{\psi(x)}(d,\bar{e})),$$

given that φ and ψ have the same free variables.

Proof is similar to the proof of the Proposition 4.

Another plausible property of R is

Existence $\exists d M \models \varphi[x/d,\bar{d}] \Rightarrow \exists d R_{\varphi(x)}(d,\bar{d})$

Proposition 6 *In the logic Existence corresponds to*

$$\exists x\varphi \rightarrow \neg\Pi x(\varphi,\neg\varphi)$$

Proof The proof is obvious: let in the canonical model $\varphi(d,\bar{d})$. Then $\exists x\varphi(x,\bar{d}) \in \Sigma$. By the axiom, $\neg\Pi x(\varphi,\neg\varphi)$. By the construction of Σ, it means that for some d' $\varphi[x/d',\bar{d}] \in \Sigma$ and for all formulas $\Pi x(\varphi(x,\bar{d}),\theta) \rightarrow \theta[x/d']$ is in Σ, that is, $R_{\varphi(x)}(d',\bar{d})$. □

Proposition 7 *The logic* Min_2 *plus*

$\Pi x(\varphi, \psi) \rightarrow \forall x(\varphi \rightarrow \psi)$, *where* x *is not free in* φ,

is complete for the class of models for the strong quantifiers in which $R_{\varphi(x)} = D^{n+1}$ *if* φ *is a formula with* n *free variables and* x *is not free in* φ.

Proof In the canonical model, $R_{\varphi(x)}(d, \bar{d})$ if for all formulas θ, $\Pi x(\varphi(x, \bar{d}), \theta) \rightarrow (\varphi(d, \bar{d}) \rightarrow \theta[x/d]) \in \Sigma$. Let x be not free in φ and for some θ $\Pi x(\varphi(\bar{d}), \theta) \in \Sigma$. Then $\forall x(\varphi(\bar{d}), \theta) \in \Sigma$ and, since x is not free in φ, $\varphi(\bar{d}) \rightarrow \forall x\theta \in \Sigma$. It means that for every d, $\Pi x(\varphi(\bar{d}), \theta) \rightarrow (\varphi \rightarrow \theta[x/d]) \in \Sigma$, that is, $\forall d R_{\varphi(x)}(d, \bar{d})$, for any sequence $\bar{d}$. This means that $R_{\varphi(x)}(d, \bar{d})$ holds for all $n+1$-tuples, i.e. $R_{\varphi(x)} = D^n$. □

Proposition 8 *The formula*

$$\Pi x(\varphi, \Pi y(\psi, \chi)) \rightarrow \Pi y(\psi, \Pi x(\varphi, \chi))$$

(x is not free in ψ) is derivable in Min_2.

Proof Due to AV, we can assume that y is not free in φ.

1. $\Pi y(\psi(y), \forall x(\Pi y(\psi(y), \chi) \rightarrow \chi))$ — Iteration
2. $\forall x(\Pi y(\psi(y), \chi) \rightarrow \chi) \rightarrow (\Pi x(\varphi(x), \Pi y(\psi(y), \chi)) \rightarrow \Pi x(\varphi(x), \chi))$ — Monotonicity
3. $\Pi y(\psi(y), \Pi x(\varphi(x), \Pi y(\psi, \chi)) \rightarrow \Pi x(\varphi(x), \chi))$ — 1,2, Monotonicity
4. $\Pi x(\varphi, \Pi y(\psi, \chi))$ — assumption
5. $\forall y(\psi(y) \rightarrow \Pi x(\varphi, \Pi y(\psi, \chi)))$ — 4
6. $\Pi y(\psi, \psi) \rightarrow \Pi y(\psi, \Pi x(\varphi, \Pi y(\psi, \chi)))$ — 5, Monotonicity
7. $\Pi y(\psi, \Pi x(\varphi, \Pi y(\psi, \chi)))$ — 6, Reflexivity
8. $\Pi y(\psi, \Pi x(\varphi, \Pi y(\psi, \chi)) \wedge (\Pi x(\varphi, \Pi y(\psi, \chi)) \rightarrow \Pi x(\varphi, \chi)))$ — 3,7, Distributivity
9. $\Pi y(\psi, \Pi x(\varphi, \chi))$ — 8, Monotonicity
10. (4) $\rightarrow$ (9) (y is not free in the assumption)

