institute for logic, language and computation

MAARTEN MARX, SZABOLCS MIKULÁS,
ISTVÁN NÉMETI[1]

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Institute for Logic, Language and Computation (ILLC)
University of Amsterdam
Plantage Muidergracht 24
NL-1018 TV Amsterdam
The Netherlands
e-mail: illc@fwi.uva.nl

1. Math. Inst., Hung. Acad. of Sci.

Taming Arrow Logic

MAARTEN MARX SZABOLCS MIKULÁS
Dept. of Math. and Comp., Univ. of Amsterdam

ISTVÁN NÉMETI
Math. Inst., Hung. Acad. of Sci.

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Abstract

In this paper, we introduce a general technology, called *taming*, for finding well-behaving versions of well-investigated logics. Further, we state completeness, decidability, definability and interpolation results for a multimodal logic, called *arrow logic*, with additional operators such as the *difference operator*, and *graded modalities*. Finally, we give a completeness proof for a strong version of arrow logic.

1 Taming

In this section, we argue that it is important to find nice (complete, decidable, etc.) versions of logics, and introduce a technology to achive this goal.

1.1 Why to tame?

There are interesting and well-investigated logics that do not behave in a nice way in some respects. Examples are the undecidability of classical first-order logic, *FOL*, and the incompleteness and undecidability of several versions of arrow logic, *AL*, cf. Def.2.1.

One may argue that some of these features are necessary, e.g., in *FOL* we can build up whole mathematics, so *FOL* must have a high complexity. However, *FOL* has several other applications when decidability would be a desirable property. For instance, [8] proposes relativized versions of *FOL* as "modal fragments" of classical logic. These relativized versions have nicer properties than *FOL* itself, cf. [20].

Our other example is arrow logic. *AL* as defined in [9] is intended to be the core of logical systems for reasoning about dynamic aspects of the subject matter of our thinking, e.g., properties of processes, actions and programs. Thus one of the basic intended areas for applications of *AL* is computer science. There decidability of *AL* is clearly a desirable property. The most interesting connective of *AL* is composition, and if it is an associative operator, then *AL* is undecidable. Moreover, any non-trivial extension or strengthening of associative *AL* is undecidable, cf. [3]. Thus it is natural to consider non-associative versions of *AL*. Indeed, most of them are decidable and complete, cf. below.

On the other hand, we would like the expressive power of the nice logic to be rather large. To achive this, one can strengthen the logic by introducing new connectives without losing the nice properties. Below we will give examples how to do this.

1.2 How to tame modal logics?

To answer this question we should understand what causes the undesirable properties of various modal logics.

Let $L(\mathsf{K})$ be a modal logic defined by a class K of Kripke frames. We can define a first-order language using the accessibility relations R_c of K as $n+1$-ary predicates for every n-ary modality c of L. We take all the substructures in the first-order model-theoretic sense of elements of K, cf. [11]. Then we get a class $\mathbf{Sub}\mathsf{K}$ of frames. We call the logic $L(\mathbf{Sub}\mathsf{K})$ the *core* of $L(\mathsf{K})$. If we consider the universal first-order theory of K, then it coincides with that of $\mathbf{Sub}\mathsf{K}$. And we get rid of the existential frame conditions of K, which may be useful, cf. below. See also [10].

There are several reasons to consider $L(\mathbf{Sub}\mathsf{K})$. First, $L(\mathbf{Sub}\mathsf{K})$ is relatively close to $L(\mathsf{K})$, since all universal frame conditions are preserved. Second, in many cases, the existential frame conditions are responsible for the unnice behaviour. So there is a chance that $L(\mathbf{Sub}\mathsf{K})$ has nicer properties than $L(\mathsf{K})$. Let us give two examples.

(1) Consider classical first-order logic as a modal logic, cf. [25] and [27]. Then the frame condition corresponding to the commutativity of the quantifiers $\exists v_i \exists v_j \varphi \rightarrow \exists v_j \exists v_i \varphi$ is

$$\forall x, y, z((\mathsf{R}_j x, z \ \&\ \mathsf{R}_i z, y) \Rightarrow \exists z'(\mathsf{R}_i x, z' \ \&\ \mathsf{R}_j z', y)).$$

In [20], it is argued that the above condition is the reason for undecidability of *FOL*. In [20] and [16] there are several decidable logics lacking this condition.

(2) Our other example is arrow logic. If we consider the associative version of AL, then the frames satisfy

$$\forall x, y, z, u, v((\mathsf{C}x, y, z \ \&\ \mathsf{C}y, u, v) \Rightarrow \exists w(\mathsf{C}x, u, w \ \&\ \mathsf{C}w, v, z))$$

corresponding to $(\varphi \bullet \psi) \bullet \chi \rightarrow \varphi \bullet (\psi \bullet \chi)$. Associativity is the reason for both undecidability and incompleteness of associative AL, cf. [1], [3] and Thm.3.2 below.

Thus $L(\mathbf{Sub}\mathsf{K})$ may be a nice logic. Moreover, if we consider the class $\mathsf{Alg}(L(\mathsf{K}))$ of algebras[1] corresponding to the logic $L(\mathsf{K})$, cf. [6] and [7], then we can get the class $\mathsf{Alg}(L(\mathbf{Sub}\mathsf{K}))$ by a well-known and well-investigated operation called *relativization*, cf. [13]. In many cases, this yields a class of algebras with nicer properties than the original class, reflecting the fact that $L(\mathbf{Sub}\mathsf{K})$ has nicer properties than $L(\mathsf{K})$ does.

Although this procedure described above may yield nicer logics, usually it is not satisfactory in itself. The situation is like taming a lion by pulling out all of its teeth. That is, $L(\mathbf{Sub}\mathsf{K})$ may be remarkably weaker than $L(\mathsf{K})$. For instance, there may be connectives that are not definable any more. The larger expressive power has obvious advantages. Beside that, the stronger logic may have nicer properties as well. For instance, the existence of the universal modality ensures that the logic has a deduction term, i.e., deduction theorem holds. See [24] for more detail and motivation for strenghtening. Thus, the problem of strengthening $L(\mathbf{Sub}\mathsf{K})$ naturally arises.

We will show two ways to do that. First, we may try to find a class K' of frames such that $\mathsf{K} \subset \mathsf{K}' \subset \mathbf{Sub}\mathsf{K}$ and $L(\mathsf{K}')$ still has nice properties. In this way we may get back some of the expressive power of $L(\mathsf{K})$. Example is pair arrow logic, cf. below.

