Institute for Language, Logic and Information

A NOTE ON THE INTERPRETABILITY LOGIC OF FINITELY AXIOMATIZED THEORIES

Maarten de Rijke

ITLI Prepublication Series
for Mathematical Logic and Foundations ML-90-07

University of Amsterdam

The ITLI Prepublication Series

1986
86-01 The Institute of Language, Logic and Information
86-02 Peter van Emde Boas — A Semantical Model for Integration and Modularization of Rules
86-03 Johan van Benthem — Categorial Grammar and Lambda Calculus
86-04 Reinhard Muskens — A Relational Formulation of the Theory of Types
86-05 Kenneth A. Bowen, Dick de Jongh — Some Complete Logics for Branched Time, Part I Well-founded Time, Forward looking Operators
86-06 Johan van Benthem — Logical Syntax

1987
87-01 Jeroen Groenendijk, Martin Stokhof — Type shifting Rules and the Semantics of Interrogatives
87-02 Renate Bartsch — Frame Representations and Discourse Representations
87-03 Jan Willem Klop, Roel de Vrijer — Unique Normal Forms for Lambda Calculus with Surjective Pairing
87-04 Johan van Benthem — Polyadic quantifiers
87-05 Víctor Sánchez Valencia — Traditional Logicians and de Morgan's Example
87-06 Eleonore Oversteegen — Temporal Adverbials in the Two Track Theory of Time
87-07 Johan van Benthem — Categorial Grammar and Type Theory
87-08 Renate Bartsch — The Construction of Properties under Perspectives
87-09 Herman Hendriks — Type Change in Semantics: The Scope of Quantification and Coordination

1988

Logic, Semantics and Philosophy of Language:
LP-88-01 Michiel van Lambalgen — Algorithmic Information Theory
LP-88-02 Yde Venema — Expressiveness and Completeness of an Interval Tense Logic
LP-88-03 Year Report 1987
LP-88-04 Reinhard Muskens — Going partial in Montague Grammar
LP-88-05 Johan van Benthem — Logical Constants across Varying Types
LP-88-06 Johan van Benthem — Semantic Parallels in Natural Language and Computation
LP-88-07 Renate Bartsch — Tenses, Aspects, and their Scopes in Discourse
LP-88-08 Jeroen Groenendijk, Martin Stokhof — Context and Information in Dynamic Semantics
LP-88-09 Theo M.V. Janssen — A mathematical model for the CAT framework of Eurotra
LP-88-10 Anneke Kleppe — A Blissymbolics Translation Program

Mathematical Logic and Foundations:
ML-88-01 Jaap van Oosten — Lifschitz' Realizabiility
ML-88-02 M.D.G. Swaen — The Arithmetical Fragment of Martin Löf's Type Theories with weak Σ-elimination
ML-88-03 Dick de Jongh, Frank Veltman — Provability Logics for Relative Interpretability
ML-88-04 A.S. Troelstra — On the Early History of Intuitionistic Logic
ML-88-05 A.S. Troelstra — Remarks on Intuitionism and the Philosophy of Mathematics

Computation and Complexity Theory:
CT-88-01 Ming Li, Paul M.B.Vitanyi — Two Decades of Applied Kolmogorov Complexity
CT-88-02 Michiel H.M. Smid — General Lower Bounds for the Partitioning of Range Trees
CT-88-03 Michiel H.M. Smid, Mark H. Overmars, Leen Torenvliet, Peter van Emde Boas — Maintaining Multiple Representations of Dynamic Data Structures
CT-88-04 Dick de Jongh, Lex Hendriks, Gerard R. Renardel de Lavalette — Computations in Fragments of Intuitionistic Propositional Logic
CT-88-05 Peter van Emde Boas — Machine Models and Simulations (revised version)
CT-88-06 Michiel H.M. Smid — A Data Structure for the Union-find Problem having good Single-Operation Complexity
CT-88-07 Johan van Benthem — Time, Logic and Computation
CT-88-08 Michiel H.M. Smid, Mark H. Overmars, Leen Torenvliet, Peter van Emde Boas — Multiple Representations of Dynamic Data Structures
CT-88-09 Theo M.V. Janssen — Towards a Universal Parsing Algorithm for Functional Grammar
CT-88-10 Edith Spaan, Leen Torenvliet, Peter van Emde Boas — Nondeterminism, Fairness and a Fundamental Analogy
CT-88-11 Sieger van Denneheuvel, Peter van Emde Boas — Towards implementing RL

Other prepublications:
X-88-01 Marc Jumelet — On Solovay's Completeness Theorem

1989

Logic, Semantics and Philosophy of Language:
LP-89-01 Johan van Benthem — The Fine-Structure of Categorial Semantics
LP-89-02 Jeroen Groenendijk, Martin Stokhof — Dynamic Predicate Logic, towards a compositional, non-representational semantics of discourse
LP-89-03 Yde Venema — Two-dimensional Modal Logics for Relation Algebras and Temporal Logic of Intervals
LP-89-04 Johan van Benthem — Language in Action
LP-89-05 Johan van Benthem — Modal Logic as a Theory of Information
LP-89-06 Andreja Prijatelj — Intensional Lambek Calculi: Theory and Application
LP-89-07 Heinrich Wansing — The Adequacy Problem for Sequential Propositional Logic
LP-89-08 Víctor Sánchez Valencia — Peirce's Propositional Logic: From Algebra to Graphs
LP-89-09 Zhisheng Huang — Dependency of Belief in Distributed Systems

