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INVESTIGATIONS INTO CLASSICAL LINEAR LOGIC

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Abstract

We investigate for which elementary fragments $\subseteq \{\rightharpoonup, \rightharpoondown, {}^{\perp}, \otimes, \sqcup, \sqcap, \oplus, \mathbf{1}, \perp, \top, \mathbf{0}, !, ?\}$ of classical linear logic interpolation holds. The first thing we need is a cut elimination theorem. Girard gives an unconventional, complicated proof in [Gir87], but a more standard proof is also possible. A method due to Dragalin [Dra87] is adapted to **CL**; this method yields also strong normalization. The first part proves cut elimination and strong normalization; the second part contains proofs for interpolation for some fragments of classical linear logic; the third part is concerned with another question, of which the answer uses a corollary of cut elimination: decidability.

1 Prologue

Linear logic is a new subject, that attracts unconventional minds. Exploring an untouched area is exciting enough, but sooner or later a lot of more conventional work has to be done. I am concerned here with some down to earth questions about classical linear logic. In this report they are answered by (long) existing methods. I found it a pleasure to see that a decidability argument for relevance logics by Dunn ([Dun86]) was general enough to be of use in the case of linear logic; I thank prof. Johan van Benthem for drawing my attention towards it; it was prof. A.S. Troelstra who gave me the proof of cut elimination with strong normalization in classical logic by Dragalin ([Dra87]). There are also some results about interpolation in this paper; for some kinds of fragments of linear logic interpolation holds, but the question whether interpolation holds for fragments where negation is not definable and where implication is present, is left open. These fragments are interesting, because the Lambek Calculus falls into these.

2 Cut Elimination and Strong Normalization for CL

2.1 Introduction

In [Dra87] Dragalin gives a nice method to prove cut elimination together with strong normalization for classical predicate logic. I shall use this method, with a few adaptations, for **CL**.

Let me first describe the system **CL**. We use a sequent calculus, of course. Language: formulae are built up from proposition letters $p_1, p_2, \ldots$, and constant symbols $\mathbf{1}, \perp, \top, \mathbf{0}$, by means of the unary connective ${}^{\perp}$, and binary connectives $\rightharpoonup, \rightharpoondown, \otimes, \sqcup, \sqcap, \oplus$, and modalities ! and ? . The constant symbol $\perp$ for falsum is different from the unary connective ${}^{\perp}$ for negation. Sequents are denoted by $\Gamma \vdash \Delta$, where Γ, Δ are multisets of formulae. The sequent $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ will be interpreted as $A_1 \otimes \cdots \otimes A_n \Longrightarrow B_1 \sqcup \cdots \sqcup B_m$ (the connectives $\otimes$ and $\sqcup$ are interpreted by associative and commutative operations). The axioms and rules are as follows:

p $$A \vdash A$$

1 l $$\frac{\Gamma \vdash \Delta}{\Gamma, \mathbf{1} \vdash \Delta}$$ **1 r** $$\vdash \mathbf{1}$$

⊥ l $$\bot \vdash$$ **⊥ r** $$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \bot, \Delta}$$

0(l) $$\Gamma, \mathbf{0} \vdash \Delta$$ **⊤(r)** $$\Gamma \vdash \top, \Delta$$

⊗ l $$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}$$ **⊗ r** $$\frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \otimes B, \Delta_1, \Delta_2}$$

⊔ l $$\frac{\Gamma_1, A \vdash \Delta_1 \quad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \sqcup B \vdash \Delta_1, \Delta_2}$$ **⊔ r** $$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \sqcup B, \Delta}$$

⊓$\mathbf{l}_{1,2}$ $$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \sqcap B \vdash \Delta} \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \sqcap B \vdash \Delta}$$ **⊓ r** $$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \sqcap B, \Delta}$$

⊕ l $$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta}$$ **⊕$\mathbf{r}_{1,2}$** $$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta} \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta}$$

$^\perp$ l $$\frac{\Gamma \vdash A, \Delta}{\Gamma, A^\perp \vdash \Delta}$$ **$^\perp$ r** $$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^\perp, \Delta}$$

⇀ l $$\frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \rightharpoonup B \vdash \Delta_1, \Delta_2}$$ **⇀ r** $$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightharpoonup B, \Delta}$$

⇁ l $$\frac{\Gamma, B \vdash A, \Delta}{\Gamma, A \rightharpoondown B \vdash \Delta}$$ **⇁ r** $$\frac{\Gamma_1, A \vdash \Delta_1 \quad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \rightharpoondown B, \Delta_1, \Delta_2}$$

!$\mathbf{l}_{a,b}$ $$\frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta}$$ **!r** $$\frac{!\Gamma \vdash C}{!\Gamma \vdash !C}$$ **!c** $$\frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta}$$

?$\mathbf{r}_{a,b}$ $$\frac{\Gamma \vdash \Delta}{\Gamma \vdash ?A, \Delta} \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash ?A, \Delta}$$ **?l** $$\frac{C \vdash ?\Delta}{?C \vdash ?\Delta}$$ **?c** $$\frac{\Gamma \vdash ?A, ?A, \Delta}{\Gamma \vdash ?A, \Delta}$$

cut $$\frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2, A \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}$$

The proof proceeds as follows:

1. Describe reductions on proofs, where a reduction removes a cut at the cost of cuts of a lesser degree; or it permutes a cut with a rule application in one of its premises.

2. Show that if a proof is not reducible, then it is cut free (also called *normal*).

3. Show that every reduction sequence terminates.

In order to perform 1. , we carry out an extensive diagnosis of the cut rule application; for every different case we supply a primitive reduction. After that we can easily see that every non-normal proof has a terminating reduction sequence, so, together with 2. , this establishes the eliminability of the cut rule. In order to prove 3. , we define an auxiliary notion of inductive proof and a measure for the complexity of cut applications. Then we show that every proof is inductive, by induction on the rank of the cut formula and the complexity of the cut application, nested in that order. It follows easily from the definition of inductive proof, that an inductive proof is strongly normalizable. However, there is certain restriction on the kind of admissible reductions, which has to do with contractions. Let me illustrate this point. Consider an application of cut of the following form:

$$\dfrac{\dfrac{\overset{\vdots}{\Gamma_1 \vdash ?A, ?A, \Delta_1}}{\Gamma_1 \vdash ?A, \Delta_1} \qquad \overset{\vdots}{?A \vdash ?\Delta_2}}{\Gamma_1 \vdash \Delta_1, ?\Delta_2}\ \text{cut}$$

This should reduce to

$$\dfrac{\dfrac{\dfrac{\overset{\vdots}{\Gamma_1 \vdash ?A, ?A, \Delta_1} \quad \overset{\vdots}{?A \vdash ?\Delta_2}}{\Gamma_1 \vdash ?A, \Delta_1, ?\Delta_2}\ \text{cut} \qquad \overset{\vdots}{?A \vdash ?\Delta_2}}{\Gamma_1 \vdash \Delta_1, ?\Delta_2, ?\Delta_2}\ \text{cut}}{\vdots ?_c \vdots \atop \Gamma_1 \vdash \Delta_1, ?\Delta_2}$$

But then we can reduce ad infinitum by permuting both cuts that are shown, in wich process the figure does not change. For a way out of this complication, I shall follow [Dra87]: we add stronger (but *derivable*) cut rules:

cut $$\dfrac{\Gamma_1 \vdash (?A)^n, \Delta_1 \qquad \Gamma_2, ?A \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}$$

and

cut $$\dfrac{\Gamma_1 \vdash !A, \Delta_1 \qquad \Gamma_2, (!A)^n \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}$$

Then we can apply cut to the premiss of the contraction.
There is another lack of freedom: when a premiss of an application of cut ends itself in a cut. Then we do not allow a permutation of these two cuts, since that could go on for ever. This may not seem a substantial lack of freedom, but it complicates case 3e. in the list below.

