Institute for Language, Logic and Information

RANDOMNESS IN SET THEORY

Michiel van Lambalgen

ITLI Prepublication Series
for Mathematical Logic and Foundations ML-90-10

University of Amsterdam

The ITLI Prepublication Series

1986

86-01 The Institute of Language, Logic and Information
86-02 Peter van Emde Boas — A Semantical Model for Integration and Modularization of Rules
86-03 Johan van Benthem — Categorial Grammar and Lambda Calculus
86-04 Reinhard Muskens — A Relational Formulation of the Theory of Types
86-05 Kenneth A. Bowen, Dick de Jongh — Some Complete Logics for Branched Time, Part I Well-founded Time, Forward looking Operators
86-06 Johan van Benthem — Logical Syntax

1987

87-01 Jeroen Groenendijk, Martin Stokhof — Type shifting Rules and the Semantics of Interrogatives
87-02 Renate Bartsch — Frame Representations and Discourse Representations
87-03 Jan Willem Klop, Roel de Vrijer — Unique Normal Forms for Lambda Calculus with Surjective Pairing
87-04 Johan van Benthem — Polyadic quantifiers
87-05 Víctor Sánchez Valencia — Traditional Logicians and de Morgan's Example
87-06 Eleonore Oversteegen — Temporal Adverbials in the Two Track Theory of Time
87-07 Johan van Benthem — Categorial Grammar and Type Theory
87-08 Renate Bartsch — The Construction of Properties under Perspectives
87-09 Herman Hendriks — Type Change in Semantics: The Scope of Quantification and Coordination

1988

Logic, Semantics and Philosophy of Language:

LP-88-01 Michiel van Lambalgen — Algorithmic Information Theory
LP-88-02 Yde Venema — Expressiveness and Completeness of an Interval Tense Logic
LP-88-03 Year Report 1987
LP-88-04 Reinhard Muskens — Going partial in Montague Grammar
LP-88-05 Johan van Benthem — Logical Constants across Varying Types
LP-88-06 Johan van Benthem — Semantic Parallels in Natural Language and Computation
LP-88-07 Renate Bartsch — Tenses, Aspects, and their Scopes in Discourse
LP-88-08 Jeroen Groenendijk, Martin Stokhof — Context and Information in Dynamic Semantics
LP-88-09 Theo M.V. Janssen — A mathematical model for the CAT framework of Eurotra
LP-88-10 Anneke Kleppe — A Blissymbolics Translation Program

Mathematical Logic and Foundations:

ML-88-01 Jaap van Oosten — Lifschitz' Realizabiility
ML-88-02 M.D.G. Swaen — The Arithmetical Fragment of Martin Löf's Type Theories with weak Σ-elimination
ML-88-03 Dick de Jongh, Frank Veltman — Provability Logics for Relative Interpretability
ML-88-04 A.S. Troelstra — On the Early History of Intuitionistic Logic
ML-88-05 A.S. Troelstra — Remarks on Intuitionism and the Philosophy of Mathematics

Computation and Complexity Theory:

CT-88-01 Ming Li, Paul M.B.Vitanyi — Two Decades of Applied Kolmogorov Complexity
CT-88-02 Michiel H.M. Smid — General Lower Bounds for the Partitioning of Range Trees
CT-88-03 Michiel H.M. Smid, Mark H. Overmars, Leen Torenvliet, Peter van Emde Boas — Maintaining Multiple Representations of Dynamic Data Structures
CT-88-04 Dick de Jongh, Lex Hendriks, Gerard R. Renardel de Lavalette — Computations in Fragments of Intuitionistic Propositional Logic
CT-88-05 Peter van Emde Boas — Machine Models and Simulations (revised version)
CT-88-06 Michiel H.M. Smid — A Data Structure for the Union-find Problem having good Single-Operation Complexity
CT-88-07 Johan van Benthem — Time, Logic and Computation
CT-88-08 Michiel H.M. Smid, Mark H. Overmars, Leen Torenvliet, Peter van Emde Boas — Multiple Representations of Dynamic Data Structures
CT-88-09 Theo M.V. Janssen — Towards a Universal Parsing Algorithm for Functional Grammar
CT-88-10 Edith Spaan, Leen Torenvliet, Peter van Emde Boas — Nondeterminism, Fairness and a Fundamental Analogy
CT-88-11 Sieger van Denneheuvel, Peter van Emde Boas — Towards implementing RL

