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An Algebraic Appreciation of Diagrams

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An algebraic appreciation of diagrams*

Jerry Seligman†

1 Are diagrams terms?

At least since Frege, it has been widely acknowledged that the concepts of function and argument are indispensable tools in linguistics. This idea is most clearly and forcefully expressed by Montague in [1]. Simply put, Montague claims that algebraic semantics is completely general: the semantic values of syntactic parts of a linguistic symbol are related to the semantic value of the whole in the same way that the denotation of subterms of a term are related to the denotation of the whole term. This follows almost directly from the assumption that the symbol is unambiguous and Frege's principle of compositionality. Any unambiguous symbol is uniquely analysed as the result of composing its syntactic parts together in a way that determines the meaning of the whole from the meaning of the parts. Consequently, any symbol can be seen to have the abstract term structure '$F(S_1, \ldots, S_n)$', where $S_1, \ldots, S_n$ are the term structures of its principal components, and F stands for the mode in which they are composed. By the principle of compositionality, F can be given an algebraic interpretation, as the function mapping the semantic-values of $S_1, \ldots, S_n$ to the semantic-value of the whole symbol.

It is very tempting to suppose that the Frege-Montague view of semantics applies quite generally, not just to language but to all forms of symbolic representation. After all, the approach has a great track record. In computer science, the method of using algebraic specification-languages has proved a powerful tool in the analysis of data-structures and programming languages. In linguistics, especially in semantics, an allegiance to the Frege-Montague approach has inspired many of the advances of the last thirty years. Moreover, the approach is very robust—quite often apparent detractors can be brought back into the fold if a sufficiently abstract view is taken (see Janssen's [2]).

One would expect simple diagrammatic systems of representation, such as Venn diagrams and Euler circles, to be ideal candidates for an analysis along Frege-Montague lines. Each diagram is composed from a finite number of diagrammatic objects, such as circles, ellipses, crosses and shading, perhaps with a few simple annotations; and the meaning of the diagram is clearly composed from the meaning of its parts. Two such diagrams are shown in Figure 1.

Indeed, one might try to argue, as follows, that *all* diagrammatic systems can be given the abstract syntax of terms. First, fix a co-ordinate frame for each diagram—for instance, that provided by measurement in inches, vertically and horizontally, from the mid-point of the paper. Associate each component of the diagram with its position within that frame. For each integer n, and each n-tuple σ of co-ordinates, let F_σ stand for the syntactic operation of drawing a diagram by placing its ith argument at co-ordinate position σ_i. Together with a range of atomic sym-

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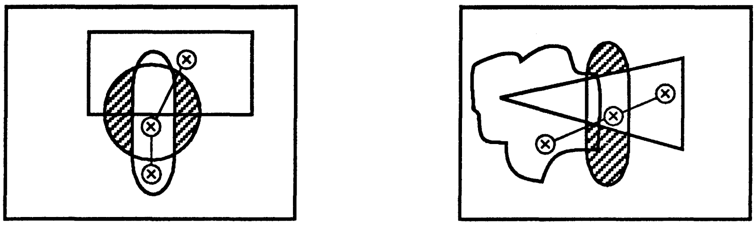


Figure 1: Homeomorphic diagrams

bols of various sizes, this is enough to specify an abstract term structure for every diagram.

The proposal fails because it is manifestly at odds with our *use* of diagrams. Imagine that the diagram on the left of Figure1 were to be transformed imperceptibly; the above analysis would assign a distinct syntactic structure to the new diagram, even though we take it to be syntactically unchanged. The compulsion that the two diagrams are syntactically equivalent is as basic as the compulsion that two inscriptions of the same word are instances of a common syntactic type.

Clearly, there must be some partition of the uncountably many syntactic forms into those which we regard as syntactically equivalent. The crucial difference between diagrams and linear systems of representation lies in how this partition is to be made.

A characteristic feature of linear systems, such as written language, is that their symbols can be divided into *segments* in a way which is invariant across all instances. For example, this sentence can be divided into ten words. Any other inscription of the sentence can also be divided into ten words, and the sentences will be syntactically identical if and only if they match word-for-word.[1] Once such a segmentation is given, the possibility of following the Frege-Montague line is opened: at the very worst, one can regard each symbol as a term built from primitive symbols and the operation of concatenation.

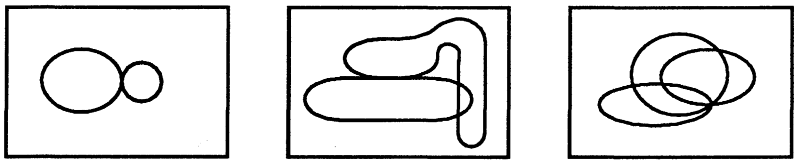
By contrast, the task of finding a useful segmentation of diagrams is quite hopeless in all part the most trivial cases. Indeed the fact that primitive components of a diagram overlap is often of the utmost importance in determining the meaning of the whole. The uncountably many arrangements of diagrammatic objects on the page must be partitioned in a different way.

A moment's thought suggests a viable alternative: the structure of most diagrams is invariant under many transformations of the plane, such as enlargement, rotation, reflection, and even more general topological transformations; so perhaps we can partition diagrams into classes which are closed under a given set of transformations. Put slightly differently, the proposal is that the syntactic type of a diagram is an *invariant* of some class of transformations.

Exactly which transformations are chosen will depend on the diagrammatic system we are analysing. In this paper, we will consider only the very simplest of diagrammatic systems, in which syntactic type is taken to be a topological invariant. Venn diagrams and Euler circles fall into this category, because their meaning depends only on facts about whether or not one diagrammatic objects overlap with another, and not on the size or shape of the object. However, we must not be too quick to *identify* syntactic type using semantic equivalence. Although—we claim—any adequate semantic theory of Venn diagrams should ensure that semantic-value

1. Of course, this is an idealization. In phonology, the problem of providing a segmentation of natural speech is very difficult, and some argue for a "tiered" approach in which several stacked segments are required for a correct analysis. Nonetheless, it is almost universally accepted that spoken language has a segmental syntax at some level of analysis.

Figure 2: Basic diagrams: connected and not



(i) (ii) (iii)

Figure 3: Three deviant diagrams

is a topological invariant, it need not be the same invariant as syntactic type. Just as in language, there are many semantically equivalent diagrams which are not naturally analysed as being of the same syntactic type.

We adopt the most liberal approach to diagram syntax by identifying syntactic type with invariance under homeomorphism. By this criterion, the diagrams in Figure 1 are syntactically equivalent.

2 Basic diagram syntax

Although we believe that the methods adopted in this paper can be applied to any diagrammatic system in which syntactic type is identified with invariance under homeomorphism, we will only consider the simplest of such system, in which the only diagrammatic objects are simple closed curves.[2]

Definition 2.1 A *(basic) diagram* $D = \langle \Box_D, \bigcirc_D \rangle$ consists of a rectangular region $\Box_D$ of the real plane, and a finite set $\bigcirc_D$ of simple closed curves inside $\Box_D$, such that the following conditions hold.

(i) No two curves intersect without crossing.
(ii) No two curves intersect at more than finitely many points.
(iii) No three curves intersect at the same point.

The region $\Box_D$ is called the *rectangle* of D, and each member of $\bigcirc_D$ is called a *curve* of D. D is said to be *connected* iff each pair of curves intersects at exactly two points or not at all.

Examples of connected, and disconnected basic diagrams are shown in Figure 2, on the left and right, respectively. The labels 'A', 'B' and 'C', are not considered part of the diagram itself; they are merely annotations which enable us to refer to the curves in the text. Diagrams which fail to satisfy conditions (i) to (iii) are shown in Figure 3.

Notation Given a function $f\colon A \to B$, we often need to refer to the associated function mapping each subset X of A to its image under f, namely $\{f(x) \mid x \in X\}$.

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2. A *simple closed curve* is a homeomorph of the unit circle.

To avoid notational clutter, we call this function f, and rely on the reader to discern which function is meant. Likewise, we use f^{-1} to refer both to the function mapping each subset Y of B to its inverse image under f, namely $\{x \in A \mid f(x) \in Y\}$, and to the inverse of f, if it has one. To support this convention, we banish sets which contain a subset as a member.

Definition 2.2 Given a diagram D and a homeomorphism h of the plane, the *image of* D *under* h, written $h(D)$, is the pair $\langle h(\Box_D), \{h(c) \mid c \in \bigcirc_D\}\rangle$.

If D is a diagram and $h(\Box_D)$ is rectangular then hD is also a diagram; moreover, if D is connected, so is hD. Thus the class of diagrams (and the class of connected diagrams) is "almost" closed under homeomorphic images. We could lift the restriction to rectangular rectangles—and with it the "almost" of the previous statement—but the need for such restrictions is often present in more complicated diagrammatic systems of representation, and so it would be somewhat artificial to do so.

Definition 2.3 Diagrams D_1 and D_2 are said to be *syntactically equivalent* iff there is a homeomorphism h such that $D_2 = hD_1$.

