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Basic Arrow Logic with Relation Algebraic Operators

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1 Introduction

In this paper we will investigate the interconnections between two approaches in Arrow Logic. The first one is developed basically by D. Vakarelov in, for example [8]. It studies multigraphs which may be presented as structures of the following kind: $(Ar, Po, \mathbf{1}, \mathbf{2})$, where Ar is a set of arrows, Po—a set of points, and $\mathbf{1}$, $\mathbf{2}$ are two functions from the set of arrows to that of points. These two functions give for every arrow its beginning and its end, respectively. Using these functions we may define four relations between arrows in the following way: for x, $y \in Ar$, $R_{ij}xy \Leftrightarrow i(x) = j(y)$. For example two arrows are in the relation R_{11} if their beginnings are the same, and similarly for the other cases. Using these relation we can define the modal frame $(Ar, \{R_{ij} \mid i, j = 1, 2\})$. It is easy to see that such frames have the properties:

$$
\begin{array}{ll}
(\rho ii) & R_{ii}xx; \\
(\sigma ij) & R_{ij}xy \Rightarrow R_{ji}yx; \\
(\tau ijk) & R_{ij}xy \,\&\, R_{jk}yz \Rightarrow R_{ik}xz;
\end{array}
$$

From now on we will call frames satisfying these properties *arrow frames*. In section 2 we show the connections between the two kinds of structures and briefly develop the modal theory of arrow frames. Using the above relations between arrows we can define others in various ways and study the modal theories of the frames that we get in this manner. Examples of this will be given shortly.

The second approach we consider was proposed by J. van Benthem in [2], and further investigated by Y. Venema, M. Marx and also a group of Hungarian logicians (cf. [9], [5], [6]). The main topics of discussion in it are frames of the form (W, C, F, I), where $C \subseteq W^3$, $F \subseteq W^2$, and $I \subseteq W$. These relations connections between members of W are interpreted as composition of arrows, converse arrow, and identity. The object of study are logics which are intermediate between the minimal logic of these structures and the logic as rich as the theory of binary relations.

The prime concern is to develop them in such a way that will enable us to prove that these logics have some nice properties as decidability, etc.

As was mentioned it is possible to incorporate the relations from the second approach to Arrow Logic in the first one. Since the relations—composition, converse and identity—provide a specific configuration of the arrows which are in one of them, then some connections emerge between the sets of relations of both approaches. The connections that we have in mind are the following

$$
\begin{array}{llll}
(\delta) & Ix & \Leftrightarrow & R_{21}xx, \\
(\varphi) & Fxy & \Leftrightarrow & R_{21}xy \,\&\, R_{12}xy, \\
(\gamma) & Cxyz & \Leftrightarrow & R_{11}xy \,\&\, R_{21}yz \,\&\, R_{22}zx,
\end{array}
$$

Combining the coincidence relations R_{ij} of the first approach with the relation-algebraic ones of the second approach we will get a new kind of frames which satisfy the above stated conditions (δ), (φ), (γ). We call such kind of frames *relational arrow frames.* Our aim in this paper is to axiomatize the modal theory of the relational arrow frames. This will enable us to assert that the objects we consider are arrows in the usual meaning of the word i.e. object having a beginning and an end, and still we will have the machinery to speak of the connections typical of the theory of binary relations.

After providing the short survey of the modal theory of arrow frames the paper further on is organized as follows. In section 3 we develop the axiomatic system of the modal theory of relational arrow frames and show the corresponding first-order conditions of the axioms, as well as briefly dwell on the soundness of the system.

In section 4 we go in great detail through the completeness proof. Since it turns out to be very long this section is divided into several subsections emphasizing on the main stages of the proof.

Finally, in section 5 we provide some final remarks and suggestions for using our method for proving completeness in other situations.

2 Arrow frames. Basic Arrow Logic

By an *arrow structure* we mean any system $S = (Ar, Po, \mathbf{1}, \mathbf{2})$, where

- Ar is a nonempty set, whose elements are called arrows,
- Po is a nonempty set, whose elements are called points. We assume also that $Ar \cap Po = \emptyset$.
- $\mathbf{1}$ and $\mathbf{2}$ are total functions from Ar to Po associating to each arrow x the following two points: $\mathbf{1}(x)$, the first point (beginning) of x, and $\mathbf{2}(x)$, the last point (end) of x.

We also put the restriction that the arrow structures should satisfy the property: For each point A there exists an arrow x such that $A = \mathbf{1}(x)$ or $A = \mathbf{2}(x)$. As mentioned in the introduction, from arrow structures we can get modal frames by taking the set of arrows Ar and defining the relations between arrows: $R_{ij}xy \leftrightarrow \mathrm{i}(x) = \mathrm{j}(y)$. These arrow frames satisfy the properties (ρii), (σij), (τijk) from the introduction.

Now we have seen that from every arrow structure $\mathcal{A}$ we can define an arrow frame which we designate by $S(\mathcal{A})$. An arrow frame $\mathcal{M}$ for which there is an arrow structure $\mathcal{A}$ such that $\mathcal{M} = S(\mathcal{A})$ is called $\mathcal{M}$ *standard.* A main result now is the following:

Theorem 2.1 *Every arrow frame is standard.*

Proof. See [8] Theorem 1.7. $\dashv$

This theorem in effect allows us to investigate the properties of only one class of these structures. Since arrow frames are much more suitable for modal study we will now axiomatize the modal theory of arrow frames.

The language in which we are going to define our logics consists of the following elements:

- VAR – a denumerable set of proposition variables,

- $\neg, \wedge, \vee$ – classical propositional connectives,
- $\langle ij \rangle$, $i, j = 1, 2$ – four modal operations,
- $(,)$ – parentheses.

The definition of the set of all formulas Frm for Lis the usual one. We will also use the following abbreviations: $A \to B = \neg A \vee B$, $A \leftrightarrow B = (A \to B) \wedge (B \to A)$, $\langle ij \rangle A = \neg [ij] \neg A$. The general semantics for that language is the Kripke semantics under relational structures. In the next lemma we show how we can characterize the properties of arrow frames.

Lemma 2.2 *Let $\Omega = (W, R_{ij})$ be any frame. Then Ω has the property in the left side of the following table iff the modal formula in the right side is true in Ω.*

$\forall x (R_{ii}xx)$	$[ii]A \to A$
$\forall x, y (R_{ij}xy \Rightarrow R_{ji}yx)$	$A \vee [ij]\neg[ji]A$
$\forall x, y, z (R_{ij}xy \,\&\, R_{jk}yz \Rightarrow R_{ik}xz)$	$[ik]A \to [ij][jk]A$

Table 1: Modal formulas corresponding to the first-order conditions (ρ_{ii}), (σ_{ij}), (τ_{ijk})

Next we will present the axiomatic system of the Basic Arrow Logic (**BAL**), the logic that characterizes arrow frames as we will show shortly.

Axioms of BAL

(Bool) All or enough boolean tautologies
(K[ij]) $[ij](A \to B) \to ([ij]A \to [ij]B)$
(KC) $[C](A \to B) \to ([C]A \to [C]B)$
(Pii) $[ii]A \to A$
(Sij) $A \vee [ij]\neg[ji]A$
(Tijk) $[ik]A \to [ij][jk]A$

Rules of BAL

$$\text{(MP)}\ \frac{A,\ A \to B}{B}\ \text{(Modus Ponens)} \quad (N_{ij})\ \frac{A}{[ij]A}\ \text{(Necessitation for } [ij])$$

Now from the definition of arrow frame we know that every axiom of **BAL** is valid in every arrow frame, so we get at once that **BAL** is sound in the class of all arrow frames. To prove that it is also complete we will use the method of canonical models.

Let L be an arbitrary logic in the language of **BAL** which is an extension of the logic **BAL** i.e. every theorem of **BAL** is a theorem of L. By $\Omega^L = (W^L, R^L_{ij},)$ we mean the canonical frame for the logic L. Then using the correspondences in Table 1 we can infer that Ω^L satisfies the first-order conditions in the left side of Table 1. In this way we have that Ω^L is am arrow frame.

So, we may now conclude that the canonical frame for **BAL** is an arrow frame and this using standard modal techniques allows us to derive that **BAL** is complete in that class of frames.

3 Axiomatization of the logic BALR

Our goal in this section is to present the axiomatic system of the logic **BALR**—the Basic Arrow Logic with relation algebraic connectives added to it. To formulate the axiomatic system of this logic we will use an extension of the language **BAL** by adding to it the following new elements.

- Id – a propositinal constant,
- $\langle F \rangle$ – new unary modalities,
- $\{\bullet\}$ – new binary modality,

To define the semantics of our language let us consider structure $\mathcal{W} = \langle W, \{R_{ij} \mid i,j = 1,2\}, C, F, I \rangle$, where

$$\begin{array}{rcl} W \text{ is} & \text{a} & \text{set (of arrows)} \\ R_{ij} & \subseteq & W \times W, \text{ for every } i,j = 1,2 \\ C & \subseteq & W \times W \times W \\ F & \subseteq & W \times W \\ I & \subseteq & W, \end{array}$$

and a valuation of the propositional variables in W i.e. a function $V : VAR \to \mathcal{P}(W)$. Define the truth value of a formula in the model $\mathcal{M} =< \mathcal{W}, V >$ at a given point designated by $\mathcal{M}, x \models A$ as follows:

$$\begin{array}{l} \mathcal{M}, x \models p \Leftrightarrow x \in V(p) \\ \mathcal{M}, x \models \neg A \Leftrightarrow \mathcal{M}, x \not\models A \\ \mathcal{M}, x \models (A \wedge B) \Leftrightarrow \mathcal{M}, x \models A \text{ and } \mathcal{M}, x \models B \\ \mathcal{M}, x \models \langle ij \rangle A \Leftrightarrow (\exists y \in W) : R_{ij}xy \text{ and } \mathcal{M}, y \models A \\ \mathcal{M}, x \models Id \Leftrightarrow Ix \\ \mathcal{M}, x \models \langle F \rangle A \Leftrightarrow (\exists y \in W) : Fxy \text{ and } \mathcal{M}, y \models A \\ \mathcal{M}, x \models (A \bullet B) \Leftrightarrow (\exists y, z \in W) : Cxyz \text{ and } \mathcal{M}, y \models A \text{ and } \mathcal{M}, z \models B \end{array}$$

The notions of truth of a formula in a model and in a frame are defined in the usual way.

