Cut Might Cautiously*

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Abstract

This note is on cautious cut elimination for one of Veltman's might logics. Syntactically, the logic is presented as an extension of a sequent system for classical propositional logic (hence: CPL). I show that this extension preserves the completeness and decidability of CPL. The proof has cautious cut elimination as a corollary. I also give a rather general syntactic proof of cautious cut elimination. It states that any 'base' logic which has a reflexive, monotone consequence relation that allows cautious cut to be eliminated preserves cautious cut elimination when extended to a might logic.

1 Introduction

The last decade has shown a growing concern with aspects of language interpretation which do not fit nicely within the truth conditional paradigm. This led to the idea that the focus should be on the context change potential of a sentence rather than on its truth conditions. Well-known systems that embody this philosophy are the semantics for anaphora, and the dynamic logics for studying computer programs. In this article I investigate a logic for the phrase 'it might be', which developed within this tradition. It is

*This paper is inspired by a talk of Frank Veltman on completeness theorems for might logics. The sections 2–4 elaborate on his notes, now published as Groeneveld and Veltman 1994 (section 3). I would like to thank him, Tim Fernando, and Willem Groeneveld for discussions. An earlier version of the paper was read at the workshop on Proof Theory and Natural Language (SOAS, London) organized by Ruth Kempson. The remarks from the audience were helpful. This work is part of the PIONIER project 'Reasoning with Uncertainty' (NWO grant PGS–22–262).

introduced by Veltman (1991) as a first step towards more complex systems to handle defaults.

Besides more formal concerns, Veltman tries to explain why (1a) is an acceptable continuation of (1) while (1b) is not.

(1) Somebody is knocking at the door...
   a. It might be John... It is Mary.
   b. *It is Mary... It might be John.

The example shows that the acceptability of sentences with 'might' depends on the place at which they occur within a text.[1] There are different ways in which the phenomenon can be analyzed, but the logic designed by Veltman does so by:

- interpreting sentences as update functions over a set of information states;
- introducing a notion of logical consequence which is sensitive to the order of the premisses.

As a result of this set up 'might' can be seen as a kind of metalogical devise: it tests whether the formula within its scope can be consistently added to the preceding discourse. If so the information carried by that discourse is preserved unaltered, but otherwise the 'might' sentence yields the discourse inconsistent.

Formally, the structure sensitivity of a consequence relation can be mirrored by a sequent system which uses, e.g., 'cautious' versions of cut and monotony. Groeneveld and Veltman 1994 introduce such sequent systems for several might logics, and show them to be sound and complete with respect to what I call concrete models. Their proofs use cautious cut, so the question arises whether this rule can be eliminated. In this article I concentrate on the Update-Test might logic, and give a semantic as well as a syntactic proof of cautious cut elimination. To this end I introduce a new sequent system, which basically adds two rules for the might operator to a sequent system for CPL. The semantic proof of cautious cut elimination is obtained as a corollary to a completeness proof for the system relative to abstract models for the might logic. Syntactically the result follows from the fact that any 'base' logic that has a reflexive, monotone consequence

[1] Tense is another important factor: in case of 'it might have been' (1b) is acceptable while (1a) is not.

relation and which allows cautious cut to be eliminated preserves cautious cut elimination when extended to a might logic.

Some of the results presented here are also proved by Van Eijck and De Vries (1992) by means of a Hoare logic, and a translation into S5.

2 Syntax and Semantics

In this section I define the syntax and semantics of **M**, a propositional logic with formulas of the form '$\mathrm{M}\varphi$' corresponding to 'it might be the case that φ'. The semantics of **M** is much like that of the Update-Test might logic in Groeneveld and Veltman 1994. But the notion of model used is more general.

2.1 Syntax

The syntax of **M** is defined on top of a standard propositional language over a set of propositional letters $\mathcal{P} := \{\mathrm{p}_1, \ldots, \mathrm{p}_\mathrm{n}, \ldots\}$. It discerns $\mathcal{L}_0$ and $\mathcal{L}_1$ formulas in order to preclude iterations of the might-operator.

Definition 2.1 (syntax)

$\mathcal{L}_0 \quad F_0 ::= p_\mathrm{i} \mid \neg F_0 \mid (F_0 \wedge F_0)$

$\mathcal{L}_1 \quad F_1 ::= F_0 \mid \mathrm{M}F_0$

Semantically 'might' is interpreted as an operator which tests for consistency, a metaproperty. So the fact that M only occurs as an outermost operator corresponds to a strict division between the object language $\mathcal{L}_0$ and the metalanguage $\mathcal{L}_1$. Below, formulas of the form $\mathrm{M}\varphi$ are called M-formulas.

2.2 Semantics

The semantics of **M** specifies the update function associated with a formula, not its truth conditions. More precisely, a formula φ denotes a function from information states to information states. We therefore have to stipulate, among other things, which information structures will be used.

