Institute for Language, Logic and Information

EXTENDING THE LAMBDA CALCULUS WITH SURJECTIVE PAIRING IS CONSERVATIVE

Roel de Vrijer

ITLI Prepublication Series
for Mathematical Logic and Foundations ML-89-02

University of Amsterdam

Institute for Language, Logic and Information
Instituut voor Taal, Logica en Informatie

EXTENDING THE LAMBDA CALCULUS WITH SURJECTIVE PAIRING IS CONSERVATIVE

Roel de Vrijer
Department of Philosophy, University of Amsterdam
Department of Mathematics and Computer Science, Free University
de Boelelaan 1081, 1081 HV Amsterdam, email: rdv@cs.vu.nl

To appear in the Proceedings of the
Symposium on Logic in Computer Science (LICS)
june 5-8, 1989, Asilomar, California

Received March 1989

Correspondence to:

Faculteit der Wiskunde en Informatica
(Department of Mathematics and Computer Science)
Roetersstraat 15
1018WB Amsterdam

or

Faculteit der Wijsbegeerte
(Department of Philosophy)
Nieuwe Doelenstraat 15
1012CP Amsterdam

Extending the Lambda Calculus with Surjective Pairing is Conservative

ROEL DE VRIJER

Department of Mathematics and Computer Science, Free University, de Boelelaan 1081, 1081 HV Amsterdam, The Netherlands; Department of Philosophy, University of Amsterdam, Nieuwe Doelenstraat 15, 1012 CP Amsterdam, The Netherlands
email: rdv@cs.vu.nl

ABSTRACT. We consider the equational theory $\lambda\pi$ of λ-calculus extended with constants π, π_0, π_1 and axioms for surjective pairing: $\pi_0(\pi XY) = X$, $\pi_1(\pi XY) = Y$, $\pi(\pi_0 X)(\pi_1 X) = X$. The reduction system that one obtains by reading the equations as reductions (from left to right) is not Church-Rosser. Despite of this failure, we obtain a syntactic consistency proof of $\lambda\pi$. Moreover, it is shown that $\lambda\pi$ is a conservative extension of the pure λ-calculus.*

§1. Introduction

1.1. Let $\lambda\pi$ be the extension of the pure λ-calculus, λ, with the constants π, π_0 and π_1 and with the following axioms, which express that π, with the projections π_0 and π_1, is a surjective pairing:

$$\begin{array}{lll} & \pi_0(\pi XY) & = X \\ & \pi_1(\pi XY) & = Y \\ \textbf{SP}: & \pi(\pi_0 X)(\pi_1 X) & = X \end{array}$$

The set of $\lambda\pi$-terms will be denoted by $\Lambda\pi$, the set of pure λ-terms by Λ.

The extension of λ-calculus with surjective pairing was first studied by Mann [1973] and Barendregt [1974]. Barendregt showed that surjective pairing is not definable within the pure λ-calculus. Mann's principle concern was the connection between category theory and proof theory. This connection is also the subject of the recent monograph, Lambek & Scott [1986]; their C-monoids do correspond to the type free λ-calculus with surjective pairing. We should also mention the work in Curien [1986] and Hardin [1987] on a system called "Strong Categorical Combinatory Logic", which is designed for the implementation of functional programming languages. The system contains λ-calculus with surjective pairing.

A reduction system that corresponds to $\lambda\pi$ results if one adds to usual β-reduction the contraction rules:

$$\begin{array}{lll} \pi_0: & \pi_0(\pi XY) & \to X, \\ \pi_1: & \pi_1(\pi XY) & \to Y, \text{ and} \\ \pi^c: & \pi(\pi_0 X)(\pi_1 X) & \to X. \end{array}$$

We call this system $\lambda\pi^c$, for "classical" $\lambda\pi$, as the rules π_0, π_1 and π^c seem to have been widely accepted as the natural derivatives of the axioms for surjective pairing. It was shown by Klop [1980] that the Church-Rosser theorem for $\lambda\pi^c$ does not hold. See also Barendregt [1981] (chapter 15) and Hardin [1986].

The original Church-Rosser theorem was the first method to establish the consistency of the type-free λ-calculus. Because of the failure of Church-Rosser for $\lambda\pi^c$, this road seems now to be blocked for the case of $\lambda\pi$. Nevertheless, the consistency of $\lambda\pi$ can be shown by the construction of models; see e.g. Scott [1975] or Barendregt [1981] (exercise 18.5.12).

The question remains then, whether the consistency of $\lambda\pi$ cannot also be established by purely syntactic means. This paper aims at such a syntactic consistency proof for $\lambda\pi$. We will supply $\lambda\pi$ with a reduction relation that is somewhat different from that of $\lambda\pi^c$; and show the resulting system to satisfy a weak form of the Church-Rosser property, which is, however, still strong enough to yield consistency. Moreover, it will follow that $\lambda\pi$ is a conservative extension of λ.

1.2. $\lambda\pi^c$ modified

In this section we define, in an ad hoc manner, the modified system $\lambda\pi^{lr}$ and formulate a claim from which our results can be shown to follow. An intuitive motivation for the system $\lambda\pi^{lr}$ will be given in §2.

1.2.1. DEFINITION. The reduction relation $\geq$ of $\lambda\pi^{lr}$ is the least compatible, reflexive and transitive relation on $\Lambda\pi$, satisfying:

$$\begin{array}{lll} \boldsymbol{\beta}: & (\lambda x.M)N & \geq (x{:=}N)M; \\ \pi_0: & \pi_0(\pi X_0 X_1) & \geq X_0; \\ \pi_1: & \pi_1(\pi X_0 X_1) & \geq X_1; \\ \mathbf{l}: & \pi(\pi_0 X)Y & \geq X, \text{ provided that } \pi_1 X = Y; \\ \mathbf{r}: & \pi Y(\pi_1 X) & \geq X, \text{ provided that } \pi_0 X = Y. \end{array}$$

l and **r** stand for "left" and "right". The conditions on the rules **l** and **r** are given in terms of = of $\lambda\pi$ and are therefore independent of $\geq$. As a matter of fact, the equivalence relation generated by $\geq$ coincides with the convertibility relation = of $\lambda\pi$. So there is no need to distinguish conversion in $\lambda\pi^{lr}$ (or $\lambda\pi^c$) from

* This paper is a revision and abridgment of chapter 1 of the author's dissertation. I would like to thank H.P. Barendregt, L.S. van Benthem Jutting, D.H.J. de Jongh and W. Peremans for stimulating support and for useful advise during the writing of the dissertation.

conversion in $\lambda\pi$. Note that the rules **l** and **r** both imply the rule π^c: $\pi(\pi_0 X)(\pi_1 X) \geq X$.

Now in order to state our main result on $\lambda\pi^{lr}$, we need the equivalence relation $\approx$. For the background of $\approx$ see section 3.5.

1.2.2. DEFINITION. By $\approx$ we denote the least compatible equivalence relation on $\Lambda\pi$, satisfying the clause

$$X_0 = Y_0, X_1 = Y_1 \Rightarrow \pi X_0 X_1 \approx \pi Y_0 Y_1.$$

In effect, $\approx$ disregards replacement of occurrences of subterms under the influence of a π by convertible ones. Since there are no π's there, on Λ the relation $\approx$ is just syntactic identity ($\equiv$).

Now the Church-Rosser property for $\lambda\pi^{lr}$ will be established modulo $\approx$, that is, in the form of claim 1.2.3.

1.2.3. CLAIM (CR/$\approx$). *If* $\lambda\pi \vdash M = N$, *then there exist* $\approx$-*equivalent* Q_0 *and* Q_1, *such that* $M \geq Q_0$ *and* $N \geq Q_1$. (See the diagram.)

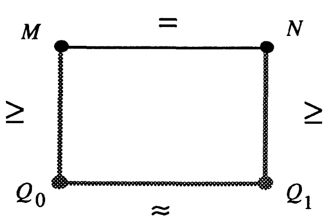


FIGURE 1

From this central claim our main results, the conservativity and hence the consistency of $\lambda\pi$, follow at once. For assume that $\lambda\pi \vdash M = N$ for $M, N \in \Lambda$. Find Q_0 and Q_1 as in the diagram. Then, as $\geq$-reduction cannot introduce constants which were not already present, all terms on the reduction sequences $M \geq Q_0$ and $N \geq Q_1$ must be in Λ, in particular $Q_0, Q_1 \in \Lambda$. Hence $Q_0 \equiv Q_1$ and the reductions $M \geq Q_0$ and $N \geq Q_1$ can only use $\boldsymbol{\beta}$. So M and N are convertible in λ as well.

Thus we established as a corollary to 1.2.3:

1.2.4. MAIN RESULTS. (i) $\lambda\pi$ *is a conservative extension of* λ, *i.e. if* $M, N \in \Lambda$, *then* $\lambda\pi \vdash M = N \Rightarrow \lambda \vdash M = N$.
(ii) $\lambda\pi$ *is consistent.*

In Klop & de Vrijer [1989] it is shown that 1.2.3 can be used to settle yet another issue: uniqueness of normal forms (UN) for $\lambda\pi^c$. UN is the property of two normal forms being convertible only if they are syntactically identical. (In the presence of the Church-Rosser property a triviality.) In Klop [1980] several systems that are not Church-Rosser, for example the system $\lambda\delta$ which is to be discussed below, are proved yet to satisfy UN. For $\lambda\pi^c$, however, UN remained open. It can now be proved with the help of 1.2.3.

The proof of claim 1.2.3 is complicated; it is not the straightforward type of Church-Rosser proof, such as e.g. one which proceeds by defining and then glueing together elementary diagrams. The traditional techniques do play an indirect role though, in the analysis of some auxiliary reduction systems.

1.3. Plan of the paper

It is the principal aim of this paper, to survey the proof of 1.2.3. The accent will be on its global structure and on the new techniques used. Because of space limitations, we have to be somewhat sketchy and we will skip over many technical details. A full proof can be found in chapter 1 of de Vrijer [1987].

§2 is meant as a motivational spring-board; we try to give some intuitive motivation for the system $\lambda\pi^{lr}$ and sketch a tentative proof idea. In §3 we combine more heuristics with the basic definitions. On that basis a further outline of the rather technical §§4 and 5 is given in section 3.7.

For an assessment of the results presented here and for some related open problems consult Klop & de Vrijer [1989].

