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There exist exactly two Maximal Strictly Relevant Extensions of the Relevant Logic R^*

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There exist exactly two maximal strictly relevant extensions of the relevant logic R*

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Abstract

In [60] N. Belnap presented an 8-element matrix for the relevant logic R with the following property: if in a implication $A \to B$ the formulas A and B do not have a common variable then there exists a valuation v such that $v(A \to B)$ does not belong to the set of designated elements of this matrix. Below we present a 6-element matrix with the same properties and prove that the logics generated by these two matrices are all maximal extensions of the relevant logic R which have the *relevance property*: if $A \to B$ is provable in such a logic then A and B have a common propositional variable.

1 Preliminaries. C_R-matrices.

Let a set of propositional variables $p, q, r, \ldots$ be given and let F be the set of propositional formulae built up from propositional variables by means of the connectives: $\to$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction) and $\neg$ (negation). The Anderson and Belnap logic R with relevant implication (cf. A. R. Anderson, N. D. Belnap [75]) is defined as the subset of propositional formulae of F which are provable from the set of axiom schemes indicated below, by application of the rule of Modus Ponens (MP; $A, A \to B/B$) and the Rule of Adjunction ($A, B/A \wedge B$):

$A1.$ $A \to A$
$A2.$ $(A \to B) \to ((B \to C) \to (A \to C))$
$A3.$ $A \to ((A \to B) \to B)$
$A4.$ $(A \to (A \to B)) \to (A \to B)$
$A5.$ $A \wedge B \to A$
$A6.$ $A \wedge B \to B$
$A7.$ $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
$A8.$ $A \to A \vee B$
$A9.$ $B \to A \vee B$

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$A10.\quad (A \to B) \wedge (C \to B) \to (A \vee C \to B)$
$A11.\quad (A \wedge (B \vee C)) \to ((A \wedge B) \vee C)$
$A12.\quad (A \to \neg B) \to (B \to \neg A)$
$A13.\quad \neg\neg A \to A.$

Lemma 1 *The formulae listed below are theses of R:*
(t1) $(p \to q) \wedge (r \to s) \to (p \wedge r \to q \wedge s)$,
(t2) $(p \to q) \wedge (r \to s) \to (p \vee r \to q \vee s)$,
(t3) $(p \vee q \to r) \to (p \to r)$,
(t4) $(p \to q \wedge r) \to (p \to r)$,
(t5) $(p \to (q \to r)) \to (q \to (p \to r))$.

A *matrix* is a pair $\langle \mathbf{A}, \nabla_{\mathbf{A}} \rangle$ where $\mathbf{A}$ is an algebra while $\nabla_{\mathbf{A}}$ is a subset of the domain of $\mathbf{A}$. To the logic R and its extensions we can associate a set of so-called C_R-matrices (cf. W. Dziobiak [83]); their characterization is given by the following

Theorem 2 (W. Dziobiak (83), L. Maximowa (73)) *Let* $\mathbf{A} = \langle A, \to, \wedge, \vee, \neg \rangle$ *be an algebra similar to F and let* $\nabla_{\mathbf{A}}$ *be a subset of A. Then the following conditions are equivalent:*
(i) $\langle \mathbf{A}, \nabla_{\mathbf{A}} \rangle$ *is a* C_R*-matrix,*
(ii) $\langle A, \wedge, \vee \rangle$ *is a distributive lattice with* $\wedge$ *and* $\vee$ *as its meet and join, respectively, and* $\nabla_{\mathbf{A}}$ *is a filter on A with the property: for all* $a, b \in A, a \wedge b = a$ *iff* $a \to b \in \nabla_{\mathbf{A}}$*; and moreover, the following conditions are satisfied for all x,y,z of A,*
(c1) $(x \to y) \leq (y \to z) \to (x \to z)$,
(c2) $x \leq (x \to y) \to y$,
(c3) $x \to (x \to y) \leq x \to y$,
(c4) $(x \to y) \wedge (x \to z) \leq x \to (y \wedge z))$,
(c5) $(x \to z) \wedge (y \to z) \leq (x \vee y) \to z$,
(c6) $x \to \neg y \leq y \to \neg x$,
(c7) $\neg\neg x = x$,
where $\leq$ *is ordering of the lattice* $\langle A, \wedge, \vee \rangle$.

Let us add some additional properties of C_R-matrices:

Lemma 3 (L. Maximowa (73)) *Let* $\langle \mathbf{A}, \nabla_{\mathbf{A}} \rangle$ *be a* C_R*-matrix and let the relation* $\leq$ *be defined as follows:* $x \leq y$ *iff* $x \to y \in \nabla_{\mathbf{A}}$.
Then the relation $\leq$ *satisfes the following implications and inequalities:*
(i) *if* $x \in \nabla_{\mathbf{A}}$ *then* $x \to y \leq y$
(ii) *if* $x \leq y$ *then* $y \to z \leq x \to z$
(iii) *if* $x \leq y$ *then* $z \to x \leq z \to y$
(iv) $x \to \neg x \leq \neg x$.

Let us moreover quote a lemma and a proposition proved by W. Dziobiak, which are important for our further investigations. Let $\langle \mathbf{A}, \nabla_{\mathbf{A}} \rangle$ be a C_R-matrix and let $X \subseteq A$. By $[X)$ we shall denote the least filter on $\mathbf{A}$ containing X. Moreover, a filter ∇ on $\mathbf{A}$ will be called *normal* iff $\nabla_{\mathbf{A}} \subseteq \nabla$.

We have first

Lemma 4 (W. Dziobiak (83)) *Let $\mathcal{A} = \langle \mathbf{A}, \nabla_{\mathbf{A}} \rangle$ be a C_R-matrix. Then*
(i) $\nabla_{\mathbf{A}} = [\{a \to a : a \in A\})$,
(ii) If $\mathbf{A}$ *is generated by elements* $a_0, \ldots, a_{n-1}$ *then*
$\nabla_{\mathbf{A}} = [\bigwedge_{i < n} (a_i \to a_i))$.

It is known that the set of all C_R-matrices forms a variety (cf. W. Dziobiak [83])[1]; algebras which belong to this variety can be called R-algebras. Observe that (cf. Lemma 4) each R-algebra determines a C_R-matrix. Moreover, the logic R is algebraizable (cf. W. J. Blok and D. Pigozzi [89]), thus in particular the lattice of congruences of each R-algebra is isomorphic to the lattice of its normal filters (cf. also W. Dziobiak [83]).
However, the notion of a filter of designated elements plays a fundamental role in this paper and thus we decided to exercise the notion of C_R-matrix rather than the notion of R-algebra.

Now we have the following

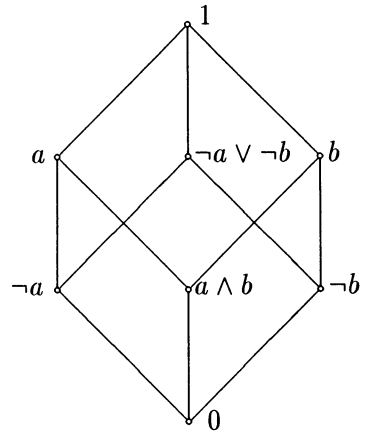
Theorem 5 (W. Dziobiak, unpublished) *Each finitely generated C_R-matrix has the least and the greatest element which form a C_R-matrix isomorphic to the two-element matrix* $\mathbf{2}$.

Proof: Let us denote by $\mathbf{R}$ the variety of C_R-matrices and by $\mathbf{2}$ the two-element C_R-matrix. It is known that $\mathbf{2} \in \mathbf{R}$. Let $\mathbf{F}_{\mathbf{R}}(n)$ be the $\mathbf{R}$-free algebra over the set n of free generators. Of course $\mathbf{2} \in H(\mathbf{F}_{\mathbf{R}}(n))$ for each natural n. Thus there exists a normal filter ∇ on $\mathbf{F}_{\mathbf{R}}(n)$ such that $\mathbf{2} \cong (\mathbf{F}_{\mathbf{R}}(n))/\nabla$. Since $\mathbf{2}$ is finite, by the Rival-Sands Theorem (cf. I. Rival and B. Sands [78]) the filter ∇ is a principal filter, e.g. $\nabla = [\{a\})$ for some a. But $[\{a\})$ is a proper normal filter (because $\mathbf{2}$ is not trivial) thus there exists a b such that $b \notin [\{a\})$ and in consequence $b \wedge a \leq a$, but $b \wedge a \neq a$. Thus there exist an element below the element a. It is easy to observe that there exists exactly one such an element, because $\mathbf{2} \cong (\mathbf{F}_{\mathbf{R}}(n))/\nabla$. Let us denote it by 0. Without any difficulties we can prove that 0 is the least element in $\mathbf{F}_{\mathbf{R}}(n)$; similarly the element $1 = \neg 0$ is the greatest element. Now let $\mathbf{A}$ be a finitely generated (e.g. n-generated) C_R-algebra. Of course, $\mathbf{A} \in H(\mathbf{F}_{\mathbf{R}}(n))$, i.e $\mathbf{A} \cong (\mathbf{F}_{\mathbf{R}}(n))/\nabla$, thus $\mathbf{A}$ must contain 1 and 0.