Definition 2.2 (information structures, models, updates)
An information structure is a Boolean algebra (hence: BA) together with a family of operators $F_i : I^n \longrightarrow I$*:*[2]

$\mathcal{I} := \langle I, ^\mathrm{c}, \wedge, \top, F_i \rangle$

[2] Cf. Van Benthem 1991c and Kanazawa 1994a,b for a more general relational notion of information structure. Also, I use basic facts concerning BA's without much notice.

A model is an information structure $\mathcal{I} := \langle I, ^{c}, \wedge, \top, \mu, [\![p]\!]^{\mathcal{I}} \rangle_{p \in \mathcal{P}}$ *with:*

i) $\mu : I \times I \longrightarrow I$ *so that:*[3]

$$\mu(i,j) = \begin{cases} i & \text{if } i \wedge j \neq \bot \\ \bot & \text{otherwise.} \end{cases}$$

ii) $[\![p]\!]^{\mathcal{I}} : I \longrightarrow I$, *for each* $p \in \mathcal{P}$, *so that:*

a.	$i[\![p]\!] \subseteq i$	*introspective*
b.	*If* $i \subseteq j$ *then* $i[\![p]\!] \subseteq j[\![p]\!]$	*monotone*
c.	*If* $i \subseteq j[\![p]\!]$ *then* $i \subseteq i[\![p]\!]$	*stable*

Here and elsewhere I omit the superscript $^{\mathcal{I}}$ *when no confusion is likely. The argument is placed before the function so as not to disturb the order among the formulas in case of the sequences introduced below.*

With each formula φ *we associate an update function* $[\varphi]^{\mathcal{I}} : I \longrightarrow I$ *as follows.*

$$\begin{array}{lll} \text{a.} & i[p] & = i[\![p]\!] \\ \text{b.} & i[\neg\varphi] & = i - i[\varphi] \\ \text{c.} & i[\varphi \wedge \psi] & = i[\varphi] \wedge i[\psi] \\ \text{d.} & i[\mathrm{M}\varphi] & = \mu(i, i[\varphi]) \end{array}$$

Finally, a sequence of formulas $\sigma_1 \ldots \sigma_n$ *is associated with an update function:* $[\sigma_1 \ldots \sigma_n]^{\mathcal{I}} : I \longrightarrow I$ *by means of the following induction:*

$$i[\sigma_1 \ldots \sigma_n]^{\mathcal{I}} = (i[\sigma_1 \ldots \sigma_{n-1}]^{\mathcal{I}})[\sigma_n]^{\mathcal{I}}$$

Although the semantics sustains different notions of consequence, we confine ourselves to the following:

Definition 2.3 (logical consequence)
Let $\mathcal{I}$ *be a model and let* Π, τ *be a sequence of* $\mathcal{L}_1$*-formulas.*

$$\Pi \models^{\mathcal{I}} \tau \text{ iff: } i[\Pi, \tau] = i[\Pi] \text{ for each } i \in \mathcal{I}$$

We say that τ *is a consequence of* Π—$\Pi \models \tau$—*iff* $\Pi \models^{\mathcal{I}} \tau$ *for each model* $\mathcal{I}$.

[3] A more abstract interpretation of μ treats it as a functional of type $\mathrm{I}^{\mathrm{I}} \times \mathrm{I} \longrightarrow \mathrm{I}$ with:

$$\mu(\mathrm{F}, i) = \begin{cases} i & \text{if } \mathrm{F}(i) \neq \bot \\ \bot & \text{otherwise} \end{cases}$$

Cf. Kanazawa 1994b for a partial alternative without $\bot$.

Some natural questions concerning the consequence relation are: What are its structural properties? And how does it relate to classical consequence? That the present relation is sensitive to the order of the premisses is already shown by (2–3) indicating that permutation no longer holds. But it lacks an unconditional form of monotonicity too. For instance, due to idempotency $p \models p$ is true in each model. But not so for $p, \mathrm{M}\neg p \models p$. Therefore, left-monotonicity fails for the sequential variants. Right-monotonicity, on the other hand, is valid for $\models$. Cf. Veltman 1991, Groeneveld and Veltman 1994.

Before proving **M** to be sound, complete and decidable, I discuss some basic facts of the system.

2.2.1 Basic Facts

Perhaps the most prominent feature of definition 2.2 is that all formulas are interpreted as operators on a BA. This move to a higher level enables a uniform definition of interpretation. But $\mathcal{L}_0$ formulas could also be interpreted as elements of a BA, as they normally are. Proposition 2.4, adapted from Veltman 1991, shows that the value of $[\varphi]$ at an information state, φ in $\mathcal{L}_0$, can be defined in terms of the element $\top[\varphi]$ (cf. also Van Benthem 1989).

Proposition 2.4
For all $\varphi \in \mathcal{L}_0$ and all information states i: $i[\varphi] = i \wedge \top[\varphi]$.