Next we consider a special class of frames, which satisfies some first-order conditions on the relations. Specifically, we call a frame $\mathcal{W} =< W, \{R_{ij} \mid i,j = 1,2\}, C, F, I >$ *standard* if the following properties are satisfied:

- $< W, \{R_{ij} \mid i,j = 1,2\} >$ is an arrow frame,
- for every $x \in W$: $Ix \Leftrightarrow R_{21}xx$,
- for every $x, y \in W$: $Fxy \Leftrightarrow R_{21}xy \;\&\; R_{12}xy$,
- for every $x, y, z \in W$: $Cxyz \Leftrightarrow R_{11}xy \;\&\; R_{22}xz \;\&\; R_{21}yz$.

We will axiomatize the theory of standard frames, or the set of all formulas valid in every standard frame. We present the axiomatic system that is needed and after that investigate briefly the first-order properties on frames that are forced by its axioms.

Axioms of BALR

General axioms

(Bool)	All or enough boolean tautologies
(K[ij])	$[ij](A \to B) \to ([ij]A \to [ij]B)$
(KF)	$[F](A \to B) \to ([F]A \to [F]B)$
(KC1)	$A{\blacksquare}(B \to C) \to (A{\blacksquare}B \to A{\blacksquare}C)$
(KC2)	$(B \to C){\blacksquare}A \to (B{\blacksquare}A \to C{\blacksquare}A)$

Axioms governing R_{ij}

(FR12) $\langle F\rangle A \rightarrow \langle 12\rangle A$

(FR21) $\langle F\rangle A \rightarrow \langle 21\rangle A$

(CR11) $A \bullet B \rightarrow \langle 11\rangle A$

(CR22) $A \bullet B \rightarrow \langle 22\rangle B$

(CR21) $\neg([21]A \bullet \neg A)$

Axioms governing I

(I1) $\langle F\rangle Id \rightarrow Id$

(I2) $\neg(Id \wedge (Id \bullet \neg Id))$

(I3) $\neg(Id \wedge (\neg Id \bullet Id))$

Axioms governing F

(F1) $Id \rightarrow (A \rightarrow \langle F\rangle A)$

(F2) $A \vee [F]\neg[F]A$

(F3) $\langle F\rangle\langle F\rangle\langle F\rangle A \rightarrow \langle F\rangle A$

(F4) $Id \rightarrow \neg([F]\neg A \bullet A)$

(F5) $[F]A \bullet Id \rightarrow [F]A$

(F6) $Id \bullet [F]A \rightarrow [F]A$

Basic Arrow Logic axioms

(Pii) $[ii]A \rightarrow A$

(Sij) $A \vee [ij]\neg[ji]A$

(Tijk) $[ik]A \rightarrow [ij][jk]A$

Axioms governing C

(C1) $A \wedge \langle 1i\rangle(Id \wedge B) \rightarrow (Id \wedge B) \bullet A$

(C2) $A \wedge \langle 2i\rangle(Id \wedge B) \rightarrow A \bullet (Id \wedge B)$

(C3) $(Id \bullet A) \bullet B \rightarrow A \bullet B$

(C4) $(A \bullet Id) \bullet B \rightarrow A \bullet B$

(C5) $A \bullet (Id \bullet B) \rightarrow A \bullet B$

(C6) $A \bullet (B \bullet Id) \rightarrow A \bullet B$

(C7) $Id \bullet (A \bullet B) \rightarrow A \bullet B$

(C8) $(A \bullet B) \bullet Id \rightarrow A \bullet B$

(C9) $\langle F\rangle A \bullet \neg(A \bullet B) \rightarrow \neg B$

(C10) $\neg(A \bullet B) \bullet \langle F\rangle B \rightarrow \neg A$

(C11) $\langle F\rangle\langle F\rangle(A \bullet B) \rightarrow A \bullet B$

(C12) $\neg(B \bullet (Id \wedge \neg(A \bullet B)) \wedge \langle F\rangle A)$

(C13) $\neg((Id \wedge \neg(A \bullet B)) \bullet B) \wedge \langle F\rangle A)$

Rules of BALR

$$(\text{MP})\ \frac{A,\ A \rightarrow B}{B}\ (\text{Modus Ponens}),\quad (\text{N[ij]})\ \frac{A}{[ij]A}\ (\text{Necessitation for [ij]}),$$

$$(\text{NC1})\ \frac{A}{A \bullet B}\quad (\text{NC2})\ \frac{B}{A \bullet B}\ (\text{Necessitation for } \bullet).$$

The axiom system just presented may seem rather formidable, so we will try next to justify it a little. Of course, the full justification will be provided by our completeness proof, but for the moment we will at least show what all the above axioms amount to in terms of properties of the frames.

Lemma 3.1 *Let $\mathcal{W} = (W, R_{ij}, Id, F, C)$ be any frame. Then $\mathcal{W}$ has the property in the left side of Table 2 iff the modal formula in the right side is valid in $\mathcal{W}$.*

$\forall x, y\,(Fxy \Rightarrow R_{12}xy)$	$\langle F\rangle A \rightarrow \langle 12\rangle A$
$\forall x, y\,(Fxy \Rightarrow R_{21}xy)$	$\langle F\rangle A \rightarrow \langle 21\rangle A$
$\forall x, y, z\,(Cxyz \Rightarrow R_{11}xy)$	$A \bullet B \rightarrow \langle 11\rangle A$
$\forall x, y, z\,(Cxyz \Rightarrow R_{22}xz)$	$A \bullet B \rightarrow \langle 22\rangle B$
$\forall x, y, z\,(Cxyz \Rightarrow R_{21}yz)$	$\neg([21]A \bullet \neg A)$

Table 2: Correspondence for the axioms governing R_{ij}.

Lemma 3.2 *Let $\mathcal{W} = (W, R_{ij}, Id, F, C)$ be any frame. Then $\mathcal{W}$ has the property in the left side of Table 3 iff the modal formula in the right side is valid in $\mathcal{W}$.*

$\forall x, y \in W(Ix \,\&\, Fxy \Rightarrow Iy)$	$\langle F\rangle Id \to Id$
$\forall x, y, z \in W(Cxyz \,\&\, Ix \,\&\, Iy \Rightarrow Iz)$	$\neg(Id \wedge (Id \bullet \neg Id))$
$\forall x, y, z \in W(Cxyz \,\&\, Ix \,\&\, Iz \Rightarrow Iy)$	$\neg(Id \wedge (\neg Id \bullet Id))$

Table 3: Correspondence for the axioms governing I.

Lemma 3.3 *Let $\mathcal{W} = (W, R_{ij}, Id, F, C)$ be any frame. Then $\mathcal{W}$ has the property in the left side of Table 4 iff the modal formula in the right side is valid in $\mathcal{W}$.*

$\forall x \in W(Ix \Rightarrow Fxx)$	$Id \to (A \to \langle F\rangle A)$
$\forall x, y(Fxy \Rightarrow Fyx)$	$A \vee [F]\neg[F]A$
$\forall x, y, z, t\,(Fxy \,\&\, Fyz \,\&\, Fzt \Rightarrow Fxt)$	$\langle F\rangle\langle F\rangle\langle F\rangle A \to \langle F\rangle A$
$\forall x, y, z\,(Ix \,\&\, Cxyz \Rightarrow Fyz)$	$Id \to \neg([F]\neg A \bullet A)$
$\forall x, y, z, u\,(Cxyz \,\&\, Fxu \,\&\, Iz \Rightarrow Fyu)$	$[F]A \bullet Id \to [F]A$
$\forall x, y, z, u\,(Cxyz \,\&\, Fxu \,\&\, Iy \Rightarrow Fzu)$	$Id \bullet [F]A \to [F]A$

Table 4: Correspondence for the axioms governing F.

