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ISOMORPHISMS AND NON-ISOMORPHISMS OF GRAPH MODELS

Harold Schellinx

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ISOMORPHISMS AND NON-ISOMORPHISMS OF GRAPH MODELS

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Isomorphisms and non-isomorphisms of graph models

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In this paper the existence or non-existence of isomorphic mappings between graph models for the untyped lambda calculus is studied. It is shown that Engeler's $\mathbf{D}_A$ is completely determined, up to isomorphism, by the cardinality of its 'atom-set' A. A similar characterization is given for a collection of graph models of the $\mathcal{P}\omega$-type; from this some propositions regarding automorphisms are obtained. Also we give an indication of the complexity of the first-order theory of graph models by showing that the second-order theory of first-order definable elements of a graph model is first-order expressable in the model.

1. Introduction

Among the set-theoretical models for the untyped lambda calculus that were introduced in the seventies and early eighties there is a class of which the members are particularly easy to describe. We will refer to these as *graph models*. They can be characterized as follows.

1.1 DEFINITION: A *graph model* is a pair $(\mathcal{P}(X), \bullet)$, where $\mathcal{P}(X)$ is the powerset of some infinite set X and $\bullet$ a binary operation on $\mathcal{P}(X)$ defined by means of an embedding $(\cdot,\cdot)$ into X from the Cartesian product $X^{<\omega} \times X$ of the collection of finite subsets of X and X such that, for all $a, b \in \mathcal{P}(X)$, we have

$$a \bullet b = \{m \mid \exists \mu \subseteq b.(\mu, m) \in a\}. \qquad \square$$

In the literature graph models appear in two main variants. The first construction is originally and independently due to G. Plotkin and D.S. Scott and uses the set $\mathbb{N}$ of natural numbers. An embedding $(\cdot,\cdot) : \mathbb{N}^{<\omega} \times \mathbb{N} \hookrightarrow \mathbb{N}$ is obtained by means of an injective (but not necessarily surjective) coding $p : \mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$ of pairs of natural numbers as natural numbers and a bijective coding $e : \mathbb{N} \hookrightarrow \mathbb{N}^{<\omega}$ of finite sets of natural numbers by natural numbers, as follows:

$$\mathbb{N}^{<\omega} \times \mathbb{N} \ni (\alpha, m) \longmapsto p(e^{-1}(\alpha), m) \in \mathbb{N}. \tag{1}$$

Putting e_n for $e(n)$ we then can rewrite the definition of application through an embedding by codings p and e as:

$$x \bullet y = \{m \mid \exists e_n \subseteq y.p(n,m) \in x\}. \tag{2}$$

1.2 DEFINITION: A $\mathcal{P}(\mathbb{N})$-*structure* is a graph model $[p,e] := (\mathcal{P}(\mathbb{N}), \bullet)$ with application defined by the codings p and e as in (2).
$\mathcal{P}\omega$ denotes the structure $[p^\star, e^\star]$ defined as follows:
for all $n, m \in \mathbb{N} : p^\star(n,m) = \frac{1}{2}(n+m)(n+m+1) + m$;
for all $n \in \mathbb{N} : e^\star_n = \{k_0, k_1, \ldots, k_{m-1}\}$ iff $n = \sum_{i<m} 2^{k_i} \quad (k_i \neq k_j \text{ if } i \neq j)$;
$e^\star_0 = \emptyset$. □

Note that the codings $p^\star$ and $e^\star$ defining $\mathcal{P}\omega$ are such that a natural number always is smaller than any code of a finite set containing it and greater than or equal to the left and right projections of the pair it encodes.

1.3 DEFINITION: Injective codings p of pairs of natural numbers as natural numbers and bijective codings e of finite sets of natural numbers by natural numbers are called *basic codings* if and only if (iff)

$$\forall k. \quad x \in e_k \Longrightarrow x < k;$$

$$\forall m, n. \quad m, n \leq p(n,m).$$ □

For all basic codings, by definition the empty set is coded by 0; also $p(0,0) = 0$ whenever 0 is in the range of p.

Iterated pairing $p(x_1, p(x_2, p(x_3, \ldots, p(x_k, y)\ldots)))$ will be written as $p(x_1, x_2, x_3, \ldots, x_k, y)$.

A second paradigmatic construction of graph models, due to E. Engeler, is such that actually $X^{<\omega} \times X \subseteq X$: we may take the identity mapping as our embedding.

1.4 DEFINITION: Let A be any non-empty set of which none of the elements is written as a pair $(-,-)$. Then put:

$$G_0(A) := A$$
$$G_{n+1}(A) := G_n(A) \cup (G_n(A)^{<\omega} \times G_n(A))$$
$$G(A) := \bigcup_{n \in \mathbb{N}} G_n(A).$$

$\mathbf{D}_A$ will denote the graph model $(\mathcal{P}(G(A)), \bullet)$. □

So $G(A)$ is the smallest set $X \supset A$ such that for all finite $\beta \subseteq X$ and $b \in X$ we have that $(\beta, b) \in X$.

In view of the construction described above graph models sometimes are referred to as *Plotkin-Scott-Engeler (PSE)-algebras* (see [Lo]).

Any graphmodel $(\mathcal{P}(X), \bullet)$ is a model for the untyped lambda calculus, because $(\mathcal{P}(X), \subseteq)$ is a reflexive cpo through the continuous (w.r.t. the Scott-topology) mappings $F : \mathcal{P}(X) \longrightarrow [\mathcal{P}(X) \to \mathcal{P}(X)]$ and $G : [\mathcal{P}(X) \to \mathcal{P}(X)] \longrightarrow \mathcal{P}(X)$ given by

$$F(a)(b) = a \bullet b$$

and

$$G(f) = \{(\beta, b) \mid b \in f(\beta)\}.$$

For more details we refer the reader to e.g. chapter 5 of [Ba].

Now, given this huge amount of models (one for each pair of codings (p, e) and one for each non-empty set A), one naturally asks whether and in which sense they are different or alike.

As any graph model is an applicative structure we may apply the usual definition of homomorphic and isomorphic mappings.

1.5 DEFINITION: Given two graph models $(\mathcal{P}(X), \bullet)$ and $(\mathcal{P}(Y), \star)$ we will call a 1-1 mapping ψ of $\mathcal{P}(X)$ onto $\mathcal{P}(Y)$ an *(applicative) isomorphism* iff we have for all $a, b \in \mathcal{P}(X)$:

$$\psi(a \bullet b) = \psi(a) \star \psi(b).$$

If such a mapping ψ can be found, we say that the graph models are *isomorphic*. This is written as $(\mathcal{P}(X), \bullet) \cong (\mathcal{P}(Y), \star)$. □

We can talk about applicative structures in a first-order language $\mathcal{L}$, defined in terms of a countably infinite set of variables, the usual logical symbols (parentheses, connectives, quantifiers), a binary relation symbol $=$ for identity and a binary function symbol $\mathcal{A}$ for application. (Actually we will never use the symbol $\mathcal{A}$ but always write

"xy" for "$\mathcal{A}(x,y)$"; for nested applications we adopt the convention of association to the left, i.e. $wxyz := ((wx)y)z$.)

As any applicative structure is an $\mathcal{L}$-model the concept of "applicative isomorphism" coincides with that of "isomorphism of $\mathcal{L}$-models". So for applicative structures $(A,\bullet)$ and $(B,*)$ we have

$$(A,\bullet) \cong (B,*) \Longrightarrow (A,\bullet) \equiv (B,*), \tag{3}$$

meaning that whenever two applicative structures are isomorphic, they are elementary equivalent as $\mathcal{L}$-models (i.e. an $\mathcal{L}$-sentence φ is valid in $(A,\bullet)$ iff it is valid in $(B,*)$; this will be written as $(A,\bullet) \models \varphi$ iff $(B,*) \models \varphi$).

The main part of this paper is devoted to questions of existence or non-existence of isomorphic mappings between graph models. Furthermore attention will be paid to the first-order theory of graph models which, because of (3), often provides us with a means to quickly refute isomorphism.

As far as we know in the literature notes as to the isomorphism of graph models are limited to the observation that representatives of Engeler's variant are never isomorphic to $\mathcal{P}(\mathbb{N})$-structures defined through *surjective* codings (see e.g. [Lo], or chapter 20 of [Ba]). The proofs however seem to blur the basic reason for this non-isomorphism. We will give a simple argument in the next section. After this we show that the lattice-structure of a graph model $\mathcal{M}$ is first-order definable in $\mathcal{M}$. We use this to characterize isomorphic mappings between graph models and to show that Engeler's $\mathbf{D}_A$ is, up to isomorphism, completely determined by the cardinality of the set A. We then end the section by showing the second-order theory of first-order definable elements of a graph model to be first-order expressable in the model.

Baeten en Boerboom showed in [Ba,Bo] how changes in the codings determining a graph model of the Plotkin/Scott-type may change the set of equations between λ-terms valid in the model. In the third section we use the characterization of isomorphism obtained to take a look at the relation between codings and the possibility of isomorphism between $\mathcal{P}(\mathbb{N})$-structures. We introduce the concept of *frame* for such structures and use it to show the existence of uncountable many non-isomorphic representatives of the Plotkin/Scott-variant. Also we show structures definable by means of 'rather well-behaved' codings to be, up to isomorphism, completely determined by their frames. From this we obtain some propositions regarding automorphisms of $\mathcal{P}(\mathbb{N})$-structures. Finally we note that many properties of frames of $\mathcal{P}(\mathbb{N})$-structures are in fact first-order properties. This enables us e.g. to show the existence of uncountable many non-elementary equivalent represenatives of graph models of the Plotkin/Scott-type.

The work in this paper was part of the author's master's thesis, written under supervision of prof. Anne Troelstra, who posed the initial questions and read and commented upon many of the drafts and preliminary answers. Also we'd like to

thank Kees Doets and Johan van Benthem for their useful suggestions regarding the first-order theories. We are indebted to Ingemarie Bethke, whose proposition on the $\mathcal{L}$-definability of the lattice-structure of graph models plays a major rôle.

2. Isomorphism between graph models

Atoms

The next lemma will be of use in what follows.

2.1 LEMMA: *Let $(\mathcal{P}(X), \bullet)$ be a graph model. Then, for all $a, b, c \in \mathcal{P}(X)$:*

(i) $b \subseteq c \Rightarrow a \bullet b \subseteq a \bullet c$,
$b \subseteq c \Rightarrow b \bullet a \subseteq c \bullet a$;

(ii) $(a \cup b) \bullet c = (a \bullet c) \cup (b \bullet c)$,
$a \bullet (b \cup c) \supseteq (a \bullet b) \cup (a \bullet c)$;

(iii) $(a \cap b) \bullet c \subseteq (a \bullet c) \cap (b \bullet c)$,
$a \bullet (b \cap c) \subseteq (a \bullet b) \cap (a \bullet c)$.

Proof: Left to the reader. □

Though any embedding gives rise to a structure that is a lambda-model, generally different embeddings will result in models with different properties.

