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SOLUTION OF A PROBLEM OF DAVID GUASPARI

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0. Introduction

Guaspari (1983) discusses the fact, first proved by Kent (1973) that, although the modal formula $p \wedge \Box p$ has the property that under any arithmetical interpretation as in Solovay (1976) it implies, provably in **PA** or any other reasonably strong arithmetic, its own provability, nevertheless arithmetical interpretations can be found under which it is not equivalent to a Σ_1-sentence. In fact, he showed that arithmetical interpretations of $p \wedge \Box p$ can be found arbitrarily high up in the arithmetical hierarchy. Thus, the fact that $A \rightarrow \Box A$ is provable does in no way entail that A is a Σ_1-sentence. He stated as a question (Guaspari 1983, Question 6.5., p.788) whether, for the modal logic **L** (**PRL** in Smoryński 1985, also **GL** or **G**), the modal formulae which are essentially Σ_1 in this sense with respect to **PA** are exactly those A for which A is, provably in **L**, equivalent to a (possibly empty) disjunction of $\Box$-formulae, and, furthermore, that in the Guaspari-Solovay (1979) system **R**, which extends **L** with symbols for witness comparisons, the essentially Σ_1-formulae are the ones which are, provably in **R**, equivalent to a (possibly empty) disjunction of conjunctions of $\Box$-formulae and witness comparison formulae. Albert Visser (1987, Theorem 11.4) answered the question positively for the formulae of **L** with respect to **PA**. Moreover, he suggested to us that the interpretability logic **ILM** (see Visser 1990, or de Jongh/Veltman 1990) might be useful in attacking the full problem. Indeed, using the arithmetic completeness results of Hájek-Montagna (1989) for **ILM**, reading $\rhd$ as Π_1-conservativity, (as well as their system **IRM** which extends **ILM** with witness comparison symbols) we have been able to solve the full problem as expected for **R** with respect to any RE theory extending $\mathbf{I\Sigma_1}$. The result of Hájek-Montagna is a strengthening of the arithmetic completeness theorem of Berarducci (1989) /Shavrukov (1989).

1. Guaspari's conjecture for L.

We first prove the conjecture for the simple case of **L**, because this will make the method clearer in the more complicated case of **R**. For the definition of **ILM**-frame and **ILM**-model one may consult Visser (1990), de Jongh-Veltman (1990), or Berarducci (1989).

1.1 Definition. The **L**-formula A is *essentially* Σ_1 *with respect to* the (arithmetical) theory T iff, for each interpretation * of the modal language into T, there exists a Σ_1-sentence S such that $\vdash_T A^* \leftrightarrow S$.

1.2 Theorem. If A is a **L**-formula and T is RE extending $\mathbf{I\Sigma_1}$, then A is essentially Σ_1 with respect to T, if, for some $B_1, \ldots, B_n$ ($n \geq 0$), $\vdash_L A \leftrightarrow \bigvee_{i \leq n} \Box B_i$.

The implication from the right to the left is, of course, trivial. The other direction is an immediate consequence of the following three lemmas. We write mR for set of the R-successors of m.

1.3 Lemma. If A is an **L**-formula such that, for no $B_1, \ldots, B_n$ ($n \geq 0$), $\vdash_L A \leftrightarrow \bigvee_{i \leq n} \Box B_i$, then there exist **L**-models $\mathbf{M_1}$ and $\mathbf{M_2}$, with roots m_1 and m_2 respectively, such that $m_1 \Vdash A$ and $m_2 \Vdash \neg A$ and $m_2R \hookrightarrow m_1R$ (m_2R is embeddable as a generated submodel in m_1R).

1.4 Lemma. If A is an **L**-formula and there exist **L**-models $\mathbf{M_1}$ and $\mathbf{M_2}$ with roots m_1 and m_2 respectively such that $m_1 \Vdash A$ and $m_2 \Vdash \neg A$ and $m_2R \hookrightarrow m_1R$, then, for propositional letters p, q not occurring in A, $\nvdash_{ILM} p \rhd q \to p \wedge A \rhd q \wedge A$.

1.5 Lemma. If A is an **L**-formula and, for propositional letters p, q not occurring in A, $\nvdash_{ILM} p \rhd q \to p \wedge A \rhd q \wedge A$, then A is not essentially Σ_1 with respect to any RE theory containing $\mathbf{I\Sigma_1}$.

We prove these propositions in the reverse order.