Lemma 3.4 *Let $\mathcal{W} = (W, R_{ij}, Id, F, C)$ be any frame. Then $\mathcal{W}$ has the property in the left side of Table 5 iff the modal formula in the right side is valid in $\mathcal{W}$.*

$\forall x, y \in W(R_{1i}xy \,\&\, Iy \Rightarrow Cxyx)$	$A \wedge \langle 1i\rangle(Id \wedge B) \to (Id \wedge B) \bullet A$
$\forall x, y \in W(R_{2i}xy \,\&\, Iy \Rightarrow Cxxy)$	$A \wedge \langle 2i\rangle(Id \wedge B) \to A \bullet (Id \wedge B)$
$\forall x, y, z, u, v \in W(Cxyz \,\&\, Cyuv \,\&\, Iu \Rightarrow Cxvz)$	$(Id \bullet A) \bullet B \to A \bullet B$
$\forall x, y, z, u, v \in W(Cxyz \,\&\, Cyuv \,\&\, Iv \Rightarrow Cxuz)$	$(A \bullet Id) \bullet B \to A \bullet B$
$\forall x, y, z, u, v \in W(Cxyz \,\&\, Czuv \,\&\, Iu \Rightarrow Cxyv)$	$A \bullet (Id \bullet B) \to A \bullet B$
$\forall x, y, z, u, v \in W(Cxyz \,\&\, Czuv \,\&\, Iv \Rightarrow Cxyu)$	$A \bullet (B \bullet Id) \to A \bullet B$
$\forall x, y, z, u, v \in W(Cxyz \,\&\, Cuvx \,\&\, Iv \Rightarrow Cuyz)$	$Id \bullet (A \bullet B) \to A \bullet B$
$\forall x, y, z, u, v \in W(Cxyz \,\&\, Cuxv \,\&\, Iu \Rightarrow Cuyz)$	$(A \bullet B) \bullet Id \to A \bullet B$
$\forall x, y, z, u \in W(Cxyz \,\&\, Fyu \Rightarrow Czux)$	$\langle F\rangle A \bullet \neg(A \bullet B) \to \neg B$
$\forall x, y, z, u \in W(Cxyz \,\&\, Fzu \Rightarrow Cyxu)$	$\neg(A \bullet B) \bullet \langle F\rangle B \to \neg A$
$\forall x, y, z, u, v \in W(Cxyz \,\&\, Fxu \,\&\, Fuv \Rightarrow Cvyz)$	$\langle F\rangle\langle F\rangle(A \bullet B) \to A \bullet B$
$\forall x, y, z, u \in W(Cxyz \,\&\, Fxu \,\&\, Iy \Rightarrow Cyzu)$	$\neg(B \bullet (Id \wedge \neg(A \bullet B)) \wedge \langle F\rangle A)$
$\forall x, y, z, u \in W(Cxyz \,\&\, Fxu \,\&\, Iz \Rightarrow Czuy)$	$\neg((Id \wedge \neg(A \bullet B)) \bullet B) \wedge \langle F\rangle A)$

Table 5: Correspondence for the axioms governing C relation.

Now we can state the main result of the paper.

Theorem 3.5 (Soundness and Completeness of BALR) *For every formula A the following two assertions are equivalent:*

- *A is a theorem of* **BALR**
- *A is valid in every standard frame*

Proof. The completeness part is proved in the next section. As for the soundness part, it is easy to check that all the properties on the left side of the tables 1, 2, 3, 4 are true in every standard frame, so the corresponding axioms are valid in standard frames. The fact that the rules of inference preserve validity concludes the argument. ⊣

4 Completeness Proof for the logic BALR

Now we begin the proof of completeness of **BALR** with respect to standard frames. First let $\mathcal{L} =< W^L,\ R^L_{ij},\ I^L,\ F^L,\ C^L >$ be the canonical frame of the logic **BALR**. It is easy to see that this frame is not standard so we cannot use in this case the canonical construction for proving completeness. Nonetheless, we have the following lemma the proof of which follows standard lines:

Lemma 4.1 *The axioms of* **BALR** *are canonical.*

What this means is that every property in the left side of the tables 1, 2, 3 is satisfied in the canonical frame. From now on when we say that we use a property, for example (I1), we will mean the property governed by the axiom (I1) and similarly for the other cases. Also, since **BALR** is an extension of the logic **BAL** from [8], the restriction $\langle W^L,\ R^L_{ij}\rangle$ of the canonical frame of the logic **BALR** is an arrow frame. These facts will be used extensively in our construction.

Now a few words as to how are we going to proceed: the idea is to build a standard frame $\mathcal{A}$ such that the canonical frame $\mathcal{L}$ will a p-morphic image of $\mathcal{A}$. In this way the modal theory of $\mathcal{A}$ will be the same as that of the canonical frame but since $\mathcal{A}$ is standard we will have our completeness result.

Further on our exposition will consist of several subsections which try to emphasize the most important parts of the completeness proof.

4.1 The construction

In this section we proceed to build a frame which as we will show in the following subsections will be standard and also every formula which is not a theorem will be refuted in it. Strictly speaking, what we are going to build is not directly modal frame, but first we will build an arrow structure $\mathcal{S} = \langle Po,\ Ar,\ 1,\ 2\rangle$ in the sense of [8] and a mapping $\mathcal{T}$ from the set Ar to the set W^L. Having such an arrow structure we can easily build a standard frame $\mathcal{F}(\mathcal{S}) = \langle Ar,\ R_{ij},\ I,\ F,\ C\rangle$ from it in the following way:

- Ar is the set of arrows from the arrow structure $\mathcal{S}$,
- for every $x, y \in Ar : R_{ij}xy \Leftrightarrow i(x) = j(y)$,
- for every $x \in Ar : Ix \Leftrightarrow 1(x) = 2(x)$,
- for every $x, y \in Ar : Fxy \Leftrightarrow 1(x) = 2(y) \,\&\, 2(x) = 1(y)$,
- for every $x, y, z \in Ar : Cxyz \Leftrightarrow 1(x) = 1(y) \,\&\, 2(x) = 2(z) \,\&\, 2(y) = 1(z)$.

This frame $\mathcal{F}(\mathcal{S})$ will be our frame $\mathcal{A}$. And the bulk of the proof will be to show that the mapping $\mathcal{T}$ is a p-morphism from $\mathcal{A}$ to $\mathcal{L}$.

We will build the arrow structure $\mathcal{S}$ in steps. First, let $\kappa = 2^\omega$ and suppose we have two disjoint sets L^P and L^A each of cardinality at least κ^ω. We use these two sets to form the sets Po and Ar, when we need to add a point or an arrow we will take it from these sets. We will use the following conventions for naming the objects we deal with:

small latin letters from the beginning of the alphabet a, b, $c, \ldots$ for points
small latin letters from the end of the alphabet x, y, $z, \ldots$ for arrows
big latin letter from the end of the alphabet X, Y, $Z, \ldots$ for maximal theories
the letters i, j, k possibly with subscripts for the elements of the set $\{1, 2\}$

One more preliminary point, in the course of the construction we will also define a funcion H from the set Ar to the set of all words over the alphabet $\{p_x \mid x \in L^A\} \cup \{l, r, f, d\}$, where we impose the restriction that $\{p_x \mid x \in L^A\} \cap \{l, r, f, id\} = \emptyset$. This function will give us what we call ***a history*** of the creation of an arrow, i.e. since in our construction we will be adding new arrows which will extend other ones, then this function will show us exactly how an arrow appeared. This may sound too vague at the moment but we hope it will become clear in the course of the construction. We will also define a function P which for a given arrow will give us its parent if it has one, or the arrow from which it appeared in the construction. Now the construction:

Beginning of the Construction

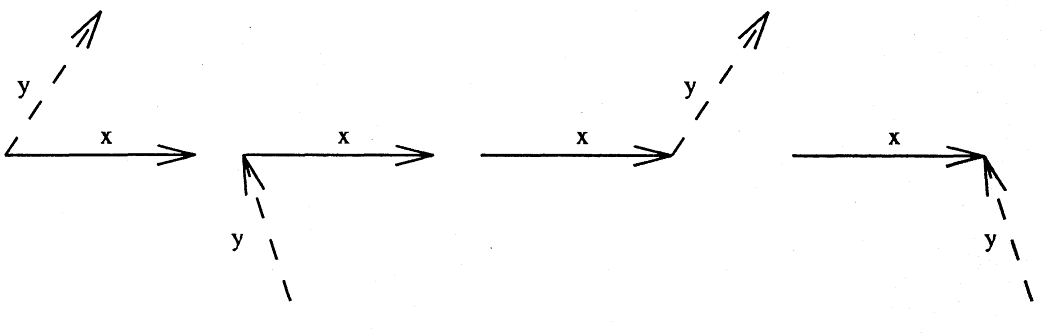
Step 0
For every maximal theory X do the following:

- if $Id \notin X$, then we remove two distinct elements a, b from L^P, and add them to Po_0; and we also remove one element x from L^A and add it to Ar_0; further we define $1(x) = a$, $2(x) = b$, $\mathcal{T}(x) = X$, and $H(x) = p_x$,
- if $Id \in X$, then we remove one element a from L^P and add it to Po_0; and we also remove one element x from L^A and add it to Ar_0; again we define $1(x) = 2(x) = a$, $\mathcal{T}(x) = X$, and $H(x) = id$.

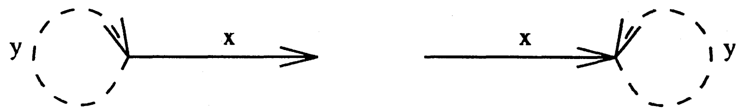
With that step 0 is finished, the sets Po_0 and Ar_0 are defined and so are the functions $\mathcal{T}$ and H on Ar_0. For the arrows in Ar_0 the function P is not defined.

Step $n + 1$
For every arrow x in $Ar_n \backslash Ar_{n-1}$ (let $Ar_{-1} = \emptyset$) do the following.
Since x in Ar_n, the functions 1, 2, $\mathcal{T}$, H are all defined for x. For every maximal theory Y such that $R^L_{ij}\mathcal{T}(x)Y$ or $F^L\mathcal{T}(x)Y$ and for every two maximal theories Y, Z such that $C^L\mathcal{T}(x)YZ$ carry out one of the cases below depending on the relation.

(ij_1) Suppose $R^L_{ij}\mathcal{T}(x)Y$ and $Id \notin Y$. Then we remove one element a from L^P and add it to Po_{n+1}; we also remove one element y from L^A one element y and add it to Ar_{n+1}; further we define $j(y) = i(x)$, $k(y) = a$, where $k \neq j$, $\mathcal{T}(y) = Y$, and $H(y) = p_y$, and $P(y) = x$. What we have done can be seen in the picture below. Since we will draw a lot of pictures from now on, we will make some conventions: an arrow drawn with a solid line is the arrow we work with, and an arrow drawn with a dashed line is one we add at the current step.