Other prepublications:

X-88-01 Marc Jumelet — On Solovay's Completeness Theorem

1989

Logic, Semantics and Philosophy of Language:

LP-89-01 Johan van Benthem — The Fine-Structure of Categorial Semantics
LP-89-02 Jeroen Groenendijk, Martin Stokhof — Dynamic Predicate Logic, towards a compositional, non-representational semantics of discourse
LP-89-03 Yde Venema — Two-dimensional Modal Logics for Relation Algebras and Temporal Logic of Intervals
LP-89-04 Johan van Benthem — Language in Action
LP-89-05 Johan van Benthem — Modal Logic as a Theory of Information
LP-89-06 Andreja Prijatelj — Intensional Lambek Calculi: Theory and Application
LP-89-07 Heinrich Wansing — The Adequacy Problem for Sequential Propositional Logic
LP-89-08 Víctor Sánchez Valencia — Peirce's Propositional Logic: From Algebra to Graphs
LP-89-09 Zhisheng Huang — Dependency of Belief in Distributed Systems

Mathematical Logic and Foundations:

ML-89-01 Dick de Jongh, Albert Visser — Explicit Fixed Points for Interpretability Logic
ML-89-02 Roel de Vrijer — Extending the Lambda Calculus with Surjective Pairing is conservative
ML-89-03 Dick de Jongh, Franco Montagna — Rosser Orderings and Free Variables
ML-89-04 Dick de Jongh, Marc Jumelet, Franco Montagna — On the Proof of Solovay's Theorem
ML-89-05 Rineke Verbrugge — Σ-completeness and Bounded Arithmetic
ML-89-06 Michiel van Lambalgen — The Axiomatization of Randomness
ML-89-07 Dirk Roorda — Elementary Inductive Definitions in HA: from Strictly Positive towards Monotone
ML-89-08 Dirk Roorda — Investigations into Classical Linear Logic
ML-89-09 Alessandra Carbone — Provable Fixed points in $I\Delta_0+\Omega_1$

Computation and Complexity Theory:

CT-89-01 Michiel H.M. Smid — Dynamic Deferred Data Structures
CT-89-02 Peter van Emde Boas — Machine Models and Simulations
CT-89-03 Ming Li, Herman Neuféglise, Leen Torenvliet, Peter van Emde Boas — On Space Efficient Simulations
CT-89-04 Harry Buhrman, Leen Torenvliet — A Comparison of Reductions on Nondeterministic Space
CT-89-05 Pieter H. Hartel, Michiel H.M. Smid, Leen Torenvliet, Willem G. Vree — A Parallel Functional Implementation of Range Queries
CT-89-06 H.W. Lenstra, Jr. — Finding Isomorphisms between Finite Fields
CT-89-07 Ming Li, Paul M.B. Vitanyi — A Theory of Learning Simple Concepts under Simple Distributions and Average Case Complexity for the Universal Distribution (Prel. Version)
CT-89-08 Harry Buhrman, Steven Homer, Leen Torenvliet — Honest Reductions, Completeness and Nondeterminstic Complexity Classes
CT-89-09 Harry Buhrman, Edith Spaan, Leen Torenvliet — On Adaptive Resource Bounded Computations
CT-89-10 Sieger van Denneheuvel — The Rule Language RL/1
CT-89-11 Zhisheng Huang, Sieger van Denneheuvel, Peter van Emde Boas — Towards Functional Classification of Recursive Query Processing

Other Prepublications:

X-89-01 Marianne Kalsbeek — An Orey Sentence for Predicative Arithmetic
X-89-02 G. Wagemakers — New Foundations: a Survey of Quine's Set Theory
X-89-03 A.S. Troelstra — Index of the Heyting Nachlass
X-89-04 Jeroen Groenendijk, Martin Stokhof — Dynamic Montague Grammar, a first sketch
X-89-05 Maarten de Rijke — The Modal Theory of Inequality
X-89-06 Peter van Emde Boas — Een Relationele Semantiek voor Conceptueel Modelleren: Het RL-project