The other way of strengthening is to introduce new connectives to the logic without losing the nice properties. Examples are the universal modality $\diamond$, the difference operator D, and the graded modalities $\langle n \rangle$ that can be added to PAL and to $PAL_{\{r,s\}}$ without losing decidability, cf. Thm.3.4 below.

[1]In case of modal logics, this class can be defined as the class of subalgebras of products of complex algebras.

2 Taming arrow logic

First we give the definitions of several versions of arrow logic. We will concentrate on its pair version and tame it. For more on arrow logic see [9], [16], [17] and [26].

Definition 2.1 Arrow logic, *AL, is defined as follows. Its connectives are the Booleans, the* identity *constant* id, *a unary connective* $\otimes$ *called* converse, *and a binary connective* $\bullet$ *called* composition. *The set of formulas is built up in the usual way using a denumerable set of parameters (or propositional variables).*

A structure $\langle W, \mathsf{C}, \mathsf{F}, \mathsf{I}\rangle$ *is called an an* arrow frame *if* W *is a non-empty set,* I *is a unary,* F *is a binary, and* C *is a ternary relation on* W. *An* arrow model *is an arrow frame together with a valuation* v *assigning value to atomic formulas.* Truth *of a formula* φ *at a world* w *in a model* $\langle W, \mathsf{C}, \mathsf{F}, \mathsf{I}, v\rangle$, *in symbols* $w\Vdash_v\varphi$, *is defined as follows:*

- $w\Vdash_v p \stackrel{\text{def}}{\Longleftrightarrow} w \in v(p)$ *for propositional variable* p,
- $w\Vdash_v \neg\varphi \stackrel{\text{def}}{\Longleftrightarrow}$ *not* $w\Vdash_v\varphi$,
- $w\Vdash_v \varphi\wedge\psi \stackrel{\text{def}}{\Longleftrightarrow} w\Vdash_v\varphi \;\&\; w\Vdash_v\psi$,
- $w\Vdash_v \varphi\bullet\psi \stackrel{\text{def}}{\Longleftrightarrow} (\exists w', w'' \in W)\mathsf{C}w, w', w'' \;\&\; w'\Vdash_v\varphi \;\&\; w''\Vdash_v\psi$,
- $w\Vdash_v \otimes\varphi \stackrel{\text{def}}{\Longleftrightarrow} (\exists w' \in W)\mathsf{F}w, w' \;\&\; w'\Vdash_v\varphi$,
- $w\Vdash_v \mathsf{id} \stackrel{\text{def}}{\Longleftrightarrow} \mathsf{I}w$.

Validity *of a formula in a model,* $\langle W, \mathsf{C}, \mathsf{F}, \mathsf{I}, v\rangle \models \varphi$, *and the* (global) semantical consequence relation, $\Gamma \models \varphi$, *are defined in the usual way:*

$$\langle W, \mathsf{C}, \mathsf{F}, \mathsf{I}, v\rangle \models \varphi \stackrel{\text{def}}{\Longleftrightarrow} (\forall w \in W) w\Vdash_v\varphi$$

and $\Gamma \models \varphi$ *iff in every model validating* Γ, φ *is valid.*[2]

Pair arrow logic *PAL is defined as follows. Its syntax is the same as that of AL. A structure* $\langle W, \mathsf{C}_W, \mathsf{F}_W, \mathsf{I}_W\rangle$ *is a* pair frame *if the following holds. The universe* W *is a binary relation* $W \subseteq U\times U$ *for some non-empty set* U, *called the* base *of the frame, and the accessibility relations* C_W, F_W, *and* I_W *are relational composition, relation converse, and identity restricted to* W. *That is,*

- $\mathsf{C}_W\langle x, x'\rangle, \langle y, y'\rangle, \langle z, z'\rangle \stackrel{\text{def}}{\Longleftrightarrow} x = y \;\&\; x' = z' \;\&\; y' = z$,
- $\mathsf{F}_W\langle x, x'\rangle, \langle y, y'\rangle \stackrel{\text{def}}{\Longleftrightarrow} x = y' \;\&\; x' = y$,
- $\mathsf{I}_W\langle x, x'\rangle \stackrel{\text{def}}{\Longleftrightarrow} x = x'$.

PAL_{sq} *denotes the* square version *of PAL, where the universes of the frames are Cartesian squares* $W = U\times U$.

Let s, r, t *abbreviate 'symmetry', 'reflexivity' and 'transitivity', respectively, and* $H \subseteq \{r, s, t\}$. *The logic* PAL_H *is defined as PAL with the following modification. A frame for* PAL_H *is a binary relation* W *satisfying the conditions in* H. *Thus for* $H = \emptyset$, $PAL_H = PAL$. *We will call these logics the* relativized versions *of arrow logic.*

[2]Note that the semantical consequence relation can be defined *locally*, i.e., using truth at a world instead of validity in a model as well.

Let us consider the strongest pair arrow logic PAL_{sq}. In the above definition of PAL_{sq}, we required that the universes of the frames are Cartesian squares. Transitivity of the universe ensures that composition is associative, i.e., the following is a valid formula:

$$(\varphi \bullet \psi) \bullet \chi \leftrightarrow \varphi \bullet (\psi \bullet \chi).$$

Associativity causes both incompleteness and undecidability of *PAL*, cf. [1] and [3]. Thus, to find nicer versions of *PAL* we should apply the "non-square approach", i.e., allow frames with non-square universes.