§2. Introduction to the main proof

2.1. Digression

One of the complications that arises in an attempted proof of Church-Rosser for $\lambda\pi^c$ stems from the fact that the metavariable X occurs twice in the π^c-redex $\pi(\pi_0 X)(\pi_1 X)$, thus causing the redex to be unstable under reduction in one of the X's; the reduction rule π^c is not left linear. In order to isolate this phenomenon, Hindley proposed in 1973 to study the system $\lambda\delta$ which results by extending λ with a single constant δ and the following simplified form of the π^c-rule:

δ: $\qquad \delta XX \rightarrow X.$

This system was proved by Klop to be not Church-Rosser. As a matter of fact, his counterexample for Church-Rosser in $\lambda\pi^c$ is a direct translation of that for $\lambda\delta$.

All the same, $\lambda\delta$ can very well serve as a toy system for illustrating some of the ideas which lie behind our main proof. Observe that the contraction rule **δ**: $\delta XX \rightarrow X$ may be conceived of as a restricted form of the more liberal rule:

$\boldsymbol{\delta}^l$: $\qquad X = Y \vdash \delta XY \rightarrow X,$

which, in contrast to **δ**, is stable under reduction (i.e., a descendant of a δ^l-redex is still a δ^l-redex). It is easy to prove that $\boldsymbol{\delta}^l$, in combination with $\boldsymbol{\beta}$ is Church-Rosser. Now, somewhat surprisingly, $\rightarrow$ can, without the Church-Rosser property being spoiled, be extended further by the rule

$\boldsymbol{\delta}^r$: $\qquad X = Y \vdash \delta XY \rightarrow Y.$

For under this further extension of $\rightarrow$ the convertibility relation generated remains the same. Hence the Church-Rosser result for $\lambda\delta^l$ carries over immediately to $\lambda\delta^{lr}$: a common reduct of $\lambda\delta^{lr}$-convertible terms can be found already by using only β- and δ^l-reduction.

There is a general principle at stake here: the Church-Rosser problem for a more extended reduction relation can be reduced to a more restricted one, as long as the restricted reduction is strong enough to generate the original convertibility relation.

2.2. Back to $\lambda\pi$

Cannot the same method be applied to $\lambda\pi$? Indeed the rules **l** and **r** of $\lambda\pi^{lr}$ are extensions of the trouble causing contraction rule π^c: $\pi(\pi_0 X)(\pi_1 X) \geq X$, by which left linearity is restored. Establishing Church-Rosser for $\lambda\pi^{lr}$ minus **r** remains now problematic, however, owing to a case of overlap. For assume $\pi_1 X = Y$ and consider the following diagram.

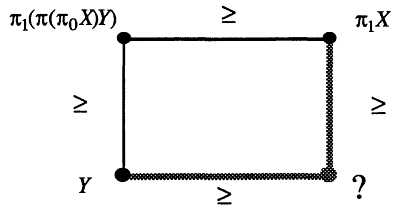


FIGURE 2

Indeed $\pi_1(\pi(\pi_0 X)Y) \geq \pi_1 X$ by **l** and $\pi_1(\pi(\pi_0 X)Y) \geq Y$ by π_1; but now it is in general not at all clear how to find a common reduct of $\pi_1 X$ and Y.

In the next few sections we describe how we cope with this problem; as a matter of fact, the type of overlap depicted in Fig. 2, together with our designation to stick to the proof strategy of §2, determines for a good deal the general shape of our proof.

2.2.1. INTERMEZZO. There are still other cases of overlap—between the contraction rules **l** and **r** on the one hand and π_0 and π_1 on the other—that must be taken into account in the proof.
(i) Under the provision that $\pi_1 X = Y$; the term $\pi_0(\pi(\pi_0 X)Y)$ reduces to $\pi_0 X$ in two different ways: by applying rule **l** to $\pi(\pi_0 X)Y$, or by applying rule π_0 to the whole term. In the theory of CRS's, systems in which there is only this harmless kind of overlap (two different rules, but with the same end result) are called weakly non-ambiguous. See further section 4.1.
(ii) Under the provision that $Y = Z$; the term $\pi(\pi_0(\pi XY))Z$ reduces to πXZ by applying rule π_0 to $\pi_0(\pi XY)$, and to πXY by applying rule **l** to the whole term. Here the situation is more serious. A mechanism for dealing with it wil be introduced in section 3.4.

2.3. Tentative heuristics

Recall that the rules **l** and **r** are both liberalized variants of π^c and that therefore both would do to generate the intended convertibility relation = of $\lambda\pi$ (with $\boldsymbol{\beta}$ and the projection rules of course). The only reason why it would not make sense to replace the reduction rule π^c altogether by say **l**, is the possibility of clashes with π_1 of the kind described in Fig. 2. But can we not both have the cake and eat it, by as a rule liberalizing π^c to **l**, but in all cases that a π_1-clash threatens reinterpreting it as **r**?

This rough proposal leads to a first approximation of our proof of the Church-Rosser property for $\geq$ of $\lambda\pi^{lr}$. Assume that M and N are convertible. Then there exists a conversion of M and N with, apart from $\boldsymbol{\beta}$, π_0 and π_1, only applications of the surjectivity rule π^c. (That is, a conversion in $\lambda\pi^c$.) Now, for the purpose of finding a common reduct of M and N, interpret applications of the surjectivity rule to redexes of which the residuals can be predicted to come under the influence of an occurrence of π_1 as instances of **r**, other ones as instances of **l**.

Matters are more complicated, though. For, on the basis of which information to choose between the rules **l** and **r**? And why would the requisite choice be uniform in the different residuals of one and the same redex? It is even clear that in general this is not the case; redexes and their residuals can be dispersed under reduction, some ending under the influence of a π_0, some under a π_1, some remaining "free" forever. To cope with this possibility of lack of information, we will introduce in an extension of $\Lambda\pi$ a new device (viz. that of bookkeeping pair), that will allow us to handle occurrences of redexes and their residuals simultaneously under different assumptions on the order in which projections will eventually act on them.

§3. The systems with bookkeeping pairs

Before extending the set of terms with bookkeeping pairs we first introduce labels. They serve as a formal tool for managing information of the character: which projection constants has an occurrence of a subterm to 'expect' and in which order? Then, formally defining the terms of the system $\lambda\pi p^*$ will still take several steps (sections 3.2/5). For reduction in $\lambda\pi p^*$ see 3.6.

3.1. Labels

Each subterm occurrence Σ (and hence all redexes) in a given term M will be supplied with a label—denoted as $\mathcal{L}(\Sigma)$, see definition 3.3.4—consisting of a sequence of zeros and ones and depending only on the context of occurrence of Σ and on the label of M.

3.1.1. DEFINITION. (i) The set L of *labels* consists of all finite sequences of the symbols 0 and 1 (including the empty sequence, denoted by <>), and the infinite sequences that become eventually constant 0. The symbol ∞ is used for the infinite constant sequence 000... .
(ii) On L a partial order is defined by:

$$\alpha \leq \beta \iff \exists\gamma.\ \beta = \alpha\gamma$$

(if $\alpha \leq \beta$ we say that β *extends* α).

The (partial) information which is coded in a label should be interpreted as follows: if the n'th digit of $\mathcal{L}(\Sigma)$ is i, then the n'th element of a sequence of projections, of length at least n, acting on (any descendant of) Σ, will always be π_i. (NB. In $C[\pi_0(\pi_0(\pi_1 N))]$ three projections—or a sequence of projections of length three—are said to act on N, of which the first one is π_1 and the second and third ones are π_0.)

The labels will make it possible to stipulate restrictions on the rules **l** and **r** with the effect that problematic situations of the kind of Fig. 2 are avoided (cf. section 2.3).

3.2. Bookkeeping pairs

Three kinds of contexts can be distinguished according to the above, namely those admitting of rule **l** (label of the form 0α), those admitting of rule **r** (label 1α), and those which do not (yet) carry enough information to settle the issue. Now in order to ensure that reduction has not to be obstructed in the latter case, the systems **λπp** and **λπp*** will be defined, incorporating as a formal device pairs of the form $\ulcorner M_0, M_1 \urcorner$, with the stipulation that M_0 and M_1 are treated as if their context provided them respectively with label 0 and with label 1. Formally: $\mathcal{L}(M_0) = 0$ and $\mathcal{L}(M_1) = 1$. In this manner e.g. a contraction of the π-redex $\pi(\pi_0 X)(\pi_1 X)$ can be simultaneously dealt with as an instance of **l** and of **r** in the respective left and right components of the "bookkeeping pair" $\ulcorner \pi(\pi_0 X)(\pi_1 X), \pi(\pi_0 X)(\pi_1 X) \urcorner$. And when in the process of reduction descendants of this pair end up in a context which is more determined, the bookkeeping pair can be canceled: only that component is kept, of which the label is consistent with the extended information carried by the new context.

For technical reasons, the concept of bookkeeping pair will be made a little more general still: an indexed bookkeeping pair $\ulcorner M_0, M_1 \urcorner_\alpha$ is admitted in an α-context (label α); it artificially extends the information already carried by the context with one digit: $\mathcal{L}(M_0) = \alpha 0$ and $\mathcal{L}(M_1) = \alpha 1$.

The resulting system **λπp** will roughly be obtained by extending $\Lambda\pi$ with bookkeeping pairs, and incorporating the reduction rules $\boldsymbol{\beta}$, $\boldsymbol{\pi_0}$, $\boldsymbol{\pi_1}$, **l** and **r**, along with a rule **p** which in an α-context allows an occurrence M to be replaced by the bookkeeping pair $\ulcorner M, M \urcorner_\alpha$. In **λπp*** a mechanism for the cancellation of superfluous bookkeeping pairs is added.

3.3. Labeled terms

In this section we define labeled terms with bookkeeping pairs and assign labels to occurrences of subterms.

3.3.1. DEFINITION. (i) Consider the extension of $\Lambda\pi$ which is obtained by adding to its rules of term formation, λ-abstraction and application, the extra rule to construct from M_0 and M_1 for each finite label α (here called the *index*), the *bookkeeping pair* $\ulcorner M_0, M_1 \urcorner_\alpha$.

(ii) Define the function φ, which maps terms with bookkeeping pairs back to $\Lambda\pi$ by deleting all second coordinates, by induction:

$$
\begin{array}{lll}
\varphi(M) & \equiv & M, \text{ if } M \text{ is a variable or a constant} \\
\varphi(\lambda x.M) & \equiv & \lambda x.\varphi(M) \\
\varphi(MN) & \equiv & \varphi(M)\varphi(N) \\
\varphi(\ulcorner M_0, M_1 \urcorner_\alpha) & \equiv & \varphi(M_0)
\end{array}
$$

Then:

(iii) the bookkeeping pair forming rule is restricted to the condition that $\boldsymbol{\lambda\pi} \vdash \varphi(M_0) = \varphi(M_1)$;

(iv) the set $\Lambda\pi q$ of *labeled terms* consists of the pairs $<\alpha, M>$, where $\alpha \in L$ is a label and M a well-formed term, according to clause (iii). Notation: $\alpha.M$.

