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EXTENSION OF LIFSCHITZ' REALIZABILITY TO HIGHER ORDER ARITHMETIC, AND A SOLUTION TO A PROBLEM OF F. RICHMAN

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Abstract. F. Richman raised the question whether the following principle of second order arithmetic is valid in intuitionistic higher order arithmetic **HAH**:

$\forall X\,[\forall x(x\in X \vee \neg x\in X) \wedge \forall Y\,(\forall x(x\in Y \vee \neg x\in Y) \rightarrow \forall x(x\in X \rightarrow x\in Y) \vee \forall x \neg(x\in X \wedge x\in Y))$

$\rightarrow \exists n \forall x(x\in X \rightarrow x=n)]$

and if not, whether assuming Church's Thesis CT and Markov's Principle MP would help. Blass & Scedrov gave models of **HAH** in which this principle, which we call RP, is not valid, but their models do not satisfy either CT or MP.

In this paper a realizability topos Lif is constructed in which CT and MP hold, but RP is false (It is shown, however, that RP is derivable in **HAH**+CT+MP+ECT_0, so RP holds in the effective topos). Lif is a generalization of a realizability notion invented by V. Lifschitz. Furthermore, Lif is a subtopos of the effective topos.

Key words and phrases : **HAH**, realizability, tripos, topos

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Extension of Lifschitz' realizability to Higher Order Arithmetic, and a solution to a problem of F. Richman

Jaap van Oosten

§1. Introduction

Blass & Scedrov (1986) is about the following principle of Second Order Arithmetic:

$$\text{RP} \quad \forall X[\forall x(x\in X \vee \neg x\in X) \wedge \forall Y(\forall x(x\in Y \vee \neg x\in Y) \to \forall x(x\in X \to x\in Y) \vee \forall x\neg(x\in X \wedge x\in Y)) \to \exists n\forall x(x\in X \to x=n)]$$

We have christened this principle RP from Richman's Principle: F. Richman, who needed this principle for an application in constructive algebra, raised the question whether it is constructively valid. Blass & Scedrov showed that it is not, by giving a topological model and a sheaf model in which RP does not hold.

Apparently this did not quite settle the matter, for the authors write: "Our models do not satisfy further conditions imposed by Richman, namely Church's Thesis and Markov's Principle, so the full conjecture remains an open problem".

We will exhibit a realizability topos in which CT and MP are valid, but which refutes RP (so ¬RP holds in our model). The topos is a generalisation of a notion of realizability invented by V. Lifschitz (1979). This realizability is studied further in Van Oosten (1990). It may surprise the reader that such a topos can satisfy CT, since Lifschitz designed his realizability in order to refute the schema CT_0:

$$CT_0 \quad \forall x\exists y\, Axy \to \exists f\forall x\exists z\,(Tfxz \wedge Ax(Uz))$$

However, this first order schema is, in the presence of function variables, a consequence of two others:

$$\text{CT} \quad \forall f{:}\mathbb{N}\to\mathbb{N}\,\exists e{:}\mathbb{N}\,\forall x{:}\mathbb{N}\,\exists y{:}\mathbb{N}\,(Texy \wedge Uy=f(x))$$

(Church's Thesis), and a choice principle:

$$AC_{00} \quad \forall x{:}\mathbb{N}\,\exists y{:}\mathbb{N}\,Axy \to \exists f{:}\mathbb{N}\to\mathbb{N}\,\forall x{:}\mathbb{N}\,Ax(fx)$$

In our model, CT holds but AC_{00} fails.

This paper consists of two parts. In §2 some details about Lifschitz' realizability are recalled and we refute a first-order version RP_0 of RP. RP_0 is (provably in **HA**) Kleene-realizable and equivalent to RP in Higher Order intuitionistic Arithmetic **HAH**+CT, so **HAH**+CT+MP+$ECT_0 \vdash$ RP, where MP and ECT_0 are the schemata:

MP $\forall x{:}\mathbb{N}\,(Ax\vee\neg Ax)\wedge\neg\neg\exists x{:}\mathbb{N}\,Ax \to \exists x{:}\mathbb{N}\,Ax$

ECT_0 $\forall x{:}\mathbb{N}\,(Ax\to\exists y{:}\mathbb{N}\,Bxy) \to \exists z{:}\mathbb{N}\,\forall x{:}\mathbb{N}\,(Ax\to\exists w{:}\mathbb{N}\,(Tzxw\wedge BxUw))$

(ECT_0 is a first-order schema, so A and B are first-order arithmetical formulas with the proviso that A is built up from Σ^0_1-formulas using only $\to$, $\forall$ and $\wedge$; MP is unrestricted).

It is nice to see two realizabilities neatly separated (in the context of **HAH**+CT+MP) by a mathematical axiom originating from constructive algebra.