3 Basic diagram semantics

Basic diagrams may be interpreted by taking each curve to represent a class of individuals. A diagram is true under an interpretation just in case the set-theoretic relationships between the classes are as portrayed. For example, the diagram shown on the right of Figure 2 is true under an interpretation just in case every member of the class represented by the curve labelled 'B' is also a member of the class represented by the curve labelled 'A'. The curve labelled 'C' may be interpreted as any class whatsoever, without effecting the truth-value of the diagram.

We can sharpen the account of how a diagram receives a truth-value with the aid of a simple thought-experiment. To see if a diagram is true under an interpretation, imagine placing each individual on the diagram in such a way that it is surrounded by a curve if and only if it is a member of the class represented by that curve. If this can be done, the diagram is true under the interpretation; if not, it is false. We invite the reader to check that this method agrees with common sense.

Interpretations of diagrams are modelled using structures called classifications.[3]

Definition 3.1 A *classification* $\boldsymbol{A}$ consists of two sets $tok(\boldsymbol{A})$ and $typ(\boldsymbol{A})$, whose elements are called *tokens* and *types*, respectively, and a binary relation of *classification* between them. We write $a :_{\boldsymbol{A}} \alpha$ to mean that token $a \in tok(\boldsymbol{A})$ is classified by type $\alpha \in typ(\boldsymbol{A})$, dropping the subscripted '$\boldsymbol{A}$' when no ambiguity can arise. The *extension* $\underline{\alpha}$ of a type $\alpha \in typ(\boldsymbol{A})$ is defined by $\underline{\alpha} = \{a \in tok(\boldsymbol{A}) \mid a : \alpha\}$.

A generic example of a classification is a relational structure, all of whose relations are unary—indeed the reader may prefer to think of all classifications in this way, and may interpret our definitions and results accordingly. [4]

Definition 3.2 Given a diagram D, an *interpretation* of D consists of a classification $\boldsymbol{A}$, together with a function $f{:}\bigcirc_D \to typ(\boldsymbol{A})$. The diagram is *true* under the

3. The theory of classifications is developed by Barwise and Seligman in a series of papers, the most recent being [3]
4. The advantage of using classifications is that we will be defining structure-preserving maps between classifications which do not generalize easily to arbitrary relational structures.

interpretation, written $\boldsymbol{A}, f \models D$, iff there is a function $g\colon tok(A) \to \square_D$ such that for each $a \in tok(A)$, and each $c \in \bigcirc_D$,

$a :_A f(c)$ iff $g(a)$ is surrounded by c. [5]

A function g satisfying the above is called a *witnessing function* of the interpretation.

Our informal account of interpretation can be recaptured by regarding the type $f(c)$ as the class represented by the curve c of D, whose members are the elements of $f(c)$. The function g witnesses the imaginary placement of individuals in our thought-experiment.

In support of our definition of syntactic equivalence, we show that interpretations commute with homeomorphisms.

Notation Given functions $f\colon A \to B$ and $g\colon B \to C$, the function $gf\colon A \to C$ is the composition of f and g. Given a set A, the function $\mathrm{id}(A)\colon A \to A$ is the identity function on A.

Claim 3.3 *Given a diagram D and a homeomorphism h, if $\langle \boldsymbol{A}, f\rangle$ is an interpretation of D then $\langle \boldsymbol{A}, fh^{-1}\rangle$ is an interpretation of hD, and*

$\boldsymbol{A}, f \models D$ *iff* $\boldsymbol{A}, fh^{-1} \models hD$.

PROOF: Given a homeomorphism h, a simple closed curve c, and a point p, p is surrounded by c iff $h(p)$ is surrounded by $h(c)$. The rest follows from the definitions. QED

4 Diagram classifications and links

We will now work towards an alternative characterization of the syntax and semantics of basic diagrams using some concepts from the theory of classifications and links. The first step is to see diagrams as classifications.

Definition 4.1 For any diagram D, $\boldsymbol{D}$ is the classification defined by: $tok(\boldsymbol{D}) = \square_D$, $typ(\boldsymbol{D}) = \bigcirc_D$, and for each point $p \in \square_D$ and each curve $c \in \bigcirc_D$, $p :_D c$ iff p is surrounded by c.

The next step is to see interpretations as links between a diagram classification and the classification into which the diagram is interpreted.

Definition 4.2 Given classifications $\boldsymbol{A}$ and $\boldsymbol{B}$, and functions $f\colon typ(\boldsymbol{A}) \to typ(\boldsymbol{B})$ and $g\colon tok(\boldsymbol{B}) \to tok(\boldsymbol{A})$, the pair $\langle f, g\rangle$ is an *S-link* from $\boldsymbol{A}$ to $\boldsymbol{B}$, written $f, g\colon \boldsymbol{A} \rightleftarrows \boldsymbol{B}$, iff for each $b \in tok(\boldsymbol{B})$ and each $\alpha \in typ(\boldsymbol{A})$, $g(a) :_A \alpha$ iff $a :_B f(\alpha)$. [6]

Claim 4.3 $\boldsymbol{A}, f \models D$ *iff there is a g such that* $f, g\colon \boldsymbol{D} \rightleftarrows \boldsymbol{A}$.

PROOF: Direct from the definitions. QED

This characterization puts the following, purely link-theoretic question in focus: given a function $f\colon typ(\boldsymbol{A}) \to typ(\boldsymbol{B})$, what property must f have for there to be a function $g\colon tok(\boldsymbol{A}) \to tok(\boldsymbol{B})$ such that $f, g\colon \boldsymbol{A} \rightleftarrows \boldsymbol{B}$, and when is the choice of g uniquely determined? For the reminder of this section, we will work towards an answer. First, we introduce a useful abbreviation.

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5. A point in the plane is *surrounded by* a simple closed curve iff it lies in the (bounded) open region bounded by the curve.

6. In the language of [3], an 'S-link' is a Strongly Sound link with functionality '$\rightleftarrows$', i.e., it is 'S'-shaped.

Definition 4.4 Given classifications $\boldsymbol{A}$ and $\boldsymbol{B}$, a function $f\colon typ(\boldsymbol{A}) \to typ(\boldsymbol{B})$ is a *partial S-link* from $\boldsymbol{A}$ to $\boldsymbol{B}$ iff there is a $g\colon tok(\boldsymbol{B}) \to tok(\boldsymbol{A})$ such that $f, g\colon \boldsymbol{A} \rightleftarrows \boldsymbol{B}$.

To sharpen Claim 4.3, we would like to show that there is a one-one correspondence between true interpretations and S-links. In general, it is not true that a partial S-link $f\colon typ(\boldsymbol{A}) \to \boldsymbol{B}$ has a unique extension to an S-link $f, g\colon \boldsymbol{A} \rightleftarrows \boldsymbol{B}$. However, we can find another classification $\boldsymbol{A}_\sim$, such that if f uniquely determines and is determined by an S-link from $\boldsymbol{A}_\sim$ into $\boldsymbol{B}$.

Definition 4.5 Given a classification $\boldsymbol{A}$, elements $a, b \in tok(\boldsymbol{A})$ are *indistinguishable*, written $a \sim b$, iff for each $\alpha \in typ(\boldsymbol{A})$, $a :_A \alpha$ iff $b :_A \alpha$. The *indistinguishability class of* a, written $[a]_\sim$, is the set of elements of $tok(\boldsymbol{A})$ which are indistinguishable from a. The $\sim$-quotient of $\boldsymbol{A}$ is the classification $\boldsymbol{A}_\sim$ with $tok(\boldsymbol{A}_\sim) = \{[a]_\sim \mid a \in tok(\boldsymbol{A})\}$, $typ(\boldsymbol{A}_\sim) = typ(\boldsymbol{A})$, and $[a]_\sim :_{A_\sim} \alpha$ iff $a :_A \alpha$.

Theorem 4.6 *If f is a partial S-link from $\boldsymbol{A}$ to $\boldsymbol{B}$ then there is an S-link $f, f_*\colon \boldsymbol{A}_\sim \rightleftarrows \boldsymbol{B}$, and for any S-link $f, g\colon \boldsymbol{A}_\sim \rightleftarrows \boldsymbol{B}$, f is a partial S-link from $\boldsymbol{A}$ to $\boldsymbol{B}$ and $f_* = g$.*

To prove Theorem 4.6 we will need to establish a few elementary properties of links and $\sim$-quotients. First, note that S-links compose in the obvious way: if $f, g\colon \boldsymbol{A} \rightleftarrows \boldsymbol{B}$ and $f', g'\colon \boldsymbol{B} \rightleftarrows \boldsymbol{C}$ then $f'f, gg'\colon \boldsymbol{A} \rightleftarrows \boldsymbol{C}$. Also, for any classification $\boldsymbol{A}$, there is an *identity link*, $\mathrm{id}(typ(\boldsymbol{A})), \mathrm{id}(tok(\boldsymbol{A}))\colon \boldsymbol{A} \rightleftarrows \boldsymbol{A}$. Finally, if $f, g\colon \boldsymbol{A} \rightleftarrows \boldsymbol{B}$ and both f and g are bijections, then $\langle f, g\rangle$ is called an *isoinfomorphism* and $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be *isoinfomorphic*, written $\boldsymbol{A} \leftrightarrows \boldsymbol{B}$. As expected, the compositions $\langle f^{-1}f, gg^{-1}\rangle$ and $\langle ff^{-1}g^{-1}g\rangle$ are the identity links on $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively.

