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LIFSHITZ' REALIZABILITY

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Abstract. In Lifschitz 1979 a variation on Kleene's realizability for **HA** is presented, with a different clause for the existential quantifier. Lifschitz showed soundness for this realizability, and showed furthermore that $CT_0! \nvdash CT_0$. In this paper a formalized version is considered, for which an axiomatization is given. An extension to **HAS** is given, as well as an analogue for realizability for functions. Finally, the construction of an "effective topos" for this realizability is sketched.

Key words and phrases: realizability, **HA**, **HAS**, **EL**, tripos.

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Introduction.

In Lifschitz 1979 a realizability interpretation for **HA** is given which differs from Kleene's realizability only in the clause for the existential quantifier.

A somewhat more complex coding of finite sets of natural numbers by numbers is given: let V_e, the finite set coded by e, be defined by

$$V_e \equiv \{x \leq j_2 e \mid \{j_1 e\}(x)\uparrow\}.$$

Here j_1, j_2 are the first and second projections of the inverse of a bijective primitive recursive pairing-function $j:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$, $\{.\}$ denotes partial recursive application.

Lifschitz put $e\,\underline{r}\,\exists x Ax \equiv V_e \neq \varnothing \;\&\; \forall g \in V_e (j_2 g\,\underline{r}\,A j_1 g)$. His aim was to show that the schema CT_0 is really stronger than $CT_0!$, where

CT_0 $\quad\forall x\exists y Axy \to \exists z \forall x(\{z\}(x)\downarrow \& Ax\{z\}(x))$

$CT_0!$ $\quad\forall x\exists! y Axy \to \exists z \forall x(\{z\}(x)\downarrow \& Ax\{z\}(x))$.

The basic idea for the proof of this is that there can't be an effective procedure which produces, given that $V_e \neq \varnothing$, an element of V_e; on the other hand, there <u>is</u> such a procedure working on all e for which V_e is a singleton.

For, if there were a code g such that $V_e \neq \varnothing \Rightarrow \{g\}(e)\downarrow \& \{g\}(e) \in V_e$, and W_f and W_h are two disjoint, recursively inseparable r.e. sets, find a recursive function F such that

$$\forall x[\{F(x)\}(0) \simeq \{f\}(x) \;\&\; \{F(x)\}(1) \simeq \{h\}(x)].$$

Then always $V_{j(F(x),1)} \neq \varnothing$, so $\{g\}(j(F(x),1)) \in V_{j(F(x),1)}$ and g serves to construct a recursive separation between W_f and W_h.

(If V_e is a singleton then one simply waits until $\{j_1 e\}(x)$ has been computed for all $x \leq j_2 e$ save one; the remaining one must be the element of V_e.)

In this paper we will be concerned with the following questions: Can Lifschitz' realizability be formalized? Can we give an adequate axiomatization? Can we extend it to higher-order systems like **HAS**? Is there an analogon to Kleene's realizability for functions? Can it be put into the framework of tripos theory?

These questions can be answered affirmatively; however, to formalize the proof of soundness we seem to need to extend these systems somewhat. Lifschitz' proof that **HA** is sound for his realizability hinges on some lemmas that can't be formalized in **HA**. For this we seem to need two extra principles. One is Markov's Principle for primitive recursive predicates:

M_{PR} $\neg\neg\exists nAn\rightarrow\exists nAn$, for A primitive recursive.

The other one is:

$\Psi(e)$ $\forall n\neg[lth(n)=j_2e+1\&\forall i\leq j_2e\,T(j_1e,i,(n)_i)]\rightarrow\exists i\leq j_2e\,\forall n\,\neg T(j_1e,i,n)$,

which can be read as: if there is no witness for $V_e=\emptyset$, then V_e must contain an element. An equivalent formulation would be:

$\neg\neg\exists i\leq y\forall nA(i,z,n)\rightarrow\exists i\leq y\forall nA(i,z,n)$ for primitive recursive A.

(Let us show this. One has to see:

$\forall n\neg(lth(n)=j_2e+1\ \&\ \forall i\leq j_2eT(j_1e,i,(n)_i))\leftrightarrow\neg\neg\exists i\leq j_2e\forall n\neg T(j_1e,i,n)$,

and use a standard Kleene normal form for $\Pi^0{}_1$-predicates.

Now $\leftarrow$ is trivial because $(lth(n)=j_2e+1\ \&\ \forall i\leq j_2eT(j_1e,i,(n)_i))$ of course implies $\neg\exists i\leq j_2e\forall n\neg T(j_1e,i,n)$.

For $\rightarrow$: suppose $\neg\exists i\leq j_2e\forall n\neg T(j_1e,i,n)$, then $\forall i\leq j_2e\neg\forall n\neg T(j_1e,i,n)$, so $\forall i\leq j_2e\neg\neg\exists nT(j_1e,i,n)$. And this implies $\neg\neg\forall i\leq j_2e\exists nT(j_1e,i,n)$ because of $\vdash\forall i\leq y\neg\neg\exists nT(z,i,n)\rightarrow\neg\neg\forall i\leq y\exists nT(z,i,n)$ (induction on y).

Now $\neg\neg\forall i\leq j_2e\exists nT(j_1e,i,n)$ gives at once $\neg\neg\exists n\forall i\leq j_2eT(j_1e,i,(n)_i)$, so $\neg\forall n\neg\forall i\leq j_2eT(j_1e,i,(n)_i)$, contradiction. Conclusion $\neg\neg\exists i\leq j_2e\forall n\neg T(j_1e,i,n)$.)

It is easy to show that, w.r.t. **EL**, $\forall e\Psi(e)$ is equivalent to some form of König's Lemma(see §3).

In the following, $V_e\neq\emptyset$ will be an abbreviation for $\exists x(x\leq j_2e\&\forall n\neg T(j_1e,x,n))$.

We define formulas $x\underline{r}A$, for Lifschitz' $\underline{r}$, in the obvious way.

The formalization of Lifschitz' soundness proof is completely straightforward.

§0. *Formalization of Lifschitz' realizability*

Lemma 0.1. There is a total recursive function b such that

$$\mathbf{HA}\vdash\forall a\forall y(y\in V_{b(a)}\leftrightarrow y=a).$$

Lemma 0.2. There is a partial recursive function ϕ such that

$$\mathbf{HA}\vdash\forall e(\exists x\forall y(y\in V_e\leftrightarrow y=x)\rightarrow\phi(e)\downarrow\&\phi(e)\in V_e).$$

The proofs are easy.

Lemma 0.3. There is a partial recursive function Φ such that

$$\mathbf{HA}+M_{PR}+\forall e\Psi(e)\vdash\forall e,f[\forall g\in V_e\{f\}(g)\downarrow\rightarrow\Phi(e,f)\downarrow\ \&$$
$$\&\ \forall h(h\in V_{\Phi(e,f)}\leftrightarrow\exists g\in V_e(h=\{f\}(g)))].$$

Proof. $\exists g\in V_e(h=\{f\}(g))\equiv\exists g\leq j_2e(\forall n\neg T(j_1e,g,n)\&\exists m(T(f,g,m)\&Um=h))$, which is, given that $\forall g\leq j_2e(\forall n\neg T(j_1e,g,n)\rightarrow\exists mT(f,g,m))$, equivalent to $\exists g\leq j_2e\forall n[\neg T(j_1e,g,n)\&(T(f,g,n)\rightarrow Un=h)]$, or $\exists g\leq j_2e\forall n\neg T(\chi(e,h,f),g,n)$ for a suitable primitive recursive χ; by $\forall e\Psi(e)$, $\exists g\leq j_2e\forall n\neg T(\chi(e,h,f),g,n)$ is equivalent to $\forall n\neg(\mathrm{lth}(n)=j_2e+1\ \&\forall i\leq j_2eT(\chi(e,h,f),i,(n)_i))$, or to $\forall n\neg T(\chi'(e,f),h,n)$ for suitable $\chi'(e,f)$; let Φ be $j(\chi'(e,f),\kappa)$ with $\kappa=\max\{Un|n=\min_z(T(j_1e,l,z)\vee T(f,l,z)),l\leq j_2e\}$. Note that this is defined, by M_{PR}.

Lemma 0.4. There is a total recursive function γ such that

$$\mathbf{HA}+M_{PR}+\forall e\Psi(e)\vdash\forall e\forall h(h\in V_{\gamma(e)}\leftrightarrow\exists g\in V_e(h\in V_g)).$$

Proof. $\exists g\in V_e(h\in V_g)$ is $\exists g\leq j_2e(\forall n\neg T(j_1e,g,n)\ \&\ h\leq j_2g\ \&\ \forall n\neg T(j_1g,h,n))$ or $\exists g\leq j_2e\forall n\neg T(\pi(e,h),g,n)$ for suitable π; which by $\forall e\Psi(e)$ is equivalent to $\forall n\neg(\mathrm{lth}(n)=j_2e+1\ \&\forall i\leq j_2eT(\pi(e,h),i,(n)_i))$ or $\forall n\neg T(\pi'(e),h,n)$ for suitable π'; so if we take $\gamma(e):=j(\pi'(e),\max\{j_2g|g\leq j_2e\})$, then γ satisfies the lemma.

Lemma 0.5. For every formula A in the language of **HA** there is a p-term $\lambda x.\chi_A(x)$ (which may contain variables occurring free in A) such that

$$\mathbf{HA}+M_{PR}+\forall e\Psi(e)\vdash\forall e(V_e\neq\emptyset\&\forall f\in V_e(f\underline{r}A)\rightarrow\chi_A(e)\downarrow\&\chi_A(e)\underline{r}A).$$

Lemma 0.6. For every theorem A of **HA** there is a number n such that

$$\mathbf{HA}+M_{PR}+\forall e\Psi(e)\vdash n\underline{r}A.$$

Lemmas 0.5 and 0.6 are immediate formalizations of Lifschitz' lemmas 5 and 6.

§1. *Characterization of Lifschitz' realizability.*

The following lemma gives a more manageable form to Lifschitz' realizability.