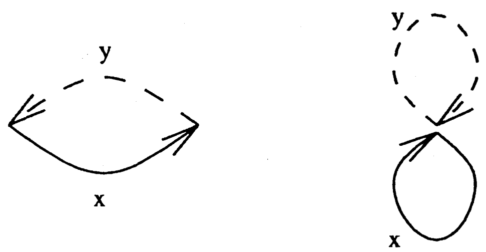
(ij_2) Suppose $R^L_{ij}\mathcal{T}(x)Y$ and $Id \in Y$. Then we remove one element y from L^A and add it to Ar_{n+1}; further we define $j(y) = i(x)$, $k(y) = j(y)$, $\mathcal{T}(y) = Y$, and $H(y) = id$, and $P(y) = x$. The picture is



($conv$) Suppose $F^L\mathcal{T}(x)Y$. Then we remove one element y from L^A and add it to Ar_{n+1}; further we define $1(y) = 2(x)$, $2(y) = 1(x)$, $\mathcal{T}(y) = Y$, and $P(y) = x$. As for $H(y)$ the definition is:

$$H(y) = \begin{cases} H(x)f & \text{if } 1(x) \neq 2(x) \\ id & \text{if } 1(x) = 2(x) \end{cases}$$

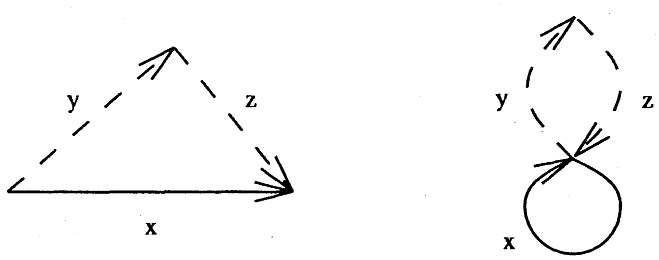
The possible pictures are



($comp_1$) Suppose $C^L\mathcal{T}(x)YZ$ and $Id \notin Y$ and $Id \notin Z$. Then we remove one element a from L^P and add it to Po_{n+1}; also we remove two elements from L^A and add them to Ar_{n+1}; further we define $1(y) = 1(x)$, $2(y) = a$, $1(z) = a$, $2(z) = 2(x)$ and $\mathcal{T}(y) = Y$, $\mathcal{T}(z) = Z$, and $P(y) = P(z) = x$. As for the function H it is defined as follows:

$$H(y) = \begin{cases} p_y & \text{if } 1(x) \neq 2(x) \\ p_z f & \text{if } 1(x) = 2(x) \end{cases}, \quad H(z) = p_z.$$

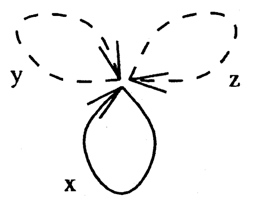
And visually we have one of the situations depending on whether $1(x) = 2(x)$ or not:

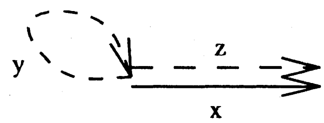


($comp_2$) Suppose $C^L\mathcal{T}(x)YZ$ and $Id \in Y$. Then we remove two elements y, z from L^A and add them to Ar_{n+1}; further we define $1(y) = 1(x)$, $2(y) = 1(x)$, $1(z) = 1(x)$, $2(z) = 2(x)$ and $\mathcal{T}(y) = Y$, $\mathcal{T}(z) = Z$, and $P(y) = P(z) = x$. As for the function H it is defined as follows:

$$h(y) = id, \quad H(z) = \begin{cases} H(x)l & \text{if } 1(x) \neq 2(x) \\ id & \text{if } 1(x) = 2(x) \end{cases}$$

And now the situations are

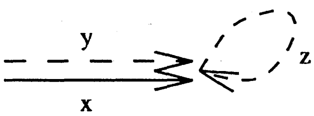


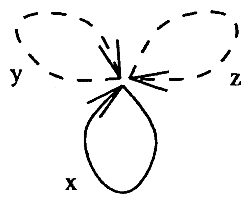


($comp_3$) And, finally, suppose $C^L\mathcal{T}(x)YZ$ and $Id \in Z$. Then we remove two elements from L^A and add them to Ar_{n+1}; further we define $1(y) = 1(x)$, $2(y) = 2(x)$, $1(z) = 1(x)$, $2(z) = 2(x)$ and $\mathcal{T}(y) = Y$, $\mathcal{T}(z) = Z$, and $P(y) = P(z) = x$. As for the function H it is defined as follows:

$$H(y) = \begin{cases} H(x)r & \text{if } 1(x) \neq 2(x) \\ id & \text{if } 1(x) = 2(x) \end{cases}, \quad l(z) = id.$$

And again using pictures we have





End of the Construction

Now we define the sets Ar and Po as follows:

$$Po = \bigcup_{n<\omega} Po_n \quad \text{and} \quad Ar = \bigcup_{n<\omega} Ar_n$$

Now probably some explanation about the construction is in order. Basically we have followed reasonable pattern by adding arrows only when they are necessary because of a similar relations in the canonical frame. We had to be careful a bit in order to preserve the exact spacial configuration between the arrows we add. What is more in the function H we have tried to keep the exact way some arrows have emerged, i.e. consider case ($comp_2$), then the function H for the arrow z is defined in a way that allows us to see that this arrow z appeared from x and also z and x are in a definite spacial configuration.

Now we will begin to show that the function $\mathcal{T}$ is a p-morphism from the relational arrow frame generated by $\langle Po, Ar, 1, 2\rangle$ to the canonical frame.

4.2 $\mathcal{T}$ is a p-morphism

The function $\mathcal{T}$ satisfies the following properties from the definition of p-morphism. For every arrow $x \in Ar$, and for every two maximal theories Y, Z, we have

- if $R^L_{ij}\mathcal{T}(x)Y$, then there is an arrow y such that $R_{ij}xy$ and $\mathcal{T}(y) = Y$,
- if $F^L\mathcal{T}(x)Y$, then there is an arrow y such that Fxy and $\mathcal{T}(y) = Y$,
- if $C^L\mathcal{T}(x)YZ$, then there are arrows y and z such that $Cxyz$ and $\mathcal{T}(y) = Y$, and $\mathcal{T}(z) = Z$.

This can be clearly seen from the way we did our construction, in a way it was done with the idea the function $\mathcal{T}$ to satisfy exactly these properties. Furthermore, the function $\mathcal{T}$ is onto i.e. for every maximal theory X, there is an arrow x, such that $\mathcal{T}(x) = X$. So, for $\mathcal{T}$ to be really a p-morphism from $\mathcal{A}$ to $\mathcal{L}$, it remains to prove that it is also a homomorphism. We will do that by showing that $\mathcal{T}$ is a homomorphism for each of the relations I, R_{ij}, F, C in a separate subsubsection.

$\mathcal{T}$ is a homomorphism for I

Lemma 4.2 *For every $x \in Ar$ the property holds:*

$$\textit{if } 1(x) = 2(x), \textit{ then } Id \in \mathcal{T}(x)$$

Proof. At step 0 i.e. for the arrows in Ar_0 the property holds by construction. Assume that it holds at step n i.e. for every arrow x in Ar_n, if $1(x) = 2(x)$, then $Id \in \mathcal{T}(x)$. We must prove that the same is true for the arrows added at step $n+1$. Let us call arrows for which the beginning and the end coincide *identity arrows.* We consider the different cases.

(ij_1) In this case no new identity arrows are created.

(ij_2) Now we create an identity arrow y, but also by definition we have $Id \in \mathcal{T}(y)$.

($conv$) Suppose that for the arrow y that is created in this case we have $1(y) = 2(y)$, using the definition we conclude that $1(P(y)) = 2(P(y))$. From the induction hypothesis we then have $Id \in \mathcal{T}(P(y))$. But then from $F^L\mathcal{T}(P(y))\mathcal{T}(y)$ using property (I1) we have $Id \in \mathcal{T}(y)$.

($comp_1$) In this case no new identity arrows are created.

($comp_2$) Now two new arrows y, z are created. We have $1(y) = 2(y)$, but also $Id \in \mathcal{T}(y)$, so with this arrow everything is in order. Suppose that $1(z) = 2(z)$, then by definition we must have $1(P(z)) = 2(P(z))$. So by the induction hypothesis $Id \in \mathcal{T}(P(z))$. Now we can use the fact that $C^L\mathcal{T}(P(z))\mathcal{T}(y)\mathcal{T}(z)$ together with (I2) to get $Id \in \mathcal{T}(z)$.

($comp_3$) Similar to the previous case using (I3). ⊣

We notice now that because of the special way we defined the function l we have that for every arrow x: $1(x) = 2(x)$ if and only if $l(x) = i$. So lemma 4.2 can be restated using this function.

$\mathcal{T}$ is a homomorphism for R_{ij}

Now we will prove that $\mathcal{T}$ is a homomorphism w.r.t. the R_{ij} relations. Further on we will make essential use of certain structural properties of our frame which we summarize in the lemma below.