2.2 DEFINITION: Let $\mathcal{M} = (\mathcal{P}(X), \bullet)$ be a graph model defined by means of an embedding $(\cdot, \cdot) : X^{<\omega} \times X \hookrightarrow X$. An *atom* of the model is an element of X that is not in the range of $(\cdot, \cdot)$. The set of atoms in $\mathcal{M}$ is denoted by $\mathcal{AT}(\mathcal{M})$. A graph model $\mathcal{M}$ is *atom-free* iff the embedding that defines it is onto. Elements of the model are called atom-free iff they do not contain atoms. □

2.3 LEMMA: *Let $\mathcal{M} = (\mathcal{P}(X), \bullet)$ be a graph model. Then $a \bullet x = \emptyset$ for all $x \in \mathcal{P}(X)$ iff $a \subseteq \mathcal{AT}(\mathcal{M})$.*

Proof: If $a \subseteq \mathcal{AT}(\mathcal{M})$ then $a \bullet x = \emptyset$ by definition of $\bullet$ and $\mathcal{AT}(\mathcal{M})$. Conversely, if $c \in a$ and $c \notin \mathcal{AT}(\mathcal{M})$ we have $c = (\beta, b)$ for some $\beta \in X^{<\omega}, b \in X$. But then $a \bullet \beta \ni b$. □

2.4 PROPOSITION: *Let $\mathcal{M}, \mathcal{N}$ be graph models. Suppose $\mathcal{AT}(\mathcal{M})$ is finite and $|\mathcal{AT}(\mathcal{N})| > |\mathcal{AT}(\mathcal{M})|$. Then $\mathcal{M} \not\equiv \mathcal{N}$.*

Proof: Let $\Phi(x, z)$ be the $\mathcal{L}$-formula $\forall y.xy = z$. Suppose $|\mathcal{AT}(\mathcal{M})| = n$, for some $n \in \mathbb{N}$, and consider the $\mathcal{L}$-formula $\Psi_n(t) := \Phi(t, t) \wedge$ *"there are precisely*

$2^n - 1$ *elements* u *such that* $u \neq t$ *and* $\Phi(u,t)$". By lemma 2.3 the empty set satisfies Ψ_n in $\mathcal{M}$, so $\mathcal{M} \models \exists t \Psi_n(t)$. If for some b we have that in $\mathcal{N}$ it is true that $b \bullet y = b$, for all y, then write $b = b' \cup a$, where $a \subseteq \mathcal{AT}(\mathcal{N})$ and $b' \cap \mathcal{AT}(\mathcal{N}) = \emptyset$. Let c be an arbitrary subset of $\mathcal{AT}(\mathcal{N})$, $c \neq a$. We have, by lemmas 2.1 and 2.3, for all y,

$$(b' \cup c) \bullet y = b,$$

and if c_1, c_2 are two different subsets of $\mathcal{AT}(\mathcal{N})$, then $b' \cup c_1 \neq b' \cup c_2$. So in $\mathcal{N}$ there are at least $2^{\mathcal{AT}(\mathcal{N})} - 1 \geq 2^{n+1} - 1$ elements $u \neq b$ such that $u \bullet y = b$, for all y. But this means that $\mathcal{N} \not\models \exists t \Psi_n(t)$. □

Note that in a $\mathcal{P}(\mathbb{N})$-structure, whenever p is a bijective coding of pairs, the embedding as given by (1) is onto. Consequently the graph model $[p, e]$ will be atom-free, for all bijective codings p, e. This as opposed to $\mathbf{D}_A$, for which by definition $\mathcal{AT}(\mathbf{D}_A) = A \neq \emptyset$. The following then is immediate from proposition 2.4.

2.5 COROLLARY: *For all* $A \neq \emptyset$ *and all bijective codings* :

$$\mathbf{D}_A \not\cong [p, e].$$ □

It also follows that for all graph models $\mathcal{M}, \mathcal{N}$ such that at least one of the atom-sets is finite, we have $\mathcal{M} \cong \mathcal{N} \Longrightarrow |\mathcal{AT}(\mathcal{M})| = |\mathcal{AT}(\mathcal{N})|$.

This last implication turns out to be true for *arbitrary* atom-sets and in fact this will be our proposition 2.8. But for its proof we need a deeper understanding of the expressive power of first-order sentences over graph models.

The lattice structure of graph models is $\mathcal{L}$-definable

In any graph model $(\mathcal{P}(X), \bullet)$ the (operations of taking) unions and intersections of two elements are representable: just take $\mathbf{u} = \{(\beta, (\gamma, d)) \mid d \in \beta \cup \gamma\}$, $\mathbf{i} = \{(\beta, (\gamma, d)) \mid d \in \beta \cap \gamma\}$ and check that for all x, y we have $\mathbf{u} \bullet x \bullet y = x \cup y$ and $\mathbf{i} \bullet x \bullet y = x \cap y$.

2.6 LEMMA: *Let* $\mathcal{M} = (\mathcal{P}(X), \bullet)$ *be a graph model. Then*

$$\mathcal{M} \models \exists u \exists i \exists t \exists b \bigwedge_{k=1}^{5} \varphi_k,$$

where

$$\varphi_1 := \forall x (ixx = x \wedge uxx = x);$$

$\varphi_2 := \forall x \forall y(ixy = iyx \wedge uxy = uyx);$
$\varphi_3 := \forall x(ixb = b \wedge uxt = t \wedge ixt = x \wedge uxb = x);$
$\varphi_4 := \forall x \forall y \forall z(ixy = x \rightarrow (i(zx)(zy) = zx \wedge i(xz)(yz) = xz));$
$\varphi_5 := \forall x \forall y \forall z.uxyz = u(xz)(yz).$

Proof: Take $t = X, b = \emptyset$ and $u = \mathbf{u}, i = \mathbf{i}$ as above. It is easy to verify that now $\varphi_1, \ldots, \varphi_5$ are satisfied. □

The next proposition as well as its proof were kindly communicated to us by Ingemarie Bethke.

2.7 PROPOSITION: *Let $\mathcal{M} = (\mathcal{P}(X), \bullet)$ be a graph model. If $\varphi_1, \ldots, \varphi_5$ as given in the previous lemma are satisfied by $\mathbf{u}, \mathbf{i}, \mathbf{t}, \mathbf{b} \in \mathcal{P}(X)$, then $\mathbf{t} = X, \mathbf{b} = \emptyset$. Moreover $\mathbf{u} \bullet x \bullet y = x \cup y$ and $\mathbf{i} \bullet x \bullet y = x \cap y$, for all $x, y \in \mathcal{P}(X)$.*

Proof: First we show:

$$\forall x, y \quad (\mathbf{i} \bullet x \bullet y = x \Longrightarrow x \subseteq y). \tag{$\star$}$$

Suppose not. Take $x_0, y_0 \in \mathcal{P}(X)$ such that $\mathbf{i} \bullet x_0 \bullet y_0 = x_0$ and $x_0 \not\subseteq y_0$. Let $p \in x_0 \backslash y_0$ and put $z := \{(\{p\}, k) \mid k \in \mathbf{t}\}$. As $\mathbf{t}$ and $\mathbf{b}$ satisfy $\varphi_4, \varphi_2, \varphi_3$ we have

$$\mathbf{t} = z \bullet x_0 = \mathbf{i} \bullet (z \bullet x_0) \bullet (z \bullet y_0) = \mathbf{i} \bullet \mathbf{t} \bullet \emptyset = \mathbf{i} \bullet \emptyset \bullet \mathbf{t} = \emptyset.$$

So, by φ_3 and lemma 1.1 (i), for all x we have $x = \mathbf{i} \bullet x \bullet \mathbf{t} \subseteq \mathbf{i} \bullet x \bullet \mathbf{b} = \mathbf{b}$, i.e. $\mathbf{b} = X$. Then, again using 1.1, we find that for all x, y,

$$x = \mathbf{u} \bullet x \bullet \mathbf{b} \supseteq \mathbf{u} \bullet x \bullet y \qquad \text{(by } \varphi_3\text{)},$$

and

$$y = \mathbf{u} \bullet y \bullet \mathbf{b} \supseteq \mathbf{u} \bullet x \bullet y \qquad \text{(by } \varphi_3, \varphi_2\text{)}.$$

So

$$x \cap y \supseteq \mathbf{u} \bullet x \bullet y \supseteq \mathbf{u} \bullet (x \cap y) \bullet (x \cap y) = x \cap y \qquad \text{(by } \varphi_1\text{)}.$$

This means that $\mathbf{u} \bullet x \bullet y = x \cap y$ for all x, y.
Now take $v, v', w \in X, v \neq v'$ and put

$$q := \{(\{v\}, w)\}, r := \{(\{v'\}, w)\}, s := \{v, v'\}.$$

Then

$$\mathbf{u} \bullet q \bullet r \bullet s = (q \cap r) \bullet s = \emptyset \bullet s = \emptyset.$$

On the other hand, by φ_5 :

$$\mathbf{u} \bullet q \bullet r \bullet s = \mathbf{u} \bullet (q \bullet s) \bullet (r \bullet s) = (q \bullet s) \cap (r \bullet s) = \{w\}.$$

This is a contradiction.

But then by $(\star)$ and φ_3 we have $\mathbf{b} \subseteq x$ for all x; so $\mathbf{b} = \emptyset$. Also $x \subseteq \mathbf{t}$ for all x; this means that $\mathbf{t} = X$.

Then , for all x, y: $x = \mathbf{i} \bullet x \bullet \mathbf{t} \supseteq \mathbf{i} \bullet x \bullet y$ and $y = \mathbf{i} \bullet y \bullet \mathbf{t} \supseteq \mathbf{i} \bullet x \bullet y$, whence

$$x \cap y \supseteq \mathbf{i} \bullet x \bullet y \supseteq \mathbf{i} \bullet (x \cap y) \bullet (x \cap y) = x \cap y.$$

Also, for all x, y: $x = \mathbf{u} \bullet x \bullet \mathbf{b} \subseteq \mathbf{u} \bullet x \bullet y$ and $y = \mathbf{u} \bullet y \bullet \mathbf{b} \subseteq \mathbf{u} \bullet x \bullet y$ whence

$$x \cup y \subseteq \mathbf{u} \bullet x \bullet y \subseteq \mathbf{u} \bullet (x \cup y) \bullet (x \cup y) = x \cup y. \qquad \square$$

From proposition 2.7 it follows that also set-inclusion is $\mathcal{L}$-definable, as $x \subseteq y$ will be true in the graph models $\mathcal{M} = (\mathcal{P}(X), \bullet)$ iff $\mathcal{M} \models x \cap y = x$. This in turn leads us to the fact that we may express in $\mathcal{L}$ that $a \in \mathcal{P}(X)$ is a singleton: "*a is a singleton*" iff

$$\mathcal{M} \models \mathbf{One}(a),$$

where $\mathbf{One}(x)$ is the $\mathcal{L}$-formula

$$x \neq \emptyset \wedge \forall y (x \subseteq y \rightarrow y = x \vee y = \emptyset).$$

We therefore conclude that the atom-set $\mathcal{AT}(\mathcal{M})$ of a graph model $\mathcal{M} = (\mathcal{P}(X), \bullet)$ is $\mathcal{L}$-definable: it suffices (see lemma 2.3) to put

$$\mathbf{Atom}(a) \leftrightarrow \mathbf{One}(a) \wedge \forall x.ax = \emptyset.$$

From this we obtain

2.8 PROPOSITION: *Let $\mathcal{M}, \mathcal{N}$ be graphmodels and $\mathcal{M} \cong \mathcal{N}$. Then $|\mathcal{AT}(\mathcal{M})| = |\mathcal{AT}(\mathcal{N})|$.*

Proof: Let ψ be an applicative isomorphism $\mathcal{M} \hookrightarrow \mathcal{N}$. Then $\mathcal{M} \models \mathbf{Atom}(a)$ iff $\mathcal{N} \models \mathbf{Atom}(\psi(a))$, i.e. ψ induces a 1-1 mapping of $\mathcal{AT}(\mathcal{M})$ onto $\mathcal{AT}(\mathcal{N})$. $\square$

A characterization of isomorphisms between graph models

From the remarks above it is also immediate that isomorphisms of graphmodels $(\mathcal{P}(X), \bullet)$ and $(\mathcal{P}(Y), \star)$ are monotonic w.r.t. the set-inclusion relation, and induce a 1-1 mapping of the singleton-sets in $\mathcal{P}(X)$ onto the singleton-sets in $\mathcal{P}(Y)$. So we have

2.9 PROPOSITION: *Let $\psi : (\mathcal{P}(X), \bullet) \hookrightarrow (\mathcal{P}(X), \star)$ be an isomorphism of graph models. Then $\psi(x) = \bigcup \{\psi(\{a\}) \mid a \in x\}$, for all $x \in \mathcal{P}(X)$.* $\square$

We leave it to the reader to convince her/himself that in fact it follows that an isomorphism of graph models is a homeomorphism w.r.t. the Scott-topologies of the cpo's involved and that each such homeomorphism between $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(\mathcal{Y}), \subseteq)$ may be considered as being the natural extension of a bijective mapping $X \hookrightarrow Y$.

2.10 DEFINITION: If a 1-1 mapping of $\mathcal{P}(X)$ onto $\mathcal{P}(Y)$ canonically extends the bijection $\phi : X \hookrightarrow Y$ we will denote it by $\tilde{\phi}$. So in that case for all $x \in \mathcal{P}(X)$:

$$\tilde{\phi}(x) = \{\phi(a) \mid a \in x\}. \qquad \square$$

By proposition 2.9 we have that $\psi : (\mathcal{P}(X), \bullet) \hookrightarrow (\mathcal{P}(Y), *)$ is an applicative isomorphism only if $\psi = \tilde{\phi}$ for some 1-1 mapping ϕ of X onto Y.

Now let $\mathcal{M} = (\mathcal{P}(X), \bullet)$ be a graph model for which the application $\bullet$ is defined through an embedding $(\cdot, \cdot) : X^{<\omega} \times X \hookrightarrow X$.

Let $\mathcal{N} = (\mathcal{P}(Y), *)$ be a graph model for which the application $*$ is defined through an embedding $\langle \cdot, \cdot \rangle : Y^{<\omega} \times Y \hookrightarrow Y$.