A classification is related to its $\sim$-quotient by the following lemma.

Lemma 4.7 *Given classifications $\boldsymbol{A}$ and $\boldsymbol{B}$, there are functions μ_A and η_A such that*

(i) if $f, g\colon \boldsymbol{A}_\sim \rightleftarrows \boldsymbol{B}$ then $f, \eta_A g\colon \boldsymbol{A} \rightleftarrows \boldsymbol{B}$

(ii) if $f, g\colon \boldsymbol{A} \rightleftarrows \boldsymbol{B}$ then $f, \mu_A g\colon \boldsymbol{A}_\sim \rightleftarrows \boldsymbol{B}$

PROOF: Let $\mu_A\colon tok(\boldsymbol{A}) \to tok(\boldsymbol{A}_\sim)$ be the function mapping each $a \in tok(\boldsymbol{A})$ to its indistinguishability class, and let $\eta_A\colon tok(\boldsymbol{A}_\sim) \to tok(\boldsymbol{A})$ be a function which selects a representative of each indistinguishability class, so that $\mu_A\eta_A$ is the identity function on $tok(\boldsymbol{A}_\sim)$. From these definitions, it is easy to see that $\mathrm{id}(typ(\boldsymbol{A})), \mu_A\colon \boldsymbol{A}_\sim \rightleftarrows \boldsymbol{A}$ and $\mathrm{id}(typ(\boldsymbol{A})), \eta_A\colon \boldsymbol{A} \rightleftarrows \boldsymbol{A}_\sim$. The two parts of the lemma follow by composition of S-links. QED

Unlike classifications in general, $\sim$-quotients have the useful property that any partial S-link from $\boldsymbol{A}_\sim$ is uniquely extendable to an S-link.

Lemma 4.8 *If $f, g_1\colon \boldsymbol{A}_\sim \rightleftarrows \boldsymbol{B}$ and $f, g_2\colon \boldsymbol{A}_\sim \rightleftarrows \boldsymbol{B}$ then $g_1 = g_2$.*
PROOF: By Lemma 4.7(i), $f, \eta_A g_1\colon \boldsymbol{A} \rightleftarrows \boldsymbol{B}$ and $f, \eta_A g_2\colon \boldsymbol{A} \rightleftarrows \boldsymbol{B}$. So for each $b \in tok(\boldsymbol{B})$ and each $\alpha \in typ(\boldsymbol{A})$, $\eta_A g_1(b) :_A \alpha$ iff $b :_B f(\alpha)$ iff $\eta_A g_2(b) :_A \alpha$, showing that $\eta_A g_1(b) \sim \eta_A g_2(b)$. Thus $\mu_A\eta_A g_1(b) = \mu_A\eta_A g_2(b)$. But $\mu_A\eta_A = \mathrm{id}(tok(\boldsymbol{A}))$, and so $g_1(b) = g_2(b)$. QED

PROOF OF THEOREM 4.6: Given that $\boldsymbol{A}, f \models D$, the existence of g follows from Claim 4.3 and Lemma 4.7(ii), the uniqueness from Lemma 4.8. The converse follows from Claim 4.3 and Lemma 4.7(i). QED

5 Consistent and tautologous diagrams

The results of the previous section provide us with a way of investigating the logical properties of diagrams.

Definition 5.1 A diagram is *consistent* iff it is true under at least one interpretation, and *tautologous* iff it is true under every interpretation.

Observe that every diagram can be interpreted in itself: by Claim 4.3, the existence of the identity link on $\boldsymbol{D}$ establishes that $\boldsymbol{D}, \mathrm{id}(\mathrm{O}_D) \models D$. Thus:

Corrollary 5.2 *Every diagram is consistent.*

And, by composition of links, we can see that S-links between interpretations are truth-preserving.

Corrollary 5.3 *If $\boldsymbol{A}, f \models D$ and f' is a partial S-link from $\boldsymbol{A}$ to $\boldsymbol{B}$ then $\boldsymbol{B}, f'f \models D$.*

As promised, Theorem 4.6 yields a characterization of the true interpretations of diagram D as S-links from $\boldsymbol{D}_\sim$.

Corrollary 5.4 *If $\boldsymbol{A}, f \models D$ then $f, f_*\colon \boldsymbol{D}_\sim \rightleftarrows \boldsymbol{A}$ is the unique S-link from $\boldsymbol{D}_\sim$ to $\boldsymbol{A}$ extending f. Moreover, for any S-link $f, g\colon \boldsymbol{D}_\sim \rightleftarrows \boldsymbol{A}$, $\boldsymbol{A}, f \models D$.*

Thus, for any diagram D, the classification $\boldsymbol{D}_\sim$ contains all the information necessary to evaluate the truth of D under an interpretation. It is finite and very easy to inspect, especially when D is connected, because then the tokens of $\boldsymbol{D}_\sim$ are just the smallest regions bounded by curves of the diagram.

An important aspect of our analysis is that S-links can be used to study both the semantics and the syntax of diagrams in a uniform way. The bridge is made by exploring the idea that one diagram can be interpreted in the diagram-classification of another.

Definition 5.5 Given two diagrams D and D', a function $f\colon \mathrm{O}_D \to \mathrm{O}_{D'}$ is a *diagram-homomorphism* from D to D' iff f is a partial S-link from $\boldsymbol{D}$ to $\boldsymbol{D}'$. A diagram-homomorphism is a *diagram-isomorphism* iff it is a bijection and its inverse is also a diagram-homomorphism. Diagrams D and D' are *isomorphic* iff there is an isomorphism from D to D'.

The diagrams in Figure 2 are isomorphic, with an isomorphism given by associating curves with the same labels. To check whether two diagrams D and D' are isomorphic it is often advisable to look for an isoinfomorphism between $\boldsymbol{D}_\sim$ and $\boldsymbol{D}_\sim$. This is clearly sufficient, but the existence of such an link is also necessary.

Claim 5.6 *Diagrams D and D' are isomorphic iff $\boldsymbol{D}_\sim \rightleftarrows \boldsymbol{D}'_\sim$.*
PROOF: Suppose $f\colon \mathrm{O}_D \to \mathrm{O}_{D'}$ is an isomorphism from D to D'. By Theorem 4.6, $f, f_*\colon \boldsymbol{D}_\sim \rightleftarrows \boldsymbol{D}'$, and $f^{-1}, f_*^{-1}\colon \boldsymbol{D}'_\sim \rightleftarrows \boldsymbol{D}$. So, by Lemma 4.7 and composition of links, $f\mathrm{id}(\mathrm{O}_D)f^{-1}\mathrm{id}(\mathrm{O}_D'), f_*\eta_{D'}f_*^{-1}\eta_D\colon \boldsymbol{D}_\sim \rightleftarrows \boldsymbol{D}_\sim$. But $f\mathrm{id}(\mathrm{O}_D)f^{-1}\mathrm{id}(\mathrm{O}_D') = \mathrm{id}(\mathrm{O}_D)$ and so $\langle f\mathrm{id}(\mathrm{O}_D)f^{-1}\mathrm{id}(\mathrm{O}_D'), f_*\eta_{D'}f_*^{-1}\eta_D\rangle$ must be the identity link on $\boldsymbol{D}_\sim$, by Lemma 4.8. Thus $f_*\eta_{D'}f_*^{-1}\eta_D = \mathrm{id}(tok(\boldsymbol{D}_\sim))$. A similar argument shows that $f_*^{-1}\eta_D f_*\eta_{D'} = \mathrm{id}(tok(\boldsymbol{D}'_\sim))$, and so $f_*^{-1}\eta_D = (f_*\eta_{D'})^{-1}$. The required isoinfomorphism is therefore $\langle f, f_*\eta_{D'}\rangle$. The converse is trivial. QED

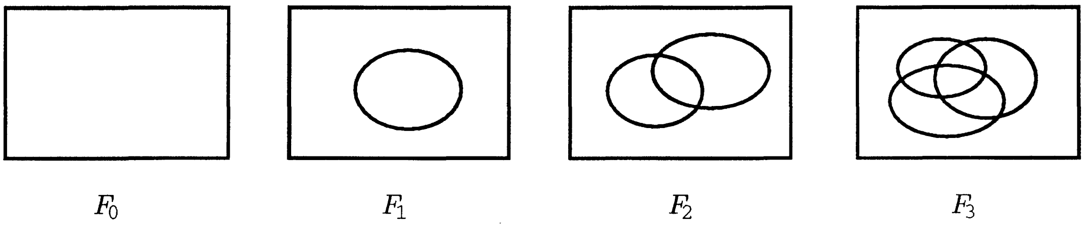


Figure 4: Free diagrams

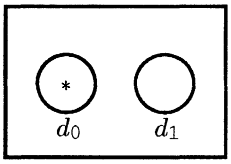


Figure 5: The diagram D_*

The following two Lemmas follow straight from Claim 4.3 and the definition of homomorphism.