Lemma 4.3 *Suppose $x \in Ar_{n+1}$, then*

(i) *For some i: $i(x) \in Po_n$.*

(ii) *If for some i: $i(x) \in Po_n$, then there is j, such that $i(x)$=$j(P(x))$,*

(iii) *If for some $y \in Ar_{n+1}$, $i(x) = j(y) \in Po_{n+1}$, then $i = 1$ and $j = 2$, or $i = 2$ and $j = 1$.*

Proof. For the cases (i) and (ii) simply use the special way we have added arrows in the construction. For (iii) notice that the situation that is wanted is possible only in case ($comp_1$). ⊣

Lemma 4.4 *For every* $x \in Ar_{n+1}$:

$$\text{if } R_{ij}xP(x), \text{ then } R^L_{ij}\mathcal{T}(x)\mathcal{T}(P(x)).$$

Proof. Again we should go through the possible cases. For (ij_1) and (ij_2) what we want is true by definition, for the other cases use properties (FR12), (FR21), (CR11), and (CR22). ⊣

Lemma 4.5 *In the canonical frame* $\mathcal{L}$ *the property holds for* $i \neq j$:

$$\text{if } I^L x, \text{ then } R^L_{ij}xx.$$

Proof. Suppose $I^L x$, then using property (F1) we get $F^L xx$. Applying the properties (FR12) and (FR21) we get the result. ⊣

Before proceeding with the next property of the frame we have built, let us notice that if $x \in Ar_n$, then $1(x) \in Po_n$ and $2(x) \in Po_n$. We will use this property extensively.

Lemma 4.6 *For every* $x, y \in Ar$, *if* $R_{ij}xy$, *then* $R^L_{ij}\mathcal{T}(x)\mathcal{T}(y)$

Proof. Again we proceed by induction on the structure S. At step 0 the only relations that are created are of the kind $R_{ii}xx$ inwhich case we can use the property (Pii), or of the kind $R_{ij}xx$, where $i \neq j$. In the latter case we have $Id \in \mathcal{T}(x)$, so we can use lemma 4.5 to conclude $R_{ij}xy$.

Now suppose that for all arrows in Ar_n the assertion is true. Assume that $x \in Ar_{n+1}$ and $R_{ij}xy$. There are different cases to be considered depending on whether y is in Ar_n or not. Suppose first that $y \in Ar_n$. Then from $i(x) = j(y) \in Po_n$ and lemma 4.3 we derive that there is an i_1 such that $i(x) = i_1(P(x))$. Now from lemma 4.4 we know that $R^L_{ii_1}\mathcal{T}(x)\mathcal{T}(P(x))$. Further we have $i_1(P(x)) = j(y)$, and both y and $P(x)$ are in Ar_n. So from the induction hypothesis we get $R^L_{i_1j}\mathcal{T}(P(x))\mathcal{T}(x)$. Now applying the fact that the restriction of the canonical frame to the relations R_{ij} is an arrow frame, we have what is desired.

Now suppose that $y \in Ar_{n+1}\backslash A_n$, then we have two other different cases depending on whether $i(x) = j(y) \in Po_{n+1}$. If this point is in fact in Po_{n+1}, then the only possibility is that x and y are obtained from $P(x)$ in the case $(comp_1)$, and we use lemma 4.3 and property (CR21). In the other case we proceed as before using lemma 4.3 to find i_1 and j_1, such that $R_{ii_1}xP(x)$ and $R_{jj_1}yP(y)$; using lemma 4.4 twice, the induction hypothesis, and the properties of arrow frames we conclude $R_{ij}xy$. ⊣

$\mathcal{T}$ is a homomorphism for F

We have now established that $\mathcal{T}$ is a homomorphism for I and for R_{ij}. Next comes F, i.e. we must prove that if Fxy, then $F^L\mathcal{T}(x)\mathcal{T}(y)$.

Using the history function H which associates with every arrow some word over the alphabet we introduced in the beginning of the construction, we will define some more relations between the arrows. For an arbitrary arrow x we we use the following definitions:

$H(x)'$ is the first letter of $H(x)$,
$H(x)^f$ is the number of ocurrences of the letter f in $H(x)$, and
$|H(x)|$ is the length of the word $H(x)$.

Suppose that x and y are not identity arrows, then we define the relations:

$$x \rightrightarrows y \text{ iff } \begin{cases} H(x)' = H(y)', \text{ and} \\ H(x)^f + H(y)^f \text{ is even} \end{cases} \quad \text{and } x \leftrightarrows y \text{ iff } \begin{cases} H(x)' = H(y)', \text{ and} \\ H(x)^f + H(y)^f \text{ is odd.} \end{cases}$$

We say that an arrow x *connects* the points a and b if $1(x) = a$, $2(x) = b$, or $1(x) = b$, $2(x) = a$. For the moment notice that if $x \in Ar_{n+1}$ and $1(x), 2(x) \in Po_n$, then the arrow $P(x)$ connects the points $1(x)$ and $2(x)$. Using this, and a similar induction as before, we can prove:

Lemma 4.7 *For every two arrows x, y: $H(x)' = H(y)'$ iff x and y connect the same two points.*

Now we are in a position to see exactly why all these words and relations we defined are needed.

Lemma 4.8 *For every two arrows x, y the following two assertions hold:*

(i) *if $x \rightrightarrows y$, then $R_{11}xy$ and $R_{22}xy$,*

(ii) *if $x \leftrightarrows y$, then $R_{12}xy$ and $R_{21}xy$.*

Proof. We use induction on $|H(x)| + |H(y)|$ for both assertions. Since for every arrow x we have that $|H(x)| \geq 1$, our basic case should be $|H(x)| + |H(y)| = 2$. It follows that $|H(x)| = 1$ and $|H(y)| = 1$. Then $H(x) = p_x$ and $H(y) = p_y$, As $x \rightrightarrows y$ we must have $p_x = p_y$, and hence $x = y$, which proved (i).The case (ii) is trivial in such a situation.

Now suppose $|H(x)| + |H(y)| = k > 2$, and that for $|H(x)| + |H(y)| < k$ both assertions are true. Then, one of $|H(x)|$ or $|H(y)|$ is greater than 2. Let that be $|H(x)|$, so $|H(x)| = \alpha q$, where α is not the empty word and $q \in \{l, r, f\}$. For (i) suppose that $x \rightrightarrows y$. We have to consider different cases:

Case 1 $q = l$. Then the arrow x appeared in the construction from $P(x)$ in case $(comp_2)$. Let $z = P(x)$, then we have $H(z) = \alpha$, and since α is not empty, then $\alpha' = H(y)'$ and $\alpha^f + H(z)^f$ is even, so $z \rightrightarrows y$. Notice also that $|\alpha| + |H(y)| < k$, so we can apply the induction hypothesis for (i) which gives $R_{11}zy$ and $R_{22}zy$. But by definition we have $R_{11}xz$ and $R_{22}xz$ from which we can conclude that $R_{11}xy$ and $R_{22}xy$.

Case 2 $q = r$. Similar to the above case.

Case 3 $q = f$. Then the arrow x appeared in the construction from $P(x)$ either in case $(conv)$ or in case $(comp_1)$. What is important is that in both cases we have an arrow z such that $H(z) = \alpha$. But now we have that $\alpha' = H(y)'$ and $\alpha^f + H(z)^f$ is odd, so $z \leftrightarrows y$. Furthermore $|\alpha| + |H(y)| < k$, so we can apply the induction hypothesis for (ii), which gives $R_{12}zy$ and $R_{21}zy$. But by definition we have $R_{12}xz$ and $R_{21}xz$ from which we can conclude that $R_{11}xy$ and $R_{22}xy$.

In case when $x \leftrightarrows y$, we reason along the same lines as above. $\dashv$

What we are trying to do is to characterize the relation F in some other way which will be more appropriate for proving properties of this relation using the technique of the above proof. Now we have defined a relation, namely $\leftrightarrows$, which is similar to F but it concerns only non-identity arrows and we have to account for identity arrows as well. Consider the following relation between two arrows x and y:

$$x \infty y \text{ iff } \begin{cases} H(x) = H(y) = id, \text{ and} \\ \text{for some } i, j \colon i(x) = j(y) \end{cases}$$

In addition we define for arbitrary arrows x and y

$$Exy \text{ iff } 1(x) = 1(y) \text{ and } 2(x) = 2(y).$$

Now we are ready for the characterizations of F and E.

Lemma 4.9 *For every two arrows x and y the following assertions hold:*

(i) $Exy \Leftrightarrow x \rightrightarrows y$ *or* $x \infty y$,

(ii) $Fxy \Leftrightarrow x \leftrightarrows y$ *or* $x \infty y$.

Proof.

($\leftarrow$) We will only prove the first item. The proof of item (ii) is similar. Suppose first that $x \infty y$, then x and y are identity arrows and for some i, j : $R_{ij}xy$. From the first we have that $1(x) = 2(x)$ and $1(y) = 2(y)$. We should consider the four different cases, depending on the values of i, j. Since they are similar, we will consider just one of them. Suppose $i = 1$ and $j = 1$. Then we have $1(x) = 1(y)$, and must prove that $2(x) = 2(y)$. But that easily follows from the above equalities. Next suppose that $x \rightrightarrows y$. But then we need only apply lemma 4.8.