1990 SEE INSIDE BACK COVER

Instituut voor Taal, Logica en Informatie
Institute for Language, Logic and Information

Faculteit der Wiskunde en Informatica
(Department of Mathematics and Computer Science)
Plantage Muidergracht 24
1018TV Amsterdam

Faculteit der Wijsbegeerte
(Department of Philosophy)
Nieuwe Doelenstraat 15
1012CP Amsterdam

RANDOMNESS IN SET THEORY

Michiel van Lambalgen
Department of Mathematics and Computer Science
University of Amsterdam

ITLI Prepublications
for Mathematical Logic and Foundations
ISSN 0924-2090

Received October 1990

Research supported by the
Netherlands Organization for Scientific Research (NWO)

Randomness in set theory

Michiel van Lambalgen
Dept. of Mathematics
University of Amsterdam
email: michiell@fwi.uva.nl

0 INTRODUCTION The fact that ZF set theory does not decide various easily expressible statements concerning the reals and sets of reals, leads one to search for additional axioms. Most axioms proposed so far use concepts which are formalizable in set theory and they express new, in some cases even plausible, properties of these concepts. Typical examples include large cardinal axioms, Martin's axiom, or the axiom of determinacy. But, as Kreisel pointed out, one might also think of a different way of introducing axioms, this time involving new primitives:

> Let us try to expand the language of set theory, that is, add symbols for new primitive notions, and look for axioms in the wider language which are evident (for the notions given). They may imply set theoretic propositions, i.e. propositions in the language L_E of set theory, which are not. Prima facie the case for this project is overwhelming. [Kreisel then goes on to discuss one such proposal.]
> A more interesting, but also more problematic, expansion in the literature [Kruse 1967] concerns the primitive predicate of being a "*random sequence*". Myhill formulated axioms for the notion l.c. which, however, are not altogether plausible. As he himself observed, the *intuitive* notion of random sequence is certainly *not* extensional; specifically, if we think of a sequence as being produced by a die, it is random, but this does not ensure that some element of the sequence differs from, say, 1; also, it is not intuitively clear that there is a collection of (intensional) sequences such that (i) for each it is decided whether or not the sequence is random and (ii) the usual axioms of set theory are satisfied by the collection. So, once again, we have here an idea for an expansion rather than an effective use of an expansion ([1968], 100-1).

In the following pages we shall study various ways of introducing random sequences in set theory, and their consequences. As Kreisel remarks, it is a moot point whether the intuitive notion of randomness can be captured extensionally. We therefore pursue two different lines:

- an intensional approach, where one studies essentially only quantification over random sequences by means of a generalized quantifier;
- an extensional approach, where one adds a new predicate for random sequences.

The main conclusion will be that, once one accepts the axioms Q for quantification over random sequences given in section 1, and moreover classical logic, the two approaches are essentially equivalent (theorem 2.4). The consequences of the axioms that have

been obtained so far, are all related to AC; e.g. in section 3 we show that Lebesgue measure has a translation invariant extension to all sets of reals (theorem 3.6).
To appreciate the potential significance of this result, recall that, in modern probability theory as codified by Kolmogorov, one proceeds by setting up first the apparatus of σ–algebras and σ–additive measures; afterwards the notions of independence and randomness are defined. The results of this article indicate that one can profitably reverse this procedure: we start with the notions of randomness and independence as true primitives, not even formalizable in set theory; it is then shown that one can dispense with the usual apparatus.