The core of PAL_{sq} is the completely relativized pair arrow logic *PAL*, cf. [16]. Since we got rid of the existential conditions by relativization, associativity does not hold in *PAL*. To see more clearly why associativity does not hold in (many of) the relativized versions of arrow logic, we introduce the following notation: let $[\![\varphi]\!] \stackrel{\text{def}}{=} \{w \in W : w \Vdash_v \varphi\}$, and for $X, Y \subseteq W$, $X \circ_W Y \stackrel{\text{def}}{=} \{\langle w, w'\rangle \in W : \exists w''(\langle w, w''\rangle \in X \;\&\; \langle w'', w'\rangle \in Y)\}$. Then, in a model for PAL_H with universe $W \subseteq U \times U$,

$$[\![\varphi \bullet \psi]\!] = [\![\varphi]\!] \circ_W [\![\psi]\!] = ([\![\varphi]\!] \circ_{U\times U} [\![\psi]\!]) \cap W.$$

That is, to get the meaning of composition in relativized models, we have to intersect the unrelativized meaning with the universe of the model. And there are non-transitive relations W such that $\langle w, w'\rangle \in W$, and for some x, $\langle w, x\rangle \in [\![\varphi]\!] \circ_W [\![\psi]\!]$ and $\langle x, w'\rangle \in [\![\chi]\!]$, while $[\![\psi]\!] \circ_W [\![\chi]\!] = \emptyset$. That is, $[\![(\varphi \bullet \psi) \bullet \chi]\!] \neq \emptyset = [\![\varphi \bullet (\psi \bullet \chi)]\!]$.

Although the relativized version *PAL* behave much nicer than the square version PAL_{sq}, cf. Thm.3.2, its expressive power is remarkably weaker. Applying the first technique of strengthening we may consider pair frames with reflexive and/or symmetric universes. Then we get the logics PAL_H ($H \subseteq \{r, s\}$). These logics still have the nice properties, cf. Thm.3.2.

In PAL_{sq} there are connectives that are not definable in PAL_H ($H \subseteq \{r, s\}$). Consider the universal modality $\Diamond$ interpreted as:

$$[\![\Diamond\varphi]\!] \stackrel{\text{def}}{=} \{w \in W : (\exists w' \in W) w' \in [\![\varphi]\!]\}.$$

In PAL_{sq}, $\Diamond\varphi$ can be defined as $\top \bullet \varphi \bullet \top$ while in PAL_H it is not definable, cf. [2]. The $\Diamond$ is really useful, since a deduction term is definable using $\Diamond$, cf. [24].

To strengthen our relativized logics, we may re-introduce connectives which were definable in the square logic, or we may even add new connectives. We will add the *difference operator* D to PAL_H, obtaining PAL_H^{diff}. The interpretation of D is:

$$[\![\mathsf{D}\varphi]\!] \stackrel{\text{def}}{=} \{w \in W : (\exists w' \in W) w \neq w' \;\&\; w' \in [\![\varphi]\!]\}.$$

Note that $\mathsf{D}\varphi$ is definable in PAL_{sq} by $(\top \bullet \varphi \bullet \neg\mathsf{id}) \vee (\neg\mathsf{id} \bullet \varphi \bullet \top)$, cf. [26], and that $\Diamond\varphi$ is definable by D: $\mathsf{D}\varphi \vee \varphi$. We will also add the so-called *graded modalities* $\langle n\rangle$ for $n \in \omega \setminus 1$:

$$[\![\langle n\rangle\varphi]\!] \stackrel{\text{def}}{=} \begin{cases} W & \text{if } |[\![\varphi]\!]| \geq n \\ \emptyset & \text{otherwise.} \end{cases}$$

We will denote this new logic by PAL_H^{grad}. Note that $\langle n\rangle$ is definable in PAL_{sq} iff $n \in \{1, 2, 3\}$, cf. [16], and that D and $\langle 1\rangle$, $\langle 2\rangle$ are definable by each other. For motivation concerning graded modalities, their application in computer science, epistemic and probability logic we refer to [14].

We will prove that adding D to $PAL_{\{r,s\}}$ does not ruin completeness, cf. Thm.3.3. The same holds for decidability, and we can even add all the $\langle n\rangle$ to PAL_H ($t \notin H$) without losing decidability, cf. Thm.3.4.

Let us conclude this section with some remarks.

Note that, for $n > 1$, $\langle n \rangle$ is not a modality in the following sense: it does not distribute over disjunction, i.e., the following is not a valid formula:

$$\langle n \rangle(\varphi \vee \psi) \leftrightarrow (\langle n \rangle \varphi \vee \langle n \rangle \psi).$$

However, we can add modalities $\Diamond_n$ to PAL_H such that the two logics become equivalent, cf. [5]. The interpretation of $\Diamond_n$ is:

$$[\![\Diamond_n(\varphi_0, \dots, \varphi_{n-1})]\!] \stackrel{\text{def}}{=} \begin{cases} W & \text{if } (\exists w_0, \dots, w_{n-1} \in W)(\forall i \in n) w_i \in [\![\varphi_i]\!] \;\&\; (\forall j \neq i) w_i \neq w_j \\ \emptyset & \text{otherwise.} \end{cases}$$

It is easy to see that $\Diamond_n$ distributes over disjunction in each of its argument.

As we mentioned above, there are relativized versions of FOL that behave nicely. We will not deal with this problem here. However, FOL_3^2, the 3 variable fragment of FOL with binary predicates is equivalent with PAL_{sq}, cf. [13]. Thus, whenever we obtain results about PAL_{sq}, these results apply to FOL_3^2 as well.

3 A Survey of Results

In this section we collect results available about relativized and strengthened versions of arrow logic.

First we define what we mean by a Hilbert-style inference system or calculus. For the other metalogical notions we refer to [6], [7], [11] and [16].

Definition 3.1 *The set of* formula schemes *is defined as the set of formulas using a set* A_n $(n \in \omega)$ *of* formula variables *instead of the set of parameters. An* instance *of a formula scheme is a formula given by uniformly substituting formulas for the formula variables.*

An inference rule *is*

$$\frac{B_1, \dots, B_n}{B_0}$$

with formula schemes $B_0, \dots, B_n$. *An* instance *of an inference rule is given by uniformly substituting formulas in the formula schemes occurring in the rule.*

By a Hilbert-style calculus *or* inference system, *usually denoted as* $\vdash$, *we mean a* finite *set of formula schemes, called* axiom schemes, *and inference rules.*

Let $\Gamma \cup \{\varphi\}$ *be a set of formulas. We say that* φ *is* derivable *from* Γ *by the calculus* $\vdash$, $\Gamma \vdash \varphi$, *if the following holds. There is a finite sequence of formulas* $\langle \varphi_0, \dots, \varphi_n \rangle$ *such that* φ_n *is* φ *and* $\forall i \in n$

- $\varphi_i \in \Gamma$ *or*
- φ_i *is an instance of an axiom scheme (an* axiom *for short) of* $\vdash$ *or*
- *there are* $j_0, \dots, j_k < i$ *and an inference rule of* $\vdash$ *such that* $\frac{\varphi_{j_0}, \dots, \varphi_{j_k}}{\varphi_i}$ *is an instance of this rule.*

Given a logic and its semantical consequence relation $\models$, *we say that* $\vdash$ *is* strongly sound, *and* strongly complete, *respectively, for* $\models$ *if the following holds:*

- $\Gamma \vdash \varphi \Rightarrow \Gamma \models \varphi$,
- $\Gamma \models \varphi \Rightarrow \Gamma \vdash \varphi$

for any set $\Gamma \cup \{\varphi\}$ of formulas. When Γ is empty, we use the adverb weakly *instead of strongly.*

Since the frames of $PAL_{\{r,s,t\}}$ are disjoint unions of frames of PAL_{sq}, the two logics are equivalent. Thus all the (negative) results about $PAL_{\{r,s,t\}}$ below apply to PAL_{sq} as well.