Proof Induction on the structure of φ. For the atomic case we use the fact that $[p]$ is introspective, monotone, and stable. □

As a corollary to proposition 2.4 we see to what extent the constraints on $[\![p]\!]$ are preserved under the definition of $[\varphi]$:

Corollary 2.5
For all formulas φ, $[\varphi]$ is introspective and monotone. It is stable in case φ is $\mathcal{L}_0$. □

A further consequence of proposition 2.4 is that $[\varphi]$ is idempotent: $i[\varphi][\varphi] = i[\varphi]$, for all φ and all $i \in \mathcal{I}$. Proposition 2.4 does not hold for M-formulas. But observe that in that case $i[\mathrm{M}\varphi] = i$ if $i[\varphi] \neq \bot$, while $i[\mathrm{M}\varphi] = \bot$ otherwise; due to introspectivity.

Proposition 2.6 collects some useful properties concerning sequences of formulas.

Proposition 2.6
For all sequences Π, Π', *and all* $i \in \mathcal{I}$*:*

i) $\bot[\Pi]^{\mathcal{I}} = \bot$
ii) $i[\Pi, \Pi']^{\mathcal{I}} = (i[\Pi]^{\mathcal{I}})[\Pi']^{\mathcal{I}}$
iii) $i[\Pi, \sigma, \Pi'] \subseteq i[\Pi, \Pi']$
iv) $i[\Pi] \subseteq i[\Pi_0]$
v) $i[\Pi] \neq \bot$ *iff* $i[\Pi^*] = i[\Pi_0^*] \neq \bot$ *for each initial segment* Π^* *of* Π.

Π_0 *refers to the sequence of* $\mathcal{L}_0$*-formulas which results from erasing the M-formulas in* Π.

Proof We only prove (v). One direction is clear, so assume $i[\Pi] \neq \bot$. First note that no initial segment Π^* has $i[\Pi^*] = \bot$. For otherwise $i[\Pi] = \bot$ by (i–ii). As to the remaining claim, we discern two cases in an induction on the length of Π^*.

- $\Pi^* \equiv \Lambda, \varphi$ with $\varphi \in \mathcal{L}_0$. Then: $i[\Lambda, \varphi] =_{\text{i.h.}} i[\Lambda_0, \varphi] = i[(\Lambda, \varphi)_0]$.
- $\Pi^* \equiv \Lambda, \mathrm{M}\varphi$. It is an immediate consequence of $i[\Lambda, \varphi] = \bot$ that $i[\Pi^*] = \bot$, which we know to be impossible. So $i[\Lambda, \varphi] \neq \bot$, and therefore $i[\Lambda, \mathrm{M}\varphi] = i[\Lambda] =_{\text{i.h.}} i[\Lambda_0] = i[(\Lambda, \mathrm{M}\varphi)_0]$. □

We finish this section with a comparison of the models introduced here with the concrete models defined by Veltman (1991).

2.2.2 Concrete Models

Veltman considers concrete models, which are based on the following assumptions:

- a world is a finite set of proposition letters (the atomic facts that hold true in it).
- an information state is a set of worlds (the worlds compatible with this information).
- a model should contain as many information states as possible.

More in particular, Veltman uses models of the form:

$$\langle \wp\wp(\mathcal{P}), {}^c, \cap, \wp(\mathcal{P}), \mu, [\![p_i]\!] \rangle_{p \in \mathcal{P}}$$

with $\mathcal{P}$ a finite set of proposition letters, and $[\![p]\!]$ defined by:

$$i[\![p]\!] = i \cap \{j \in \wp(\mathcal{P}) : p \in j\}$$

It is almost immediate that $[\![p]\!]$ is introspective, monotone, and stable. This means that concrete models are a special case of the models given by definition 2.2. By abuse of notation I denote these models by $\mathcal{P}$, and allow $\mathcal{P}$ to be infinite.

Concrete models have the great advantage of turning an $\mathcal{L}_0$-semantics into one for $\mathcal{L}_1$-sentences. Since a world in an information state is equivalent to an valuation $m : \mathcal{P} \longrightarrow \{0,1\}$, the concrete models are built by taking the power of the set of models for CPL. Given the $\mathcal{L}_0$-models, a concrete model contains all possible information states which can be obtained from them. By contrast, definition 2.2 allows models of this kind to consists of a field over a *subset* of the set of all $\mathcal{L}_0$-models.[4] Proposition 2.7 has some properties of updating concrete information states with a sequence Γ of $\mathcal{L}_0$-formulas in terms of their models.