The next theorem tells us that $\mathcal{M}$ and $\mathcal{N}$ are isomorphic as applicative structures iff there is a 1-1 mapping ϕ of X onto Y such that the following diagram commutes:

$$\begin{array}{ccccc} X^{<\omega} & \times & X & \xrightarrow{(\cdot,\cdot)} & X \\ \downarrow \tilde{\phi} & & \downarrow \phi & & \downarrow \phi \\ Y^{<\omega} & \times & Y & \xrightarrow{\langle \cdot,\cdot \rangle} & Y \end{array}$$

2.11 THEOREM: *A mapping* $\psi : (\mathcal{P}(X), \bullet) \hookrightarrow (\mathcal{P}(Y), *)$ *is an applicative isomorphism iff*

(i) $\psi = \tilde{\phi}$ *for some 1-1 mapping* ϕ *of* X *onto* Y;

(ii) $\forall \beta \in X^{<\omega} \forall b \in X. \quad \phi((\beta, b)) = \langle \tilde{\phi}(\beta), \phi(b) \rangle$.

Proof: ($\Leftarrow$) For all $M, N \in \mathcal{P}(X)$ we have:

$$\begin{aligned} \psi(M \bullet N) &= \tilde{\varphi}(M \bullet N) \\ &= \{\varphi(b) \mid \exists \beta \subseteq N.(\beta, b) \in M\} \\ &= \{\varphi(b) \mid \exists \tilde{\varphi}(\beta) \subseteq \tilde{\varphi}(N).\varphi((\beta, b)) \in \tilde{\varphi}(M)\} \\ &= \{\varphi(b) \mid \exists \tilde{\varphi}(\beta) \subseteq \tilde{\varphi}(N).\langle \tilde{\varphi}(\beta), \varphi(b) \rangle \in \tilde{\varphi}(M)\} \\ &= \tilde{\varphi}(M) * \tilde{\varphi}(N) = \psi(M) * \psi(N). \end{aligned}$$

($\Rightarrow$) Let ψ be an applicative isomorphism. We already saw that then necessarily $\psi = \tilde{\phi}$ for some 1-1 mapping ϕ of X onto Y .

Now suppose there is a $\beta \in X^{<\omega}$ such that there exists an element $b \in X$ for which $\phi((\beta, b)) \neq \langle \tilde{\phi}(\beta), \phi(b) \rangle$. Then there will be a finite set β having this property while every proper subset of β does not.

(i) First let $\phi((\beta,b))$ be in the range of $\langle\cdot,\cdot\rangle$, i.e. $\phi((\beta,b)) = \langle\alpha,c\rangle \neq \langle\tilde{\phi}(\beta),\phi(b)\rangle$. Now calculate:

$$\tilde{\phi}(\{(\beta,b)\} \bullet \beta) = \tilde{\phi}(\{b\}) = \{\phi(b)\}.$$

As $\tilde{\phi}$ is an applicative isomorphism we also have:

$$\tilde{\phi}(\{(\beta,b)\} \bullet \beta) = \{\langle\alpha,c\rangle\} * \tilde{\phi}(\beta) = \begin{cases} \{c\}, & \text{if } \alpha \subseteq \tilde{\phi}(\beta); \\ \emptyset, & \text{otherwise.} \end{cases}$$

If $c \neq \phi(b)$ this is a contradiction. So $c = \phi(b)$. But then α has to be a proper subset of $\tilde{\phi}(\beta)$, say $\alpha = \tilde{\phi}(\beta')$. Clearly, if $\tilde{\phi}(\beta')$ is a proper subset of $\tilde{\phi}(\beta)$, then β' is a proper subset of β. So $\phi((\beta',b)) = \langle\alpha,c\rangle = \phi((\beta,b))$, which contradicts the injectivity of ϕ.

(ii) If $\phi((\beta,b)) = a$, and a is not in the range of $\langle\cdot,\cdot\rangle$ (i.e. a is an atom), then $\tilde{\phi}(\{(\beta,b)\} \bullet \beta) = \{\phi(b)\}$, contradicting $\{a\} * \tilde{\phi}(\beta) = \emptyset$. □

2.12 COROLLARY: *The correspondence between embeddings $X^{<\omega} \times X \hookrightarrow X$ and the applications they define on $\mathcal{P}(X)$ is 1-1.*

Proof : For, suppose $(\cdot,\cdot)$ defines the application $\bullet$ on $\mathcal{P}(X)$, the application $*$ on $\mathcal{P}(X)$ is defined by $\langle\cdot,\cdot\rangle$ and $\bullet = *$. Then the identity mapping is an applicative isomorphism. So, by theorem 2.11, $\forall\alpha \in X^{<\omega}\forall a \in X.(\alpha,a) = \langle\alpha,a\rangle$. Therefore $(\cdot,\cdot) = \langle\cdot,\cdot\rangle$. □

The following now is immediate.

2.13 PROPOSITION: *For all non-empty sets A, B:*

$$\mathbf{D}_A \cong \mathbf{D}_B \quad \textit{iff} \quad |A| = |B|.$$

Proof: ($\Rightarrow$) As $\mathcal{AT}(\mathbf{D}_A) = A$ and $\mathcal{AT}(\mathbf{D}_B) = B$, this is just (a special case of) proposition 2.8. (Note that 2.8 may also be obtained as a corollary to theorem 2.11 : by 2.11 an applicative isomorphism $\mathcal{M} \hookrightarrow \mathcal{N}$ maps the pairs of $\mathcal{M}$ 1-1 onto the pairs of $\mathcal{N}$; consequently the isomorphism induces a 1-1 mapping of $\mathcal{AT}(\mathcal{M})$ onto $\mathcal{AT}(\mathcal{N})$.)
($\Leftarrow$) By definition of Engeler's graph model any 1-1 mapping of A onto B can be extended to a 1-1 mapping of $G(A)$ onto $G(B)$ in accordance with condition (ii) of theorem 2.11. This extension then obviously defines an applicative isomorphism. □

2.14 COROLLARY: *For all non-empty sets A, B such that* $\min\{|A|,|B|\}$ *is finite:*

$$\mathbf{D}_A \cong \mathbf{D}_B \quad \textit{iff} \quad \mathbf{D}_A \equiv \mathbf{D}_B \quad \textit{iff} \quad |A| = |B|.$$

Proof: By propositions 2.13 and 2.4. □

The second-order theory of an $\mathcal{L}$-definable element of a graph model is $\mathcal{L}$-expressable.

Corollary 2.14 leaves open the question of elementary equivalence of Engeler's graph models $\mathbf{D}_A$ for *infinite* atom-sets A.

Now note that, as our language $\mathcal{L}$ is countable, there are $2^{\aleph_0}$ ways to assign different first-order theories to $\mathbf{D}_A$. The collection of all $\mathbf{D}_A$'s forms a proper class, so there are certainly more than continuum-many non-isomorphic $\mathbf{D}_A$. We conclude that there have to be infinite sets A and B such that $\mathbf{D}_A \equiv \mathbf{D}_B$, but $\mathbf{D}_A \not\cong \mathbf{D}_B$.

It is however not the case that $\mathbf{D}_A$ and $\mathbf{D}_B$ are elementary equivalent for *all* infinite sets A and B: the construction of $\mathbf{D}_A$ as a powerset enables us to use first-order sentences over $\mathbf{D}_A$ to express essentially *second*-order properties of the set A.

Let us first, more generally, show that given a graph model $\mathcal{M} = (\mathcal{P}(X), \bullet)$ we can find representations in $\mathcal{M}$ for all n-ary relations over $\mathcal{L}$-definable subsets of X.

2.15. LEMMA: *Let $\mathcal{M} = (\mathcal{P}(X), \bullet)$ be a graph model with application defined through $(\cdot,\cdot)$, $n > 0$ and $R \subseteq B^n$, where $B \subseteq X$ is definable by some $\mathcal{L}$-formula $\phi_B(x)$ (i.e. $a \subseteq B$ iff $\mathcal{M} \models \phi_B(a)$). Then there is an element $\nu_R \in \mathcal{P}(X)$ such that*

$$\mathcal{M} \models \bigwedge_{i=1}^{n} \phi_B(a_i) \wedge \bigwedge_{i=1}^{n} \mathbf{One}(a_i) \wedge \nu_R \bullet a_1 \neq \emptyset \wedge \nu_R \bullet a_1 \bullet a_2 \neq \emptyset \wedge \ldots$$

$$\ldots \wedge \nu_R \bullet a_1 \bullet a_2 \bullet \ldots \bullet a_{n-1} \neq \emptyset \wedge \nu_R \bullet a_1 \bullet a_2 \bullet \ldots \bullet a_{n-1} \bullet a_n = \emptyset \qquad (4)$$

iff $a_i = \{b_i\}$ and $(b_1, \ldots, b_n) \in R$.

Proof: We proceed by induction on n. For $n = 1$, put

$$\nu_R = X \backslash \big\{(\{a\}, m), (\emptyset, m) \mid a \in R, m \in X\big\}.$$

Suppose our claim has been proved for $n > 0$ and let $R \subseteq X^{n+1}$. Then define

$$R_x := \{a_1 \mid \exists a_2, \ldots, a_{n+1}(a_1, a_2, \ldots, a_{n+1}) \in R\} \subseteq X,$$

$$R_y := \{(a_2, \ldots, a_{n+1}) \mid \exists a_1(a_1, a_2, \ldots, a_{n+1}) \in R\} \subseteq X^n.$$

By induction hypothesis there exists $\nu_{R_y} \in \mathcal{P}(X)$ satisfying (4). Now put

$$\nu_R := \big\{(\{a\}, m) \mid a \in R_x, m \in \nu_{R_y}\big\}.$$

□

Conversely, for each $k \in \mathcal{P}(X)$, we have that

$$\{(a_1,\ldots,a_n) \mid \mathcal{M} \models \bigwedge_{i=1}^{n} \phi_B(a_i) \wedge \bigwedge_{i=1}^{n} \mathbf{One}(a_i) \wedge k \bullet a_1 \neq \emptyset \wedge \ldots$$

$$\ldots \wedge k \bullet a_1 \bullet \ldots \bullet a_{n-1} \neq \emptyset \wedge k \bullet a_1 \bullet \ldots \bullet a_{n-1} \bullet a_n = \emptyset\}$$

determines a relation $R \subseteq B^n$.

Let $\mathcal{L}_2^-$ be the second-order language obtained from $\mathcal{L}^-$ (i.e. $\mathcal{L}$ *without* the binary function symbol for application) by adding for all $n > 0$, all $i \in \mathbb{N}$, second-order variables X_i^n as well as quantification over these variables. Let $B \subseteq X$ be definable by means of an $\mathcal{L}$-formula ϕ_B.

Now define $\mathcal{L}$-sentences $\mathcal{R}_B^n(x)$:

$$\mathcal{R}_B^n(x) := \forall y_1,\ldots,y_n\big(\bigwedge_{i=1}^{n} \phi_B(y_i) \wedge \bigwedge_{i=1}^{n} \mathbf{One}(y_i) \wedge x \bullet y_1 \neq \emptyset \wedge \ldots$$

$$\ldots \wedge x \bullet y_1 \bullet \ldots \bullet y_{n-1} \neq \emptyset \wedge x \bullet y_1 \bullet \ldots \bullet y_{n-1} \bullet y_n = \emptyset\big),$$

and a translation $(\cdot)^B : \mathcal{L}_2^- \to \mathcal{L}$ as follows :

$$\begin{aligned}
(x_i)^B &:= x_{2i} \\
(X_i^n)^B &:= \mathcal{R}_B^n(x_{2i+1}) \\
(x_i = x_j)^B &:= x_{2i} = x_{2j} \wedge \mathbf{One}(x_{2i}) \wedge \mathbf{One}(x_{2j}) \wedge \phi_B(x_{2i}) \wedge \phi_B(x_{2j}) \\
(X_i^n = X_j^n)^B &:= \mathcal{R}_B^n(x_{2i+1}) \leftrightarrow \mathcal{R}_B^n(x_{2j+1}) \\
(X_i^n(x_{i_1},\ldots,x_{i_n}))^B &:= \mathcal{R}_B^n(x_{2i+1})[(x_{i_1})^B/y_1,\ldots,(x_{i_n})^B/y_n] \\
(\chi \wedge \psi)^B &:= \chi^B \wedge \psi^B \\
(\neg\chi)^B &:= \neg(\chi)^B \\
(\forall x_i.\chi)^B &:= \forall(x_{2i})\big((\mathbf{One}(x_{2i}) \wedge \phi_B(x_{2i})) \to (\chi)^B\big) \\
(\forall X_i^n.\chi)^B &:= \forall x_{2i+1}(\chi)^B.
\end{aligned}$$

Then the following holds:

2.16 PROPOSITION: *Let $\mathcal{M} = (\mathcal{P}(X), \bullet)$ be a graph model and let $B \subseteq X$ be definable by means of an $\mathcal{L}$-formula ϕ_B. Then, for all $\mathcal{L}_2^-$-formula ψ,*

$$B \models \psi[R_1,\ldots,R_n,b_1,\ldots,b_m] \quad \text{iff} \quad \mathcal{M} \models (\psi)^B[\nu_{R_1},\ldots,\nu_{R_n},\{b_1\},\ldots,\{b_m\}],$$

where $R_i \subseteq B^{k_i}$ *for some* $k_i > 0, b_i \in B$ *and* ν_{R_i} *as in 2.15.*

Proof: Induction on ψ. □

In particular we have that the second-order theory of $\mathcal{L}$-definable elements of a graphmodel $\mathcal{M}$ is expressable in $\mathcal{M}$ by means of $\mathcal{L}$-sentences.