The various parts of the following theorem have been proven by H. Andréka, R. Kramer, R. Maddux, M. Marx and I. Németi. For precise reference we refer to [16].

Theorem 3.2 *Let $H \subseteq \{r,s,t\}$ be arbitrary. Then*

1. *PAL_H has a strongly sound and strongly complete Hilbert-style calculus iff $t \notin H$;*
2. *PAL_H is decidable iff $t \notin H$;*
3. *PAL_H has the Craig interpolation property iff $t \notin H$;*
4. *PAL_H has the Beth definability property iff $t \notin H$.*

We prove the following theorem in the next section. An algebraic proof can be found in [18].

Theorem 3.3 *For $H = \{r,s\}$, PAL_H^{diff} has a strongly sound and strongly complete Hilbert-style calculus.*

We conjecture that the above theorem remains true for any $H \subseteq \{r,s\}$. For the case $t \in H$, [1] gives a negative answer.

The next theorem is due to H. Andréka, Sz. Mikulás and I. Németi, its proof can be found in [4].

Theorem 3.4 *The graded logic PAL_H^{grad} and the difference logic PAL_H^{diff} are decidable iff $t \notin H$.*

Unfortunately, the Craig interpolation and Beth definability properties are not preserved after strengthening. M. Marx, I. Németi and I. Sain showed that PAL_H^{diff} and PAL_H^{grad} do not have the above two properties for any $H \subseteq \{r,s,t\}$, see [16].

4 Proof of the Completeness Thorem

Below, we will give an axiomatization for the class of reflexive and symmetric pair frames with the difference operator $\mathsf{KD}^{rel}_{setRS}$. From this, by a standard modal-logical argument, it follows that there is a strongly sound and strongly complete calculus for the arrow logic $PAL^{diff}_{\{r,s\}}$ defined by this class.

We will use the fact that the logic $PAL_{\{r,s\}}$ has a strongly sound and strongly complete calculus, cf. Thm.3.2. Let K^{rel}_{setRS} denote the class of symmetric and reflexive pair frames. The following set of formulas axiomatizes K^{rel}_{setRS}, cf. Prop.4.1 below.

$$
\begin{array}{ll}
(A_1) & ((\varphi \wedge \mathsf{id}) \bullet \psi) \bullet \chi \leftrightarrow (\varphi \wedge \mathsf{id}) \bullet (\psi \bullet \chi) \\
(A_2) & \varphi \bullet \mathsf{id} \leftrightarrow \varphi \\
(A_3) & \otimes\otimes\varphi \leftrightarrow \varphi \\
(A_4) & \otimes(\varphi \bullet \psi) \leftrightarrow \otimes\psi \bullet \otimes\varphi \\
(A_5) & \otimes\varphi \bullet \neg(\varphi \bullet \psi) \rightarrow \neg\psi
\end{array}
$$

Now we enumerate the frame conditions corresponding to the above arrow-logical formulas.

$$\begin{array}{ll}
(C_1) & \forall xyzv(\mathsf{C}xyz\&\mathsf{C}xvx\&\mathsf{I}v \iff \mathsf{C}xyz\&\mathsf{C}yvy\&\mathsf{I}v) \\
(C_2) & \forall x\exists y(\mathsf{I}z \ \& \ \mathsf{C}xxz) \\
(C_{2'}) & \forall xyz(\mathsf{C}xyz \ \& \ \mathsf{I}z \Rightarrow x = y) \\
(C_3) & \forall x\exists! y(\mathsf{F}xy \ \& \ \mathsf{F}yx) \\
(C_4) & \forall xyz(\exists w(\mathsf{F}xw \ \& \ \mathsf{C}wyz) \iff \exists y'z'(\mathsf{F}y'y \ \& \ \mathsf{F}z'z \ \& \ \mathsf{C}xz'y')) \\
(C_5) & \forall xyzv((\mathsf{C}xyz\&\mathsf{F}yv) \Rightarrow \mathsf{C}zvx)
\end{array}$$

Define $\mathsf{K}^{rel}_{rIRS} \stackrel{\text{def}}{=} \{\mathcal{F} = \langle W, \mathsf{C}, \mathsf{F}, \mathsf{I}\rangle : \mathcal{F} \models (C_1) - (C_5)\}$. For K a class of frames we use $\mathbf{Zig}\mathsf{K}$ to denote the class of all zigzagmorphic images of members of K, for the definition of zigzagmorphism (under the name of p-morphism) we refer to [12].

Proposition 4.1 $\mathsf{K}^{rel}_{rIRS} = \{\mathcal{F} = \langle W, \mathsf{C}, \mathsf{F}, \mathsf{I}\rangle : \mathcal{F} \models (A_1) - (A_5)\} = \mathbf{Zig}\mathsf{K}^{rel}_{setRS}$.

Proof: This follows from the proof of Theorem 5.20 in Maddux [15]. **qed**

Consequences. If an arrow frame satisfies conditions $(C_1) - (C_5)$ then there are three total functions living in this frame (cf. Prop.4.2 below). They are defined as follows.

$$\begin{array}{lcl}
\mathsf{f}x = y & \stackrel{\text{def}}{\iff} & \mathsf{F}xy \\
x_l = y & \stackrel{\text{def}}{\iff} & \mathsf{C}xyx \ \& \ \mathsf{I}y \\
x_r = y & \stackrel{\text{def}}{\iff} & \mathsf{C}xxy \ \& \ \mathsf{I}y
\end{array}$$

So, $\mathsf{f}x$ gives us the converse arrow of x, and the functions x_l and x_r (l for left and r for right) give us the left and the right "endpoints" of an arrow.