In §3 we show that Lifschitz' realizability has an extension to Higher Order Arithmetic + CT + MP. The topos we will describe is a subtopos of the effective topos.

§2. *Lifschitz' realizability and* RP

Let us agree on some notation. Primitive recursive pairing and divorcing functions will be denoted by $\langle\rangle$, $()_0$, $()_1$ respectively. We write $x\bullet y$ instead of $\{x\}(y)$ for partial recursive application. We use [z] instead of Lifschitz' V_z:

$[z] = \{x\leq(z)_1 \mid (z)_0\bullet x\uparrow\}$

For a proof of the following proposition, the reader is referred to Lifschitz 1979 or Van Oosten 1990.

2.1. Proposition. i) There is no partial recursive function F such that for all z, if [z] is nonempty then $F(z)\downarrow$ and $F(z)\in[z]$.

ii) There is a partial recursive function δ such that for all z, whenever [z] is a singleton, then $\delta(z)\downarrow$ and $\delta(z)\in[z]$.

iii) There is a total recursive function γ such that for all z, $[\gamma(z)]=\{g \mid \exists f\in[z]\,(g\in[f])\}$.

iv) There is a partial recursive function G such that for all e and f, if $\forall x\in[f]\,(e\bullet x\downarrow)$ then $G(e,f)\downarrow$ and $[G(e,f)]=\{e\bullet x \mid x\in[f]\}$.

Sometimes we just write something like $\Lambda yw.\bigcup\{[g\bullet h] \mid g\in[y], h\in[w]\}$, writing set notation for the (standard) code, and trusting the reader with the ability to construct such recursive functions with the help of proposition 2.1. For definitions 2.2 and 2.3 as well as proposition 2.4 see Van Oosten 1990.

2.2. Definition. Lifschitz' realizability can be defined as follows:

$x\,\mathbf{r}\,t{=}s$ if $[x]\neq\emptyset$ and $t=s$

$x\,\mathbf{r}\,A\wedge B$ if $[x]\neq\emptyset$ and $\forall y\in[x]\,((y)_0\,\mathbf{r}\,A$ and $(y)_1\,\mathbf{r}\,B)$

$x\,\mathbf{r}\,A\to B$ if $[x]\neq\emptyset$ and $\forall y\in[x]\,\forall z\,(z\,\mathbf{r}\,A \Rightarrow y\bullet z\downarrow$ and $y\bullet z\,\mathbf{r}\,B)$

$x\,\mathbf{r}\,\forall xA(x)$ if $[x]\neq\emptyset$ and $\forall y\in[x]\,\forall n\,(y\bullet n\downarrow$ and $y\bullet z\,\mathbf{r}\,A(n))$

$x\,\mathbf{r}\,\exists xA(x)$ if $[x]\neq\emptyset$ and $\forall y\in[x]\,((y)_1\,\mathbf{r}\,A((y)_0))$

2.3. Definition. i) The class of $B\Sigma_2^0$-negative formulas is inductively defined as follows:

- Σ_1^0-formulas are $B\Sigma_2^0$-negative;
- Formulas of the form $\exists y \leq t\, A$, with A a Π_1^0-formula and t a term not containing y, are $B\Sigma_2^0$-negative;
- $B\Sigma_2^0$-negative formulas are closed under $\forall, \rightarrow$ and $\wedge$.

ii) ECT_L is the following first-order schema:

(ECT_L) $\quad \forall x (Ax \rightarrow \exists y Bxy) \rightarrow \exists z \forall x (Ax \rightarrow z{\bullet}x\downarrow \wedge \exists w(w \in [z{\bullet}x]) \wedge \forall w \in [z{\bullet}x]\, Bxw)$,

with the condition that Ax must be a $B\Sigma_2^0$-negative formula.

2.4. Proposition. Every instance of the schema ECT_L is realizable. Markov's Principle is realizable.

In fact, ECT_L axiomatizes formalised **r**-realizability over a suitable extension of **HA**, but that does not concern us here. Let us now assume that Lifschitz' realizability has an extension to Higher Order Arithmetic (**HAH**) which satisfies CT and MP. Every decidable subset of $\mathbb{N}$ may be identified with the set of zeroes of some function $f: \mathbb{N} \rightarrow \mathbb{N}$, and by CT all such functions are recursive, so in **HAH**+CT the principle RP is equivalent to the first-order axiom:

RP_0 $\quad \forall e\, [\forall x \exists y Texy \wedge \forall f(\forall x \exists y Tfxy \rightarrow \forall x(e{\bullet}x{=}0 \rightarrow f{\bullet}x{=}0) \vee \forall x \neg(e{\bullet}x{=}0 \wedge f{\bullet}x{=}0))$
$\quad\quad \rightarrow \exists n \forall x(e{\bullet}x{=}0 \rightarrow x{=}n)]$