Proposition 2.7
Let i be a state in $\mathcal{P}$. Set $m(\Gamma) = m(\bigwedge \Gamma)$, m a valuation. We have:

i) $\top[\Gamma] = \{m \in \mathcal{P} \longrightarrow \{0,1\} : m(\Gamma) = 1\}$
ii) $i[\Gamma] = \{m \in i : m(\Gamma) = 1\}$
iii) $i[\Gamma] = i$ *iff* $i \subseteq \top[\Gamma]$ *iff for all* $m \in i : m(\Gamma) = 1$
iv) $\top[\Gamma] = \bot$ *iff* $\Gamma \vdash_{\mathrm{cpl}}$

PROOF Immediate from proposition 2.4, and the completeness theorem for classical propositional logic. □

In case $i[\varphi] = i$, i *accepts* φ (Veltman 1991, 1). Proposition 2.7 (iii) shows that acceptance generalizes truth: i accepts φ iff φ is valid in i. In a sense, this is dual to $i \neq \bot$ accepting $\mathrm{M}\varphi$. For this is so iff there is an $m \in i$ which makes φ true.

As a simple application of proposition 2.7, we describe the difference between (1a,b). Consider p = 'somebody is knocking on the door', q = 'it is John', and r = 'it is Mary'. Since both p and q, and p and r are consistent with each other, there are valuations m_1 and m_2 with $m_1(r) \neq m_2(r) = 1$ so that:

(2) $\{m_1, m_2\}[p, \mathrm{M}q, r] = \{m_1, m_2\}[p, r] = \{m_2\}$

On the other hand, r is inconsistent with the sequence p, q. Therefore:

(3) $i[p, q, \mathrm{M}r] = \bot$

[4] A field is a non-empty set of sets which is closed under intersection and complementation.

for each i. (Of course, the argument assumes p to be about just one person.)

The above facts prove useful in establishing completeness and decidability for **M**, which is the topic of section 4. But first I give the sequent system M.

3 The System M

System M combines a sequent system for the object-language $\mathcal{L}_0$ with one for the meta-language $\mathcal{L}_1$.[5] More in particular, M consists of:

- classical logical rules for the constants $\neg$ and $\wedge$;
- two logical rules for the 'might' operator;
- monotonicity, contraction, and permutation for $\mathcal{L}_0$-sequents;
- reflexivity and cautious cut for $\mathcal{L}_1$-sequents.

Conventions The letters $\varphi,\ \psi,\ \chi, \ldots$ vary over $\mathcal{L}_0$-formulas, and $\Delta,\ \Gamma, \ldots$ over finite, possibly empty sequences of $\mathcal{L}_0$-formulas. $\mathcal{L}_1$-formulas are denoted by $\sigma,\ \tau,\ \mu, \ldots$, and finite, possibly empty sequences of such formulas by $\Pi,\ \Lambda, \ldots$. The letters may carry sub- or superscripts. The set $\textsc{prop}(\Pi)$ consists of the proposition letters used to built Π. Π_0 refers to the sequence of $\mathcal{L}_0$-formulas, which results from erasing the M-formulas in Π. A sequent is a pair $\langle \Pi, \sigma \rangle$. The sequent $\langle \Pi, \tau \rangle$ is derivable iff $\Pi \vdash \tau$ can be derived from instances of the axioms and the rules of M.

The classical part

The classical part is restricted to $\mathcal{L}_0$-sequents.

Logical rules

$$\frac{\Gamma, \varphi_{\mathrm{i}} \vdash \chi}{\Gamma, \varphi_1 \wedge \varphi_2 \vdash \chi}\ \mathrm{L}^{\mathrm{i}}_{\wedge} \qquad \frac{\Gamma \vdash \varphi_1 \quad \Gamma \vdash \varphi_2}{\Gamma \vdash \varphi_1 \wedge \varphi_2}\ \mathrm{R}_{\wedge}$$

$$\frac{\Gamma \vdash \varphi}{\Gamma, \neg\varphi \vdash}\ \mathrm{L}_{\neg} \qquad \frac{\Gamma, \varphi \vdash}{\Gamma \vdash \neg\varphi}\ \mathrm{R}_{\neg}$$

$$\frac{\Gamma, \varphi \vdash \chi}{\Gamma, \neg\neg\varphi \vdash \chi}\ \mathrm{L}_{\neg\neg}$$

[5] At present, the combination of logics is studied by several people, e.g., by Gabbay using his LDS framework and his fibring semantics.

Structural rules

$$\frac{\Gamma, \Delta \vdash \chi}{\Gamma, \varphi, \Delta \vdash \chi}\ \text{mon} \quad \frac{\Gamma, \varphi, \varphi, \Delta \vdash \chi}{\Gamma, \varphi, \Delta \vdash \chi}\ \text{contr} \quad \frac{\Gamma, \varphi, \psi, \Delta \vdash \chi}{\Gamma, \psi, \varphi, \Delta \vdash \chi}\ \text{perm}$$

The $\mathcal{L}_1$-part

Logical rules

$$\frac{\Pi, \Lambda \vdash \tau}{\Pi, \mathrm{M}\varphi, \Lambda \vdash \tau}\ \mathrm{L_m} \quad \frac{\Delta, \varphi \vdash \chi\ (\chi \textit{ in } \Gamma, \psi)}{\Delta, \mathrm{M}\varphi, \Gamma \vdash \mathrm{M}\psi}\ \mathrm{M}$$

Structural rules

$$\frac{}{\sigma \vdash \sigma}\ \text{refl} \quad \frac{\Pi \vdash \sigma \quad \Pi, \sigma, \Lambda \vdash \tau}{\Pi, \Lambda \vdash \tau}\ \text{cautious cut}$$

With a view to natural language semantics it seems less than ideal to distinguish between levels of language. But logically it is proficient. For example, the completeness and decidability results below directly extend well-known facts concerning classical logic.