Note that φ is idempotent and that $\Lambda\pi$ is the set of its fixed points. φ can also be extended to the set of labeled terms $\Lambda\pi q$ by simply putting $\varphi(\alpha.M) = \varphi(M)$. Conversely, the function ψ: $\Lambda\pi \to \Lambda\pi q$ that is defined by $\psi(M) = {<>}.M$, is a natural embedding of $\Lambda\pi$ into $\Lambda\pi q$.

We adopt the general policy to omit labels whenever possible without danger of confusion. That is, we may write M instead of $\alpha.M$, both when the label α is already known and when it is not relevant for the discussion. If C[ ] is a subcontext of M, then as a subcontext of $\alpha.M$ it would officially be denoted by α.C[ ], but without danger of being misunderstood we can speak of the subcontext C[ ] of $\alpha.M$.

3.3.2. DEFINITION. (i) The convertibility relation of **λπ** is extended to terms with bookkeeping pairs by adding the rule:

$\ulcorner M_0, M_1 \urcorner_\alpha = M_0$ (of course provided that $\ulcorner M_0, M_1 \urcorner_\alpha$ is well-formed according to definition 3.3.1(iii)).

(ii) Conversion in **λπq** just neglects the labels, i.e., we define

$\boldsymbol{\lambda\pi q} \vdash \alpha.M = \beta.N \Leftrightarrow M = N$ according to (i).

As said in section 3.1, the label of an occurrence of N in $\alpha.M$ will depend on α and on the context C[ ] of N in M—not on the actual form of N itself. Therefore we first define a function $\mathcal{L}$ which assigns to each subcontext C[ ] of $\alpha.M$ a label $\mathcal{L}$(C[ ]) and then identify $\mathcal{L}(N)$ with $\mathcal{L}$(C[ ]). The notation $\mathrm{C}[\ ]_\beta$ is used if we want to indicate (implicitly) that $\mathcal{L}(\mathrm{C}[\ ]) = \beta$. Such a context is called a β-context. (So $\alpha.\mathrm{C}[\ ]_\beta$ is a β- and not an α-context.)

For some occurrences (and contexts) the label will remain undefined. Undefined labels are indicated by the symbol $\uparrow$.

3.3.3. DEFINITION. The label of the hole in α.C[ ], notation $\mathcal{L}$(C[ ]), is defined by induction on (the number of symbols in) C[ ]. In advance we stipulate that subcontexts of contexts with undefined label also have their label undefined: if $\mathcal{L}(\mathrm{D}[\ ]) = \uparrow$, then $\mathcal{L}(\mathrm{D}[\mathrm{E}[\ ]]) = \uparrow$ as well. Other cases are taken care of by the following clauses:

$$
\begin{array}{llll}
\mathcal{L}(\alpha.[\ ]) & = & \alpha & \\
\mathcal{L}(\mathrm{D}[\lambda x.[\ ]]) & = & <> & \\
\mathcal{L}(\mathrm{D}[[\ ]Q]) & = & \infty & \\
\mathcal{L}(\mathrm{D}[\pi_i[\ ]]_\beta) & = & i\beta, & \text{for } i = 0 \text{ or } 1 \\
\mathcal{L}(\mathrm{D}[\pi[\ ]Q]_\beta) & = & \gamma, & \text{if } \beta = 0\gamma \\
 & & \uparrow, & \text{if } \beta = 1\gamma \\
 & & <>, & \text{if } \beta = <> \\
\mathcal{L}(\mathrm{D}[\pi Q[\ ]]_\beta) & = & \gamma, & \text{if } \beta = 1\gamma \\
 & & \uparrow, & \text{if } \beta = 0\gamma \\
 & & <>, & \text{if } \beta = <>
\end{array}
$$

$\ell(\mathrm{D}[Q[\,]]) = \langle\rangle$, in other cases with the hole in argument position

$\ell(\mathrm{D}[\ulcorner[\,], Q\urcorner_\beta]) = \beta 0$

$\ell(\mathrm{D}[\ulcorner Q, [\,]\urcorner_\beta]) = \beta 1$

3.3.4. DEFINITION. If P occurs in $\alpha.M$ in the context C[ ], then the label of that occurrence is defined by $\ell(P) = \ell(\mathrm{C}[\,])$.

If $\ell(P) = \beta$, then $\beta.P$ can be viewed as a labeled subterm of $\alpha.M$ and by abuse of language we will speak of the occurrence $\beta.P$. Table 1 may be helpful in computing labels of occurrences in labeled terms.

TABLE 1

N	$\ell(N)$	$\ell(P)$	$\ell(Q)$
C[P]	$\uparrow$	$\uparrow$	
$\lambda x.P$	β	$\langle\rangle$	
$\pi_i P$	β	$i\beta$	
πPQ	0β	β	$\uparrow$
	1β	$\uparrow$	β
	$\langle\rangle$	$\langle\rangle$	$\langle\rangle$
PR	β	∞	
RP (not one of the cases above)	β	$\langle\rangle$	
$\ulcorner P, Q\urcorner_\gamma$	β	$\gamma 0$	$\gamma 1$

In $\alpha.M$ the occurrence M itself of course has label $\ell(M) = \alpha$. Other occurrences are always the direct suboccurrence of an intermediate occurrence N of M of smaller depth. In the table the label of P (and Q) can be looked up, given the form and the label of N. For some occurrences the label remains undefined; in the table this is denoted by $\uparrow$. (It is assumed that $\beta \neq \uparrow$).

3.3.5. CONVENTION. All relations $\rightarrow_a$ on labeled (pre-)terms that will be met in the sequel respect labels. That is, they satisfy the implication

$$\alpha.M \rightarrow_a \beta.N \Rightarrow \alpha = \beta$$

So in accordance with our policy to suppress labels as much as possible, we will further write $\alpha.M \rightarrow_a N$ instead of $\alpha.M \rightarrow_a \alpha.N$.

The usual notion of compatibility has to be adapted to the presence of labels.

3.3.6. DEFINITION. Let A be a subset of $\Lambda\pi$q. The relation $\rightarrow_a$ on A is called *compatible (with respect to A)*, if

$$\alpha.M \rightarrow_a N \;\&\; \alpha \leq \beta \;\&\; \mathrm{C}[M]_\beta \in \mathrm{A} \Rightarrow \mathrm{C}[M]_\beta \rightarrow_a \mathrm{C}[N]_\beta$$

3.4. The equivalences ~ and ≈

We are not yet done with the definitions of $\lambda\pi$p- and $\lambda\pi$p*-terms. Two more steps are needed, in this and in the following section. Here equivalence classes of terms will be formed by disregarding occurrences with an undefined label. We now first give the motivation for this manœuvre, which forms also the background of the mysterious role played by the relation ≈ in section 1.2.

Recall the second case of overlap that was pointed out in Intermezzo 2.2.1. It is illustrated in the following diagram.

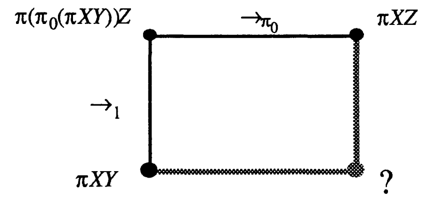


FIGURE 3

It must be assumed here that $\pi_1(\pi XY) = Z$, that is, $Y = Z$ and, by the intended context restrictions on **l**, that we are in a 0α-context. Now the question is: how to find a common reduct of the respective one step reducts πXY and πXZ under the single assumption that Y and Z are convertible?

Our solution is drastic: we declare πXY and πXZ to be equivalent (under the given assumptions) and thereby just stop worrying. The equivalence relation ~ will be defined, which disregards up to convertibility the component X_i of the pair $\pi X_0 X_1$ in contexts with label $j\alpha$, $j \neq i$; that is, disregards just those subterm occurrences with label $\uparrow$. Then in **λπp** and **λπp*** terms will be considered modulo ~.

Just as in the case of **l** and **r**, the context restrictions will give protection against the possibility of interference of a wrong projection rule. (Matters would not work out all right if we declared $\pi_1(\pi XY) \sim \pi_1(\pi XZ)$ if $Y = Z$, for reasons similar to those described in section 2.2 with respect to **l** and **r**.)

3.4.1. DEFINITION. (i) ~ is the compatible equivalence relation on $\Lambda\pi$q which is generated by the clauses:

$$Y = Y' \Rightarrow 0.\pi XY \sim \pi XY',$$
$$Y = Y' \Rightarrow 1.\pi YX \sim \pi Y'X.$$

(ii) $\Lambda\pi\mathrm{q}^{\sim} = \Lambda\pi\mathrm{q}/\!\sim$.

We will from now on consider the elements of $\Lambda\pi$q as representatives for their respective equivalence classes, and thereby pretend to work in $\Lambda\pi\mathrm{q}^{\sim}$. That is, we operate with ~-equivalence in the same way as it is normally done with α-equivalence. Of course it then has to be checked that all relevant predicates and operations on terms respect this equivalence relation. In this short version of the paper we ignore this complication.

Note that ~ is a context dependent refinement of the equivalence relation ≈ on $\Lambda\pi$ defined in 1.2.2. In section 1.2 we showed that the Church-Rosser result for $\boldsymbol{\lambda\pi}^{\mathbf{lr}}$ modulo ≈ was sufficient for our purpose.

3.5. Canceling

In the informal account given in section 3.2, it is assumed that an α-bookkeeping pair $\ulcorner M_0, M_1 \urcorner_\alpha$ occurs in an α-context. That is, it would be natural to consider only those terms that are *canonical* according to the following definition.

3.5.1. DEFINITION. The set $\Lambda\pi p^*$ of *canonical* terms is defined as the subset of $\Lambda\pi q^{\sim}$ consisting of all terms which meet the requirement that α-bookkeeping pairs occur only in α-contexts:

$$\Sigma \equiv \ulcorner M_0, M_1 \urcorner_\alpha \;\Rightarrow\; \ell(\Sigma) = \alpha.$$

Unfortunately, however, the set $\Lambda\pi p^*$ is not closed under the reduction rules which are intended for the system $\lambda\pi p^*$. For example, we have $\langle\rangle.(\lambda y.yx)\ulcorner y,y\urcorner \in \Lambda\pi p^*$ but $\langle\rangle.\ulcorner y,y\urcorner x \notin \Lambda\pi p^*$, the latter term being a β-reduct of the former one.

It will turn out that the worst thing that can happen is that an α-bookkeeping pair shows up in a β-context with $\beta > \alpha$. That is, the result of a reduction step from a canonical term will always remain within the set $\Lambda\pi p$ defined below.