2.2 Preliminaries

In an application of a rule, the formulae that match the $\Gamma, \Gamma_1, \Gamma_2, \Delta, \Delta_1, \Delta_2$ are called *side formulae*, the others that occur in the conclusion *major formulae*, and the others that occur in the premises *minor formulae.*

Lemma 2.1 *Every proof of a sequent $\Gamma, \mathbf{1} \vdash \Delta$ (resp. $\Gamma \vdash \bot, \Delta$) can be transformed into a proof of $\Gamma \vdash \Delta$ with exactly the same structure, but with the difference that all occurrences of $\mathbf{1}$ (resp. $\bot$) that are connected to the occurrence in the conclusion, are removed.*

PROOF: The occurrences of $\mathbf{1}$ ($\bot$) in question are introduced by $\mathbf{1l}$($\bot\mathbf{r}$) , skip those introductions. The rest of the occurrences are merely side formulae, it is not harmful to remove them.

□

2.3 Case analysis of cut applications

Consider the two premises

$$(l)\ \Gamma_1 \overset{\vdots}{\vdash} A, \Delta_1 \qquad\qquad (r)\ \Gamma_2, A \overset{\vdots}{\vdash} \Delta_2$$

of a cut application. The following list of cases and subcases is complete:

1. (l) or (r) is an axiom of the form $A \vdash A$
2. (l) is $\vdash \mathbf{1}$ or (r) is $\bot \vdash$
3. in at least one of (l) and (r) all occurrences of A involved in the cut are not major formulae; the last rule applied there, is:
   a. $\mathbf{0}$ or $\top$
   b. a rule with one premiss, not **!r**, not **?l**
   c. a parallel rule with two premises: $\otimes$**r**, $\sqcup$**l**, $\rightharpoondown$**l**, $\rightharpoondown$**r**; (**cut** is excluded)
   d. a sequential rule with two premises: $\sqcap$**r**, $\oplus$**l**
   e. **!r** or **?l**
4. in both (l) and (r) A is major formula; in at least one of (l) and (r) the last rule applied is
   a. $\mathbf{?r_a}$ or $\mathbf{!l_a}$
   b. $\mathbf{?r_b}$ or $\mathbf{!l_b}$
   c. $\mathbf{?_c}$ or $\mathbf{!_c}$
5. in both (l) and (r) A is major formula; and in both A is introduced according its principal connective $\otimes$, $\sqcup$, $\sqcap$, $\oplus$, $\rightharpoondown$, $\rightharpoondown$ or $^\perp$. So here there is only one occurrence of A at either side involved in the cut.

2.4 Primitive reductions

According the distinctions above we shall give reductions of proofs that end with a cut application. If there are symmetrical cases, we treat only one representative.

1.

$$\frac{A \vdash A \quad \overset{\vdots}{\Gamma_2, A \vdash \Delta_2}}{\Gamma_2, A \vdash \Delta_2}\,\text{cut} \quad \rightsquigarrow \quad \overset{\vdots}{\Gamma_2, A \vdash \Delta_2}$$

2.

$$\frac{\vdash \mathbf{1} \quad \overset{\vdots}{\Gamma_2, \mathbf{1} \vdash \Delta_2}}{\Gamma \vdash \Delta}\,\text{cut} \quad \rightsquigarrow \quad \overset{\vdots}{\Gamma \vdash \Delta} \text{ by lemma 2.1}$$

3a.

$$\frac{\Gamma_1, \mathbf{0} \vdash A, \Delta_1 \quad \overset{\vdots}{\Gamma_2, A \vdash \Delta_2}}{\Gamma_1, \Gamma_2, \mathbf{0} \vdash \Delta_1, \Delta_2}\,\text{cut} \quad \rightsquigarrow \quad \Gamma_1, \Gamma_2, \mathbf{0} \vdash \Delta_1, \Delta_2$$

3b.

$$\frac{\dfrac{\overset{\vdots}{\Gamma \vdash A, \Delta}}{\Gamma_1 \vdash A, \Delta_1}\,\text{rx} \quad \overset{\vdots}{\Gamma_2, A \vdash \Delta_2}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\,\text{cut} \quad \rightsquigarrow \quad \frac{\dfrac{\overset{\vdots}{\Gamma \vdash A, \Delta} \quad \overset{\vdots}{\Gamma_2, A \vdash \Delta_2}}{\Gamma, \Gamma_2 \vdash \Delta_2, \Delta}\,\text{cut}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\,\text{rx}$$

3c.

$$\frac{\dfrac{\overset{\vdots}{\Gamma_1^1 \vdash \varphi, A, \Delta_1^1} \quad \overset{\vdots}{\Gamma_1^2 \vdash \psi, \Delta_1^2}}{\Gamma_1^1, \Gamma_1^2, \vdash \varphi \circ \psi, A, \Delta_1^1, \Delta_1^2}\,\text{rx} \quad \overset{\vdots}{\Gamma_2, A \vdash \Delta_2}}{\Gamma_1^1, \Gamma_1^2, \Gamma_2 \vdash \varphi \circ \psi, \Delta_1^1, \Delta_1^2, \Delta_2}\,\text{cut} \quad \rightsquigarrow$$

$$\frac{\dfrac{\overset{\vdots}{\Gamma_1^1 \vdash \varphi, A, \Delta_1^1} \quad \overset{\vdots}{\Gamma_2, A \vdash \Delta_2}}{\Gamma_1^1, \Gamma_2, \vdash \varphi, \Delta_1^1, \Delta_2}\,\text{cut} \quad \overset{\vdots}{\Gamma_1^2 \vdash \psi, \Delta_1^2}}{\Gamma_1^1, \Gamma_1^2, \Gamma_2 \vdash \varphi \circ \psi, \Delta_1^1, \Delta_1^2, \Delta_2}\,\text{rx}$$

For this reduction it is immaterial wether the φ, ψ, and $\varphi \circ \psi$ occur left or right.

3d.

$$\frac{\dfrac{\overset{\vdots}{\Gamma_1 \vdash \varphi, A, \Delta_1} \quad \overset{\vdots}{\Gamma_1 \vdash \psi, A, \Delta_1}}{\Gamma_1 \vdash \varphi \circ \psi, A, \Delta_1}\,\text{rx} \quad \overset{\vdots}{\Gamma_2, A \vdash \Delta_2}}{\Gamma_1, \Gamma_2 \vdash \varphi \circ \psi, \Delta_1, \Delta_2}\,\text{cut} \quad \rightsquigarrow$$

$$\frac{\dfrac{\overset{\vdots}{\Gamma_1 \vdash \varphi, A, \Delta_1} \quad \overset{\vdots}{\Gamma_2, A \vdash \Delta_2}}{\Gamma_1, \Gamma_2 \vdash \varphi, \Delta_1, \Delta_2}\,\text{cut} \quad \dfrac{\overset{\vdots}{\Gamma_1 \vdash \psi, A, \Delta_1} \quad \overset{\vdots}{\Gamma_2, A \vdash \Delta_2}}{\Gamma_1, \Gamma_2 \vdash \psi, \Delta_1, \Delta_2}\,\text{cut}}{\Gamma_1, \Gamma_2 \vdash \varphi \circ \psi, \Delta_1, \Delta_2}\,\text{rx}$$

For this reduction it is immaterial wether the φ, ψ, and $\varphi \circ \psi$ occur left or right.

