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FROM STRICTLY POSITIVE TOWARDS MONOTONE

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Elementary Inductive Definitions in HA: from Strictly Positive towards Monotone

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Abstract

A study of elementary inductive definitions (e.i.d.) in **HA**. Strictly positive e.i.d. have closure ordinals $\leq \omega$, and define predicates that are already definable in **HA**. We enlarge this class by adding so-called J-operators, for example $\neg\neg$. E.i.d. in this larger class have closure ordinals up to $\omega + \omega$, but they are conservative over **HA** w.r.t. definability.

1 Introduction

We shall consider as inductive definitions formulae in the language of **HA** expanded with a single one place predicate variable P, containing at most one numerical variable free. The meaning of such an inductive definition $A(P,x)$ is the least fixed-point of $A(P,x)$, i.e. a predicate P^A satisfying

(i): $\forall x(A(P^A,x) \leftrightarrow P^A x)$

(ii): $\forall x(A(Q,x) \rightarrow Qx) \rightarrow \forall x(P^A x \rightarrow Qx)$.

So the inductive definition specifies the closure conditions of the predicate it defines. The question is: for which $A(P,x)$ can we justify the existence of such a P^A? If $A(P,x)$ is *monotone*, i.e.

$$\forall x(Qx \rightarrow Rx) \rightarrow \forall x(A(Q,x) \rightarrow A(R,x)),$$

then we can approximate P^A from below; define

$$P^A_0 x \quad :\Longleftrightarrow A(\lambda x.\bot, x)$$

$$P^A_{\beta+1} x \quad :\Longleftrightarrow A(P^A_\beta, x)$$

$$P^A_\lambda x \quad :\Longleftrightarrow \exists \mu{<}\lambda P^A_\mu x \ \text{, lim } \lambda$$

$$P^A_\infty x \quad :\Longleftrightarrow \exists \mu P^A_\mu x$$

Note that for monotone $A(P,x)$ (i)$\leftarrow$ is redundant: we have $A(P^A,x) \rightarrow P^A x$ by (i)$\rightarrow$, then by monotonicity we get $A(A(P^A,.),x) \rightarrow A(P^A,x)$, and finally by (ii) $P^A x \rightarrow A(P^A,x)$.

Classically P^A exists and is equal to the least fixed-point of $A(P,x)$. An *elementary* inductive definition (e.i.d.) is an inductive definition without an unbounded universal quantifier occurring in front of a positive subformula containing P, and without an unbounded existential quantifier in front of a negative subformula containing P; the inductive definition must be monotone. Classically we know that for e.i.d. the approximation closes up at or before stage ω, so $P^A_\infty = P^A_\omega$. Intuitionistically, this is only true (in general) for strictly positive inductive definitions, i.e. formulae $A(P,x)$ built up from atomic formulae Pt, from **HA**-formulae φ (these do not contain P), by means of $\exists, \forall y{<}s$, $\wedge$, $\vee$.
Now we want to solve the following problems

(i): give neat ordinal bounds for arbitrary e.i.d., not only for the strictly positive ones

(ii): prove or refute: e.i.d. enhance the expressive power of **HA**.

I have no complete answer to these questions. I will describe special extensions of the class of strictly positive e.i.d. ,which do not enhance the expressive power of **HA**, while those e.i.d. may have a closure ordinal up to $\omega+\omega$. Those extensions are made by closing the strictly positive formulae under new operations, like $\neg\neg$. When we allow arbitrary monotone formulae, these problems look rather intractable. In particular, implication (with negative antecedent and positive consequent) seems rather tough.

Acknowledgement

This article is a partial answer to a question, posed by Kreisel in [Kre63, p.3.25]. I am indebted to prof. A.S. Troelstra for remembering it, and pointing it out to me.

Convention

Throughout this article the symbols $\leftrightarrow$ resp. $\rightarrow$ and $\Longleftrightarrow$ resp. $\Longrightarrow$ stand for *provable* equivalence resp. consequence in a formal system. But only $\leftrightarrow$ and $\rightarrow$ are used as connectives in a formal language, while $\Longleftrightarrow$ and $\Longrightarrow$ denote equivalence resp. consequence relations between formulae.