2 The Belnap matrix $\mathcal{M}_8$.

Denote by $\mathcal{M}_8$ the matrix $\langle\langle\{0, a, \neg a, b, \neg b, a \wedge b, \neg a \vee \neg b, 1\}, \to, \wedge, \vee, \neg\rangle, \{a, b, a \wedge b, 1\}\rangle$ whose lattice operations $\wedge$ and $\vee$ are defined here as it is shown in the following diagram:

[1] An equational characterization of this variety can be found e.g.in: J. M. Font and G. Rodriguez [90].



and whose operations $\to$ and $\neg$ are defined by the following tables:

$\to$	0	a	$\neg a$	b	$\neg b$	$a \wedge b$	$\neg a \vee \neg b$	1
0	1	1	1	1	1	1	1	1
a	0	a	$\neg a$	0	0	0	$\neg a$	1
$\neg a$	0	a	a	0	0	0	a	1
b	0	0	0	b	$\neg b$	0	$\neg b$	1
$\neg b$	0	0	0	b	b	0	b	1
$a \wedge b$	0	a	$\neg a$	b	$\neg b$	$a \wedge b$	$\neg a \vee \neg b$	1
$\neg a \vee \neg b$	0	0	0	0	0	0	$a \wedge b$	1
1	0	0	0	0	0	0	0	1

x	$\neg x$
0	1
a	$\neg a$
b	$\neg b$
$a \wedge b$	$\neg a \vee \neg b$

It is not difficult to prove that $\mathcal{M}_8$ is a C_R matrix; it is just the matrix presented by N. Belnap in [60]. We have changed, however, the symbols used by N. Belnap in [60]; since the algebra of this matrix is 2-generated, we have reduced the number of basic symbols to two symbols to make the lattice connections and negation connections in this matrix more suggestive.
For this matrix the following is true:

Theorem 6 (Belnap (60)) *If in $A \to B$ the sets of variables of A and B are disjoint then there exists a valuation h^v such that $h^v(A \to B)$ does not belong to the set of designated elements of $\mathcal{M}_8$.*

To prove this Theorem it suffices to note that the valuation function h^v can be defined as the homomorphic extension of a function v defined as follows: if p_i occurs in A then we put $v(p_i) = a$ or $v(p_i) = \neg a$, and if p_i occurs in B then we put $v(p_i) = b$ or $v(p_i) = \neg b$. It is easy to check that $h^v(A \to B)$ cannot belong to the set of designated elements of $\mathcal{M}_8$.

Let us observe that this proof is based on the fact that $\mathcal{M}_8$ has two disjoint and $\leq$-incomparable submatrices with universes $\{a, \neg a\}$ and $\{b, \neg b\}$, respectively (they are isomorphic to the matrix **2**, of course).

This observation justifies introducing the following notion. Let $\langle \mathbf{A}, \nabla_A \rangle$ be a C_R-matrix and let $\mathbf{B}_1$, $\mathbf{B}_2$ be subalgebras of the algebra $\mathbf{A}$; let $\leq$ be the partial order which defines the lattice of the algebra $\mathbf{A}$. The subalgebras $\mathbf{B}_1$, $\mathbf{B}_2$ will be called *$\leq$-incomparable* (i.e. incomparable with respect to the relation $\leq$) if for any a,b, if $a \in B_1, b \in B_2$, then neither $a \leq b$ nor $b \leq a$[2].

An implication $A \rightarrow B$ will said to be a *relevant implication* if the intersection of the set of variables occuring in A and the set of variables occuring in B is nonempty; in the opposite case this implication is said to be *non-relevant*. Moreover, we will say that the *relevance principle* hold for a logic L if L does not contain any non-relevant implication. Thus Belnap's Theorem just quoted above states that the *matrix $\mathcal{M}_8$ falsifies all non-relevant implications* and that if a logic L is contained in the logic determined by the Belnap's matrix $\mathcal{M}_8$ then the relevance principle holds for L.

The definition of the next notion needs some assumptions. Let C_R-matrix $\langle \mathbf{C}, \nabla_C \rangle$ contain two $\leq$-incomparable submatrices: $\langle \mathbf{C}_1, \nabla_{\mathbf{C}_1} \rangle$ and $\langle \mathbf{C}_2, \nabla_{\mathbf{C}_2} \rangle$. Moreover, let each non-relevant implication $A \rightarrow B$ be falsified in $\langle \mathbf{C}, \nabla_{\mathbf{C}} \rangle$ by any valuation h^v which is the homomorphic extension of a valuation v such that $v(p_i) \in C_1$ if p_i is a variable which occurs in the formula A and $v(p_j) \in C_2$ if p_j occurs in the formula B. Then the submatrices $\langle \mathbf{C}_1, \nabla_{\mathbf{C}_1} \rangle$ and $\langle \mathbf{C}_2, \nabla_{\mathbf{C}_2} \rangle$ will be called *falsifying submatrices* of the matrix C_R-matrix $\langle \mathbf{C}, \nabla_C \rangle$.

Lemma 7 (on the matrix $\mathcal{M}_8$) *Let $\mathcal{C} = \langle \mathbf{C}, \nabla_C \rangle$ be a C_R-matrix. Let $\leq$ be the partial ordering relation which defines the lattice of the algebra $\mathbf{C}$ and let the algebra $\mathbf{C}$ has two $\leq$-incomparable subalgebras $\mathbf{A}$, $\mathbf{B}$ with units and zero's; let us denote by $a, \neg a$ the unit and the zero of $\mathbf{A}$ and by $b, \neg b$ - the unit and the zero of $\mathbf{B}$ and let $a \neq \neg a$ and $b \neq \neg b$. Then if $\mathbf{C}$ satisfies the equalities:*
$(a \rightarrow b) = (b \rightarrow a) = (b \rightarrow \neg a) = (\neg a \rightarrow b) = \neg a \wedge \neg b$,
then the submatrix of the matrix $\mathcal{C}$ generated by the elements a, b (i.e. the matrix $\langle [a, b]_C, \nabla_{[a,b]_C} \rangle$) is isomorphic to the matrix $\mathcal{M}_8$.

Proof: Let $[a, b]_C$ be the subalgebra of the algebra $\mathbf{C}$ generated by a, b. By the assumptions it is known that $\neg a < a, \neg b < b$ and moreover the following equalities hold in $\mathbf{C}$ (thus in $[a, b]_C$ as well): $a \rightarrow a = \neg a \rightarrow a = \neg a \rightarrow \neg a = a; a \rightarrow \neg a = \neg a$ and $b \rightarrow b = \neg b \rightarrow b = \neg b \rightarrow \neg b = b, b \rightarrow \neg b = \neg b$. By the W. Dziobiak's theorem (cf. Lemma 4. above) the filter $\nabla_{[a,b]_C}$ of designated elements of the matrix $\langle [a, b]_C, \nabla_{[a,b]_C} \rangle$ is the filter $[(a \rightarrow a) \wedge (b \rightarrow b))_{[a,b]_C} = [a \wedge b)_{[a,b]_C}$.

The algebra $[a, b]_C$ contains, of course, the elements $a, \neg a, b, \neg b, \neg a \wedge \neg b, a \wedge \neg b, \neg a \wedge b, a \vee b, a \vee \neg b, \neg a \vee b, \neg a \vee \neg b, \neg(a \wedge b)$. Since the subalgebras $[a]_C, [b]_C$ of the algebra $\mathbf{C}$ are two-element $\leq$-incomparable algebras, the lattice operations and the operation $\neg$ reduce the number of elements of the algebra $[a, b]_C$ to the following eight ones: $a, \neg a, b, \neg b, \neg a \wedge$

[2] It is not excluded that two C_R-subalgebras of a C_R-algebra are disjoint, but $\leq$-comparable; cf. e.g. the subalgebras $\{1, 0\}$ and $\{a, \neg a\}$ of the algebra of the matrix $\mathcal{M}_8$

$\neg b, a \wedge b, \neg a \vee \neg b, a \vee b$; the remaining ones from the list above are equal to some of those eight, e.g. $a \wedge \neg b = \neg a \wedge b = \neg a \wedge \neg b; \neg a \vee \neg b = \neg(a \wedge b)$ etc. It can be shown very easily that the lattice (with the "negation" $\neg$) of the algebra $[a, b]_C$ is isomorphic to the lattice of the algebra of the matrix $\mathcal{M}_8$.

We will show that if the equalities listed in the Lemma are satisfied by the elements a, b then the set $\{a, \neg a, b, \neg b, \neg a \wedge \neg b, a \wedge b, \neg a \vee \neg b, a \vee b\}$ is closed under the operation $\rightarrow$ and the "table of values" for the operation $\rightarrow$ is just the "table of values" of the operation $\rightarrow$ in the matrix $\mathcal{M}_8$.

To abbreviate the calculations let us denote by 0 the element $\neg a \wedge \neg b$ and by 1 the element $a \vee b$; of course, we will sometimes write $a \rightarrow b, b \rightarrow a$ etc. instead of 0, if necessary; similarly, we will sometimes write $a \vee b, \neg(a \rightarrow b)$ etc. instead of 1.

Let us begin with proving the following useful equality:
(*) $1 = (1 \rightarrow 1) = (0 \rightarrow 0)$
i.e. the equality
(*)$a \vee b = a \vee b \rightarrow a \vee b = (a \rightarrow b) \rightarrow (a \rightarrow b) = \neg(a \rightarrow b)$.