3.5.2. DEFINITION. $\Lambda\pi p$ is the set of terms meeting the requirement that for every occurrence Σ,

$$\Sigma \equiv \ulcorner M_0, M_1 \urcorner_\alpha \;\Rightarrow\; \alpha \le \ell(\Sigma).$$

Remind that from the point of view of section 3.2, the informal meaning of an α-bookkeeping pair $\ulcorner M_0, M_1 \urcorner_\alpha$ is something like "M_i if the context would have a label of the form αiα'". Now it does not make much sense to leave the two options i = 0 or 1 open in a context with a label that actually already extends α. The auxiliary reduction relation $\hookrightarrow$ on $\Lambda\pi p$ can be used to cancel such superfluous bookkeeping pairs.

By repeated canceling, terms from $\Lambda\pi p$ can be projected back to $\Lambda\pi p^*$; this is accomplished by the operation $(\,)^*$.

3.5.3. DEFINITION. (i) The one step reduction relation $\hookrightarrow$, called *canceling*, is defined as the compatible closure, in $\Lambda\pi p$, of the contraction rule:

can: $\quad \alpha i.\ulcorner M_0, M_1 \urcorner_\alpha \hookrightarrow M_i$, for i = 0 or 1.

(ii) The *canonical form* of a term $\alpha.M \in \Lambda\pi p$, notation $(\alpha.M)^*$, is the (unique) $\hookrightarrow$-normal form of $\alpha.M$.
(iii) The relation on $\Lambda\pi p$ which transforms M into M^* in one step is denoted by $\hookrightarrow^*$; so we have $M \hookrightarrow^* M^*$.

3.5.4. DEFINITION. For a relation $\to_a$ on $\Lambda\pi p$ we introduce the notations $\to_a{}^*$ and $\to_a{}^{1/2}$ for first $\to_a$ and then some canceling:
(i) $\to_a{}^*$ denotes the restriction to $\Lambda\pi p^*$ of $\to_a + \hookrightarrow^*$;
(ii) $\to_a{}^{1/2}$ denotes $\to_a + \twoheadrightarrow$.
(iii) $\twoheadrightarrow_a{}^*$ denotes the transitive reflexive closure of $\to_a$.

In other words, if $\to_a$ is a reduction relation that may lead outside $\Lambda\pi p^*$ (i.e. disturb canonicalness), then an $\to_a{}^*$-step is: start with a canonical term, perform an $\to_a$-step and project the result back to $\Lambda\pi p^*$ using the operation $(\,)^*$. An $\to_a{}^{1/2}$-step does not necessarily cancel all superfluous bookkeeping pairs in the reduct, but possibly some.

The following algebraic notions are useful in studying $\to_a{}^*$ via $\to_a$.

3.5.5. DEFINITION. Let $\to_a$ be a relation on $\Lambda\pi p$. Then $\to_a$ is called

(i) *∗-projectable* if $M \to_a N \;\Rightarrow\; M^* \twoheadrightarrow_a{}^* N^*$;
(ii) *∗-compatible* if $\alpha.M \to_a N \;\&\; \alpha \le \beta \;\Rightarrow$
$\qquad C[M]_\beta{}^* \twoheadrightarrow_a{}^* C[N]^*$;
(iii) *½-projectable* if $M \to_a N \;\&\; M \hookrightarrow M_0 \Rightarrow$
$\qquad (\exists N_0)(M_0 \twoheadrightarrow_a N_0 \;\&\; N \twoheadrightarrow N_0)$;
(iv) *½-compatible* if $\alpha.M \to_a N \;\&\; \alpha \le \beta \;\Rightarrow$
$\qquad (\exists Q)(C[M]_\beta \twoheadrightarrow_a Q \;\&\; C[N] \twoheadrightarrow Q)$.

All relations $\to_a \subseteq \Lambda\pi p^* \times \Lambda\pi p^*$ are trivially ∗-projectable. As to the connection between the different kinds of compatibility, observe that ½-compatibility is a weaker property than plain compatibility. Furthermore it is obvious that for ∗-projectable relations compatibility implies ∗-compatibility. The following lemma provides an instrument for proving ∗-projectability and ∗-compatibility.

3.5.6. LEMMA. (i) *Let $\to_a$ be ½-projectable. Then both $\to_a$ and $\twoheadrightarrow_a$ are ∗-projectable.*
(ii) *If $\to_a$ is ½-projectable and ½-compatible, then $\to_a$ and $\twoheadrightarrow_a$ are ∗-compatible.*

3.6. The systems $\lambda\pi p$ and $\lambda\pi p^*$

Finally we are in the position to define reduction in the systems $\lambda\pi p$ and $\lambda\pi p^*$. Note the context restrictions on the rules **l** and **r**; they comply with the intuitive plan initiated in section 2.3.

3.6.1. DEFINITION. (i) The one step reduction relation $\to_\pi$ is the compatible closure, in $\Lambda\pi p$, of the contraction rules:

β:	$\alpha.(\lambda x.M)N \;\to\; (x{:=}N)M$;	
π_0:	$\alpha.\pi_0(\pi X_0 X_1) \;\to\; X_0$;	
π_1:	$\alpha.\pi_1(\pi X_0 X_1) \;\to\; X_1$;	
l:	$0\alpha.\pi(\pi_0 X)Y \;\to\; X$,	provided that $\pi_1 X = Y$;
r:	$1\alpha.\pi Y(\pi_1 X) \;\to\; X$,	provided that $\pi_0 X = Y$.

(ii) The one step reduction relation $\to_p$ is the least relation on $\Lambda\pi p$ that satisfies the "contraction" rule:

p: $\quad C[X]_\alpha \;\to\; C[\ulcorner X, X \urcorner_\alpha]$, provided that X is not already a bookkeeping pair itself and α is finite.

(iii) Then $\to$, the one step reduction relation of $\lambda\pi p$, is defined as the union:

$$\to \;=\; \to_\pi \cup \to_p.$$

3.6.2. DEFINITION. The system $\lambda\pi p^*$ is the reduction system $\langle\Lambda\pi p^*, \to^*\rangle$; with the one step reduction relation $\to^*$ defined as the composition $\to + \hookrightarrow^*$:

$$M \to^* N \;\Leftrightarrow\; (\exists P)(M \to P \;\&\; P \hookrightarrow^* N).$$

Moreover, $\twoheadrightarrow^*$ denotes the reflexive transitive closure of $\rightarrow^*$.

We mention two basic technical properties of reduction in $\lambda\pi p$ and $\lambda\pi p^*$. Lemma 3.6.3(i) ensures the stability of redexes under reduction. With (ii) it is possible to establish properties of $\rightarrow^*$ via $\rightarrow$.

3.6.3. LEMMA. (i) If $M \rightarrow N$ or $M \hookrightarrow N$ and Σ' is a descendant in N of an occurrence Σ in M, then $\ell(\Sigma') \geq \ell(\Sigma)$.
(ii) $\rightarrow$ and $\twoheadrightarrow$ are $*$-compatible

The usefulness of $\lambda\pi p^*$ for our original problem concerning $\lambda\pi$ rests on the fact that $\lambda\pi$ can be seen as a subsystem of $\lambda\pi p^*$. First, via the embedding ψ of section 3.3, the set $\Lambda\pi$ is included in $\Lambda\pi p^*$: terms without bookkeeping pairs are always canonical. Second, conversion in $\lambda\pi$ can be carried out, via the detour of $\rightarrow^*$, in $\lambda\pi p^*$ as well. This in spite of the weakening of the reduction relation by the context restrictions on the rules. We prove this now. The equivalence relation generated by $\rightarrow^*$ is denoted by $=^*$.

3.6.4. THEOREM. *Let* $M, N \in \Lambda\pi$. *Then*

$$\lambda\pi \vdash M = N \;\Rightarrow\; <>.M =^* N.$$

PROOF. Induction on deductions in $\lambda\pi$. The only interesting deduction step is an application of the surjectivity axiom $\pi(\pi_0 X)(\pi_1 X) = X$; the other axioms are already included in $\lambda\pi p^*$ as the rules π_0, π_1 and $\boldsymbol{\beta}$, independent of context. So it suffices to prove $<>.C[\pi(\pi_0 X)(\pi_1 X)]_\alpha =^* C[X]$ for any C[ ] and X (in $\Lambda\pi$). Now if $\alpha \neq <>$, this can be concluded simply by an application of **l** (if α=0α') or **r** (if α=1α'). In case of α = <>, the =*-equivalence can be established via the introduction of a bookkeeping pair:

$$\begin{aligned} C[\pi(\pi_0X)(\pi_1X)] &\rightarrow_p^* C[\ulcorner \pi(\pi_0X)(\pi_1X), \pi(\pi_0X)(\pi_1X) \urcorner_{<>}] \\ &\rightarrow_\pi^* C[\ulcorner X, \pi(\pi_0X)(\pi_1X) \urcorner_{<>}] \\ &\rightarrow_\pi^* C[\ulcorner X, X \urcorner_{<>}] \\ &\leftarrow_p^* C[X]. \end{aligned}$$

The rules used were respectively **p**, **l**, **r**, and again **p**. □

3.7. Recapitulation and outline

In the above heuristic explanations the system $\lambda\pi p^*$ was arrived at in an attempt to design a variant of $\lambda\pi^{lr}$ such that

(i) the modified reduction relation would be a restriction of that of $\lambda\pi^{lr}$;
(ii) the conversion relation of $\lambda\pi^{lr}$ would be retained;
(iii) we would be able to prove the (modified) Church-Rosser theorem for the restricted reduction relation.

This in order to be able to use the principle formulated in the last paragraph of 2.1 for deriving the CR/≈ theorem for $\lambda\pi^{lr}$. Now, since we had to introduce labels and bookkeeping pairs in $\lambda\pi p^*$, matters have become much more complicated than they were in the case of $\lambda\delta^{lr}$ and $\lambda\delta^{l}$. Yet it will turn out that the pattern of reasoning that was illustrated in section 2.1 can be used for $\lambda\pi^{lr}$ and $\lambda\pi p^*$ as well.

What it all amounts to, is establishing claim 3.7.1, consisting of three propositions, that match respectively with the requirements (i), (ii) and (iii) above. Then CR/≈ for $\lambda\pi^{lr}$ (i.e. claim 1.2.3) follows at once (as corollary 3.7.2). The proof outline of claim 3.7.1 is in fact an outline of the rest of the paper.