Lemma 5.7 *If $\boldsymbol{A}, f' \models D'$ and $f{:}\,\mathrm{O}_D \rightarrow \mathrm{O}_{D'}$ is a homomorphism then $\boldsymbol{A}, f'f \models D$.*

Lemma 5.8 *$f{:}\,\mathrm{O}_D \rightarrow \mathrm{O}_{D'}$ is a homomorphism iff $\boldsymbol{D}', f \models D$.*

Definition 5.9 A diagram D is *free* iff for each diagram D', every function from O_D to $\mathrm{O}_{D'}$ is a homomorphism. Examples of free diagrams are given in Figure 4.

Theorem 5.10 *A diagram is tautologous iff it is free.*
PROOF: Suppose D is a tautologous diagram. For any diagram D' and any function $f{:}\,\mathrm{O}_D \rightarrow \mathrm{O}_{D'}$, $\boldsymbol{D}', f \models D$, so f is a homomorphism, by Lemma 5.8. Thus D is free.

Conversely, suppose D is free and $\langle \boldsymbol{A}, f\rangle$ is an interpretation of D. For each $a \in tok(\boldsymbol{A})$, let f_a be the function from O_D to the curves of the diagram D_* depicted in Figure 5, defined by

$$f_a(c) = \begin{cases} d_0 & \text{if} \quad a : f(c) \\ d_1 & \text{otherwise} \end{cases}$$

f_a is a homomorphism, because D is free, and so there is a function g_a such that $f_a, g_a{:}\,\boldsymbol{D} \rightleftarrows \boldsymbol{D}_*$. Now define the function $g{:}\ tok(\boldsymbol{A}) \rightarrow tok(\boldsymbol{D})$ by $g(a) = g_a(*)$, where '$*$' is as marked in Figure 5. Noting that $g(a) :_D c$ iff $* :_{D_*} f_a(c)$ iff $a :_{\boldsymbol{A}} f(c)$, we can conclude that $f, g{:}\,\boldsymbol{D} \rightleftarrows \boldsymbol{A}$, and so $\boldsymbol{A}, f \models D$. QED

Corrollary 5.11 *Tautologous diagrams with the same number of curves are isomorphic.*
PROOF: QED
If D and D' are tautologous diagrams with the same number of curves then there is a bijection f from O_D to $\mathrm{O}_{D'}$. By Theorem 5.10, D and D' are both free, so f and f^{-1} are both homomorphisms, and so f is an isomorphism.

This shows that the diagrams F_0, F_1, F_2 and F_3, shown in Figure 4, are the unique tautologous diagrams with 0, 1, 2, and 3 curves, respectively, up to isomorphism. In fact, these diagrams are also the unique tautologous, connected diagrams in their size, up to *syntactic equivalence*. Unfortunately, we will see later that the series cannot be continued.

Definition 5.12 Diagram D' is an *extension* of diagram D, written $D' \supseteq D$, iff $\Box_D = \Box_{D'}$ and $\bigcirc_D \subseteq \bigcirc_{D'}$. We also say that D is a *subdiagram* of D'.

Diagram D' is a *free extension* of diagram D, written $D' \geq D$, iff $D' \supseteq D$ and for any diagram D'' and any function $f{:}\bigcirc_{D'} \rightarrow \bigcirc_{D''}$ if the restriction of f to $\bigcirc_D$ is a homomorphism then so is f. D is *simple* iff it is a free extension only of itself.

Corrollary 5.13 *D is tautologous iff $D \geq F_0$.*
PROOF: For any $f{:}\bigcirc_D \rightarrow \bigcirc_{D'}$, the restriction of f to $\bigcirc_{F_0} = \emptyset$ is just the empty function, and so is a homomorphism. Thus $D \geq F_0$ iff D is free. The result then follows from Theorem 5.10. QED

6 Covers

A different kind of characterization of partial S-links (and so of true interpretations and diagram homomorphisms) can be obtained by looking for the structural property they preserve.

Definition 6.1 Given a classification $\boldsymbol{A}$ and sets $\Sigma, \Sigma' \subseteq typ(\boldsymbol{A})$, let $[\Sigma, \Sigma'] = \bigcap_{\alpha \in \Sigma} \underline{\alpha} - \bigcup_{\alpha' \in \Sigma'} \underline{\alpha'}$. The pair $\langle \Sigma, \Sigma' \rangle$ is a *cover* iff $[\Sigma, \Sigma'] = \emptyset$. A *subcover* of a pair $\langle \Sigma, \Sigma' \rangle$ is any cover $\langle \Sigma_0, \Sigma'_0 \rangle$ such that $\Sigma_0 \subseteq \Sigma$ and $\Sigma'_0 \subseteq \Sigma'$. A function $f{:}\, typ(A) \rightarrow typ(B)$ is said to *preserve covers* from $\boldsymbol{A}$ to $\boldsymbol{B}$ iff the image under f of each cover in $\boldsymbol{A}$ is a cover in $\boldsymbol{B}$.

Theorem 6.2 *A function $f{:}\, typ(\boldsymbol{A}) \rightarrow typ(\boldsymbol{B})$ is a partial S-link iff it preserves covers.*
PROOF: If f is a partial S-link then there is a g such that $f, g{:}\, \boldsymbol{A} \rightleftarrows \boldsymbol{B}$. For any $\Sigma, \Sigma' \subseteq typ(\boldsymbol{A})$, we will show that if $b \in [f\Sigma, f\Sigma']$ then $g(b) \in [\Sigma, \Sigma']$, which is enough to show that f preserves covers. So, for any $b \in [f\Sigma, f\Sigma']$ and any $\alpha \in typ(\boldsymbol{A})$,

if $\alpha \in \Sigma$ then $b \in \underline{f(\alpha)}$, so $b \in \underline{\alpha}$, and

if $\alpha' \in \Sigma'$ then $b \notin \underline{f(\alpha')}$, so $b \notin \underline{\alpha'}$.

Conversely, if f preserves covers, we define a function $g{:}\, tok(\boldsymbol{B}) \rightarrow tok(\boldsymbol{A})$ as follows. For each $b \in tok(\boldsymbol{B})$, let $\Sigma_b^+ = \{\beta \in typ(\boldsymbol{B}) \mid b :_B \beta\}$ and let $\Sigma_b^- = typ(\boldsymbol{B}) - \Sigma_b^+$. Note that $b \in [\Sigma_b^+, \Sigma_b^-]$. Also note that $ff^{-1}\Sigma_b^+ \subseteq \Sigma_b^+$ and $ff^{-1}\Sigma_b^- \subseteq \Sigma_b^-$, and so $b \in [ff^{-1}\Sigma_b^+, ff^{-1}\Sigma_b^-]$. But then $\langle f^{-1}\Sigma_b^+, f^{-1}\Sigma_b^- \rangle$ is not a cover, because f preserves them. So pick any $a \in [f^{-1}\Sigma_b^+, f^{-1}\Sigma_b^-]$ and let $g(b) = a$. We leave it to the reader to check that, for each $\alpha \in typ(\boldsymbol{A})$, $g(b) :_A \alpha$ iff $b :_B f(\alpha)$. QED

The theorem may be applied to both interpretations and homomorphisms.

Corrollary 6.3 *$\boldsymbol{A}, f \models D$ iff f preserves covers.*

Corrollary 6.4 *$f{:}\bigcirc_D \rightarrow \bigcirc_{D'}$ is a homomorphism iff f preserves covers from $\boldsymbol{D}$ to $\boldsymbol{D'}$.*

Corrollary 6.5 *D is a tautologous diagram iff $\boldsymbol{D}$ has no non-trivial covers.*
PROOF: For every interpretation $\langle \boldsymbol{A}, f\rangle$ of D, f preserves trivial covers, so the result follows from Corollary 6.3. QED

Covers are also useful for studying free extensions.

Lemma 6.6 *D' is a free extension of D iff each non-trivial cover in D' has a subcover in D.*

Theorem 6.7 *Every diagram is the free extension of a unique simple diagram.*

PROOF: Let core(D) be the subdiagram of D with $\bigcirc_{\text{core}(D)} = \bigcap \{\bigcirc_{D'} \mid D' \leq D\}$. We show that core($D$) $\leq D$, and so establish that core(D) is the smallest subdiagram of D with this property. Suppose $\langle C, C'\rangle$ is a non-trivial cover in D with no proper subcover. We call such a cover a *minimal pair* of D. For all $D' \leq D$, $\langle C, C'\rangle$ has a subcover in D', by Lemma 6.6. By minimality, this subcover can only be $\langle C, C'\rangle$ itself; and so $\langle C, C'\rangle$ is also a cover in core(D), by construction of the latter. We have shown that every minimal cover in D is also a cover in core(D). The subcover-order is clearly well-founded, so every cover in D contains a minimal subcover in D, and hence also in core(D); thus core(D) $\leq D$, by Lemma 6.6 again. Now core(D) is clearly simple, and every $D' \leq D$ is a free extension of core(D), so core(D) is the only simple subdiagram of D having D as a free extension. QED

Definition 6.8 For each diagram D, let core(D) $= \langle \square_D, \bigcap \{\bigcirc_{D'} \mid D' \leq D\}\rangle$ be the unique simple subdiagram of D having D as a free extension.