2.5. Proposition. $HA+RP_0+ECT_L+MP$ is inconsistent.

Proof. Assuming $\forall x \exists y Texy$, $\forall f(\forall x \exists y Tfxy \rightarrow \forall x(e{\bullet}x{=}0 \rightarrow f{\bullet}x{=}0) \vee \forall x \neg(e{\bullet}x{=}0 \wedge f{\bullet}x{=}0))$ is equivalent to:

C(e) $\quad \forall f(\forall x \exists y Tfxy \rightarrow \forall xyz(Texy \wedge Tfxz \wedge Uy{=}0 \rightarrow Uz{=}0)$
$\quad\quad\quad \vee \forall xyz \neg(Texy \wedge Uy{=}0 \wedge Tfxz \wedge Uz{=}0))$,

which is equivalent to a $B\Sigma_2^0$-negative formula; we may apply ECT_L to RP_0 which would give a z such that:

(1) $\forall e[\forall x \exists y Texy \wedge C(e) \rightarrow z{\bullet}e\downarrow \wedge \exists w(w \in [z{\bullet}e]) \wedge \forall w \in [z{\bullet}e] \forall x(e{\bullet}x{=}0 \rightarrow x{=}w)]$,

which means the existence of a z such that:

(2) $\forall e[\forall x \exists y Texy \wedge C(e) \rightarrow z{\bullet}e\downarrow \wedge \exists w \leq z{\bullet}e \forall x(e{\bullet}x{=}0 \rightarrow x{=}w)]$,

and this is contradictory: suppose z as in (2). Let, by the recursion theorem, e be such that:

$e{\bullet}x \simeq$ $\quad$ 1 if $\neg Tzex$
$\quad\quad\quad$ 0 else.

Then $z{\bullet}e$ is defined. For if not, then $\forall x(e{\bullet}x{=}1)$ and C(e) clearly holds, so $z{\bullet}e\downarrow$, contradiction; so $\neg\neg(z{\bullet}e\downarrow)$; apply MP. Furthermore, C(e) holds, for if f codes a total function, we only have to look at $f{\bullet}(\mu x.Tzex)$ to decide which of the two possibilities holds. But $\exists w \leq z{\bullet}e \forall x(e{\bullet}x{=}0 \rightarrow x{=}w)$ is obviously false, since if Tzex, then $z{\bullet}e < x$ (for any standard coding).

2.6. Proposition. $\mathbf{HA+ECT_0} \vdash \mathrm{RP}_0$

Proof. We argue in $\mathbf{HA+ECT_0}$. Suppose $\forall x\exists yTexy \wedge \forall f(\forall x\exists yTfxy \rightarrow \forall x(e\bullet x=0 \rightarrow f\bullet x=0) \vee \forall x\neg(e\bullet x=0 \wedge f\bullet x=0))$. By ECT_0, there is a z such that, for all f, if $\forall x\exists yTfxy$ then $z\bullet f\downarrow$ and:

i) $z\bullet f=0 \rightarrow \forall x(e\bullet x=0 \rightarrow f\bullet x=0)$

ii) $z\bullet f\neq 0 \rightarrow \forall x\neg(e\bullet x=0 \wedge f\bullet x=0)$

iii) $\neg\exists xy\,(x\neq y \wedge e\bullet x=0 \wedge e\bullet y=0)$.

Use the recursion theorem to find a code f such that:

$$f\bullet x \simeq \begin{cases} 1 \text{ if } \forall y\leq x\,\neg T(z,f,y) \\ 0 \text{ if } T(z,f,x) \wedge Ux=0 \\ 1 \text{ if } \exists y<x\,(T(z,f,y) \wedge Uy=0)) \\ 0 \text{ if } \exists y\leq x\,(T(z,f,y) \wedge Uy\neq 0)). \end{cases}$$

Then f codes a total function, so $z\bullet f\downarrow$. Say $T(z,f,x)$. Two possibilities:

a) $Ux=0$. Then by i), $\forall y(e\bullet y=0 \rightarrow f\bullet y=0)$. But the only zero of f is x. So $\forall y(e\bullet y=0 \rightarrow y=x)$.

b) $Ux\neq 0$. Then by ii), $\forall y\neg(e\bullet y=0 \wedge f\bullet y=0)$. But $\forall y\geq x\,(f\bullet y=0)$. So $\forall y(e\bullet y=0 \rightarrow y<x)$.

In both cases, $\exists n\forall y\,(e\bullet y=0 \rightarrow y=n)$ (in case b, check $e\bullet y$ for all $y<x$. Use iii)).