□

8.3 Completeness proof for Min_1 + D

Proof We shall show that the following axioms are derivable:

Restricted Substitution for the First Argument $\vdash \forall x(\varphi(x) \leftrightarrow \psi(x)) \wedge \Pi x(\varphi, \chi) \rightarrow \Pi x(\psi, \chi)$, given that $FV(\varphi) \cup FV(\chi) = FV(\psi) \cup FV(\chi)$;

Proof

1. $\forall x(\varphi \leftrightarrow \psi)$ — assumption
2. $\forall x((\varphi \rightarrow \chi) \leftrightarrow (\psi \rightarrow \chi))$ — 1

3. $\Pi x(\top, \varphi \to \chi) \leftrightarrow \Pi x(\top, \psi \to \chi)$, given that $FV(\top) = FV(\varphi) \cup FV(\chi) = FV(\psi) \cup FV(\chi)$ — 2, Restricted Monotonicity twice

4. $\Pi x(\varphi, \chi) \leftrightarrow \Pi x(\psi, \chi)$ — 3, D

AS' $\vdash \Pi x(\varphi, \psi) \to \Pi x(\varphi \wedge \chi, \psi)$, given that $FV(\chi) \subseteq FV(\varphi) \cup FV(\psi)$

Proof

1. $\Pi x(\varphi, \psi)$ — assumption
2. $\Pi x(\top, \varphi \to \psi)$, $FV(\top) = FV(\varphi) \cup FV(\psi)$ — 1, D
3. $\forall x((\varphi \to \psi) \to (\varphi \wedge \chi \to \psi))$
4. $\Pi x(\top, \varphi \wedge \chi \to \psi)$, $FV(\top) \cup FV(\varphi) \cup FV(\psi) = FV(\top) \cup FV(\varphi) \cup FV(\psi) \cup FV(\chi)$ — 2, 3, Restricted Monotonicity,
5. $\Pi x(\varphi \wedge \chi, \psi)$ — 4, D, $FV(\top) = FV(\varphi) \cup FV(\psi) \cup FV(\chi)$

AD' $\vdash \Pi x(\varphi, \psi) \wedge \Pi x(\chi, \psi) \to \Pi x(\varphi \vee \chi \to \psi)$, given that $FV(\varphi) \subseteq FV(\chi) \cup FV(\psi)$ or $FV(\chi) \subseteq FV(\varphi) \cup FV(\psi)$.

Proof

1. $\Pi x(\varphi, \psi)$ — assumption
2. $\Pi x(\chi, \psi)$ — assumption
3. $\Pi x(\top_1, \varphi \to \psi)$, $FV(\top_1) = FV(\varphi) \cup FV(\psi)$ — 1, Definability
4. $\Pi x(\top_2, \chi \to \psi)$, $FV(\top_2) = FV(\chi) \cup FV(\psi)$ — 2, Definability
5. $FV(\top_1) \subseteq FV(\top_2)$ or $FV(\top_2) \subseteq FV(\top_1)$ — assumption
6. $\Pi x(\top_1 \wedge \top_2, \varphi \to \psi)$ — 3, 5, AS'
7. $\Pi x(\top_1 \wedge \top_2, \chi \to \psi)$ — 4, 5, AS'
8. $\Pi x(\top_1 \wedge \top_2, (\varphi \to \psi) \wedge (\chi \to \psi))$ — 6,7, Restr.Distr.
9. $\Pi x(\top_1 \wedge \top_2, \varphi \vee \chi \to \psi)$ — 8, Restr.Monotonicity
10. $\Pi x(\varphi \vee \chi, \psi)$ — 9, Definability ($FV(\top_1 \wedge \top_2) = FV(\varphi) \cup FV(\psi) \cup FV(\chi)$)

Restricted Anti-Monotonicity for the First Argument $\forall x(\varphi \to \psi) \wedge \Pi x(\psi, \chi) \to \Pi x(\varphi, \chi)$, given that $FV(\psi) \cup FV(\chi) = FV(\varphi) \cup FV(\chi)$.