Finally, we leave the proofs of the following three claims to the reader.

Claim 6.9 *core(D) is the largest simple subdiagram of D.*

Claim 6.10 *Given diagrams D and D', the following are equivalent.*
(i) There is a homomorphism from D' to a free extension of D.
(ii) There is a homomorphism from core(D') to D.

Claim 6.11 *core(D) is isomorphic to core(D') iff there are homomorphisms from core(D) to D' and from core(D') to D.*

7 Constructing diagrams

The association of a diagram D with the classification $\boldsymbol{D}$ respects our notion of syntactic equivalence in the following sense.

Claim 7.1 *Syntactically equivalent diagrams are isomorphic.*
PROOF: If D and D' are syntactically equivalent then there is a homeomorphism h such that $hD = D'$. It is easy to see that $h^{-1}, h\colon \boldsymbol{D}' \rightleftarrows \boldsymbol{D}$ is an isoinfomorphism. QED

However, the converse of Claim 7.1 is not true, even if we restrict it to free diagrams which are connected. The isomorphic, free, connected diagrams shown in Figure 6 are not syntactically equivalent. This is a shame, because Corollaries 5.11

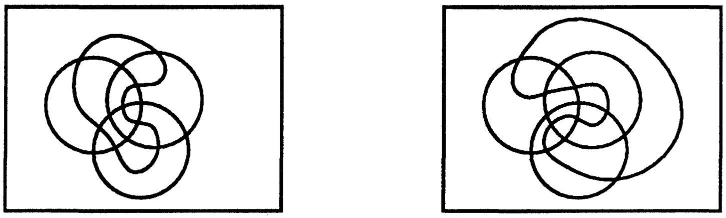
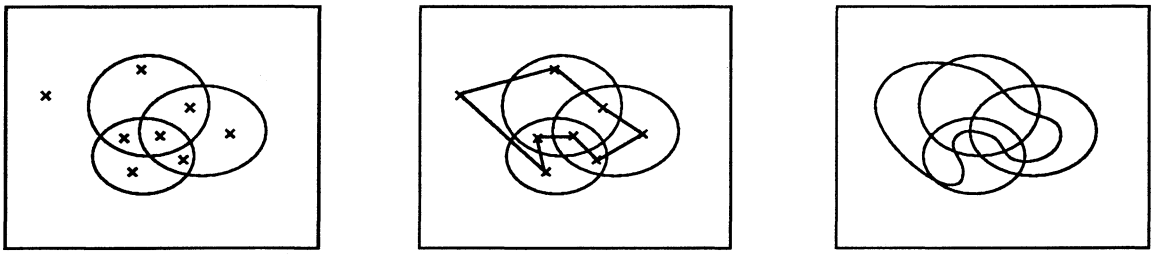


Figure 6: Isomorphic, free, connected diagrams, which are not syntactically equivalent.



(a) (b) (c)

Figure 7: Freely extending a diagram

and 6.5 suggest a strategy for constructing all free diagrams. If for each n we can construct one free diagram with n curves then we can be sure that it is the unique free diagram with n curves, up to isomorphism. Unfortunately, that is not enough to guarantee syntactic equivalence.

To construct the tautologous diagrams we must therefore be a little more cunning that one might have expected. First, we show that any diagram can be freely extended by one curve.

Construction 7.2 Recall that any finite number of points inside a rectangular region can be joined by a simple closed curve lying inside the region. The following is an algorithm for constructing such a curve. If there is only one point then a (small enough) circle passing through the point will do. If there is more than one point then there are two cases, depending on whether or not the points are collinear. If they are, then they lie on a line of a finite length, which forms the side of a (small enough) rectangle. If they are not collinear, draw a line through each pair of points in the collection. Pick one of the resulting minimal regions (a convex polygon) and call its centre c. No two points of the collection are collinear with c. Order the points according to the size of the angle a line drawn from the point to c makes with the horizontal. Connect the points with straight lines in the order just determined, to form a simple polygon—of course, any closed curve connecting the points in this order will do as well.

Now suppose D is a basic diagram, to which we wish to add a curve freely. Pick points in each $\sim$-class of D, none of which lie on one of the curves. Join the selected points them with a simple closed curve, as shown above. Because D is a basic, there are only a finite number of points of intersection, so we can always draw a curve which avoids them all and intersects the old curves in only a finite number of places; this guarantees that the resulting diagram is also basic.

Figure 7(a) shows a three-curve diagram with selected points marked by crosses. In (b), the crosses have been joined by straight lines, following the above algorithm. In (c), a smother curve is chosen.

Claim 7.3 *If D' is constructed from D in accordance with Construction 7.2 then it is a free extension of D.*

PROOF: By construction, each $\sim$-class E of D contains a point p lying on the new curve. The point p was selected so that it does not lie on any of the curves of D, and so it is contained in the interior of E, which is an open set. The new curve passes through p and so it divides the interior of E (and hence E also) into two non-empty regions. Thus if c is the new curve, neither $\underline{c} \cap E$ nor $E - \underline{c}$ are empty. Consequently, if $\langle \Gamma, \Gamma' \rangle$ is a cover of D' then either $c \in \Gamma \cap \Gamma'$ and so $\langle \Gamma, \Gamma' \rangle$ is trivial, or $\langle \Gamma - \{c\}, \Gamma' - \{c\} \rangle$ is a cover of D. The result follows by Lemma 6.6. QED

Construction 7.2 gives us a method of constructing free diagrams of each size, thus characterizing the free diagrams up to isomorphism. To improve on this result, we need to show that every free diagram is syntactically equivalent to one constructed in this way. To do this, we must look a little more closely at the construction.

Definition 7.4 Given a diagram D, a sequence $p_1, \ldots, p_n$ of points in $\Box_D$ is a *blueprint* of D iff

1. for each $i \neq j < n$, $p_i \neq p_j$
2. $p_1 = p_n$
3. for each $i < n$, there is a simple curve with endpoints p_i and p_{i+1}, and which crosses one curve of D exactly once.[7]

Blueprints $p_1, \ldots, p_n$ and $q_1, \ldots, q_m$ of D are *equivalent* iff $n = m$ and for each $i < n$, there is a curve with endpoints p_i and q_i which does not intersect any of the curves of D.

Given any simple closed curve c in $\Box_D$, we say that $p_1, \ldots, p_n$ is a *blueprint of* c iff for every segment s of c bounded by (but not containing) points of intersection with curves of D, there is an $i < n$ such that p_i lies on s.

Construction 7.5 Given a diagram D and a blueprint $p_1, \ldots, p_n$. For each $i < n$, there is a curve c_i with endpoints p_i and p_{i+1}, and which crosses exactly one curve of D exactly once. Draw a closed curve c by joining up the curves $c_1, \ldots, c_{n-1}$.

Construction 7.5 provides a finer degree of control than Construction 7.2, but it has the drawback that it may not produce a basic diagram, because the constructed curve may not be a simple closed curve—it may intersect itself. Nonetheless, it is more general.

Claim 7.6 *Given a diagram D and a curve $c \in \bigcirc_D$. D results from the diagram $D' = \langle \Box_D, \bigcirc_D - \{c\} \rangle$ by an application of Construction 7.5.*

PROOF: Divide c into segments $s_1, \ldots, s_n$ bounded by, but not containing, points of intersection between c and curves of D'. (There are only a finite number of segments, because c crosses any other curve of D at most finitely many times, and there are only a finite number of curves in D—both restrictions imposed by the definition of basic diagram.) For each $i < n$, select a point p_i on s_i. The sequence $p_1, \ldots, p_n$ is a blueprint for c, and so c can be constructed from it by an application of Construction 7.5. QED

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7. A simple curve is a homeomorph of the closed unit interval.

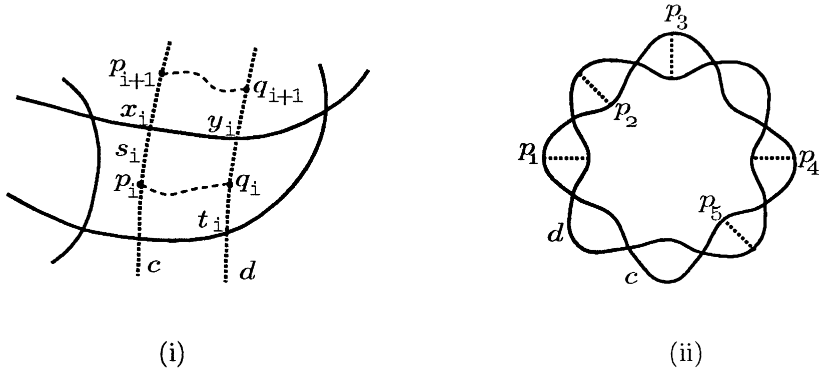


Figure 8: Equivalent extensions

The following theorem gives the condition under which successful applications of Construction 7.5 yield syntactically equivalent diagrams.