Lemma 1.1. Define a realizability $\mathbf{r'}$ by the following clauses:

1) $x\mathbf{r'}\ t=s \equiv V_x\neq\emptyset\ \&\ \forall y\in V_x\ \ t=s$ (y not in $t=s$!)
2) $x\mathbf{r'}\ A\&B \equiv V_x\neq\emptyset\ \&\ \forall y\in V_x\ \ j_1y\mathbf{r'}A\ \&\ j_2y\mathbf{r'}B$
3) $x\mathbf{r'}\ A\to B \equiv V_x\neq\emptyset\ \&\ \forall y\in V_x\ \ \forall w(w\mathbf{r'}A\to\{y\}(w)\downarrow\ \&\ \{y\}(w)\mathbf{r'}B)$
4) $x\mathbf{r'}\ \forall zAz \equiv V_x\neq\emptyset\ \&\ \forall y\in V_x\ \ \forall n(\{y\}(n)\downarrow\ \&\ \{y\}(n)\mathbf{r'}A(n))$
5) $x\mathbf{r'}\ \exists zAz \equiv V_x\neq\emptyset\ \&\ \forall y\in V_x\ \ j_2y\mathbf{r'}A(j_1y)$

Then for every formula A in the language of **HA** there are recursive functions ϕ_A and ψ_A (they may contain variables occurring free in A) such that

$\mathbf{HA}+M_{PR}+\forall e\Psi(e) \vdash \forall e(e\underline{\mathbf{r}}A\to\phi_A(e)\downarrow\ \&\ \phi_A(e)\mathbf{r'}A)$

$\mathbf{HA}+M_{PR}+\forall e\Psi(e) \vdash \forall e(e\mathbf{r'}A\to\psi_A(e)\downarrow\ \&\ \psi_A(e)\underline{\mathbf{r}}A)$,

where $\underline{\mathbf{r}}$ denotes Lifschitz' realizability. (Note the form of the clauses: apart from a prefix $V_x\neq\emptyset\ \&\ \forall y\in V_x$, it is just the Kleene clauses.)

Proof.Definition of ϕ_A and ψ_A and proof of the lemma simultaneously by induction on A. The notation is from the lemmas in §0. Following Lifschitz we write g^* for $\lambda f.\Phi(f,g)$, where Φ is as in lemma 0.3.

i) $\phi_{t=s}(e) \equiv b(e)$

$\psi_{t=s}(e) \equiv 0.$

ii) $\phi_{A\&B}(e) \equiv b(j(\phi_A(j_1e),\phi_B(j_2e)))$

$\psi_{A\&B}(e) \equiv j(\chi_A((\psi_A\cdot j_1)^*(e)),\chi_B((\psi_B\cdot j_2)^*(e))).$

iii) $\phi_{A\to B}(e) \equiv b(\lambda h.\phi_B(e(\psi_A(h)))).$

$\psi_{A\to B}(e) \equiv \chi_{A\to B}(g^*(e))$, where $g\equiv\lambda f.\lambda a.\psi_B(f(\phi_A(a))).$

iv) $\phi_{\forall xAx}(e) \equiv b(\lambda n.\phi_A[^n/_x](\{e\}(n))).$

$\psi_{\forall xAx}(e) \equiv \chi_{\forall xAx}(g^*(e))$, where $g\equiv\lambda f.(\lambda n.\psi_A[^n/_x](\{f\}(n))).$

v) $\phi_{\exists xAx}(e) \equiv g^*(e)$ with $g\equiv\lambda f.j(j_1f,\phi_A[j_1f/_x](j_2f))$

$\psi_{\exists xAx}(e) \equiv g^*(e)$ with $g\equiv\lambda f.j(j_1f,\psi_A[j_1f/_x](j_2f))$. We trust that the reader will be able to carry out the proof by himself.

Definition. Let Γ be the class of formulas inductively generated by the clauses:

1) $\Sigma^0{}_1$-formulas are in Γ;
2) Formulas of form $\exists x \leq y\, Ax$, with $A \in \Pi^0{}_1$, are in Γ;
3) Γ is closed under $\forall$, $\rightarrow$ and $\wedge$.

As Γ will play a role similar to that of the "almost negative" formulas in §3.2 of Troelstra 1973, which could be termed $\Sigma^0{}_1$-negative, let us call Γ-formulas "$\Sigma^0{}_2$-negative".

Lemma 1.2. (cf. l.c., 3.2.11) For every $\Sigma^0{}_2$-negative formula $A(\mathbf{a})$ (with free variables $\mathbf{a}$) there is a partial recursive function ψ_A satisfying

i) $\mathbf{HA}+M_{PR}+\forall e \Psi(e) \vdash \exists u(u\mathbf{r}'A) \rightarrow A$

ii) $\mathbf{HA}+M_{PR}+\forall e \Psi(e) \vdash A(\mathbf{a}) \rightarrow \psi_A(\mathbf{a})\downarrow \;\&\; \psi_A(\mathbf{a})\mathbf{r}'A$.

Proof. We prove i) and ii) simultaneously by induction on A.

1) Suppose A is $\Sigma^0{}_1$: $A \equiv \exists y By$, B prime; then $u\mathbf{r}'A$ is $V_u \neq \emptyset \;\&\; \forall f \in V_u\; V_{j_2 f} \neq \emptyset \;\&\; \forall h \in V_{j_2 f} B(j_1 f)$ which clearly implies A; for ii) take $\psi_A \equiv b(j(\chi_B, b(0)))$ where $\chi_B \simeq \mu x.Bx$. For then, A implies $\chi_B\downarrow$ and $b(0)\mathbf{r}'B(\chi_B)$, so $\psi_A \mathbf{r}' \exists x Bx$.

2) Suppose $A \equiv \exists x \leq t\, Bx$, x not in t, B is $\Pi^0{}_1$; say $B \equiv \forall y Cxy$, then $u\mathbf{r}'A$ is equivalent to

(∗) $V_u \neq \emptyset \;\&\; \forall h \in V_u\; V_{j_2 h} \neq \emptyset \;\&\; \forall k \in V_{j_2 h} \forall n[\{k\}(n)\downarrow \;\&\; \{k\}(n)\mathbf{r}'(j_1 h \leq t \;\&\; Cj_1 hn)]$ which implies $V_u \neq \emptyset \;\&\; \forall h \in V_u\; V_{j_2 h} \neq \emptyset \;\&\; \forall n\, Cj_1 hn$ which implies A. For ii) let e be such that $A \equiv V_e \neq \emptyset$, and let u such that $V_u = \{j(j_1 h, b(\lambda n.b(0))) \mid h \in V_e\}$; then $V_e \neq \emptyset$ implies (∗) for u.

3) We will only do the case $A \equiv B \rightarrow C$; the other cases are left to the reader. $u\mathbf{r}'A$ is $V_u \neq \emptyset \;\&\; \forall h \in V_u \forall x(x\mathbf{r}'B \rightarrow \{h\}(x)\downarrow \;\&\; \{h\}(x)\mathbf{r}'C)$. Now if B then $\psi_B \mathbf{r}'B$ so $\forall h \in V_u\, \{h\}(\psi_B)\downarrow \;\&\; \{h\}(\psi_B)\mathbf{r}'C$; so if χ is such that $V_{\chi(u)} = \{\{h\}(\psi_B) \mid h \in V_u\}$ then $\phi(\chi(u))\mathbf{r}'C$, so C; But if $B \rightarrow C$ then $b(\lambda u.\psi_C)\mathbf{r}'B \rightarrow C$, for suppose $u\mathbf{r}'B$, then B, so C, so $\psi_C \mathbf{r}'C$.

<u>Remark</u>. So the $\Sigma^0{}_2$-negative formulas are the "self-realizing" formulas for this realizability. As a quick glance reveals that formulas of form $x\mathbf{r}'A$ are $\Sigma^0{}_2$-negative, this realizability is idempotent.

Furthermore, since $\forall e\Psi(e)$ is also Σ^0_2-negative, as well as M_{PR}, we see that the soundness theorem for **HA** for this realizability can be extended to **HA**$+M_{PR}+\forall e\Psi(e)$.

We now introduce a principle analogous to ECT_0. Consider

ECT_L $\quad \forall x(Ax\to\exists yBxy) \to \exists z\forall x(Ax\to\{z\}(x)\downarrow \& V_{\{z\}(x)}\neq\emptyset \& \forall h\in V_{\{z\}(x)}Bxh)$,

for A Σ^0_2-negative.

Lemma 1.3.(cf. Troelstra 1973,3.2.15) ECT_L is **r'**-realizable.

Proof. Suppose u**r'** $\forall x(Ax\to\exists yBxy)$. This is:

$V_u\neq\emptyset \& \forall f\in V_u\forall n(\{f\}(n)\downarrow \& V_{\{f\}(n)}\neq\emptyset \& \forall h\in V_{\{f\}(n)}\forall w(w\mathbf{r'}An\to\{h\}(w)\downarrow \& V_{\{h\}(w)}\neq\emptyset$
$\& \forall k\in V_{\{h\}(w)}(j_2k\mathbf{r'}Bnj_1k)))$.

Let us simplify a bit. Let u' be such that $\forall n(\{u'\}(n)\downarrow \& V_{\{u'\}(n)}=\cup(V_{\{f\}(n)}| f\in V_u))$, then $\forall h\in V_{\{u'\}(n)}\forall w(w\mathbf{r'}An\to\{h\}(w)\downarrow \& V_{\{h\}(w)}\neq\emptyset \& \forall k\in V_{\{h\}(w)}(j_2k\mathbf{r'}Bnj_1k))$. Put $\beta\equiv\{h\}(\psi_A(n))$, u" such that $V_{\{u"\}(n)}=\cup(V_\beta| h\in V_{\{u'\}(n)})$, then
$\forall w(w\mathbf{r'}An\to\{u"\}(n)\downarrow \& V_{\{u"\}(n)}\neq\emptyset \& \forall k\in V_{\{u"\}(n)}(j_2k\mathbf{r'}Bnj_1k))$. It is clear that u" can be obtained recursively in u.