3e. In $\Gamma_1 \overset{\vdots}{\vdash} A, \Delta_1$ the last rule applied cannot be **!r** if A is not major. So we have the following situation:

$$\dfrac{\dfrac{\overset{\vdots}{C \vdash (?A)^n, ?\Delta_1}}{?C \vdash (?A)^n, ?\Delta_1} \qquad \overset{\vdots}{\Gamma_2, ?A \vdash \Delta_2}}{?C, \Gamma_2 \vdash ?\Delta_1, \Delta_2}\ \text{cut}$$

If we try to permute the cut with the **?l** then we encounter the problem that after the cut we do not have a good premiss for **?l**. So, in this case we are forced to permute on the other premiss. But that could be a problem in three cases:

1. the situation in the second premiss is the mirror image of the first premiss. But then **!r** must be the last rule applied there, and $?A$ should begin with a !, which is not so.
2. the second premiss ends with a cut. Then we do not provide any reduction for this cut, but there is at least one other cut to apply a reduction to.
3. in the second premiss $?A$ was just introduced. But then we have a situation in which it is possible to permute on the left premiss !:

$$\dfrac{\dfrac{\overset{\vdots}{C \vdash (?A)^n, ?\Delta_1}}{?C \vdash (?A)^n, ?\Delta_1} \qquad \dfrac{\overset{\vdots}{A \vdash ?\Delta_2}}{?A \vdash ?\Delta_2}}{?C \vdash ?\Delta_1, ?\Delta_2}\ \text{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\vdots}{C \vdash (?A)^n, ?\Delta_1} \qquad \overset{\vdots}{?A \vdash ?\Delta_2}}{C \vdash ?\Delta_1, ?\Delta_2}\ \text{cut}}{?C \vdash ?\Delta_1, ?\Delta_2}$$

4a. We have $?\mathbf{r}_a$ in (l) or $!\mathbf{l}_a$ in (r). Note that in (r) $?A$ was just introduced.
Case 1: $?A$ occurs in Δ_1 and such an occurrence is involved in the cut:

$$\dfrac{\dfrac{\overset{\vdots}{\Gamma_1 \vdash (?A)^{n+1}, \Delta_1}}{\Gamma_1 \vdash ?A, (?A)^{n+1}, \Delta_1} \qquad \overset{\vdots}{?A \vdash ?\Delta_2}}{\Gamma_1 \vdash \Delta_1, ?\Delta_2}\ \text{cut} \quad \rightsquigarrow \quad \dfrac{\overset{\vdots}{\Gamma_1 \vdash (?A)^{n+1}, \Delta_1} \qquad \overset{\vdots}{?A \vdash ?\Delta_2}}{\Gamma_1 \vdash \Delta_1, ?\Delta_2}\ \text{cut}$$

Case 2: otherwise we have the following situation:

$$\dfrac{\dfrac{\overset{\vdots}{\Gamma_1 \vdash \Delta_1}}{\Gamma_1 \vdash ?A, \Delta_1} \qquad \overset{\vdots}{?A \vdash ?\Delta_2}}{\Gamma_1 \vdash \Delta_1, ?\Delta_2}\ \text{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\vdots}{\Gamma_1 \vdash \Delta_1}}{\vdots\ ?\mathbf{r}_a\ \vdots}}{\Gamma_1 \vdash \Delta_1, ?\Delta_2}$$

4b. We have $?\mathbf{r}_b$ in (l) or $!\mathbf{l}_b$ in (r). Note that in (r) $?A$ was just introduced.
Case 1: $?A$ occurs in Δ_1 and such an occurrence is involved in the cut:

$$\dfrac{\dfrac{\overset{\vdots}{\Gamma_1 \vdash A, (?A)^{n+1}, \Delta_1}}{\Gamma_1 \vdash ?A, (?A)^{n+1}, \Delta_1} \qquad \dfrac{\overset{\vdots}{A \vdash ?\Delta_2}}{?A \vdash ?\Delta_2}}{\Gamma_1 \vdash \Delta_1, ?\Delta_2}\ \text{cut} \quad \rightsquigarrow$$

$$\cfrac{\cfrac{\overset{\vdots}{\Gamma_1 \vdash A, (?A)^{n+1}, \Delta_1} \quad \overset{\vdots}{?A \vdash ?\Delta_2}}{\Gamma_1 \vdash A, \Delta_1, ?\Delta_2}\,\text{cut} \qquad \overset{\vdots}{A \vdash ?\Delta_2}}{\cfrac{\Gamma_1 \vdash \Delta_1, ?\Delta_2, ?\Delta_2}{\cfrac{\vdots ?_c \vdots}{\Gamma_1 \vdash \Delta_1, ?\Delta_2}}}\,\text{cut}$$

Case 2: Otherwise we have the following situation:

$$\cfrac{\cfrac{\overset{\vdots}{\Gamma_1 \vdash A, \Delta_1}}{\Gamma_1 \vdash ?A, \Delta_1} \qquad \cfrac{\overset{\vdots}{A \vdash ?\Delta_2}}{?A \vdash ?\Delta_2}}{\Gamma_1 \vdash \Delta_1, ?\Delta_2}\,\text{cut} \quad \rightsquigarrow \quad \cfrac{\overset{\vdots}{\Gamma_1 \vdash A, \Delta_1} \quad \overset{\vdots}{A \vdash ?\Delta_2}}{\Gamma_1 \vdash \Delta_1, ?\Delta_2}\,\text{cut}$$

4c. We have $?_c$ in (l) or $!_c$ in (r). We have the following situation (note that in (r) $?A$ was just introduced):

$$\cfrac{\cfrac{\overset{\vdots}{\Gamma_1 \vdash ?A, ?A, (?A)^n, \Delta_1}}{\Gamma_1 \vdash ?A, (?A)^n, \Delta_1} \qquad \overset{\vdots}{?A \vdash ?\Delta_2}}{\Gamma_1 \vdash \Delta_1, ?\Delta_2}\,\text{cut} \quad \rightsquigarrow \quad \cfrac{\overset{\vdots}{\Gamma_1 \vdash ?A, ?A, (?A)^n, \Delta_1} \quad \overset{\vdots}{?A \vdash ?\Delta_2}}{\Gamma_1 \vdash \Delta_1, ?\Delta_2}\,\text{cut}$$

5. We combine the proofs Π_1 of (l) and Π_2 of (r) into $c(\Pi_1, \Pi_2)$:

	Π_1	Π_2
$\perp$	trivial	

$\otimes$

$$\cfrac{\overset{\vdots}{\Gamma_1^0 \vdash A, \Delta_1^0} \qquad \overset{\vdots}{\Gamma_1^1 \vdash B, \Delta_1^1}}{\Gamma_1^0, \Gamma_1^1, \vdash A \otimes B, \Delta_1^0, \Delta_1^1} \qquad \cfrac{\overset{\vdots}{\Gamma_2, A, B \vdash \Delta_2}}{\Gamma_2, A \otimes B \vdash \Delta_2}$$

$$c(\Pi_1, \Pi_2) = \cfrac{\overset{\vdots}{\Gamma_1^1 \vdash B, \Delta_1^1} \qquad \cfrac{\overset{\vdots}{\Gamma_1^0 \vdash A, \Delta_1^0} \qquad \overset{\vdots}{\Gamma_2, A, B \vdash \Delta_2}}{\Gamma_1^0, \Gamma_2, B \vdash \Delta_1^0, \Delta_2}}{\Gamma_1^0, \Gamma_1^1, \Gamma_2 \vdash \Delta_1^0, \Delta_1^1, \Delta_2}$$

$\sqcup$ analogously

$\rightharpoonup$, $\rightharpoondown$ easy

$\sqcap$

$$\cfrac{\overset{\vdots}{\Gamma_1 \vdash A, \Delta_1} \qquad \overset{\vdots}{\Gamma_1 \vdash B, \Delta_1}}{\Gamma_1 \vdash A \sqcap B, \Delta_1} \qquad \cfrac{\overset{\vdots}{\Gamma_2, A \vdash \Delta_2}}{\Gamma_2, A \sqcap B \vdash \Delta_2}$$

$$c(\Pi_1, \Pi_2) = \cfrac{\overset{\vdots}{\Gamma_1 \vdash A, \Delta_1} \qquad \overset{\vdots}{\Gamma_2, A \vdash \Delta_2}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}$$

the other case likewise

$\oplus$ analogously

2.5 Strong normalization

A (one step) reduction of a proof Π is a proof Σ, obtained by applying an appropriate primitive reduction to an instance of the cut rule in Π. Notation $\Pi > \Sigma$ or $\Sigma < \Pi$.