Theorem 7.7 *Let D be a diagram, and let c and d two simple closed curves in $\Box_D$, which are not contained in $\bigcirc_D$, and do not lie on any of the intersection points of D, but which have equivalent blueprints. If c and d also surround the same curves of D, then the diagrams $\langle \Box_D, \bigcirc_D \cup \{c\}\rangle$ and $\langle \Box_D, \bigcirc_D \cup \{c\}\rangle$ are syntactically equivalent.*
PROOF: Suppose $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ are equivalent blueprints of c and d, respectively. For each $i < n$, there are segments s_i of c and t_i of d which are bounded by, but do not contain, points of intersection with the curves of D. The points p_i and q_i lie on s_i and t_i, respectively; and there is a curve with endpoints p_i and q_i, which does not cross any of the curves of D. Figure 8(i) depicts the situation.

Let x_i be the point of intersection of c with a curve of D between p_i and p_{i+1}, whose existence and uniqueness is implied by the fact that $p_1, \ldots, p_n$ is a blueprint of c. Likewise, let y_i be the point of intersection of d with a curve of D, between q_i and q_{i+1}.

Claim 1 The points x_i and y_i lie on the same curve of D and no other curves of D intersect the region $p_i p_{i+1} q_{i+1} q_i$.
PROOF OF CLAIM: First note that x_i and y_i are the only points of curves of D to cross the boundary of the region. This follows from observations already made, stemming from the fact that the corner points of the region lie on equivalent blueprints of the curves c and d. Thus the curve of D which enters the region at x_i must either terminate inside the region, or else leave at either x_i or y_i. The first possibility is excluded because every curve of d is closed, and the second is excluded because no curve of D is self-intersecting. The only remaining possibility is that the curve entering at x_i leaves at y_i, which is just to say that x_i and y_i lie on the same curve. No other curve can enter the region, because x_i and y_i are the only possible entry points, and neither c nor d lies on an intersection point of D (by hypothesis). Finally, if there were any curve lying entirely within the region, it would be surrounded by either c or d, but not by both, which is forbidden by hypothesis.

If we now consider the whole curves c and d, it is clear that they are related in the manner depicted in Figure 8(ii). In other words, the region bounded by c and d is a "pinched" annulus crossed by a finite number of segments of curves of D. By a standard extension theorem (?), any homeomorphism of c onto d which maps p_i to q_i and x_i to y_i (for $0 < i < n$), can be extended to a homeomorphism of the plane which maps c to d, while keeping the rectangle and curves of D fixed. QED

Corollary 7.8 *If D_c and D_d are free extensions of D which result from the addition of curves c and d, and c and d have equivalent blueprints, then D_c is syntactically equivalent to D_d.*
PROOF: If D_c and D_d are free extensions of D then c and d must *intersect* all the curves of D, and so there are none which they *surround.* QED

Corollary 7.9 *There are countably many syntactically non-equivalent diagrams.*
PROOF: In any diagram D there are only countably many non-equivalent blueprints. So, by Theorem7.7, D has only countably many extensions with one extra curve. The result follows by induction on the number of curves in a diagram. QED

We will now see how to enumerate basic diagrams, up to syntactic equivalence.

Definition 7.10 Let D be a diagram. Blueprints $p_1, \ldots, p_n$ and $q_1, \ldots, q_m$ in D are *co-extensive* iff for each $i < n$ there is a $j < m$ and a curve with endpoints p_i and q_j, and which does not intersect any curve of D. We define the *order* of a blueprint in D by induction:

(0) A blueprint in D has order 0 iff there is no shorter, co-extensive blueprint in D.

(n) A blueprint in D has order $n+1$ iff it does not have order n but every shorter, co-extensive blueprint has order n or less.

An extension D' of D has *order* n iff it has one extra curve, and that curve has a blueprint of order n.

Note that if we restrict our attention to connected diagrams, every one-curve extension has order 0. Unfortunately, we do not yet know whether there are free connected diagrams of every size; but we can be certain that there are always extensions of order 0.

Corollary 7.11 *Every diagram has a finite, positive number of syntactically non-equivalent, one-curve extensions of order n.*

Corollary 7.12 *There are a finite number of syntactically non-equivalent, free diagrams with n curves.*

Thus, by enumerating blueprints, we can enumerate the tautologous diagrams, up to syntactic equivalence.

8 Valid diagrammatic arguments

Our next goal is a link-theoretic characterization of valid arguments using diagrams. It is not sufficient to represent a diagrammatic argument just as a sequence of premise diagrams and a conclusion. In addition, we need to know which curves in the conclusion are intended to represent the same class as curves in the premises. In other words, we must record the connections between curves in the premises and conclusion which establish co-reference. In informal use, these connections are indicated by gestures, labels, or simply by the fact that a concrete image standing for the conclusion is arrived at by a process of modifying concrete images of the premises in a way that leaves some of the original curves in tact.

The easiest way to model co-reference between curves is to pretend that all the curves in both premises and conclusion are labelled.[8]

Definition 8.1 Let L be a set. An *L-labelled diagram* is a pair $\langle D, \lambda\rangle$, consisting of a diagram D together with a function $\lambda: \bigcirc_D \to L$. It is *properly-labelled* iff λ is one-one. $\langle D, \lambda\rangle$ is *syntactically equivalent* to $\langle D', \lambda'\rangle$ iff there is a homeomorphism h such that $hD = D'$ and $\lambda = \lambda' h$.

Let L be a fixed countably infinite set of labels. We draw labelled diagrams by writing the label $\lambda(c)$ next to the curve c, in the expected way. Our decision not to regard the labels as part of the diagram itself is reflected in the definition of syntactic equivalence: we only require the curves with the same label to be preserved under homeomorphism, not the labels themselves. Two properly-labelled diagrams are shown in Figure 2. An improperly labelled diagram is shown in Figure 9.

Definition 8.2 An *L-interpretation* is a classification $\boldsymbol{A}$ with $typ(\boldsymbol{A}) = L$. An L-labelled diagram $\langle D, \lambda\rangle$ is *true* under an L-interpretation $\boldsymbol{A}$, written $\boldsymbol{A} \models \langle D, \lambda\rangle$ iff $\boldsymbol{A}, \lambda \models D$. An L-labelled diagram is *consistent* iff it is true under at least one L-interpretation, and *tautologous* iff it is true under all.

Given a set Δ of L-labelled diagrams and an L-labelled diagram $\langle D, \lambda\rangle$, we say that $\langle D, \lambda\rangle$ is a *consequence* of Δ, and write $\Delta \models \langle D, \lambda\rangle$, iff every L-interpretation under which each labelled diagram in Δ is true is one under which $\langle D, \lambda\rangle$ is also true. Labelled diagrams $\langle D, \lambda\rangle$ and $\langle D, \lambda'\rangle$ are *logically equivalent* iff $\langle D, \lambda\rangle \models \langle D, \lambda'\rangle$ and $\langle D', \lambda'\rangle \models \langle D, \lambda\rangle$

Claim 8.3 *Given an L-model $\boldsymbol{A}$, and an L-labelled diagram $\langle D, \lambda\rangle$, the following are equivalent:*

1. $\boldsymbol{A} \models \langle D, \lambda\rangle$,
2. $\lambda, \lambda_*: \boldsymbol{D}_\sim \rightleftarrows \boldsymbol{A}$,
3. *λ preserves covers.*

PROOF: From Theorems 4.6 and 6.2. QED

In order to use this theorem to extend the results of the previous sections, we need some way of relating arbitrary interpretations of a diagram to L-models. To do this, it is convenient to make use of another concept from the theory of links.

Definition 8.4 Given a classification $\boldsymbol{A}$, and a relation $r \subseteq typ(\boldsymbol{A}) \times B$, we define *the coherent projection of $\boldsymbol{A}$ along r*, written $r\boldsymbol{A}$, as follows. For each $a \in tok(\boldsymbol{A})$, define

$$a^+ = \{\beta \in B \mid \exists \alpha \in typ(\boldsymbol{A}) \quad a :_A \alpha \quad \text{and} \quad \langle \alpha, \beta\rangle \in r\}$$
$$a^- = \{\beta \in B \mid \exists \alpha \in typ(\boldsymbol{A}) \quad a \not:_A \alpha \quad \text{and} \quad \langle \alpha, \beta\rangle \in r\}$$

Let $r\boldsymbol{A}$ be the classifications with $tok(r\boldsymbol{A}) = \{a \in tok(\boldsymbol{A}) \mid a^+ \cap a^- = \emptyset\}$, $typ(r\boldsymbol{A}) = \text{rng}(r)$, and $a :_{rA} \beta$ iff $\beta \in a^+$.

Lemma 8.5 *For any labelled diagram $\langle D, \lambda\rangle$, let $\langle \lambda\rangle$ be the graph of λ. Then $\lambda, \text{id}(\square_D): \boldsymbol{D} \rightleftarrows \langle\lambda\rangle \boldsymbol{D}$*

PROOF: By construction. QED

8. Another strategy is to represent the co-reference relation between curves (See Shin's [4]). It is fairly easy to see that this is equivalent.