Now choose z with $\forall x\, V_{\{z\}(x)}=j_1[V_{\{u"\}(x)}]$, γ' such that $V_{\gamma'(m)}=\{k| j(m,k)\in V_{\{u"\}(x)}\}$, $\gamma"$ such that $V_{\gamma"(m)}=\{\lambda y.\phi(\gamma'(m))\}$. Then we have $V_{\gamma"(m)}\neq\emptyset$, and if $g\mathbf{r'}m\in V_{\{z\}(x)}$ then $m\in V_{\{z\}(x)}$ (since this is Σ^0_2-negative), so $V_{\gamma'(m)}\neq\emptyset \& \forall k\in V_{\gamma'(m)}k\mathbf{r'}Bxm$, so $\phi(\gamma'(m))\mathbf{r'}Bxm$. Let $\gamma\equiv b(\gamma')$, then
$V_\gamma\neq\emptyset \& \forall l\in V_\gamma\forall m(\{l\}(m)\downarrow \& V_{\{l\}(m)}\neq\emptyset \& \forall p\in V_{\{l\}(m)}\forall g(g\mathbf{r'}(m\in V_{\{z\}(x)})\to\{p\}(g)\downarrow \&$
$\{p\}(g)\mathbf{r'}Bxm))$, which is
$\gamma\mathbf{r'}\forall h(h\in V_{\{z\}(x)}\to Bxh)$. The rest is easy.

Theorem 1.4(cf. l.c. 3.2.18; characterization of r'-realizability).

i) **HA**$+M_{PR}+\forall e\Psi(e)+ECT_L\vdash A \leftrightarrow \exists x(x\mathbf{r'}A)$;

ii)**HA**$+M_{PR}+\forall e\Psi(e)\vdash \exists x(x\mathbf{r'}A) \Leftrightarrow$ **HA**$+M_{PR}+\forall e\Psi(e)+ECT_L\vdash A$.

Proof. i) is proved by induction on A. As usual, the only non-trivial steps are $A\equiv B\to C$ and (similar) $A\equiv\forall yBy$.

Now $(B\to C) \leftrightarrow \forall x(x\mathbf{r'}B\to\exists y(y\mathbf{r'}C)) \leftrightarrow \exists z\forall x(x\mathbf{r'}B\to\{z\}(x)\downarrow \& V_{\{z\}(x)}\neq\emptyset$
$\& \forall y\in V_{\{z\}(x)}(y\mathbf{r'}C)) \leftrightarrow \exists z\forall x(x\mathbf{r'}B\to\{z\}(x)\downarrow \& \{z\}(x)\mathbf{r'}C) \leftrightarrow \exists x(x\mathbf{r'}B\to C)$. We leave the other case to the reader.

The proof of ii) (using i)) is completely analogous to 3.2.18 of Troelstra 1973.

Remarks on ECT_L. i) $ECT_L!$ is equivalent to a schema which resembles $ECT_0!$ except for the condition that A can be taken Σ^0_2-negative. We see that this schema is consistent relative to **HA**, whereas ECT_0 w.r.t. Σ^0_2-negative formulas is not: if W_e and W_f are disjoint, recursively inseparable r.e. sets, let F be such that $\forall x\,\{F(x)\}(0) \simeq \{e\}(x)$, $\{F(x)\}(1) \simeq \{f\}(x)$, then $V_{j(F(x),1)} \neq \emptyset$ for all x, so let $Ax \equiv V_{j(F(x),1)} \neq \emptyset$ (Σ^0_2-negative), $Bxy \equiv y \in V_{j(F(x),1)}$. Any z as in the conclusion of the schema will give a recursive separation between W_e and W_f.

ii) The example given in 3.2.20 of Troelstra 1973 ($A \equiv \exists yTxxy \vee \neg\exists yTxxy$, $B \equiv (z=0 \to \exists yTxxy \;\&\; z=1 \to \neg Txxy)$) shows that the restriction to Σ^0_2-negative formulas cannot be dropped.

iii) We can define a **q'**-realizability corresponding to **r'**-realizability by the clauses:

1) $x\mathbf{q}'\ t=s \equiv V_x \neq \emptyset \;\&\; \forall y \in V_x\ t=s$

2) $x\mathbf{q}'\ A\&B \equiv V_x \neq \emptyset \;\&\; \forall y \in V_x\ j_1 y \mathbf{q}' A \;\&\; j_2 y \mathbf{q}' B$

3) $x\mathbf{q}'\ A \to B \equiv V_x \neq \emptyset \;\&\; \forall y \in V_x\ \forall w(w\mathbf{q}'A \to \{y\}(w)\downarrow \;\&\; \{y\}(w)\mathbf{q}'B) \;\&\; A \to B$

4) $x\mathbf{q}'\ \forall zAz \equiv V_x \neq \emptyset \;\&\; \forall y \in V_x\ \forall n(\{y\}(n)\downarrow \;\&\; \{y\}(n)\mathbf{q}'A(n))$

5) $x\mathbf{q}'\ \exists zAz \equiv V_x \neq \emptyset \;\&\; \forall y \in V_x\ j_2 y \mathbf{q}' A(j_1 y)$

Proposition 1.5. $\mathbf{HA}+M_{PR}+\forall e\Psi(e) \vdash A \Rightarrow \mathbf{HA}+M_{PR}+\forall e\Psi(e) \vdash n\mathbf{q}'A$ for some n; $\mathbf{HA}+M_{PR}+\forall e\Psi(e) \vdash y\mathbf{q}'A \to A$; If A is Σ^0_2-negative, $\mathbf{HA}+M_{PR}+\forall e\Psi(e) \vdash A \to \psi_A \mathbf{q}'A$ for ψ_A as in lemma 1.2.

Proof. The first statement is proved by a routine induction on lengths of deductions in **HA**; the reader may wish to consult Theorem 3.2.4 of Troelstra 1973. The other two statements are proved by induction on A.

Corollary 1.6. $\mathbf{HA}+M_{PR}+\forall e\Psi(e)$ obeys the following rule:
$\vdash \forall x(Ax \to \exists yBxy) \Rightarrow \exists z \vdash \forall x(Ax \to \{z\}(x)\downarrow \;\&\; V_{\{z\}(x)} \neq \emptyset \;\&\; \forall h \in V_{\{z\}(x)} Bxh)$, for A Σ^0_2-negative.

§2. *Extension of Lifschitz' realizability to* ***HAS***.

The extension of Kleene's realizability to **HAS**, described in Troelstra 1973, is given by the simple clauses:

$x\mathbf{r}\,(t_0,...,t_{n-1})\in X \equiv (t_0,...,t_{n-1},x)\in X^*$

$x\mathbf{r}\,\forall XA(X) \equiv \forall X^*\, x\mathbf{r}A(X)$

$x\mathbf{r}\,\exists XA(X) \equiv \exists X^*\, x\mathbf{r}A(X)$,

where $X\to X^*$ is an operation that assigns to each n-ary set variable X a n+1-ary set variable X^* from a fresh stock of variables.

As a consequence, this extension satisfies the Uniformity Principle:

UP $\forall X\exists nA(X,n)\to\exists n\forall XA(X,n)$.

Now this cannot work for Lifschitz' realizability, because in that case we would have all realizability clauses equal for both interpretations except for the clause for the numerical existential quantifier; but this quantifier can be eliminated in **HAS**, because of the equivalence

$$\exists yA(y) \leftrightarrow \forall X(\forall y(Ay\to X)\to X),$$

that holds in systems based on second-order logic with full comprehension. So then these two interpretations would be the same, quod non. However, combined with lemma 1.1, this idea suggests the following extension:

6) $x\mathbf{r}'\,(t_0,...,t_{n-1})\in X \equiv V_x\neq\emptyset\ \&\ \forall y\in V_x\ (t_0,...t_{n-1},y)\in X^*$

7) $x\mathbf{r}'\,\forall XA(X) \equiv V_x\neq\emptyset\ \&\ \forall y\in V_x\ \forall X^*\ y\mathbf{r}'A(X)$

8) $x\mathbf{r}'\,\exists XA(X) \equiv V_x\neq\emptyset\ \&\ \forall y\in V_x\ \exists X^*\ y\mathbf{r}'A(X)$.

Theorem 2.1. $\mathbf{r}'$ is a sound realizability for $\mathbf{HAS}+\forall e\Psi(e)+M_{PR}$.

Proof. The verification of the rules for second-order predicate logic does not pose any problem. For instance, if $\psi(y)\mathbf{r}'\,A(y)\to B$, y not in B, and $x\mathbf{r}'\exists yAy$, where A and B are arbitrary formulas in the language of **HAS**, then $V_x\neq\emptyset$ & $\forall y\in V_x\ j_2y\mathbf{r}'A(j_1y)$, so $V_x\neq\emptyset\ \&\ \forall y\in V_x\forall h\in V_{\psi(j_1y)}\ \{h\}(j_2y)\downarrow\ \&\ \{h\}(j_2y)\mathbf{r}'B$. Let χ be such that $V_\chi=\{\{h\}(j_2y)|h\in V_{\psi(j_1y)},\ y\in V_x\}$, then $\phi(\chi(x))\mathbf{r}'B$, so $b(\lambda x.\phi(\chi(x)))\ \mathbf{r}'\ \exists yAy\to B$, where b and ϕ are as defined in lemmas 0.1 and 0.4.

For the comprehension schema:

CA $\exists X\forall y(y\in X\leftrightarrow Ay)$,

first note that the following holds:

$(*)\ V_k\neq\emptyset\ \&\ \forall l\in V_k\exists k'(l\in V_{k'}\ \&\ k'\mathbf{r}'A)\to k\mathbf{r}'A$ (Trivial from the definition of $\mathbf{r}'$-

realizability). Now x $\mathbf{r}'$ $\exists X\forall y(y\in X\leftrightarrow Ay)$ means

$$V_x\neq\emptyset\ \&\ \forall f\in V_x\exists X^*\forall y[\ \{f\}(y)\downarrow\ \&\ \forall k$$

(o) $$(V_k\neq\emptyset\ \&\ \forall l\in V_k\ (y,l)\in X^*\rightarrow\{j_1(\{f\}(y))\}(k)\downarrow\&\ \{j_1(\{f\}(y))\}(k)\mathbf{r}'Ay\ \ \&$$
$$k\mathbf{r}'Ay\rightarrow t=\{j_2(\{f\}(y))\}(k)\downarrow\ \&\ V_t\neq\emptyset\ \&\ \forall l\in V_t\ (y,l)\in X^*)].$$

Now let $V_x=\{f\}$, with f such that $\{j_1(\{f\}(y))\}(k)=\{j_2(\{f\}(y))\}(k)=k$; and if $X^*=\{(y,l)|\exists k(k\mathbf{r}'Ay\ \&\ l\in V_k)\}$, then (o) is easily verified for f, x, and X^*, using (∗).