Proof of lemma 1.5. Hájek-Montagna (1989) strengthen the Berarducci (1989) /Shavrukov (1989) arithmetic completeness theorem for **ILM** by proving that **ILM** embodies exactly the principles envalidated by any RE theory T extending $\mathbf{I\Sigma_1}$, if $(A \rhd B)^*$ is taken to be "each Π_1-consequence of $T+B^*$ is a consequence of $T+A^*$". Assume A is essentially Σ_1 with respect to some RE theory T containing $\mathbf{I\Sigma_1}$. Reason in T: Assume $(p \rhd q)^*$, i.e. $T+q^* \vdash P \Rightarrow T+p^* \vdash P$ for any $P \in \Pi_1$. Assume moreover that $T+q^*+A^* \vdash P$ for $P \in \Pi_1$. Then $T+q^* \vdash A^* \to P$, and $A^* \to P \in \Pi_1$. So, $T+p^* \vdash A^* \to P$, and $(p \rhd q \to p \wedge A \rhd q \wedge A)^*$ has been shown. By the Hájek-Montagna arithmetic completeness completeness theorem $\vdash_{ILM} p \rhd q \to p \wedge A \rhd q \wedge A$. ⊠

Proof of lemma 1.4. Assume models as described exist. We define a new model **M** as follows. Assume $\mathbf{M_1}$ and $\mathbf{M_2}$ to be disjoint.

(1) Take the union of $\mathbf{M_1}$ and $\mathbf{M_2}$.

(2) Append a new root m below m_1 and m_2.

(3) Take S_m to be the reflexive, transitive closure of $R \cup < m_1, m_2 >$.

An **ILM**-frame is obtained, because it is obvious that **M** is an **IL**-model and with regard to the additional requirement for M just note that $m_1 S_m m_2$ is the only non-trivial case in which S_m applies and, if $m_2 Rw$, then $w \in m_2R$, i.e. $m_1 R w$. Taking fresh propositional letters p and q and having them forced respectively only on m_1 and m_2 gives a model **M** on which $m \Vdash p \rhd q$, $m \nVdash p \wedge A \rhd q \wedge A$. ⊠

Proof of lemma 1.3. Assume A is an **L**-formula such that, for no $B_1, \ldots, B_n$ ($n \geq 0$), $\vdash_L A \leftrightarrow \bigvee_{i \leq n} \Box B_i$. Bring A into disjunctive normal form. Since $\vdash_L \Box(B \wedge C) \leftrightarrow \Box B \wedge \Box C$ only one $\Box$-formula need occur positively in each disjunct. Now proceed to execute the following operations on A as long as they are applicable.

(1) If necessary take care by reshuffling that the first disjunct of A does not consist only of a $\Box$-formula. We can write A then as $(S \wedge \overline{S}) \vee R$ with S a $\Box$-formula and $\overline{S}$ a conjunction of $\Diamond$-formulae and pure Booleans. Note that, if the disjunction is empty we are in the case n=0: $\vdash_L A \leftrightarrow \bot$ and, if all conjunctions consist of only one $\Box$-formula the assumption would be falsified.

(2) If $\vdash_L \neg(S \wedge \overline{S})$, replace A by R which is obviously equivalent to it.

(3) If $\vdash_L S \wedge \neg\overline{S} \rightarrow R$, replace A by $S \vee R$ which is obviously equivalent to it.

When (1)-(3) can no longer be applied A is of the form $(S \wedge \overline{S}) \vee R$ with $\overline{S}$ non-empty and $\nvdash_L \neg(S \wedge \overline{S})$ and $\nvdash_L S \wedge \neg\overline{S} \rightarrow R$.

We now have a model $\mathbf{M_2}$ with root $m_2 \Vdash S \wedge \neg\overline{S} \wedge \neg R$, i.e. $m_2 \Vdash \neg A$, and a disjoint model $\mathbf{M_3}$ with root $m_1 \Vdash S \wedge \overline{S}$. We form $\mathbf{M_1}$ by taking the union of $\mathbf{M_3}$ with m_2R and connecting m_1 by R with the elements of m_2R. It is clear that also in $\mathbf{M_1}$ the root $m_1 \Vdash S \wedge \overline{S}$, since

(i) if S is $\Box B$, all members of m_1R as well as m_2R force B,

(ii) if $\neg\Box C \in \overline{S}$, then $\neg C$ is forced by some member of m_1R,

(iii) if p_i or $\neg p_i \in \overline{S}$, then nothing changes.