2.17 COROLLARY: *There are infinite sets* A, B *for which* $\mathbf{D}_A \not\equiv \mathbf{D}_B$.

Proof: Given lemma 2.15, it is routine to express in $\mathcal{L}$ that ν represents a dense linear ordering of the atom-set A and that ν has a gap (i.e. there is an $x \subseteq A$ which is bounded above, but has no ***least*** upper bound). As is well-known, every countable dense linear ordering has gaps, while there are dense linear orderings of the continuum without gaps; so taking A and B to be sets such that $|A| = \aleph_0$ and $|B| = 2^{\aleph_0}$ we clearly have $\mathbf{D}_A \not\equiv \mathbf{D}_B$. □

In any graph model $\mathcal{M} = (\mathcal{P}(X), \bullet)$ the infinite set X is $\mathcal{L}$-definable (proposition 2.7) and by proposition 2.16 we can embed the second-order theory of X in the first-order theory of $\mathcal{M}$. As the second-order theory of X is very complex (it contains e.g. true arithmetic), this gives us a glimpse of the complexity of the first-order theory of graph models.

Finally we note that, given some infinite set X and an embedding $(\cdot,\cdot) : X^{<\omega} \times X \hookrightarrow X$ that is second-order definable over X, we have a converse to proposition 2.16: in the *second-order* theory over X we may define (an isomorphic copy of) the graphmodel $\mathcal{M} = (\mathcal{P}(Y), \bullet)$ (with Y any set such that $|Y| = |X|$ and the application $\bullet$ defined by (some suitable variation on) the embedding $(\cdot,\cdot)$) in which we then may interpret the *first-order* theory of $\mathcal{M}$. This can be done e.g. for all models of the $\mathbf{D}_A$-type: (an isomorphic copy of) $\mathbf{D}_A$ is second-order definable over $G(A)$, or, for infinite A, even over the atom set A.

On the other hand, given some infinite X, for trivial reasons there are uncountably many embeddings $X^{<\omega} \times X \hookrightarrow X$ that are *not* second-order definable over X; so in general the converse to 2.16 will not hold.

3. $\mathcal{P}(\mathbb{N})$ - structures

A simulation of $\mathbf{D}_A$ in $\mathcal{P}(\mathbb{N})$

In the previous section we saw that for surjective codings p of natural numbers a $\mathcal{P}(\mathbb{N})$-structure $[p, e]$ can never be isomorphic to a graph model $\mathbf{D}_A$, for the simple reason that surjective codings result in atom-free models. However, non-surjective codings give rise to $\mathcal{P}(\mathbb{N})$-structures that are *not* atom-free. The next proposition shows that this enables us to consider Engeler's graph model $\mathbf{D}_A$, for countable A, as a $\mathcal{P}(\mathbb{N})$-structure.

3.1 PROPOSITION: *Let e be a basic coding of finite sets of natural numbers, and let $A \neq \emptyset$ be countable. There is a (non-surjective) coding q_A of pairs of natural numbers such that $\mathbf{D}_A \cong [q_A, e]$.*

Proof: (1) First let A be finite, say $A = \{a_0, a_1, \ldots, a_{k-1}\}$, ($a_i = a_j$ iff $i = j$). Define a mapping $q_k : \mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$ by

$$q_k(n,m) = p^\star(n,m) + k = \tfrac{1}{2}(n+m)(n+m+1) + m + k,$$

for all $n, m \in \mathbb{N}$. Note that the range of q_k is $\mathbb{N}\backslash\{0, 1, \ldots, k-1\}$.

Then define inductively a mapping $\psi : G(A) \hookrightarrow \mathbb{N}$ by

(i) $\psi(a_i) = i$, for all $a_i \in G_0(A)$;

(ii) Suppose ψ has been defined for all $x \in G_n(A)$. Then, for $x = (\beta, b) \in G_{n+1}(A)\backslash G_n(A)$, put $\psi((\beta, b)) = q_k\big(e^{-1}(\{\psi(x) \mid x \in \beta\}), \psi(b)\big)$.

An easy induction (note that $n, m < q_k(n,m)$ and use that e is a basic coding) shows that ψ is a well-defined 1-1 mapping of $G(A)$ onto $\mathbb{N}$.

$$\begin{array}{ccccccc}
G(A)^{<\omega} & = & G(A)^{<\omega} & \times & G(A) & \subset & G(A) \\
\downarrow \tilde{\psi} & & & & \downarrow \psi & & \downarrow \psi \\
\mathbb{N}^{<\omega} & \xleftarrow{e} & \mathbb{N} & \times & \mathbb{N} & \xrightarrow{q_k} & \mathbb{N}
\end{array}$$

By the definition of ψ the diagram commutes. So $\tilde{\psi}$ is an applicative isomorphism $\mathbf{D}_A \hookrightarrow [q_k, e]$ (theorem 2.11).

(2) Next let A be countably infinite, say $A = \{a_0, a_1, \ldots\}$, where $(a_i)_{i \in \mathbb{N}}$ is an enumeration of A without repetitions.

Define a mapping $q_\omega : \mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$ by $q_\omega(n,m) = p^\star(n,m) + n + m + 1$, for all $n, m \in \mathbb{N}$. Note that the range of q_ω is $\mathbb{N}\backslash\{\frac{1}{2}m(m+1) + m \mid m \in \mathbb{N}\}$.

Then define inductively a mapping $\psi : G(A) \hookrightarrow \mathbb{N}$ by

(i) $\psi(a_i) = \frac{1}{2}i(i+1) + i$, for all $a_i \in G_0(A)$;

(ii) Suppose ψ has been defined for all $x \in G_n(A)$. For $x = (\beta, b) \in G_{n+1}(A)\backslash G_n(A)$, put $\psi((\beta, b)) = q_\omega\big(e^{-1}(\{\psi(X) \mid x \in \beta\}), \psi(b)\big)$.

Again, by induction ($n, m < q_\omega(n,m)$ and e is a basic coding), it is easy to show that ψ is a well-defined 1-1 mapping of $G(A)$ onto $\mathbb{N}$, and by the definition of ψ and theorem 2.11 again $\tilde{\psi}$ is an applicative isomorphism $\mathbf{D}_A \hookrightarrow [q_\omega, e]$. □

Note that if $[p, e]$ is such that $|\mathcal{AT}([p,e])| = |A|$, for some non-empty countable set A, it will not always be the case that $[p, e]$ is isomorphic to $\mathbf{D}_A$: one easily constructs a non-surjective coding p such that $p(0, m) = m$ for some $m \in \mathbb{N}$. If $\tilde{\psi}$ were an applicative isomorphism $\mathbf{D}_A \hookrightarrow [p,e]$ (e a basic coding) this would mean that

$$m = p(0,m) = p(e^{-1}(\emptyset), m) = \psi(\emptyset, \psi^{-1}(m)),$$

so $\psi^{-1}(m) = (\emptyset, \psi^{-1}(m))$, which clearly is impossible in $\mathbf{D}_A$.

Isomorphism

By theorem 2.11 two $\mathcal{P}(\mathbb{N})$-structures $[p,e]$ and $[p',e']$ are isomorphic iff there are $\chi, \psi \in S(\mathbb{N})$ (i.e. the set of all 1-1 mappings of $\mathbb{N}$ onto $\mathbb{N}$), such that the following diagram commutes:

$$\begin{array}{ccccccc}
\mathbb{N}^{<\omega} & \xleftarrow{e} & \mathbb{N} & \times & \mathbb{N} & \xrightarrow{p} & \mathbb{N} \\
\downarrow{\scriptstyle\tilde{\psi}} & & \downarrow{\scriptstyle\chi} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\
\mathbb{N}^{<\omega} & \xleftarrow{e'} & \mathbb{N} & \times & \mathbb{N} & \xrightarrow{p'} & \mathbb{N}
\end{array}$$

The next proposition just spells out the conditions corresponding to the commuting of the diagram.

3.2 PROPOSITION: $[p,e] \cong [p',e']$ *iff there are* $\chi, \psi \in S(\mathbb{N})$ *such that*
(i) $\forall n,m.\ \ p'(\chi(n), \psi(m)) = \psi(p(n,m))$,
(ii) $\forall n.\ \ e'_{\chi(n)} = \{\psi(x) \mid x \in e_n\}$. □

The mapping χ accounts for a certain amount of redundancy: whereas the correspondence between *embeddings* $\mathbb{N}^{<\omega} \times \mathbb{N} \hookrightarrow \mathbb{N}$ and the applications they define on $\mathcal{P}(\mathbb{N})$ is 1-1 (see corollary 2.12), this is not true for the correspondence between applications and *pairs of codings*: different codings may very well induce the same embedding $\mathbb{N}^{<\omega} \times \mathbb{N} \hookrightarrow \mathbb{N}$.

3.3 DEFINITION: If codings p, e and p', e' induce the same embedding $\mathbb{N}^{<\omega} \times \mathbb{N} \hookrightarrow \mathbb{N}$ (i.e. whenever the identity mapping is an isomorphism of the corresponding $\mathcal{P}(\mathbb{N})$-structures) we call $[p,e]$ and $[p',e']$ *compatible.* (Notation: $[p,e] \sim [p',e']$). □

This situation is represented in the following commuting diagram:

$$\begin{array}{ccccccc}
\mathbb{N}^{<\omega} & \xleftarrow{e} & \mathbb{N} & \times & \mathbb{N} & \xrightarrow{p} & \mathbb{N} \\
\| & & \downarrow{\scriptstyle\chi} & & \| & & \| \\
\mathbb{N}^{<\omega} & \xleftarrow{e'} & \mathbb{N} & \times & \mathbb{N} & \xrightarrow{p'} & \mathbb{N}
\end{array}$$

Clearly compatibility of $\mathcal{P}(\mathbb{N})$-structures just boils down to a re-indexing of the finite sets:

3.4 PROPOSITION: $[p,e] \sim [p',e']$ *iff there is a* $\chi \in S(\mathbb{N})$ *such that for all* $n,m \in \mathbb{N}$:
(i) $e'_n = e_{\chi(n)}$,
(ii) $p'(n,m) = p(\chi(n),m)$. □

Proposition 3.4 shows that the indexing of the finite sets is not very relevant for the properties of the corresponding $\mathcal{P}(\mathbb{N})$-structure, in the sense that given any model $[p,e]$ and any bijective coding e' of finite sets there is a coding p' of pairs such that $[p',e'] \sim [p,e]$. (Note however that the properties of p' may be very different from those of p: e.g. if p is a basic coding generally p' is not.)

The same is not true for the coding of pairs: given $[p,e]$ and a coding p' we may have that $[p,e] \not\cong [p',e']$, for all e'. We will encounter an example below.

It also follows from 3.4 that, given some $[p,e]$, there are conti-nuum-many different $\mathcal{P}(\mathbb{N})$-structures compatible with $[p,e]$ ('different' meaning that they are defined by means of different codings). As the remainder of this section will show, in fact we have *for each* $\psi \in S(\mathbb{N})$ continuum-many different $\mathcal{P}(\mathbb{N})$-structures $[p',e']$ such that $\tilde{\psi}$ is an isomorphism $[p,e] \hookrightarrow [p',e']$.