The verification of extensionality

EXT $\quad Ay\ \&\ y=x\rightarrow Ax,$

is completely trivial, which concludes the proof.

§3. *A Lifschitz analogon to realizability for functions.*

Description of EL. The language of **EL** contains, in addition to the language of **HA**, variables for functions, an application operator Ap, a recursor R and abstraction operators λx. for every number variable x, such that the following hold:

1) function variables are functors (i.e. terms for functions);
2) function constants are functors (for example, the constants for all primitive recursive functions);
3) If ϕ is a functor and t a term then $Ap(\phi,t)$, always written $\phi(t)$, is a term;
4) R is a functor;
5) If t, t' are terms and ϕ is a functor then $R(t,\phi,t')$ is a term;
6) If t is a term and x a number variable then $\lambda x.t$ is a functor.

The non-logical axioms and rules of **EL** are:

λ-CON: $\quad (\lambda x.t)(t')=t[^{t'}/_x]$, and

R-ax: $\quad R(t,\phi,0)=t$ and $R(t,\phi,St')=\phi(R(t,\phi,t'),t')$.

$QF\text{-}AC_{00}$: $\quad \forall x\exists yAxy\rightarrow\exists\alpha\forall xA(x,\alpha x)$, for A quantifier-free.

EL is discussed extensively in Troelstra 1973, as well as Kleene's function-realizability for **EL**, based on partial continuous application. Let us fix some notation.

$\bar{\alpha}0\equiv\langle\ \rangle$; $\bar{\alpha}(k+1)\equiv\bar{\alpha}k*\langle\alpha(k)\rangle$ where $\langle\ \rangle$ denotes the empty sequence, and $*$

concatenation of finite sequences.
$\langle n\rangle^{[m]} \equiv \mu\sigma.(\mathrm{lth}(\sigma)=m \;\&\; \forall i<m\; \sigma_i=n)$.
$\beta(\alpha)\downarrow$ means $\exists x\,(\beta(\overline{\alpha}x)\neq 0)$, and $\beta(\alpha)\equiv\beta(\overline{\alpha}(\mu z.\beta(\overline{\alpha}z)>0))\dot{-}1$;
$\beta|\alpha\downarrow$ means $\forall x\,\beta(\langle x\rangle * \alpha)\downarrow$, and $\beta|\alpha \equiv \lambda x.\beta(\langle x\rangle * \alpha)$;
$[n]$ will stand for $\lambda x.n$.
$\supset$ will stand for the partial ordering on finite sequences;
$\sigma\supset\tau$ means that σ is an initial segment of τ.
$\alpha\in\sigma$ says $\forall i<\mathrm{lth}(\sigma)\,(\sigma_i=\alpha(i))$;
$\beta\leq\alpha$ is $\forall i(\beta(i)\leq\alpha(i))$;
$j_i\alpha\equiv\lambda x.j_i(\alpha(x))$, for $i=1,2$.

The obvious analogon in the language of functions of the coding V_e is to put $V_\alpha\equiv\{\beta\leq j_2\alpha \mid j_1\alpha(\beta)\uparrow\}=\{\beta\leq j_2\alpha\mid\forall n\beta(\overline{j_1\alpha}n)=0\}$.
If we read the principle $\forall e\Psi(e)$ from §0 as: ((there is no witness n for $V_e=\emptyset$) $\rightarrow V_e\neq\emptyset$), then the analogous principle in the language of **EL** is:
$\forall n\,\neg\forall\sigma\,[(\mathrm{lth}(\sigma)=n \;\&\forall i<n\;\; \sigma_i\leq j_2\alpha(i))\rightarrow\exists\tau\supset\sigma\; j_1\alpha(\tau)>0]\rightarrow\exists\beta\leq j_2\alpha\,\forall n\; j_1\alpha(\overline{\beta}n)=0$,
which amounts to a version of König's Lemma.
In fact, if we put $P(\sigma)\equiv\forall i<\mathrm{lth}(\sigma)\exists m<\mathrm{lth}(\sigma)T(j_1e,(\sigma)_i,m)$, then $\Psi(e)$ is in **EL** equivalent to $\forall n\exists\sigma[\mathrm{lth}(\sigma)=n+1\&\forall i\leq n(\sigma)_i\leq j_2e\,\&\neg P(\sigma)]\rightarrow\exists\beta\leq[j_2e]\forall n\neg P(\overline{\beta}(n+1))$.
(For in **EL** one has: $\exists i\leq j_2e\forall n\neg T(j_1e,i,n)\leftrightarrow\; \exists\beta\leq[j_2e]\forall n\neg P(\overline{\beta}(n+1))$,
and in **HA**: $\forall n\neg(\mathrm{lth}(n)=j_2e+1 \;\&\; \forall i\leq j_2eT(j_1e,i,(n)_i))\leftrightarrow\forall n\neg\forall i\leq j_2e\exists m\leq nT(j_1e,i,m)\leftrightarrow$ $\forall n\exists i\leq j_2e\forall m\leq n\neg T(j_1e,i,m)\leftrightarrow\forall n\exists\sigma[\mathrm{lth}(\sigma)=n+1 \;\&\; \forall i\leq n(\sigma)_i\leq j_2e \;\&\; \neg P(\sigma)]$. From this the equivalence easily follows.)
To prove the appropriate closure properties of the sets V_α, we will work in the theory **EL**+MP_{QF}+KL_{QF}, where MP_{QF} denotes Markov's Principle w.r.t. quantifier-free formulas and KL_{QF} will be:

KL_{QF} $\qquad \forall n\,\exists\sigma\,(\mathrm{lth}(\sigma)=n \,\&\forall i<n(\sigma_i\leq\alpha(i))\;\&R\sigma)\rightarrow$
$\qquad\qquad \exists\beta\,\forall n\,(\beta(n)\leq\alpha(n)\;\&\; R(\overline{\beta}n))$, for R quantifier-free.

We see, using the equivalent formulation of $\forall e\Psi(e)$ given in §0 and the well-known fact that every finitely branching tree can be encoded as a subtree of e.g. the binary tree, that $\forall e\Psi(e)$ is actually equivalent (in **EL**) to KL_{QF}.
Observe that KL_{QF}+$MP_{QF}\vdash FAN_{QF}$, where FAN_{QF} is the schema:

FAN_{QF} $\qquad \forall\beta\leq\alpha\,\exists nR\overline{\beta}n\rightarrow\exists z\,\forall\beta\leq\alpha\,\exists n\leq z\,R\overline{\beta}n$, R quantifier-free.

Also note that KL_{QF}+QF-AC_{00} is sufficient to prove KL for $\Sigma^0{}_1$-formulas R.

(Suppose $\forall n\exists\sigma[\mathrm{lth}(\sigma)=n \ \&\ \forall i<n(\sigma)_i\le\alpha(i) \ \&\ \exists m R(\sigma,m)]$, so $\forall n\,\exists m\exists\sigma[\mathrm{lth}(\sigma)=n \ \&$ $\forall i<n(\sigma)_i\le\alpha(i) \ \&\ R(\sigma,m)]$, so (QF-$AC_{00}$) $\exists\alpha_1\alpha_2\forall n\,[\mathrm{lth}(\alpha_1(n))=n \ \&\ \forall i<n(\alpha_1(n))_i\le\alpha(i) \ \&$ $R(\alpha_1(n),\alpha_2(n))]$, so $\exists\alpha_2\forall n\exists\sigma\,[\mathrm{lth}(\sigma)=n \ \&\ \forall i<n(\sigma)_i\le\alpha(i) \ \&\ R(\sigma,\alpha_2(n))]$ which gives with KL_{QF}: $\exists\alpha_2\exists\beta\forall n[\beta(n)\le\alpha(n) \ \&\ R(\overline{\beta}(n),\alpha_2(n))]$, so $\exists\beta\forall n[\beta(n)\le\alpha(n) \ \&\ \exists m R(\overline{\beta}(n),m)]$.)
In the following, we will denote MP_{QF}, KL_{QF} and FAN_{QF} simply by MP, KL and FAN, respectively.
We will make use of the expressions "p-term" and "p-functor" as in Kleene 1969.

Definition. We define for every formula A a formula $\alpha \,\underline{r}\, A$ with $\alpha\notin FV(A)$ and $FV(\alpha \,\underline{r}\, A) \subset \{\alpha\}\cup FV(A)$ as follows:

1) $\alpha \,\underline{r}\ A \equiv A$ for A atomic;
2) $\alpha \,\underline{r}\ A\&B \equiv j_1\alpha \,\underline{r}\, A \ \&\ j_2\alpha \,\underline{r}\, B$;
3) $\alpha \,\underline{r}\ A\to B \equiv \forall\beta(\beta \,\underline{r}\, A \to \alpha|\beta\downarrow \ \&\ \alpha|\beta \,\underline{r}\, B)$;
4) $\alpha \,\underline{r}\ \forall x A x \equiv \forall n\,(\alpha|[n]\downarrow \ \&\ \alpha|[n] \,\underline{r}\, An)$;
5) $\alpha \,\underline{r}\ \exists x A x \equiv V_\alpha\neq\emptyset \ \&\ \forall\gamma\in V_\alpha(j_2\gamma \,\underline{r}\, A(j_1\gamma(0)))$;
6) $\alpha \,\underline{r}\ \forall\beta A(\beta) \equiv \forall\beta\,(\alpha|\beta\downarrow \ \&\ \alpha|\beta \,\underline{r}\, A(\beta))$;
7) $\alpha \,\underline{r}\ \exists\beta A(\beta) \equiv V_\alpha\neq\emptyset \ \&\ \forall\gamma\in V_\alpha(j_2\gamma \,\underline{r}\, A(j_1\gamma))$.

The proof that **EL** is sound for this realizability, goes completely parallel to the proof of §0.