Mathematical Logic and Foundations:
ML-89-01 Dick de Jongh, Albert Visser — Explicit Fixed Points for Interpretability Logic
ML-89-02 Roel de Vrijer — Extending the Lambda Calculus with Surjective Pairing is conservative
ML-89-03 Dick de Jongh, Franco Montagna — Rosser Orderings and Free Variables
ML-89-04 Dick de Jongh, Marc Jumelet, Franco Montagna — On the Proof of Solovay's Theorem
ML-89-05 Rineke Verbrugge — Σ-completeness and Bounded Arithmetic
ML-89-06 Michiel van Lambalgen — The Axiomatization of Randomness
ML-89-07 Dirk Roorda — Elementary Inductive Definitions in HA: from Strictly Positive towards Monotone
ML-89-08 Dirk Roorda — Investigations into Classical Linear Logic
ML-89-09 Alessandra Carbone — Provable Fixed points in $I\Delta_0+\Omega_1$

Computation and Complexity Theory:
CT-89-01 Michiel H.M. Smid — Dynamic Deferred Data Structures
CT-89-02 Peter van Emde Boas — Machine Models and Simulations
CT-89-03 Ming Li, Herman Neuféglise, Leen Torenvliet, Peter van Emde Boas — On Space Efficient Simulations
CT-89-04 Harry Buhrman, Leen Torenvliet — A Comparison of Reductions on Nondeterministic Space
CT-89-05 Pieter H. Hartel, Michiel H.M. Smid, Leen Torenvliet, Willem G. Vree — A Parallel Functional Implementation of Range Queries
CT-89-06 H.W. Lenstra, Jr. — Finding Isomorphisms between Finite Fields
CT-89-07 Ming Li, Paul M.B. Vitanyi — A Theory of Learning Simple Concepts under Simple Distributions and Average Case Complexity for the Universal Distribution (Prel. Version)
CT-89-08 Harry Buhrman, Steven Homer, Leen Torenvliet — Honest Reductions, Completeness and Nondeterminstic Complexity Classes
CT-89-09 Harry Buhrman, Edith Spaan, Leen Torenvliet — On Adaptive Resource Bounded Computations
CT-89-10 Sieger van Denneheuvel — The Rule Language RL/1
CT-89-11 Zhisheng Huang, Sieger van Denneheuvel, Peter van Emde Boas — Towards Functional Classification of Recursive Query Processing

Other Prepublications:
X-89-01 Marianne Kalsbeek — An Orey Sentence for Predicative Arithmetic
X-89-02 G. Wagemakers — New Foundations: a Survey of Quine's Set Theory
X-89-03 A.S. Troelstra — Index of the Heyting Nachlass
X-89-04 Jeroen Groenendijk, Martin Stokhof — Dynamic Montague Grammar, a first sketch
X-89-05 Maarten de Rijke — The Modal Theory of Inequality
X-89-06 Peter van Emde Boas — Een Relationele Semantiek voor Conceptueel Modelleren: Het RL-project

1990 SEE INSIDE BACK COVER

Instituut voor Taal, Logica en Informatie
Institute for Language, Logic and Information

Faculteit der Wiskunde en Informatica
(Department of Mathematics and Computer Science)
Plantage Muidergracht 24
1018TV Amsterdam

Faculteit der Wijsbegeerte
(Department of Philosophy)
Nieuwe Doelenstraat 15
1012CP Amsterdam

A NOTE ON THE INTERPRETABILITY LOGIC OF FINITELY AXIOMATIZED THEORIES

Maarten de Rijke
Department of Mathematics and Computer Science
University of Amsterdam

ITLI Prepublications
for Mathematical Logic and Foundations
ISSN 0924-2090

Received August 1990

Research supported by the
Netherlands Organization for Scientific Research (NWO)

A Note on the Interpretability Logic of Finitely Axiomatized Theories

Maarten de Rijke*

Department of Mathematics and Computer Science
University of Amsterdam

July 1990

1 Introduction

In [5] Albert Visser introduces a logic *ILP* in a modal language $\mathcal{L}(\Box, \rhd)$ with a unary operator $\Box$, to be interpreted arithmetically as provability, and a binary operator $\rhd$, to be interpreted arithmetically as relative interpretability over some fixed theory U. In [6] he shows that *ILP* completely axiomatizes all schemata about provability and relative interpretability that are provable in Σ_1^0-sound finitely axiomatized sequential theories U that extend $\mathrm{I}\Delta_0 + \mathrm{SupExp}$. In this paper we give a somewhat different proof of this result; we also present a complete axiomatization, called ILP^ω, of all *true* such schemata.

The main difference between Visser's proof of the arithmetical completeness of *ILP* and ours, is that we use infinite Kripke-like models, instead of finite ones, to find arithmetical interpretations for unprovable modal formulas. The models we use are variations on the *tail models* for provability logic as developed by Albert Visser (cf. [4]). The advantage of using these infinite models is two-fold. First of all, it allows us to set up things in such a way, that we can prove the arithmetical completeness of *ILP* and ILP^ω (almost) in one go.

To understand the second advantage, recall that the arithmetical sentences needed to prove the arithmetical completeness of some given logic Λ are usually found by embedding models of Λ into arithmetic. If these models are finite, the embedding will only be partial, in the following sense. Consider a formula $A(\vec{p})$ as a polynomial in the truth values of the *p*s, and suppose that $[B]$ is a representation in arithmetic of the extension of B in a given model. To justify the use of the phrase 'embedding into arithmetic' we want the equivalence $A([\vec{p}]) \leftrightarrow [A(\vec{p})]$ to be provable in our arithmetical theory, for all formulas A.

*Research supported by the Netherlands Organization for Scientific Research (NWO).

But, assuming that our arithmetical theory is Σ_1^0-sound, this is not possible when we are working with finite models: for if $\mathcal{M}$ is such a model then for some n, $\mathcal{M} \models \Box^n\bot \leftrightarrow \Box^{n+1}\bot$. By using infinite models we will be able to obtain complete embeddings.