It is convenient to have explicit symbols in our language corresponding to the two defined functions. Define $\mathsf{s}_0^1\,\varphi \leftrightarrow (\mathsf{id} \wedge \varphi) \bullet \top$ and $\mathsf{s}_1^0\,\varphi \leftrightarrow \top \bullet (\mathsf{id} \wedge \varphi)$. Also define their conjugates dom and ran as follows: $\mathsf{dom}\varphi \leftrightarrow (\mathsf{id} \wedge (\varphi \bullet \top))$ and $\mathsf{ran}\varphi \leftrightarrow (\mathsf{id} \wedge (\top \bullet \varphi))$. Their meaning is given by the following equations. This is easy to see by writing out the definitions.

$$\begin{array}{rcl}
[\![\mathsf{s}_0^1\,\phi]\!] & = & \{x : x_l \in [\![\phi]\!]\} \\
[\![\mathsf{dom}\phi]\!] & = & \{x_l : x \in [\![\phi]\!]\} \\
[\![\mathsf{s}_1^0\,\phi]\!] & = & \{x : x_r \in [\![\phi]\!]\} \\
[\![\mathsf{ran}\phi]\!] & = & \{x_r : x \in [\![\phi]\!]\} \\
[\![\otimes\,\phi]\!] & = & \{x : \mathsf{f}x \in [\![\phi]\!]\}
\end{array}$$

Proposition 4.2 *Every arrow frame which satisfies the conditions $(C_1) - (C_5)$ also satisfies conditions $(T_0) - (T_5)$ below.*

$$\begin{array}{ll}
(T_0) & \mathsf{f},\ (.)_l \text{ and } (.)_r \text{ are total functions and } \mathsf{f} \text{ is idempotent} \\
(T_1) & \mathsf{I}x \Rightarrow x = \mathsf{f}(x) = x_l = x_r \\
(T_2) & x_l = (\mathsf{f}x)_r \text{ and } x_r = (\mathsf{f}x)_l \\
(T_3) & \mathsf{C}xyz \Rightarrow x_l = y_l \ \& \ y_r = z_l \ \& \ z_r = x_r \\
(T_4) & \forall xyzv(\mathsf{C}xyz\&\mathsf{F}zv \Rightarrow \mathsf{C}yxv) \\
(T_5) & \forall xyz(\mathsf{C}xyz \ \& \ \mathsf{I}x \Rightarrow \mathsf{F}zy)
\end{array}$$

Proof: Cf. [16]. **qed**

Axioms for D on pair–frames.

(C_6)	$\mathsf{R}xy \Rightarrow \mathsf{R}yx$	(A_6)	$\varphi \wedge \mathsf{D}\psi \rightarrow \mathsf{D}(\psi \wedge \mathsf{D}\varphi)$
(C_7)	$\mathsf{R}xy\ \&\ \mathsf{R}yz \Rightarrow x = z$ or $\mathsf{R}xz$	(A_7)	$\mathsf{DD}\varphi \rightarrow (\varphi \vee \mathsf{D}\varphi)$
(C_8)	$\mathsf{F}xy \Rightarrow (x = y$ or $\mathsf{R}xy)$	(A_8)	$\otimes\varphi \rightarrow \Diamond\varphi$
(C_9)	$\mathsf{C}xyz \Rightarrow (x = y$ or $\mathsf{R}xy)\ \&\ (x = z$ or $\mathsf{R}xz)$	(A_9)	$\varphi \bullet \psi \rightarrow \Diamond\varphi \wedge \Diamond\psi$
(C_{10})	$x_l = y_l\ \&\ y_r = z_l \Rightarrow \mathsf{C}xyz$ or $\mathsf{R}x_r z_r$	(A_{10})	$\mathsf{s}_0^1(\mathsf{dom}(\varphi \bullet \mathsf{dom}\psi)) \rightarrow \varphi \bullet \psi \vee \mathsf{s}_1^0\,\mathsf{Dran}\psi$
(C_{11})	$\mathsf{R}xy \iff \mathsf{R}x_l y_l$ or $\mathsf{R}x_r y_r$	(A_{11})	$\mathsf{D}\varphi \leftrightarrow \mathsf{s}_0^1\,\mathsf{Ddom}\varphi \vee \mathsf{s}_1^0\,\mathsf{Dran}\varphi$

Consequences. In the proof of the lemma below, we will use that the following theorems are derivable from conditions $(C_1) - (C_{10})$. Conditions (D_1) and (D_2) are just variants of (C_{10}). (D_3) and (D_4) express the fact that if $x_l = y_l$ and $x_r = y_r$ and one of the two pairs is R-irreflexive, then x equals y.

Proposition 4.3 *The following theorems follow from conditions* $(C_1) - (C_{10})$,

(D_1)	$\neg\mathsf{R}x_l y_l\ \&\ y_r = z_l\ \&\ z_r = x_r \Rightarrow \mathsf{C}xyz$	(D_3)	$x_l = y_l\ \&\ \neg\mathsf{R}x_r y_r \Rightarrow x = y$
(D_2)	$x_l = y_l\ \&\ \neg\mathsf{R}y_r z_l\ \&\ z_r = x_r \Rightarrow \mathsf{C}xyz$	(D_4)	$\neg\mathsf{R}x_l y_l\ \&\ x_r = y_r \Rightarrow x = y$

Proof: Use $(C_{10}), (C_5), (T_4)$ and (T_2) to show (D_1) and (D_2). Assume the antecedent of (D_3). Use (T_1) and (T_2) to derive that $(x_l)_l = x_l\ \&\ \neg\mathsf{R}x_r(\mathsf{f}y)_l\ \&\ (\mathsf{f}y)_r = (x_l)_r$. Then (D_2) implies that $\mathsf{C}x_l x\mathsf{f}y$ and (T_5) that $\mathsf{ff}y = x$. But then, idempotence of f implies $x = y$. (D_4) follows easily from (D_3). **qed**

Define the following class of arrow frames expanded with a binary relation R:

$$\mathsf{KD}^{rel}_{rlRS} \stackrel{\text{def}}{=} \{\mathcal{F} = \langle W, \mathsf{C}, \mathsf{f}, \mathsf{I}, \mathsf{R}\rangle : \mathcal{F} \models (C_1) - (C_{11})\}$$

where R interprets D. That is, in a model with valuation v,

$$x \Vdash_v \mathsf{D}\varphi \stackrel{\text{def}}{\iff} (\exists y \in W)\mathsf{R}xy\ \&\ y \Vdash_v \varphi.$$

Proposition 4.4 *The calculus defined by the* K*-axioms*[3], $(A_1) - (A_{11})$, *and the rules modus ponens and universal generalization is strongly sound and strongly complete with respect to the arrow logic defined by the class* KD^{rel}_{rlRS} *of frames.*

Proof: This follows from the Sahlqvist form of the axioms, cf. [16] and [25]. **qed**

The main lemma

Let (AU) denote the frame condition $(x \neq y \Rightarrow \mathsf{R}xy)$.