Lemma 2.2 *If no reduction applies to a derivation* Π *then* Π *is cut free.*

PROOF: As long there is a cut in Π, then it falls in one of the cases listed above; in all those cases a reduction is described, either on the designated cut, or on a related cut (cf. case 3e.).

□

Let us define a few notions, in order to get a measure of complexity for cut applications. Define, for Π a derivation terminating in a cut with premises Π_1 and Π_2:

$$a(\Pi_i) = \begin{cases} 0 \text{ if the cut formula is just introduced, but not by } ?_{ab}, !_{ab}, \mathbf{p}, \mathbf{1r}, \bot\mathbf{l}, \top, \mathbf{0} \\ 1 \text{ otherwise} \end{cases}$$

$$a(\Pi) = a(\Pi_1) + a(\Pi_2)$$

$$r(\Pi) = \text{the number of symbols in the cut formula}$$

Definition 2.1 *We define the notion* inductive proof *by induction:*

(1) $A \vdash A$; $\vdash \mathbf{1}$; $\bot \vdash$; $\Gamma, \mathbf{0} \vdash \Delta$; $\Gamma \vdash \top, \Delta$ *are inductive proofs;*

(2) $\dfrac{\cdots \Gamma_i \vdash \Delta_i \cdots}{\Gamma \vdash \Delta}$ x $\neq$ cut *is inductive if all premises are inductive;*

(3) $\Pi = \dfrac{\Pi_1 \quad \Pi_2}{\Gamma \vdash \Delta}$ cut *is inductive if every* $\Sigma < \Pi$ *is inductive.*

Note that for any Π there are only finitely many $\Sigma < \Pi$. For inductive derivations Π we define the size ind($\cdot$) by (the cases match the cases in definition 2.1):

(1) $\text{ind}(\Pi) = 1$;

(2) $\text{ind}(\Pi) = \sum_i \text{ind}(\Pi_i) + 1$;

(3) $\text{ind}(\Pi) = \sum_{\Sigma<\Pi} \text{ind}(\Sigma) + 1$.

Lemma 2.3 *If* Π *is inductive, and* $\Sigma < \Pi$, *then* Σ *is inductive.*

PROOF: Induction on the structure of Π: if Π is inductive by clause (1) or (2) then Σ is of that form, and the result follows easily from induction hypothesis. If Π is inductive by clause (3) then it follows by the definition of inductive.

□

Lemma 2.4 *Every inductive proof is strongly normalizing*

PROOF: Induction on $\text{ind}(\Pi)$. If $\text{ind}(\Pi) = 1$ then no reductions are possible. If Π is inductive by clause (2) then every reduction is inside one premiss. Apply induction hypothesis. If Π is inductive by clause (3) then the result is built into the definition.

□

The following lemma is the crucial step towards strong normalization.

Lemma 2.5 *If in a proof that ends with a cut, the premises are inductive, then* Π *is inductive.*

PROOF: Define a complexity of cut applications as follows: given the application with premises $\Gamma_1 \vdash A, \Delta_1$ and $\Gamma_2, A \vdash \Delta_2$ then

$$h(\Pi) := \omega \cdot a(\Pi) + \text{ind}(\Pi_1) + \text{ind}(\Pi_2).$$

We induct first on $r(\Pi)$, and inside on $h(\Pi)$.
Case 1. Σ arises by reducing a cut in Π_1 or in Π_2. Then we see that $a(\Sigma) \leq a(\Pi)$ and by definition of $\text{ind}(\cdot)$:$\text{ind}(\Sigma) < \text{ind}(\Pi)$. So $h(\Sigma) < h(\Pi)$ and Σ has inductive premises by lemma 2.3. Then by h-induction hypothesis, Σ is inductive.
Case 2. Σ arises by reducing the last cut of Π. Then we inspect all possible primitive reductions.
Reduction 1. Σ is one of the premises, and thus inductive by assumption.
Reduction 2. Σ is nearly one of the premises; it is easy to verify that the removal of **1** (resp. $\perp$) does not affect inductiveness.(See the proof of lemma 2.1).
Reduction 3a. Σ is inductive by clause (1).
Reduction 3b. The situation is (schematically)

$$\frac{\dfrac{\Pi_1'}{\Pi_1} \quad \Pi_2}{\Pi}\,\text{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Pi_1' \quad \Pi_2}{\Pi'}\,\text{cut}}{\Sigma}$$

The original cut has complexity

$$h(\Pi) = \omega \cdot (1 + a(\Pi_2)) + \text{ind}(\Pi_1) + \text{ind}(\Pi_2)$$

and the new one

$$h(\Pi') = \omega \cdot (a(\Pi') + a(\Pi_2)) + \text{ind}(\Pi_1') + \text{ind}(\Pi_2)$$

Now $a(\Pi') \leq 1$ and $\text{ind}(\Pi_1') < \text{ind}(\Pi_1)$ so $h(\Pi') < h(\Pi)$. So by h-induction hypothesis Π' is inductive, and then by definition Σ is inductive.
Reduction 3c, 3d and 3e. In the same way as in case 3b: One verifies easily that the new cuts have lower h-values then the original one, and concludes that the resulting proof is inductive. For clarity I show case 3d.:

$$\frac{\dfrac{\Pi_1' \quad \Pi_1''}{\Pi_1} \qquad \Pi_2}{\Pi}\,\text{cut} \quad \rightsquigarrow \quad \frac{\dfrac{\Pi_1' \quad \Pi_2}{\Pi_1^a}\,\text{cut} \qquad \dfrac{\Pi_1'' \quad \Pi_2}{\Pi_1^b}\,\text{cut}}{\Sigma}$$

$$h(\Pi) = \omega \cdot (1 + a(\Pi_2)) + \text{ind}(\Pi_1) + \text{ind}(\Pi_2)$$
$$h(\Pi_1^a) = \omega \cdot (a(\Pi_1') + a(\Pi_2)) + \text{ind}(\Pi_1') + \text{ind}(\Pi_2)$$
$$h(\Pi_1^b) = \omega \cdot (a(\Pi_1'') + a(\Pi_2)) + \text{ind}(\Pi_1'') + \text{ind}(\Pi_2)$$

Now $\text{ind}(\Pi_1'), \text{ind}(\Pi_1'') < \text{ind}(\Pi_1)$ and $a(\Pi_1'), a(\Pi_1'') < 1$ so $h(\Pi_1^a), h(\Pi_1^b) < h(\Pi)$. By h-induction hypothesis Π_1^a and Π_1^b are inductive, and by definition Σ is inductive.
Reduction 4a. Case 1 is easy, like 3b. Case 2 is easy, because the cut disappears.
Reduction 4b. Case 1. We have the following situation:

$$\dfrac{\dfrac{\Pi_1'}{\Pi_1} \quad \dfrac{\Pi_2'}{\Pi_2}}{\Pi} \text{ cut} \quad \rightsquigarrow \quad \begin{array}{c} \dfrac{\dfrac{\Pi_1' \quad \Pi_2}{\Pi'} \text{ cut} \quad \Pi_2'}{\underline{\Pi''}} \text{ cut} \\ \vdots \\ \overline{\Sigma} \end{array}$$

It is easy to see that the upper cut has lower h-complexity; so Π' is inductive. The lower cut has lower rank (its cut formula is A, while the cut formula of the original cut is $?A$), so by r-induction hypothesis Π'' is inductive. So Σ is inductive.
Case 2: easy.
Reduction 4c. Easy.
Reduction 5. Every new cut has lower rank, apply r-induction hypothesis.