The rest of this paper is organized as follows: in §2 the systems ILP and ILP^ω are introduced; in §3 we review the arithmetical notions we need and assumptions we make for our completeness results. Then, in §4, we state and prove the arithmetical completeness of ILP and ILP^ω.

Finally, a word on prerequisites: we assume that the reader is familiar with the discussion of systems and arithmetization in [7].

2 The systems ILP and ILP^ω

The provability logic L is propositional logic plus all instance of the schemas $\Box(A \to B) \to (\Box A \to \Box B)$, $\Box A \to \Box\Box A$ and $\Box(\Box A \to A) \to \Box A$; its rules of inference are A, $A \to B/B$ (Modus Ponens), and $A/\Box A$ (Necessitation). Let $\mathcal{L}(\Box, \rhd)$ denote the language of interpretability logic. The interpretability logic ILP extends L with all instances of the following schemas:

$(J1)$	$\Box(A \to B) \to A \rhd B$;	$(J4)$	$A \rhd B \to (\Diamond A \to \Diamond B)$;
$(J2)$	$(A \rhd B) \wedge (B \rhd C) \to A \rhd C$;	$(J5)$	$A \rhd \Diamond A$;
$(J3)$	$(A \rhd C) \wedge (B \rhd C) \to (A \rhd C)$;	(P)	$A \rhd B \to \Box(A \rhd B)$.

ILP^ω has as axioms all theorems of ILP plus all instances of the schema of *reflection*: $\Box A \to A$; its sole rule of inference is Modus Ponens. Since $ILP \vdash \Box A \leftrightarrow \neg A \rhd \bot$, we may consider '$\Box$' to be defined in terms of '$\rhd$'.

ILP has been shown to be modally complete with respect to two kinds of (finite) models, notably w.r.t. Veltman models for ILP in [1], and w.r.t. Friedman models for ILP in [6].

Definition 2.1 A *Friedman tail model* is a tuple $\mathcal{M} = \langle \omega, 0, Q, P, \Vdash \rangle$ with

1. $Q \subseteq \omega^2$ is transitive, irreflexive and tree-like;
2. $P \subseteq Q$ is given by a set $X \subseteq \omega$ such that $0 \in X$, and $xPy \Leftrightarrow xQy$ and $y \in X$, and such that $y \in X$, yPz implies $yQz'Pz$, for some z';
3. if $xQyPz$ then xPz;
4. if $n \neq 0$ then $0Qn$, and if $0 \neq nQm$ then $n > m$;
5. there is an $N \in \omega$ such that
    (a) for every n, $m \geq N$, if $m < n$ then nQm;
    (b) for every $n \geq N$, if for some k, $n = 2k + N$ then mPn for all $m > n$;
    (c) for every $n \geq N$, $N \Vdash p$ iff $n \Vdash p$ iff $0 \Vdash p$.

An N which satisfies 5 is called a *tail element*. We define $R := Q \circ P$, i.e., xRy iff $\exists z\, xQzPy$. $\Vdash$ satisfies the usual clauses, with R as the accessibility relation for '$\Box$', and

$$x \Vdash A \rhd B \iff \forall u\,(xQu \Rightarrow (\exists y\,(uPy \wedge y \Vdash A) \Rightarrow \exists z\,(uPz \wedge z \Vdash B))).$$

Finally, if $\mathcal{M}$ is a Friedman tail model, and A a formula. Then $[\![A]\!]_{\mathcal{M}} := \{x \in \mathcal{M} : x \Vdash A\}$.

Definition 2.2 We introduce two operators $\triangle_p$, $\triangle_q$ with forcing conditions $x \Vdash \triangle_p A$ iff for all y with xPy, $y \Vdash A$, and $x \Vdash \triangle_q A$ iff for all y with xQy, $y \Vdash A$. We write ∇_p, ∇_q for $\neg\triangle_p\neg$, $\neg\triangle_q\neg$ respectively. $\mathcal{L}(\triangle_p, \triangle_q)$ denotes the language with the two new operators.

Define a translation $(\cdot)^\tau : \mathcal{L}(\Box, \rhd) \longrightarrow \mathcal{L}(\triangle_p, \triangle_q)$ as follows: $(\cdot)^\tau$ is the identity on proposition letters and the constants $\top$, $\bot$, and it commutes with the Boolean connectives; furthermore $(A \rhd B)^\tau := \triangle_q(\nabla_p A^\tau \to \nabla_p B^\tau)$.

We write $\tau\mathcal{L}(\Box, \rhd)$ for the image of $\mathcal{L}(\Box, \rhd)$ under τ, and define $\tau\mathcal{L}(\Box, \rhd)^*$ to be the sublanguage of $\mathcal{L}(\triangle_p, \triangle_q)$ in which $\triangle_q$ occurs only in front of implications of the form $\nabla_p C \to \nabla_p D$. Clearly, then, $\tau\mathcal{L}(\Box, \rhd) \subseteq \tau\mathcal{L}(\Box, \rhd)^*$.

Remark 2.3 $\mathcal{L}(\triangle_p, \triangle_q)$ is in fact the language of the bi-modal provability logic PRL_1 discussed in [3] (with the modal operators interpreted as *tableaux provability* instead of ordinary provability). Using $(\cdot)^\tau$ and 2.7 one easily verifies that PRL_1 is a conservative extension of ILP.