Observe that in the context of the classical structural rules cautious cut, referred to as 'ccut', is equivalent to the familiar cut rule.

$$\frac{\Gamma \vdash \varphi \quad \Delta, \varphi, \Delta' \vdash \psi}{\Delta, \Gamma, \Delta' \vdash \psi}\ \text{cut}$$

To be precise, monotonicity and ccut imply cut, while contraction, permutation and cut imply ccut. This means that M_0, i.e., M restricted to $\mathcal{L}_0$-sequents, is a sound and complete sequent calculus for classical logic.

Fact 3.1
$\Gamma \vdash_{\mathrm{m}_0} \varphi \Leftrightarrow \Gamma \models_{\mathrm{cpl}} \varphi$

Fact 3.2
$\Gamma \vdash_{\mathrm{cpl}} \varphi \Rightarrow \Gamma \vdash_{\mathrm{m}} \varphi$

As to the $\mathcal{L}_1$-part, rule M allows φ, Γ, and ψ to be empty. Therefore, both rules in (4) are instances of M.

(4) $$\frac{\Delta, \varphi \vdash \psi}{\Delta, \mathrm{M}\varphi \vdash \mathrm{M}\psi} \quad \frac{\Delta \vdash \chi\ (\chi \textit{ in } \Gamma, \psi)}{\Delta, \Gamma \vdash \mathrm{M}\psi}$$

The first rule shows that M generalizes the familiar normality rule.

Proposition 3.3 (soundness)
The system M is sound: if $\Pi \vdash \tau$ *then* $\Pi \models \tau$.

Proof By way of example, we prove the soundness of rule M. So assume $\Delta, \varphi \models^{\mathcal{I}} \gamma$ for each γ in Γ, ψ. Pick $i \in \mathcal{I}$. In case $i[\Delta, \varphi] = \bot$ it is clear that $i[\Delta, \mathrm{M}\varphi, \Gamma, \mathrm{M}\psi] = i[\Delta, \mathrm{M}\varphi, \Gamma]$. So let $i[\Delta, \varphi] \neq \bot$. By assumption $i[\Delta, \varphi] = i[\Delta, \varphi, \Gamma, \psi]$. So $i[\Delta, \varphi, \Gamma, \psi] \neq \bot$ and hence $i[\Delta, \Gamma, \psi] \neq \bot$. Therefore (*) $i[\Delta, \Gamma, \mathrm{M}\psi] = i[\Delta, \Gamma]$. And since $i[\Delta, \varphi] \neq \bot$, also (**) $i[\Delta, \mathrm{M}\varphi] = i[\Delta]$. But (*) and (**) imply $i[\Delta, \mathrm{M}\varphi, \Gamma, \mathrm{M}\psi] = i[\Delta, \mathrm{M}\varphi, \Gamma]$. The choice of i was arbitrary, so: $\Delta, \mathrm{M}\varphi, \Gamma \models^{\mathcal{I}} \mathrm{M}\psi$. □

Corollary 3.4 *The relations* $\models_{\mathrm{cpl}}$ *and* $\models$ *coincide on* $\mathcal{L}_0$*-formulas.*

Proof If $\Gamma \models^{\mathcal{I}} \varphi$ for all $\mathcal{I}$, then in particular $\Gamma \models^{\mathcal{P}} \varphi$, for the concrete model $\mathcal{P}$. From this, $\Gamma \models_{\mathrm{cpl}} \varphi$ follows by proposition 2.7 (iii). Conversely: $\Gamma \models_{\mathrm{cpl}} \varphi$, (completeness) $\Gamma \vdash_{\mathrm{cpl}} \varphi$, (fact 3.2) $\Gamma \vdash_{\mathrm{m}} \varphi$, (soundness) $\Gamma \models \varphi$. □

Corollary 3.5
The system M is conservative over CPL: *If* $\Gamma \vdash_{\mathrm{m}} \varphi$ *then* $\Gamma \vdash_{\mathrm{cpl}} \varphi$*, for* Γ *and* φ *in* $\mathcal{L}_0$. □

4 Completeness and Decidability

Here I prove that **M** is complete and decidable for $\models$ along the lines of Groeneveld and Veltman 1994. The present completeness proof is mainly semantical, and does not use ccut. So we obtain ccut elimination as a corollary. A consistency lemma is proved first. It uses the formula $\sigma^{\bullet}$, which is σ with its initial M erased.