Now, clearly, $m_1 \Vdash \neg(S \wedge \overline{S}) \wedge \neg R$. So, $m_1 \Vdash \neg A$, whereas $m_1 \Vdash S \wedge \overline{S}$, so $m_2 \Vdash A$. Also $m_2R \subsetneq m_1R$. ⊠

2. Extending the method to the system R.

To extend the result of Section 1 to **R**, we will have to replace lemmas 1.3 and 1.4 by slightly more complicated lemmas. Lemma 1.5 can stand as it is, except that the proof now needs arithmetic and modal completeness of **IRM** instead of **ILM** (Hájek-Montagna 1989). (Of course, this implies that the whole result is behept with the usual weakness associated with the arithmetical completeness result for **R**; the proof predicate has to be allowed to vary over arbitrary standard proof predicates, see Guaspari-Solovay 1979, or Smoryński 1985). The theorem will get the following form.

2.1 Theorem If A is a **R**-formula such that, for no
$B_{ij}, B'_{ij'}, B''_{ij'}, B^{\circ}_{ij''}, B^{\circ\circ}_{ij''}, (0 \leq i \leq n \geq 0, 0 \leq j \leq n_i, 0 \leq j' \leq n'_i, 0 \leq j'' \leq n''_i)$,
$\vdash_{\mathbf{R}} A \leftrightarrow \bigvee_{i \leq n}(\bigwedge_{j \leq n_i} \Box B_j \vee \bigwedge_{j \leq n_i'} \Box B'_{ij} \prec \Box B''_{ij} \vee \bigwedge_{j \leq n_i''} \Box B^{\circ}_{ij} \preccurlyeq \Box B^{\circ\circ}_{ij})$, then, for $p, q \notin A$, we have $\nvdash_{\mathbf{IRM}} p \rhd q \rightarrow p \wedge A \rhd q \wedge A$, and hence A is not essentially Σ_1 with respect to any arithmetic containing $\mathbf{I\Sigma_1}$.

The lemmas replacing lemmas 1.3 and 1.4 are as follows.

2.2 Lemma. Let A satisfy the conditions of theorem 2.1. Let Φ be the smallest adequate set containing A. Then there are two A-sound models $\mathbf{M_1}$ and $\mathbf{M_2}$ with nodes $m_1 \vDash A$ and $m_2 \vDash \neg A$ respectively such that
(i) $m_2R \subsetneq m_1R$,
(ii) Each $\Box B \prec \Box C$ or $\Box B \preccurlyeq \Box C$ in Φ that is forced in m_1 is also forced in m_2.

2.3 Lemma. If A is such that $\mathbf{M_1}$ and $\mathbf{M_2}$ as in lemma A' exist, then, for $p, q \notin A$, we have $\nvdash_{\mathbf{IRM}} p \rhd q \rightarrow p \wedge A \rhd q \wedge A$.

Proof of lemma 2.2. The differences with lemma 1.3 are explained as follows. In the first place we have to get A-sound models and cannot expect $m_1 \Vdash A$ and $m_2 \Vdash \neg A$ to be the respective roots of the models $\mathbf{M_1}$ and $\mathbf{M_2}$. This is no source of trouble however, and in that respect we can prove lemma 1.3 essentially in the same way.

The second difference is that, when we want to use the revised lemma 1.3 in a revised lemma 1.4, m_1 cannot be R-connected to m_2R, unless witness comparison formulas are preserved from m_1 to m_2R and this is in no way guaranteed if we simply follow the previous procedure. This difficulty can be overcome by starting with a *full* disjunctive normal form for A, i.e. each $\Box$-formula, $\prec$-formula and each $\preccurlyeq$-formula from the adequate set Φ occurs, either positively, or negatively in each disjunct.

So, let us bring A into a full disjunctive normal form. If we do that, then we first note that negative occurrences of $\preccurlyeq$-formulae or $\prec$-formulae can be left out. Suppose e.g. that $\neg(\Box B \prec \Box C)$ occurs as a conjunct in some disjunct A_i. Then, either

(i) $\neg\Box B$ and $\neg\Box C$ occur in A_i and $\neg(\Box B \prec \Box C)$ logically follows, or

(ii) $\neg\Box B$ and $\Box C$ occur in A_i and $\neg(\Box B \prec \Box C)$ logically follows, or

(iii) $\Box B$ and $\Box C$ occur in A_i. Then, because we have a full disjunctive normal form, $\Box C \preccurlyeq \Box B$ has to occur in A_i and $\neg(\Box B \prec \Box C)$ logically follows.