Proposition 2.4 *Let $\mathcal{M}$ be a Friedman tail model, and let $A \in \mathcal{L}(\Box, \rhd)$. Then for all $n \in \mathcal{M}$, $n \Vdash A \leftrightarrow A^\tau$.*

Proposition 2.5 *Let $\mathcal{M}$ be a Friedman tail model in which P is given by some set X. Let $\triangle_q B \in \tau\mathcal{L}(\Box, \rhd)^*$. If $n \in X$ and $n \Vdash \triangle_q B$ then $n \Vdash B$.*

Proof. If $\triangle_q B \in \tau\mathcal{L}(\Box, \rhd)^*$ then B has the form $\nabla_p C \to \nabla_p D$. Moreover, if $n \in X$ and nPm then nRm. These observations yield the result. QED.

Proposition 2.6
1. *Let $A \in \mathcal{L}(\Box, \rhd) \cup \mathcal{L}(\triangle_p, \triangle_q)$. Then $[\![A]\!]_{\mathcal{M}}$ is either finite or cofinite.*
2. *$0 \Vdash A$ iff for some N and all $n \geq N$, $n \Vdash A$;*
3. *$0 \nVdash A$ iff for some N and all $n \geq N$, $n \nVdash A$.*

Theorem 2.7 *Let $A \in \mathcal{L}(\Box, \rhd)$. Then*

1. *$ILP \vdash A$ iff for every Friedman tail model $\mathcal{M}$, and all $n \in \mathcal{M}$, $n \Vdash A$;*
2. *$ILP^\omega \vdash A$ iff for every Friedman tail model $\mathcal{M}$, $0 \Vdash A$.*

Proof. The first claim is immediate from [6, Theorem 8.1]. To prove the second one, note first that 0 forces the theorems of ILP in any Friedman tail model. Closure under Modus Ponens is trivial. Assume that $0 \Vdash \Box A$, then for all n

with $0Rn$, $n \Vdash A$. So $[\![A]\!]_{\mathcal{M}}$ is infinite, and hence, by 2.6, cofinite. Thus $0 \Vdash A$, again by 2.6.

Next assume that ILP^{ω}, then, obviously, $ILP \nvdash \mathrm{T}(A) \rightarrow A$, where $\mathrm{T}(A) := \bigwedge_{\Box B \in \mathrm{S}(A)}(\Box B \rightarrow B) \wedge \bigwedge_{C \rhd D \in \mathrm{S}(A)}(C \rightarrow \Diamond C)$, and $\mathrm{S}(A)$ is the set of subformulas of A. So by 1 there is a tail model $\mathcal{M}$ such that for some tail element N in $\mathcal{M}$, $N \Vdash \mathrm{T}(A) \wedge \neg A$. An easy induction now establishes that for $C \in \mathrm{S}(A)$ the following facts hold: (1) if $N \Vdash C$ then for all m with mRN, $m \Vdash C$; and (2) if $N \nVdash C$ then for all m with mRN, $m \nVdash C$. So $0 \nVdash A$. QED.

3 Arithmetical completeness: preliminaries

To prove the arithmetical completeness of ILP^{ω} we want to use several results from [6]. To be able to do so, we only consider arithmetical theories that satisfy a number of conditions to be given now. (Details about the notions used below may be looked up in [2], [5], [6] and [7].)

Officially we will be working in a relational version of the language of arithmetic, in which successor, addition and multiplication are (2-, 3- and 3-place) relation symbols. We will, however, *pretend* that we are working with function symbols. We assume that the theories T we consider are given by an R_1^+-formula $\alpha_T(x)$ having just x free plus the relevant information on what the set of natural numbers of T is; α_T gives the set of codes of non-logical axioms of the theory (cf. [7]). We also assume that the numbers of T satisfy $\mathrm{I}\Delta_0 + \Omega_1$, and that T is finitely axiomatized and sequential.

Wilkie and Paris [7] show that $\mathrm{I}\Delta_0 + \Omega_1$ is a completely adequate theory for arithmetizing syntax. E.g., if T is a theory satisfying the assumptions made above, we can formalize in $\mathrm{I}\Delta_0 + \Omega_1$ (as an R_1^+-formula) $\mathrm{Proof}_T(x, y)$, which represents the relation 'x is a proof of the formula y from T'. We further define $\mathrm{Prov}_T(y) := \exists x\, \mathrm{TabProof}_T(x, y)$.

One of the key results needed to prove our arithmetical completeness results, is a result by Friedman, extended by Visser, that gives a characterization of interpretability in terms of consistency. To state it we need a notion of cut free proof. We follow [6] in choosing tableaux provability. We write $\mathrm{TabProof}_T(x, y)$ for (a formalization of) the relation 'x is a tableau proof of the formula y from T'. Furthermore, $\mathrm{TabProv}_T(y) := \exists x\, \mathrm{TabProof}_T(x, y)$, and $\mathrm{TabCon}_T(\ulcorner\varphi\urcorner) := \neg\mathrm{TabProv}_T(\ulcorner\neg\varphi\urcorner)$. Using this notation we can state the Friedman-Visser characterization as follows: let U be finitely axiomatized and sequential, and let Interp_U denote (a formalization of) relative interpretability over U, then $\mathrm{I}\Delta_0 + \mathrm{Exp}$ proves

$$\mathrm{Interp}_U(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) \leftrightarrow \mathrm{TabProv}_{Exp}(\ulcorner\mathrm{TabCon}_U(\ulcorner\varphi\urcorner) \rightarrow \mathrm{TabCon}_U(\ulcorner\psi\urcorner)\urcorner).$$

A proof of this result may be found in [6].

4 Arithmetical completeness: the main result

For the remainder of this paper, let U be a Σ^0_1-sound extension of $I\Delta_0 + \mathrm{SupExp}$ that satisfies all the requirements from §3.