Lemma 4.1 (consistency lemma)
If $\Pi \not\vdash \tau$*, then* $\top^{\mathcal{P}}[\neg \bigwedge(\Pi_0, \tau^{\bullet}), \Pi'] \neq \bot$*, for each initial segment* Π' *of* Π *and each* $\mathcal{P} \supseteq$ PROP(Π, τ).

Proof Let $\mathcal{P}$ be a model of the relevant kind and set $i = \top^{\mathcal{P}}[\neg \bigwedge(\Pi_0, \tau^{\bullet})]$. We use induction on the length of $\Pi' \equiv \Lambda, \sigma$.

Observe that it is sufficient to prove $i[\Lambda, \sigma^{\bullet}] \neq \bot$. For if $\sigma^{\bullet} \equiv \sigma$ we are done. Whereas if $\sigma \equiv \mathrm{M}\varphi$ we have: $i[\Pi'] = i[\Lambda, \mathrm{M}\varphi] = i[\Lambda] \neq_{\mathrm{i.h.}} \bot$. So assume $i[\Lambda, \sigma^\bullet] = \bot$. By the induction hypothesis and proposition 2.6 (v): $i[\Lambda_0, \sigma^\bullet] = \bot$. According to proposition 2.7 and the definition of i:

$$\neg \bigwedge(\Pi_0, \tau^\bullet), \Lambda_0, \sigma^\bullet \vdash_{\mathrm{cpl}}$$

So by classical reasoning and fact 3.2: $\Lambda_0, \sigma^\bullet \vdash \bigwedge(\Pi_0, \tau^\bullet)$. Let $\Pi \equiv \Pi', \Pi''$. $\mathrm{R}_\wedge$ yields: $\Lambda_0, \sigma^\bullet \vdash \gamma$ for each γ in $\Pi_0'', \tau^\bullet$. Rule M: $\Lambda_0, \sigma, \Pi_0'' \vdash \tau$. (To be precise: if $\sigma \in \mathcal{L}_0$, φ in M is assumed to be empty, and similarly for τ.) $\mathrm{L_m}$ proves: $\Pi \vdash \tau$, a contradiction. □

Theorem 4.2 (completeness)
If $\Pi \models \tau$ *then* $\Pi \vdash \tau$.

Proof Assume $\Pi \not\vdash \tau$, and let $\mathcal{P} \supseteq \mathrm{PROP}(\Pi, \tau)$. Set $i = \top^{\mathcal{P}}[\neg \bigwedge(\Pi_0, \tau^\bullet)]$. By the consistency lemma and proposition 2.6 (v):

$$i[\Pi] = i[\Pi_0] = \top[\neg\tau^\bullet, \Pi_0] \neq \bot$$

So: $i[\Pi, \tau^\bullet] = \top[\neg\tau^\bullet, \Pi_0, \tau^\bullet] = \bot$. Whether or not $\tau^\bullet = \tau$, we get $i[\Pi] \neq \bot = i[\Pi, \tau]$, and therefore: $\Pi \not\models^{\mathcal{P}} \tau$. □

A check of the above proofs gives some corollaries.

Corollary 4.3
M *is sound and complete with respect to the model* $\mathcal{P}$*, with* $\mathcal{P}$ *the proposition letters used to generate the formulas.* □

Corollary 4.4
Let $\langle \Pi, \tau \rangle$ *be a sequent and* $\mathcal{P} = \mathrm{PROP}(\Pi, \tau) : \Pi \vdash \tau$ *iff* $\Pi \models^{\mathcal{P}} \tau$. □

Corollary 4.5 (decidability)
The logic **M** *is decidable.*

Proof In order to check whether or not $\Pi \vdash \tau$ it suffices to search the finitely many states of $\mathrm{PROP}(\Pi, \tau)$. As soon as a countermodel is found we know $\Pi \not\vdash \tau$, but otherwise: $\Pi \vdash \tau$. □

Corollary 4.6 (ccut elimination)
The cautious cut rule can be eliminated from M. □

PROOF If at all, cautious cut is only used in the classical part, where it is eliminable. □

I have been careful in presenting M as an extension of classical propositional logic, and the same can be done for the other might logics in Groeneveld and Veltman 1994. But to what extent does this approach generalize to other 'base' logics? That is, could we frame the above result as a preservation result of the form: for each complete $\mathcal{L}_0$-logic of a certain kind, there is a might logic which is complete as well (and similarly for other properties). The main point seems to be to find a generalization of the concrete models. I leave this question open. A similar question can be asked with respect to cautious cut elimination. But here we need not bother about semantical issues since the result can be proved syntactically. The next section has such a preservation result.

5 Cautious Cut Elimination

In this section we forget about set-theoretic interpretations and confine ourselves to syntactic methods. In particular we shall prove the following theorem.