3.7.1. CLAIM. (i) *If* $M \in \Lambda\pi$, *then* $\lambda\pi p^* \vdash \psi(M) \twoheadrightarrow^* N \;\Rightarrow$ $(\exists N' \in \Lambda\pi)(\lambda\pi^{lr} \vdash M \geq N' \;\&\; N' \approx \varphi(N))$;
(ii) $\lambda\pi \vdash M = N \;\Rightarrow\; \lambda\pi p^* \vdash \psi(M) =^* \psi(N)$;
(iii) $\lambda\pi p^*$ *is Church-Rosser.*

PROOF OUTLINE. (i) In order to translate reduction sequences of $\lambda\pi p^*$ into $\lambda\pi^{lr}$, what has to be done roughly, is eliminating the bookkeeping pairs from the $\rightarrow^*$-reduction sequence. This involves a postponement argument in the spirit of standardization: we use the fact that reduction steps which consist in the contraction of a redex that occurs within a bookkeeping pair, can be moved to the end of a reduction sequence. See §5.
(ii) This was already established as theorem 3.6.4.
(iii) The proof of the Church-Rosser theorem for $\lambda\pi p^*$ is still rather nasty; it will be sketched in §4. □

3.7.2. COROLLARY (CR/≈).
$\lambda\pi \vdash M = N \;\Rightarrow\; (\exists K', K'')(K' \approx K'' \;\&\; M \geq K' \;\&\; N \geq K'')$.

PROOF. Suppose that $\lambda\pi \vdash M = N$. Then by 3.7.1(ii) also $\lambda\pi p^* \vdash \psi(M) =^* \psi(N)$ and it follows by 3.7.1(iii) that $\psi(M)$ and $\psi(N)$ must have a common $\rightarrow^*$-reduct K. We can then apply 3.7.1(i) (twice) to find $K', K'' \in \Lambda\pi$, such that $M \geq K'$, $N \geq K''$ and $K' \approx K'' \approx \varphi(K)$. □

§4. The Church-Rosser theorem for $\lambda\pi p^*$

About all well known proofs of the Church-Rosser theorem have the same global structure. An auxiliary one-step reduction relation $\rightarrow_1$ is defined, which, instead of just contracting a single redex, consists in an immediate jump to the complete development with respect to an arbitrary set of redexes. One then proves that $\rightarrow_1$ satisfies the diamond property. From that, the Church-Rosser property for $\rightarrow$ can be deduced at once; it suffices to verify the obvious inclusions $\rightarrow \subseteq \rightarrow_1$ and $\rightarrow_1 \subseteq \twoheadrightarrow$. The differences between the various proofs lie mainly in the way $\rightarrow_1$ is arrived at—by a Tait-Martin Löf type direct definition for example, or via the finite developments theorem—and, correspondingly, in the proof of the diamond property for $\rightarrow_1$.

It is essential in this kind of set up, that residuals of redexes are redexes again, of the same type as the ancestor. Under $\rightarrow^*$, however, this is not always the case; it is possible that the constants involved in an existing π-redex become separated by the bookkeeping pair which is introduced in a $\rightarrow_p^*$-step. (Example: $<>.\pi_0(\pi X_0 X_1) \rightarrow_p^* \pi_0 \ulcorner \pi X_0 X_1, \pi X_0 X_1 \urcorner_0$.)

We deal with this problem by segregating $\rightarrow_p^*$-reduction from the other, substantial, reduction rules. Thus we exploit two complementary concepts of fast one step reduction. In the first place there is $\rightarrow_1{}^*$, derived from $\rightarrow_\pi{}^*$ in a more or less standard way. In addition, in section 4.2 a notion $\rightarrow_s$ of "simplifying" p*-reduction will be defined. It is the restriction of $\twoheadrightarrow_p{}^*$ obtained by requiring that in the end term no redexes are disturbed by occurrences of bookkeeping pairs. Then the role of the "one step" reduction relations in the traditional Church-Rosser proofs is played here by the relation $\rightarrow_+$, defined as the composition $\rightarrow_1{}^* + \rightarrow_s$. Accordingly we shall prove the diamond property for $\rightarrow_+$. The structure of the proof is best described by way of the following diagram.

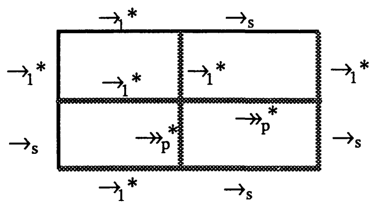


FIGURE 4

The task of establishing this diagram can be divided into three parts, corresponding to the different rectangles in the diagram. The first, left upper rectangle asks for a more or less standard treatment, using the notion of marked reduction and the finite developments theorem. Some complications are caused by the ambiguity of the contraction rules (cf. section 4.1). The treatment of p-reduction that settles the fourth, right downmost rectangle is straightforward. The identical second and third rectangles ask for some more ingenuity (cf. section 4.2).

In section 4.3 Church-Rosser for $\rightarrow^*$ can then be concluded. For, though we do not have $\rightarrow^* \subseteq \rightarrow_+$, it will turn out that the convertibility relation $=_+$ which is generated by $\rightarrow_+$, coincides with $=^*$. As $\rightarrow_+ \subseteq \twoheadrightarrow^*$ does hold all right the "conversion" version of the Church-Rosser theorem follows.

4.1. Marked $\rightarrow_\pi$-reduction

In this section we prove $\rightarrow_\pi{}^*$-Church-Rosser by the method which uses the finite developments theorem to arrive at $\rightarrow_1$, and marked reduction in order to encode developments with respect to sets of redexes (compare Barendregt [1981], Ch.11, §2). Due to the ambiguity of the rules some adjustments have to be made, however.

As a matter of fact, by considering terms in $\Lambda\pi p$ modulo ~-equivalence, we tailored **λπp** minus p-reduction as a weakly regular CRS with (stable) conditions. Church-Rosser for regular Term Rewriting Systems with conditions of a kind bearing some resemblance to the ones here encountered is proved in Bergstra & Klop [1986]. The authors express the belief that their results will carry over to weakly regular TRS's as well. Quite in general, the opinion seems to prevail that the Church-Rosser theorem and related results for regular CRS's generalize easily to the weakly regular case. Accordingly, it may be worthwhile to call attention to the complications described below in defining a coherent notion of residual.

4.1.1. Recall the cases of overlap between the contraction rules $\boldsymbol{\pi_0}$ and **l** (and $\boldsymbol{\pi_1}$ and **r**) that were signalized in Intermezzo 2.2.1. Observe that the π_0's in the respective reducts $\pi_0 X$ in 2.2.1(i) descend from a different ancestor in the original term. The same is true of the π's in πXY and πXZ in 2.2.1(ii). This awkward subtlety gives rise to a serious problem in tracing a possible third redex in which one of these constants is involved.

We sketch the situation in the following diagram. (Only $\boldsymbol{\pi_0}$ and **l** are covered, but the case of $\boldsymbol{\pi_1}$ and **r** is completely analogous.)

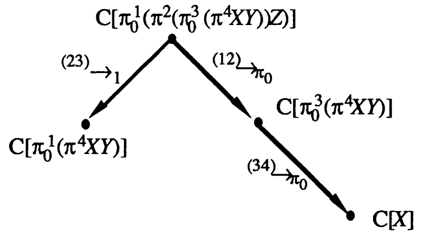


FIGURE 5

It must be assumed of course that $Y = Z$.

We have attached numerals to the constants involved in the reductions and for this occasion indicate a redex by the combination of numerals attached to the constants that constitute the redex in question. (That is, e.g. the redex $\pi_0{}^3(\pi^4XY)$ is indicated as 34.) So in the original term of the diagram we can distinguish redexes 12, 23 and 34.

Now it is clear that in the result $C[\pi_0{}^3(\pi^4XY)]$ of reduction step (12) the redex 34 is residual of the redex 34 in the original term, whereas in the result $C[\pi_0{}^1(\pi^4XY)]$ of (23) no such residual exists. In the usual notation: $34/(12) = 34$ and $34/(23) = \emptyset$. Hence it appears that the reduction sequences (23) and (12) + (34) are both complete developments with respect to the set of redexes $\{12, 23, 34\}$. But the end terms $C[\pi_0(\pi XY)]$ and $C[X]$ are not the same. In order to obtain yet an unique complete development we declare $\pi_0{}^1(\pi^4XY)$ to be a *virtual residual* of the redexes 12 and 34 under (23).

4.1.2. TERMINOLOGY. Call the occurrences of constants that are required to constitute a redex the *critical constants* of that redex. (E.g. in $\pi_0(\pi\pi_1\pi_1)$ the critical constants are π_0 and π; in general the critical constants are the ones that are displayed in (left hand sides of) the contraction rules, cf. 3.6.1.) Notice that the redexes Σ and Δ in *M overlap* if they share one critical constant. Now, given a set $\mathfrak{R}$ of redex occurrences in M, we define an $\mathfrak{R}$-*chain* to be a maximal set $\{\Sigma_1, ..., \Sigma_n\} \subseteq \mathfrak{R}$, $n > 0$, such that for each $i < n$, Σ_i and Σ_{i+1} overlap. The $\mathfrak{R}$-chains form a partition of $\mathfrak{R}$. Since β-redexes have no overlap, they constitute an $\mathfrak{R}$-chain each on their own. An *inner redex* of $\mathfrak{R}$ is one which overlaps with two other redexes in $\Re$; these redexes will then belong to the same $\Re$-chain. As explicated in 4.1.1, contraction of an inner redex of $\Re$ leaves only a *virtual residual* of its immediate neighbour redexes.

We now give a formalization of all this by way of appropriate concepts of marked term and marked reduction.

4.1.3. DEFINITION. The pair $<M, \Re>$ is represented by the marked term which is obtained by attaching primes ('), apostrophes (') and inverse apostrophes (') to λ's and critical constants occurring in M in the following way.

(i) the initial λ of a redex $(\lambda x.N_0)N_1$ in $\Re$ is primed (result: $(\lambda'x.N_0)N_1$),

(ii) the leftmost critical constant of each $\Re$-chain gets ' (result: $...\pi_{(i)}$'...),

(iii) the rightmost critical constant of each $\Re$-chain gets an ' (result: ...$\pi_{(i)}$'...),

(iv) all other critical constants of π-redexes in $\Re$ are primed $(...\pi_{(i)}'...)$.

The marked terms that are thus obtained as representing pairs $<M, \Re>$, constitute the set $\Lambda'\pi p$. If $M \in \Lambda'\pi p$, then $|M|$ is the term that results by deleting the marks from M. The restriction of $\Lambda'\pi p$ to M's with canonical $|M|$'s is denoted by $\Lambda'\pi p^*$.

COMMENT. An $\Re$-chain in M can be recognized in the representing marked variant of M in $\Lambda'\pi p$ because all its critical constants are marked. Such a chain of marked $\pi_{(i)}$'s is called a π-chain. The apostrophes play the role of begin (') and end (') markers for π-chains. This feature is necessary for marking the kind of difference that exists e.g. between the marked terms π_0'$(\pi'(\pi_0'(\pi'XY))Z)$ and π_0'$(\pi'(\pi_0$'$(\pi'XY))Z)$, the first one representing one $\Re$-chain of length 3, the second two $\Re$-chains, each of length 1.