3.5 DEFINITION: Let $[p,e]$ be some $\mathcal{P}(\mathbb{N})$-structure. For each $\psi \in S(\mathbb{N})$ we define $[p^\psi, e^\psi]$ by

$$\forall n,m \in \mathbb{N}. \quad p^\psi(n,m) = \psi(p(n,\psi^{-1}(m)))$$

and

$$\forall n \in \mathbb{N}. \quad e^\psi_n = \tilde{\psi}(e_n) = \{\psi(x) \mid x \in e_n\}.$$ □

Clearly p^ψ and e^ψ are well-defined. Observe that $a \in \mathcal{AT}([p,e])$ iff $\psi(a) \in \mathcal{AT}([p^\psi,e^\psi])$. By proposition 3.2 $\tilde{\psi}$ is an applicative isomorphism $[p,e] \hookrightarrow [p^\psi,e^\psi]$.

Now suppose $\tilde{\phi}$ is an applicative isomorphism $[p,e] \hookrightarrow [p',e']$. Let $\diamond$ be the application on $\mathcal{P}(\mathbb{N})$ defined through p^ϕ, e^ϕ; let the application defined through p,e be $\bullet$, and $*$ the application defined through p',e'. Then, for all $a,b \in \mathcal{P}(\mathbb{N}) : a \diamond b = \tilde{\phi}(\tilde{\phi}^{-1}(a) \bullet \tilde{\phi}^{-1}(b)) = a * b$, i.e. $[p^\phi,e^\phi] \sim [p',e']$.

This gives us

3.6 PROPOSITION: $[p,e] \cong [p',e']$ *iff* $[p',e'] \sim [p^\psi,e^\psi]$ *for some* $\psi \in S(\mathbb{N})$. □

Frames

In section 2 we showed that members of the class of Engeler's graph models $\mathbf{D}_A$ are up to isomorphism determined by the cardinality of their atom-sets. We cannot possibly expect to have the same characterization for the collection of all $\mathcal{P}(\mathbb{N})$-structures. Of course it is necessary for isomorphism that $|\mathcal{AT}([p,e])| = |\mathcal{AT}([p',e'])|$, but in general this will be far from sufficient; indeed, it is an easy exercise to produce codings p and p' such that $|\mathcal{AT}([p,e^\star])| = |\mathcal{AT}([p',e^\star])|$, but for no $\chi, \psi \in S(\mathbb{N})$ conditions (i) and (ii) of proposition 3.2 are fulfilled (see e.g. the remarks following 3.1).

3.7 DEFINITION: Let $[p,e]$ be some $\mathcal{P}(\mathbb{N})$-structure. For all $k \geq 1$ and each $\overline{n}_k := (n_1, \ldots, n_k) \in \mathbb{N}^k$ we define the *$\overline{n}_k$-frame* $\mathcal{F}^{[p,e]}_{\overline{n}_k}$ of $[p,e]$ by

$(x_1, \ldots, x_k, y) \in \mathcal{F}^{[p,e]}_{\overline{n}_k} \subseteq \mathbb{N}^{k+1}$ iff
- (i) $p(x_1, \ldots, x_k, y) = y$;
- (ii) for all $1 \leq i \leq k : |e_{x_i}| = (\overline{n}_k)_i$;
- (iii) $p(z_j, z_{j+1}, \ldots, z_{j+(k-j)}, y) \neq y$, for all $1 < j \leq k$, for all $0 \leq i \leq k-j$, for all z_{j+i}.

By $\mathbf{X}_i\mathcal{F}^{[p,e]}_{\overline{n}_k}$ $(1 \leq i \leq k)$ and $\mathbf{Y}\mathcal{F}^{[p,e]}_{\overline{n}_k}$ we denote the $k+1$ projections of the $\overline{n}_k$-frame of $[p,e]$:

$$\mathbf{X}_i\mathcal{F}^{[p,e]}_{\overline{n}_k} := \{x_i \mid (x_1, \ldots, x_i, \ldots, x_k, y) \in \mathcal{F}^{[p,e]}_{\overline{n}_k}\};$$
$$\mathbf{Y}\mathcal{F}^{[p,e]}_{\overline{n}_k} := \{y \mid (x_1, \ldots, x_k, y) \in \mathcal{F}^{[p,e]}_{\overline{n}_k}\}.$$

By the *frame* $\mathcal{F}^{[p,e]}$ of $[p,e]$ we mean the union of all $\overline{n}_k$-frames and the atom-set:

$$\mathcal{F}^{[p,e]} := \bigcup_k \bigcup_{\overline{n}_k} \mathcal{F}^{[p,e]}_{\overline{n}_k} \quad \cup \quad \mathcal{AT}([p,e]).$$

We will write $\mathbf{Y}\mathcal{F}^{[p,e]}, \mathbf{X}\mathcal{F}^{[p,e]}$ for the obvious unions of projections. □

Observe that for all $\overline{n}_k$, all $1 \leq i \leq k$:

$$|\mathbf{X}_i\mathcal{F}^{[p,e]}_{\overline{n}_k}| \leq |\mathbf{Y}\mathcal{F}^{[p,e]}_{\overline{n}_k}| = |\mathcal{F}^{[p,e]}_{\overline{n}_k}|.$$

Also, if $\overline{n}_k \neq \overline{m}_{k'}$, then $\mathbf{Y}\mathcal{F}^{[p,e]}_{\overline{n}_k} \cap \mathbf{Y}\mathcal{F}^{[p,e]}_{\overline{m}_{k'}} = \emptyset$.

Note, that the *set* $\mathcal{F}^{[p,e]}$ is independent of the coding e of finite sets. Still we will always explicitly mention e; thus we implicitly provide $\mathcal{F}^{[p,e]}$ with a 'memory' of its constituents.

The frame of a $\mathcal{P}(\mathbb{N})$-structure may be empty, it may be very big, all depending on the specific coding at hand. As an example the reader may verify that for $\mathcal{P}\omega$ $(= [p^\star, e^\star])$ we have $\mathcal{F}^{[p^\star,e^\star]} = \{(0,0)\}$.

The next lemma explains our interest in these frames.

3.8 LEMMA: *Let $\tilde{\psi} : [p,e] \hookrightarrow [p',e']$ be an isomorphism. Then, for all $k \geq 1$, all $\overline{n}_k \in \mathbb{N}^k$:*

$$|\mathbf{X}_i \mathcal{F}^{[p,e]}_{\overline{n}_k}| = |\mathbf{X}_i \mathcal{F}^{[p',e']}_{\overline{n}_k}|, \quad 1 \leq i \leq k;$$
$$|\mathbf{Y} \mathcal{F}^{[p,e]}_{\overline{n}_k}| = |\mathbf{Y} \mathcal{F}^{[p',e']}_{\overline{n}_k}|;$$

$$\forall x \in \mathbf{X}_i \mathcal{F}^{[p,e]}_{\overline{n}_k} : (e')^{-1}\{\psi(y) \mid y \in e_x\} \in \mathbf{X}_i \mathcal{F}^{[p',e']}_{\overline{n}_k}.$$

Proof: Take $(x_1, \ldots, x_k, y) \in \mathcal{F}^{[p',e']}_{\overline{n}_k}$, i.e. $p(x_1, \ldots, x_k, y) = y$ and $|e_{x_i}| = (\overline{n}_k)_i$, for all $1 \leq i \leq k$.
Let $\chi(x_i) := (e')^{-1} \circ \tilde{\psi} \circ e(x_i)$. As $\tilde{\psi}$ is an isomorphism we then have $\psi(y) = p'(\chi(x_1), \ldots, \chi(x_k), \psi(y))$ and $e'_{\chi(x_i)} = \{\psi(y) \mid y \in e_{x_i}\}$.
From this the claims of the lemma follow directly. □

So the importance of the frame of a $\mathcal{P}(\mathbb{N})$-structure is that it determines sets which, like the sets of atoms, are in a way invariant under isomorphic mappings. Comparing frames of $\mathcal{P}(\mathbb{N})$-structures therefore provides us with a means to quickly refute the existence of isomorphisms.

As an example, consider the code r, obtained, like $p^\star$, by enumerating $\mathbb{N} \times \mathbb{N}$ along the diagonals, but in the opposite direction:

$$r(n,m) = \tfrac{1}{2}(n+m)(n+m+1) + n.$$

One easily checks that $\mathcal{F}^{[r,e]} = \{(0,0),(0,1)\}$, for any bijective coding e of finite sets. As on the other hand we have that $\mathcal{F}^{[p^\star,e']} = \{(0,0)\}$ for all bijective codings e' of finite sets, we conclude , by lemma 3.8, that for all e, e': $[p^\star, e'] \not\cong [r, e]$.

We will now apply lemma 3.8 to produce continuum-many non-isomorphic $\mathcal{P}(\mathbb{N})$-structures, so the number of non-isomorphic $\mathcal{P}(\mathbb{N})$-structures is *at least* $2^{\aleph_0}$. As this number also obviously is *at most* $2^{\aleph_0}$ we have

3.9 PROPOSITION: *There are $2^{\aleph_0}$ non-isomorphic $\mathcal{P}(\mathbb{N})$-structures.*

Proof: We'll exhibit $2^{\aleph_0}$ $\mathcal{P}(\mathbb{N})$-structures with different frames. For this there are many possibilities. One of them is the following:

Take $e^\star$ for the coding of finite sets.

Let $\{\pi_i \mid i \in \mathbb{N}\}$ be the set of all prime-numbers without repetition. Define a mapping $s : \mathbb{N} \hookrightarrow \mathbb{N}$ by

$$s(n) = \begin{cases} 0, & \text{if } n = 0; \\ \sum_{i=0}^{n-1} 2^{\pi_i}, & \text{if } n > 0. \end{cases}$$

Note that $|e^\star_{s(n)}| = n$, for all $n \in \mathbb{N}$.

Let A be any subset of $\mathbb{N}_{>0}$. Now define a coding p_A of pairs as follows:

* $p_A(0,0) = 0$;
* if $a \in A$ put $p_A(s(a), s(a)) = s(a)$;
* for the remaining pairs (n, m) and numbers k put $p_A(n, m) = k$ in such a way that $n \leq k$ and $m < k$.

For each subset A of $\mathbb{N}_{>0}$ the resulting code p_A will be a (basic) coding of pairs of natural numbers as natural numbers. Now, if $A, B \subseteq \mathbb{N}_{>0}$ and $A \neq B$, then we may assume that there is a number $b \in B \backslash A$. But by definition then we have that $\mathbf{Y}\mathcal{F}_b^{[p_B, e^\star]} = \{s(b)\}$, while $\mathbf{Y}\mathcal{F}_b^{[p_A, e^\star]} = \emptyset$. Therefore, by 3.8, $[p_A, e^\star] \not\cong [p_B, e^\star]$, which finishes the proof. □

$\mathcal{P}(\mathbb{N})$-structures through basic codings

The sets 'invariant' under isomorphic images as provided by the frame of a $\mathcal{P}(\mathbb{N})$-structure suggest the idea of trying to characterize $\mathcal{P}(\mathbb{N})$-structures up to isomorphism precisely by their frames. For this we would have to prove some kind of converse to lemma 3.8. Clearly then first of all it is necessary to be able to define an isomorphism 'near the frames'. This notion is made precise in the following definition.

3.10 DEFINITION: Let $[p, e], [p', e']$ be $\mathcal{P}(\mathbb{N})$ -structures. We will call the frames $\mathcal{F}^{[p,e]}$ and $\mathcal{F}^{[p',e']}$ *similar* (written as $\mathcal{F}^{[p,e]} \sim \mathcal{F}^{[p',e']}$) iff there exist (possibly not everywhere defined) 1-1 mappings ψ and χ such that

(o) ψ is defined on $\mathcal{AT}([p,e])$ and onto $\mathcal{AT}([p',e'])$;

(i) ψ is defined on $\mathbf{Y}\mathcal{F}^{[p,e]}$ and onto $\mathbf{Y}\mathcal{F}^{[p',e']}$; χ is defined on $\mathbf{X}\mathcal{F}^{[p,e]}$;

(ii) for all i, all $x \in \mathbf{X}_i\mathcal{F}^{[p,e]}_{\overline{n}_k}$:

$\psi(y)$ is defined for all $y \in e_x$ and $\chi(x) = (e')^{-1}\{\psi(y) \mid y \in e_x\} \in \mathbf{X}_i\mathcal{F}^{[p',e']}_{\overline{n}_k}$;

(iii) if $\psi(n)$ is defined and $n = p(x_1, \ldots, x_k, m)$, then $\chi(x_i)$ $(1 \leq i \leq m)$ and $\psi(m)$ are defined and $\psi(n) = p'(\chi(x_1), \ldots, \chi(x_k), \psi(m))$.