We now give a brief description of the contents of the paper. In section 1 we study the effect of adding a generalized quantifier Q with the intuitive meaning of "almost certain" to ZF. The most natural way to do so (allowing arbitrary sets as parameters in formulas with Q) leads to a refutation of AC. A strong restriction (allowing only real and ordinal parameters) still refutes $V \neq L$. The extensional approach to randomness is studied in section 2. It so happens that this approach is strongly related to that of section 1 via the elimination of the quantifier Q given in van Lambalgen [1990]. In section 3 we introduce some further plausible axioms for randomness, which are analogous to the density and data axioms for the (intuitionistic) theory of lawless sequences. A consequence of the new axioms is the existence of a translation invariant extension of Lebesgue measure to all sets of reals, and furthermore the nonexistence of free ultrafilters on the powerset of ω. In section 4 we briefly indicate why, as a consequence of the preceding results, the addition of the theory of lawless sequences to intuitionistic set theory is no longer a conservative extension (as opposed to the situation in, say, intuitionistic second order arithmetic). The Epilogue, 5, contains some philosophical remarks. In an appendix, section 6, we compare our approach with Freiling's "Axioms of Symmetry" [1986]. Freiling shows how some simple axioms on throwing n random darts forces the cardinality of the continuum to be $\geq \aleph_n$. The main result here is, roughly, that the size of $2^{\aleph_0}$ is related to the number of iterations of Q that one allows. In this section we also comment on axioms for randomness proposed by Myhill in 1963.
It is only fair to say that this work owes part of its inspiration to Freiling [1986] and Simms [1989]. In particular Freiling's emphasis on the fundamental importance of symmetry for randomness, and Simms suggestion that Fubini's theorem is related to extensions of measures have been instrumental. Furthermore I would like to thank Domenico Zambella for permission to include some of his results and Georg Kreisel for helpful correspondence.

1 QUANTIFICATION OVER RANDOM SEQUENCES. When we generate a random sequence, there is prima facie nothing certain about the result, except for the finite segment already generated. In this sense, random sequences behave as lawless sequences and are indeed often adduced as examples of the latter. However, on an intuitve level we are also almost certain, or practically certain, of many other properties: that relative frequencies converge, that the same holds in suitably chosen subsequences etc. The fact that practical certainty is not absolute certainty is what renders an extensional theory of randomness difficult. But we have a different option: formalizing the properties of "practically certain" itself and incorporating this notion in ZF.

So suppose we have some stochastic mechanism X that randomly produces infinite binary sequences x. For X one could take, e.g., a fair coin. We add to the language of set theory a generalized quantifier Q, where the intended interpretation of $Qx\phi(x)$ is: "if x is randomly generated, it is practically certain that $\phi(x)$". We shall state the axioms first, and comment on them afterwards. Unless specified otherwise, we assume that the variables bound by Q run over 2^{ω}. We first fix the logic to be used:

Q0 Axioms and inference rules of classical predicate logic.

The following five properties, expressing that Q is a nonprincipal filter, are fairly immediate.

Q1 $\neg Qx\ x \neq x$

Q2 $Qy\ x \neq y$

Q3 $Qx\phi(..,x,..) \rightarrow Qy\phi(..,y,..)$ provided y is free for x in ϕ

Q4 $Qx\phi \wedge \forall x(\phi \rightarrow \psi) \rightarrow Qx\psi$

Q5 $Qx\phi \wedge Qx\psi \rightarrow Qx(\phi \wedge \psi)$

So far, there is no difference between Q and the filter quantifiers studied in Kaufmann [1983] and Kakuda [1989]. The crucial property is

Q6 $QxQy\phi \leftrightarrow QyQx\phi$.

COMMENTS ON THE AXIOMS

Q0 This choice of a logic is not altogether obvious, since we have expressly introduced random sequences as objects about which we have partial information only. The force of Q0 will become clear when we discuss Q4.

Q1 When read classically, this axiom states that there is some randomly generated x. Without such a weak existence axiom, the enterprise would trivialize. But notice that Q1 is still *very* weak; the following density axiom also seems to be justified: if a set A has positive measure, then $\neg Qx(x \in A \rightarrow x \neq x)$. This stronger axiom will be considered in section 3.

Q2 This means that it is almost certain that randomly generated sequences differ from some given sequence. This axiom is related to Kreisel's observation that "if we think of a sequence as being produced by a die, it is random, but this does not ensure that some element of the sequence differs from, say, 1"; for y, take the sequence 111... Here, intensionality makes itself felt; we only require that it is *almost* certain that a randomly produced x differs from y.

Q3 This is a syntactic expedient only.

Q5 states that the intersection of two almost certain events is again almost certain (though, intuively, perhaps slightly less so).