Remark: The topological model of Blass & Scedrov satisfies $\neg\neg$RP but does not even validate the weaker axiom in which the decidability condition on Y is dropped:

$$\forall X\,[\forall x(x\in X \vee \neg x\in X) \wedge \forall Y\,(\forall x(x\in X \rightarrow x\in Y) \vee \forall x\neg(x\in X \wedge x\in Y)) \rightarrow \exists n\forall x(x\in X \rightarrow x=n)]$$

This schema will be valid in our model, since our model satisfies the Uniformity Principle:

(UP) $\quad \forall X\exists yA(X,y) \rightarrow \exists y\forall XA(X,y)$

so if $\forall Y\,(\forall x(x\in X \rightarrow x\in Y) \vee \forall x\neg(x\in X \wedge x\in Y))$, then $\forall Y\,\forall x(x\in X \rightarrow x\in Y) \vee \forall Y\,\forall x\neg(x\in X \wedge x\in Y)$ and X is the empty set.

§3. Extension of Lifschitz' realizability to HAH+CT+MP

The construction of a topos for Lifschitz' realizability relies heavily on the methods developed in Hyland, Johnstone & Pitts 1980 (abbreviated HJP 1980). Since not every reader will be familiar with that article and we try to be as self-contained as possible, we will recapitulate the main results as we go along. The reader will find it helpful to bear in mind the analogies with the construction of a topos of Ω-sets, where Ω is a complete Heyting algebra, as laid out in Fourman & Scott 1979. We construct a set Σ with a binary operation $\Rightarrow$ on it, such that the following hold:

1) For every set X, the set Σ^X forms a Heyting pre-algebra (i.e. it has all the properties of a Heyting algebra except that the underlying order is a preorder), where the Heyting implication is given by $\Rightarrow$: for $\phi,\psi\in\Sigma^X$, $\phi\Rightarrow\psi = \lambda x.(\phi(x)\Rightarrow\psi(x))$;

2) For every function between sets $f{:}X\rightarrow Y$ the map $\Sigma^f\colon \Sigma^Y\rightarrow\Sigma^X$, defined by composition with

f: $(\Sigma^f(\psi))(x)=\psi(f(x))$, which preserves Heyting implication by 1), is an order-preserving map and has both a left and a right adjoint $\exists f$ and $\forall f$, respectively: $\Sigma^X \to \Sigma^Y$ (i.e. for $\phi \in \Sigma^X$ and $\psi \in \Sigma^Y$: $\Sigma^f(\psi) \vdash \phi$ iff $\psi \vdash \forall f(\phi)$ and $\exists f(\phi) \vdash \psi$ iff $\phi \vdash \Sigma^f(\psi)$);

3) These adjoints satisfy the *Beck-condition* : if $f:X \to Y$ is a set map and Z a set, then the two maps $\forall(\pi_X) \circ \Sigma^{Z\times f}: \Sigma^{Z\times Y} \to \Sigma^{Z\times X} \to \Sigma^X$ and $\Sigma^f \circ \forall(\pi_Y): \Sigma^{Z\times Y} \to \Sigma^Y \to \Sigma^X$ are isomorphic maps of preorders (π_X and π_Y denote projections). A similar condition, for the left adjoint, then holds automatically.

In the terminology of HJP, we are going to define a *canonically presented Sets-tripos.*

3.1. Definition.

i) Let J be $\{e \in \mathbb{N} \mid [e] \neq \emptyset\}$. Σ is the set of all subsets H of J that satisfy the following conditions:

a) H is extensional, i.e. if $e \in H$ then $[f]=[e]$ implies $f \in H$;

b) H is closed under finite unions: if $[e]=[f]\cup[g]$ with $f \in H$ and $g \in H$, then $e \in H$.

ii) For $G,H \in \Sigma$ we put $G \Rightarrow H = \{e \in J \mid \forall f \in [e] \forall x \in G\ (f \bullet x\downarrow\ \&\ f \bullet x \in H)\}$. The reader sees at once that $G \Rightarrow H$ is well-defined, i.e. $G \Rightarrow H \in \Sigma$.

iii) Let X be a set. We define a preorder $\vdash$ on Σ^X by: $\phi \vdash \psi$ iff $\cap\{\phi(x) \Rightarrow \psi(x) \mid x \in X\}$ is nonempty.

Let us show that $\vdash$ is a preorder. We reserve the letter β for (a code of) the total recursive function such that $[\beta(e)]=\{e\}$. Then $\beta(\Lambda x.x) \in \cap\{\phi(x) \Rightarrow \phi(x) \mid x \in X\}$, so $\vdash$ is reflexive. Now suppose $\phi \vdash \psi$ and $\psi \vdash \chi$, say $e \in \cap\{\phi(x) \Rightarrow \psi(x) \mid x \in X\}$ and $f \in \cap\{\psi(x) \Rightarrow \chi(x) \mid x \in X\}$. If $y \in \phi(x)$ then $\forall a \in [e]$ $a \bullet y \in \psi(x)$, so (since $\psi(x) \in \Sigma$) $\cup\{[a \bullet y] \mid a \in [e]\} \in \psi(x)$; with proposition 2.1, $\cup\{[a \bullet y] \mid a \in [e]\}$ is $[G(e,y)]$ for a suitable partial recursive G. Similarly, $\cup\{[b \bullet G(e,y)] \mid b \in [f]\} \in \chi(x)$ and again, $\cup\{[b \bullet G(e,y)] \mid b \in [f]\}$ is $[G'(e,f,y)]$ for a partial recursive G'. So if g is a code for $\Lambda y.G'(e,f,y)$, then $\beta(g) \in \cap\{\phi(x) \Rightarrow \chi(x) \mid x \in X\}$, i.e. $\phi \vdash \chi$ and $\vdash$ is transitive.