2 Examples

2.1 Closure at $\omega+1$

An e.i.d. that closes up at stage $\omega+1$ (exactly). Let C be a nonrecursive RE - set, say

$$x \in C \leftrightarrow \exists z Texz\ ;\ \text{assume}\ Texz \rightarrow x \leq z.$$

Define, assuming that pairing is surjective :

$$A(P,\langle x,z\rangle) :\Longleftrightarrow \exists m \leq z\, Texm \vee P\langle x, z+1\rangle.$$

Then

$$P_0^A\langle x,z\rangle \iff \exists m \leq z\, Texm$$

$$P_1^A\langle x,z\rangle \iff \exists m \leq z\, Texm \vee P_0^A\langle x, z+1\rangle \iff \exists m \leq z+1 Texm$$

$$\vdots$$

$$P_k^A\langle x,z\rangle \iff \exists m \leq z+k\, Texm$$

$$\vdots$$

$$P_\omega^A\langle x,z\rangle \iff \exists m Texm \iff x \in C.$$

We see quickly that $P_\omega^A = P_{\omega+1}^A$ and $P_k^A \neq P_\omega^A$. The last inequality follows from the fact that C is infinite and $Texz \to x \leq z$. Now we define, following [Kre63, pp. 3.6 and 3.24]:

$$B(P,x) \quad :\Longleftrightarrow A(P,x) \vee \neg\neg Px.$$

Then, for all $n < \omega, P_n^B x \leftrightarrow P_n^A x$, and P_n^A is recursive.
PROOF:

$$P_0^B x \quad \iff P_0^A x \vee \neg\neg\bot \iff P_0^A x \text{ and clearly } P_0^A \text{ is recursive.}$$

$$P_{n+1}^B x \quad \iff A(P_n^B, x) \vee \neg\neg P_n^B x \quad \iff \text{ind hyp}$$

$$A(P_n^A, x) \vee \neg\neg P_n^A x \quad \iff \text{def, ind hyp}$$

$$P_{n+1}^A x \vee P_n^A x \quad \iff P_{n+1}^A x, \text{ and } P_{n+1}^A \text{ is recursive.}$$

□

Consider now P_ω^B, $P_{\omega+1}^B$ and $P_{\omega+2}^B$:

$$P_\omega^B x \quad \iff \exists n P_n^B x \iff \exists n P_n^A x \iff P_\omega^A x.$$

$$P_{\omega+1}^B x \quad \iff B(P_\omega^B, x) \iff A(P_\omega^A, x) \vee \neg\neg P_\omega^A x$$

$$\iff P_\omega^A x \vee \neg\neg P_\omega^A x \iff \neg\neg P_\omega^A x \not\Longleftrightarrow P_\omega^A x \text{ ,for } P_\omega^A \text{ is nonrecursive.}$$

$$P_{\omega+2}^B x \quad \iff A(P_{\omega+1}^B, x) \vee \neg\neg P_{\omega+1}^B x \iff A(\neg\neg P_\omega^A, x) \iff \neg\neg\neg\neg P_\omega^A x$$

$$\iff \neg\neg P_\omega^A x \text{ because } A(\neg\neg P_\omega^A, x) \implies \neg\neg A(P_\omega^A, x) \implies \neg\neg P_\omega^A x.$$

It is possible to construe e.i.d. $C(P,x)$ that close up at stage $\omega + \omega$, by exploiting this trick.