Proof of (*): Since $a \rightarrow b \leq a \rightarrow b$, $a \rightarrow b \leq a \rightarrow (a \rightarrow b)$, (by ($t5$), Lemma 1) $a \leq (a \rightarrow b) \rightarrow (a \rightarrow b)$, i.e. $a \leq a \vee b \rightarrow a \vee b$. Similarly we get $b \leq (a \rightarrow b) \rightarrow (a \rightarrow b)$, i.e. $b \leq a \vee b \rightarrow a \vee b$. These inequalities entail $a \vee b \leq (a \vee b) \rightarrow (a \vee b)$, i.e. $1 \leq (1 \rightarrow 1)$. For the converse inequality, let us observe that $(a \vee b) \rightarrow (a \vee b) \leq (a \vee b)$ because $a \vee b \in \nabla_{[a,b]_C}$ (cf. Lemma 3), i.e. $(1 \rightarrow 1) \leq 1$. Since the De Morgan laws are valid in R, $1 \rightarrow 1 = 0 \rightarrow 0$.

Now we will fill the "table of values" for the operation $\rightarrow$.

a) Values for $0 \rightarrow x$.

1. $0 \rightarrow 0 = 1$.
Proof: Cf. (*) above.
2. $0 \rightarrow \neg a = 1$.
Proof: (i) By the Principle of Transposition we have $0 \rightarrow \neg a = a \rightarrow 1$, thus $a \rightarrow 1 \leq 1$ because $a \in \nabla_{[a,b]_C}$.
(ii) By the proof of (*), $a \leq 1 \rightarrow 1$, thus $1 \leq a \rightarrow 1$, and it is the second inequality we need.
3. $0 \rightarrow \neg b = 1$
Proof: See the case 2.
4. $0 \rightarrow a = 1$.
Proof: (i) By the Principle of Transposition we have $0 \rightarrow a = \neg a \rightarrow 1$. Since $\neg a \leq a$, by the proof of (*) we have $\neg a \leq 1 \rightarrow 1$, thus $1 \leq \neg a \rightarrow 1$.
(ii) Since in each C_R-algebra the inequality $x \wedge \neg y \leq \neg(x \rightarrow y)$ is satisfied, $0 = 0 \wedge \neg a \leq \neg(0 \rightarrow a)$, thus $(\neg a \rightarrow 1) \leq 1$.
5. $0 \rightarrow b = 1$.
Proof: See the case 4.
6. $0 \rightarrow (a \wedge b) = 1$.
Proof: (i) Since $a \wedge b \leq a$, $0 \rightarrow a \wedge b \leq 0 \rightarrow a$ (cf. Lemma 3), i.e. (by 4.above) $0 \rightarrow a \wedge b \leq 1$.
(ii) By Lemma 1, ($t2$) we have $(0 \rightarrow a) \wedge (0 \rightarrow b) \leq (0 \rightarrow a \wedge b)$, thus by 4. and 5. (cf. above) $1 \leq (0 \rightarrow a \wedge b)$.
7. $0 \rightarrow \neg(a \wedge b) = 1$.

Proof: (i) By the Principle of Transposition we have $0 \to \neg(a \wedge b) = a \wedge b \to 1$. Since $a \wedge b \in \nabla_{[a,b]_C}$, $a \wedge b \to 1 \leq 1$.
(ii) The inequalities proved in the proof of (*) imply the inequality $a \wedge b \leq 1 \to 1$, thus $1 \leq a \wedge b \to 1$.
8. $0 \to 1 = 1$.
Proof: (i) As in the proof of 4. (cf. above) we have $\neg a \leq 1 \to 1, \neg b \leq 1 \to 1$, thus $\neg a \wedge \neg b \leq 1 \to 1$, thus $1 \leq 0 \to 1$.
(ii) We have $0 = (a \to b) \wedge (a \to b) \leq \neg((a \to b) \to \neg(a \to b)) = \neg(0 \to 1)$, i.e. $0 \leq \neg(0 \to 1)$, thus $0 \to 1 \leq 1$, and it finishes the proof of part a).

b) The values for $\neg a \to x$. The first five cases are obvious.
1. $\neg a \to 0 = \neg a \to (b \to \neg a) = b \to a = 0$.
2. $\neg a \to \neg a = a$.
3. $\neg a \to \neg b = 0$.
4. $\neg a \to a = a$.
5. $\neg a \to b = 0$.
6. $\neg a \to a \wedge b = 0$.
Proof: (i) By thesis $(t1)$ of Lemma 1, $(\neg a \to a) \wedge (\neg a \to b) \leq (\neg a \to a \wedge b)$, thus $0 \leq \neg a \to a \wedge b$.
By $(t3)$, $(\neg a \to a \wedge b) \leq \neg a \to b$, i.e. $\neg a \to a \wedge b \leq 0$.
7. $\neg a \to \neg(a \wedge b) = a \wedge b \to a = a$.
Proof: (i) Since $a \wedge b \in \nabla_{[a,b]_C}$, $a \wedge b \to a \leq a$.
Since $a \wedge b \leq a, a \to a \leq a \wedge b \to a$, i.e. $a \leq a \wedge b \to a$ (cf. Lemma 3).
8. $\neg a \to 1 = 0 \to a = 1$.
Proof: Cf. a) 4.

c) The values for $\neg b \to x$.
1. $\neg b \to 0 = \neg b \to (a \to \neg b) = a \to b = 0$.
2. $\neg b \to \neg a = a \to b = 0$.
3. $\neg b \to \neg b = b$.
4. $\neg b \to a = 0$.
5. $\neg b \to b = \neg b$.
6. $\neg b \to a \wedge b = 0$.
Proof: See the the case b) 6.
7. $\neg b \to \neg(a \wedge b) = b$.
Proof: See the case b) 7.
8. $\neg b \to 1 = 0 \to b = 1$.
Proof: Cf. a) 5.

d) The values for $a \to x$.
1. $a \to 0 = a \to (b \to a) = b \to (a \to a) = 0$.
2. $a \to \neg a = \neg a$.
3. $a \to \neg b = 0$.
4. $a \to a = a$.
5. $a \to b = 0$.

6. $a \to a \wedge b = 0$.
Proof: (i) $a \wedge b \leq b$ implies $a \to a \wedge b \leq a \to b$, i.e. $a \to a \wedge b \leq 0$.
(ii) By thesis ($t1$) of Lemma 1 we have $0 = (a \to a) \wedge (a \to b) \leq (a \to a \wedge b)$.
7. $a \to \neg(a \wedge b) = a \wedge b \to \neg a = \neg a$.
Proof: (i) Since $a \wedge b \in \nabla_{[a,b]_C}$, $a \wedge b \to \neg a \leq \neg a$.
(ii) Since $a \wedge b \leq a$, $a \to \neg a \leq a \wedge b \to \neg a$ (cf. Lemma 3).
8. $a \to 1 = 1$.
Proof: (i) $a \in \nabla_{[a,b]_C}$, thus $a \to 1 \leq 1$.
(ii) By an inequality used in the proof of (*), $a \leq 1 \to 1$, thus $1 \leq a \to 1$.

e) The values for $b \to x$.
1. $b \to 0 = 0$.
2. $b \to \neg a = 0$.
3. $b \to \neg b = \neg b$.
4. $b \to a = 0$.
5. $b \to b = b$.
6. $b \to a \wedge b = 0$.
Proof: Similar to d) 6.(cf. above).
7. $b \to \neg(a \wedge b) = \neg b$.
Proof: Similar to d) 7.
8. $b \to 1 = 1$.
Proof: As in the case d) 8.

f) The values for $a \wedge b \to x$.
1. $a \wedge b \to 0 = 0$
Proof: (ii) Since $a \wedge b \in \nabla_{[a,b]_C}$, $a \wedge b \to 0 \leq 0$.
(ii) By the inequalities used in the proof of (*), $a \wedge b \leq 0 \to 0$, thus $0 \leq a \wedge b \to 0$.
2. $a \wedge b \to \neg a = a \to \neg(a \wedge b) = \neg a$.
Proof: By d) 7.
3. $a \wedge b \to \neg b = \neg b$.
Proof: Cf. e) 7.
4. $a \wedge b \to a = \neg a \to \neg(a \wedge b) = a$.
Proof: Cf. b) 7.
5. $a \wedge b \to b = b$.
Proof: Cf. c) 7.
6. $a \wedge b \to a \wedge b = a \wedge b$.
Proof: (i) $a \wedge b \in \nabla_{[a,b]_C}$, thus $a \wedge b \to a \wedge b \leq a \wedge b$.
(ii) Since $a \wedge b$ is the generator of the filter of the designated elements of the submatrix we consider, $a \wedge b \leq a \wedge b \to a \wedge b$.
7. $a \wedge b \to \neg(a \wedge b) = \neg(a \wedge b)$.
Proof: (i) $a \wedge b \in \nabla_{[a,b]_C}$, thus $a \wedge b \to \neg(a \wedge b) \leq \neg(a \wedge b)$ (cf. Lemma 3).
(ii) By the inequality $a \wedge b \leq a \wedge b \to a \wedge b$ (cf. the proof of the previous equality) we have $a \wedge b \leq \neg(a \wedge b) \to \neg(a \wedge b)$, thus $\neg(a \wedge b) \leq a \wedge b \to \neg(a \wedge b)$.
8. $a \wedge b \to 1 = 1$.
Proof: (i) $a \wedge b \to 1 \leq 1$, because $a \wedge b \in \nabla_{[a,b]_C}$.