($\rightarrow$) We consider two major cases, the one when at least one of the arrows is an identity-arrow and the one when neither of them is. Suppose that the arrow x is an identity-arrow, and also that Fxy. Then y is an identity-arrow, and we have for some i, j: $R_{ij}xy$. So by definition we have $x \infty y$, and we are finished. Everything is the same if we have Exy. Now suppose that neither is an identity arrow. Since x and y connect the same two points, we know from lemma 4.7 that $H(x)' = H(y)'$. For both (i) and (ii) we have to prove the other item from the definition of $\rightrightarrows$ and $\leftrightarrows$, respectively. We do that by induction on $|H(x)| + |H(y)|$ for both items simultaneously. If $|H(x)| + |H(y)| = 2$, then we have, as before, $x = y$. Then since $H(x)^f + H(y)^f = 0$, we have $x \rightrightarrows y$, so item (i) is true. In this case item (ii) is trivial. Next assume that $|H(x)| + |H(y)| = k > 2$, and for $|H(x)| + |H(y)| < k$ both assertions are true. Then one of $|H(x)|$ or $|H(y)|$ is greater than 2. Let that be $|H(x)|$, so $|H(x)| = \alpha q$, where α is not the empty word and $q \in \{l, r, f\}$. For (i) suppose that Exy. We have to consider different cases:

Case 1 $q = l$. Then the arrow x appeared in the construction from $P(x)$ in case ($comp_2$). Let $z = P(x)$, then we have $H(z) = \alpha$, and since we have $1(x) = 1(z)$ and $2(x) = 2(z)$, we can conclude $1(y) = 1(z)$ and $2(y) = 2(z)$. Notice also that $|\alpha| + |H(y)| < k$, so we can apply the induction hypothesis for (i). Then we have that $z \rightrightarrows y$ or $z \infty y$, but since y is not an identity arrow, then $z \infty y$ is not true, so we have $z \rightrightarrows y$. By definition that means that $\alpha^f + H(y)^f$ is even, but then since $H(x) = \alpha l$, $H(x)^f + H(y)^f$ is even so $x \rightrightarrows y$.

Case 2 $q = r$. Similar to the above case.

Case 3 $q = f$. Then the arrow x appeared in the construction from $P(x)$ either in case ($conv$) or in case ($comp_1$). What is important is that in both cases we have an arrow z such that $H(z) = \alpha$ and $H(x) = \alpha f$. Also, in both cases we have $1(x) = 2(z)$ and $2(x) = 1(z)$. So from Exy, we derive that $1(y) = 2(z)$ and $2(y) = 1(z)$. Since $H(z) = \alpha$, and $|\alpha| + |H(y)| < k$, we can apply the induction hypothesis to get, like before, $z \leftrightarrows y$. So $H(z)^f + H(y)^f$ is odd, and $H(x)^f + H(y)^f$ is even, hence $x \rightrightarrows y$.

If we have Fxy, then we can consider exactly the same cases and the reasoning is similar. $\dashv$

Lemma 4.10 *In the canonical frame $\mathcal{L}$ the property holds for every two elements X, Y of W:*

$$\text{if } I^L X,\ I^L Y, \text{ and for some } i, j \text{ we have } R^L_{ij} XY, \text{ then } F^L XY.$$

Proof. We consider two cases. Suppose first that $i = 1$. Then using property (C1), we have $C^L XYX$. Further, applying property (F4) gives us $F^L YX$, from which we conclude $F^L XY$ by the symmetry of F. The case where $i = 2$ is similar but now we must first use property (C2). $\dashv$

Lemma 4.11 *For every two arrows x, y the following holds:*

$$\textit{if } Ix,\ Iy, \textit{ and for some } i,j \textit{ we have } R_{ij}xy, \textit{ then } F^L \mathcal{T}(x)\mathcal{T}(y)$$

Proof. From lemmas 4.5 and 4.6 we have $I^L\mathcal{T}(x)$, $I^L\mathcal{T}(x)$, and $R^L_{ij}\mathcal{T}(x)\mathcal{T}(y)$. It remains to use lemma 4.10. $\dashv$

Lemma 4.12 *For every two non-identity arrows x and y, such that $H(x) = H(y)f$, holds that $F^L\mathcal{T}(x)\mathcal{T}(y)$.*

Proof. From the construction we know that the arrow x appeared either when extending the arrow y in case (*conv*), and the assertion is true by definition, or when extending some identity arrow in case ($comp_1$) and we can apply property (F4) to conclude the result. $\dashv$

Lemma 4.13 *For every three arrows x, y, u the following assertions hold:*

(i) *if Exy and $F^L\mathcal{T}(y)\mathcal{T}(u)$, then $F^L\mathcal{T}(x)\mathcal{T}(u)$,*

(ii) *if Exy and $F^L\mathcal{T}(u)\mathcal{T}(y)$, then $F^L\mathcal{T}(u)\mathcal{T}(x)$,*

(iii) *if Fxy, then $F^L\mathcal{T}(x)\mathcal{T}(y)$.*

Remark. Before beginning the proof let us just notice that if we prove this lemma, then we will have proved that $\mathcal{T}$ is in fact a homomorphism with respect to the relation F.

Proof. Notice that item (ii) can be derived from (i) using the symmetry of F^L, so we will be concerned from now on only with items (i) and (iii).We will use our characterizations of the relations E and F from lemma 4.9. Let us first suppose that x and y are identity arrows and $x \infty y$. Then by definition we have Ix, Iy, and for some i, j: $R_{ij}xy$, so by lemmas 4.5 and 4.6 we have $I^L\mathcal{T}(x)$, $I^L\mathcal{T}(x)$, and $R^L_{ij}\mathcal{T}(x)\mathcal{T}(y)$.

(i) From $F^L\mathcal{T}(y)\mathcal{T}(u)$ and $I^L\mathcal{T}(y)$ we have $I^L\mathcal{T}(u)$. Also it is easily shown that for some i_1, j_1 : $R^L_{i_1j_1}\mathcal{T}(x)\mathcal{T}(u)$. Now apply lemma 4.10.

(ii) Similar to item (i).

(iii) Direct from lemma 4.10 and the fact that $\mathcal{T}$ is a homomorphism for I.

Now suppose that x and y are not identity arrows. In this case we will use as before induction on $|H(x)| + |H(y)|$ for the three items. For the induction basis suppose $|H(x)| + |H(y)| = 2$, then we have $x = y$, and the first two cases are tautologies, the third one is trivial. Next suppose $|H(x)| + |H(y)| = k > 2$, and assume that for $|H(x)| + |H(y)| < k$ both assertions are true. Then one of $|H(x)|$ or $|H(y)|$ is greater than 2. Let that be $|H(x)|$, so $|H(x)| = \alpha q$, where α is not the empty word and $q \in \{l, r, f\}$.

(i) We must consider the different cases depending on which the letter q is.

Case 1. $q = l$. Then the arrow x appeared in the construction from $P(x)$ in case ($comp_2$). Let $z = P(x)$. We have $H(z) = \alpha$, and also $|\alpha|+|H(y)| < k$, and $z \rightrightarrows y$. Then applying the induction hypothesis will give us $F^L\mathcal{T}(z)\mathcal{T}(u)$. From the construction we have that there is an identity-arrow v, that appeared together with x, and $C^L\mathcal{T}(z)\mathcal{T}(v)\mathcal{T}(x)$. Then we can use property (F5) to conclude $F^L\mathcal{T}(x)\mathcal{T}(u)$.

Case 2. $q = r$. Similar to the above case, using (F6) instead of (F5).

Case 3. $q = f$. Now we have that $\alpha^f + H(y)^f$ is odd, so Fzy. For z and y we can apply the induction hypothesis for (iii) to conclude $F^L\mathcal{T}(z)\mathcal{T}(y)$. From lemma 4.12, we have $F^L\mathcal{T}(x)\mathcal{T}(z)$. Combined with $F^L\mathcal{T}(y)\mathcal{T}(u)$ and property (F3) this gives the desired result.

(iii) Again we must consider some cases for the letter q. They are similar to the ones above.

Case 1. $q = l$. Then the arrow x appeared in the construction from $P(x)$ in case ($comp_2$). Let $z = P(x)$. Applying the induction hypothesis for (i) for z and y will give us $F^L\mathcal{T}(z)\mathcal{T}(y)$. From the construction we have that there is an identity-arrow v, that appeared together with x, and $C^L\mathcal{T}(z)\mathcal{T}(v)\mathcal{T}(x)$. Then we can use property (F6) to conclude $F^L\mathcal{T}(x)\mathcal{T}(y)$.

Case 2. $q = r$. Similar to the above case, using property (F5) instead of (F6).

Case 3. $q = f$. From lemma 4.12, $F^L\mathcal{T}(x)\mathcal{T}(z)$. Now, $\alpha^f + l(y)^f$ is even, so Ezy. For z and y we can apply the induction hypothesis for (ii) to conclude $F^L\mathcal{T}(x)\mathcal{T}(y)$. ⊣

$\mathcal{T}$ is a homomorphism for C

Finally, we show that $\mathcal{T}$ is a homomorphism for C i.e.

$$\text{if } Cxyz, \text{ then } C^L\mathcal{T}(x)\mathcal{T}(y)\mathcal{T}(z).$$

This takes quite a few steps. We will first prove the result assuming that one of the arrows in the composition is an identity arrow, leaving the other case until later.

Let us first prove some preliminary properties using the characterization of E.

Lemma 4.14 *For every four arrows x, y, u, and v the following properties hold:*

(i) *if Exy and $C^L\mathcal{T}(u)\mathcal{T}(y)\mathcal{T}(v)$, then $C^L\mathcal{T}(u)\mathcal{T}(x)\mathcal{T}(v)$,*

(ii) *if Exy and $C^L\mathcal{T}(u)\mathcal{T}(v)\mathcal{T}(y)$, then $C^L\mathcal{T}(u)\mathcal{T}(v)\mathcal{T}(x)$,*

(iii) *if Exy and $C^L\mathcal{T}(y)\mathcal{T}(u)\mathcal{T}(v)$, then $C^L\mathcal{T}(x)\mathcal{T}(u)\mathcal{T}(v)$.*

Proof. By lemma 4.9 we will have to consider two cases. Suppose that we have $x \infty y$. For (i) suppose also that $C^L\mathcal{T}(u)\mathcal{T}(y)\mathcal{T}(v)$. Then from property (CR11) we get $R^L_{11}\mathcal{T}(u)\mathcal{T}(y)$. Also $R^L_{11}\mathcal{T}(x)\mathcal{T}(y)$ from lemmas 4.9 and 4.6. So by properties (Tijk) and (Sij) $R^L_{11}\mathcal{T}(u)\mathcal{T}(x)$. Since we have $I^L x$, using property (C1) we conclude $C^L\mathcal{T}(u)\mathcal{T}(x)\mathcal{T}(u)$. Now we can apply property (C5) to conclude $C^L\mathcal{T}(u)\mathcal{T}(x)\mathcal{T}(v)$. For item (ii) we reason similarly but in the end we use property (C4). Now for item (iii), suppose $C^L\mathcal{T}(y)\mathcal{T}(u)\mathcal{T}(v)$. From that, using $I^L\mathcal{T}(y)$ and property (F4), we get $F^L\mathcal{T}(u)\mathcal{T}(v)$. As before we can make sure that $C^L\mathcal{T}(u)\mathcal{T}(x)\mathcal{T}(u)$ from which by property (C9) we conclude $C^L\mathcal{T}(x)\mathcal{T}(u)\mathcal{T}(v)$.