We say that ψ and χ *witness the similarity* of $\mathcal{F}^{[p,e]}$ and $\mathcal{F}^{[p',e']}$. □

The reader may verify that from (i) - (iii) it follows that χ is onto $\mathbf{X}\mathcal{F}^{[p',e']}$ and $m \in \mathbf{Y}\mathcal{F}^{[p,e]}_{\overline{n}_k}$ implies $\psi(m) \in \mathbf{Y}\mathcal{F}^{[p',e']}_{\overline{n}_k}$.

A converse to lemma 3.8 should state that a partial isomorphism as given by frame-similarity of the $\mathcal{P}(\mathbb{N})$-structures $[p, e]$ and $[p', e']$ can be extended to an isomorphism. Now, unfortunately, in general this is not true. Though frame similarity of course is necessary for isomorphism (if $\tilde{\psi}$ is an isomorphism $[p, e] \hookrightarrow [p', e']$ then ψ and $(e')^{-1} \circ \tilde{\psi} \circ e$ witness the similarity of $\mathcal{F}^{[p,e]}$ and $\mathcal{F}^{[p',e']}$), it still is not sufficient. The reason is that there are 'invariants' not captured by our concept of frame. We will clarify this by giving an example. From this it also will be clear that extending the concept in order to capture *all* possible invariants really is not feasible.

Let $[p, e]$ be a $\mathcal{P}(\mathbb{N})$-structure of which the code p of pairs includes the following data:

$$p(2,3) = 1, p(1,3) = 2, p(4,2) = 3.$$

Then p gives rise to the repetitive pattern

$$1 = p(2,3) = p(2,4,2) = p(2,4,1,3) = p(2,4,1,4,2) = p(2,4,1,4,1,3) = \dots$$

Let $[p', e']$ be a second $\mathcal{P}(\mathbb{N})$-structure and suppose $\psi, \chi \in S(\mathbb{N})$ satisfy the conditions of proposition 3.2 . Then the reader may easily convince him/herself that this pattern is mapped to a similar 'p'-pattern': the sets of all such 'cycles' also will in some sense be invariant under isomorphic mappings. These invariants are not accounted for by our concept of frame.

On the other hand, if we restrict our attention to a collection of codes that are moderately well-behaved, then we *do* have the desired converse to 3.8. And even more.

For this we will consider the collection of $\mathcal{P}(\mathbb{N})$-structures defined by means of *basic codings*, i.e. the bijective codings e of finite sets satisfy $x \in e_k \Longrightarrow x < k$, and for the codings p of pairs we have $m, n \leq p(n, m)$ (see the introduction).

Observe that restricting ourselves to basic codings excludes the occurence of the above-mentioned repetitions. Also, $\mathcal{P}(\mathbb{N})$-structures $[p, e]$ defined through basic codings have comparatively small frames: if $k > 1$, then $\mathcal{F}^{[p,e]}_{\overline{n}_k} = \emptyset$; for, suppose $p(x_1, \dots, x_k, y) = y$, then $p(x_2, \dots, x_k, y) \leq y$ implies $p(x_2, \dots, x_k, y) = y$. Therefore, in fact already $p(x_k, y) = y$.

3.11 THEOREM: *Let $[p, e]$ and $[p', e']$ be defined through basic codings. Then*

$$[p, e] \cong [p', e'] \quad \textit{iff} \quad \mathcal{F}^{[p,e]} \sim \mathcal{F}^{[p',e']}.$$

Moreover, every isomorphism is induced by a unique extension of witnesses of frame similarity.

Proof: As we already made ample remarks as to ($\Rightarrow$), we will confine ourselves to a detailed proof of ($\Leftarrow$).

Let the (partial) mappings ψ and χ witness frame-similarity of the $\mathcal{P}(\mathbb{N})$-structures $[p, e]$ and $[p', e']$.

If $\psi(x)$ is defined, put $\phi(x) := \psi(x)$; if $\chi(x)$ is defined, put $\xi(x) := \chi(x)$.

We will show that the fact that $[p, e]$ and $[p', e']$ are defined through basic codings allows us to extend ϕ and ξ inductively to mappings in $S(\mathbb{N})$, in a way that is uniquely determined by the conditions given in proposition 3.2 as being necessary and sufficient for isomorphism of $[p, e]$ and $[p', e']$.

(1) First we show that $\phi(n)$ and $\xi(n)$ can be defined in accordance with proposition 3.2 for all $n \in \mathbb{N}$, in a unique way.

As p is a basic coding either $(0,0)$ or $0 \in \mathcal{F}^{[p,e]}$. So $\xi(0) = 0 (= \chi(0))$ and $\phi(0) = \psi(0)$, which is in accordance with 3.2 by definition of frame similarity. Suppose now we already succeeded in defining $\phi(k)$ and $\xi(k)$ for all $k < n$. Then we may assume that $\psi(n)$ is not defined (for otherwise $\phi(n) = \psi(n)$, etcetera). But then $n \notin \mathbf{Y}\mathcal{F}^{[p,e]}$, so $n = p(n_1, n_2)$ for some $n_2 < n$ and $\phi(n_2)$ is defined by induction hypothesis. Also we may assume that $\chi(n)$ is not defined (otherwise $\xi(n) = \chi(n)$, etcetera). Then, in order to keep up with the conditions of proposition 3.2, we have to put

$$\xi(n) := (e')^{-1}\{\phi(y) \mid y \in e_n\},$$

which is well-defined by induction hypothesis, as $y \in e_n$ implies $y < n$. As $n_1 \leq n$ we therefore have that $\xi(n_1)$ is defined, either by induction hypothesis or because $n_1 = n$. To go on in accordance with 3.2 we necessarily have to put

$$\phi(n) := p'(\xi(n_1), \phi(n_2)).$$

(2) Next we show that for the mappings ϕ and ξ thus defined we have, for all n, for all $k, k' \leq n$:

(i) $\xi(k) = \xi(k') \vee \phi(k) = \phi(k') \Longrightarrow k = k'$;

(ii) $\phi(k') \in \mathbf{Y}\mathcal{F}^{[p',e']} \Longrightarrow k' \in \mathbf{Y}\mathcal{F}^{[p,e]}$.

We once more proceed by induction. The claims are trivial for $n = 0$. Therefore let $n > 0$ and suppose (i) and (ii) have been proven for all $k, k' < n$. Now let $k, k' \leq n$. From $\xi(k) = \xi(k')$ we have that

$$e'_{\xi(k)} = \{\phi(z) \mid z \in e_k\} = \{\phi(y) \mid y \in e_{k'}\} = e'_{\xi(k')}.$$

But, as $a \in e_k$ implies $a < k$, we find $k = k'$ by induction hypothesis and injectivity of e'.

Now let $\phi(k') \in \mathbf{Y}\mathcal{F}^{[p',e']}$. Suppose $k' = p(n,y) \notin \mathbf{Y}\mathcal{F}^{[p,e]}$; then $\phi(k') = p'(\xi(n), \phi(y)) = \phi(y) \in \mathbf{Y}\mathcal{F}^{[p',e']}$ and $y < k'$. Therefore $y \in \mathbf{Y}\mathcal{F}^{[p,e]}$ by induction hypothesis; let's say that $y = p(m,y)$.

Then $\phi(y) = p'(\xi(m), \phi(y)) = p'(\xi(n), \phi(y))$, so $\xi(m) = \xi(n)$. This implies $m = n$ by the preceding part of our proof. So $k' = y$, contradicting our assumption.

Finally suppose $\phi(k) = \phi(k')$.

($\star$) If $k \in \mathcal{AT}([p,e])$ or $k' \in \mathcal{AT}([p,e])$, then $\phi(k) = \phi(k') \in \mathcal{AT}([p',e'])$. It is easy to see that this is possible only if $\phi(k) = \psi(k)$ and $\phi(k') = \psi(k')$; then also $\psi(k) = \psi(k') \in \mathcal{AT}([p',e'])$, so $k = k'$ by injectivity of ψ.

($\star$) If $k \in \mathbf{Y}\mathcal{F}^{[p,e]}$ or $k' \in \mathbf{Y}\mathcal{F}^{[p,e]}$ then $\phi(k) = \phi(k') \in \mathbf{Y}\mathcal{F}^{[p',e']}$. By (ii) then k *and* $k' \in \mathbf{Y}\mathcal{F}^{[p,e]}$. Therefore $\phi(k) = \psi(k)$, $\phi(k') = \psi(k')$ and $\psi(k) = \psi(k')$, so again $k = k'$ by injectivity of ψ.

($\star$) Otherwise $k = p(k_1, k_2), k' = p(z_1, z_2)$ and $k_2 < k, z_2 < k'$. The result then follows by induction hypothesis and the first part.

(3) We finish the proof by showing that ϕ and ξ are onto $\mathbb{N}$.

In order to do so, suppose there is an $n \in \mathbb{N}$ such that $\phi(k) \neq n$, for all $k \in \mathbb{N}$. Let n be minimal with that property. Then of course $n = p'(n_1, n_2)$ for some $n_1, n_2 \in \mathbb{N}$. Also necessarily $n_2 < n$, so by minimality of n we have a k_2 such that $\phi(k_2) = n_2$. Suppose $e'_{n_1} = \{x_1, \ldots, x_m\}$. For $x_i \in e'_{n_1}$ we have $x_i < n_1 \leq n$, so again by minimality of n there are y_i such that $\phi(y_i) = x_i$, for all $1 \leq i \leq m$. Say $\{y_1, \ldots, y_m\} = e_{k_1}$. As ξ and ϕ have been defined in accordance with the conditions given by proposition 3.2 we have $e'_{\xi(k_1)} = \{\phi(y) \mid y \in e_{k_1}\} = e'_{n_1}$. So $\xi(k_1) = n_1$ and $\phi(p(k_1, k_2)) = p'(n_1, n_2) = n$. This is a contradiction.

The surjectivity of ξ follows directly from that of ϕ. □

Automorphisms

Applications of theorem 3.11 can be found in the proofs of the following two propositions on automorphisms of $\mathcal{P}(\mathbb{N})$-structures.

Obviously, for all codings p and e, the identity mapping is an applicative automorphism of $[p, e]$. For some it is also the only one.

3.12 PROPOSITION: *There are no non-trivial automorphisms of $\mathcal{P}\omega$.*

Proof: $\mathcal{P}\omega$ is defined through the basic codings $p^\star$ and $e^\star$. Furthermore we have that $\mathcal{F}^{[p^\star, e^\star]} = \{(0,0)\}$. The partial mappings χ and ψ defined by $\chi(0) = \psi(0) = 0$ witness the similarity of $\mathcal{F}^{[p^\star, e^\star]}$ and $\mathcal{F}^{[p^\star, e^\star]}$. Any automorphism of $\mathcal{P}\omega$ necessarily extends these witnesses. By theorem 3.11 such an extension is unique. □

On the other hand it is certainly not true that all $\mathcal{P}(\mathbb{N})$-structures have trivial automorphism-groups. Let S_n, for $n \in \mathbb{N}_{>0}$, denote the symmetric group of n elements, and write $Aut([p, e])$ for the automorphism-group of a $\mathcal{P}(\mathbb{N})$-structure $[p, e]$. We then have

3.13 PROPOSITION: *Suppose $\mathcal{G}$ is an arbitrary direct product $\prod_{i=1}^{\infty} G_i$ of groups G_i such that each factor G_i is either (isomorphic to) S_n for some $n \in \mathbb{N}_{>0}$, or (isomorphic to) $S(\mathbb{N})$. Then there is a basic coding $p_\mathcal{G}$ such that $Aut([p_\mathcal{G}, e^\star]) \cong \mathcal{G}$.*

Proof: We apply a strategy similar to the one followed in the proof of proposition III.9. Again, $\{\pi_i \mid i \in \mathbb{N}\}$ is the set of all prime-numbers without repetition and s a mapping $\mathbb{N} \hookrightarrow \mathbb{N}$ defined by

$$s(n) = \begin{cases} 0, & \text{if } n = 0; \\ \displaystyle\sum_{i=0}^{n-1} 2^{\pi_i}, & \text{if } n > 0. \end{cases}$$

Let $\mathcal{G} = \prod_{i=1}^{\infty} G_i$ be fixed.

To produce a code $p_\mathcal{G}$ such that $Aut([p_\mathcal{G}, e^\star]) \cong \mathcal{G}$ we proceed as follows:

- $\star$ $p_\mathcal{G}(0,0) = 0$; $p_\mathcal{G}(0, \pi_j) = \pi_j$, for all $j \in \mathbb{N}$;
- $\star$ then, for all $i \in \mathbb{N}_{>0}$:
  if $G_i \cong S_n$, put $p_\mathcal{G}\big(s(i), s(i) + \pi_j\big) = s(i) + \pi_j$, for $j = 0, \ldots, n-1$;
  if $G_i \cong S(\mathbb{N})$, put $p_\mathcal{G}\big(s(i), s(i) + \pi_j\big) = s(i) + \pi_j$, for all $j \in \mathbb{N}$;
- $*$ for the remaining pairs (n, m) and numbers k put $p_A(n, m) = k$ in such a way that $n \leq k$ and $m < k$.