(ii) We have $a \wedge b \leq 1 \to 1$ (cf. the proof of (*)), thus $1 \leq a \wedge b \to 1$.

g) The values for $\neg(a \wedge b) \to x$.

1. $\neg(a \wedge b) \to 0 = 1 \to a \wedge b = 0$.

Proof: (i) By ($t2$) (Lemma 1) d) 6. and e) 6. we have $0 = (a \to a \wedge b) \wedge (b \to a \wedge b) \leq (a \vee b) \to (a \wedge b) = 1 \to a \wedge b$.

(ii) By Lemma 1, ($t4$) and d) 6. we have $1 \to (a \wedge b) \leq (a \vee b) \to (a \wedge b) \leq a \to a \wedge b = 0$.

2. $\neg(a \wedge b) \to \neg a = a \to a \wedge b = 0$.

Proof: Cf. d) 6.

3. $\neg(a \wedge b) \to \neg b = 0$.

Proof: Cf. e) 6.

4. $\neg(a \wedge b) \to a = 0$.

Proof: Cf. b) 6.

5. $\neg(a \wedge b) \to b = 0$.

Proof: Cf. c) 6.

6. $\neg(a \wedge b) \to (a \wedge b) = (\neg a \vee \neg b) \to (a \wedge b) = 0$.

Proof: (i) Using the same inequalities as in g) 1., by c) 6. we have $0 = (\neg a \to a \wedge b) \wedge (\neg b \to a \wedge b) \leq (\neg a \vee \neg b) \to a \wedge b)$.

(ii) $(\neg a \vee \neg b) \to a \wedge b \leq \neg a \to a \wedge b = 0$.

7. $\neg(a \wedge b) \to \neg(a \wedge b) = a \wedge b$.

Proof: Cf. f) 6.

8. $\neg(a \wedge b) \to 1 = 0 \to a \wedge b = 1$.

Proof: Cf. a) 6.

h) The values for $1 \to x$.

1. $1 \to 0 = 0$.

Proof: (i) $1 \to 0 \leq 0$, because $1 \in \nabla_{[a,b]_C}$.

(ii) Since by (*) $1 \leq 0 \to 0$, $0 \leq 1 \to 0$.

2. $1 \to \neg a = a \to 0 = 0$.

Proof: Cf. d) 1.

3. $1 \to \neg b = 0$.

Proof: Cf. e) 1.

4. $1 \to a = 0$.

Proof: Cf. b) 1.

5. $1 \to b = 0$.

Proof: Cf. c) 1.

6. $1 \to a \wedge b = 0$.

Proof: Cf. g) 1.

7. $1 \to \neg(a \wedge b) = 0$.

Proof: Cf. f) 1.

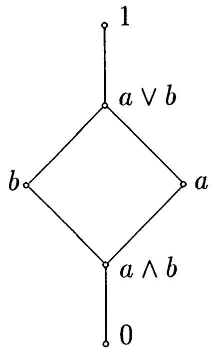
8. $1 \to 1 = 1$.

Proof: Cf. (*),

and it finishes the proof of the Lemma.

3 The matrix $\mathcal{M}_6$.

Denote now by $\mathcal{M}_6$ the matrix $\langle\langle\{0, a, b, a \wedge b, a \vee b, 1\}, \rightarrow, \wedge, \vee, \neg\rangle, \{a, b, a \wedge b, a \vee b, 1\}\rangle$ whose lattice operations $\wedge$ and $\vee$ are defined as it is shown in the following diagram:



and whose operations $\rightarrow$ and $\neg$ are defined by the following tables:

$\rightarrow$	0	$a \wedge b$	a	b	$a \vee b$	1
0	1	1	1	1	1	1
$a \wedge b$	0	$a \wedge b$	a	0	$a \vee b$	1
a	0	0	a	0	a	1
b	0	0	0	b	b	1
$a \vee b$	0	0	0	0	$a \wedge b$	1
1	0	0	0	0	0	1

x	$\neg x$
0	1
a	a
b	b
$a \wedge b$	$a \vee b$

We have the following

Theorem 8 *(i) $\mathcal{M}_6$ is a C_R-matrix;*
(ii) If in $A \rightarrow B$ sets of variables of A and B are disjoint then there exists a valuation v such that $v(A \rightarrow B) = 0$ in $\mathcal{M}_6$.

To prove part (ii) it suffices to note that the valuation function h^v can be defined as the homomorphic extension of a function v defined as follows: if p_i occurs in A then we put $v(p_i) = a$ and if p_i occurs in B then we put $v(p_i) = b$. It is easy to check that $h^v(A \rightarrow B) = 0$.
Similarly as in the case of the matrix $\mathcal{M}_8$ the proof of the property (ii) is based on the fact that $\mathcal{M}_6$ has two one-element (i.e. trivial)[3] $\leq$-incomparable submatrices: $\{a\}$ and $\{b\}$.

[3] A similar 6-element C_R-matrix (called "crystal") was used by P. B. Thistlewaite, M. A. Mc Robbie and R. K. Meyer in their book [88]. However, the "negation" operation in the "crystal" is defined in a quite different way: $\neg a = b$ and $\neg a = b$, thus their "crystal" does not contain trivial subalgebras.

Moreover, the matrix $\mathcal{M}_6$ is the least matrix which has two one-element $\leq$-incomparable falsifying submatrices. Namely, we have

Proposition 9 *Let $\mathcal{A} = \langle \mathbf{A}, \nabla_{\mathbf{A}} \rangle$ be a C_R-matrix and let the lattice of the algebra $\mathbf{A}$ be defined by the partial ordering relation $\leq$. Then if $\mathcal{A}$ contains two trivial $\leq$-incomparable falsifying submatrices then the matrix $\mathcal{M}_6$ is a submatrix of $\mathcal{A}$.*

Proof: Let us denote the universes of these trivial submatrices by $\{a\}$, $\{b\}$, respectively. Of course, $a = \neg a, b = \neg b, a \to a = a, b \to b = b$. Let us consider the subalgebra of $\mathcal{A}$ generated by the elements a, b. Of course, it will contain (besides of a, b) at least the following elements: $a \vee b, a \wedge b, a \to b, b \wedge a, \neg(a \to b), \neg(b \to a)$. It is clear that (cf. Lemma 4 (W. Dziobiak [83]), see above) the filter of designated elements of this submatrix is generated by $a \wedge b$. Since (by the Principle of Transposition) $a \to b = b \to a$ and $a \to b \leq a, b \to a \leq a$ (cf. Lemma 3, see above), $a \to b < a \wedge b$, because (by the assumption that the trivial submatrices $\{a\}$ and $\{b\}$ are falsifying submatrices) $a \to b$ does not belong to the filter of designated elements of the algebra $\mathcal{A}$.

In the following we will show that (i) the submatrix of $\mathcal{A}$ generated by a, b consists of the following elements: $a, b, a \wedge b, a \vee b, a \to b, \neg(a \to b)$, (ii) the element $a \to b$ is the least and the element $\neg(a \to b)$ is the greatest element of the algebra of this submatrix and (iii) the submatrix in question is isomorphic to $\mathcal{M}_6$.
Let us observe first that the following inequalities hold for our elements a, b of the matrix $\mathcal{A}$:
(1) $a \leq (a \to b) \to (a \to b)$,
(2) $b \leq (a \to b) \to (a \to b)$,
(3) $a \wedge b \leq (a \to b) \to (a \to b)$,
(4) $a \vee b \leq (a \to b) \to (a \to b)$,
(5) $\neg(a \to b \leq (a \to b) \to (a \to b)$
For the proof of (1) and (2) - cf. the proof of the equality (*) in Lemma 7; (3) and (4) follow from (1) and (2). The proof for (5) is the following sequence

$$\begin{aligned} a \leq (a \to b) \to (a \to b) \quad & iff \quad (a \to b) \leq a \to (a \to b) \\ & iff \quad (a \to b) \leq a \to (a \to (b \to b)) \\ & iff \quad (a \to b) \leq a \to (b \to (a \to b)) \\ & iff \quad (a \to b) \leq a \to (\neg(a \to b) \to b)) \\ & iff \quad (a \to b) \leq \neg(a \to b) \to (a \to b) \\ & iff \quad \neg(a \to b) \leq (a \to b) \to (a \to b)). \end{aligned}$$

The inequalities (1) - (5) imply the following

Claim. *The element $\neg(a \to b)$ is an upper bound for the elements $a, b, a \vee b, a \wedge b$, and $a \to b$.*

To prove Claim let us note that by the inequalities (1) - (5) (cf. above) the element $(a \to b) \to (a \to b)$ is an upper bound for the elements $a, b, a \vee b, a \wedge b, \neg(a \to b)$. By the assumptions concerning the matrix $\mathcal{A}$ we have $a \to b < a \wedge b$, so (by (3)) we have the following inequality: $(a \to b) \leq (a \to b) \to (a \to b)$, thus $(a \to b) \to (a \to b)$ is an upper bound for the elements we consider. Now it suffices to prove that $\neg(a \to b) = (a \to b) \to (a \to b)$. So since $a \to b \leq a \wedge b, a \vee b \leq \neg(a \to b)$, thus $\neg(a \to b) \in \nabla_{\mathbf{A}}$ and in consequence $\neg(a \to b)$

is a designated element of the submatrix we consider. It is known (cf. Lemma 3, see above) that if $x \in \nabla$ then $x \to y \leq y$ thus $\neg(a \to b) \to \neg(a \to b) \leq \neg(a \to b)$ and by the Principle of Transposition we get $(a \to b) \to (a \to b) \leq \neg(a \to b)$. Now we join the last inequality with the inequality (5) from the previous Lemma and get $\neg(a \to b) = (a \to b) \to (a \to b)$. It finishes the proof of our Claim.