We now define operations of conjunction and disjunction on Σ: for $G,H \in \Sigma$ put:

$G \wedge H = \{e \in J \mid \forall f \in [e]\ ((f)_0 \in G\ \&\ (f)_1 \in H)\}$

$G \vee H = \{e \in J \mid \forall f \in [e]\ ((f)_0=0 \text{ and } (f)_1 \in G) \text{ or } ((f)_0=1 \text{ and } (f)_1 \in H)\}$

3.2. Proposition. Let, on Σ^X,

$\phi \wedge \psi = \lambda x.(\phi(x) \wedge \psi(x))$,

$\phi \vee \psi = \lambda x.(\phi(x) \vee \psi(x))$,

$\phi \Rightarrow \psi = \lambda x.(\phi(x) \Rightarrow \psi(x))$,

$\top = \lambda x.J$,

$\bot = \lambda x.\emptyset$. Then Σ^X forms with this structure a Heyting prealgebra.

Proof. For instance, $\beta(\Lambda x.\cup\{[(a)_0] \mid a \in [x]\}) \in \cap\{(\phi \wedge \psi)(x) \Rightarrow \phi(x) \mid x \in X\}$ (Use definition 3.1). All calculations are similar.

3.3. Proposition. Let f: $X \to Y$ be a function between sets and $\phi \in \Sigma^X$. Define $\forall f(\phi)$ and $\exists f(\phi)$ in

Σ^Y by:
$\forall f(\phi)(y) = \{e \in J \mid \forall x \in X \forall h \in J\,(f(x)=y \Rightarrow \forall g \in [e]\ (g \bullet h\downarrow\ \&\ g \bullet h \in \phi(x)))\}$
$\exists f(\phi)(y) = \{e \in J \mid \forall h \in [e] \exists x \in X\,(f(x)=y\ \&\ h \in \phi(x))\}$
Then $\forall f$ and $\exists f$ are order-preserving maps: $\Sigma^X \to \Sigma^Y$ and respectively right and left adjoint to Σ^f.

Proof. By way of example, we show for $\phi \in \Sigma^X$ and $\psi \in \Sigma^Y$: $\Sigma^f(\psi) \vdash \phi$ in Σ^X iff $\psi \vdash \forall f(\phi)$ in Σ^Y (In particular, this will show that $\forall f$ is order-preserving). So let $e \in \bigcap\{\psi(f(x)) \Rightarrow \phi(x) \mid x \in X\}$ and $y \in Y$, $a \in \psi(y)$. If $f(x)=y$ and $h \in J$, then $\bigcup\{[g \bullet a] \mid g \in [e]\} \in \phi(x)$ so for $e' = \beta(\Lambda a.\beta(\Lambda h.\bigcup\{[g \bullet a] \mid g \in [e]\}))$ we have that $e' \in \bigcap\{\psi(y) \Rightarrow \forall f(\phi)(y) \mid y \in Y\}$; conversely, if $e' \in \bigcap\{\psi(y) \Rightarrow \forall f(\phi)(y) \mid y \in Y\}$ and $x \in X$, $a \in \psi(f(x))$, then $\bigcup\{[f \bullet a] \mid f \in [e']\} \in \forall f(\phi)(f(x))$ so $\bigcup\{[(f \bullet a) \bullet \beta(0)] \mid f \in [e']\} \in \phi(x)$, etc.