Our first aim is to embed Friedman tail models into U. To do so we fix $\mathcal{M} = \langle \omega, 0, Q, P, \Vdash \rangle$ to be a tail model; we assume that P is given by a set X as in item 2 of the definition of a Friedman tail model. Define as formulas in the language of U:

$$(x \in [\![A]\!]_{\mathcal{M}}) := \begin{cases} \bigvee\{(x = \underline{i}) : i \Vdash A\}, & \text{if } [\![A]\!]_{\mathcal{M}} \text{ is finite} \\ \bigwedge\{(x \neq \underline{i}) : i \nVdash A\}, & \text{if } [\![A]\!]_{\mathcal{M}} \text{ is cofinite.} \end{cases}$$

It is easily verified that $I\Delta_0 + \mathrm{Exp}$ proves

- $(x \in [\![A]\!]_{\mathcal{M}}) \wedge (x \in [\![B]\!]_{\mathcal{M}}) \leftrightarrow (x \in [\![A \wedge B]\!]_{\mathcal{M}})$;
- $(x \in [\![A]\!]_{\mathcal{M}}) \vee (x \in [\![B]\!]_{\mathcal{M}}) \leftrightarrow (x \in [\![A \vee B]\!]_{\mathcal{M}})$;
- $x \notin [\![A]\!]_{\mathcal{M}} \leftrightarrow x \in [\![\neg A]\!]_{\mathcal{M}}$.

Using the Recursion Theorem we define a Solovay-like function H as follows:

$$\begin{aligned} H(0) &= 0 \\ H(x+1) &= \begin{cases} y, & \text{if } H(x)Py \text{ and } \mathrm{TabProof}_U(x+1, \ulcorner L \neq y \urcorner) \\ y, & \text{if } H(x)Qy \text{ and } \mathrm{TabProof}_{Exp}(x+1, \ulcorner L \neq y \urcorner) \\ H(x), & \text{otherwise} \end{cases} \\ L &= \text{the limit of } H. \end{aligned}$$

We leave it to the reader to check (or to look up in the proof of [6, Theorem 8.2]) that the formula '$H(x) = u$' is $\Delta_0(2^x)$, and that for any x, y

1. $I\Delta_0 + \mathrm{Exp} \vdash xQy \to H(x)\underline{Q}H(y)$;
2. $I\Delta_0 + \mathrm{Exp} \vdash$ 'L exists';
3. $I\Delta_0 + \mathrm{Exp} \vdash L = x \leftrightarrow \exists y\,(H(y) = x) \wedge \forall uv\,(H(u) = x \wedge v > u \to H(v) = x)$;
4. $L = 0$.

Definition 4.1 We define the representation $[A]_{\mathcal{M}}$ of $[\![A]\!]_{\mathcal{M}}$ in the language of U by $[A]_{\mathcal{M}} := (L \in [\![A]\!]_{\mathcal{M}}$.

Let g be any function that takes the proposition letters from $\mathcal{L}(\Box, \rhd)$ (or $\mathcal{L}(\Delta_p, \Delta_q)$) to sentences in the language of arithmetic. Then the *arithmetical interpretation* $\langle\cdot\rangle_g$ of $\mathcal{L}(\Box, \rhd) \cup \mathcal{L}(\Delta_p, \Delta_q)$ into the language of arithmetic is defined by

$$\begin{aligned} \langle p \rangle_g &:= [p]_g & \langle \Box A \rangle_g &:= \mathrm{Prov}_U(\ulcorner \langle A \rangle_g \urcorner) \\ \langle \bot \rangle_g &:= \text{`}0 = 1\text{'} & \langle A \rhd B \rangle_g &:= \mathrm{Interp}_U(\ulcorner \langle A \rangle_g \urcorner, \ulcorner \langle B \rangle_g \urcorner) \\ \langle \neg A \rangle_g &:= \neg \langle A \rangle_g & \langle \Delta_p A \rangle_g &:= \mathrm{TabProv}_U(\ulcorner \langle A \rangle_g \urcorner) \\ \langle A \wedge B \rangle_g &:= \langle A \rangle_g \wedge \langle B \rangle_g & \langle \Delta_q A \rangle_g &:= \mathrm{TabProv}_{Exp}(\ulcorner \langle A \rangle_g \urcorner). \end{aligned}$$

In case $g(p) = [p]_{\mathcal{M}}$ for some model $\mathcal{M}$, we write $\langle\cdot\rangle_{\mathcal{M}}$ for $\langle\cdot\rangle_g$.

Proposition 4.2 *Let* $\psi \in \Pi_2^0$. *Then* $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash \mathrm{TabProv}_{Exp}(\ulcorner\psi\urcorner) \to \mathrm{TabProv}_U(\ulcorner\psi\urcorner)$.

Proof. Cf. [6, Lemma 8.2]. QED.

Proposition 4.3 $U \vdash L \in X$.

Proof. Reason in U: by our earlier remarks the limit L exists. So assume $L = \underline{i} \notin X$. Then, by the definition of H, $i > 0$ and $\mathrm{TabProv}_{Exp}(\ulcorner L \neq \underline{i}\urcorner)$. Recall that U extends $\mathrm{I}\Delta_0 + \mathrm{SupExp}$. By [6, Consequence 7.3.7], $\mathrm{I}\Delta_0 + \mathrm{SupExp}$ proves Π_2^0-reflection for $\mathrm{I}\Delta_0 + \mathrm{Exp}$. (This is in fact the only place where we really need U to be an extension of $\mathrm{I}\Delta_0 + \mathrm{SupExp}$.) Therefore, in U we have $L \neq \underline{i}$—a contradiction. QED.

Lemma 4.4 *Let* $A \in \tau\mathcal{L}(\Box, \rhd)^*$. *Then* $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash [A]_{\mathcal{M}} \leftrightarrow \langle A\rangle_{\mathcal{M}}$.