Lemma 4.5 (i). *Each* $\mathcal{F} \in \mathsf{KD}^{rel}_{rlRS}$ *consists of a disjoint union of frames satisfying* (AU). (ii). *Each* $\mathcal{F} \in \mathsf{KD}^{rel}_{rlRS}$ *which satisfies* (AU) *is a zigzagmorphic image of some* $\mathcal{G} \in \mathsf{KD}^{rel}_{setRS}$.

Proof: (i). Let $\mathcal{F} = \langle W, \mathsf{C}, \mathsf{f}, \mathsf{I}, \mathsf{R}\rangle \in \mathsf{KD}^{rel}_{rlRS}$. Define a binary relation $\equiv$ on W as follows

$$x \equiv y \stackrel{\text{def}}{\iff} x = y \vee \mathsf{R}xy$$

[3] This set of axioms includes enough propositional tautologies, and the formulas ensuring that the modalities distribute over disjunction.

Conditions (C_6) and (C_7) imply that $\equiv$ is an equivalence relation. We denote the equivalence class of x by $\bar{x} \stackrel{\text{def}}{=} \{y \in W : x \equiv y\}$. Define for each equivalence class a frame $\mathcal{F}_{\bar{x}} \stackrel{\text{def}}{=} \langle \bar{x}, \mathsf{C}', \mathsf{f}', \mathsf{I}', \mathsf{R}' \rangle$ such that the relations are the restrictions to $\bar{x}$. We claim that each $\mathcal{F}_{\bar{x}} \models (AU)$ and $\mathcal{F}$ is a disjoint union of the system of frames $\{\mathcal{F}_{\bar{x}} : x \in F\}$, by which we prove part (i) of the lemma. The first part of the claim is immediate, for the second it suffices to show that each $\mathcal{F}_{\bar{x}}$ is a subframe of $\mathcal{F}$ generated by $\bar{x}$, which is precisely the point of conditions (C_8) and (C_9).

(ii). The proof of part (ii) consists of two steps, corresponding to the two things which can go wrong with the accessibility relation of the difference operator. First we show that $\mathcal{F}$ is a zigzagmorphic image of a pair frame expanded with a relation R which satisfies (AU). In the second step we make the R relation irreflexive, thereby turning it into the inequality relation. These two steps are given in the schema below.

		STEP I		**STEP II**	
full language	$\mathcal{F} \in \mathsf{KD}^{rel}_{rIRS}$	$\stackrel{l}{\longleftarrow}$	$\mathcal{G}_{pair}(V) \in \mathsf{KD}^{rel}_{rIRS}$	$\stackrel{p}{\longleftarrow}$	$\mathcal{H}_{pair}(H) \in \mathsf{KD}^{rel}_{setRS}$
	$\vdots$		$\vdots$		
D-free reduct	$\mathcal{F}^* \in \mathsf{K}^{rel}_{rIRS}$	$\stackrel{l^*}{\longleftarrow}$	$\mathcal{G}^*_{pair}(V^*) \in \mathsf{K}^{rel}_{setRS}$		

Let $\mathcal{F} = \langle W, \mathsf{C}, \mathsf{f}, \mathsf{I}, \mathsf{R} \rangle \in \mathsf{KD}^{rel}_{rIRS}$ satisfy (AU). By Proposition 4.1 we may assume that the R-free reduct $\mathcal{F}^*$ of $\mathcal{F}$ is a zigzagmorphic image, say by function l^*, of a pair frame $\mathcal{G}^*_{pair}(V^*) = \langle V^*, \mathsf{C}_{V^*}, \mathsf{f}_{V^*}, \mathsf{I}_{V^*} \rangle$ for some reflexive and symmetric relation V^* with base U^*.

STEP I The problem with the representation $\mathcal{G}^*_{pair}(V^*)$ is that it may contain two different pairs x and y which get mapped to the same point in $\mathcal{F}$ which is not R reflexive. This will prevent extending the zigzagmorphism l^* to one for R as well. We will create a new pair frame $\mathcal{G}_{pair}(V)$ where this problem is eliminated.

Define an equivalence relation $\equiv$ on the base U^* as follows:

$$(\forall u, v \in U^*) : u \equiv v \stackrel{\text{def}}{\Longleftrightarrow} u = v \text{ or } \neg \mathsf{R} l^*\langle u, u \rangle, l^*\langle v, v \rangle$$

Claim 1 (i). $\equiv$ is an equivalence relation.
(ii). $u \equiv v \Rightarrow l^*\langle u, u \rangle = l^*\langle v, v \rangle$

Proof of Claim: Immediate because $\mathcal{F} \models (AU)$. QED

Define

$$\begin{aligned} U &\stackrel{\text{def}}{=} U^*/\equiv \\ V &\stackrel{\text{def}}{=} \{\langle u/\equiv, v/\equiv \rangle \in U \times U : \langle u, v \rangle \in V^*\} \end{aligned}$$

Define a function $l : V \longrightarrow F$ as $l\langle u/\equiv, v/\equiv \rangle \stackrel{\text{def}}{=} l^*\langle u', v' \rangle$ for some $u' \in u/\equiv$ and $v' \in v/\equiv$. Note that, by the definition of V, for every pair $\langle u/\equiv, v/\equiv \rangle \in V$ there exist a pair $\langle u', v' \rangle \in V^*$ such that $u \equiv u'$ and $v \equiv v'$. Hence l is defined for every element in V. The next claim states that this is a real definition.