□ (first example)

2.2 Closure at $\omega+\omega$

We give an e.i.d. with closure ordinal $\omega+\omega$. Let $\langle\ldots\rangle$ be a coding of sequences of natural numbers. Let $A(P,x)$ be an e.i.d. that defines a nonrecursive $P^A = P^A_\omega$,while the P^A_k are recursive (cf. the first example); in addition, let $P^A \subseteq \{\langle x\rangle \mid x \in \mathbb{N}\}$, and let $A(P,x)$ be insensitive to numbers outside this set, i.e.

$$A(P,x) \;\leftrightarrow A(\lambda y.Py \wedge \exists z\,(\langle z\rangle = y), x)$$

Define

$$B(P,x) :\equiv \;(A(P,x) \wedge \mathrm{lh}\,x = 1) \vee \exists y \exists z(Py \wedge \neg\neg A(P,z) \wedge \mathrm{lh}\,z = 1 \wedge x = y \star z)$$

Then $P^B = P^B_{\omega+\omega}$, by the following lemmas, whose proofs are not particularly interesting and not too difficult. Sometimes I use set- theoretic notation like $x \in P^A_\omega$ for $P^A_\omega x$.

Lemma 2.1 $P^B_\omega = \{\langle x_1,\ldots,x_k\rangle \mid k \in \mathbb{N}, \langle x_i\rangle \in P^A_\omega, i = 1,\ldots,k\}$

Lemma 2.2

$$\begin{aligned} P^B_{\omega+n} = \{\langle x_1,\ldots,x_k\rangle \mid\; & k > 0 \wedge \langle x_1\rangle \in P^A_\omega \\ & \wedge \forall i \in \{1,\ldots,k \dot{-} n\}\ \langle x_i\rangle \in P^A_\omega \\ & \wedge \forall i \in \{k \dot{-} (n+1),\ldots,k\}\ \langle x_i\rangle \in \neg\neg P^A_\omega\} \end{aligned}$$

Lemma 2.3 $x \in P^B_{\omega+n+1} \not\Leftrightarrow x \in P^B_{\omega+n}$

Lemma 2.4

$$P^B_{\omega+\omega} = \bigcup_{n\in\omega} P^B_{\omega+n} = \{\langle x_1,\ldots,x_k\rangle \mid k{>}0 \wedge \langle x_1\rangle{\in}P^A_\omega \wedge \langle x_2\rangle,\ldots,\langle x_k\rangle{\in}\neg\neg P^A_\omega\}$$

It is clear from this construction, that the closure ordinal of B cannot be proved to be less than $\omega+\omega$.

3 J-operators

The following definition is meant as a generalization of the $\neg\neg$-operator (cf. [FS73, pp.324–334]):

Definition 3.1 *A* J-operator *is an operator $J(\cdot)$, on* **HA**-*formulae , that is* **HA**-*definable, and that satisfies:*

(i): $Q \to J(Q)$	*(increasing)*
(ii): $J(Q \wedge R) \leftrightarrow J(Q) \wedge J(R)$	*($\wedge$-distributive)*
(iii): $J(J(Q)) \to J(Q)$	*(idempotent)*
Note that from (ii)($\to$) follows:	
(iv): $(Q \to R) \to (J(Q) \to J(R))$	*(monotone).*

We do not allow J to have free variables.

Definition 3.2 : *P[P] is the class of strictly positive formulae, i.e.:*

- *Pt, t a term, is a formula of P[P]*
- *a formula φ of the language of* **HA** *is a formula of P[P]*
- *P[P] is closed under $\exists, \forall^{<}, \wedge, \vee$.*

P(J)[P], J a J-operator, is defined analogously, except that P(J)[P] is also closed under J.

Fact 3.1 *For $A(P,x) \in \mathcal{P}[P,x], P^A = P^A_\omega$ is* **HA**-*definable.* See [TvD88, Vol I,pp.145-152].

Theorem 3.2 *For $A(P,x) \in \mathcal{P}(J)[P,x], P^A = P^A_{\omega+\omega}$ is* **HA**-*definable.*

Before giving the proof, I will supply some technical lemmas and hint at the idea behind the proof.