Theorem 5.1
Let $\vdash_0$ be a consequence relation for $\mathcal{L}_0$-sequents which is reflexive, and closed under monotony and cautious cut. Extend the language to an $\mathcal{L}_1$-language as in definition 2.1, and extend $\vdash_0$ to $\vdash_1$ for the $\mathcal{L}_1$-language by closure under the rules M, L_{m}, and cautious cut. If $\vdash_0$ has cautious cut elimination, then so has $\vdash_1$.

Note that we need not assume reflexivity for $\mathcal{L}_1$-sequents, since it can be derived by means of M. This is handy, for it means that in the ccut-free variant of M reflexivity need not be considered as 'might' introducing.

PROOF OF THEOREM 5.1 As in case of $\vdash_{\mathrm{m}}$ the relation $\vdash_0$ will contain logical and structural rules for $\mathcal{L}_0$-sequents. But the use of these rules is blocked after an application of M or $\mathrm{L_m}$. This means that if ccut is applied to ccut-free premisses that part of a derivation will have the following structure:

$$\dfrac{\dfrac{\dfrac{\Delta, \varphi \vdash_0 \gamma \ (\gamma \text{ in } \Gamma, \sigma^\bullet)}{\Delta, \mathrm{M}\varphi, \Gamma \vdash \sigma}\ (\mathrm{M})}{\Pi \vdash \sigma}\ (\mathrm{L_m})^* \qquad \dfrac{\dfrac{\Delta', \psi \vdash_0 \chi \ (\chi \text{ in } \Gamma', \tau^\bullet)}{\Delta', \mathrm{M}\psi, \Gamma' \vdash \tau}\ (\mathrm{M})}{\Pi, \sigma, \Lambda \vdash \tau}\ (\mathrm{L_m})^*}{\Pi, \Lambda \vdash \tau}\ \text{ccut}$$

Here, (M) indicates that M is applied at most once, and $(\mathrm{L_m})^*$ that $\mathrm{L_m}$ is used finitely many (possibly zero) times. Given this general form we prove ccut-elimination as follows.

Let $\mathcal{D}$ be a derivation in M. If $\mathcal{D}$ is ccut-free we are done. Otherwise select an occurrence of ccut with cutt-free premisses. If this occurrence lies within the $\mathcal{L}_0$-part of $\mathcal{D}$ we know by assumption how to eliminate it. But if the ccut is applied to $\mathcal{L}_1$-sequents we discern four cases.

Case I: There are no applications of M above the cut. Then, the situation is:

$$\dfrac{\dfrac{\Pi_0 \vdash_0 \varphi}{\Pi \vdash \varphi}\,(\mathrm{L_m})^* \quad \dfrac{\Pi_0, \varphi, \Lambda_0 \vdash_0 \psi}{\Pi, \varphi, \Lambda \vdash \psi}\,(\mathrm{L_m})^*}{\Pi, \Lambda \vdash \psi}\,\mathrm{ccut}$$

This can be reduced to:

$$\dfrac{\dfrac{\Pi_0 \vdash_0 \varphi \quad \Pi_0, \varphi, \Lambda_0 \vdash_0 \psi}{\Pi_0, \Lambda_0 \vdash \psi}\,\mathrm{ccut}}{\Pi, \Lambda \vdash \psi}\,(\mathrm{L_m})^*$$

Here ccut occurs in the $\mathcal{L}_0$-part and is hence eliminable.

Case II: In deriving the right-hand side premiss of the ccut, M is applied once. The situation is:

$$\dfrac{\dfrac{\Pi_0 \vdash_0 \chi}{\Pi \vdash \chi}\,(\mathrm{L_m})^* \quad \dfrac{\dfrac{\Pi_0' \vdash_0 \gamma\ (\gamma \text{ in } \Lambda_0', \tau^\bullet)}{\Pi_0', \mathrm{M}\varphi, \Lambda_0' \vdash \tau}\,\mathrm{M}}{\Pi, \chi, \Lambda \vdash \tau}\,(\mathrm{L_m})^*}{\Pi, \Lambda \vdash \tau}\,\mathrm{ccut}$$

We discern two subcases. When χ occurs in Λ_0' we obtain a ccut-free derivation from the left premiss by deleting this occurrence. But when χ occurs in Π_0' we have $\Pi_0' \equiv \Pi_0, \chi, \Pi_0''$ for some Π_0'' (and hence $\Lambda_0 \equiv \Pi_0'', \Lambda_0'$). Then the above can be reduced to:

$$\dfrac{\dfrac{\dfrac{\Pi_0 \vdash_0 \chi \quad \Pi_0, \chi, \Pi_0'' \vdash_0 \gamma\ (\gamma \text{ in } \Lambda_0', \tau^\bullet)}{\Pi_0, \Pi_0'', \varphi \vdash \gamma\ (\gamma \text{ in } \Lambda_0', \tau^\bullet)}\,\mathrm{ccut}}{\Pi_0, \Pi_0'', \mathrm{M}\varphi, \Lambda_0' \vdash \tau}\,\mathrm{M}}{\Pi, \Lambda \vdash \tau}\,(\mathrm{L_m})^*$$

Again, these ccuts are eliminable by assumption.