4.1.4. DEFINITION. $\rightarrow_{\pi'}$ is the one step reduction relation on $\Lambda'\pi p$ which is derived from $\rightarrow_\pi$ by restricting:

- β-contraction to redexes of which the initial λ is primed: $(\lambda'x.N_0)N_1 \rightarrow_{\pi'} (x:=N_1)N_0$, and
- the rules π_0, π_1, **l** and **r** to redexes of which the critical constants are marked, in that order, by either ' and ', ' and ', ' and ', or ' and '.

If an outermost critical constant of a π-chain of length at least four is involved in the contraction, its mark (' or ') is passed on to the leftmost (or rightmost) critical constant in the residual π-chain. If the original π-chain contained only one or two redexes (two or three critical constans respectively), no residuals remain. Hence in the case of three critical constants, the mark of the single critical constant that is not involved in the contraction (it must be either ' or ') is canceled in the reduct.

EXAMPLES. π_0'$(\pi'(\pi_0'(\pi'XY))Y) \rightarrow_{\pi'} \pi_0$'$(\pi'XY)$ (in three ways);

π_0'$(\pi'(\pi_0$'$(\pi'XY))Y) \rightarrow_{\pi'} \pi_0$'$(\pi'XY)$ (in two ways);

π_0'$(\pi'(\pi_0'X)(\pi_1X)) \rightarrow_{\pi'} \pi_0X$ (in two ways).

4.1.5. THEOREM (FD!). *For each* $M \in \Lambda'\pi p$:

(i) *the number of steps in an* $\rightarrow_{\pi'}$*-sequence is finite;*

(ii) *there exists a unique* $\rightarrow_{\pi'}$*-normal form N such that* $M \twoheadrightarrow_{\pi'} N$;

(iii) *The same results hold for* $\rightarrow_{\pi'}{}^*$ *in* $\Lambda'\pi p^*$.

PROOF. It is not difficult to adapt the standard proofs. The stability of the context sensitive reduction rules **l** and **r**, needed for the construction of elementary diagrams in (ii), is guaranteed by lemma 3.6.3(i). For (iii) one can make use of the fact that $\rightarrow_{\pi'}$ etc. are $*$-compatible. □

4.1.6. DEFINITION. (i) For $M \in \Lambda'\pi p$, the *height* h(M) is the number of reduction steps in a $\rightarrow_{\pi'}$-reduction sequence from M of maximal length.

(ii) Denote the unique $\rightarrow_{\pi'}$-normal form of $M \in \Lambda'\pi p$ by CD(M). Then the one step reduction relation $\rightarrow_1$ on $\Lambda\pi p$ is defined by

$$M \rightarrow_1 N \Leftrightarrow (\exists M' \in \Lambda'\pi p)(|M'| \equiv M \ \& \ \mathrm{CD}(M') \equiv N).$$

(iii) The height function h*, assigns to each $M \in \Lambda'\pi p^*$ the length of the longest $\rightarrow_{\pi'}{}^*$-sequence from M and the function CD* assigns to $M \in \Lambda'\pi p^*$ its unique $\rightarrow_{\pi'}{}^*$-normal form.

Of course one has h*(M) ≤ h(M) and CD*(M) ≡ CD(M)*.

4.1.7. LEMMA. (i) $\rightarrow_1$ is self commuting.
(ii) $\rightarrow_1{}^*$ is self commuting.

PROOF. (i) Standard, using 4.1.5.
(ii) This follows easily from (i), once it is established that $\rightarrow_1$ is $*$-projectable. □

4.2. Adding p-reduction

Even in canonical terms it is still possible that bookkeeping pairs stand in the way and obstruct reduction. An example of this is the term $\diamond.\pi_0\ulcorner\pi X_0X_1, Y\urcorner_0$, with the blocked potential redex $\pi_0(\pi X_0X_1)$.

One consequence is that $\Lambda'\pi p$ is not closed under the rule of bookkeeping pair introduction **p** (cf. 3.6.1(ii)). A new bookkeeping pair might break a π-chain and thereby disturb a marked redex. An example would be the p-step

$$\alpha.\pi_0\text{'}(\pi'XY) \rightarrow_p \pi_0\text{'}\ulcorner\pi'XY, \pi'XY\urcorner_{0\alpha}.$$

Therefore we work on $\Lambda'\pi p$ with a restricted version $\rightarrow_{p'}$ of $\rightarrow_p$. It must be kept in mind then, that not just any p-reduction sequence can be lifted to a p'-sequence.

4.2.1. DEFINITION. For $M \in \Lambda'\pi p$ we define:

$$M \rightarrow_{p'} N \Leftrightarrow M \rightarrow_p N \ \& \ N \in \Lambda'\pi p.$$

Now it turns out that an obstructing bookkeeping pair can always be removed by performing an appropriate $\rightarrow_p{}^*$-reduction step. The pertinent cases are covered in the following lemma.

4.2.2. LEMMA. *The following are derived rules in* $\lambda\pi p^*$.

$$C[\pi\ulcorner X_0,X_1\urcorner_\alpha Y]_{0\alpha} \rightarrow_p{}^* C[\ulcorner \pi X_0 Y, \pi X_1 Y\urcorner_{0\alpha}],$$
$$C[\pi Y\ulcorner X_0,X_1\urcorner_\alpha]_{1\alpha} \rightarrow_p{}^* C[\ulcorner \pi Y X_0, \pi Y X_1\urcorner_{1\alpha}],$$
$$C[\pi_i\ulcorner X_0,X_1\urcorner_{i\alpha}]_\alpha \rightarrow_p{}^* C[\ulcorner \pi_i X_0, \pi_i X_1\urcorner_\alpha].$$

PROOF. Each of the rules follows by just doubling the whole occurrence displayed and subsequently canceling the descendants of the original bookkeeping pair. □

The effect of these reduction steps is that the bookkeeping pair is as it were opened to that part of the expression which acts upon it as a function. That way a redex may be constituted, of which the ingredients were still separated before the simplification was performed. (An example of this would be a simplifying p*-step

$$<>.\pi_0\ulcorner \pi X_0 X_1, Y\urcorner_0 \rightarrow_p{}^* \ulcorner \pi_0(\pi X_0 X_1), \pi_0 Y\urcorner_{<>}.)$$

Normal forms under the derived rules of 4.2.2 are called p-*simple*. In p-simple terms no π-redexes are blocked anymore. It is easy to verify that for each $M \in \Lambda\pi p^*$ a p-simple form can be reached by performing a finite number of p*-steps of the above kind. As a matter of fact this normal form is unique. This does not interest us here, however. Rather, for the purpose of establishing Church-Rosser for $\rightarrow^*$, the following auxiliary one step reduction relation of non-unique p-simplification will turn out to be very useful.

4.2.3. DEFINITION. The one step reduction relation $\rightarrow_s$ on $\Lambda\pi p^*$ is defined by:

$$M \rightarrow_s N \iff M \twoheadrightarrow_p{}^* N \ \& \ N \text{ is p-simple.}$$

4.2.4. LEMMA. (i) *For all* $M \in \Lambda\pi p^*$ *there exists a term* N, *such that* $M \rightarrow_s N$. *This* N *is in general not unique.*
(ii) $\twoheadrightarrow_p{}^* + \rightarrow_s = \rightarrow_s$.

Now the missing rectangles of Fig. 4 are filled in by the following lemmas. The proofs of 4.2.5 are very technical.

4.2.5. LEMMA. (i) $\twoheadrightarrow_{\pi'}{}^*$ *and* $\twoheadrightarrow_{p'}{}^*$ *commute.*
(ii) *On* $\Lambda'\pi p^*$ *we have* $\rightarrow_s \subseteq \twoheadrightarrow_{p'}{}^*$.

4.2.6. LEMMA. (i) $\rightarrow_p{}^*$ *is Church-Rosser.*
(ii) $M \rightarrow_1{}^* N \ \& \ M \rightarrow_s P \Rightarrow (\exists Q)(N \twoheadrightarrow_p{}^* Q \ \& \ P \rightarrow_1{}^* Q)$.
(iii) $M \twoheadrightarrow_p{}^* N \ \& \ M \twoheadrightarrow_p{}^* P \Rightarrow (\exists Q)(N \rightarrow_s Q \ \& \ P \rightarrow_s Q)$.

PROOF. (i) Rather straightforward.
(ii) $M \rightarrow_1{}^* N$ means that an M_0 can be chosen such that $|M_0| \equiv M$ and $M_0 \twoheadrightarrow_{\pi'}{}^* N$. Lift $M \rightarrow_s P$ to $M_0 \rightarrow_s P_0$ (with $|P_0| \equiv P$). By 4.2.5(ii) then $M_0 \twoheadrightarrow_{p'}{}^* P_0$ holds too. So 4.2.5(i) can be applied to find a Q such that the diagram in Fig.6(a) holds. Since N has no marks ($\in \Lambda\pi p^*$), neither has Q. So $P_0 \twoheadrightarrow_{\pi'}{}^* Q$ is a complete development and hence $P \rightarrow_1{}^* Q$.
(iii) The proof is depicted in Fig. 6(b). Constructing A is $\rightarrow_p{}^*$-Church-Rosser. Q is found from P by the existence of p-simple forms. Then B follows (twice) because $\twoheadrightarrow_p{}^* + \rightarrow_s = \rightarrow_s$. □

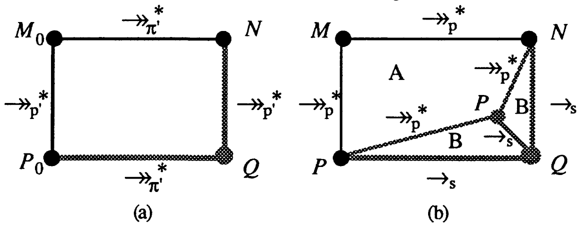


FIGURE 6

4.3. Church-Rosser

By now we have gathered all the ingredients of Fig. 4. We proceed as it was already indicated in the introduction to §4.

4.3.1. DEFINITION. (i) The one step reduction relation $\rightarrow_+$ on $\Lambda\pi p^*$ is defined by

$$\rightarrow_+ = \rightarrow_1{}^* + \rightarrow_s.$$

(ii) $=_+$ is the equivalence relation which is generated by $\rightarrow_+$.

4.3.2. LEMMA. (i) $\rightarrow_+ \subseteq \twoheadrightarrow^*$.
(ii) $=_+$ *and* $=^*$ *coincide.*

PROOF. (i) is trivial, since $\rightarrow_1{}^* \subseteq \twoheadrightarrow_\pi{}^*$ and $\rightarrow_s \subseteq \twoheadrightarrow_p{}^*$. One half of (ii), namely $=_+ \subseteq =^*$, is immediate by (i). Since $=^*$ is generated by $\rightarrow_p{}^*$ and $\rightarrow_\pi{}^*$, it suffices for the other half to establish (a) and (b) below.