Lemma 3.1. There is a p-functor β_1, such that
$EL+KL+MP\vdash\forall\alpha(V_\alpha$ is a singleton$\to\beta_1|\alpha\downarrow \& \ \beta_1|\alpha\in V_\alpha)$.

Proof. Write $B_\alpha\equiv\{\beta|\ \beta\le j_2\alpha\}$.
If $V_\alpha=\{\beta\}$ then for every n and m such that $m\le j_2\alpha(n)$ and $m\neq\beta(n)$, a finite computation suffices to show that $j_1\alpha(\gamma)\downarrow$, for every γ such that $\gamma\in\overline{\beta}n*\langle m\rangle$ and $\gamma\in B_\alpha$. For, $\{\gamma\in B_\alpha \mid \gamma\in\overline{\beta}n*\langle m\rangle\}$ is a finitely branching tree. (Here, of course, we are using FAN.)

Now $\forall\gamma\in B_\alpha(\ \gamma\in\overline{\beta}n*\langle m\rangle \Rightarrow j_1\alpha(\gamma)\downarrow)$ holds for every $m\le j_2\alpha(n)$ save one; a finite computation shows this and the remaining $m\le j_2\alpha(n)$ must be equal to $\beta(n)$.

Lemma 3.2. There is a p-functor β_2, such that
$EL+KL+MP\vdash\forall\alpha\,(\beta_2|\alpha\downarrow \ \&\ V_{\beta_2|\alpha}=\{\alpha\})$.

Proof. Let γ be such that $\forall\alpha\,((\gamma|\alpha)(\sigma)=0 \leftrightarrow \alpha\in\sigma)$; take β_2 such that $\forall\alpha\,(\beta_2|\alpha = j(\gamma|\alpha,\alpha))$.

The following sublemma, trivial as it may be, greatly simplifies the proofs of the lemmas thereafter, and will be applied frequently.

Sublemma 3.1. Let $A(\beta)$ and $C(\beta,\gamma)$ be formulas such that:
1) there is a p-functor ψ such that $A(\beta)\vdash \psi|\beta\downarrow \;\&\; \forall\gamma(C(\beta,\gamma)\rightarrow \gamma\leq\psi|\beta)$;
2) $A(\beta)\vdash C(\beta,\gamma) \leftrightarrow \forall n\, D(\beta,\gamma,n)$, where D is a prime formula.
Then there is a p-functor Φ such that:
$\mathbf{EL+KL+MP}\vdash A(\beta) \rightarrow \Phi|\beta\downarrow \;\&\; \forall\gamma(\gamma\in V_{\Phi|\beta} \leftrightarrow C(\beta,\gamma))$.

Proof. If D is the prime formula from 2), there is a prime formula $D'(\beta,\sigma)$ such that $D(\beta,\gamma,n)$ is equivalent to $D'(\beta,\overline{\gamma}n)$. Now let χ be defined as follows: $\chi(\sigma) = 0$ if $D'(\beta,\sigma)$; $\chi(\sigma) = 1$ else.
Now put $\Phi:=\Lambda\beta.j(\chi,\psi)$, where ψ is the functor from condition 1).

Lemma 3.3. There is a p-functor β_3 such that
$\mathbf{EL+KL+MP}\vdash \forall\alpha\,(\beta_3|\alpha\downarrow \& \; V_{\beta_3|\alpha} = \cup_{\gamma\in V_\alpha} V_\gamma)$.

Proof. We apply sublemma 3.1.
$\varepsilon\in \cup_{\gamma\in V_\alpha} V_\gamma \rightarrow \varepsilon\leq \max\,\{j_2\gamma|\,\gamma\leq j_2\alpha\}$ is easy to see. Furthermore, the formula $\beta\in \cup_{\gamma\in V_\alpha} V_\gamma$ is equivalent to $\exists\gamma\leq j_2\alpha\,\forall n\,(j_1\alpha(\overline{\gamma}n)=0 \;\&\; \beta(n)\leq j_2\gamma(n) \;\&\; j_1\gamma(\overline{\beta}n)=0)$ which is, modulo KL and MP, equivalent to
$\forall n\exists\sigma[\mathrm{lth}(\sigma)=n \;\&\; j_1\alpha(\sigma)=0 \;\&\; \forall k<n(\sigma_k\leq j_2\alpha(k) \;\&\; (\overline{\beta}n)_k\leq j_2(\sigma_k) \;\&\; (\overline{\beta}k<n \rightarrow j_1(\sigma_{\overline{\beta}k})=0)]$
which is a formula of the form required in condition 2) of the sublemma.

Lemma 3.4. There is a p-functor Φ such that
$\mathbf{EL+KL+MP}\vdash \forall\phi,\beta[\forall\alpha(\alpha\in V_\beta\rightarrow\phi|\alpha\downarrow)\rightarrow\Phi|(\phi,\beta)\downarrow \;\&$
$\forall\alpha(\alpha\in V_{\Phi|(\phi,\beta)}\leftrightarrow\exists\gamma(\gamma\in V_\beta \;\&\; \alpha=\phi|\gamma))]$.
In other words: $V_\beta\subseteq \mathrm{dom}(\phi) \rightarrow \phi|[V_\beta]=V_{\Phi|(\phi,\beta)}$.
In the following, for p-functors ϕ, we will abbreviate ϕ^* for the p-functor $\Lambda\beta.\Phi(\phi,\beta)$.

Proof. Again, we check the conditions of sublemma 3.1.

1) Suppose $\forall\alpha(\alpha\in V_\beta \to \phi|\alpha\downarrow)$. So:

$$\forall x\, \forall\alpha\leq j_2\beta\, (\forall n(j_1\beta(\bar{\alpha}n)=0)\to(\phi|\alpha)(x)\downarrow),$$

which is equivalent to

$$\forall x\, \forall\alpha\leq j_2\beta\, \neg\neg\exists n(j_1\beta(\bar{\alpha}n)\neq 0 \vee \phi(\langle x\rangle * \bar{\alpha}n)\neq 0)$$

which by MP is equivalent to

$$\forall x\, \forall\alpha\leq j_2\beta\, \exists n\, (j_1\beta(\bar{\alpha}n)\neq 0 \vee \phi(\langle x\rangle * \bar{\alpha}n)\neq 0\,)$$

which in turn, by FAN, is equivalent to

$$\forall x\, \exists n\, \forall\alpha\leq j_2\beta\, \exists z\leq n\, (j_1\beta(\bar{\alpha}z)\neq 0 \vee \phi(\langle x\rangle * \bar{\alpha}z)\neq 0\,).$$

Note that the part following $\forall x\,\exists n$ is actually quantifier-free, so define ψ by

$$\psi \equiv \lambda x.\mu n.[\forall\alpha\leq j_2\beta\, \exists z\leq n\, (j_1\beta(\bar{\alpha}z)\neq 0 \vee \phi(\langle x\rangle * \bar{\alpha}z)\neq 0)\,].$$

Let $\Phi(x,z)$ be $(\phi|\alpha)(x)$, if $\phi(\langle x\rangle * \bar{\alpha}z)\neq 0$ (and otherwise, for example, undefined). Now put

$\eta(x)$ $\quad = \max\{\Phi(x,z) \mid z\leq\psi(x)\ \&\ z$ witnesses $(\phi|\alpha)(x)\downarrow\}$;
$\quad\quad\quad = 0$ if this set is empty;

then $\chi \equiv \lambda x.\eta(x)$ is the required upper bound.

2) Now $\gamma\in\phi|[V_\beta]$ is, modulo $V_\beta\subseteq dom(\phi)$, equivalent to a $\Pi^0{}_1$-formula: for, $\exists\delta\in V_\beta(\gamma=\phi|\delta)$ is equivalent to

$\exists\delta\forall n(\delta(n)\leq j_2\beta(n)\ \&\ j_1\beta(\bar{\delta}n)=0\ \&\ \exists z(\forall k<z\, \phi(\langle n\rangle * \bar{\delta}k)=0\ \&\ \phi(\langle n\rangle * \bar{\delta}z)=\gamma(n)+1)),$

which, modulo $\forall\alpha(\alpha\in V_\beta\to\phi|\alpha\downarrow)$, is equivalent to

$\exists\delta\forall n[\delta(n)\leq j_2\beta(n)\ \&\ j_1\beta(\bar{\delta}n)=0\ \&\ \forall z((\forall k<z\, \phi(\langle n\rangle * \bar{\delta}k)=0\ \&\ \phi(\langle n\rangle * \bar{\delta}z)>0)\to$
$\phi(\langle n\rangle * \bar{\delta}z)=\gamma(n)+1)],$

and this is, in view of the boundedness of δ, in **EL**+KL+MP equivalent to a $\Pi^0{}_1$-formula, by the kind of derivation we have seen before.

Lemma 3.5. For every formula A in the language of **EL** there is a p-functor χ_A, which may contain free variables occurring in A, such that

$$\mathbf{EL}+KL+MP\vdash\forall\beta\,[V_\beta\neq\varnothing\ \&\ \forall\alpha\in V_\beta\,(\alpha \sqsubseteq A) \to \chi_A|\beta\downarrow\ \&\ \chi_A|\beta \sqsubseteq A].$$

Proof. χ_A is defined by induction on the logical complexity of A:

1) $\chi_A \equiv$ $\quad [1]$ if A, $\chi_A \equiv [0]$ if $\neg A$, for A atomic.
Remember that $[0]|\alpha\uparrow$ for every α.

2) $\chi_A \equiv$ $\quad \Lambda\beta.j(\chi_B|j_1{}^*\beta,\ \chi_C|j_2{}^*\beta)$ if $A \equiv B\&C$.

For suppose $V_\beta \neq \emptyset$, $\forall\alpha\in V_\beta\,(\alpha\,\underline{r}\,B\&C)$, then
$j_1|[V_\beta]=V_{j_1*\beta}$ (lemma 4)$\neq\emptyset$ and $\forall\alpha\in V_{j_1*\beta}(\alpha\underline{r}B)$, so $\chi_B|j_1*\beta\downarrow$ and
$\chi_B|j_1*\beta\,\underline{r}\,B$; analogously for C.