□ lemma 2.5

Corollary 2.6 *Every derivation is inductive.*

Theorem 2.7

(1) The system **CL** *is equivalent to the system* **CL** *without* **cut***;*

(2) Every sequence of reductions, applied to a **CL***-proof, terminates.*

PROOF: From corollary 2.6, lemma 2.4 (2) follows immediately. From this and lemma 2.2 follows that every proof can be transformed to a cut free proof of the same sequent.

□

This theorem is not an adamant result on strong normalization, because of the lack of freedom in applying the primitive reductions and the way contraction is treated.

3 Interpolation in CL

3.1 One-sided calculus, elementary fragments

We are interested in interpolation properties of fragments of **CL**. A *fragment* of **CL** is a subset of the formulae of **CL**. Let $\mathcal{F}$ be a fragment of **CL**. We say that *interpolation holds for* $\mathcal{F}$ if for all provable sequents $\Gamma \vdash \Delta$ in $\mathcal{F}$ there is a formula $M \in \mathcal{F}$ such that $\Gamma \vdash M$ and $M \vdash \Delta$ are provable and the *material* of M is a subset of the material of Γ and Δ, i.e. every proposition letter that occurs in M occurs both in Γ and Δ.
In order to study interpolation in fragments containing negation and implication , it is convenient to work with an one-sided calculus $\mathbf{CL}^1$. The language of $\mathbf{CL}^1$ consists of the symbols $\mathbf{0}, \mathbf{1}, \perp, \top, p_1, p_1^{\perp}, p_2, p_2^{\perp}, \ldots, \rightharpoonup, \multimap, \sqcap, \oplus, \otimes, \sqcup, !, ?$. Negation is defined:

$$
\begin{array}{llll}
(p_i)^\perp & :\equiv p_i^\perp & (p_i^\perp)^\perp & :\equiv p_i \\
(\mathbf{1})^\perp & :\equiv \perp & (\perp)^\perp & :\equiv \mathbf{1} \\
(\mathbf{0})^\perp & :\equiv \top & (\top)^\perp & :\equiv \mathbf{0} \\
(A \sqcap B)^\perp & :\equiv A^\perp \oplus B^\perp & (A \oplus B)^\perp & :\equiv A^\perp \sqcap B^\perp \\
(A \otimes B)^\perp & :\equiv A^\perp \sqcup B^\perp & (A \sqcup B)^\perp & :\equiv A^\perp \otimes B^\perp \\
(A \rightharpoonup B)^\perp & :\equiv A^\perp \rightharpoondown B^\perp & (A \rightharpoondown B)^\perp & :\equiv A^\perp \rightharpoonup B^\perp \\
(!A)^\perp & :\equiv ?A^\perp & (?A)^\perp & :\equiv !A^\perp
\end{array}
$$

Remark

In fact, $\rightharpoonup, \rightharpoondown$ are definable in the other connectives:

$$A \rightharpoonup B \equiv A^\perp \sqcup B$$

$$A \rightharpoondown B \equiv A^\perp \otimes B$$

The axioms and rules are:

p $\quad \vdash A, A^\perp$

1 $\quad \vdash \mathbf{1}$ $\qquad\qquad$ $\perp \quad \dfrac{\vdash \Gamma}{\vdash \perp, \Gamma}$

$\top \quad \vdash \top, \Gamma$

$\otimes \quad \dfrac{\vdash A, \Gamma_1 \qquad \vdash B, \Gamma_2}{\vdash A \otimes B, \Gamma_1, \Gamma_2}$ $\qquad\qquad$ $\sqcup \quad \dfrac{\vdash A, B, \Gamma}{\vdash A \sqcup B, \Gamma}$

$\sqcap \quad \dfrac{\vdash A, \Gamma \qquad \vdash B, \Gamma}{\vdash A \sqcap B, \Gamma}$ $\qquad\qquad$ $\oplus_{1,2} \quad \dfrac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma} \qquad \dfrac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}$

$\rightharpoonup \quad \dfrac{\vdash A^\perp, B, \Gamma}{\vdash A \rightharpoonup B, \Gamma}$ $\qquad\qquad$ $\rightharpoondown \quad \dfrac{\vdash A^\perp, \Gamma_1 \qquad \vdash B, \Gamma_2}{\vdash A \rightharpoondown B, \Gamma_1, \Gamma_2}$

$?_{a,b,c} \quad \dfrac{\vdash \Gamma}{\vdash ?A, \Gamma} \qquad \dfrac{\vdash A, \Gamma}{\vdash ?A, \Gamma} \qquad \dfrac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}$ $\qquad$ $! \quad \dfrac{\vdash C, ?\Gamma}{\vdash !C, ?\Gamma}$

An *elementary fragment* $\mathcal{F}$ is a subset of the set of formulae of **CL** based on a subset $C_\mathcal{F} \subseteq \{\mathbf{0}, \mathbf{1}, \perp, \top, \rightharpoonup, \rightharpoondown, \sqcap, \oplus, \otimes, \sqcup, ^\perp, !, ?\}$ i.e. the constants of $C_\mathcal{F}$ and the proposition letters are in F, and F is closed under the connectives and modalities of $C_\mathcal{F}$. We distinguish between three kinds of elementary fragments:

Type I: elementary fragments without $^\perp$ (negation) and without $\rightharpoonup, \rightharpoondown$

Type II: elementary fragments with negation, or $\rightharpoonup$ together with $\perp$ (falsum), or $\rightharpoondown$ together with **1**

Type III: other elementary fragments

So negation is defined or definable in type II fragments, and in type III fragments negation is absent, while one of $\rightharpoonup$ $\rightharpoondown$ must be present there.
If $\mathcal{F}$, based on $C_{\mathcal{F}}$ is an elementary fragment, then $\mathcal{F}^* := \{A^\perp \mid A \in \mathcal{F}\}$ is called the *De Morgan dual* of $\mathcal{F}$. It is the elementary fragment based on $C^*_{\mathcal{F}} := \{c^* \mid c \in C_{\mathcal{F}}\}$ where

$$\begin{array}{ll}
\mathbf{0}^* :\equiv \top, & \top^* :\equiv \mathbf{0}, \\
\mathbf{1}^* :\equiv \bot, & \bot^* :\equiv \mathbf{1}, \\
\sqcap^* :\equiv \oplus, & \oplus^* :\equiv \sqcap, \\
\otimes^* :\equiv \sqcup, & \sqcup^* :\equiv \otimes, \\
\rightharpoonup^* :\equiv \rightharpoondown, & \rightharpoondown^* :\equiv \rightharpoonup, \\
!^* :\equiv ?\,, & ?^* :\equiv !, \\
{}^{\perp *} :\equiv {}^{\perp}. &
\end{array}$$

Notation

$$\vdash \Gamma' \overset{N}{\sharp} \Gamma'' \;:\Longleftrightarrow \text{(i)} \vdash_{\mathbf{CL}} \Gamma', \Gamma'', \quad \text{(ii)} \vdash_{\mathbf{CL}} \Gamma', N, \quad \text{(iii)} \vdash_{\mathbf{CL}} N^\perp, \Gamma''$$

and (iv) the material of N is contained in Γ' and in Γ''.
When we work in a two sided calculus, we define

$$\Gamma \overset{N}{\vdash} \Delta \quad :\Longleftrightarrow \text{(i)}\ \Gamma \vdash_{\mathbf{CL}} \Delta, \quad \text{(ii)}\ \Gamma \vdash_{\mathbf{CL}} N, \quad \text{(iii)}\ N \vdash_{\mathbf{CL}} \Delta$$

and (iv) the material of N is contained in Γ and in Δ.