Claim 2 l is well defined, i.e., for every $\langle u, v \rangle, \langle u', v' \rangle \in V^*$, if $u \equiv u'$ and $v \equiv v'$ then $l^*\langle u, v \rangle = l^*\langle u', v' \rangle$.

PROOF OF CLAIM: Suppose $\langle u,v\rangle, \langle u',v'\rangle \in V^*$ and $u \equiv u'$ and $v \equiv v'$. We have four cases, according to whether $u = u'$ and $v = v'$. If $u = u'$ and $v = v'$ the statement is trivial. So assume otherwise:

Case 2: $[u \neq u' \;\&\; v \neq v']$. Then the definition of $\equiv$ implies that $\neg \mathsf{R} l^*\langle u,u\rangle, l^*\langle u',u'\rangle$ and $\neg \mathsf{R} l^*\langle v,v\rangle l^*\langle v',v'\rangle$. Since l^* is a zigzagmorphism for the relational operators, this means that we have $\neg \mathsf{R}(l^*\langle u,v\rangle)_l, (l^*\langle u',v'\rangle)_l$ and $\neg \mathsf{R}(l^*\langle u,v\rangle)_r, (l^*\langle u',v'\rangle)_r$. Then condition (C_{11}) implies that $\neg \mathsf{R} l^*\langle u,v\rangle, l^*\langle u',v'\rangle$, so by (AU), $l^*\langle u,v\rangle = l^*\langle u',v'\rangle$.

Case 3 and 4: $[u = u' \;\&\; \neg \mathsf{R} l^*\langle v,v\rangle, l^*\langle v',v'\rangle]$ and $[\neg \mathsf{R} l^*\langle u,u\rangle, l^*\langle u',u'\rangle \;\&\; v = v']$. These cases are solved in a similar way, but now using conditions (D_3) and (D_4). QED

To finish the first step of the proof, define an accessibility relation R_V on the pair frame $\mathcal{G}_{pair}(V)$ as $\mathsf{R}_V xy \overset{\text{def}}{\iff} \mathsf{R} l(x) l(y)$. Call this frame $\mathcal{G} = \langle V, \mathsf{C}_V, \mathsf{f}_V, \mathsf{I}_V, \mathsf{R}_V\rangle$, where $\mathsf{C}_V, \mathsf{f}_V$ and I_V are relational composition, converse and identity restricted to V, respectively. The next claim states that we have accomplished our first goal.

Claim 3 (i). V is a reflexive and symmetric relation.
(ii). $\mathcal{G} \models x \neq y \Rightarrow \mathsf{R}_V xy$
(iii). The function l is a zigzagmorphism from $\mathcal{G}$ onto the frame $\mathcal{F}$.

PROOF OF CLAIM: (i). Obvious.
(ii). We will denote $u/\!\equiv$ by $\bar{u}$. Suppose $\neg \mathsf{R}_V \langle \bar{u},\bar{v}\rangle, \langle \bar{u'},\bar{v'}\rangle$ for some $\langle \bar{u},\bar{v}\rangle, \langle \bar{u'},\bar{v'}\rangle \in V$. We have to show that $\bar{u} = \bar{u'}$ and $\bar{v} = \bar{v'}$. We compute:

$\neg \mathsf{R}_V \langle \bar{u},\bar{v}\rangle, \langle \bar{u'},\bar{v'}\rangle$	$\iff$	(using well-definedness of l)
$\neg \mathsf{R} l^*\langle u,v\rangle, l^*\langle u',v'\rangle$	$\overset{(C_{11})}{\iff}$	(using that l^* is a zigzagmorphism)
$\neg \mathsf{R} l^*\langle u,u\rangle, l^*\langle u',u'\rangle \;\&\; \neg \mathsf{R} l^*\langle v,v\rangle, l^*\langle v',v'\rangle$	$\Rightarrow$	(definition of $\equiv$)
$\bar{u} = \bar{u'}$ and $\bar{v} = \bar{v'}$		

(iii). All steps in this proof, except homomorphism of C_V, are straightforward, since equivalent pairs are mapped to the same place, cf. Claim 2. We show that l is a homomorphism for C_V. Suppose $\langle \bar{u},\bar{v}\rangle, \langle \bar{u},\bar{w}\rangle, \langle \bar{w},\bar{v}\rangle \in V$. We have to show that $\mathsf{C} l\langle \bar{u},\bar{v}\rangle, l\langle \bar{u},\bar{w}\rangle, l\langle \bar{w},\bar{v}\rangle$ holds. By definition of V, we have $u,u',v,v',w,w' \in U^*$, $\{\langle u,v\rangle, \langle u',w\rangle, \langle w',v'\rangle\} \subseteq V^*$ and $u \equiv u', w \equiv w'$ and $v \equiv v'$. By the definition of l it is sufficient to show that $\mathsf{C} l^*\langle u,v\rangle, l^*\langle u',w\rangle, l^*\langle w',v'\rangle$ holds.

There are several cases, depending on why the points are equivalent. One easy case is this. If $u = u', w = w'$ and $v = v'$, then, since l^* is a homomorphism, we have $\mathsf{C} l^*\langle u,v\rangle, l^*\langle u,w\rangle, l^*\langle w,v\rangle$. In all other cases, for at least one of the three pairs of equivalent points, the reflexive pairs at those points are mapped to an R irreflexive arrow. For these cases, we need condition (C_{10}) and the fact that $\mathcal{F} \models (AU)$. The next claim helps out.

Claim 4 If $\mathcal{F} \in \mathsf{KD}^{rel}_{rlRS}$ and $\mathcal{F} \models (AU)$ then

$$\mathcal{F} \models \quad [x_l = y_l \;\&\; y_r = z_l \;\&\; z_r = x_r \;\&\; (\neg \mathsf{R} x_l y_l \vee \neg \mathsf{R} y_r z_l \vee \neg \mathsf{R} z_r x_r)] \Rightarrow \mathsf{C} xyz$$

In words: if $x_l = y_l \;\&\; y_r = z_l \;\&\; z_r = x_r$ and at least one of the pairs $\langle x_l, y_l\rangle, \langle y_r z_l\rangle, \langle z_r, x_r\rangle$ is R irreflexive, then x can be decomposed into y and z.