The reader is invited to check himself that the Beck-condition holds. This completes the construction of our tripos. We now indicate how to interpret many-sorted intuitionistic predicate logic without equality in this tripos: *sorts* are interpreted by sets (for a sort σ, we write $[\![\sigma]\!]$ for the set that interprets σ); *predicates* with variables $x_1,\dots,x_n$ of sorts $\sigma_1,\dots,\sigma_n$ are interpreted as elements of Σ^X where $X = [\![\sigma_1]\!] \times \dots \times [\![\sigma_n]\!]$; *formulas* are then formed using the Heyting pre-algebra structure and the maps $\exists f$ and $\forall f$; for example: suppose our language has two sorts σ and τ, a unary predicate $R(x^\sigma)$ and a two-place predicate $S(x^\sigma, y^\tau)$. Let $X = [\![\sigma]\!]$, $Y = [\![\tau]\!]$, $\phi \in \Sigma^X$ interprets R and $\psi \in \Sigma^{X \times Y}$ interprets S. Let π_1 and π_2 be the projections from $X \times Y$ to X and Y respectively; then the formula $\exists y(R(x) \to S(x,y))$ is interpreted by the element $\exists(\pi_1)(\Sigma^{\pi_1}(\phi) \Rightarrow \psi)$ of Σ^X. This interpretation can be extended to languages with function symbols: function symbols f of sort σ which take arguments of sort τ are interpreted as functions $[\![f]\!]: [\![\tau]\!] \to [\![\sigma]\!]$; if, for instance, R is a predicate of one variable of sort σ and $\phi \in \Sigma^{[\![\sigma]\!]}$ interprets R, then the formula $R(f(y^\tau))$ is interpreted by $\Sigma^{[\![f]\!]}(\phi) \in \Sigma^{[\![\tau]\!]}$. Constants of sort σ are elements of $[\![\sigma]\!]$ (functions: $1 \to [\![\sigma]\!]$).

We say that a formula *holds* in the tripos (under a given interpretation of sorts, predicates and functions) if its interpretation is (isomorphic to) the top element in the Heyting algebra it belongs to.

Intuitionistic many-sorted predicate logic without equality is sound for this interpretation (this is Lemma 2.1 in HJP 1980); this is important because logical calculations play a large role in the construction of our topos as well as the verification of axioms in it.

Now we have to rely on the reader's gullibility (or his willingness to read HJP 1980) for the fact that the category we now construct is a topos. Objects are pairs (X,=) with X a set and = an element of $\Sigma^{X \times X}$, such that the formula $(x=x' \to x'=x) \wedge (x=x' \wedge x'=x'' \to x=x'')$ holds; morphisms $(X,=) \to (Y,=')$ are equivalence classes of elements $F \in \Sigma^{X \times Y}$ such that the formula:
$(F(x,y) \to x=x \wedge y='y) \wedge (F(x,y) \wedge x=x' \wedge y='y' \to F(x',y')) \wedge (F(x,y) \wedge F(x,y') \to y='y') \wedge (x=x \to \exists y F(x,y))$ holds; two such F,G being equivalent iff the formula $F(x,y) \to G(x,y)$ holds (note, that this is an equivalence relation!). If $F \in \Sigma^{X \times Y}$ represents a morphism $(X,=) \to (Y,=')$ and $G \in \Sigma^{Y \times Z}$ represents a morphism $(Y,=') \to (Z,='')$ then the interpretation of $\exists y(F(x,y) \wedge G(y,z))$ represents a morphism: $(X,=) \to (Z,='')$, the composition of [F] and [G]. Checking associativity is

an easy exercise in logic.
The resulting category is a topos, which we call Lif (from: Lifschitz' realizability).

We now want to establish a relation between Lif and Hyland's topos Eff (the "effective topos", see Hyland 1982). The underlying tripos of Eff is the system of Heyting pre-algebras $P(\mathbb{N})^X$, with preordering $\phi \vdash \psi$ iff there is e such that for all x and all $a \in \phi(x)$, $e \bullet a\downarrow$ & $e \bullet a \in \psi(x)$.

3.4. Proposition. Let $\Psi_+(X)$: $\Sigma^X \to P(\mathbb{N})^X$ be defined by composition with the inclusion: $\Sigma \to P(\mathbb{N})$. Then:
i) $\Psi_+(X)$ is order-preserving;
ii) $\Psi_+(X)$ has a left adjoint $\Psi^+(X)$, that preserves finite meets;
iii) The map $\Psi^+(X) \circ \Psi_+(X)$ is isomorphic to the identity on Σ^X.

Proof. i) Trivial. ii) Let $\Psi = \Psi^+(1)$: $P(\mathbb{N}) \to \Sigma$ be defined by $\Psi(A) = \{e \in J \mid [e] \subseteq A\}$ and let $\Psi^+(X)$ be composition with Ψ. Suppose $\phi \in P(\mathbb{N})^X$, $\psi \in \Sigma^X$ and $\phi \vdash \Psi_+(X)(\psi)$ in $P(\mathbb{N})^X$, say for all x and all $a \in \phi(x)$, $e \bullet a\downarrow$ & $e \bullet a \in \Psi_+(X)(\psi)(x) = \psi(x)$. Then if $x \in X$ and $b \in \Psi^+(X)(\phi)(x)$, so $[b] \subseteq \phi(x)$, then $\bigcup\{[e \bullet h] \mid h \in [b]\} \in \psi(x)$. So for all x, $\beta(\Lambda b.\bigcup\{[e \bullet h] \mid h \in [b]\}) \in \Psi^+(X)(\phi)(x) \Rightarrow \psi(x)$, so $\Psi^+(X)(\phi) \vdash \psi$ in Σ^X. Conversely, if for all x, $e \in \Psi^+(X)(\phi)(x) \Rightarrow \psi(x)$, then [e] is nonempty and for every $g \in [e]$, for every $x \in X$ and $a \in \phi(x)$, $g \bullet \beta(a)$ is defined and in $\Psi_+(X)(\psi)(x)$. So $\phi \vdash \Psi_+(X)(\psi)$.
iii) left to the reader.