3.2 Interpolation for type I fragments

Theorem 3.1 *For every type I fragment $\mathcal{F}$ interpolation holds.*

PROOF: We work in a two sided calculus. Let a cut-free proof Π of $\Gamma \vdash \Delta$ be given, $\Gamma, \Delta \subseteq \mathcal{F}$. We construct an interpolant for $\Gamma \vdash \Delta$ by giving interpolants for the axioms in Π, and constructing interpolants for the conclusions from interpolants from the premises of every application of a rule in Π.

p $\quad A \overset{A}{\vdash} A$

1l $\quad \dfrac{\Gamma \overset{N}{\vdash} \Delta}{\Gamma, \mathbf{1} \overset{N}{\vdash} \Delta}$ $\qquad$ **1r** $\quad \overset{\mathbf{1}}{\vdash} \mathbf{1}$

⊥l $\quad \bot \overset{\bot}{\vdash}$ $\qquad$ **⊥r** $\quad \dfrac{\Gamma \overset{N}{\vdash} \Delta}{\Gamma \overset{N}{\vdash} \bot, \Delta}$

0 $\quad \Gamma, \mathbf{0} \overset{\mathbf{0}}{\vdash} \Delta$ $\qquad$ $\top$ $\quad \Gamma \overset{\top}{\vdash} \top, \Delta$

$$\otimes l \quad \frac{\Gamma, A, B \overset{N}{\vdash} \Delta}{\Gamma, A \otimes B \overset{N}{\vdash} \Delta} \qquad \otimes r \quad \frac{\Gamma_1 \overset{N}{\vdash} A, \Delta_1 \quad \Gamma_2 \overset{M}{\vdash} B, \Delta_2}{\Gamma_1, \Gamma_2 \overset{N \otimes M}{\vdash} A \otimes B, \Delta_1, \Delta_2}$$

$$\sqcup l \quad \frac{\Gamma_1, A \overset{N}{\vdash} \Delta_1 \quad \Gamma_2, B \overset{M}{\vdash} \Delta_2}{\Gamma_1, \Gamma_2, A \sqcup B \overset{N \sqcup M}{\vdash} \Delta_1, \Delta_2} \qquad \sqcup r \quad \frac{\Gamma \overset{N}{\vdash} A, B, \Delta}{\Gamma \overset{N}{\vdash} A \sqcup B, \Delta}$$

$$\sqcap l_{1,2} \quad \frac{\Gamma, A \overset{N}{\vdash} \Delta}{\Gamma, A \sqcap B \overset{N}{\vdash} \Delta} \quad \frac{\Gamma, B \overset{N}{\vdash} \Delta}{\Gamma, A \sqcap B \overset{N}{\vdash} \Delta} \qquad \sqcap r \quad \frac{\Gamma \overset{N}{\vdash} A, \Delta \quad \Gamma \overset{M}{\vdash} B, \Delta}{\Gamma \overset{N \sqcap M}{\vdash} A \sqcap B, \Delta}$$

$$\oplus l \quad \frac{\Gamma, A \overset{N}{\vdash} \Delta \quad \Gamma, B \overset{M}{\vdash} \Delta}{\Gamma, A \oplus B \overset{N \oplus M}{\vdash} \Delta} \qquad \oplus r_{1,2} \quad \frac{\Gamma \overset{N}{\vdash} A, \Delta}{\Gamma \overset{N}{\vdash} A \oplus B, \Delta} \quad \frac{\Gamma \overset{N}{\vdash} B, \Delta}{\Gamma \overset{N}{\vdash} A \oplus B, \Delta}$$

$$!l_{a,b} \quad \frac{\Gamma \overset{N}{\vdash} \Delta}{\Gamma, !A \overset{N}{\vdash} \Delta} \quad \frac{\Gamma, A \overset{N}{\vdash} \Delta}{\Gamma, !A \overset{N}{\vdash} \Delta} \qquad !r \quad \frac{\Gamma \overset{N}{\vdash} C}{\Gamma \overset{!N}{\vdash} !C} \qquad !c \quad \frac{\Gamma, !A, !A \overset{N}{\vdash} \Delta}{\Gamma, !A \overset{N}{\vdash} \Delta}$$

$$?r_{a,b} \quad \frac{\Gamma \overset{N}{\vdash} \Delta}{\Gamma \overset{N}{\vdash} ?A, \Delta} \quad \frac{\Gamma \overset{N}{\vdash} A, \Delta}{\Gamma \overset{N}{\vdash} ?A, \Delta} \qquad ?l \quad \frac{C \overset{N}{\vdash} ?\Delta}{?C \overset{?N}{\vdash} ?\Delta} \qquad ?c \quad \frac{\Gamma \overset{N}{\vdash} ?A, ?A, \Delta}{\Gamma \overset{N}{\vdash} ?A, \Delta}$$

All verifications are easy, in particular thar the interpolation formula stays in the given fragment.

□

3.3 Interpolation for type II fragments

Theorem 3.2 *For every type II fragment $\mathcal{F}$ interpolation holds.*

PROOF: We try again an induction on a cut-free proof, but we have to "load" the induction hypothesis. Suppose, for provable $\vdash \Gamma$ in $\mathcal{F}$, that for *every partition* Γ', Γ'' of Γ there is an $N \in \mathcal{F}$ such that $\vdash \Gamma' \overset{N}{\sharp} \Gamma''$.(This is not so easy to express in the two sided calculus).

Let Π be cut-free proof of $\vdash \Gamma$. Then an induction on the length of Π shows that every sequent of Π is in $\mathcal{F} \cup \mathcal{F}^*$, but $\mathcal{F}^* = \mathcal{F}$ because negation is definable in $\mathcal{F}$. That is also the reason why we do not have to consider linear implication $\multimap$ and its De Morgan dual $\multimapinv$.

The following list shows how to construct an interpolant N such that $\vdash \Gamma' \overset{N}{\sharp} \Gamma''$. Remark: the position of $\sharp$ in the premises is dictated by the position of $\sharp$ in the conclusion, while the formula above $\sharp$ in the conclusion is constructed from the formula(e) above the $\sharp$ in the premises. So the construction requires an upward walk followed by a downward walk through Π.