Let us return to the proof of the Proposition. Since - as we just proved - $a \to b$ is the lower bound and $\neg(a \to b)$ the upper bound of this set, we will sometimes denote them by 0 and 1, respectively. It is clear that the set $\{a, b, a \vee b, a \wedge b, a \to b, \neg(a \to b)\}$ is closed under the lattice operations $\wedge$ and $\vee$ and under the operation $\neg$. We will prove now that this set is closed under the operation $\to$ and that the submatrix generated by a, b is isomorphic to $\mathcal{M}_6$. To show it we will fill row by row the "table of values" for the operation $\to$.

a) Values for $0 \to x$.

1. $0 \to 0 = 1$.
Proof: Cf. above.
2. (c) $0 \to (a \wedge b) = 1$.
Proof: (i) $(a \to b) \to (a \wedge b) \leq a \vee b \to \neg(a \to b) \leq \neg(a \to b)$ (because $a \vee b \in \nabla_{\mathbf{A}}$), i.e. $(a \to b) \to a \wedge b \leq \neg(a \to b)$.
(ii) Since $a \vee b \leq \neg(a \to b), \neg(a \to b) \to \neg(a \to b) \leq a \vee b \to \neg(a \to b)$ (cf. Lemma 3), the Principle of Transposition and the inequality (5) (cf. above) $\neg(a \to b) \leq (a \to b) \to (a \wedge b)$.
3. $0 \to a = 1$.
Proof: (i) $a \to b \leq a$, thus $(a \to b) \to (a \to b) \leq (a \to b) \to a$ (cf. Lemma 3) i.e. $\neg(a \to b) \leq (a \to b) \to a$.
(ii) $(a \to b) \to a \leq a \to \neg(a \to b) \leq \neg(a \to b)$ (because $a \in \nabla_{\mathbf{A}}$), thus $(a \to b) \to a \leq \neg(a \to b)$.
4. $0 \to b = 1$.
Proof: As in the case 3.
5. $0 \to a \vee b = 1$.
Proof: (i) Since $a \wedge b \in \nabla_{\mathbf{A}}, a \wedge b \to \neg(a \to b) \leq \neg(a \to b)$ i.e. $(a \to b) \to a \vee b \leq \neg(a \to b)$.
(ii) We have $a \wedge b \leq \neg(a \to b)$, thus by (5), cf. above, $a \wedge b \leq \neg(a \to b) \to \neg(a \to b)$, thus $\neg(a \to b) \leq (a \to b) \to (a \vee b)$.
6. $0 \to 1 = 1$
Proof: (i) Since $x \to \neg x \leq \neg x$ for all x of A, $(a \to b) \to \neg(a \to b) \leq \neg(a \to b)$.
(ii) Since $(a \to b) \leq \neg(a \to b)$ (cf. (5) and the proof of Claim, cf. above), $(a \to b) \leq \neg(a \to b) \to \neg(a \to b)$, thus $\neg(a \to b) \leq (a \to b) \to \neg(a \to b)$.

b) The values for $a \wedge b \to x$.

1. $a \wedge b \to 0 = 0$.
Proof: (i) $a \wedge b \to 0 \leq 0$, because $a \wedge b \in \nabla_{\mathbf{A}}$.
(ii) We have $a \wedge b \leq 1$ i.e. $a \wedge b \leq (a \to b) \to (a \to b)$ iff $a \to b \leq a \wedge b \to (a \to b)$ i.e. $0 \leq a \wedge b \to 0$.
2. $a \wedge b \to a \wedge b = a \wedge b$.
Proof: (i) Since $a \wedge b \in \nabla_{\mathbf{A}}, a \wedge b \to a \wedge b \leq a \wedge b$.

(ii) Since $a \wedge b$ is the generator of the filter of designated elements of the submatrix in question, $a \wedge b \leq a \wedge b \rightarrow a \wedge b$.

3. $a \wedge b \rightarrow a = a$.

Proof: (i) $a \wedge b \rightarrow a \leq a$ because $a \wedge b \in \nabla_{\mathbf{A}}$.

(ii) Since $a \wedge b \leq a, a \rightarrow a \leq a \wedge b \rightarrow a$ by Lemma 3.

4. $a \wedge b \rightarrow b = b$.

Proof: As for 3., cf. above.

5. $a \wedge b \rightarrow a \vee b = a \vee b$.

Proof: (i) Since $a \wedge b \in \nabla_{\mathbf{A}}, a \wedge b \rightarrow a \vee b \leq a \vee b$.

(ii) Since $a \wedge b \leq a \wedge b \rightarrow a \wedge b$ (cf. the part (ii) of the proof of b) 2., cf. above), $a \wedge b \leq a \vee b \rightarrow a \vee b$ (by the Principle of Transposition), thus $a \vee b \leq a \wedge b \rightarrow a \vee b$.

6. $a \wedge b \rightarrow 1 = 1$.

Proof: (i) $a \wedge b \rightarrow 1 \leq 1$, because $a \wedge b \in \nabla_{\mathbf{A}}$.

(ii) $a \wedge b \leq \neg(a \rightarrow b) \rightarrow \neg(a \rightarrow b)$ (cf. (5) above), thus $\neg(a \rightarrow b) \leq (a \wedge b) \rightarrow \neg(a \rightarrow b)$, i.e. $1 \leq a \wedge b \rightarrow 1$.

c). The values for $a \rightarrow x$.

1. $a \rightarrow 0 = 0$.

Proof: (i) Since $a \in \nabla_{\mathbf{A}}$, $a \rightarrow 0 \leq 0$.

(ii) $a \leq (a \rightarrow b) \rightarrow (a \rightarrow b)$ (cf. (5) above), thus $a \rightarrow b \leq a \rightarrow (a \rightarrow b)$.

2. $a \rightarrow a \wedge b = 0$.

Proof: (i) Since $b \leq (a \vee b), (a \vee b) \rightarrow a \leq b \rightarrow a$, thus $a \rightarrow a \wedge b \leq a \rightarrow b$.

(ii) By $b \rightarrow a \leq b \rightarrow a$ we get (*) $b \leq (b \rightarrow a) \rightarrow a$ and by $b \rightarrow a \leq a$ (i.e. by $b \rightarrow a \leq a \rightarrow a$) we have (**) $a \leq (b \rightarrow a) \rightarrow a$. By (*) and (**) we get $a \vee b \leq (b \rightarrow a) \rightarrow a$, i.e. $a \vee b \leq a \rightarrow \neg(a \rightarrow b)$, thus $a \leq a \vee b \rightarrow \neg(a \rightarrow b)$ and in consequence $a \leq (a \rightarrow b) \rightarrow a \wedge b$ and at last $a \rightarrow b \leq a \rightarrow a \wedge b$.

3. $a \rightarrow a = a$ - by the assumption.

4. $a \rightarrow b = 0$.

5. $a \rightarrow a \vee b = a$.

Proof: $a \wedge b \rightarrow a = a$ (cf. b) 2.).

6. $a \rightarrow 1 = 1$.

Proof: (i) $a \rightarrow 1 \leq 1$, because $a \in \nabla_{\mathbf{A}}$.

(ii) Since $a \leq \neg(a \rightarrow b) \rightarrow \neg(a \rightarrow b)$ (cf. the inequality (1), see above), $\neg(a \rightarrow b) \leq a \rightarrow \neg(a \rightarrow b)$.

d) The values for $b \rightarrow x$ - can be determined as in c).

e) The values for $a \vee b \rightarrow x$.

1. $a \vee b \rightarrow 0 = 0$.

Proof: (i) $a \vee b \rightarrow 0 \leq 0$, because $a \vee b \in \nabla_{bfA}$.

(ii) Cf. the inequality (4) above.

2. $a \vee b \rightarrow a \wedge b = 0$.

Proof: (i) Since $(p \vee q \rightarrow r) \rightarrow (p \rightarrow r) \wedge (q \rightarrow r) \in R, a \vee b \rightarrow a \wedge b \leq (a \rightarrow a \wedge b) \wedge (b \rightarrow a \wedge b)$,

so by c) 2. $a \vee b \to a \wedge b \leq (a \to b) \wedge (a \to b)$, and in consequence $a \vee b \to a \wedge b \leq (a \to b)$ i.e. $a \vee b \to a \wedge b \leq 0$.
(ii) Conversely, since $a \to b \leq a \to b$, by c) 2. $a \to b \leq a \to a \wedge b$, thus (*)$a \leq (a \to b) \to a \wedge b$. Similarly we get (**) $b \leq (a \to b) \to a \wedge b$, so by (*) and (**) we have $a \vee b \leq (a \to b) \to a \wedge b$, thus $(a \to b) \leq (a \vee b) \to a \wedge b$.
3. $a \vee b \to a = o$.
Proof: $a \to a \wedge b = 0$ (cf. c) 2.); we apply the Transposition Principle.
4. $a \vee b \to b = a \to b$.
Proof: As in the previous case.
5. $a \vee b \to a \vee b = a \wedge b$.
Proof: We apply the Transposition Principle to $a \wedge b \to a \wedge b = a \wedge b$ (cf. b) 2.).
6. $a \vee b \to 1 = 1$.
Proof: By a) 2.

f) The values for $1 \to x$.
1. $1 \to 0 = 0$.
Proof: (i) $1 \in \nabla_{\mathbf{A}}$, thus $1 \to 0 \leq 0$.
(ii) Since $0 \to 0 = 1$ (cf. a) 1. above), $1 \leq 0 \to 0$, thus $0 \leq 1 \to 0$.
2. $1 \to a \wedge b = 0$.
Proof: Cf. e) 1.
3. $1 \to a = 0$.
Proof: Cf. c) 1.
4. $1 \to b = 0$.
Proof: As in case 3.
5. $1 \to a \vee b = 0$.
Proof: Cf. b) 1.
6. $1 \to 1 = 1$.
Proof: Cf. a) 1.