Now we will treat the case when $x \rightrightarrows y$. We will use induction on $|H(x)|+|H(y)|$ for the three items. The basis of the induction when $|H(x)| + |H(y)| = 2$ is trivial since in that case $x = y$.

Now suppose $|H(x)| + |H(y)| = k > 2$ and assume that for $|H(x)| + |H(y)| < k$ the three items are true. Then one of $|H(x)|$ or $|H(y)|$ is greater than 2. Let that be $|H(x)|$, so $|H(x)| = \alpha q$, where α is not the empty word and $q \in \{l, r, f\}$. Since α is not the empty word, we have that there is an arrow z such that $H(z) = \alpha$. For (i) suppose also that $C^L \mathcal{T}(u)\mathcal{T}(y)\mathcal{T}(v)$. Again we must consider that cases for the letter q.

Case 1. $q = l$. Then the arrow x appeared in the construction from $P(x)$ in case ($comp_2$). The there must be an arrow w such that $I^L \mathcal{T}(w)$ and $C^L \mathcal{T}(z)\mathcal{T}(w)\mathcal{T}(x)$. We have that Ezy, and since also $|\alpha| + |H(y)| < k$, applying the induction hypothesis gives $C^L \mathcal{T}(u)\mathcal{T}(z)\mathcal{T}(v)$. Now applying property (C3) we conclude that $C^L \mathcal{T}(u)\mathcal{T}(x)\mathcal{T}(v)$.

Case 2. $q = r$. Similar to the above case, but in the end use property (C4).

Case 3. $q = f$. Now we have $\alpha^f + H(y)^f$ is odd, so Fzy. From lemma 4.12, $F^L \mathcal{T}(x)\mathcal{T}(z)$ and also from lemma 4.13 (iii), $F^L \mathcal{T}(x)\mathcal{T}(z)$. Using property (C9) twice gives the desired result.

For (ii) the reasoning is similar, but in the different cases we use properties (C5), (C6), and twice (C10), respectively. For (iii) again the same by using properties (C7), (C8), and (C11). $\dashv$

Now we can start proving that $\mathcal{T}$ is homomorphism with respect to C.

Lemma 4.15 *Suppose x, y, and z are three arrows, two of which are identity arrows, and suppose also $Cxyz$, then $C^L \mathcal{T}(x)\mathcal{T}(y)\mathcal{T}(z)$.*

Proof. By considering the various cases it is easy to see that if $Cxyz$ holds and if two of the arrows x, y, and z are identity arrows, then the third one is also an identity arrow. From the definition of C, we have $R_{11}yx$. Then by lemma 4.6, $R^L_{11}\mathcal{T}(y)\mathcal{T}(x)$. Now using property (C1) gives us $C^L \mathcal{T}(y)\mathcal{T}(x)\mathcal{T}(y)$. From $R_{21}yx$, we get $R^L_{21}\mathcal{T}(y)\mathcal{T}(z)$. This, together with $I^L \mathcal{T}(y)$ and $I^L \mathcal{T}(y)$ and lemma 4.10 gives us $F^L \mathcal{T}(y)\mathcal{T}(z)$. Now we can apply property (C10) to conclude $C^L \mathcal{T}(x)\mathcal{T}(y)\mathcal{T}(z)$. $\dashv$

Lemma 4.16 *Suppose x, y, and z are three arrows, one of which is an identity arrow, and suppose also $Cxyz$, then $C^L \mathcal{T}(x)\mathcal{T}(y)\mathcal{T}(z)$.*

Proof. We may safely assume that exactly one of the arrows is an identity arrow because of lemma 4.15. and consider the different cases depending on which of the arrows is the identity one.

(y) Suppose first that Iy, then from the definition of C and $R_{21}yy$ we have that Exz. Also from $R_{11}xy$, Iy and property (C1) we may conclude that $C^L \mathcal{T}(x)\mathcal{T}(y)\mathcal{T}(x)$. And that together with lemma 4.14 (ii) gives us $C^L \mathcal{T}(x)\mathcal{T}(y)\mathcal{T}(z)$.

(z) Assume Iz. Then the situation is similar to the above but we must use property (C2) and lemma 4.14 (i).

(x) The last case is when Ix holds. Then from the definition of C and $R_{21}xx$ we have that Fyz, so by lemma 4.13 (iii) $F^L \mathcal{T}(y)\mathcal{T}(z)$. From $R_{11}yx$ and Ix we have that $C^L \mathcal{T}(y)\mathcal{T}(x)\mathcal{T}(y)$. Now applying property (C10) we get $C^L \mathcal{T}(x)\mathcal{T}(y)\mathcal{T}(z)$. $\dashv$

What we proved so far is that if $Cxyz$ holds and at least one of the arrows x, y, and z is an identity arrow, then we indeed have that the maximal theories connected by the function $\mathcal{T}$ to these arrows are in the relation C^L in the canonical frame. What should be proved next is that if $Cxyz$ holds and none of the arrows is an identity arrow, then the same conclusion holds. So from now on we will be concerned only with non-identity arrows. Let us introduce one terminological convention: if $Cxyz$, then we say that x is the *composition* of y and z, y is the *first arrow* of the composition, and z is the *second* one. Further let us notice that if $Cxyz$ and neither of the arrows x, y, and z is an identity arrow, then the three arrows are different. Let us also agree that from now on when we say arrow we will mean non-identity arrow.

Lemma 4.17 *For every maximal theories X, Y, Z, U, and V in the canonical frame the next three items hold:*

(i) *if $C^L XYZ$, $F^L XU$, and $F^l YV$, then $C^L VZU$,*

(ii) *if $C^L XYZ$, $F^L XU$, and $F^L ZV$, then $C^L VUZ$,*

(iii) *if $C^L XYZ$, $F^L XU$, $F^L YV$, and $F^L ZW$, then $C^L UWV$*

Proof. By applying twice either property (C8) or (C9). ⊣

Next we define a new relation between the arrows and proceed with the assertion that will finish the whole thing off:

$$Sxyz \text{ iff } 1(y) = 2(x) \,\&\, 2(y) = 1(z) \,\&\, 2(z) = 1(x)$$

Lemma 4.18 *For every three arrows x, y, and z the following two assertions hold:*

(i) *If $Cxyz$, then $C^L \mathcal{T}(x)\mathcal{T}(y)\mathcal{T}(z)$.*

(ii) *If $Sxyz$, then there is an arrow u such that $Cuyz$ or $Cuzx$ or $Cuxy$.*

Remark. Item (ii) probably seems unnatural in its disjunstive conclusion, but that is so because the relation S in effect says that there is a cycle between the arrows x, y, and z, so we can cyclically move the places of x, y, and z in the relation. The property that we need is that if $Sxyz$, then at least one of x, y, and z has a converse that is in composition with the other two but we do not know which, that is why have to take care of the three possibilities.

Proof. We proceed by induction on the construction. We prove the following:

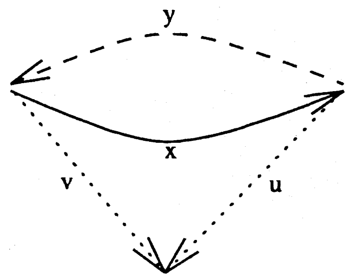
- $\forall n \forall x, y, z \in Ar_n(Cxyz \Rightarrow C^L \mathcal{T}(x)\mathcal{T}(y)\mathcal{T}(z))$,
- $\forall n \forall x, y, z \in Ar_n(Sxyz \Rightarrow \exists u \in Ar_n(Cuyz \vee Cuzx \vee Cuxy))$.

The induction base where $n = 0$ is trivial, since in Ar_0 there are no three arrows x, y, z such that $Cxyz$ or $Sxyz$. Next suppose that both assertions are true for every x, y, $z \in Ar_n$ and we proceed to show that they are true also for Ar_{n+1}. We must consider only the case where at least one of the arrows in the composition or in the cycling relation is added when extending Ar_n to Ar_{n+1}. Suppose that the arrow $x \in Ar_n$ and we will show that for every arrow that we add in the construction when extending x, both properties are preserved.

(ij_1) At this stage one new arrow is created, but its beginning (or end, depending on i, j) is not connected to another arrow, so it cannot be in the relation C or S with other arrows.

(ij_2) At this stage no new non-identity arrow is created.

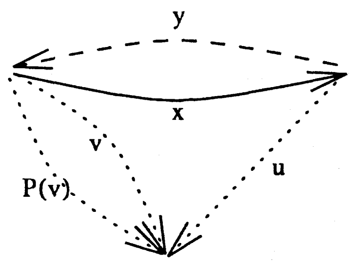
($conv$) Probably this is the most interesting case, where it should become clear why we need this S relation with the arising complications. Suppose that we create the arrow y, that is: $x = P(y)$. By the definition of the construction we also know that $F^L\mathcal{T}(x)\mathcal{T}(y)$. Further for item (i) suppose that y is in the composition relation with two arrows u and v. Now different cases appear depending on whether y is the first or the second arrow of the composition, or the composition itself. We will handle one of these cases; the other ones are similar. Suppose now that $Cuyv$, i.e. y is the first arrow in the composition. It is easiest to see what happens on a picture and we dwell formally.