Proof

1. $\forall x(\varphi \to \psi)$ — assumption
2. $\forall x(\varphi \wedge \psi \leftrightarrow \varphi)$ — from 1
3. $\Pi x(\psi, \chi)$ — assumption

4. $\Pi x(\varphi \wedge \psi, \chi)$ 3, AS' $(FV(\varphi) \subseteq FV(\psi) \cup FV(\chi))$

5. $\Pi x(\varphi, \chi)$ 2, 4, Restricted Substitution $(FV(\psi) \cup FV(\chi) = FV(\varphi) \cup FV(\chi))$.

And finally a variant of old

Iteration(3) $\vdash \Pi z(\top, \Pi x(\varphi, \psi) \to (\varphi \to \psi))$, given that $FV(\top) = FV(\varphi) \cup FV(\psi)$

Proof

1. $\Pi z(\top(z), \Pi x(\top(x), \varphi(x) \to \psi(x)) \to (\varphi(z) \to \psi(z)))$ Iteration
2. $\Pi z(\top(z), \Pi x(\varphi(x), \psi(x)) \to (\varphi(z) \to \psi(z)))$ 1,D, Restricted Monotonicity

The only difference from the completeness proof for Min_1 is that, constructing the model, for every formula $\neg\Pi x(\varphi, \psi)$ we add a new variable x' such that

(1) $\varphi[x/x']$

(2) $\neg\psi[x/x']$

(3) $\Pi x(\theta, \chi) \to (\theta[x/x'] \to \chi[x/x'])$
for all formulas θ and χ with $FV(\chi) \cup FV(\theta) = FV(\varphi) \cup FV(\psi)$.

The consistency proof for this procedure uses the new version of Iteration. It is easy to check that if it gives rise to inconsistency, then

$$\Sigma_n \vdash \Pi z(\varphi, \wedge_i \Pi x(\theta_i, \chi_i) \to (\theta_i(x') \to \chi_i(x'))) \to \Pi z(\varphi, \psi)$$

(proof goes exactly as above for Min_1). But from Iteration together with AS' and Restricted Substitution follows that

$$\vdash \Pi z(\varphi, \Pi x(\theta_i, \chi_i) \to (\theta_i \to \chi_i))$$

and by Restricted Distribution also

$$\vdash \Pi z(\varphi, \wedge_i \Pi x(\theta_i, \chi_i) \to (\theta_i \to \chi_i))$$

Thus,

$$\Sigma_n \vdash \Pi x(\varphi, \psi) :$$

a contradiction.

$R(u, \bar{z})$ in the model is defined as follows:

$$\forall \theta, \chi : FV(\theta) \cup FV(\chi) = \bar{d}, \Pi x(\theta, \chi) \in \Sigma \Rightarrow (\theta[x/u] \to \chi[x/u]) \in \Sigma$$

Proving

$$\Pi x(\varphi, \psi) \in \Sigma \Leftrightarrow \Sigma \models \Pi x(\varphi, \psi)$$

is not difficult. □

It is also possible to show that **D** is derivable from Restricted Substitution, AS' and AD' (note that Anti-Monotonicity was derived using just those axioms):

Proof

1. $\Pi x(\top, \varphi \to \psi)$ assumption
2. $\Pi x(\varphi, \varphi \to \psi)$ 1, Anti-Monotonicity

3. $\Pi x(\varphi, \psi)$ 2, Contraction

1. $\Pi x(\varphi, \psi)$ assumption

2. $\psi \rightarrow (\varphi \rightarrow \psi)$

3. $\Pi x(\varphi, \varphi \rightarrow \psi)$ 2, Restricted Monotonicity

4. $\neg\varphi \rightarrow (\varphi \rightarrow \psi)$

5. $\Pi x(\neg\varphi, \varphi \rightarrow \psi)$ 4, Tautology

6. $\Pi x(\varphi \vee \neg\varphi, \varphi \rightarrow \psi)$ 3, 5, AD' $(FV(\varphi) \subseteq FV(\varphi \vee \neg\varphi))$

7. $\Pi x(\top, \varphi \rightarrow \psi)$ 6, Restricted Substitution

It gives the completeness proof for Min_1 + Restricted Substitution + AS' + AD' with respect to the same class of models.

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