Thus the set $\{a, b, a \wedge b, a \vee b, a \to b, \neg(a \to b)\}$ is closed under all basic operations; it suffices now to compare the "table of values" for the operation $\to$ we have just filled with the "table of values" for the operation $\to$ in the matrix $\mathcal{M}_6$ to state that the matrix we have obtained in the proof of this Proposition is just the matrix $\mathcal{M}_6$. It finishes the proof of this Proposition.

As the last proposition of ths section let us note the following

Proposition 10 *The algebras of the matrices $\mathcal{M}_6$ and $\mathcal{M}_8$ are subdirectly irreducible.*

Proof: Proposition 1.5 of: W. Dziobiak [83].

4 Fundamental result.

Let us begin with the following

Lemma 11 *There does not exist a C_R-matrix $\langle \mathbf{C}, \nabla_{\mathbf{C}} \rangle$ which contains two proper submatrices which satisfy the conditions*
(i) the algebras of these submatrices are incomparable with respect to the partial order relation $\leq$ which defines the lattice of the algebra **C** *and*
(ii) the first submatrix is the trivial matrix and the second one is the two-element matrix.

Proof: Let us assume that such a matrix exists. Let us denote by $a, \neg a$ the elements of the two-element submatrix of the matrix in question (let $a \in \nabla_{\mathbf{A}}, \neg a \notin \nabla_{\mathbf{A}}$) and by b the only element of the trivial submatrix of this matrix ($b \in \nabla_{\mathbf{A}}$, of course). Note now that (since the Principle of Transposition hold for C_R-matrices) if $a \vee b = \neg a \vee b$ then $\neg a \wedge b = a \wedge b$. However in such a case the set $\{a, \neg a, b, a \vee b, \neg a \wedge b\}$ forms the well-known lattice N_5, thus the lattice of the C_R-matrix in question cannot be distributive. From $\neg a < a$ it follows now that $\neg a \wedge b < a \wedge b, \neg a \vee b < a \vee b$. Moreover $a \wedge b < b < \neg a \vee b$. But in this case the elements $\{a, a \wedge b, b, \neg a \vee b, a \vee b\}$ form the lattice N_5 which finishes the proof.

To formulate the next Proposition a new notion is useful.
A variety V of C_R-matrices will said to be a *variety with the relevance principle* if V falsifies all non-relevant implications, i.e. for each non-relevant implication $A \to B$ there exists a matrix $\mathcal{A}$ of V and a valuation h such that $h(A \to B)$ does not belong to the set of designated elements of $\mathcal{A}$.

We have now

Proposition 12 *Let V be a C_R-variety with the relevance principle. Then V contains a matrix $\mathcal{C} = \langle \mathbf{C}, \nabla_{\mathbf{C}} \rangle$ that falsifies all non-relevant implications and contains two submatrices $\mathcal{A}, \mathcal{B}$ whose algebras are 1-generated and incomparable with respect to the partial order which defines the lattice of the algebra* **C**.

Proof: Let V be a variety with the relevance principle. Let us consider the V-free algebra **C** over two generators a, b. Of course, the matrix $\langle \mathbf{C}, \nabla_C \rangle$ falsifies all non-relevant implications. Let us denote by $\leq$ the partial order which defines the lattice of the algebra **C**. Let us consider the subalgebras of **C** generated by a, b, respectively; let us denote them by $[a]_{\mathbf{C}}$ and $[b]_{\mathbf{C}}$. By Theorem 5 (cf. W. Dziobiak, unpublished, see above) both of these algebras have units and zero's; let us denote them by $1_a, 0_a, 1_b, 0_b$. Observe now that the algebras $[a]_{\mathbf{C}}, [b]_{\mathbf{C}}$ are $\leq$-incomparable. If not then there exist elements a_1, b_1 such that $a_1 \in [a]_{\mathbf{C}}$, $b_1 \in [b]_{\mathbf{C}}$ and e.g. $a_1 \leq b_1$. But in such a case $0_a \leq 1_b$ and in consequence this free algebra cannot falsify all non-relevant implications. This finishes the proof.

Let us observe here that the submatrices $[a]_{\mathbf{C}}$ and $[b]_{\mathbf{C}}$ described in this Proposition are *falsifying submatrices*.

Theorem 13 *Let V be a C_R-variety with the relevance principle. Then V contains either the Belnap matrix $\mathcal{M}_8$ or the matrix $\mathcal{M}_6$.*

Proof: We know from the previous Proposition that there exists in V a matrix $\mathcal{C} = \langle \mathbf{C}, \nabla_{\mathbf{C}} \rangle$ such that $\mathbf{C}$ contains two submatrices whose algebras are $\leq$-incomparable ($\leq$ denotes here the partial order which defines the lattice of the algebra $\mathbf{C}$); let us denote the subalgebras of these submatrices by $\mathbf{A}$ and $\mathbf{B}$, respectively. We may assume that both of these subalgebras have a greatest and a least element; let us denote these elements as follows: the unit of $\mathbf{A}$ by a, the zero of $\mathbf{A}$ by $\neg a$, and by $b, \neg b$ - the unit and the zero of the algebra $\mathbf{B}$, respectively. If $a = \neg a$ and $b = \neg b$ then (cf. Proposition 9) the matrix $\mathcal{M}_6$ belongs to V; let us assume that $a \neq \neg a$, $b \neq \neg b$.

Let us denote by $[a, b]_{\mathbf{C}}$ the subalgebra of the algebra $\mathbf{C}$ generated by the elements a, b. The algebra $[a, b]_{\mathbf{C}}$ contains in particular the following eight elements: $a, \neg a, b, \neg b, a \vee b, a \wedge b, \neg a \wedge \neg b$; it is easy to observe that the lattice operations $\wedge, \vee$ on these elements as well as the operation $\neg$ can be described here as in the algebra of the matrix $\mathcal{M}_8$; this eight-element set is closed under these lattice operations and under $\neg$. Our further investigations will concern only the operation $\rightarrow$.

Let us write down first the following obvious connections between the elements $a, \neg a$:
$a \rightarrow a = a, \neg a \rightarrow a = a, a \rightarrow \neg a = \neg a, \neg a \rightarrow \neg a = a$.
We have quite similar connections for the elements $b, \neg b$.

By the Lemma 4. (cf. W. Dziobiak (83)) the filter $[(a \rightarrow a) \wedge (b \rightarrow b))_{[a,b]_{\mathbf{C}}}$ i.e. (by the previous remark) the filter $[a \wedge b)_{[a,b]_{\mathbf{C}}}$ is the filter of designated elements of the matrix $\langle [a, b]_{\mathbf{C}}, \nabla_{[a,b]_{\mathbf{C}}} \rangle$, i.e. $\nabla_{[a,b]\ bfC} = [a \wedge b)$. By the construction of the algebra $[a, b]_{\mathbf{C}}$ the elements $a \rightarrow b, b \rightarrow a, a \rightarrow \neg b, \neg a \rightarrow b$ cannot belong to the filter $[a \wedge b)$.

Since $\neg a < a, \neg b < b$, by Lemma 3 we have the following equalities and inequalities:
(1) $b \rightarrow \neg a = a \rightarrow \neg b \leq a \rightarrow b \leq \neg a \rightarrow b = \neg b \rightarrow a$, and
$b \rightarrow \neg a = a \rightarrow \neg b \leq b \rightarrow a \leq \neg a \rightarrow b = \neg b \rightarrow a$.

The next connections, which are important for our proof we get by the following connection valid for any C_R-matrix $\langle \mathbf{C}, \nabla_{\mathbf{C}} \rangle$:
If $z \in \nabla$ then $z \rightarrow x \leq x$ for any $x \in C$
(cf. Lemma 3); thus we have the following useful inequalities:
(2) $a \rightarrow b \leq b, b \rightarrow a \leq a, a \rightarrow \neg b \leq \neg b, b \rightarrow \neg a \leq \neg a$,
which imply the next important connection
(3) $a \rightarrow \neg b = b \rightarrow \neg a \leq \neg a \wedge \neg b$.

Since the proofs of the two connections which are the basis of our proof are rather long, we present them in the form of two lemmas.

Lemma 14 *The following equalities hold in $[a, b]_C$:*
(4) $a \rightarrow b = b \rightarrow a = \neg a \rightarrow b$.