(a) $\rightarrow_p{}^* \subseteq =_+$. For suppose $M \rightarrow_p{}^* N$. By Lemma 4.2.4(i) there exists an N' such that $N \rightarrow_s N'$. Since $\rightarrow_p{}^* + \rightarrow_s = \rightarrow_s$, we have also $M \rightarrow_s N'$. Moreover $\rightarrow_s \subseteq \rightarrow_+$, because $\rightarrow_1{}^*$ is reflexive. Then $M =_+ N$ follows (via N').

(b) $\rightarrow_\pi{}^* \subseteq =_+$. If $M \rightarrow_\pi{}^* N$, then also $M \rightarrow_1{}^* N$, and $M =_+ N$ can be established, with the same reasoning as in (a), via an N' such that $N \rightarrow_s N'$. □

4.3.3. THEOREM. *The system* $\lambda\pi p^*$ (*i.e.*($\Lambda\pi p^*$, $\rightarrow^*$)) *is Church-Rosser.*

PROOF. By the lemmas 4.1.7 and 4.2.6, dealing with the respective rectangles of Fig. 4, it follows that $\rightarrow_+$ is self commuting and hence certainly Church-Rosser. So if $M =_+ N$, then M and N have a common $\rightarrow_+$-reduct. But, since $=^*$ is the same as $=_+$ (by 4.3.2(ii)) and each $\rightarrow_+$-sequence can be transformed into a $\rightarrow^*$-sequence (by 4.3.2(i)), we have then also Fig. 7. □

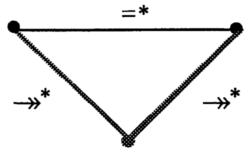


FIGURE 7

With theorem 4.3.3 we have established claim 3.7.1(iii).

§5. Translating reductions from $\lambda\pi p^*$ to $\lambda\pi^{lr}$

The part of the puzzle that is still missing, claim 3.7.1(i), will be provided in this concluding §. In order to translate reduction sequences of $\lambda\pi p^*$ into $\lambda\pi^{lr}$, we first bring them in a special form, reached by postponement of the reduction steps which consist in the contraction of a redex occurring within a bookkeeping pair. The resulting notion of e/i-reduction sequence (definition 5.1.3) has some resemblance with the concept of semi standardization, sometimes used in proofs of the standardization theorem in combinatory logic and pure λ-calculus (cf. Curry, Hindley & Seldin [1972] and Mitschke [1979]). This method, originating with Rosser [1935], inspired our proceeding in this §.

5.1. Internal and external reduction

5.1.1. DEFINITION. An occurrence is called *internal*, if it lies completely inside a bookkeeping pair, otherwise *external*.

5.1.2. DEFINITION. A reduction step (Σ): $M \to N$ is called *internal* if the contracted redex Σ is internal; notation $M \to^{i} N$. Other reduction steps are called *external* ($\to^{e}$). Formally the relation $\to^{i}$ on $\Lambda\pi p$ can be defined as the compatible closure of the reduction rule:

i: $\ulcorner M_0, M_1 \urcorner_\alpha \to^{i} N$, if $\ulcorner M_0, M_1 \urcorner_\alpha \to N$,

and $\to^{e}$ by $\to^{e} = \to \backslash \to^{i}$.

Derived notations such as $\to^{i*}$, $\twoheadrightarrow_\pi{}^{e*}$, $\to_{\pi'}{}^{i*}$, etc. are used in accordance with established usage. Moreover we use the notation $\to^{(-e)}$ for $\to \backslash \to_\pi{}^{e}$. (Note that $\to^{(-e)}$ can be conceived of either as $\to_p \cup \to_\pi{}^{i}$ or as $\to^{i} \cup \to_p{}^{e}$.)

A marked term $M \in \Lambda'\pi p$ is called *internal* if all its marked redexes are internal. Obviously, if M is internal and $M \to_{\pi'}{}^{*} N$, then $M \to_{\pi'}{}^{i*} N$ and also N is internal. We say that $M \to_1{}^{i} N$, if N is the complete development of M with respect to a set of internal redexes.

EXAMPLES. $-(\lambda x.xy)\ulcorner \pi_0(\pi xy), x \urcorner \to^{e} \ulcorner \pi_0(\pi xy), x \urcorner y \to^{i} \ulcorner x, x \urcorner y$;
$-(\lambda x.xy)\ulcorner \pi_0(\pi xy), x \urcorner \to^{i*} (\lambda x.xy)\ulcorner x, x \urcorner \to^{e*} xy$;
$-(\lambda x.xy)\ulcorner \pi_0(\pi xy), x \urcorner \to^{e*} \pi_0(\pi xy)y \to xy$;
$-0.(\lambda x.x)x \to_p{}^{e} (\lambda x.x)\ulcorner x, x \urcorner \to_\pi{}^{e*} x$.

Observe that by changing the order of reduction, an internal step may become an external one (second and third example).

5.1.3. DEFINITION. A reduction sequence in $\lambda\pi p^*$ is called *e/i*, if it consists of a number of $\to_\pi{}^{e*}$-reduction steps, followed by a number of $\to^{(-e)*}$-steps (i.e. $\to_p{}^{e*}$- and $\to^{i*}$-steps).

The result of section 5.1 will be that any $\to^{*}$-reduction sequence can be transformed into one that is e/i. It will become clear in 5.2 that e/i sequences in $\lambda\pi p^*$ can be easily translated into $\lambda\pi^{lr}$.

5.1.4. LEMMA. $M \to_1{}^{*} N \Rightarrow (\exists L)(M \twoheadrightarrow_\pi{}^{e*} L \to_1{}^{i*} N)$.

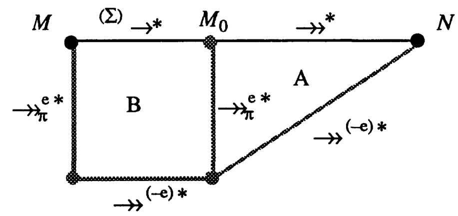
PROOF. Let $M_0 \in \Lambda'\pi p^*$ such that $|M_0| \equiv M$ and $CD^*(M_0) \equiv N$. We use induction on $h^*(M_0)$. If M_0 is internal, then $M \to_1{}^{i*} N$ and $L \equiv M$ will do. Otherwise suppose $M_0 \to_{\pi'}{}^{e*} M_1$. By the induction hypothesis there exists an L such that $|M_1| \twoheadrightarrow_\pi{}^{e*} L \to_\pi{}^{i*} N$. Since of course $M \to_\pi{}^{e*} |M_1|$ holds too, this L suffices. □

5.1.5. LEMMA.
(i) $M \to_1{}^{i*} K \to_\pi{}^{e*} N \Rightarrow (\exists L)(M \twoheadrightarrow_\pi{}^{e*} L \to_1{}^{i*} N)$
(ii) $M \to_1{}^{i*} K \twoheadrightarrow_\pi{}^{e*} N \Rightarrow (\exists L)(M \twoheadrightarrow_\pi{}^{e*} L \to_1{}^{i*} N)$

PROOF. (i) The external redex in K that is contracted to obtain N, is the residual of a unique external redex occurrence in M. Therefore $M \to_1{}^{*} N$. Apply lemma 5.1.4.
(ii) Repeat (i). □

5.1.6. THEOREM. *If $M \twoheadrightarrow^{*} N$, then there exists an e/i-reduction sequence from M to N.*

PROOF. Induction on the number of $\to^{*}$-steps in the reduction from M to N. If $M \equiv N$ there is not much to prove. So assume an M_0 which satisfies the drawn part of the following diagram.



Then the triangle A can be found by the induction hypothesis. If (Σ) is an $\to_\pi{}^{e*}$-step, that suffices. Otherwise B can be constructed using either lemma 5.1.5 (if it is a $\to_\pi{}^{i*}$-step) or a similar lemma for $\to_p{}^{*}$. □

5.2. The translation

The content of the next lemma is that a so called "main" p*-reduction, that is, a p*-step that doubles the whole term, can always be moved to the front of a reduction sequence.

5.2.1. LEMMA . Let $\alpha.M \twoheadrightarrow^{*} \ulcorner N_0, N_1 \urcorner_\alpha$. *Then either*
(i) $M \equiv \ulcorner M_0, M_1 \urcorner_\alpha$ *and* $\alpha i.M_i \twoheadrightarrow^{*} N_i$, *or*
(ii) $\alpha.M \to_p{}^{*} \ulcorner M_0, M_1 \urcorner_\alpha$ *with* $M_i \equiv \alpha i.M^*$ *and* $\alpha i.M^* \twoheadrightarrow^{*} N_i$.

PROOF. If M is a bookkeeping pair, case (i) trivially applies. If not, then of course $\alpha.M \to_p{}^{*} \ulcorner M^*, M^* \urcorner_\alpha$ by a main p*-step; and $\alpha i.M^* \twoheadrightarrow^{*} (\ulcorner N_0, N_1 \urcorner_\alpha)^* \equiv N_i$ follows from $\alpha.M \twoheadrightarrow^{*} \ulcorner N_0, N_1 \urcorner_\alpha$ by the $*$-compatibility of $\twoheadrightarrow^{*}$ and because $\alpha i.N_i$ is canonical, as it occurs in an αi-context within the canonical $\ulcorner N_0, N_1 \urcorner_\alpha$. □

5.2.2. LEMMA. *Suppose* $M \in \Lambda\pi$ *and* $\lambda\pi p^* \vdash \alpha.M \twoheadrightarrow_{\pi}^* N$. *Then* $\lambda\pi^{lr} \vdash M \geq K$ *for some* $K \in \Lambda\pi$ *such that* $\alpha.K \sim N$.

PROOF. Taking terms literally again (not modulo ~), we have $\alpha.M \twoheadrightarrow_{\pi}^* K$ for some K such that $\alpha.K \sim N$. Since $M \in \Lambda\pi$ and $\rightarrow_{\pi}$ does not create bookkeeping pairs, there will be nothing to cancel for $\hookrightarrow^*$ during this reduction, and we have $\alpha.M \twoheadrightarrow_{\pi} K$ and $K \in \Lambda\pi$. It then suffices to remind that each of the rules for $\rightarrow_{\pi}$ (cf. 3.6.1(i)) is covered by one for $\geq$ (cf. 1.2.1). In particular both **l** and **r** are included in $\lambda\pi^{lr}$ without context restrictions. □

5.2.3. THEOREM. *Suppose* $M \in \Lambda\pi$ *and* $\lambda\pi p^* \vdash \alpha.M \twoheadrightarrow^* N$. *Then* $\lambda\pi^{lr} \vdash M \geq K$ *for some* $K \in \Lambda\pi$ *such that* $K \approx \varphi(N)$.