3) $\chi_A\equiv$ $\Lambda\beta.(\Lambda\gamma.\,(\chi_C|(\psi_\gamma*|\beta)))$, where ψ_γ is such that $\forall\alpha.\ \psi_\gamma|\alpha\cong\alpha|\gamma$, if $A\equiv B\to C$.
For suppose $V_\beta\neq\emptyset$, $\forall\alpha\in V_\beta\,(\alpha\,\underline{r}\,B\to C)$, and $\gamma\,\underline{r}\,B$, then
$\psi_\gamma|[V_\beta]=V_{\psi_\gamma*|\beta}\neq\emptyset$ and $\forall\delta\in V_{\psi_\gamma*|\beta}\ \delta\underline{r}C$, so $\chi_C|(\psi_\gamma*|\beta)\downarrow$ and $\underline{r}$ C.

4) $\chi_A\equiv$ $\Lambda\beta.(\Lambda\gamma.(\chi_{A(x)}[^{\gamma(0)}/_x]|(\psi_\gamma*|\beta)))$, where ψ_γ is such that
$\forall\alpha.\psi_\gamma|\alpha=\alpha|[\gamma(0)]$, if $A\equiv\forall xA(x)$.
For suppose $V_\beta\neq\emptyset$, $\forall\alpha\in V_\beta\,(\alpha\,\underline{r}\,\forall xA(x))$, γ arbitrary, then
$\psi_\gamma|[V_\beta]=V_{\psi_\gamma*|\beta}\neq\emptyset$, $\forall\alpha\in V_{\psi_\gamma*|\beta}\ \alpha\,\underline{r}\,A(x)[^{\gamma(0)}/_x]$, so $\chi_{A(x)}[^{\gamma(0)}/_x]|(\psi_\gamma*|\beta)\downarrow$
and $\underline{r}$ $A(\gamma(0))$.

5) $\chi_A\equiv$ the functor β_3 from lemma 3, if $A\equiv\exists xB(x)$ or $\exists\alpha B(\alpha)$.

6) $\chi_A\equiv$ $\Lambda\beta.(\Lambda\gamma.\,(\chi_{B(\alpha)}[^{\gamma}/_\alpha]|(\psi_\gamma*|\beta)))$, where ψ_γ is such that $\forall\alpha.\ \psi_\gamma|\alpha=\alpha|\gamma$, if
$A\equiv\forall\alpha B\alpha$.
For if $V_\beta\neq\emptyset$, $\forall\delta\in V_\beta\,(\delta\,\underline{r}\,\forall\alpha B\alpha)$, γ arbitrary, then
$\psi_\gamma|[V_\beta]=V_{\psi_\gamma*|\beta}\neq\emptyset$, $\forall\delta\in V_{\psi_\gamma*|\beta}\ \delta\,\underline{r}\,B\alpha[^{\gamma}/_\alpha]$ so
$\chi_{B(\alpha)}[^{\gamma}/_\alpha]|(\psi_\gamma*|\beta)\ \underline{r}\ B\alpha[^{\gamma}/_\alpha]$,
etc.

Lemma 3.6. For every formula A in the language of **EL** such that **EL**$\vdash$A there is a p-functor ψ_A such that **EL**+KL+MP$\vdash\psi_A\downarrow\ \&\ \psi_A\underline{r}A$; ψ_A may contain variables occurring free in A.

Proof. This goes by induction on proofs in **EL**+KL+MP. Since our realizability differs only in the existential clauses from Kleene's, we only have to check the lemma for those rules and axioms of two-sorted predicate calculus that concern existential formulas, as well as for $QF\text{-}AC_{00}$.

It is clear that

$\Lambda\alpha.\ \beta_2|j([t],\alpha)\ \underline{r}\ A(t)\to\exists xA(x)$,

$\Lambda\alpha.\ \beta_2|j(\phi,\alpha)\ \underline{r}\ A(\phi)\to\exists\alpha A(\alpha)$, for β_2 from lemma 3.2.

Now suppose $\alpha\,\underline{r}\,A(y)\to C$, y possibly in α, not in C.
Then $\Lambda\gamma.\ \chi_C|(\psi*|\gamma)\ \underline{r}\ \exists xA(x)\to C$, where χ_C from lemma 3.5 and ψ such that
$\psi|\beta=\alpha[^{j_1\beta(0)}/y]|j_2\beta$.
For suppose $\gamma\,\underline{r}\,\exists xA(x)$, so $V_\gamma\neq\emptyset\ \&\ \forall\beta\in V_\gamma(j_2\beta\,\underline{r}\,A(j_1\beta(0))$. Then for $\beta\in V_\gamma$ we

have that $\psi|\beta \,\underline{r}\, C$, so $\forall\delta\in \psi|[V_\beta]=V_{\psi^*|\beta}\,(\delta \,\underline{r}\, C)$, so $\chi_C|(\psi^*|\gamma)\,\underline{r}\, C$.
Completely analogous for $(A(\phi)\to C)\to(\exists\alpha A(\alpha)\to C)$.

The following sublemma will be useful for the proof that $QF\text{-}AC_{00}$ is realised.

Sublemma 3.2. There is a functor χ such that

$$\mathbf{EL}\text{+KL+MP}\vdash\forall\varepsilon\,[\forall n\,(\varepsilon|[n]\downarrow\ \&\ V_{\varepsilon|[n]}\neq\varnothing)\to$$
$$\chi|\varepsilon\downarrow\ \&\ V_{\chi|\varepsilon}\neq\varnothing\ \&\ \forall\gamma\in V_{\chi|\varepsilon}\forall n\,(\gamma|[n]\downarrow\ \&\ \gamma|[n]\in V_{\varepsilon|[n]})].$$

Proof. To apply sublemma 3.1, we construct a bounded primitive recursive condition for sequences σ which says that σ is "for the time being" an initial segment of a γ such that $\forall n\,(\gamma|[n]\downarrow\ \&\ \gamma|[n]\in V_{\varepsilon|[n]})$.
Let $\sigma|[n]$ denote the maximal τ such that $\gamma|[n]\in\tau$ for all γ with $\gamma|[n]\downarrow$ and $\gamma\in\sigma$. (This is clearly primitive recursive in n and σ).
We formulate our condition $A(\varepsilon,\sigma)$ in 4 stages:

1) $\forall i<\mathrm{lth}(\sigma)\,(\sigma_i\leq i)$;
2) $\forall n<\sigma\,\forall i<\mathrm{lth}(\sigma|[n])\,(i<\mathrm{lth}(\bar\varepsilon(\mathrm{lth}(\sigma))|[n])\to(\sigma|[n])_i\leq j_2((\bar\varepsilon(\mathrm{lth}(\sigma))|[n])_i))$
   (so if $\gamma\in\sigma$ then for the time being $\gamma|[n]\leq j_2(\varepsilon|[n])$;
3) $\forall n<\sigma\,\forall i<\mathrm{lth}(\bar\varepsilon(\mathrm{lth}(\sigma))|[n])\,((\bar\varepsilon(\mathrm{lth}(\sigma))|[n])_i<\mathrm{lth}(\sigma)\to i<\mathrm{lth}(\sigma|[n]))$
   (This will ensure that $\forall m A(\varepsilon,\bar\gamma m)\to\gamma|[n]\downarrow$);
4) $\forall n<\sigma\,\forall\tau\supset\sigma|[n]\,(\tau<\mathrm{lth}(\bar\varepsilon(\mathrm{lth}(\sigma))|[n])\to j_1((\bar\varepsilon(\mathrm{lth}(\sigma))|[n])_\tau=0)$
   (So $\gamma\in V_{\varepsilon|[n]}$ if $\forall m A(\varepsilon,\bar\gamma m)$).

Now let (sublemma 3.1) δ be such that $\forall\gamma\,(\gamma\in V_\delta\leftrightarrow\forall n A(\varepsilon,\bar\gamma n)$; and put $\chi\equiv\Lambda\varepsilon.\delta$. Now if $\forall n\,(\varepsilon|[n]\downarrow\ \&\ V_{\varepsilon|[n]}\neq\varnothing)$, then there are arbitrarily long sequences σ with $A(\varepsilon,\sigma)$; with KL we conclude $V_{\chi|\varepsilon}\neq\varnothing$.

<u>$QF\text{-}AC_{00}$</u>. Let $F\equiv\forall x\,\exists y\,Axy\to\exists\alpha\,\forall x A(x,\alpha x)$ be an instance of $QF\text{-}AC_{00}$ and suppose δ realizes the premiss. Then:
$\forall n\ \delta|[n]\downarrow\ \&\ V_{\delta|[n]}\neq\varnothing\ \&\ \forall\gamma\in V_{\delta|[n]}\,(j_2\gamma\,\underline{r}\,A(n,j_1\gamma(0)))$.
Let ψ such that $\psi|\gamma\equiv j([j_1\gamma(0)],j_2\gamma)$. Then for all n: $V_{\psi^*|(\delta|[n])}=\psi|[V_{\delta|[n]}]\neq\varnothing$ (lemma 3.4), and $\forall\gamma\in V_{\psi^*|(\delta|[n])}\ j_2\gamma\,\underline{r}\,A(n,j_1\gamma(n))$. Apply sublemma 3.2 to find a χ such that

$$\forall\gamma\in V_{\chi|\delta}\,\forall n\,(\gamma|[n]\downarrow\ \&\ \gamma|[n]\in V_{\psi^*|(\delta|[n])}),$$

then this χ realizes the conclusion of F.

We now get some lemmas that are analogous to lemmas 1.1 and following.