In the language of HJP 1980, 3.4 establishes a *geometric morphism* of triposes.

3.5. Proposition. There is a geometric morphism (Ψ_*, Ψ^*): Lif$\to$Eff, which is an inclusion of toposes. The inverse image part Ψ^*: Eff$\to$Lif is given by $\Psi^*((X,=)) = (X, \Psi^+(=))$.

Again, for a proof we have to refer to HJP 1980. We will explain the terminology. Ψ_*: Lif$\to$Eff and Ψ^*: Eff$\to$Lif are functors such that Ψ_* is right adjoint to Ψ^* and Ψ^* preserves finite limits; "inclusion" means that the counit of the adjunction ε: $\Psi^* \Psi_* \to \mathrm{Id}$, is an isomorphism. As an immediate consequence:

3.6. Proposition. The natural number object in Lif is up to isomorphism given by $(\mathbb{N},=)$ with $[\![n=m]\!] = \{e \in J \mid [e] \subseteq \{n\} \cap \{m\}\} = \{e \in J \mid [e] = \{n\}\}$ if n=m, and $\emptyset$ else.

Proof. Natural numbers objects in toposes are preserved by functors that preserve 1 and have right adjoints, such as Ψ^*: diagrams of form $1 \to X \to X$ in Lif go to diagrams $1 \to \Psi_*(X) \to \Psi_*(X)$ in Eff. If N is the natural number object of Eff, then the unique map: $N \to \Psi_*(X)$ transposes under $\Psi^* \dashv \Psi_*$ to a unique map $\Psi^*(N) \to X$, and the required diagram commutes. 3.6 now follows from the characterization of the natural number object in Eff, given in Hyland 1982.

3.7. *Interpretation of arithmetic in Lif.* We can now interpret **HA** directly in Lif, as follows. We consider a language $\mathcal{L}$ with one sort σ, a function symbol for every primitive recursive function, constants n for every natural number, and one relation symbol =. This language is interpreted in the tripos underlying Lif by: $[\![\sigma]\!]=\mathbb{N}$, the function symbols and constants have the obvious interpretation, and = is interpreted by the equality defined in 3.6.

We define a translation $(-)^+$ from the language of **HA** to $\mathcal{L}$ as follows: primitive recursive function symbols and equality of **HA** are translated by the corresponding function symbols and = of $\mathcal{L}$, respectively; prime formulas are translated in the obvious way. $(-)^+$ commutes with the propositional connectives, and the clauses for the quantifiers are:

$(\forall x\phi)^+ \equiv \quad \forall x(x=x\rightarrow(\phi)^+)$

$(\exists x\phi)^+ \equiv \quad \exists x(x=x \wedge (\phi)^+)$

We say that an arithmetical sentence ϕ in the language of **HA** is *valid* in Lif iff $(\phi)^+$ is valid in the tripos, by the interpretation of $\mathcal{L}$. We have:

3.8. Proposition. An arithmetical sentence is valid in Lif iff it is realizable in the sense of definition 2.2.

Proof. One defines, by an induction on the complexity of formulas ϕ with free variables $x_1,\ldots,x_k$ primitive recursive functions t_ϕ and s_ϕ of k arguments, such that for all $e,m_1,\ldots,m_k$:
i) if e **r** $\phi(m_1,\ldots,m_k)$ then $t_\phi(m_1,\ldots,m_k)\bullet e$ is defined and is an element of $[\![(\phi)^+]\!](m_1,\ldots,m_k)$;
ii) if $e\in[\![(\phi)^+]\!](m_1,\ldots,m_k)$ then $s_\phi(m_1,\ldots,m_k)\bullet e$ is defined and **r** $\phi(m_1,\ldots,m_k)$.
If ϕ is a prime formula t=s, then $[\![(\phi)^+]\!](m_1,\ldots,m_k) = \{e\in J \mid [e]=\{t\} \text{ and } t=s\}$ so we can put:
$t_\phi \equiv \Lambda m_1,\ldots,m_k.\Lambda e.\beta(t)$; $s_\phi \equiv \Lambda m_1,\ldots,m_k.\Lambda e.\beta(0)$. (Again, β is such that $[\beta(e)]=\{e\}$)
The induction steps for the propositional connectives are trivial. If ϕ is $\forall x\psi$, then
$[\![(\phi)^+]\!](m_1,\ldots,m_k) = \{e\in J \mid \forall f\in[e]\,\forall n\in\mathbb{N}\,\forall h\in J\,(f\bullet h\downarrow\ \&\ \forall g\in[f\bullet h]\forall w([w]=\{n\}\Rightarrow g\bullet w\downarrow\ \&$ $g\bullet w\in[\![(\psi)^+]\!](m_1,\ldots,m_k,n)))\}$. So if $e\in[\![(\phi)^+]\!](m_1,\ldots,m_k)$ then $\forall f\in[e]\,\forall n\in\mathbb{N}\ (f\bullet\beta(0)\downarrow$ $\&\forall g\in[f\bullet\beta(0)]\ (g\bullet\beta(n)\downarrow\ \&\ s_\psi(m_1,\ldots,m_k,n)\bullet(g\bullet\beta(n))$ **r** $\psi(m_1,\ldots,m_k,n)))$. So if $e'=\beta(\Lambda n.\bigcup\{s_\psi(m_1,\ldots,m_k,n)\bullet(g\bullet\beta(n)) \mid g\in[f\bullet\beta(0)],f\in[e]\})$, then e' **r** $\forall x\psi$. Conversely, if e **r** $\forall x\psi(m_1,\ldots,m_k)$ then $[e]\neq\emptyset\ \&\ \forall f\in[e]\forall n\ (f\bullet n\downarrow\ \&$ $t_\psi(m_1,\ldots,m_k,n)\bullet(f\bullet n)\in[\![(\psi)^+]\!](m_1,\ldots,m_k,n))$. Let δ be as in 2.1(ii). So if e' is such that $[e'] = \{\Lambda h.\beta(\Lambda w.\ t_\psi(m_1,\ldots,m_k,\delta(w))\bullet(f\bullet\delta(w)) \mid f\in[e]\}$, then $e'\in[\![(\phi)^+]\!](m_1,\ldots,m_k)$. In both cases, e' can evidently be obtained recursively in e (use 2.1).
The induction step for the existential quantifier, equally tedious, is left to the reader.

3.9. Proposition. CT and MP are valid in Lif.

Proof. Following the characterization of exponentials in realizability toposes in 2.14 (iii) of HJP 1980, the function space $\mathbb{N}^{\mathbb{N}}$ has as underlying set $\Sigma^{\mathbb{N}\times\mathbb{N}}$. The equality is given by: $[\![F=G]\!]$ is the interpretation of $E(F)\wedge E(G)\wedge\forall xy(Fxy\leftrightarrow Gxy)$ where E(F) is the universal closure of formula

$(F(x,y)\rightarrow x=x\wedge y='y) \wedge (F(x,y)\wedge x=x'\wedge y='y'\rightarrow F(x',y') \wedge (F(x,y)\wedge F(x,y')\rightarrow y='y') \wedge$ $(x=x\rightarrow \exists yF(x,y))$. Here of course = is interpreted as the equality on $\mathbb{N}$ given in 3.6. Now if $e\in [\![E(F)]\!]$ then we can find, recursively in e, $f\in [\![\forall xy\ (F(x,y)\wedge F(x,y')\rightarrow y='y')]\!]$, $g\in [\![\forall x(x=x\rightarrow \exists yF(x,y))]\!]$ and $h\in [\![\forall xy\ (F(x,y)\rightarrow x=x\wedge y='y)]\!]$. This means $\forall x(g'\bullet x\downarrow\ \&\ [g'\bullet x]\neq\emptyset\ \&\ \forall n\in [g'\bullet x]\exists y(n\in Fxy))$ for some g. Using h, we can find a z such that $\forall x(z\bullet x\downarrow\ \&\ [z\bullet x]\neq\emptyset\ \&\ \forall n\in [z\bullet x]\exists y(n\in Fxy\wedge y=y))$. But then, using f, we know that for every x, $\{(n)_1 \mid n\in [z\bullet x]\}$ must be a singleton. Using proposition 2.1(ii) there is a w such that $\forall x(w\bullet x\downarrow\ \&\ Fx(w\bullet x)$ is nonempty)). This w then codes a total recursive function, w can be found recursively in e, and, in Lif, F is the function coded by w.
Markov's Principle is easier: note that if $e\in [\![A\vee\neg A]\!]$ then $\{(a)_0 \mid a\in [e]\}$ must be a singleton.

Remarks: 1) Externalizing the argument in the proof of 3.8 one sees that morphisms $\mathbb{N}\rightarrow\mathbb{N}$ in Lif are in bijective correspondence with total recursive functions. This shows in particular that Lif is not equivalent to a subtopos of Eff of the form "recursive in A", as studied in Phoa 1990.

2) In Lif, the notions of Cauchy real and Dedekind real coincide. This is remarkable because usually this is a consequence of the validity of certain choice axioms, which we don't have in Lif. Already the simple choice schema AC_{00} fails for arithmetical formulas $A\in\Pi^0_1$.

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