Proof of the Lemma: Since $\{[(p \to q) \wedge p] \to q\} \in R$, we have
$(\neg a \to b) \wedge \neg a \leq b$ and $(\neg b \to a) \wedge \neg b \leq a$,
i.e. $((\neg b \to a) \wedge \neg b) \vee a = a$ and $((\neg a \to b) \wedge \neg a) \vee b = b$. Since the equality $\neg b \vee a = b \vee a$ holds in $[a, b]_C$, we have:
$a = ((\neg b \to a) \wedge \neg b) \vee a = [(\neg b \to a) \vee a] \wedge (\neg b \vee a) = [(\neg b \to a) \vee a] \wedge (b \vee a) = [(\neg b \to a) \wedge b] \vee a$,
i.e. $[(\neg b \to a) \wedge b] \leq a$. In a quite similar way we get the inequality $[(\neg a \to b) \wedge a] \leq b$.
From these inequalities the following inequalities follow:
$[(\neg b \to a) \wedge b] \leq [a \wedge (\neg b \to a)]$
$[(\neg b \to a) \wedge a] \leq [b \wedge (\neg b \to a)]$,
thus $[(\neg a \to b) \wedge b] = [(\neg a \to b) \wedge a]$, because $(\neg a \to b) = (\neg b \to a)$.

Since in all C_R-algebras holds the inequality $x \to y \leq \neg x \vee y$, we have: $(\neg a \to b) \leq (a \vee b)$. Thus we have: $(\neg a \to b) = (\neg a \to b) \wedge (a \vee b) = [(\neg a \to b) \wedge a] \vee [(\neg a \to b) \wedge b] = [(\neg a \to b) \wedge a]$, i.e. $(\neg a \to b) \leq a$. In the same way we get $(\neg a \to b) \leq b$.

As we remember, in all C_R-algebras the following implication holds:
if $x \leq y$ then $z \to x \leq z \to y$ (cf. Lemma 3, see above).
By this implication and the first inequality from the last two we have: $b \to (\neg a \to b) \leq b \to a$, i.e. $\neg a \to (b \to b) \leq b \to a$, i.e. $\neg a \to b \leq b \to a$. The second inequality (i.e. $(\neg b \to a) \leq b$) implies $(\neg b \to a) \leq a \to b$, i.e. $(\neg a \to b) \leq b \to a$. Since the inequalities (1) (cf. above) hold, the Lemma has been proved.

Let us add that the inequalities shown in the proof of this Lemma imply the following useful inequality:
(5) $\neg a \to b \leq a \wedge b$.

The next connection is given by

Lemma 15 *Let the algebra $[a, b]_A$ we consider satisfy the equalities*
$(\neg a \wedge \neg b) = (a \to b) = (b \to a)$.
Then this algebra satisfies the equality:
$(a \to \neg b) = (\neg a \to b)$.

Proof of the Lemma: Let us prove the inequality $b \leq (a \vee b) \to (a \vee b)$ first; the proof of this inequality is as follows:
$\neg a \wedge \neg b \leq \neg a \wedge \neg b$ iff
$\neg a \wedge \neg b \leq a \to b$ iff
$\neg a \wedge \neg b \leq \neg b \to \neg a$ iff
$\neg a \wedge \neg b \leq \neg b \to (a \to \neg a)$ iff
$\neg a \wedge \neg b \leq a \to (a \to b)$ iff
$\neg a \wedge \neg b \leq a \to (a \to (\neg b \to b)$ iff
$\neg a \wedge \neg b \leq a \to (\neg b \to (a \to b))$ iff
$\neg a \wedge \neg b \leq a \to (\neg(a \to b) \to b)$ iff
$\neg a \wedge \neg b \leq \neg(a \to b) \to (a \to b)$ iff
$\neg a \wedge \neg b \leq a \vee b \to (a \to b)$ iff
$a \vee b \leq (\neg a \wedge \neg b) \to (\neg a \wedge \neg b)$ iff
$a \vee b \leq a \vee b \to a \vee b$.
The last inequality also implies $b \leq a \vee b \to a \vee b$. It follows from this last inequality that

$a \vee b \leq b \to a \vee b$, thus $a \leq b \to a \vee b$, i.e. $a \leq b \to \neg(a \to b)$, thus $a \leq (a \to b) \to \neg b$, and in consequence $(a \to b) \leq (a \to \neg b)$. The inverse inequality we get from inequalities (1) (cf. above).

Let us return to the proof of Theorem.

Now arises the question where the "arrow elements" of the algebra $[a,b]_{\mathbf{A}}$ are situated in relation to the remaining elements of this algebra. Since it is known that the following equalities hold in this algebra: $a \to b = b \to a = \neg b \to a = \neg a \to b$ (cf. Lemma 14) and $a \to \neg b = b \to \neg a$, we have to consider here only two "arrow elements": $a \to b$ and $a \to \neg b$.

Let us note now that the inequality (5) entails that $a \to b \leq a \wedge b$, and it is known from the construction of the algebra $[a,b]_A$ that $(a \to b)$ does not belong to the filter $[a \wedge b)$, thus we have the first important inequality

(a) $a \to b < a \wedge b$.

Moreover, we have established before that in the algebra $[a,b]_{\mathbf{A}}$ the two following interesting inequalities hold:

(b) $a \to \neg b \leq \neg a \wedge \neg b < a \wedge b$

(c) $a \to \neg b \leq a \to b$.

The remainder of the proof will be devoted to considering the question where the elements $a \to b$ and $a \to \neg b$ may be situated in the lattice of the algebra $[a,b]_{\mathbf{A}}$. Below we will prove that in each of possible cases (determined by the inequalities (a) - (c), of course) either $\mathcal{M}_8 \in HS([a,b]_{\mathbf{A}})$ or $\mathcal{M}_6 \in HS([a,b]_{\mathbf{A}})$ and that will finish the proof of the Theorem.

We have here the following possibilities:

A. $\neg a \wedge \neg b < a \to b < a \wedge b$

(it occurs - cf. below - that in this case it is not important (cf. the inequality (c)) whether $a \to \neg b = \neg a \wedge \neg b$ or $a \to b < \neg a \wedge \neg b$).

B. $\neg a \wedge \neg b = a \to b$,

(in this case $a \to \neg b = a \to b$, cf. Lemma 15).

C. $a \to b < \neg a \wedge \neg b$,

D. $a \to b$ is $\leq$-incomparable with $\neg a \wedge \neg b$, but, of course $a \to \neg b \leq a \to b < a \wedge b$.

Thus let us consider now the four cases listed above.

A. Let us assume first that $\neg a \wedge \neg b < a \to b$.

Since $a \wedge b \leq (a \to b) \to (a \to b) \leq \neg(a \to b) \vee (a \to b)$, by the inequalities we have just assumed, i.e. by $\neg a \wedge \neg b < (a \to b) < a \wedge b$ we have $\neg a \vee \neg b < \neg(a \to b) < a \vee b$, and the last two inequalities give $\neg a \vee \neg b \leq (a \to b) \vee \neg(a \to b) \leq a \vee b$. The last inequality can be strenghted to the equality $(a \to b) \vee \neg(a \to b) = a \vee b$, because $a \to b \leq a \to b$ implies $a \to b \leq a \to (a \to b)$ and it implies $a \leq (a \to b) \to (a \to b)$; similarly we show that $b \leq (a \to b) \to (a \to b)$, thus $a \vee b \leq (a \to b) \to (a \to b) \leq (a \to b) \vee \neg(a \to b)$.

Let us observe now that if $\neg a \wedge b < a \to b < a \wedge b$ then we have the obvious equalities: $a \wedge b \wedge \neg a = \neg a \wedge \neg b$, $\neg a \wedge (a \to b) = \neg a \wedge \neg a$ and $\neg a \vee (a \wedge b) = a$. If moreover $(a \to b) \vee \neg a = a$ then the elements $\{\neg a \wedge \neg a, \neg a, a, a \wedge b, a \to b\}$ form the lattice N_5, thus the lattice of the algebra $[a,b]_A$ is not distributive. However, that is not possible, so we must assume that $(a \to b) \vee \neg a < a$ and that $\neg a < (a \to b) \vee \neg a$. The second inequality is justified as follows. It is obvious that $\neg a \leq (a \to b) \vee \neg a$. But if $\neg a = (a \to b) \vee \neg a$ then on one hand $\neg a \wedge \neg b = a \wedge b \wedge \neg a$, and on the other hand $\neg a \wedge \neg b = a \wedge b \wedge ((a \to b) \vee \neg a) = (a \wedge b \wedge (a \to b)) \vee (a \wedge b \wedge \neg a) = (a \to b) \vee (\neg a \wedge \neg b) = a \to b$, i.e. $(\neg a \wedge \neg b) = (a \to b)$, which is not consistent with the assumption that $\neg a \wedge \neg b < (a \to b)$. Thus we have
$\neg a < (a \to b) \vee \neg a < a$, and
$\neg a < \neg(a \to b) \wedge a < a$.
Let us denote: $c =: (a \to b) \vee \neg a$. By the equality: $(a \to b) \vee \neg(a \to b) = a \vee b$ we have: $c \wedge \neg c = [(a \to b) \vee \neg a] \wedge [\neg(a \to b) \wedge a] = [(a \to b) \wedge \neg(a \to b) \wedge a] \vee [\neg a \wedge a \wedge \neg(a \to b)] = (\neg a \wedge \neg b \wedge a) \vee \neg a = (\neg a \wedge \neg b) \vee \neg a = \neg a$, i.e. $c \wedge \neg c = \neg a$, thus $c \vee \neg c = a$. Analogously we prove that between the elements b and $\neg b$ there exist elements $\neg b \vee (a \to b)$ and $b \wedge \neg(a \to b)$ which satisfy the inequalities
$\neg b < b \wedge \neg(a \to b) < b$, and
$\neg b < \neg b \vee (a \to b) < b$.
Let us denote: $d =: \neg b \vee (a \to b)$; we have, as above, that $d \vee \neg d = b$ and $d \wedge \neg d = \neg b$.