p	$\vdash \overset{\mathbf{1}}{\sharp} A^{\perp}, A$ $\quad \vdash A^{\perp} \overset{A}{\sharp} A$	$\vdash A^{\perp}, A \overset{\perp}{\sharp}$
	main formula left	main formula right
1	$\vdash \overset{\mathbf{1}}{\sharp} \mathbf{1}$	$\vdash \mathbf{1} \overset{\perp}{\sharp}$
$\perp$	$\dfrac{\vdash \Gamma' \overset{N}{\sharp} \Gamma''}{\vdash \Gamma', \perp \overset{N}{\sharp} \Gamma''}$	$\dfrac{\vdash \Gamma' \overset{N}{\sharp} \Gamma''}{\vdash \Gamma' \overset{N}{\sharp} \perp, \Gamma''}$
$\top$	$\vdash \Gamma', \top \overset{\mathbf{0}}{\sharp} \Gamma''$	$\vdash \Gamma' \overset{\top}{\sharp} \top, \Gamma''$
$\otimes$	$\dfrac{\vdash \Gamma', A \overset{N}{\sharp} \Gamma'' \qquad \vdash \Delta', B \overset{M}{\sharp} \Delta''}{\vdash \Gamma', \Delta', A \otimes B \overset{N \sqcup M}{\sharp} \Gamma'', \Delta''}$	$\dfrac{\vdash \Gamma' \overset{N}{\sharp} A, \Gamma'' \qquad \vdash \Delta' \overset{M}{\sharp} B, \Delta''}{\vdash \Gamma', \Delta' \overset{N \otimes M}{\sharp} A \otimes B, \Gamma'', \Delta''}$
$\sqcup$	$\dfrac{\vdash \Gamma', A, B \overset{N}{\sharp} \Gamma''}{\vdash \Gamma', A \sqcup B \overset{N}{\sharp} \Gamma''}$	$\dfrac{\vdash \Gamma' \overset{N}{\sharp} A, B, \Gamma''}{\vdash \Gamma' \overset{N}{\sharp} A \sqcup B, \Gamma''}$
$\sqcap$	$\dfrac{\vdash \Gamma', A \overset{N}{\sharp} \Gamma'' \qquad \vdash \Gamma', B \overset{M}{\sharp} \Gamma''}{\vdash \Gamma', A \sqcap B \overset{N \oplus M}{\sharp} \Gamma''}$	$\dfrac{\vdash \Gamma' \overset{N}{\sharp} A, \Gamma'' \qquad \vdash \Gamma' \overset{M}{\sharp} B, \Gamma''}{\vdash \Gamma' \overset{N \sqcap M}{\sharp} A \sqcap B, \Gamma''}$
$\oplus$	$\dfrac{\vdash \Gamma', A \overset{N}{\sharp} \Gamma''}{\vdash \Gamma', A \oplus B \overset{N}{\sharp} \Gamma''}$	$\dfrac{\vdash \Gamma' \overset{N}{\sharp} A, \Gamma''}{\vdash \Gamma' \overset{N}{\sharp} A \oplus B, \Gamma''}$
$?_{\mathrm{a,b}}$	$\dfrac{\vdash \Gamma' \overset{N}{\sharp} \Gamma''}{\vdash \Gamma', ?A \overset{N}{\sharp} \Gamma''} \qquad \dfrac{\vdash \Gamma', A \overset{N}{\sharp} \Gamma''}{\vdash \Gamma', ?A \overset{N}{\sharp} \Gamma''}$	$\dfrac{\vdash \Gamma' \overset{N}{\sharp} \Gamma''}{\vdash \Gamma' \overset{N}{\sharp} ?A, \Gamma''} \qquad \dfrac{\vdash \Gamma' \overset{N}{\sharp} A, \Gamma''}{\vdash \Gamma' \overset{N}{\sharp} ?A, \Gamma''}$
$?_{\mathrm{c}}$	$\dfrac{\vdash \Gamma', ?A, ?A \overset{N}{\sharp} \Gamma''}{\vdash \Gamma', ?A \overset{N}{\sharp} \Gamma''}$	$\dfrac{\vdash \Gamma' \overset{N}{\sharp} ?A, ?A, \Gamma''}{\vdash \Gamma' \overset{N}{\sharp} ?A, \Gamma''}$
!	$\dfrac{\vdash ?\Gamma', C \overset{N}{\sharp} ?\Gamma''}{\vdash ?\Gamma', !C \overset{?N}{\sharp} ?\Gamma''}$	$\dfrac{\vdash ?\Gamma' \overset{N}{\sharp} C, ?\Gamma''}{\vdash ?\Gamma' \overset{!N}{\sharp} !C, ?\Gamma''}$

Again, all verifications are readily made; in particular, note that whenever a connective or constant is introduced in the interpolation formula, it or its De Morgan dual occurred in the accompanying sequent, so it is in $\mathcal{F} \cup \mathcal{F}^* = \mathcal{F}$.

□

3.4 Interpolation for type III fragments

In order to prove interpolation for type three fragments, we have to be much more careful about the interpolation formula, since $\mathcal{F} \neq \mathcal{F}^*$. I have no results for this fragment yet; it is possible that interpolation does not hold for all type III fragments; my conjecture is that it does hold for fragments without ! and ?, and that there are counterexamples when ! or ? are admitted.

4 Decidability

The method described in [Dun86] to show decidability for various relevance logics can also be put to work in the case of **CL**. But for technical reasons it is better to let the method tackle a stronger system, namely $\mathbf{CL}^c$ i.e. $\mathbf{CL}^1$ with the $?_c$ rule replaced by unrestricted contraction:

$$\text{c} \; \frac{\vdash A, A, \Gamma}{\vdash A, \Gamma}$$

The problems come from the fact that in **CL** the formulae that may be contracted are syntactically different from those that may not. It causes then problems in the definition of the cognation classes; for example should $\vdash A, \Gamma$ and $\vdash A, A, \Gamma$ belong to the same cognation class? On the one hand, they are not contractions of each other (in general), on the other hand, both are reducible to $\vdash ?A, \Gamma$.

In [Dun86] the systems in consideration have intuitionistic sequents; but the place of the $\vdash$ is immaterial to the method. So I expect that this method is adaptable to *intuitionistic* linear logic as well.

The method of [Dun86] yields for each sequent a finite proof-searching tree. If a sequent is unprovable in $\mathbf{CL}^c$, then it is a fortiori unprovable in the weaker system $\mathbf{CL}^1$; if a sequent is provable in $\mathbf{CL}^c$ then the decision method yields a finite number of possible $\mathbf{CL}^{c\pm}$-proofs. If there is a $\mathbf{CL}^1$-proof, then a certain transformation of it must occur among those $\mathbf{CL}^{c\pm}$-proofs, which is a decidable question. Below, I shall consider this point in detail. It is convenient to summarize the method first.

1. Start with $\mathbf{CL}^c$. Remove the contraction rules, and allow a bit of contraction in those operational rules that do not commute with contraction. Call the resulting system $\mathbf{CL}^{c\pm}$.

2. Prove a Curry-lemma: if a sequent Δ is a contraction of a sequent Γ, and Γ is derivable in $\leq n$ steps in $\mathbf{CL}^{c\pm}$, then Δ is derivable in $\leq n$ steps in $\mathbf{CL}^{c\pm}$.

3. Define notion of cognation : two sequents are cognate, if they are contractable to the same sequent (by *unrestricted* contraction).

4. Show that there are finitely many cognation classes in every derivation (using the subformula property)

5. Use Kripke's[1] lemma to show that every branch in a irredundant complete proof searching tree is finite

6. Gather the premises for König's lemma, and conclude that the above mentioned proof searching tree is finite.

Irredundant means: containing only irredundant derivations; a derivation is irredundant if there is no sequent with sequents below it that are contractions of it.

4.1 The systems $\mathbf{CL}^{c}$ and $\mathbf{CL}^{c\pm}$.

We work in the one sided version, so $\mathbf{CL}^{c}$ is $\mathbf{CL}^{1}$ with the rule $?_c$ replaced by the rule **c**. The system $\mathbf{CL}^{c\pm}$ which has only contraction in the operational rules is as follows:

p $\quad \vdash A, A^{\perp}$

1 $\quad \vdash \mathbf{1}$ $\qquad\qquad$ $\perp \quad \dfrac{\vdash \Gamma}{\vdash \perp, \Gamma}$

$\top \quad \vdash \top, \Gamma$

$\otimes \quad \dfrac{\vdash A, \Gamma_1 \qquad \vdash B, \Gamma_2}{\vdash [A \otimes B, \Gamma_1, \Gamma_2]}$ $\qquad\qquad$ $\sqcup \quad \dfrac{\vdash A, B, \Gamma}{\vdash [A \sqcup B, \Gamma]}$

$\sqcap \quad \dfrac{\vdash A, \Gamma \qquad \vdash B, \Gamma}{\vdash [A \sqcap B, \Gamma]}$ $\qquad\qquad$ $\oplus_{1,2} \quad \dfrac{\vdash A, \Gamma}{\vdash [A \oplus B, \Gamma]} \qquad \dfrac{\vdash B, \Gamma}{\vdash [A \oplus B, \Gamma]}$