Lemma 3.7. Define a realizability **r'** by the clauses:

1) $\alpha\,\mathbf{r'}A \equiv V_\alpha \neq \emptyset \;\&\; \forall_{\beta\in V_\alpha}$ A for A atomic;
2) $\alpha\,\mathbf{r'}A\&B \equiv V_\alpha \neq \emptyset \;\&\; \forall_{\beta\in V_\alpha}$ $j_1\beta\,\mathbf{r'}\,A \;\&\; j_2\beta\,\mathbf{r'}\,B$;
3) $\alpha\,\mathbf{r'}A\to B \equiv V_\alpha \neq \emptyset \;\&\; \forall_{\beta\in V_\alpha}$ $\forall\gamma(\gamma\,\mathbf{r'}\,A \to \beta|\gamma{\downarrow} \;\&\; \beta|\gamma\,\mathbf{r'}\,B)$;
4) $\alpha\,\mathbf{r'}\forall xAx \equiv V_\alpha \neq \emptyset \;\&\; \forall_{\beta\in V_\alpha}$ $\forall n\,(\beta|[n]{\downarrow} \;\&\; \beta|[n]\,\mathbf{r'}\,An)$;
5) $\alpha\,\mathbf{r'}\,\exists xAx \equiv V_\alpha \neq \emptyset \;\&\; \forall_{\beta\in V_\alpha}$ $(j_2\beta\,\mathbf{r'}\,A(j_1\beta(0)))$;
6) $\alpha\,\mathbf{r'}\forall\beta A(\beta) \equiv V_\alpha \neq \emptyset \;\&\; \forall_{\beta\in V_\alpha}$ $\forall\gamma(\beta|\gamma{\downarrow} \;\&\; \beta|\gamma\,\mathbf{r'}\,A(\gamma))$;
7) $\alpha\,\mathbf{r'}\exists\beta A(\beta) \equiv V_\alpha \neq \emptyset \;\&\; \forall_{\beta\in V_\alpha}$ $(j_2\beta\,\mathbf{r'}\,A(j_1\beta))$.

Then for all formulas in the language of **EL** there are p-functors ϕ_A and ψ_A such that:

$$\mathbf{EL+KL+MP} \vdash \forall\alpha(\alpha\,\underline{r}\,A \to \phi_A|\alpha{\downarrow}\&\phi_A|\alpha\,\mathbf{r'}A)$$
$$\mathbf{EL+KL+MP} \vdash \forall\alpha(\alpha\,\mathbf{r'}A \to \psi_A|\alpha{\downarrow}\&\psi_A|\alpha\,\underline{r}\,A).$$

Proof. For those who are not yet asleep, we give the definitions.

i) $\phi_{t=s} \equiv \lambda\alpha.\beta_2|a$
   $\psi_{t=s} \equiv \lambda\alpha.[0]$
ii) $\phi_{A\&B} \equiv \lambda\alpha.\beta_2|j(\phi_A|j_1\alpha,\phi_B|j_2\alpha)$
   $\psi_{A\&B} \equiv \lambda\alpha.j(\chi_A|((\psi_A.j_1)^*|\alpha),\chi_B|((\psi_B.j_2)^*|\alpha))$
iii) $\phi_{A\to B} \equiv \lambda\alpha.\beta_2|(\lambda\gamma.\phi_B|(\alpha|(\psi_A|\gamma)))$
   $\psi_{A\to B} \equiv \lambda\alpha.\chi_{A\to B}|(\zeta^*|a)$, where $\zeta\equiv\lambda\beta.\lambda\gamma.\psi_B|(\beta|(\phi_A|\gamma))$
iv) $\phi_{\forall xAx} \equiv \lambda\alpha.\beta_2|(\lambda n.\phi_A[^n/_x]|(\alpha|[n]))$
   $\psi_{\forall xAx} \equiv \lambda\alpha.\chi_{\forall xAx}|(\zeta^*|\alpha)$, where $\zeta\equiv\lambda\beta.(\lambda n.\psi_A[^n/_x](\beta|[n]))$
v) $\phi_{\exists xAx} \equiv \lambda\alpha.\zeta^*|\alpha$ with $\zeta\equiv\lambda\beta.j(j_1\beta,\phi_A[j_1\beta(0)/_x]|(j_2\beta))$
   $\psi_{\exists xAx} \equiv \lambda\alpha.\zeta^*|\alpha$ with $\zeta\equiv\lambda\beta.j(j_1\beta,\phi_A[j_1\beta(0)/_x]|(j_2\beta))$
vi) $\phi_{\exists\gamma A\gamma} \equiv \lambda\alpha.\zeta^*|\alpha$ with $\zeta\equiv\lambda\beta.j(j_1\beta,\phi_A[j_1\beta/_\gamma]|(j_2\beta))$
   $\phi_{\exists\gamma A\gamma} \equiv \lambda\alpha.\zeta^*|\alpha$ with $\zeta\equiv\lambda\beta.j(j_1\beta,\phi_A[j_1\beta/_\gamma]|(j_2\beta))$
vii) $\phi_{\forall\gamma A\gamma} \equiv \lambda\alpha.\beta_2|(\lambda\delta.\phi_A[^\delta/_\gamma]|(\alpha|\delta))$
   $\psi_{\forall\gamma A\gamma} \equiv \lambda\alpha.\chi_{\forall\gamma A\gamma}(\zeta^*|\alpha)$, where $\zeta\equiv\lambda\beta.(\lambda\delta.\psi_A[^\delta/_\gamma](\beta|\delta))$.

We hope that it is clear by now how to transpose the rest of §1 to the case of **EL**; therefore we state the following lemmas without proof.

Definition. The class Γ of Σ^1_2-negative formulas is the smallest

satisfying:
i) Formulas of form $\exists\alpha A(\alpha)$ are in Γ, with A quantifier-free;
ii) Formulas of form $\exists\alpha\leq\beta\forall n A(\alpha,n)$ are in Γ, with A quantifier-free;
iii) Γ is closed under $\rightarrow,\&,\forall x,\forall\alpha$.

Lemma 3.8. For every Σ^1_2-negative formula A(a) with free variables a there is a p-functor ξ_A such that

$EL+KL+MP\vdash\exists\alpha(\alpha\mathbf{r}'A)\rightarrow A$

$EL+KL+MP\vdash A(a)\rightarrow\xi_A|a\downarrow\&\xi_A|a\mathbf{r}'A(a)$.

Corollary 3.9. EL+KL+MP is sound for **r'**.

Definition. Let GC_L be the following schema:

GC_L $\quad\forall\alpha(A\alpha\rightarrow\exists\beta B\alpha\beta)\rightarrow\exists\gamma\forall\alpha(A\alpha\rightarrow\gamma|\alpha\downarrow\&V_{\gamma|\alpha}\neq\emptyset\&\forall\zeta\in V_{\gamma|\alpha}B\alpha\zeta)$,

with the restriction that A must be Σ^1_2-negative.

Lemma 3.10. GC_L is **r'**-realizable.

Theorem 3.11. i) $EL+KL+MP+GC_L\vdash A\leftrightarrow\exists\alpha(\alpha\mathbf{r}'A)$

ii) $EL+KL+MP\vdash\exists\alpha(\alpha\mathbf{r}'A)\Leftrightarrow EL+KL+MP+GC_L\vdash A$

As a minor application of **r'**-realizability we have that $GC_L!$, so a fortiori not GC!, is not sufficient to prove GC, the principle of Generalized Continuity:

GC $\quad\forall\alpha(A\alpha\rightarrow\exists\beta B\alpha\beta)\rightarrow\exists\gamma\forall\alpha(A\alpha\rightarrow\gamma|\alpha\downarrow\&B\alpha\gamma|\alpha)$,

which is considered in Troelstra 1973 and is proven there to axiomatize Kleene's realizability based on partial continuous application.
We can do better, for the weakest well-known continuity principle without uniqueness-condition in the premiss, the schema WC-N:

WC-N $\quad\forall\alpha\exists n A(\alpha,n)\rightarrow\forall\alpha\exists n\exists m\forall\beta\in\bar{\alpha}n A(\beta,m)$,

(weak continuity for numbers), is already incompatible with KL:

Proposition 3.12. WC-N and KL are incompatible w.r.t. **EL**.

Proof. Define a functor Γ as follows:

$\Gamma(\langle\rangle) = 0$

$\Gamma(\langle\sigma\rangle * n) = 0$ if $\mathrm{lth}(\sigma) > \mathrm{lth}(n)$

$= 1$ if $\mathrm{lth}(\sigma) \leq \mathrm{lth}(n)$ & $\forall i < \mathrm{lth}(n)\ (n_i \neq 0\ \&\ n_i \neq 1)$

$= 1$ if $\mathrm{lth}(\sigma) \leq \mathrm{lth}(n)$ & $\exists i < \mathrm{lth}(n)\ (n_i = 0\ \&\ \forall j < i\ n_j \neq 1)$ & $\forall i < \mathrm{lth}(\sigma)\ \sigma_i = 0$

$= 2$ if $\mathrm{lth}(\sigma) \leq \mathrm{lth}(n)$ & $\exists i < \mathrm{lth}(n)\ (n_i = 0\ \&\ \forall j < i\ n_j \neq 1)$ & $\exists i < \mathrm{lth}(\sigma)\ \sigma_i \neq 0$

$= 1$ if $\mathrm{lth}(\sigma) \leq \mathrm{lth}(n)$ & $\exists i < \mathrm{lth}(n)\ (n_i = 1\ \&\ \forall j < i\ n_j \neq 0)$ & $\forall i < \mathrm{lth}(\sigma)\ \sigma_i = 1$

$= 2$ if $\mathrm{lth}(\sigma) \leq \mathrm{lth}(n)$ & $\exists i < \mathrm{lth}(n)\ (n_i = 1\ \&\ \forall j < i\ n_j \neq 0)$ & $\exists i < \mathrm{lth}(\sigma)\ \sigma_i \neq 1$.

Then

$(\Gamma|\alpha)(\sigma) = \Gamma(\langle\sigma\rangle * \alpha(\mathrm{lth}(\sigma))) - 1$,

is always defined.

Let γ be such that

$\forall\alpha\ \gamma|\alpha = j(\Gamma|\alpha, [1])$.

Then we have:

$\forall\alpha\,\forall n\,\exists\sigma\,(\forall i < \mathrm{lth}(\sigma)\ \sigma_i \leq 1\ \&\ (\Gamma|\alpha)(\sigma) = 0\ \&\ \mathrm{lth}(\sigma) = n\ \&\ \forall\tau \supset \sigma\ (\Gamma|\alpha)(\tau) = 0\ \&$
$\&\ \forall i,j \leq \mathrm{lth}(\sigma)\ \sigma_i = \sigma_j)$,

so with KL we conclude:

$\forall\alpha\,\exists\beta\,(\forall n\ \beta n \leq 1\ \&\ \forall n,m\ \beta n = \beta m\ \&\ \forall n\ (\Gamma|\alpha)(\bar{\beta}n) = 0)$,

in other words:

$\forall\alpha\,\exists n\,(n \leq 1\ \&\ [n] \in V_{\gamma|\alpha})$.