Proof. Induction on A. The propositional case, and the Boolean cases are immediate from the fact that the limit provably exists, and the induction hypothesis, respectively.

Suppose $A \equiv \triangle_p B$. First we assume that $[\![\triangle_p B]\!]_{\mathcal{M}}$ is cofinite. Then $[\![\triangle_p B]\!]_{\mathcal{M}} = \omega$. So $[\triangle_p B] \equiv \bigwedge\{(L \neq \underline{i} : i \Vdash \bot\} \equiv \top$. So $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash [\triangle_p B]_{\mathcal{M}}$, and hence $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash \langle\triangle_p B\rangle_{\mathcal{M}} \to [\triangle_p B]_{\mathcal{M}}$. To prove the other direction it suffices to show that $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash \mathrm{TabProv}_U(\ulcorner[B]_{\mathcal{M}}\urcorner)$. Clearly, $[\![B]\!]_{\mathcal{M}}$ is cofinite and $X \subseteq [\![B]\!]_{\mathcal{M}}$; therefore $[B]_{\mathcal{M}} \equiv \bigwedge\{(L \neq \underline{i}) : i \nVdash B\}$. Reason in $\mathrm{I}\Delta_0 + \mathrm{Exp}$: if $i \nVdash B$, then $\mathrm{TabProv}_U(\ulcorner L \neq \underline{i}\urcorner)$, because $U \vdash L \in X$. Therefore $\mathrm{TabProv}_U(\ulcorner[B]_{\mathcal{M}}\urcorner)$.

Next we assume that $[\![\triangle_p B]\!]_{\mathcal{M}}$ is finite. Let $\{j_0, \ldots, j_s\}$ be all j with $j \Vdash \triangle_p B, \neg B$. Then, if $i \nVdash \triangle_p B$, there is a $j \in \{j_0, \ldots, j_s\}$ with iPj. By the induction hypothesis it suffices to show that $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash [\triangle_p B]_{\mathcal{M}} \leftrightarrow \mathrm{TabProv}_U(\ulcorner[B]_{\mathcal{M}}\urcorner)$. Reason in $\mathrm{I}\Delta_0 + \mathrm{Exp}$:

'$\leftarrow$': Assume $\mathrm{TabProv}_U(\ulcorner[B]_{\mathcal{M}}\urcorner)$. Let $j \in \{j_0, \ldots, j_s\}$. It follows that $\mathrm{TabProv}_U(\ulcorner L \neq \underline{j}\urcorner)$. So assume that $\mathrm{TabProof}_U(p+1, \ulcorner L \neq \underline{j}\urcorner)$. If LPj then $H(p)Pj$—so $H(p+1) = j$, which is a contradiction. Therefore, $\neg LPj$, so $\bigvee\{(L = \underline{i}) : i \Vdash \triangle_p B\}$.

'$\to$': Assume $L = \underline{i}$, $i \Vdash \triangle_p B$. Then $i \neq 0$. So $\mathrm{TabProv}_{Exp}(\ulcorner L \neq \underline{i}\urcorner)$ or $\mathrm{TabProv}_U(\ulcorner L \neq \underline{i}\urcorner)$, so by 4.2 $\mathrm{TabProv}_U(\ulcorner L \neq \underline{i}\urcorner)$. We also have for some x, $H(x) = i$, and hence $\mathrm{TabProv}_U(\ulcorner\exists x\, H(x) = i\urcorner)$. This implies $\mathrm{TabProv}_U(\ulcorner iQL\urcorner)$, which entails $\mathrm{TabProv}_U(\ulcorner iPL\urcorner)$, because $U \vdash L \in X$. Finally, iPj implies $j \Vdash B$. Therefore $\mathrm{TabProv}_U(\ulcorner\bigvee\{(L = \underline{j} : J \Vdash B\}\urcorner)$.

Assume next that $A \equiv \triangle_q B$, and that $[\![\triangle_q B]\!]_{\mathcal{M}}$ is cofinite. Then $[\![\triangle_q B]\!]_{\mathcal{M}} = [\![B]\!]_{\mathcal{M}} = \omega$. So by the induction hypothesis $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash \langle B\rangle_{\mathcal{M}}$, and hence

$$\begin{aligned}\mathrm{I}\Delta_0 + \mathrm{Exp} &\vdash \mathrm{TabProv}_{Exp}(\ulcorner\langle B\rangle_{\mathcal{M}}\urcorner)\\ &\vdash \langle\triangle_q B\rangle_{\mathcal{M}} \leftrightarrow \top\\ &\vdash \langle\triangle_q B\rangle_{\mathcal{M}} \leftrightarrow [\triangle_q B]_{\mathcal{M}}.\end{aligned}$$

Next we assume that $[\![\triangle_q B]\!]_{\mathcal{M}}$ is finite. As in the case of $\triangle_p B$, let $\{ j_0, \ldots, j_s \}$ be all j with $j \Vdash \triangle_q B$, $\neg B$. Then, if $i \nVdash \triangle_q B$, there is a $j \in \{ j_0, \ldots, j_s \}$ with iQj. By the induction hypothesis it suffices to show that $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash [\triangle_q B]_{\mathcal{M}} \leftrightarrow \mathrm{TabProv}_{Exp}(\ulcorner [B]_{\mathcal{M}} \urcorner)$. Reason in $\mathrm{I}\Delta_0 + \mathrm{Exp}$:

'$\leftarrow$': This direction is analogous to the corresponding direction in the case of $\triangle_p B$.