Lemma 3.3 (Shifting J to the outside)

(i): $J(P) \vee J(Q) \rightarrow J(P \vee Q)$

(ii): $J(P) \wedge J(Q) \rightarrow J(P \wedge Q)$

(iii): $\exists x J(A(x)) \rightarrow J(\exists x A(x))$

(iv): $\forall x < t\, J(A(x)) \rightarrow J(\forall x < t A(x))$

PROOF:

(i): $\left.\begin{array}{l} P \rightarrow P \vee Q \\ Q \rightarrow P \vee Q \end{array}\right\} \overset{\text{monotonicity}}{\Longrightarrow} \left.\begin{array}{l} J(P) \rightarrow J(P \vee Q) \\ J(Q) \rightarrow J(P \vee Q) \end{array}\right\} \Longrightarrow J(P) \vee J(Q) \rightarrow J(P \vee Q)$

(ii): by $\wedge$-distributivity($\leftarrow$)

(iii): $A(x) \rightarrow \exists x A(x) \Longrightarrow J(A(x)) \rightarrow J(\exists x A(x)) \Longrightarrow \exists x J(A(x)) \rightarrow J(\exists x A(x))$

(iv): let J-SHIFT(y) denote the following schema:

$$\forall x(x < y \rightarrow J(A(x))) \rightarrow J(\forall x(x < y \rightarrow A(x))), y \notin FV(A).$$

We prove $\forall y\, J$-SHIFT(y) by induction:

$\forall x(x < 0 \rightarrow A(x))$, so by increase:$J(\forall x(x < 0 \rightarrow A(x)))$.

$\forall x(x < Sy \rightarrow J(A(x))$	$\Longrightarrow$ "**HA**"
$\forall x(x < y \rightarrow J(A(x))) \wedge J(A(y))$	$\Longrightarrow$ ind hyp
$J(\forall x(x < y \rightarrow A(x))) \wedge J(A(y))$	$\Longrightarrow$ $\wedge$-distributivity
$J(\forall x(x < y \rightarrow A(x)) \wedge J(A(y)))$	$\Longrightarrow$ "**HA** under J"
$J(\forall x(x < Sy \rightarrow A(x)))$.	

We conclude: for any term t:

$$\forall x(x < t \rightarrow J(A(x))) \rightarrow J(\forall x(x < t \rightarrow A(x))).$$

$\square$ (lemma 3.3)

The comment "**HA**" means: by reasoning in **HA**; "**HA** under J " means: by reasoning in **HA** in the scope of J ; this is justified by the fact that J is increasing and monotone.

Definition 3.3 *Let $A(P)$ be a P(J)[P]-formula. Occurrences of subformulae, used in the construction of $A(P)$,* according to the definition of *P(J)[P], are called* components.

Remark that a P(J)[P]-formula is monotone in its components, because $\exists, \forall^<, \wedge, \vee, J$ are all monotone connectives.

Lemma 3.4 *Let $A(P)$ be a P(J)[P]-formula. Let C be a component of $A(P)$ of the form $J(B(P))$, with at least one occurrence of P. Let $A'(P)$ be obtained from A by replacing that component $J(B(P))$ by $B(P)$. Then $A(P) \to J(A'(P))$. I.e.*

$$A(P) \equiv \ldots J(B(P)) \ldots$$

$$J(A'(P)) \equiv J(\ldots \; B(P) \; \ldots)$$

PROOF: Easy, by induction on the structure of $A(P)$. In fact, this is nothing else than repeatedly shifting J outwards, using the fact that a component occurs only in scopes of $\wedge, \vee, \exists, \forall^<, J$, and applying lemma 3.3.

$\square$

4 Decomposition of the approximation process

Definition 4.1 *Let $A(P,x)$ be a P(J)[P]-formula.*

$\bar{A}$:≡ *A where every J with P in its scope has been deleted;*

A^* :≡ *A where every occurrence of P in the scope of J has been replaced by $P_\omega^{\bar{A}}$; so:*

$$A(P) \equiv \ldots Ps_i \ldots J(\ldots \; Pt_j \; \ldots)$$

$$\bar{A}(P) \equiv \ldots Ps_i \ldots \quad \ldots \; Pt_j \; \ldots$$

$$A^*(P) \equiv \ldots Ps_i \ldots J(\ldots P_\omega^{\bar{A}} t_j \ldots)$$

Remark

$\bar{A} \in \mathcal{P}[P,x]$, so $P^{\bar{A}} = P_\omega^{\bar{A}}$ is **HA**-definable by the fact above; it follows that A^* is a $\mathcal{P}[P,x]$-formula, so $P^{A^*} = P_\omega^{A^*}$ is **HA**-definable too.