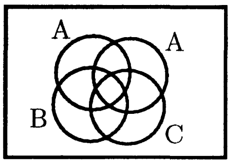


Figure 9: A tautologous labelled diagram

The classification $\langle\lambda\rangle\boldsymbol{D}$ does for labelled diagrams what $\boldsymbol{D}$ did for unlabelled diagrams.

Theorem 8.6 *Every labelled diagram is consistent.*
PROOF: Let $(\langle\lambda\rangle\boldsymbol{D})^+$ be any L-model extending $\langle\lambda\rangle\boldsymbol{D}$ (which must exist: assign arbitrary extensions to the labels not occurring in $\text{rng}(\lambda)$). The inclusion function is a partial S-link, and so by composition with the link in Lemma 8.5, we have $(\langle\lambda\rangle\boldsymbol{D})^+ \models \langle D, \lambda\rangle$. QED

Theorem 8.7 *A labelled diagram $\langle D, \lambda\rangle$ is tautologous iff $\langle\lambda\rangle\boldsymbol{D}$ has no non-trivial covers.*
PROOF: If $\langle\lambda\rangle\boldsymbol{D}$ has no non-trivial covers, then by Theorem 6.2, for any L-model $\boldsymbol{A}$, $\text{id}(\text{rng}(\lambda))$ is a partial S-link from $\langle\lambda\rangle\boldsymbol{D}$ to $\boldsymbol{A}$. By composition with the link in Lemma 8.5, $\lambda = \text{id}(\text{rng}(\lambda))\lambda$ is a partial S-link from $\boldsymbol{D}$ to $\boldsymbol{A}$. Then, by Claim 8.3, $\boldsymbol{A} \models \langle D, \lambda\rangle$.

Conversely, suppose $\langle D, \lambda\rangle$ is tautologous. For any function, $f: L \to typ(\boldsymbol{A})$, let $\boldsymbol{A}_f$ be the classification with types L, tokens $tok(\boldsymbol{A})$ and $a : l$ iff $a :_A f(l)$. Then $\langle \boldsymbol{A}_f, \lambda\rangle$ is an L-interpretation of D, and so $\boldsymbol{A}_f \models \langle D, \lambda\rangle$, because $\langle D, \lambda\rangle$ is tautologous; and so there is a function g such that $\lambda, g: \boldsymbol{D} \rightleftarrows \boldsymbol{A}_f$.

Claim 1 $f, g: \langle\lambda\rangle\boldsymbol{D} \rightleftarrows \boldsymbol{A}$

PROOF OF CLAIM: Given $a \in tok(\boldsymbol{A})$ and $l \in typ(\langle\lambda\rangle\boldsymbol{D}) = \text{rng}(\lambda)$, there is a $c \in \text{O}_D$ such that $l = \lambda(c)$. We have shown that $\lambda, g: \boldsymbol{D} \rightleftarrows \boldsymbol{A}_f$ and so $g(a) :_D c$ iff $a :_{A_f} \lambda(c) = l$, and this is the case iff $a :_A f(l)$, by the definition of $\boldsymbol{A}_f$, above. Now, by Lemma 8.5, $g(a) = \text{id}(\square_D)g(a) :_D c$ iff $g(a) :_{\langle\lambda\rangle D} l$; and we are done.

From this it follows that every $f: L \to typ(\boldsymbol{A})$ is a partial S-link from $\langle\lambda\rangle\boldsymbol{D}$ to $\boldsymbol{A}$. By Theorem 6.2, $\langle\lambda\rangle\boldsymbol{D}$ can have no non-trivial covers. QED

Corrollary 8.8 *If $\langle D, \lambda\rangle$ is a properly-labelled diagram, $\boldsymbol{D}$ is isoinfomorphic to $\langle\lambda\rangle\boldsymbol{D}$ and $\langle D, \lambda\rangle$ is a tautology iff D is a tautology.*
PROOF: By Lemma 8.5, if $\langle D, \lambda\rangle$ is a properly-labelled then $\langle\lambda, \text{id}(\square_D)\rangle$ is an isoinfomorphism. The rest follows from the Theorem 8.7 and the fact that S-links preserve covers. QED

Note that the corollary cannot be generalized to improperly-labelled diagrams. Figure 9 shows a tautologous labelled diagram which would cease to be tautologous were we to rub out the labels. We extend the notion of a diagram-homomorphism to labelled diagrams in the obvious way.

Definition 8.9 If $\langle D, \lambda\rangle$ and $\langle D', \lambda'\rangle$ are L-labelled diagrams, then f is a *labelled-diagram homomorphism* from $\langle D, \lambda\rangle$ to $\langle D', \lambda'\rangle$ iff f is a digram homomorphism from D to D' and $\lambda' f = \lambda$. A labelled-diagram homomorphism is a *labelled-diagram isomorphism* iff it is a bijection and its inverse is also a labelled-diagram homomorphism.

$\langle D'\lambda'\rangle$ is an *extension* of diagram $\langle D,\lambda\rangle$, written $\langle D',\lambda'\rangle \supseteq \langle D,\lambda\rangle$ iff $D' \supseteq D$ and λ is the restriction of λ' to $\bigcirc_D$. It is a *free extension*, written $\langle D',\lambda'\rangle \geq \langle D,\lambda\rangle$ iff, in addition, for any labelled diagram $\langle D'',\lambda''\rangle$ and any function $f{:}\bigcirc_{D'} \to \bigcirc_{D''}$ such that $\lambda'' f = \lambda'$, if the restriction of f to $\bigcirc_D$ is a homomorphism then so is f. D is *simple* iff it is a free extension only of itself.

Lemma 8.10 *If $\langle D',\lambda'\rangle$ is a free extension of $\langle D,\lambda\rangle$ then $\langle D,\lambda\rangle \models \langle D',\lambda'\rangle$*

PROOF: Left to the reader. QED

Theorem 8.11 *(For properly-labelled diagrams only.) $\langle D,\lambda\rangle \models \langle D',\lambda'\rangle$ iff there is a free extension $\langle D^*,\lambda^*\rangle$ of $\langle D,\lambda\rangle$ and a labelled-diagram homomorphism from $\langle D',\lambda'\rangle$ to $\langle D^*,\lambda^*\rangle$.*

PROOF: Suppose $\langle D,\lambda\rangle \models \langle D',\lambda'\rangle$. Let $L' = \mathrm{rng}(\lambda') - \mathrm{rng}(\lambda)$. Construct the free extension D^* of D by adding a new curve c_l for each $l \in L'$, according to Construction 7.5. Define $\lambda^*{:}\bigcirc_{D^*} \to L$ by

$$\lambda^*(c) = \begin{cases} \lambda(c) & \text{if} \quad c \in \bigcirc_D \\ l & \text{if} \quad c = c_l \text{ for some } l \in L' \end{cases}$$

This makes $\langle D^*,\lambda^*\rangle$ a free extension of $\langle D,\lambda\rangle$. From the proof of Theorem 8.6, we have $(\langle\lambda^*\rangle\boldsymbol{D}^*)^+ \models \langle D^*,\lambda^*\rangle$, and so $(\langle\lambda^*\rangle\boldsymbol{D}^*)^+ \models \langle D',\lambda'\rangle$, by hypothesis. Thus there is a g such that $\lambda', g{:}\,\boldsymbol{D}' \rightleftarrows (\langle\lambda^*\rangle\boldsymbol{D}^*)^+$. In fact, because $\mathrm{rng}(\lambda') \subseteq L' \cup \mathrm{rng}(\lambda) = \mathit{typ}(\langle\lambda^*\rangle\boldsymbol{D}^*)$, we have $\lambda', g{:}\,\boldsymbol{D}' \rightleftarrows \langle\lambda^*\rangle\boldsymbol{D}^*$. Now, we use the fact that $\langle D,\lambda\rangle$ and hence $\langle D^*,\lambda^*\rangle$ is properly-labelled: by Corollary 8.8, $\lambda^{*-1}, \mathrm{id}(\square_D){:}\,\langle\lambda^*\rangle\boldsymbol{D}^* \rightleftarrows \boldsymbol{D}^*$. By composition, $\lambda^{*-1}\lambda', g\mathrm{id}(\square_D){:}\,\boldsymbol{D}' \rightleftarrows \boldsymbol{D}^*$, and so $\lambda^{*-1}\lambda'$ is a homomorphism from $\langle D',\lambda'\rangle$ to $\langle D^*,\lambda^*\rangle$, as required.

Conversely, suppose we have a homomorphism f from $\langle D',\lambda'\rangle$ to a free extension $\langle D^*,\lambda^*\rangle$ of $\langle D,\lambda\rangle$. If $\boldsymbol{A} \models \langle D',\lambda'\rangle$ then by Lemma 8.10 $\boldsymbol{A} \models \langle D^*,\lambda^*\rangle$. So $\boldsymbol{A}, \lambda^* \models D^*$, and thus $\boldsymbol{A}, \lambda^* f \models D'$ by Lemma 5.7. Finally, $\lambda^* f = \lambda'$, and so $\boldsymbol{A} \models \langle D',\lambda'\rangle$, as required. QED

Definition 8.12 If $\langle D,\lambda\rangle$ is properly-labelled, we define $\mathrm{core}(\langle D,\lambda\rangle)$ to be the diagram $\mathrm{core}(D)$, labelled with the restriction of λ to $\bigcirc_{\mathrm{core}(D)}$.