Obviously the resulting code $p_\mathcal{G}$ will be a basic coding of pairs of natural numbers as natural numbers and, as $|e^\star_{s(n)}| = n$, we have $\mathbf{X}\mathcal{F}_n^{[p_\mathcal{G}, e^\star]} = \{s(n)\}$, for each $n \in \mathbb{N}$.

Define $\xi : \mathbf{Y}\mathcal{F}^{[p_\mathcal{G}, e^\star]} \hookrightarrow \mathbf{Y}\mathcal{F}^{[p_\mathcal{G}, e^\star]}$ through arbitrary permutations of $\mathbf{Y}\mathcal{F}_n^{[p_\mathcal{G}, e^\star]}$ for $n \geq 1$ and by $\xi(x) = x$ for all $x \in \mathbf{Y}\mathcal{F}_0^{[p_\mathcal{G}, e^\star]}$; then

$$(e^\star)^{-1}\{\xi(y) \mid y \in e^\star_{s(i)}\} = (e^\star)^{-1}\{\xi(\pi_0), \ldots, \xi(\pi_{i-1})\} = s(i).$$

Put $\chi(x) = x$ for all $x \in \mathbf{X}\mathcal{F}^{[p_\mathcal{G}, e^\star]}$. It now is clear that ξ and χ witness the similarity of $\mathcal{F}^{[p_\mathcal{G}, e^\star]}$ and $\mathcal{F}^{[p_\mathcal{G}, e^\star]}$. By theorem 3.11 we can extend ξ and χ uniquely to an automorphism of $[p_\mathcal{G}, e^\star]$.

Conversely, as $\mathbf{X}\mathcal{F}_n^{[p_\mathcal{G}, e^\star]} = \{s(n)\}$, given any automorphism $\tilde{\phi}$, the witness $(e^\star)^{-1} \circ \tilde{\phi} \circ e^\star$ necessarily maps $s(n)$ to $s(n)$, for all $n \in \mathbb{N}$; it is easy to see that this is possible only if $\phi(x) = x$ for all $x \in \mathbf{Y}\mathcal{F}_0^{[p_\mathcal{G}, e^\star]}$. Also, $\tilde{\phi}$ will induce a permutation of $\mathbf{Y}\mathcal{F}_n^{[p_\mathcal{G}, e^\star]}$, for all $n \geq 1$. □

In what follows we will characterize the mappings ψ of $\mathbb{N}$ onto $\mathbb{N}$ such that $\tilde{\psi}$ is an applicative automorphism of $[p, e]$, for some codings p and e.

3.14 DEFINITION: Let ψ be any bijective mapping $\mathbb{N} \hookrightarrow \mathbb{N}$ and $k \in \mathbb{N}$. By the *ψ-order of* k (notation: $\lfloor k \rfloor_\psi$) we will mean the smallest natural number $n > 0$ such that $\psi^n(k) = k$; if such a number does not exist we put $\lfloor k \rfloor_\psi = \infty$. An *$n$-orbit in* ψ is any set $\{\psi^k(a) \mid k \in \mathbb{N}\}$, where $a \in \mathbb{N}$ and $\lfloor a \rfloor_\psi = n \quad (n \in \mathbb{N}_{>0}) \cup \{\infty\})$.

□

3.15 LEMMA: *Suppose for some codings p and e the mapping $\tilde{\psi}$ is an applicative automorphism of $[p, e]$. Then, for all $n \in \mathbb{N}_{>0} \cup \{\infty\}$, there is an n-orbit in ψ iff there are infinitely many distinct n-orbits in ψ.*

Proof: If $\tilde{\psi}$ is an applicative automorphism of $[p, e]$, then, by proposition 3.2, there is a bijective mapping $\chi : \mathbb{N} \hookrightarrow \mathbb{N}$ such that

$$\forall n \in \mathbb{N}. e_{\chi(n)} = \{\psi(x) \mid x \in e_n\}, \tag{1}$$

$$\forall n, m \in \mathbb{N}. \psi(p(n, m)) = p(\chi(n), \psi(m)). \tag{2}$$

Then clearly for all $k \in \mathbb{N}$ we have $e_{(\chi)^k(n)} = \{\psi^k(x) \mid x \in e_n\}$ and $\psi^k(p(n,m)) = p((\chi)^k(n), \psi^k(m))$. Note that, if $e_k = \emptyset$, then $\chi(k) = k$.

First let n be finite. Obviously it is sufficient to prove that there is an $a \in \mathbb{N}$ of ψ-order n if and only if $\lfloor a \rfloor_\psi = n$ for infinitely many different $a \in \mathbb{N}$. So let $a \in \mathbb{N}$ and $\lfloor a \rfloor_\psi = n$. We show that a can't be the only natural number of ψ-order n:

(i) if $n > 1$, then $\lfloor \psi(a) \rfloor_\psi = n$ and $\psi(a) \neq a$;

(ii) suppose $n = 1$. Let $e_s = \emptyset$. Then $\chi(s) = s$, so $\psi(p(s,a)) = p(s,a)$. But also, if $e_t = \{a\}$ then $\chi(t) = t$, so $\psi(p(t,a)) = p(t,a)$, contradicting the uniqueness of a.

So let there be $k > 1$ natural numbers a of ψ-order n. From (1) we have $(\chi)^n(a') = a'$, where $a' = e^{-1}(\{a\})$, and then (2) implies that we have at least k^2 natural numbers of ψ-order n. As $k^2 > k$ this is a contradiction.

Finally we take n to be ∞. Let $\{\psi^i(x) \mid i \in \mathbb{N}\}$ be an ∞-orbit in ψ and let $k', k \in \mathbb{N}$ be such that $\chi(k) = k \neq k'$. Then $\{p(k, \psi^i(x)) \mid i \in \mathbb{N}\}$ and $\{p((\chi)^i(k'), \psi^i(x)) \mid i \in \mathbb{N}\}$ are disjunct ∞-orbits in ψ. From this it easily follows that there can't be just finitely many distinct ∞-orbits in ψ. □

Fix some coding e of finite sets of natural numbers as natural numbers and let ψ be a bijective mapping $\mathbb{N} \hookrightarrow \mathbb{N}$. For $\tilde{\psi}$ to be an applicative automorphism of $[p,e]$ for some coding p of pairs of natural numbers as natural numbers we need a bijective mapping $\chi : \mathbb{N} \hookrightarrow \mathbb{N}$ satisfying conditions (1) and (2). This mapping of course is completely determined by (1), so given e and ψ we have χ and only need to determine whether there exists a coding p satisfying (2). For such a map to exist ψ has to fulfil the condition given in lemma 3.15. But this is not enough. For let $\lfloor a \rfloor_\chi = n$ and $\lfloor b \rfloor_\psi = m$, then $\lfloor p(a,b) \rfloor_\psi = lcm(n,m)$: for any occurring ψ-order n and any occurring χ-order m there has to be a number such that its ψ-order is the least common multiple of n and m.

3.16 PROPOSITON: *Let e be a coding of finite sets, $\psi : \mathbb{N} \hookrightarrow \mathbb{N}$ a bijective mapping and $\chi : \mathbb{N} \hookrightarrow \mathbb{N}$ the mapping determined by e and ψ through*

$$\chi(n) = e^{-1}(\{\psi(x) \mid x \in e_n\}).$$

There is a (bijective) coding p such that $\tilde{\psi}$ is an applicative automorphism of $[p,e]$ iff

(i) for each $n \in \mathbb{N} \cup \{\infty\}$ the number of n-orbits in ψ is zero or infinite;

(ii) if there is an n-orbit in ψ and an m-orbit in χ, then there is an $lcm(n,m)$-orbit in ψ.

Proof: ($\Rightarrow$) This has been done in lemma 3.15 and the remarks preceding this proposition.

($\Leftarrow$) We will give an informal argument: let e and ψ satisfy (i) and (ii) and, for all $n \in \mathbb{N}$, let E_n be an enumeration of all n-orbits in ψ; also let $P = \{a_0, a_1, a_2, \ldots\}$ be some enumeration of $\mathbb{N} \times \mathbb{N}$ (all enumerations of course without repetitions).

Say $a_0 = (b, b')$. Determine $\lfloor b \rfloor_\chi$, $\lfloor b' \rfloor_\psi$ and $lcm(\lfloor b \rfloor_\chi, \lfloor b' \rfloor_\psi) =: n$. Then put $p(a_0) = k$, where k is the first number in the first n-orbit in ψ enumerated in E_n. Delete a_0 from P. Next put $p(\chi(b), \psi(b')) = \psi(k)$ and delete $(\chi(b), \psi(b'))$ from P. Going on in this way (if $n = \infty$ also working backwards) we will eventually have found pairs for all k in this first n-orbit. Then delete that orbit from E_n. Let a_k be the first pair in P not yet deleted and repeat the procedure.
If $\lfloor x \rfloor_\psi = k$ and $e_t = \{x\}$, then $\lfloor t \rfloor_\chi = k$ and $lcm(\lfloor t \rfloor_\chi, \lfloor x \rfloor_\psi) = k$; so, as there are infinitely many k-orbits in ψ, the above procedure will infinitely many times call for such a k-orbit. Also clearly the presence of infinitely many ∞-orbits in ψ will lead to infinitely many calls for an ∞-orbit, for if $\lfloor x \rfloor_\chi = 1$ and y_1, y_2 are in different ∞-orbits, then $p(x, y_1)$ and $p(x, y_2)$ will end up in different ∞-orbits.
So eventually this procedure results in a (bijective) mapping $p : \mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$ that satisfies (2). Then $\tilde{\psi}$ is an applicative automorphism of $[p, e]$. □

Of course when there are ∞-orbits in ψ the procedure to determine p used in the proof of proposition 3.16 is highly non-constructive. On the other hand, if there are only n-orbits in ψ with n finite, then also there are no ∞-orbits in χ: for if $k = lcm\{\lfloor x \rfloor_\psi \mid x \in e_n\}$, then $(\chi)^k(n) = n$, so $\lfloor n \rfloor_\chi \leq k < \infty$. In that case the above procedure may be used to effectively determine a code p such that $\tilde{\psi}$ is an applicative automorphism of $[p, e]$, though this code is not unique: different enumerations of n-orbits and pairs will lead to different codes.

The first-order theory of $\mathcal{P}(\mathbb{N})$-structures

For $\mathcal{P}(\mathbb{N})$-structures $[p, e]$ it turns out that many properties of the codings at hand are $\mathcal{L}$-expressable in $[p, e]$. This implies that a lot of the properties of $\mathcal{P}(\mathbb{N})$-structures studied above (e.g. properties of the frames) are in fact *first-order* properties. As a result, we will show that the proof of proposition 3.9 actually gives us continuum-many non-elementary equivalent $\mathcal{P}(\mathbb{N})$-structures.

Observe that, given $[p, e]$, any coding q of pairs is representable in $[p, e]$. For let q be some coding, take

$$\rho_q := \left\{ p\big(e^{-1}(\{a\}), e^{-1}(\{b\}), q(a, b)\big) \mid a, b \in \mathbb{N} \right\} \in \mathcal{P}(\mathbb{N}).$$

Then ρ_q represents q in $[p, e]$ in the sense that for all $a, b \in \mathbb{N}$ we have $\{q(a, b)\} = \rho_q \bullet \{a\} \bullet \{b\}$.
Similarly any coding e' of finite sets is representable. For, given e', take

$$\rho_{e'} := \left\{ p\big(e^{-1}(\{n\}), x\big) \mid x \in e'_n \right\} \in \mathcal{P}(\mathbb{N}).$$

Then for all $n \in \mathbb{N}$ we have $\rho_{e'} \bullet \{n\} = e'_n$.