We distinguish whether the arrows u, v are in Ar_{n+1} or not.

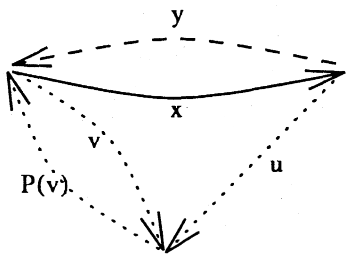
Suppose first that $u \in Ar_n$ and $v \in Ar_n$. By simple calculations using the functions 1 and 2 we can conclude that $Cvxu$. For the arrows x, u, and v we can apply the induction hypothesis to get $C^L\mathcal{T}(v)\mathcal{T}(x)\mathcal{T}(u)$. Now from $F^L\mathcal{T}(x)\mathcal{T}(y)$ and property (C8) we have $C^L\mathcal{T}(u)\mathcal{T}(y)\mathcal{T}(v)$.

Next suppose that exactly one of u, v is in Ar_{n+1}. Let that be v. We have that $2(u) \in Po_n$, and also $2(u) = 2(v)$. Further $2(x) \in Po_n$, and $1(x) = 1(v)$, so both ends of v are points that belong to Ar_n, so from earlier observation we may conclude that $P(v)$ connects the points $1(v)$ and $2(v)$. Again we must distinguish between the possible situations $EvP(v)$ and $FvP(v)$. Suppose first that we have $EvP(v)$. Now a picture is probably in order:



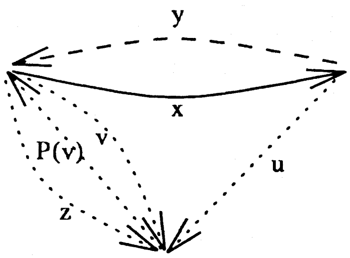
Notice that we have $CP(v)xu$, and for these three arrows we can apply the induction hypothesis, so we have $C^L\mathcal{T}(P(v))\mathcal{T}(x)\mathcal{T}(u)$. Now applying lemma 4.14 we get $C^L\mathcal{T}(v)\mathcal{T}(x)\mathcal{T}(u)$. Next from $F^L\mathcal{T}(x)\mathcal{T}(y)$ and property (C9) we conclude that $C^L\mathcal{T}(u)\mathcal{T}(y)\mathcal{T}(v)$ which is what we wanted.

Now let $FvP(v)$ be true, or we have the situation



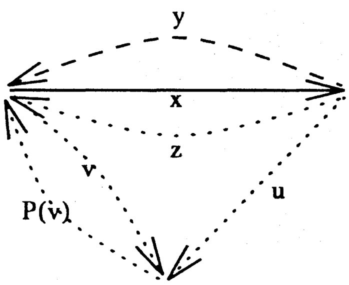
Now we have a further complication, we can in this case conclude that we have $SP(v)xu$, so using the induction hypothesis for (ii) we can conclude that there is an arrow $z \in Ar_n$, such that $Czxu$ or $CzuP(v)$ or $CzP(v)x$. Again we must deal with these cases one at a time.

($Czxu$) In this cases the situation looks like

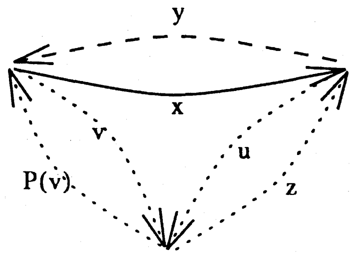


Now from $SP(v)xu$ and $Czxu$ by simple calculations using the functions 1 and 2 we conclude that $FzP(v)$. Since we are also in case that $FvP(v)$ we get that Ezv. From the induction hypothesis for (i) since x, u, $z \in Ar_n$ we have $C^L\mathcal{T}(z)\mathcal{T}(x)\mathcal{T}(u)$, by lemma 4.14 $C^L\mathcal{T}(v)\mathcal{T}(x)\mathcal{T}(u)$. Next as before use $F^L\mathcal{T}(x)\mathcal{T}(y)$ and property (C9) to conclude that .

($CzuP(v)$) Now using arguments as before we conclude that the relations below are true: $C^L\mathcal{T}(z)\mathcal{T}(u)\mathcal{T}(P(v))$, $C^L\mathcal{T}(u)\mathcal{T}(z)\mathcal{T}(v)$, and $C^L\mathcal{T}(u)\mathcal{T}(y)\mathcal{T}(v)$.



($CzP(v)x$) In this case we get $C^L\mathcal{T}(z)\mathcal{T}(P(v))\mathcal{T}(x)$. We can further conclude that Fxz and apply lemma 4.17 to get $C^L\mathcal{T}(u)\mathcal{T}(y)\mathcal{T}(v)$.

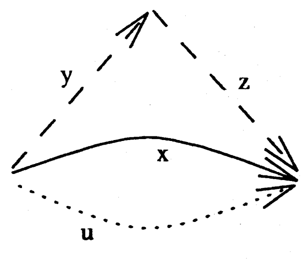


The case when $u \in Ar_{n+1}$ and $v \in Ar_n$ is similar.

Now let us consider that case when both u and v are in Ar_{n+1}. We have that $2(u) = 2(v)$, if we assume that $2(u) = 2(v) \in Po_{n+1}$ then we arrive at a contradiction with lemma 4.3. Then this point is in Ar_n, hence as before we can conclude that $P(u)$ connects the points $1(u)$ and $2(u)$ and something similar for $P(v)$. Then we can continue along the above lines considering the cases when $EuP(u)$ or $FuP(u)$ and $EvP(v)$ or $FvP(v)$. With that the case when y is the first arrow in the possible composition is finished. The cases when y is the second arrow in the composition, and the composition itself are treated as the first case with some small modifications at times. With all that item (i) is proved in this case.

Now for item (ii) suppose that for some arrows u, v it holds that either $Syuv$ or $Suyv$ or $Suvy$. In any of these cases we can compute using Fyx, that $Cxuv$ is true, and in any case at least one of the members of the disjunction is true.
So the case when extending an arrow by adding its converse is finished and we should say that the worst is behind.

($comp_1$) First let us deal with item (i). In this case we add two new arrows y and z such that $C^L\mathcal{T}(x)\mathcal{T}(y)\mathcal{T}(z)$. Now since there is no other arrow connected at the point $2(y)$ but z, if y or z is to be in the composition relation with other arrows, then y and z are the first and second arrow of such a composition, or the only possible case is for some previous arrow to have $Cuyz$. Again a picture helps:



But we can compute from $Cuyz$ that Exu, and this together with $C^L\mathcal{T}(x)\mathcal{T}(y)\mathcal{T}(z)$ and lemma 4.14 gives us $C^L\mathcal{T}(u)\mathcal{T}(y)\mathcal{T}(z)$.

Now for (ii) suppose that $Suyz$ or $Szuy$ or $Szyu$, then since by the construction $Cxyz$ and in any of the above cases item (ii) holds.

($comp_2$) In this case we create only one new non-identity arrow y, and it is such that Exy. Now for (i) suppose that for some arrows u, v we have either $Cyuv$, or $Cuyv$, or $Cuvy$. In any of these cases we can compute that the arrow x is also in the appropriate place in a C relation with u, v. Afterwards we proceed as before i.e. considering the case whether u, v belong to

Ar_n or not. If they do not belong to this set then we consider respectively $P(u)$ and $P(v)$. The picture we get is entirely similar to the one we have treated.

For (ii) suppose that $Syuv$ or $Suyv$ or $Suvy$, then we can substitute in any of these cases x for y, apply the induction hypothesis, get an appropriate C relation and in it substitute y for x which will again bring us home.

($comp_3$) It is the same as the above case.

This completes the proof of lemma 4.18. ⊣

BALR is complete

We have seen that there is a p-morphism from the frame $\mathcal{A}$ we built to the canonical frame and furthermore $\mathcal{A}$ is standard. We may then conclude that it is possible to refute every formula that is not a theorem of **BALR** in a standard frame which establishes the completeness of this logic in the class of all standard frames. With that Theorem 3.5 is proved.

5 Conclusion

In this section we provide some concluding remarks as well as suggestions for further investigations. In this work we saw how two approaches to Arrow Logic can be brought together and combined in a unifying framework. Although to arrive at a satisfactory result cost great efforts and attention mainly due to problems of undefinability the important thing is that the method we used in our proof can be used in a great variety of similar situations. To be more precise let us mention some of them.

First, using similar construction we can investigate the underlying logic of multigraphs only in the language of the Amsterdam approach i.e. omitting the co-incidence relations. We feel that this can be done without great complcations. This will also provide a finishing touch to a line of reseach that aims at dropping some restrictions on our frames with the idea of getting more tractable logics. This line started with considering the full square of arrows, then this square was relativised i.e. the condition that between any two points there should be an arrow was dropped. We propose to go further and drop one more restriction, namely that saying that between any two points there should be at most one arrow.

Second, using our coincidence relations this time we can try to consider other relations between arrows. Very interesting suggestion in this direction is the one to try to characterize every possible relation between 2, 3, ... arrows with the only restriction that these arrows should form a *circle*, i.e. when considered as undirected these arrows form an closed curve. For example, the only such relations between two arrows are the relations F and E. Between three arrows there four such relations, C and S are two of them and so on. It is interesting to consider interactons between relations between one and the same number of arrows as well and those between the relations among different number of arrows.

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