The idea of the proof is emerging: instead of iterating $A(P,x)$ indefinitely, we split the process in iterations that continue at most till stage ω. In the first iteration we neglect the J-operator completely, then we administer its effect one time; the second iteration also goes on without J-operator. The reason that this suffices, is mainly the idempotency of the J-operator.

Lemma 4.1 *Let $A(P,x) \in \mathcal{P}(J)[P,x]$. Then*

(i): $P_\alpha^{\bar{A}} x \to P_\alpha^{A} x$

(ii): $J(P_\alpha^{A} x) \to J(P_\alpha^{\bar{A}} x)$.

PROOF: (i) follows from $\bar{A} \to A$, (ii) from $J(A) \to J(\bar{A})$, both by induction on α. Ad (i): A is obtained from $\bar{A}$ by replacing components B by $J(B)$. Use increase ($B \to J(B)$) and monotonicity in components. Ad (ii): this is seen as follows: by repeatedly applying lemma 3.4 we have $A \to J(\bar{A})$; then, by monotonicity $J(A) \to J(J(\bar{A}))$ and by idempotency $J(A) \to J(\bar{A})$. Let us now carry out the induction for (ii):

$\alpha = 0$	:	$J(P_0^{A} x) \overset{\text{by def}}{\equiv} J(A(\lambda x.\bot, x))$	$\Longrightarrow$ for $J(A) \to J(\bar{A})$,see above
		$J(\bar{A}(\lambda x.\bot, x)) \overset{\text{by def}}{\equiv} J(P_0^{\bar{A}}, x)$.	
$\alpha = \beta + 1$	:	$J(P_{\beta+1}^{A} x) \overset{\text{by def}}{\equiv} J(A(P_\beta^{A}, x))$	$\Longrightarrow$ for $J(A) \to J(\bar{A})$,see above
		$J(\bar{A}(P_\beta^{A}, x))$	$\Longrightarrow$ $\bar{A}$ monotone, J increasing
		$J(\bar{A}(J(P_\beta^{A}), x))$	$\Longrightarrow$ ind hyp
		$J(\bar{A}(J(P_\beta^{\bar{A}}), x))$	$\Longrightarrow$ lemma 3.4
		$J(J(\bar{A}(P_\beta^{\bar{A}}, x))$	$\Longrightarrow$ idempotency
		$J(P_{\beta+1}^{\bar{A}} x)$.	
$\lim \alpha$	:	$J(P_\alpha^{A} x) \overset{\text{by def}}{\equiv} J(\exists \beta < \alpha\, P_\beta^{A} x)$	$\Longrightarrow$ J increasing
		$J(\exists \beta < \alpha\, J(P_\beta^{A} x))$	$\Longrightarrow$ ind hyp, monotonicity of J
		$J(\exists \beta < \alpha\, J(P_\beta^{\bar{A}} x))$	$\Longrightarrow$ lemma 3.4
		$J(J(\exists \beta < \alpha\, P_\beta^{\bar{A}} x))$	$\Longrightarrow$ idempotency
		$J(P_\alpha^{\bar{A}} x)$.	

□

Lemma 4.2 *Let $A(P,x) \in \mathcal{P}(J)[P,x]$. Then*

(i): $P_\infty^{A} x \leftrightarrow P_\omega^{A^*} x$

(ii): $P_\omega^{A^*} x \leftrightarrow P_{\omega+\omega}^{A} x$

PROOF:
(i)($\to$):by induction on α we prove $P_\alpha^{A} x \to P_\omega^{A^*} x$.