'$\rightarrow$': Assume $L = \underline{i}$, $i \Vdash \triangle_q B$. Then there exists an x with $H(x) = i$. So $\mathrm{TabProv}_{Exp}(\ulcorner \exists x\, H(x) = i \urcorner)$, and hence $\mathrm{TabProv}_{Exp}(\ulcorner \bigvee \{ (L = \underline{k} : i\underline{Q}k \} \urcorner)$. Now, if $i \notin X$, then $\mathrm{TabProv}_{Exp}(\ulcorner L \neq \underline{i} \urcorner)$ by the definition of H. Therefore $\mathrm{TabProv}_{Exp}(\ulcorner \bigvee \{ (L = \underline{k}) : iQk \} \urcorner)$. Thus $\mathrm{TabProv}_{Exp}(\ulcorner \bigvee \{ (L = \underline{k}) : k \Vdash B \} \urcorner)$. And if, on the other hand, $i \in X$, then $i \Vdash \triangle_q B$ implies by 2.5 that $i \Vdash B$. Again we have $\mathrm{TabProv}_{Exp}(\ulcorner \bigvee \{ (L = \underline{k}) : k \Vdash B \} \urcorner)$. QED.

Lemma 4.5 *Let* $A \in \mathcal{L}(\Box, \rhd)$. *Then* $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash [A]_{\mathcal{M}} \leftrightarrow \langle A \rangle_{\mathcal{M}}$.

Proof. Since, by 2.4, for all $A \in \mathcal{L}(\Box, \rhd)$, and all i, $i \Vdash A \leftrightarrow A^\tau$, we trivially have $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash [A]_{\mathcal{M}} \leftrightarrow [A^\tau]_{\mathcal{M}}$. Since $A^\tau \in \tau\mathcal{L}(\Box, \rhd)^*$, we can apply Lemma 4.4 to conclude that $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash [A]_{\mathcal{M}} \leftrightarrow \langle A^\tau \rangle_{\mathcal{M}}$ $(\star)$.

By the Friedman-Visser characterization of relative interpretability over finitely axiomatized sequential theories, $\mathrm{I}\Delta_0 + \mathrm{Exp}$ proves

$$\mathrm{Interp}_U(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner) \leftrightarrow \mathrm{TabProv}_{Exp}(\ulcorner \mathrm{TabCon}_U(\ulcorner \varphi \urcorner) \rightarrow \mathrm{TabCon}_U(\ulcorner \psi \urcorner) \urcorner).$$

This characterization allows us to show by induction on A that $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash \langle A^\tau \rangle_{\mathcal{M}} \leftrightarrow \langle A \rangle_{\mathcal{M}}$. Together with $(\star)$ this yields the Lemma. QED.

We need one more definition and a proposition before we can prove the arithmetical completeness of *ILP* and ILP^ω. From now on $\mathcal{M}$ is no longer a fixed Friedman tail model.

Definition 4.6 Let $\mathcal{M}$ be a Friedman tail model. Define $\mathrm{d}_{\mathcal{M}}(k) := \sup\{ \mathrm{d}_{\mathcal{M}}(l) + 1 : kRl \}$, and

$$\mathrm{d}(A) := \begin{cases} \mu n.\, \exists \mathcal{M} \exists m\, (\mathrm{d}_{\mathcal{M}}(m) = n \wedge m \nVdash A), & \text{if such an } n \text{ exists} \\ \omega, & \text{otherwise.} \end{cases}$$

Proposition 4.7 *Let* $A \in \mathcal{L}(\Box, \rhd)$. *Then there is a function* g *that takes proposition letters to sentences in the language of* U *such that* $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash \langle A \wedge \Box A \rangle_g \leftrightarrow \mathrm{Prov}_U^{\mathrm{d}(A)}(\ulcorner 0 = 1 \urcorner)$.

Proof. If $\mathrm{d}(A) = \omega$ then $ILP \vdash A$, so any g does the trick. If $\mathrm{d}(A) < \omega$, then there is a tail model $\mathcal{M}$ with tail element N such that $\mathrm{d}_{\mathcal{M}}(N) = n$, and $N \nVdash A$. Define $g(p) := [p]_{\mathcal{M}}$. Then for every k with NRk, $k \Vdash A \wedge \Box A$. Moreover, if $k\underline{R}N$, then $k \nVdash A \wedge \Box A$. Therefore

$$\begin{aligned} \mathrm{I}\Delta_0 + \mathrm{Exp} \vdash \langle A \wedge \Box A \rangle_g &\leftrightarrow [A \wedge \Box A]_{\mathcal{M}}, \text{ by 4.5} \\ &\leftrightarrow [\Box^{\mathrm{d}(A)} \bot] \\ &\leftrightarrow \mathrm{Prov}_U^{\mathrm{d}(A)}(\ulcorner 0 = 1 \urcorner). \text{ QED.} \end{aligned}$$

Theorem 4.8 *Let $A \in \mathcal{L}(\Box, \rhd)$. Then ILP $\vdash A$ iff for every interpretation $\langle\cdot\rangle_g$, $U \vdash \langle A\rangle_g$.*

Proof. The direction from left to right is left to the reader. To prove the other one, assume that $ILP \nvdash A$. Then there is a tail model $\mathcal{M}$ and a tail element N such that $d_{\mathcal{M}}(N) = d(A) < \omega$, $N \not\Vdash A$ and $I\Delta_0 + \mathrm{Exp} \vdash \langle A \wedge \Box A\rangle_{\mathcal{M}} \leftrightarrow \mathrm{Prov}_U^{d(A)}(\ulcorner 0 = 1\urcorner)$. If $U \vdash \langle A\rangle_{\mathcal{M}}$ then $U \vdash \langle A \wedge \Box A\rangle_{\mathcal{M}}$, and hence $U \vdash \mathrm{Prov}_U^{d(A)}(\ulcorner 0 = 1\urcorner)$—contradicting our assumption that U is Σ_1^0-sound. Conclude that $U \nvdash \langle A\rangle_{\mathcal{M}}$. QED.