Corollary 8.13 *(For properly-labelled diagrams only.) $\langle D,\lambda\rangle \models \langle D',\lambda'\rangle$ iff there is a homomorphism from $\mathrm{core}(\langle D',\lambda'\rangle)$ to $\langle D,\lambda\rangle$.*

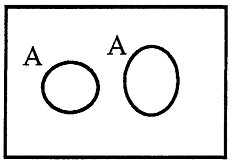
PROOF: From the theorem and Lemma 6.10. QED

Corollary 8.14 *(For properly-labelled diagrams only.) $\langle D,\lambda\rangle$ is logically equivalent to $\langle D',\lambda'\rangle$ iff $\mathrm{core}(\langle D,\lambda\rangle)$ is isomorphic to $\mathrm{core}(\langle D',\lambda'\rangle)$*

PROOF: From Corollary 8.13 and Lemma 6.11. QED

Theorem 8.11 and its corollaries do not apply to the case in which the premise is improperly labelled. A counterexample is shown in Figure 10. The diagram on the right is a consequence of the one on the left and both are simple, but there is no homomorphism from the right to the left.

We would like to extend Theorem 8.11 to a characterization of valid arguments with an arbitrary number of premises. Clearly it would be sufficient to find a way of combining diagrams $\langle D_1,\lambda_1\rangle$ and $\langle D_2,\lambda_2\rangle$ into a single diagram $\langle D_3\lambda_3\rangle$, such that



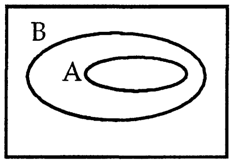


Figure 10: Improper consequence

for each $\langle D, \lambda \rangle$

$\langle D_3 \lambda_3 \rangle \models \langle D, \lambda \rangle$ iff $\langle D_1, \lambda_1 \rangle, \langle D_2, \lambda_2 \rangle \models \langle D, \lambda \rangle$

In fact, this can be done, but it is convenient to consider first a diagrammatic device which makes the construction of the combined diagram much easier: shading. But that would take us beyond the scope of this paper.

9 Depiction and Denotation

Our initial departure from the algebraic thoroughfare was motivated by syntactic considerations: the fact that syntactic equivalence between diagrams is best described using geometric concepts. But all roads lead to algebra, and we should say something about how to navigate the rest of the journey.

First, we should recall the role played by terms in algebra. It will suffice to restrict our attention to Boolean algebras, although the point is quite general. The Boolean algebras constitute a variety BA of algebras of type $\langle \wedge, \vee, \neg, 0, 1 \rangle$, which can be characterized either equationally, by the usual axioms, or as the class of algebras embeddable in a powerset algebra $\langle \mathcal{P}(S), \cap, \cup, -, \emptyset, S \rangle$, for some set S.

A *term-algebra* of type $\langle \wedge, \vee, \neg, 0, 1 \rangle$ is an algebra $\mathcal{T}[X]$ whose elements are terms built from the Boolean connectives with elements of X taken as atomic symbols, and whose operations are the corresponding *syntactic* functors, e.g., the function mapping terms t_1 and t_2 to the term '$(t_1 \wedge t_2)$'. The term-algebras play an essential role in algebra for various reasons.

First, terms are *segmentable*: they can be written down in a linear notation. This is an obvious point, but a very important one. So close is the concept of an abstract term to the concept of a concrete symbol that it is sometimes difficult to imagine one without the other.

Second, terms are *inductive*. To put it another way, terms wear their inductive structure on their sleeves. This plays an essential role in shaping the way we think about the manipulation of terms, when designing logical calculi, for example. Rules which are defined in terms of the inductive structure of terms will have a special character.

Third, terms are *free*: for any algebra A of the same type type as $\mathcal{T}[X]$, and any function $f: X \longrightarrow A$, there is a unique extension of f to a homomorphism $\hat{f}: \mathcal{T}[X] \longrightarrow \mathcal{A}$. This means that terms provide an extremely versatile means of representing the elements of another algebra; no structural information is built into a term, apart from its arity.

Fourth, terms *denote*. The sole representational role of a term is to denote an element of an algebra. On its own, a term does not make any claim about the element it denotes. To achieve go beyond denotation, one needs to combine terms in more complicated expressions, the simplest being the equations.

Equations provide the means by which the denotational powers of terms are turned into classificatory powers. An equation '$t_1 = t_2$' classifies functions $f: X \longrightarrow A$ into those that do and those that do not satisfy the condition: $\hat{f}(t_1) = \hat{f}(t_2)$.

Moreover, equations allow us to find a *free* Boolean algebra, and thereby characterize the variety BA, by purely algebraic means. We define a congruence '$\approx$'

on $\mathcal{T}[X]$ by $t_1 \approx t_2$ iff for each Boolean algebra B and each function $f: X \to B$, $\hat{f}(t_1) = \hat{f}(t_2)$. Now the quotient $\mathcal{T}[X]/\approx$ is still free with respect to the Boolean algebras—each $f: X \to B$ is uniquely extendable to a homomorphism $\hat{f}: \mathcal{T}[X]/\approx \to \mathcal{B}$—but unlike $\mathcal{T}[X]$ the quotient $\mathcal{T}[X]/\approx$ is itself a Boolean algebra.

Elements of the free Boolean Algebra $\mathcal{F}[X] = \mathcal{T}[X]/\approx$ are a perfect compromise between freedom and expressivity. All and only the equations which are satisfied by all Boolean algebras are satisfied by $\mathcal{F}[X]$. However, one important ingredient is gone: the wearing of structure on the sleeve. One cannot usually divine the structure of a congruence simply by looking at an equation; at least not in the same way that one can see the structure of a term.

By moving away from terms, we have shifted the balance from syntax to semantics—or so it would seem, if we were to make the mistake that congruence-classes of terms are the only way to represent the elements of $\mathcal{F}[X]$. However, there is another way. As mentioned earlier, the class of Boolean algebras can be characterized either using equations, or as subalgebras of concrete powerset algebras, also called "fields of sets". This result due to Birkhoff was improved by Stone, who made the remarkable discovery that the structure of these fields of sets can be specified *by entirely geometric means.*

To be a little more precise, Stone's Duality Theorem states that a Boolean algebra can be uniquely represented as a certain kind of topological space (now called a Stone space), in such a way that homomorphisms between Boolean algebras are also uniquely representable as *continuous functions* between the corresponding spaces (in the opposite direction).

The potential for diagrams would be apparent, even had they not been invented first. It only remains to fill in the details.

As was noted earlier, the essential structure of a basic diagram D is contained in the classification $\boldsymbol{D}_{\sim}$. This classification retains all the geometric structure of the diagram, without inessential details about the exact arrangement of curves in the plane. Such a classification can be used to generate both a Boolean algebra and its corresponding Stone space.

Definition 9.1 Given a classification $\boldsymbol{A}$, we let $\mathcal{B}(\boldsymbol{A})$ to be the Boolean set algebra of $\sim$-closed sets, and let $\mathcal{S}(\boldsymbol{A})$ be the topological space with points $tok(\boldsymbol{A})$ and whose open sets are the $\sim$-closed sets.

For a digram D, the space $\mathcal{S}(\boldsymbol{D}_{\sim})$ is (homeomorphic to) the Stone space of $\mathcal{B}(\boldsymbol{D}_{\sim})$. What's more, the duality between Boolean homomorphisms and continuous functions is reflected in the two components of an S-link.

Claim 9.2 *Given functions* $f: typ(\boldsymbol{D}_{\sim}) \to typ(\boldsymbol{D}'_{\sim})$ *and* $g: typ(\boldsymbol{D}'_{\sim}) \to typ(\boldsymbol{D}_{\sim})$, $f, g: \boldsymbol{D}'_{\sim} \rightleftarrows \boldsymbol{D}_{\sim}$ *iff*

(i) f *can be uniquely extended to a homomorphism* $\hat{f}: \mathcal{B}(\boldsymbol{D}_{\sim}) \to \mathcal{B}(\boldsymbol{D}'_{\sim})$

(ii) g *is a continuous function from* $\mathcal{S}(\boldsymbol{D}'_{\sim})$ *to* $\mathcal{S}(\boldsymbol{D}_{\sim})$, *and*

(iii) $\hat{f}$ *and* g *are Stone duals.*

The correspondence can be extended to much of what has gone before. In particular, D is a free diagram iff $\mathcal{B}(\boldsymbol{D}_{\sim})$ is a free Boolean algebra. The addition of labels makes the correspondence easier to state:

Claim 9.3 *If* $\langle D, \lambda \rangle$ *is a properly-labelled, free diagram, then* $\hat{\lambda}: \mathcal{B}(\boldsymbol{D}_{\sim}) \to \mathcal{F}[L]$ *is an embedding.*

This shows precisely how diagrams are analogous to terms: both provide a way of concretely representing finite information about the free Boolean algebra. It also shows how they differ: terms do it by denoting elements of the algebra and use them to define congruences; whereas diagrams do it by depicting a finite subalgebra.

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