$\alpha = 0 \quad : \quad P_0^A x \iff A(\lambda x.\bot, x) \implies P_0^{A^*} x (\text{since } \bot \to P_\omega^{\bar{A}} x) \implies P_\omega^{A^*} x.$

$\lim \alpha \quad : \quad P_\alpha^A x \iff \exists \beta < \alpha P_\beta^A x \overset{\text{indhyp}}{\implies} \exists \beta < \alpha P_\omega^{A^*} x \implies P_\omega^{A^*} x.$

For the successor case we note first that $P_\beta^{\bar{A}} t_j \to P_\omega^{\bar{A}} t_j$; this is seen as follows: for $\beta < \omega$ it follows by the fact that $\alpha < \beta \implies (P_\alpha^{\bar{A}} x \to P_\beta^{\bar{A}} x)$ (routine induction, using monotonicity of $\bar{A}$); for $\beta > \omega$ we recollect the fact that at stage ω the iteration of $\bar{A}$ has reached its fixed-point.

$\alpha = \beta + 1 \quad : \quad P_{\beta+1}^A x \implies A(P_\beta^A, x) \equiv$

$\ldots P_\beta^A s_i \ldots J(\ldots \; P_\beta^A t_j \; \ldots) \qquad \implies$ ind hyp

$\ldots P_\omega^{A^*} s_i \ldots J(\ldots \; P_\beta^A t_j \; \ldots) \qquad \implies$ increase

$\ldots P_\omega^{A^*} s_i \ldots \; J(\ldots J(P_\beta^A t_j) \ldots) \qquad \implies$ lemma 4.1(ii)

$\ldots P_\omega^{A^*} s_i \ldots \; J(\ldots J(P_\beta^{\bar{A}} t_j) \ldots) \qquad \implies$ lemma 3.4

$\ldots P_\omega^{A^*} s_i \ldots J(J(\ldots P_\beta^{\bar{A}} t_j \ldots)) \qquad \implies$ idempotency

$\ldots P_\omega^{A^*} s_i \ldots J(\ldots \; P_\beta^{\bar{A}} t_j \; \ldots) \qquad \implies$ since $P_\beta^{\bar{A}} t_j \to P_\omega^{\bar{A}} t_j$

$\ldots P_\omega^{A^*} s_i \ldots J(\ldots \; P_\omega^{\bar{A}} t_j \; \ldots) \qquad \iff$ by definition

$A^*(P_\omega^{A^*}, x) \iff P_{\omega+1}^{A^*} x \iff P_\omega^{A^*} x \qquad$ for $A^* \in \mathcal{P}[P, x]$.

(i)($\leftarrow$): by induction on n we prove: $P_n^{A^*} x \to P_{\omega+n+1}^A x$.

$n = 0 \quad : \quad P_0^{A^*} x \iff A^*(\lambda x.\bot, x) \iff$

$\ldots (\lambda x.\bot) s_i \ldots J(\ldots P_\omega^{\bar{A}} t_j \ldots) \qquad \implies$ lemma 4.1(i)

$\ldots (\lambda x.\bot) s_i \ldots J(\ldots P_\omega^A t_j \ldots) \qquad \implies$

$\ldots \; P_\omega^A s_i \; \ldots J(\ldots P_\omega^A t_j \ldots) \qquad \iff$ by definition

$A(P_\omega^A, x) \iff P_{\omega+1}^A x.$

$n + 1 \quad : \quad P_{n+1}^{A^*} x \iff A^*(P_n^{A^*}, x) \iff$

$\ldots \; P_n^{A^*} s_i \; \ldots J(\ldots \; P_\omega^{\bar{A}} t_j \; \ldots) \qquad \implies$ ind hyp

$\ldots P_{\omega+n+1}^A s_i \ldots J(\ldots \; P_\omega^{\bar{A}} t_j \; \ldots) \qquad \implies$ lemma 4.1(i)