$?_{a,b} \quad \dfrac{\vdash \Gamma}{\vdash ?A, \Gamma} \qquad \dfrac{\vdash A, \Gamma}{\vdash [?A, \Gamma]}$ $\qquad\qquad$ $! \quad \dfrac{\vdash C, ?\Gamma}{\vdash !C, ?\Gamma}$

Here $[X, \Gamma]$ means any sequent that arises from X, Γ by deleting at most one occurrence of X from Γ; $[X, \Gamma_1, \Gamma_2]$ means any sequent that arises from X, Γ_1, Γ_2 by deleting at most one occurrence of X from Γ_1 and deleting at most one occurrence of X from Γ_2, and, for each pair of occurrences $C \in \Gamma_1, C \in \Gamma_2$ deleting at most one occurrence of C.
Let us now prove Curry's lemma. Suppose $\vdash \Gamma$ is provable in $\mathbf{CL}^{c\pm}$. Let $\vdash \Delta$ be a one-step contraction. Then we shall verify that $\vdash \Delta$ is provable in $\mathbf{CL}^{c\pm}$ with smaller or equal proof-length. Then it follows by induction on the number of contractions that *every* contraction of $\vdash \Gamma$ is provable with smaller or equal proof-length.
The system $\mathbf{CL}^{c\pm}$ is set up with the explicit purpose that a Curry lemma will hold. Therefore, the verification of all the cases is trivial, if one has seen it work in the most complicated case. So I only show the $\otimes$-case.
Suppose we have

[1] What is called Kripke's lemma here is in fact a consequence of a theorem of Kruskal ([Kru60]. See also [Kru72] for the rediscovery history of this lemma. In this paper I shall refer to it as Kripke's lemma.

$$\cfrac{\cfrac{\vdash A,\Gamma_1 \quad \vdash B,\Gamma_2}{\vdash [A\otimes B,\Gamma_1,\Gamma_2]}}{\vdash \Delta}\;\mathbf{c}$$

Case 1: a corresponding contraction was already possible on one of the premises. Then apply induction hypothesis.
Case 2: otherwise, i.e. the two contrahenda are distributed over Γ_1 and Γ_2 or one of the contrahenda is the main formula of the conclusion. But then the rule $\otimes$ is constructed in such a way Δ matches $\vdash [A\otimes B,\Gamma_1,\Gamma_2]$.

□

Corollary 4.1

(i): $\vdash \Gamma$ *provable in* $\mathbf{CL}^c \iff \vdash \Gamma$ *provable in* $\mathbf{CL}^{c\pm}$.

(ii): if there is a derivation for $\vdash \Gamma$ *in* $\mathbf{CL}^{c\pm}$ *then there is an irredundant derivation for the same sequent.*

PROOF: (i)(from left to right) All rules of $\mathbf{CL}^c$ are rules of $\mathbf{CL}^{c\pm}$, except the c-rule. But if that rule is applied, we can apply Curry's lemma.
(from right to left) The rules of $\mathbf{CL}^{c\pm}$ are derived rules of $\mathbf{CL}^c$.
(ii) Induction on the length of derivations: If there is a redundancy not involving the conclusion, we can remove it by induction hypothesis. If there is a redundancy

$$\begin{array}{l}\vdots\\ \vdash \Delta\\ \vdots\\ \vdash \Gamma \quad \text{where } \vdash \Gamma \text{ is a contraction of } \vdash \Delta,\end{array}$$

then we have by Curry's lemma a derivation of $\vdash \Gamma$ in $\mathbf{CL}^{c\pm}$ of the same length as the given derivation of $\vdash \Delta$. Then by induction hypothesis we have an irredundant derivation of $\vdash \Gamma$.

□

Theorem 4.2 $\mathbf{CL}^c$ *is decidable.*

PROOF: We apply Dunn's method:
For each sequent $\vdash \Gamma$ we can build a complete $\mathbf{CL}^c$-proof searching tree. It suffices to look for cut free proofs, regardless whether cut elimination holds in $\mathbf{CL}^c$ or not. This is because we are only interested in provability in $\mathbf{CL}^1$, and thus only in $\mathbf{CL}^c$-proofs that are transformations of $\mathbf{CL}^1$-proofs, the so-called *linear* $\mathbf{CL}^c$-proofs. We shall see that this transformation carries cut free proofs to cut free proofs, and we know that cut elimination holds in $\mathbf{CL}^1$ (cf. definiton 4.1, lemmas 4.4, 4.5). From this it is clear that the proof searching tree is finitely branching.
The next thing to show is that every branch in this tree is finite. We can make another restriction on proofs: we only search for irredundant proofs. This is justified by Curry's lemma (corollary 4.1). We define: two sequents are *cognate* if they are equal as sets. Because the tree searches only for cut free proofs, for which the subformula property holds, there are only finitely many cognation classes in the complete proof searching tree. Then we can apply Kripke's lemma:

Lemma 4.3 *If a sequence of cognate sequents is irredundant (meaning that earlier sequents are never contractions of latter sequents), then the sequence is finite.*

The branches of our complete proof searching tree try to build up irredundant (cut free) proofs, from the conclusion upwards. If there was an infinite branch then it would contain infinitely many sequents from a certain cognation class. These sequents form an irredundant sequence. So the branch must be finite.
Now we have both premises for König's lemma, so the complete proof searching tree is finite.

□theorem 4.2

4.2 Decidability of $\mathbf{CL}^1$

We shall show that the decision procedure for $\mathbf{CL}^c$, that follows from the method described above, leads to a decision method for $\mathbf{CL}^1$.

Definition 4.1
A contraction of two formulae with principal connective ? is a linear *contraction.*
A $\mathbf{CL}^{c\pm}$*-proof in which all contractions are linear, is called a* linear *(*$\mathbf{CL}^{c\pm}$*-) proof. (This relates to the contractions built into the operational rules)*

Lemma 4.4 *every* $\mathbf{CL}^1$*-provable sequent has a linear* $\mathbf{CL}^{c\pm}$*-derivation.*

PROOF:
Induction on the proof-length, analogously to the proof of corollary 4.1(i)(from left to right). It is necessary to use a sharpened Curry lemma:
If a sequent Δ is a linear contraction of a sequent Γ, and Γ is derivable in $\leq n$ steps by a linear derivation in $\mathbf{CL}^{c\pm}$*, then Δ is derivable in $\leq n$ steps by a linear derivation in* $\mathbf{CL}^{c\pm}$. This is easier to prove than the Curry lemma itself, because there are less cases to consider, for main formula are not linearly contractable (except when a ?-formula is introduced).

□

Lemma 4.5 *Every linear* $\mathbf{CL}^{c\pm}$*-proof can be transformed into a* $\mathbf{CL}^1$*-proof.*

PROOF: Every contraction in a linear $\mathbf{CL}^{c\pm}$-proof can be replaced by applications of the $?_c$-rule.

□

Theorem 4.6 $\mathbf{CL}^1$ *is decidable.*

PROOF:
Let $\vdash \Gamma$ be given. By the method of theorem 4.2 we are provided with a complete, finite (possibly empty) set of $\mathbf{CL}^{c\pm}$-proofs of it. If there are no $\mathbf{CL}^{c\pm}$-proofs, then $\vdash \Gamma$ is not provable in $\mathbf{CL}^c$, and clearly the same holds for $\mathbf{CL}^1$. If there are $\mathbf{CL}^{c\pm}$-proofs for $\vdash \Gamma$ then the question whether there are linear ones among them, is decidable. If there is a linear one, then by lemma 4.5 $\vdash \Gamma$ is $\mathbf{CL}^1$-provable; if there is no such proof, then by lemma 4.4 we know that there is no $\mathbf{CL}^1$-proof for $\vdash \Gamma$.

□

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