The fact that $\rho \in \mathcal{P}(\mathbb{N})$ represents a coding of pairs can be expressed in $\mathcal{L}$: *"ρ represents a coding of pairs"* iff

$$[p, e] \models \mathbf{Code}(\rho),$$

where $\mathbf{Code}(x)$ is the $\mathcal{L}$-formula

$$\forall y \forall z (\mathbf{One}(y) \wedge \mathbf{One}(z) \longrightarrow \mathbf{One}(xyz)) \quad \wedge$$

$$\wedge \quad \forall x_1 x_2 x_3 x_4 \Big(\bigwedge_{i=1}^{4} \mathbf{One}(x_i) \wedge \mathbf{One}(x x_1 x_2) = \mathbf{One}(x x_3 x_4) \rightarrow x_1 = x_3 \wedge x_2 = x_4 \Big).$$

To show that also the fact that $\rho \in \mathcal{P}(\mathbb{N})$ represents a coding of finite sets can be expressed in $\mathcal{L}$ we only have to do a tiny bit more. Given the representability of any binary relation on $\mathbb{N}$ it is routine to write an $\mathcal{L}$-formula $\mathbf{N}(x)$ such that $[p, e] \models \mathbf{N}(\nu)$ iff *"$\nu \in \mathcal{P}(\mathbb{N})$ represents a well-ordering R of $\mathbb{N}$ such that each natural number has a smallest R-successor and each natural number, except the R-first one, has a greatest R-predecessor."*

Using e.g. this formula $\mathbf{N}(x)$ we can express in $\mathcal{L}$ that $a \in \mathcal{P}(\mathbb{N})$ is a finite set: *"$a \in \mathcal{P}(\mathbb{N})$ is finite"* iff

$$[p, e] \models \mathbf{Finite}(a),$$

where $\mathbf{Finite}(x)$ is the $\mathcal{L}$-formula

$$\exists \mu \Big(\mathbf{N}(\mu) \wedge \exists z \big(\mathbf{One}(z) \wedge \forall u (\mathbf{One}(u) \wedge u \subseteq x) \rightarrow \mu u \neq \emptyset \wedge \mu u z = \emptyset \big) \Big).$$

This is all we need to show that the fact that ρ represents a coding of finite sets can be expressed in $\mathcal{L}$: *"ρ represents a bijective coding of finite sets"* iff

$$[p, e] \models \mathbf{Setcode}(\rho),$$

where $\mathbf{Setcode}(x)$ is the $\mathcal{L}$-formula $\bigwedge_{i=1}^{3} \varphi_i$ with

$\varphi_1 := \forall y \big(\mathbf{One}(y) \rightarrow \mathbf{Finite}(xy)\big)$,
$\varphi_2 := \forall y \forall z \big(\mathbf{One}(y) \wedge \mathbf{One}(z) \wedge \mathbf{Finite}(xy) \wedge xy = xz \rightarrow y = z\big)$,
$\varphi_3 := \forall y \big(\mathbf{Finite}(y) \rightarrow \exists z. \mathbf{One}(z) \wedge xy = z\big)$.

From this we derive

3.17 THEOREM: *Let $[p, e]$ be a $\mathcal{P}(\mathbb{N})$-structure. There is an $\mathcal{L}$-formula $\mathbf{Ap}(x, y)$ that is satisfied by p', e' in $[p, e]$ iff p' and e' are (representations of) codes that define the same application on $\mathcal{P}(\mathbb{N})$ as p and e, i.e.*

$$[p, e] \models \mathbf{Ap}(p', e') \quad \text{iff} \quad [p, e] \sim [p', e'].$$

Proof: Define $\mathbf{Ap}(x,y)$ to be the $\mathcal{L}$-formula

$$\begin{aligned}&\mathbf{Code}(x) \wedge \mathbf{Setcode}(y) \quad \wedge \\ \wedge \quad &\forall a \forall b \forall m \big(\mathbf{One}(m) \wedge m \subseteq ab \leftrightarrow \\ &\quad \leftrightarrow \exists n \exists t. t \subseteq b \wedge \mathbf{One}(n) \wedge yn = t \wedge xnm \subseteq a\big).\end{aligned}$$

□

The formula $\mathbf{Ap}(x,y)$ as given by theorem 3.17 enables us to express properties of frames in $\mathcal{L}$, e.g. for each $n \in \mathbb{N}$ the fact that $\mathcal{F}_n^{[p,e]} = \emptyset$ is $\mathcal{L}$-definable. In order to see this observe that

$$\mathcal{F}_n^{[p,e]} = \emptyset \Leftrightarrow \forall p' \forall e'.[p,e] \sim [p',e'] \rightarrow \mathcal{F}_n^{[p',e']} = \emptyset.$$

So, by definition of the n-frame of $[p,e]$ we have that $\mathcal{F}_n^{[p,e]} = \emptyset$ iff

$$[p,e] \models \forall x \forall y \big(\mathbf{Ap}(x,y) \rightarrow \forall z(\mathbf{One}(z) \wedge \text{“} \mid yz \mid = n\text{”} \rightarrow \forall u.\mathbf{One}(u) \rightarrow xzu \neq u)\big).$$

The following proposition now is immediate from the proof of proposition 3.9.

3.18 PROPOSITION: *There are $2^{\aleph_0}$ non-elementary-equivalent $\mathcal{P}(\mathbb{N})$-structures.*

□

As another application of theorem 3.17 we show

3.19 PROPOSITION: *Let $[p,e]$ be a $\mathcal{P}(\mathbb{N})$-structure. The fact that $[p,e]$ is isomorphic to some $\mathcal{P}(\mathbb{N})$-structure $[p^0,e^0]$ with p^0, e^0 basic codings, is $\mathcal{L}$-definable.*

Proof: We claim that $[p,e] \cong [p^0,e^0]$, with p^0, e^0 basic codings iff there are p', e' and ρ such that in $[p,e]$ we have

$$\mathbf{Ap}(p',e') \wedge \mathbf{N}(\rho) \wedge \forall x \forall y \big((\text{“}y \in e'x\text{”} \rightarrow \rho yx) \wedge \rho x(p'xy) \wedge \rho y(p'xy)\big),$$

where $\mathbf{N}(\rho)$ as above expresses that ρ represents an ordering R of $\mathbb{N}$ of type ω.

For, let p^0, e^0 be basic codings and suppose $[p,e] \cong [p^0,e^0]$. Then this $\mathcal{L}$-sentence holds in $[p^0,e^0]$, so it holds in $[p,e]$.

Conversely, suppose the sentence holds in $[p,e]$. Then let ψ be the order-isomorphism $(\mathbb{N},R) \rightarrow (\mathbb{N},\leq)$. Now put, for all x,y:

$$e^0(x) = \tilde{\psi}(e'(\psi^{-1}(x)))$$

and

$$p^0(x,y) = \psi(p'(\psi^{-1}(x),\psi^{-1}(y))).$$

Then $[p^0,e^0] \cong [p',e'] \sim [p,e]$, and p^0 and e^0 are basic codings. □

From this we obtain

3.20 PROPOSITION: *Let $[p,e]$ be such that $[p,e] \equiv \mathcal{P}\omega$. Then $[p,e] \cong \mathcal{P}\omega$.*

Proof: Consider the following $\mathcal{L}$-sentences:

$$\varphi_0 := \exists p \exists e(\mathbf{Ap}(p,e) \wedge \exists x(\mathbf{One}(x) \wedge pxx = x) \wedge \forall y \forall z(\mathbf{One}(y) \wedge \mathbf{One}(z) \wedge pyz = z \to y = z \wedge ez = \emptyset));$$

$$\varphi_1 := \forall p \forall e\Big(\mathbf{Ap}(p,e) \to \forall x_1(\mathbf{One}(x_1) \to \exists x_2 x_3(\mathbf{One}(x_2) \wedge \mathbf{One}(x_3) \wedge px_2x_3 = x_1))\Big).$$

(φ_1 says there are no atoms.)

We leave it to the reader to check that $\mathcal{P}\omega \models \varphi_0 \wedge \varphi_1$.

As application on $\mathcal{P}\omega$ is defined through basic codings, we may assume, by the previous proposition and elementary equivalence of $[p,e]$ and $\mathcal{P}\omega$, that p and e are basic. Also, $[p,e] \models \varphi_0 \wedge \varphi_1$. We claim that from this it follows that $\mathcal{F}^{[p,e]} = \{(0,0)\}$. For φ_0 says there is $[p',e'] \sim [p,e]$ such that $p'(x,y) = y$ iff $x = y$ and $e'_y = \emptyset$; from φ_0 and the fact that p is a basic-coding we know that otherwise the frame is empty. By compatibility we have that $p(\chi(y),y) = y$ and $e_{\chi(y)} = \emptyset$ for some $\chi \in S(\mathbb{N})$. As e is a basic coding then $\chi(y) = 0$, so $\mathcal{F}_0^{[p,e]} = \{(0,y)\}$. Again by φ_0 and the fact that p is a basic coding we know that $(0,0) \in \mathcal{F}_0^{[p,e]}$. Therefore $y = 0$. Now obviously $\mathcal{F}^{[p,e]} \sim \mathcal{F}^{[p^\star,e^\star]}$ and $[p,e] \cong \mathcal{P}\omega$ by theorem 3.11. □

3.21 COROLLARY: *For all $\mathcal{P}(\mathbb{N})$-structures $[p,e]$:*

$$[p,e] \cong \mathcal{P}\omega \quad \textit{iff} \quad [p,e] \equiv \mathcal{P}\omega. \qquad \square$$

A note on categorical lambda-models

Models for the untyped lambda calculus always are combinatory algebras, i.e. there are elements $\mathbf{k}$ and $\mathbf{s}$ satisfying $\mathbf{k}xy = x$ and $\mathbf{s}xyz = xz(yz)$ for all x, y, z. If, given a combinatory algebra $\mathcal{C}$, there are *unique* elements $\mathbf{k}$,$\mathbf{s}$ that determine $\mathcal{C}$ to be also a lambda model, then $\mathcal{C}$ is called *categorical.*

In [Lo] it is shown that for graph models 'categorical' just means 'atom-free'. So e.g. all $\mathcal{P}(\mathbb{N})$-structures defined through surjective codings are categorical. It is easy to see that for categorical lambda-models $\mathcal{M}$ and $\mathcal{N}$ the categoricity enables us to define unambiguously a translation of equations between λ-terms into $\mathcal{L}$-sentences and vice versa: i.e. for any term-equation $A = B$ there is a sentence $\phi_{(A=B)} \in \mathcal{L}$ such that

$$\mathcal{M} \models \varphi_{(A=B)} \quad \text{iff} \quad \mathcal{M} \models A = B.$$

(The sentence expresses the unicity of the elements $\mathbf{k}$ and $\mathbf{s}$, replaces occurences of the constants $\mathbf{K}$ and $\mathbf{S}$ in the equation by occurrences of the $\mathcal{L}$-variables k and s, and for occurrences in the equation of variables we put universal quantification over

$\mathcal{L}$-variables.) So for categorical lambda models we have that elementary equivalence implies *equational* equivalence: in the models the same equations between (interpretations of) λ-terms are valid. Our previous observations on $\mathcal{P}(\mathbb{N})$-structures enable us to show that the converse is *not* true.

In $\mathcal{P}\omega$ (the interpretations of) two λ-terms are equal iff they have the same Böhm-tree. The proof of this fact makes use of successive approximations $(x)_n := \{m \in x \mid x \leq n\}$ of $x \in \mathcal{P}(\mathbb{N})$ and a set of so-called 'basic equations', stating properties of these approximations (see [Ba], chapter 18/19):

(i) $x = \bigcup_n (x)_n$; $((x)_n)_m = (x)_{min(n,m)}$;

(ii) $\emptyset \bullet x = \lambda x.\emptyset = \emptyset = (\emptyset)_n$;

(iii) $(x)_0 \bullet y = (x)_0 \bullet \emptyset = ((x)_0 \bullet \emptyset)_0 = (x \bullet \emptyset)_0 = (x)_0$;

(iv) $(x)_{n+1} \bullet y = (x)_{n+1} \bullet (y)_n = ((x)_{n+1} \bullet (y)_n)_n \subseteq (x \bullet (y)_n)_n$.

3.22 LEMMA: *Suppose that p and e are basic codings such that $p(0,0) = 0$, and for all x, y we have that $p(x,y) = y$ implies $x = y$. Then $[p,e]$ satisfies (i) ... (iv).*

Proof: Left to the reader. □

Take for example basic codings p, e such that $\mathcal{F}^{[p,e]} = \{(0,0),(1,1)\}$. Lemma 3.22 says that $[p,e]$ satisfies (i)...(iv). This implies that in $[p,e]$ (the interpretations of) two λ-terms are equal iff they have the same Böhm-tree. But then we have that $[p,e]$ and $\mathcal{P}\omega$ are equationally equivalent. On the other hand $\mathcal{F}^{[p^\star,e^\star]} = \{(0,0)\}$, and by our earlier observations we know that this difference in frames is a first-order property. Therefore $[p,e] \not\equiv \mathcal{P}\omega$.

So we proved

3.23 THEOREM: *There are categorical lambda-models that are equationally, but not elementary, equivalent.* □

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