Furthermore $\forall\alpha\,[\forall n\ \alpha n > 1 \rightarrow \forall\beta\,(\forall n\ \beta n \leq 1 \rightarrow \beta \in V_{\gamma|\alpha})$ &
$\exists n\,(\alpha n = 0\ \&\ \forall m \leq n\ \alpha m \neq 1) \rightarrow V_{\gamma|\alpha} = \{[0]\}$ &
$\exists n\,(\alpha n = 1\ \&\ \forall m \leq n\ \alpha m \neq 0) \rightarrow V_{\gamma|\alpha} = \{[1]\}]$ holds.

Now we cannot have:

(*) $\forall\alpha\exists n\exists m\forall\beta \in \bar{\alpha}m(n \leq 1\,\&\,[n] \in V_{\gamma|\beta})$.

For suppose so; let n and m satisfy (*) for $\alpha = [2]$.

Then if $n = 0$ and $\beta = \langle 2\rangle^{[m]} * [1]$ we would have $[0] \in V_{\gamma|\beta}$; if $n = 1$, $\beta = \langle 2\rangle^{[m]} * [0]$ then $[1] \in V_{\gamma|\beta}$,

which is a contradiction in both cases.

§4. A topos for Lifschitz' realizability.

A further generalization of Lifschitz' realizability for **HA** can be obtained with the machinery of tripos theory, developed in Hyland, Johnstone & Pitts 1980, to be abbreviated HJP 1980 hereafter. They desribe some ways of defining triposes, and how to associate a topos with each tripos. In

Hyland 1982 the topos associated with the tripos constructed out of the partial combinatory structure <$\mathbb{N}$,{.}(.)>, the "effective topos", is described and it is shown that for the natural number object in this topos, exactly those sentences hold that are realized in Kleene's sense by some natural number. We will show that a similar tripos can be defined for Lifschitz' realizability, leading to the "Lifschitz topos" giving an extension of Lifschitz' realizability to all finite types.

First of all let $\mathfrak{I}$ be $\{e\in\mathbb{N}|\ V_e\neq\varnothing\}$. We define an application | on $\mathfrak{I}$ by putting $e|f{\downarrow}$ iff $\forall h\in V_e\{h\}(f){\downarrow}$, and then $e|f$ to be a code for $\{\{h\}(f)|\ h\in V_e\}$ (Note that such a code can be obtained recursively in e and f). Let Σ consist of those $H\subset\mathfrak{I}$ that satisfy i) $e\in H,\ V_e=V_{e'}\Rightarrow e'\in H$, and ii) $e,f\in H,\ V_g=V_e\cup V_f\Rightarrow g\in H$.

We define an implication $\rightarrow$: $\Sigma\times\Sigma\rightarrow\Sigma$ by

$F\rightarrow G\equiv\{e|\ \forall f\in F\ e|f{\downarrow}\ \&\ e|f\in G\}$. One checks immediately that this is well-defined. Now we define for each set X a preorder $(\wp X,\vdash_X)$ by $\wp X\equiv\Sigma^X$, $\phi\vdash_X\psi$ iff $\cap(\phi(x)\rightarrow\psi(x)|\ x\in X)$ is nonempty; for functions f: $I\rightarrow J$ let $\wp f$: $\wp J\rightarrow\wp I$ be composition with f, and $\forall f$: $\wp I\rightarrow\wp J$ defined by

$\forall f(\psi)\equiv\lambda j.\cap([\![f(i)=j]\!]\rightarrow\psi(i)|\ i\in I)$, where $[\![f(i)=j]\!]=\mathfrak{I}$ if $f(i)=j$, and $\varnothing$ otherwise.

So $\forall f(\psi)(j)=\{e|\ \forall i\in I\forall h\in\mathfrak{I}(f(i)=j\Rightarrow e|h{\downarrow}\ \&\ e|h\in\psi(i))\}$.

As generic element σ we take $id_\Sigma\in\wp\Sigma$. Now the verification (with the help of Theorem 1.4 of HJP 1980) that this defines a tripos does not give any problem; the only difference with a tripos constructed out of the partial combinatory structure <$\mathfrak{I}$,.|.> is that we do not take the full powerset of $\mathfrak{I}$.

Proposition 4.1. Conjunction and disjunction in $\wp I$ can be defined as follows:

i) $\phi\wedge'\psi\equiv\lambda i.\{e\in\mathfrak{I}|\ \forall h\in V_e(j_1h\in\phi(i)\ \&\ j_2h\in\psi(i))\}$.

ii) $\phi\vee'\psi\equiv\lambda i.\{e\in\mathfrak{I}|\ \forall h\in V_e(j_1h=0\Rightarrow j_2h\in\phi(i)\ \&\ j_1h\neq0\Rightarrow j_2h\in\psi(i))\}$.

Moreover, for any function f:$I\rightarrow J$, existential quantification along f can be defined by:

iii) $\exists f\phi\equiv\lambda j.\{e\in\mathfrak{I}|\ \forall h\in V_e\exists i\in I(f(i)=j\ \&\ h\in\phi(i))\}$.

Proof. Apply the definitions given in Theorem 1.4 of HJP 1980. According to these,

i) $\phi\wedge\psi=\lambda i.\{e|\ \forall G\in\Sigma\forall f(f\in\phi(i)\rightarrow(\psi(i)\rightarrow G)\Rightarrow e|f{\downarrow}\ \&\ e|f\in G)\}$. Suppose $e\in\phi\wedge\psi(i)$, let G be $\phi\wedge'\psi(i)$. Let b be a total recursive function such that b(a) codes {a}. Put

$f\equiv(\lambda s.\lambda t.b(j(s,t)))$, then $f\in\phi(i)\to(\psi(i)\to G)$, so $e|f\downarrow$ & $e|f\in G = \phi\wedge'\psi(i)$, so $\lambda e.e|f\in\phi\wedge\psi(i)\to\phi\wedge'\psi(i)$ for all i. Conversely, if $e\in\phi\wedge'\psi(i)$, $f\in\phi(i)\to(\psi(i)\to G)$, then $\forall h\in V_e(f|j_1h)|j_2h\downarrow$ & $(f|j_1h)|j_2h\in G$ so if ψ is such that $\psi(e)$ codes $\cup(V_{(f|j_1h)|j_2h}|h\in V_e)$, then $\psi\in\phi\wedge'\psi(i)\to\phi\wedge\psi(i)$ for all i.

ii) $\phi\vee\psi = \lambda i.\{e|\ \forall G\in\Sigma\forall f(f\in\phi(i)\to G \wedge \psi(i)\to G) \Rightarrow e|f\downarrow$ & $e|f\in G)\}$. Now if $e\in\phi\vee\psi(i)$, let G be $\phi\vee'\psi(i)$; $h_1=b(\lambda s.j(0,s))$, $h_2=b(\lambda s.j(1,s))$; then $h_1\in\phi(i)\to G$, $h_2\in\psi(i)\to G$, so $f\equiv b(j(h_1,h_2))\in\phi(i)\to G \wedge \psi(i)\to G$, so $e|f\downarrow$ & $e|f\in G = \phi\vee'\psi(i)$.

In the other direction, if $e\in\phi\vee'\psi(i)$, $f\in\phi(i)\to G \wedge \psi(i)\to G$, let Φ be such that $\Phi(h,g) = \{j_1h\}(j_2g)$ if $j_1g=0$, $\{j_2h\}(j_2g)$ if $j_1g\neq 0$; then $\forall h\in V_f\forall g\in V_e\Phi(h,g)\downarrow$ and $\forall h\in V_f\ \{\Phi(h,g)|\ g\in V_e\}\in G$, so $\{\Phi(h,g)|\ g\in V_e,\ h\in V_f\}\in G$, and this can be coded recursively in e and f.

iii) It is enough to show that $\exists f$ is left adjoint to $\wp f$. Suppose $\exists f\phi\vdash_J\psi$, so let $e\in\cap(\exists f\phi(j)\to\psi(j)|j\in J)$, $f\in\phi(i)$; then $b(f)\in\exists f\phi(f(i))$, so $e|b(f)\downarrow$ & $e|b(f)\in\psi(f(i))$. So $\lambda f.e|b(f)\in\cap(\phi(i)\to\psi(f(i))|i\in I)$, so $\phi\vdash_I\wp f\psi$.

In the other direction, suppose $\phi\vdash_I\wp f\psi$, $e\in\cap(\phi(i)\to\psi(f(i))|i\in I)$, $f\in\exists f\phi(j)$. Then $\forall h\in V_f\exists i\in I(f(i)=j$ & $h\in\phi(i))$, so $\forall h\in V_f\ e|h\downarrow$ & $e|h\in\psi(j)$. But then, because $\psi(j)\in\Sigma$, we must have (a code for)$\cup\{V_{e|h}|\ h\in V_f\}\in\psi(j)$; so $\lambda f.\cup\{V_{e|h}|h\in V_f\}\in\cap(\exists f\phi(j)\to\psi(j)|j\in J)$, i.e. $\exists f\phi\vdash_J\psi$.

Proposition 4.2. The coproduct in the topos of $\wp$-sets may be defined as follows: $(X,=_X)\amalg(Y,=_Y)$ is $(X\amalg Y,=_{X\amalg Y})$ with

$$[\![w=_{X\amalg Y}z]\!] \equiv \begin{cases}\{e|\ \forall h\in V_e(j_1h=0\ \&\ j_2h\in[\![w=_Xz]\!])\} & \text{if } w,z\in X\\ \{e|\ \forall h\in V_e(j_1h\neq 0\ \&\ j_2h\in[\![w=_Yz]\!])\} & \text{if } w,z\in Y\\ \emptyset & \text{else.}\end{cases}$$

Proof. Straightforward verification.

Proposition 4.3. The object $(\mathbb{N},=)$ with = defined by:

$$[\![n=m]\!] \equiv \quad \{e|\ V_e=\{n\}\cap\{m\}\}$$

is a natural number object in $\wp$-sets.

It is now a matter of calculation to show the equivalence of Lifschitz' realizability with the internal logic of $\mathbb{N}$ in $\wp$-sets.

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