The proof now goes exactly as before. When the models are constructed the Σ-part of A_i is forced in both m_1 and m_2. As each $\prec$-formula or $\preccurlyeq$-formula in Φ is either present in this part, or its negation follows directly from A_i, in m_1 exactly those Σ-formulae are forced which are present in A_i, and those have to be forced in m_2 as well and, a fortiori, in its successors. This immediately implies that m_1 may be R-connected to m_2R. ⊠

Proof of lemma 2.3. The main problem is the following. We want to have a model **M** with a predecessor m of both m_1 and m_2 and an S_m-arrow from m_1 to m_2, and, in addition, we want to make the model A-sound, since, analogously to the Guaspari-Solovay completeness proof for **R**, Hájek-Montagna prove that $\vdash_{IRM}B$ iff B is valid on all B-sound Kripke models for **IRM**$^-$. (So, at first sight it would actually not seem sufficient to have A-soundness, one would expect $p \rhd q \rightarrow p \wedge A \rhd q \wedge A$-soundness to be necessary. That A-soundness will do will be shown at the end of this proof.) The only additional requirement above the obvious ones that they put on their models is that, if uS_mv, then v forces all witness comparison formulae forced in u. The requirement that an S_m-arrow should be allowed is therefore all right, since all the $\prec$- and $\preccurlyeq$-formulae forced in m_1 are forced in m_2 But as yet nothing tells us that we can give m_1 and m_2 a common predecessor, and that the resulting model can be made A-sound. Let us analyze the situation in more detail.

We add a node m before the two roots r_1 and r_2 of the A-sound models, a fortiori this node occurs before m_1 and m_2. It is completely determined which $\Box$-formulae are forced in m. The set of $\Box$-formulae in the respective roots is a subset of the ones forced in m_1 and m_2, and hence of the ones in m_1. Moreover, the ordering in r_2 agrees (on the $\Box$-formulae it forces) with the one in the r_1 (on the $\Box$-formulae r_1 forces), because both have to agree with the ordering in m_1 and m_2. Both models can only have lost with respect to m_1 and m_2, a number of

$\Box$-formulae forming a tail end of their $\prec$-ordering, otherwise their ordering would simply conflict with the one in m_1 or m_2. If $\Box B$ is forced in both roots, then, since $\Box B \rightarrow B$ is forced in both those roots (A-soundness), B will be and hence $\Box B$ will be forced in m. Therefore, simply the shortest of the two initial segments still available will be exactly forced again in m and its ordering will have to be kept, and we will make $m \Vdash$ do exactly this. Let this shortest (or equally short) initial segment be forced in r_i (i=0 or i=1). We now endow m also for the atomic formulae with the same forcing as r_i. This means that m agrees with respect to the $\Box$-, the $\prec$- and the $\preccurlyeq$-formulae and atomic formulae with r_i. There are no formulae in Φ however but Boolean combinations of those, so forcing on all of Φ agrees on both nodes. But this means that, just as in r_i, $m \Vdash \Box B \rightarrow B$ for any $\Box B \in \Phi$: the resulting model is A-sound.

This finishes the proof except for the point that to conclude immediately that $\nvdash_{IRM} p \rhd q \rightarrow p \wedge A \rhd q \wedge A$ one would need the model to be $p \rhd q \rightarrow p \wedge A \rhd q \wedge A$-sound which it isn't. This difficulty was pointed out to us by Franco Montagna who was so kind as to immediately solve the problem for us too. His argument runs as follows:

One consider the embedding * of the model **M** in arithmetic. One has:

$\vdash \boldsymbol{\ell = m} \rightarrow (\boldsymbol{\ell = m_1} \rhd \boldsymbol{\ell = m_2}) \wedge \neg(\boldsymbol{\ell = m_1} \rhd \bot)$.

$\vdash \boldsymbol{\ell = m_1} \rightarrow A^*$.

$\vdash \boldsymbol{\ell = m_2} \rightarrow \neg A^*$.

So, $\vdash (\boldsymbol{\ell = m_2} \wedge A^*) \leftrightarrow \bot$ and $\vdash \boldsymbol{\ell = m} \rightarrow (\boldsymbol{\ell = m_1} \wedge A^* \rhd \boldsymbol{\ell = m_2} \wedge A^* \leftrightarrow \boldsymbol{\ell = m_1} \rhd \bot)$.

Therefore, $\vdash \boldsymbol{\ell = m} \rightarrow \neg(\boldsymbol{\ell = m_1} \wedge A^* \rhd \boldsymbol{\ell = m_2} \wedge A^*)$.

So, if $p^* \equiv \boldsymbol{\ell = m_1}$ and $q^* \equiv \boldsymbol{\ell = m_2}$, then $\vdash \boldsymbol{\ell = m} \rightarrow (p^* \rhd q^*) \wedge \neg(p \wedge A^* \rhd q \wedge A^*)$,

from which $\nvdash_{IRM} p \rhd q \rightarrow p \wedge A \rhd q \wedge A$ is immediate. ⊠

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