Case III: M is used once in deriving the left-hand side premiss of the ccut. This case is trivial, for in the right premiss the ccut formula comes from L_m.

Case IV: The derivations of both premisses contain an application of M. Again the trivial reduction of the previous case may apply, but the situation may also be more interesting:

$$\dfrac{\dfrac{\dfrac{\Pi_0', \varphi \vdash_0 \gamma \ (\gamma \text{ in } \Pi_0'', \psi)}{\Pi_0', \mathrm{M}\varphi, \Pi_0'' \vdash \mathrm{M}\psi}\,(\mathrm{M})}{\Pi', \mathrm{M}\varphi, \Pi'' \vdash \mathrm{M}\psi}\,(\mathrm{L_m})^* \qquad \dfrac{\dfrac{\Pi_0', \Pi_0'', \psi \vdash_0 \chi \ (\chi \text{ in } \Lambda_0, \tau^\bullet)}{\Pi_0', \Pi_0'', \mathrm{M}\psi, \Lambda_0 \vdash \tau}\,(\mathrm{M})}{\Pi', \mathrm{M}\varphi, \Pi'', \mathrm{M}\psi, \Lambda \vdash \tau}\,(\mathrm{L_m})^*}{\Pi', \mathrm{M}\varphi, \Pi'', \Lambda \vdash \tau}\,\text{ccut}$$

This reduces to:

$$\dfrac{\dfrac{\dfrac{\Pi_0', \varphi \vdash_0 \chi \ (\chi \text{ in } \Pi_0'', \psi) \qquad \dfrac{\Pi_0', \Pi_0'', \psi \vdash_0 \chi \ (\chi \text{ in } \Lambda_0, \tau^\bullet)}{\Pi_0', \varphi, \Pi_0'', \psi \vdash_0 \chi \ (\chi \text{ in } \Lambda_0, \tau^\bullet)}\,\text{mon}}{\Pi_0', \varphi \vdash_0 \chi \ (\chi \text{ in } \Pi_0'', \Lambda_0, \tau^\bullet)}\,\text{ccut}}{\Pi_0', \mathrm{M}\varphi, \Pi_0'', \Lambda \vdash \tau}\,\mathrm{M}}{\Pi', \mathrm{M}\varphi, \Pi'', \Lambda \vdash \tau}\,(\mathrm{L_m})^*$$

This completes the proof of theorem 5.1. □

Note that in reducing the ccut to $\mathcal{L}_0$-sequents the might part of the proof grows at most $n+1$ steps, where n is the length of Π_0'', ψ in the last reduction. The other reductions shorten or do not alter the length of the proof.

If $\vdash_0$ in theorem 5.1 is decidable, $\vdash_1$ can be shown to be decidable too. Except for ccut, the rules of M satisfy the subformula property. So the following algorithm to check whether or not $\Pi \vdash_1 \tau$ is recursive:

i) If $\tau \in \mathcal{L}_0$ check whether $\Pi_0 \vdash_0 \tau$.
ii) If $\tau \equiv \mathrm{M}\psi$ check whether $\Pi_0', \varphi \vdash_0 \gamma$ for each γ in Π_0'', ψ, and each partition $\Pi \equiv \Pi', \mathrm{M}\varphi, \Pi''$ (φ, Π_0'', or ψ may be empty).

By assumption $\vdash_0$ is decidable, so the recipe defines a finite search space with all possible initial sequents to introduce the M-formulas in Π, τ. Therefore, $\Pi \vdash_1 \tau$ iff the algorithm finds a derivable $\mathcal{L}_0$-sequent from which $\Pi \vdash_1 \tau$ can be derived. In particular, since $\vdash_{cpl}$ is decidable this argument gives a syntactic proof of corollary 4.5.

6 Further Issues

In this section I name two topics for further study. Firstly, one would like to obtain similar results for formulas with nested occurrences of the might-operator (cf. Van Eijck and De Vries 1992). As in Veltman 1991 such nestings are not allowed here, since the reflexivity axiom would then be lost. E.g., the formula $\mathrm{M}p \wedge \neg p$ is not reflexive. One way to go would be to assume that reflexivity only holds for proposition letters, and to argue that the formulas which do not preserve this property are somehow inadmissible. For instance, the example given corresponds to the unacceptable sentence: it might be p and it isn't p. Secondly, one may wonder about the minimal algebraic structure for the $\mathcal{L}_0$-part. For instance, do we retain completeness and decidability if we generalize the structures to those of the form $\langle I, \wedge, \perp \rangle$ with $\wedge$ associative and idempotent, and $\perp$ a left and right neutral element? Kanazawa 1994b has some results in this direction for a partial version of 'might'.

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