PROOF OF CLAIM: We have shown already two cases: (D_1) and (D_2). Use (AU) to prove all other possibilities from $(C_{10}), (D_1)$ and (D_2). QED

We show with an example how this claim helps us. Suppose $\neg\mathsf{R}l^*\langle u,u\rangle l^*\langle u',u'\rangle$ and $w = w'$ and $v = v'$. Because l^* is a zigzagmorphism, we have $l^*\langle u,u\rangle = (l^*\langle u,v\rangle)_l$, and similarly for the others. This implies that

$$\neg\mathsf{R}(l^*\langle u,v\rangle)_l(l^*\langle u',w\rangle)_l \ \&\ (l^*\langle u',w\rangle)_r = (l^*\langle w',v'\rangle)_l \ \&\ (l^*\langle w',v'\rangle)_r = (l^*\langle u,v\rangle)_r$$

So by the above claim, $\mathsf{C}l^*\langle u,v\rangle, l^*\langle u',w\rangle, l^*\langle w',v'\rangle$. Hence also $\mathsf{C}l\langle\bar{u},\bar{v}\rangle, l\langle\bar{u},\bar{w}\rangle, l\langle\bar{w},\bar{v}\rangle$, which is what we had to prove. QED

STEP II Since the frame $\mathcal{G}$ constructed in the previous step is a pair frame, we only have to make sure that R_V becomes the inequality. Since $\mathcal{G} \models (AU)$, it suffices to make the R_V relation *irreflexive*. Define the following two sets:

$$\begin{array}{lcl} BAD & \stackrel{\text{def}}{=} & \{u \in U : \mathsf{R}_V\langle u,u\rangle\,\langle u,u\rangle\ \} \\ COPIES & \stackrel{\text{def}}{=} & \{\langle u',u'\rangle : u \in BAD\} \cup \{\langle v,u'\rangle, \langle u',v\rangle : \langle u,v\rangle \in V, u \in BAD \ \&\ u \neq v\} \end{array}$$

Without loss of generality we may assume that $COPIES$ is disjoint from V. Let

$$\mathcal{H} = \langle H, \mathsf{C}_H, \mathsf{f}_H, \mathsf{I}_H, \neq\rangle \in \mathsf{KD}^{rel}_{setRS}$$

be given by the set H, defined as

$$H \stackrel{\text{def}}{=} V \cup COPIES$$

and C_H, f_H, and I_H be composition, converse, and identity restricted to H. Define a function $p : H \longrightarrow V$ as the unique function such that

- p restricted to V is the identity function
- $p(\langle u',u'\rangle) \stackrel{\text{def}}{=} \langle u,u\rangle$ if $u \in BAD$
- $p(\langle u',v\rangle) \stackrel{\text{def}}{=} \langle u,v\rangle$ and $p(\langle v,u'\rangle) \stackrel{\text{def}}{=} \langle v,u\rangle$ if $u \neq v$ and $u \in BAD$.

The next claim states that for R_V we did enough, That is, we *only* copied R_V reflexive arrows.

Claim 5 (i). $(\forall x \in V) : (\mathsf{R}_V xx \iff$ there exists a copy of x in $COPIES)$.
(ii). $(\forall x, y \in H) : ((x \neq y \ \&\ p(x) = p(y)) \Rightarrow \mathsf{R}_V p(x)p(y))$

PROOF OF CLAIM: (i). Suppose $\mathsf{R}_V\langle u,v\rangle\langle u,v\rangle$ for some $\langle u,v\rangle \in V$. If $u = v$ then the claim holds by definition. So, suppose $u \neq v$. Then:

$$\begin{array}{l} \mathsf{R}_V\langle u,v\rangle\langle u,v\rangle \stackrel{(C_{11})}{\iff} \\ \mathsf{R}_V\langle u,u\rangle\langle u,u\rangle \text{ or } \mathsf{R}_V\langle v,v\rangle\langle v,v\rangle \iff \\ u \in BAD \text{ or } v \in BAD \iff \\ \langle u',v\rangle \in COPIES \text{ or } \langle u,v'\rangle \in COPIES. \end{array}$$

(ii) Follows from (i), since two pairs of H can only be mapped to the same pair in V if they are copies of each other. QED

Claim 6 p is a zigzagmorphism from $\mathcal{H}$ onto $\mathcal{G}$.

PROOF OF CLAIM: Clearly p is surjective. That p is a zigzagmorphism for R_V is immediate by Claim 5. For I and f this is straightforward to check. For C observe that, if $\{\langle u,v\rangle, \langle u,w\rangle, \langle w,v\rangle\} \subseteq H$, then either they all are in V, or one pair is in V and the other two are in $COPIES$. QED

With these two steps we have finished the proof, because our original frame $\mathcal{F}$ will be a zigzagmorphic image of the frame $\mathcal{H}$ by the function given by the composition of l and p. **qed**

Proof of Theorem 3.3: It is an easy consequence of Prop.4.4 and Lemma 4.5.

In more detail: Soundeness of the calculus defined in the formulation of Prop.4.4 is easy to check. Now assume that $\Gamma \not\vdash \varphi$. For completeness, we have to prove that $\Gamma \not\models \varphi$. By Prop.4.4, there is a frame $\mathcal{F} \in \mathsf{KD}^{rel}_{rIRS}$, a valuation v and a world w such that $\langle \mathcal{F}, v\rangle \models \Gamma$ and $w \Vdash_v \neg\varphi$. Take the subframe $\mathcal{F}'$ of $\mathcal{F}$ generated by w. By Lemma 4.5, $\mathcal{F}'$ is a zigzagmorphic image of a pair frame $\mathcal{G} \in \mathsf{KD}^{rel}_{setRS}$. Let this zigzagmorphism be denoted by l. Let the valuation v' on $\mathcal{G}$ be defined as:

$$v'(p) = \{w' : l(w') \in v(p)\}$$

for every propositional variable p. Then the model $\langle \mathcal{F}', v\rangle$ is the zigzagmorphic image of the model $\langle \mathcal{G}, v'\rangle$. Thus, for every world w' of $\mathcal{G}$ and formula ψ, $w' \Vdash_{v'} \psi$ iff $l(w') \Vdash_v \psi$. This means that $\langle \mathcal{G}, v'\rangle \models \Gamma$. On the other hand, let w' be the pre-image of w, i.e., $l(w') = w$. Then $w' \Vdash_{v'} \neg\varphi$ by $w \Vdash_v \neg\varphi$. That is, we found a pair model witnessing $\Gamma \not\models \varphi$. **qed**

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