PROOF. The proof is by induction on N. By theorem 5.1.6 there is an L such that $\alpha.M \twoheadrightarrow_{\pi}{}^{e*} L$ and $\alpha.L \twoheadrightarrow^{(-e)*} N$. Consequently, lemma 5.2.2 can be used to supply us with a $K_0 \in \Lambda\pi$ such that $\alpha.K_0 \sim L$ and $\lambda\pi^{lr} \vdash M \geq K_0$. Now, if N is already an element of $\Lambda\pi$ itself, then the (-e)-part of the e/i sequence is empty and we have $\varphi(N) \equiv N \equiv L$. So in this case K can be taken just K_0, as on $\Lambda\pi$ one has of course $\sim \subseteq \approx$. Otherwise N contains one or more bookkeeping pairs and can thus be assumed to be written as

$$\ldots\ulcorner X_1, Y_1 \urcorner_{\alpha_1} \ldots \ulcorner X_2, Y_2 \urcorner_{\alpha_2} \ldots\ldots \ulcorner X_n, Y_n \urcorner_{\alpha_n} \ldots,$$

with each maximal occurrence of a bookkeeping pair displayed as one of the $\ulcorner X_i, Y_i \urcorner_{\alpha_i}$'s. Since $\alpha.K_0 \sim L$, the second part of the e/i-sequence from M to N may be rendered as well as $\alpha.K_0 \twoheadrightarrow^{(-e)*} N$. Since this reduction proceeds completely without external reduction steps, K_0 must have a shape similar to that of N, that is

$$K_0 \equiv \ldots Z_1 \ldots Z_2 \ldots\ldots Z_n \ldots,$$

coinciding with N on the dots, and such that for each $i \leq n$:

$$\ell(Z_i) = \alpha_i \text{ and } \alpha_i.Z_i \twoheadrightarrow^{(-e)*} \ulcorner X_i, Y_i \urcorner_{\alpha_i}.$$

A subterm Z_i of $K_0 \in \Lambda\pi$ being bookkeeping pairless, lemma 5.2.1 can be applied, yielding for each i with $1 \leq i \leq n$ a reduction $\alpha_i 0.Z_i \twoheadrightarrow^* X_i$. On these we can use the induction hypothesis, and thereby obtain reductions $Z_i \geq X_i'$ in $\lambda\pi^{lr}$, with $X_i' \approx \varphi(X_i)$.

It remains to combine the reductions we established so far. By the compatibility of $\geq$ it follows that

$$\lambda\pi^{lr} \vdash K_0 \geq \ldots X_1' \ldots X_2' \ldots\ldots X_n' \ldots.$$

Define $K \equiv \ldots X_1' \ldots X_2' \ldots\ldots X_n' \ldots$. Then $K \approx \varphi(N)$ is an immediate consequence of the above by the the definition of φ (cf. 3.3.1(ii)) and the compatibility of $\approx$. Moreover, $M \geq K$ via K_0. So K satisfies the requirements of the theorem. □

With theorem 5.2.2 we have established claim 3.7.1(i) and, at last, the Church-Rosser theorem (modulo $\approx$) for $\geq$ follows as corollary 3.7.2. One then concludes the conservativity and consistency of $\lambda\pi$ (corollary 1.2.4) as demonstrated in section 1.2.

References

Barendregt, H.P. [1974], 'Pairing without conventional restraints', *Zeitschr. für Math. Logik und Grundl. der Math.* 20, pp.289-306.

Barendregt, H.P. [1981], *The Lambda Calculus* (North Holland).

Bergstra, J.A. and J.W. Klop [1986], 'Conditional Rewrite Rules: Confluence and Termination', *Journal of Computer and System Sciences* 32, p.323–362.

Curien, P.-L. [1986], *Categorical Combinators, Sequential Algorithms and Functional Programming* (Pitman, London).

Curry, H.B., J.R. Hindley and J.P. Seldin [1972], *Combinatory Logic Vol. II* (North Holland).

Hardin, Th. [1986], *Yet another counter-example to confluence in λ-calculus with couples*, preprint, INRIA, Rocquencourt.

Hardin, Th. [1987], *Résultats de confluence pour les règles fortes de la logique catégorique et les liens avec les lambda-calculs*, thèse de doctorat, Université Paris VII.

Klop, J.W. [1980], 'Combinatory reduction systems', *Mathematical Center Tracts* 129, Amsterdam.

Klop, J.W. and R.C. de Vrijer [1989], 'Unique normal forms for lambda calculus with surjective pairing', *Information and Computation* 80, no. 2, pp. 97-113.

Lambek, J. and P. Scott [1986], *Introduction to higher order categorical logic* (Cambridge University Press).

Mann, C.R. [1973], *Connections Between Proof Theory and Category Theory*, dissertation, Oxford University.

Mitschke, G. [1979], 'The standardization theorem for the λ–calculus'. *Zeitschr. für Math. Logik und Grundl. der Math.* 25, p.29–31.

Rosser, J.B., [1935], 'A mathematical logic without variables I', *Annals of Mathematics* 36, p.127–150.

Scott, D. [1975], 'Lambda Calculus and Recursion Theory'. *Proc. of the third Scandinavian Logic Symposium*, ed. S. Kanger (North Holland), pp.154-193.

Vrijer, R.C. de [1987], '*Surjective pairing and strong normalization: two themes in λ-calculus*', dissertation, Universiteit van Amsterdam

The ITLI Prepublication Series

1986

86-01	The Institute of Language, Logic and Information
86-02 Peter van Emde Boas	A Semantical Model for Integration and Modularization of Rules
86-03 Johan van Benthem	Categorial Grammar and Lambda Calculus
86-04 Reinhard Muskens	A Relational Formulation of the Theory of Types
86-05 Kenneth A. Bowen, Dick de Jongh	Some Complete Logics for Branched Time, Part I Well-founded Time, Forward looking Operators
86-06 Johan van Benthem	Logical Syntax

1987

87-01 Jeroen Groenendijk, Martin Stokhof	Type shifting Rules and the Semantics of Interrogatives
87-02 Renate Bartsch	Frame Representations and Discourse Representations
87-03 Jan Willem Klop, Roel de Vrijer	Unique Normal Forms for Lambda Calculus with Surjective Pairing
87-04 Johan van Benthem	Polyadic quantifiers
87-05 Víctor Sánchez Valencia	Traditional Logicians and de Morgan's Example
87-06 Eleonore Oversteegen	Temporal Adverbials in the Two Track Theory of Time
87-07 Johan van Benthem	Categorial Grammar and Type Theory
87-08 Renate Bartsch	The Construction of Properties under Perspectives
87-09 Herman Hendriks	Type Change in Semantics: The Scope of Quantification and Coordination

1988

Logic, Semantics and Philosophy of Language:

LP-88-01 Michiel van Lambalgen	Algorithmic Information Theory
LP-88-02 Yde Venema	Expressiveness and Completeness of an Interval Tense Logic
LP-88-03	Year Report 1987
LP-88-04 Reinhard Muskens	Going partial in Montague Grammar
LP-88-05 Johan van Benthem	Logical Constants across Varying Types
LP-88-06 Johan van Benthem	Semantic Parallels in Natural Language and Computation
LP-88-07 Renate Bartsch	Tenses, Aspects, and their Scopes in Discourse
LP-88-08 Jeroen Groenendijk, Martin Stokhof	Context and Information in Dynamic Semantics
LP-88-09 Theo M.V. Janssen	A mathematical model for the CAT framework of Eurotra
LP-88-10 Anneke Kleppe	A Blissymbolics Translation Program

Mathematical Logic and Foundations:

ML-88-01 Jaap van Oosten	Lifschitz' Realizabiility
ML-88-02 M.D.G. Swaen	The Arithmetical Fragment of Martin Löf's Type Theories with weak Σ-elimination
ML-88-03 Dick de Jongh, Frank Veltman	Provability Logics for Relative Interpretability
ML-88-04 A.S. Troelstra	On the Early History of Intuitionistic Logic
ML-88-05 A.S. Troelstra	Remarks on Intuitionism and the Philosophy of Mathematics

Computation and Complexity Theory:

CT-88-01 Ming Li, Paul M.B.Vitanyi	Two Decades of Applied Kolmogorov Complexity
CT-88-02 Michiel H.M. Smid	General Lower Bounds for the Partitioning of Range Trees
CT-88-03 Michiel H.M. Smid, Mark H. Overmars Leen Torenvliet, Peter van Emde Boas	Maintaining Multiple Representations of Dynamic Data Structures
CT-88-04 Dick de Jongh, Lex Hendriks Gerard R. Renardel de Lavalette	Computations in Fragments of Intuitionistic Propositional Logic
CT-88-05 Peter van Emde Boas	Machine Models and Simulations (revised version)
CT-88-06 Michiel H.M. Smid	A Data Structure for the Union-find Problem having good Single-Operation Complexity
CT-88-07 Johan van Benthem	Time, Logic and Computation
CT-88-08 Michiel H.M. Smid, Mark H. Overmars Leen Torenvliet, Peter van Emde Boas	Multiple Representations of Dynamic Data Structures
CT-88-09 Theo M.V. Janssen	Towards a Universal Parsing Algorithm for Functional Grammar
CT-88-10 Edith Spaan, Leen Torenvliet Peter van Emde Boas	Nondeterminism, Fairness and a Fundamental Analogy
CT-88-11 Sieger van Denneheuvel Peter van Emde Boas	Towards implementing RL

Other prepublications:

X-88-01 Marc Jumelet	On Solovay's Completeness Theorem

1989

Logic, Semantics and Philosophy of Language:

LP-89-01 Johan van Benthem	The Fine-Structure of Categorial Semantics

Mathematical Logic and Foundations:

ML-89-01 Dick de Jongh, Albert Visser	Explicit Fixed Points for Interpretability Logic
ML-89-02 Roel de Vrijer	Extending the Lambda Calculus with Surjective Pairing is conservative

Computation and Complexity Theory:

CT-89-01 Michiel H.M. Smid	Dynamic Deferred Data Structures
CT-89-02 Peter van Emde Boas	Machine Models and Simulations
CT-89-03 Ming Li, Herman Neuféglise Leen Torenvliet, Peter van Emde Boas	On Space efficient Solutions

Other prepublications:

X-89-01 Marianne Kalsbeek	An Orey Sentence for Predicative Arithmetic
X-89-02 G. Wagemakers	New Foundations. a Survey of Quine's Set Theory