Theorem 4.9 *Let $A \in \mathcal{L}(\Box, \rhd)$. Then $ILP^{\omega} \vdash A$ iff for every interpretation $\langle\cdot\rangle_g$, $\mathbb{N} \models \langle A\rangle_g$.*

Proof. Again, the direction from left to right is left to the reader. Assume, to prove the converse, that $ILP^{\omega} \nvdash A$. Then there is a Friedman tail model $\mathcal{M}$ with $0 \not\Vdash A$. By 4.5 $\mathbb{N} \models \langle A\rangle_{\mathcal{M}} \leftrightarrow [A]_{\mathcal{M}}$. Moreover, $\mathbb{N} \models L = 0$. It follows that $\mathbb{N} \models \neg\langle A\rangle_{\mathcal{M}}$. QED.

References

[1] Dick de Jongh and Frank Veltman. Provability Logics for Relative Interpretability. In: P.P. Petkov (ed.) *Mathematical Logic, Proceedings of the 1988 Heyting Conference*, Plenum Press, New York, 1990, 31–42.

[2] P. Pudlák. Cuts, Consistency and Interpretations. *Journal of Symbolic Logic* **50** (1985), 423–441.

[3] Craig Smoryński. *Self-Reference and Modal Logic*. Springer Verlag, New York, 1985.

[4] Albert Visser. The Provability Logics of Recursively Enumerable Theories Extending Peano Arithmetic at Arbitrary Theories Extending Peano Arithmetic. *Journal of Philosophical Logic* **13** (1984), 97–113.

[5] Albert Visser. Preliminary Notes on Interpretability Logic. *Logic Group Preprint Series* No. 14, Department of Philosophy, University of Utrecht, 1988.

[6] Albert Visser. Interpretability Logic. In: P.P. Petkov (ed.) *Mathematical Logic, Proceedings of the 1988 Heyting Conference*, Plenum Press, New York, 1990, 175–210.

[7] A.J. Wilkie and J.B. Paris. On the Scheme of Induction for Bounded Arithmetic Formulas. *Annals of Pure and Applied Logic* **35** (1987), 261–302.

The ITLI Prepublication Series

1990

Logic, Semantics and Philosophy of Language

LP-90-01 Jaap van der Does	A Generalized Quantifier Logic for Naked Infinitives
LP-90-02 Jeroen Groenendijk, Martin Stokhof	Dynamic Montague Grammar
LP-90-03 Renate Bartsch	Concept Formation and Concept Composition
LP-90-04 Aarne Ranta	Intuitionistic Categorial Grammar
LP-90-05 Patrick Blackburn	Nominal Tense Logic
LP-90-06 Gennaro Chierchia	The Variablity of Impersonal Subjects
LP-90-07 Gennaro Chierchia	Anaphora and Dynamic Logic
LP-90-08 Herman Hendriks	Flexible Montague Grammar
LP-90-09 Paul Dekker	The Scope of Negation in Discourse, towards a flexible dynamic Montague grammar
LP-90-10 Theo M.V. Janssen	Models for Discourse Markers
LP-90-11 Johan van Benthem	General Dynamics

Mathematical Logic and Foundations

ML-90-01 Harold Schellinx	Isomorphisms and Non-Isomorphisms of Graph Models
ML-90-02 Jaap van Oosten	A Semantical Proof of De Jongh's Theorem
ML-90-03 Yde Venema	Relational Games
ML-90-04 Maarten de Rijke	Unary Interpretability Logic
ML-90-05 Domenico Zambella	Sequences with Simple Initial Segments
ML-90-06 Jaap van Oosten	Extension of Lifschitz' Realizability to Higher Order Arithmetic, and a Solution to a Problem of F. Richman
ML-90-07 Maarten de Rijke	A Note on the Interpretability Logic of Finitely Axiomatized Theories

Computation and Complexity Theory

CT-90-01 John Tromp, Peter van Emde Boas	Associative Storage Modification Machines
CT-90-02 Sieger van Denneheuvel, Gerard R. Renardel de Lavalette	A Normal Form for PCSJ Expressions
CT-90-03 Ricard Gavaldà, Leen Torenvliet, Osamu Watanabe, José L. Balcázar	Generalized Kolmogorov Complexity in Relativized Separations
CT-90-04 Harry Buhrman, Leen Torenvliet	Bounded Reductions

Other Prepublications

X-90-01 A.S. Troelstra	Remarks on Intuitionism and the Philosophy of Mathematics, Revised Version
X-90-02 Maarten de Rijke	Some Chapters on Interpretability Logic
X-90-03 L.D. Beklemishev	On the Complexity of Arithmetical Interpretations of Modal Formulae
X-90-04	Annual Report 1989
X-90-05 Valentin Shehtman	Derived Sets in Euclidean Spaces and Modal Logic
X-90-06 Valentin Goranko, Solomon Passy	Using the Universal Modality: Gains and Questions
X-90-07 V.Yu. Shavrukov	The Lindenbaum Fixed Point Algebra is Undecidable
X-90-08 L.D. Beklemishev	Provability Logics for Natural Turing Progressions of Arithmetical Theories
X-90-09 V.Yu. Shavrukov	On Rosser's Provability Predicate
X-90-10 Sieger van Denneheuvel, Peter van Emde Boas	An Overview of the Rule Language RL/1
X-90-11 Alessandra Carbone	Provable Fixed points in $I\Delta_0+\Omega_1$, revised version
X-90-12 Maarten de Rijke	Bi-Unary Interpretability Logic