$\ldots P_{\omega+n+1}^A s_i \ldots J(\ldots \; P_\omega^A t_j \; \ldots) \qquad \implies$ monotonicity

$\ldots P_{\omega+n+1}^A s_i \ldots J(\ldots P_{\omega+n+1}^A t_j \ldots) \qquad \iff$ by definition

$$A(P^A_{\omega+n+1}, x) \iff P^A_{\omega+n+2}x.$$

Then $P^{A^*}_\omega x \iff \exists n P^{A^*}_n x \implies \exists n P^A_{\omega+n+1}x \iff P^A_{\omega+\omega}x \implies P^A_\infty x.$

(ii): see the preceeding line.

□ (lemma 4.2)

Now theorem 3.2 follows:

- closure at $\omega + \omega$:

$$A(P^A_{\omega+\omega}, x) \iff P^A_{\omega+\omega+1}x \implies P^A_\infty x \implies \text{lemma 4.2(i)}$$

$$P^{A^*}_\omega x \overset{\text{lemma 4.2(ii)}}{\implies} P^A_{\omega+\omega}x.$$

- definability:

$$P^A_\infty x \iff P^A_{\omega+\omega}x \iff P^{A^*}_\omega x \text{ and } P^{A^*}_\omega \text{ is } \mathbf{HA}\text{-definable.}$$

□ (theorem 3.2)

5 Extensions

One of the limitations of our theorem is, that there figures at most one J-operator in an e.i.d. . When we try to admit more, and proceed by repeatedly treating the J-operators in the same way as we did our single J-operators, we encounter the following difficulty: one J-operator need to be shifted outward over another, while it is not generally true that $J_1(J_2(Q)) \to J_2(J_1(Q))$. Define

$$J_2 \leq J_1 \quad :\iff J_1(J_2(Q)) \to J_2(J_1(Q)) \quad \text{read } J_2 \text{ preceeds } J_1.$$

Theorem 5.1 *For $A(P, x)$ containing two J-operators J_1 and J_2, where $J_1 \leq J_2$ or $J_2 \leq J_1$, the following holds:*

$$P^A = P^A_{\omega+\omega+\omega+\omega} \text{ is } \mathbf{HA}\text{-definable.}$$

PROOF:

Define $\bar{A}$:≡ A where every J_2 with P in its scope has been deleted;

A^* :≡ A where every occurrence of P in the scope of J_2 has been replaced by $P^{\bar{A}}_{\omega+\omega}$.

Then proceed in the same way as before.

□

I conclude with some examples of J-operators and a few easy relationships between them. The following are all J-operators:

$$
\begin{aligned}
I &= \lambda Q.Q \\
N &= \lambda Q.\neg\neg Q \\
D_R &= \lambda Q.Q \vee R \\
H_R &= \lambda Q.R \rightarrow Q \\
N_R &= \lambda Q.(Q \rightarrow R) \rightarrow R \\
N_R^{J_1} &= \lambda Q.N_R(J_1(Q)) \\
M_R^{J_1 J_2} &= \lambda Q.(J_1(Q) \rightarrow R) \rightarrow J_2(Q) \quad \text{where } J_2(Q) \rightarrow J_1(Q) \text{ for all } Q.
\end{aligned}
$$

It is not hard to establish that

$$N \leq J,\ I \leq J,\ H_{R_1} \leq H_{R_2},\ D_{R_1} \leq D_{R_2}.$$

Fact 5.2

$$J_1 \leq J_2 \iff J_1 \circ J_2 \textit{ is a J-operator.}$$

PROOF:
(only if) straightforward; the condition $J_1 \leq J_2$ is used to get idempotency for $J_1 \circ J_2$.

(if) $J_2 J_1 Q \implies$ increase, monotonicity

$J_2 J_1(J_2 Q) \implies$ increase

$J_1(J_2 J_1(J_2 Q)) \equiv (J_1 \circ J_2)(J_1 \circ J_2)Q \implies (J_1 \circ J_2)Q$ by the idempotency of $(J_1 \circ J_2)$.

□

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