Let us consider now the element $\neg a \wedge \neg d$. We have $\neg c \wedge \neg d = [a \wedge \neg(a \to b)] \wedge [b \wedge \neg(a \to b)] = a \wedge b \wedge \neg(a \to b)$.
On one hand we have $(\neg c \wedge \neg d) \wedge (a \to b) = a \wedge b \wedge \neg(a \to b) \wedge (argb) = a \wedge b \wedge \neg a \wedge \neg b = \neg a \wedge \neg b$, i.e. $(\neg c \wedge \neg d) \wedge (a \to b) = \neg a \wedge \neg b$, on the other hand - $(\neg c \wedge \neg d) \vee (a \to b) = (a \to b) \vee [a \wedge b \wedge \neg(a \to b)] = [(a \to b) \vee (a \wedge b)] \wedge [(a \to b) \vee \neg(a \to b)] = (a \wedge b) \vee (a \vee b) = a \wedge b$, i.e. $(\neg c \wedge \neg d) = a \wedge b$. In consequence we have the inequalities:
$\neg a \wedge \neg b \leq \neg c \wedge \neg d \leq a \wedge b$
and moreover $(a \to b) \neq \neg c \wedge \neg d$.

Since $\neg c \wedge \neg d \leq a \wedge b$, the filter $F = [\neg c \wedge \neg b)$ is a normal filter, thus it determines a congruence relation on the algebra $[a,b]_A$, and moreover does not contain the element $(a \to b)$. Let us investigate the quotient algebra $[a,b]_A/\Theta(F)$. We have in particular
(a) $(a \to b) \equiv (\neg a \wedge \neg b)(\Theta(F))$.
To prove it let us observe that by the assumption we have $\neg a \wedge \neg b \leq (a \to b)$, thus $(\neg a \wedge \neg b) \to (a \to b) \in [a \wedge b)$, thus $(\neg a \wedge \neg b) \to (a \to b) \in F$. On the other hand we have:

$$
\begin{aligned}
a \to b \leq a \to b \quad & \text{iff } (a \vee b) \leq (a \to b) \to (a \to b) \\
& \text{iff } (a \vee b) \leq \neg(a \to b) \to \neg(a \to b) \\
& \text{iff } \neg(a \to b) \leq a \vee b \to \neg(a \vee b) \\
& \text{iff } \neg(a \to b) \leq (a \to b) \to (\neg a \wedge \neg b).
\end{aligned}
$$

But since $\neg(a \to b) \in F$, $[(a \to b) \to \neg a \wedge \neg b)] \in F$. Thus $a \to b \equiv \neg a \wedge \neg b(\Theta(F))$.

(b) $\neg c \wedge \neg d \equiv a \wedge b(\Theta(F))$.

The proof of (b): It it obvious that $\neg(a \to b) \wedge a \wedge b \leq a \wedge b$, thus $[(a \wedge b \wedge \neg(a \to b) \to a \wedge b] \in F$. On the other hand, since each element of the form $x \to x$ belongs to the filter of designated elements of each C_R-matrix, $a \wedge b \leq [a \wedge b \wedge \neg(a \to b)] \to [a \wedge b \wedge \neg(a \to b)]$, thus $[a \wedge b \wedge \neg(a \to b)] \leq a \wedge b \to [a \wedge b \wedge \neg(a \to b)]$, thus $a \wedge b \to [a \wedge b \wedge \neg(a \to b)] \in F$, because $F = [a \wedge b \wedge \neg(a \to b))$.

Moreover, let us note that since $a \wedge b \to (a \to b) = (a \to b)$, it is not true that $(a \to b) \equiv a \wedge b(\Theta(F))$, because $a \wedge b \to (a \to b)$ does not belong to F.

It is easy to prove that $a \equiv a \wedge \neg(a \to b)(\Theta(F))$,(i.e. that $a \equiv \neg c(\Theta(F)))$, $\neg a \equiv c(\Theta(F)), b \equiv \neg d(\Theta(F)), \neg b \equiv d(\Theta(F))$ and that it is not the case that $a \equiv \neg a(\Theta(F)), b \equiv \neg b(\Theta(F))$.

As concerns the element $(a \to \neg b) = (b \to \neg a)$, we did not consider this element yet; however, since in the quotient algebra $[a, b]_A/\Theta(F)$ we have $\neg a \wedge \neg b \equiv (a \to b)(\Theta(F))$, by the connection (*) (cf. the second lemma of this proof) we know that $(a \to \neg b) \equiv (\neg a \wedge \neg b)(\Theta(F))$.

Let us consider now the subalgebra $[a/\Theta(F), b/\Theta(F)]_{[a,b]_A/\Theta(F)}$ of the algebra $[a, b]/\Theta(F)$ (i.e. the subalgebra generated by the elements $a/\Theta(F), b/\Theta(F)$). In particular we have in this subalgebra: $(a \to b) \equiv (b \to a) \equiv (a \to \neg b) \equiv (\neg a \to b) \equiv \neg a \wedge \neg b(\Theta(F))$. By Lemma 7 (on Belnap's matrix $\mathcal{M}_8$) we have that the matrix $\langle [a/\Theta(F), b/\Theta(F)]_{[a,b]_A/\Theta(F)}, [(a \wedge b)/\Theta(F)\rangle$ is isomorphic to Belnap's matrix $\mathcal{M}_8$.

This finishes the proof of the fact that if $\neg a \wedge \neg b < (a \to b) < a \wedge b$ in $[a, b]_A$ then our variety V contains the Belnap's matrix $\mathcal{M}_8$; let us add that our proof of this fact did not depend on the position of the element $a \to \neg b$ in the algebra $[a, b]_A$.

B. Let us assume now that $\neg a \wedge \neg b = a \to b$. Thus, by (*)the second lemma of this proof we have $a \to b = a \to \neg b$, and by the Lemma on Belnap's matrix $\mathcal{M}_8$ it follows that $\mathcal{M}_8$ is isomorphic to the matrix $\langle [a, b]_A, [a \wedge b)_{[a,b]_A}\rangle$.

C. Let us assume now that $(a \to b) < \neg a \wedge \neg b$.. Since $(a \to \neg b) \leq (a \to b)$, $(a \to \neg b) < \neg a \wedge \neg b$[4]. Then the filter $F = [\neg a \wedge \neg b)$ is a normal filter in $[a, b]_A$, i.e. this filter determines a congruence relation on $[a, b]_A$. Let us consider the quotient algebra $[a, b]_A/\Theta(F)$. It is obvious that $(b \to \neg b), (\neg b \to b) \in F$ and $(a \to \neg a), (\neg a \to a) \in F$, thus $a \equiv \neg a(\Theta(F)), b \equiv \neg b(\Theta(F))$. However, it is not the case that $a \equiv b(\Theta(F))$, because by the assumption $a \to b$ does not belong to F. Thus the quotient algebra $[a, b]_A/\Theta(F)$ contains two one-point falsifying subalgebras, and by Proposition 9 (on the matrix $\mathcal{M}_6$) a matrix isomorphic to the matrix $\mathcal{M}_6$ will be a submatrix of the quotient matrix $\langle [a, b]_A/\Theta(F), [(a \wedge b)/\Theta(F))_{[a,b]_A/\Theta(F)}\rangle$.

D. $a \to b$ is $\leq$-incomparable with $\neg a \wedge \neg b$. Let us note first that $a \to \neg b < \neg a \wedge \neg b$ in this case, because in general it is known that $a \to \neg b \leq \neg a \wedge \neg b$ (cf. (b)above), but if $a \to \neg b = \neg a \wedge \neg b$, then this case reduces to the case A. As in C. let us consider now the quotient algebra $[a, b]_{\mathbf{A}}/\Theta(F)$ where $F = [\neg a \wedge \neg b)$. Since neither $a \to b$ nor $a \to \neg b$ belongs to F, the quotient algebra contains two trivial falsifying subalgebras - and we can argue as in the previous case to show that the matrix $\mathcal{M}_6$ is a submatrix of the quotient matrix $\langle [a, b]_A/\Theta(F), [(a \wedge b)/\Theta(F))_{[a,b]_A/\Theta(F)}\rangle$.

This finishes the proof of the Theorem.

[4]There exists at least one C_R-algebra generated by the elements a, b such that the following equalities and inequalities hold in it: $a \to b = a \to \neg b < \neg a \wedge \neg b$.

From the last Theorem follows the following

Theorem 16 *Let L be an extension of the relevant logic R. Then the relevance principle holds for L if and only if L is either a sublogic of the logic determined by* $\mathcal{M}_6$ *or a sublogic of the logic determined by* $\mathcal{M}_8$.

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