Q4 expresses that Q is a monotone quantifier. That this is less innocent than may seem at first sight can be seen as follows. We show that $Qx\phi(x) \rightarrow \exists x\phi(x)$. Suppose $Qx\phi(x)$; assume for simplicity that ϕ does not contain other free variables. Since $\forall x(\phi(x) \rightarrow \exists y\phi(y))$, monotonicity implies $Qx\exists y\phi(y)$. Using classical logic and Q1 – 5 one can derive, when x does not occur free in $\exists y\phi(y)$: $Qx\exists y\phi(y) \rightarrow \exists y\phi(y)$. For suppose $\neg\exists y\phi(y)$, then also $\forall x(x \neq y \rightarrow \neg\exists y\phi(y))$, whence by Q4 and Q2, $Qx\neg\exists y\phi(y)$. By Q5, $Qx(\neg\exists y\phi(y) \wedge \exists y\phi(y))$, which, again by Q4, contradicts Q1. Hence $\neg\neg\exists y\phi(y)$, and so $\exists y\phi(y)$ by classical logic. To see why the existential import of Q is somewhat problematic, let us make a historical diversion. When Borel had proved (in [1909]) that almost all real numbers are absolutely normal (i.e. normal in every base), he noted that it would be of interest either to construct a concrete example of such a number or to show that none of the definable numbers is absolutely normal. He observed that, however paradoxical at first sight, the latter possibility did not contradict his result. In other words, the existential quantifier in $Qx\phi(x) \rightarrow \exists x\phi(x)$ need not have a constructive meaning.

Indeed, on an intensional view $Qx\exists y\phi(y) \rightarrow \exists y\phi(y)$ is controversial. It will be observed that, when x does not occur free in ψ, "Qx" in $Qx\psi$ can be read as a propositional modal operator, namely "it is practically certain that ψ". The statement "$Qx\psi \rightarrow \psi$" (x not free in ψ) therefore expresses a collapse of modalities: ψ is practically certain implies that ψ is true. Hence, although each of Q1 – 5 seems justified on the intensional interpretation, together with classical logic they present a picture not unlike the extensional view. It should be observed here that the addition of Q1 – 5 to ZF is conservative so that no new existential statements become provable via $Qx\exists y\phi(y) \rightarrow \exists y\phi(y)$; however, this is no longer true when one adds Q6.

Q6 So far our axioms were concerned with randomness. The last axiom implicitly introduces a new primitive notion, namely, *independence*. We think of the quantifiers Qx, Qy in a formula $QxQy\phi$ as referring to independent processes. Hence $QxQy\phi(x,y)$ means: " if x is randomly generated and y is randomly generated independently from x, then it is practically certain that $\phi(x,y)$". But if the processes generating x and y are independent, it shouldn't matter in which order we take them. (This intuition is also the motivation behind Freiling's "Axioms of Symmetry" [1986].)

The axiom system Q0 - 6 will be denoted $\mathcal{Q}$. We have seen just now that there is a certain amount of idealization involved in rendering "almost certain" formally by means of $\mathcal{Q}$; in particular, the intuitive notion of "almost certain" may not be monotonic, or satisfy classical logic. But if we adopt $\mathcal{Q}$, we can use some work of Harvey Friedman (see Steinhorn [1985a,b]), who introduced a quantifier with properties 1 - 6. We now think of the formula $Qx\phi(x)$ as meaning "$\{x \mid \phi(x)\}$ has Lebesque measure 1". On this interpretation, Q6 bears some analogy to the Fubini theorem: for Fubini implies Q6 *if we know in addition that* $\{\langle x,y\rangle \mid \phi(x,y)\}$ *is measurable.* Without this extra condition, Q6 expresses a very strong property, which is almost solely responsible for the peculiar features of set theory with random sequences. Despite the fact that there is only an analogy, we shall refer to Q6 as the "Fubini property".
Alternatively, $Qx\phi x$ can be read topologically as "$\{x \mid \phi(x)\}^c$ is first category". In this case, Q6 corresponds (in the same sense as above) to the Kuratowski - Ulam theorem (see Oxtoby [1980], Chapter 16). Friedman shows that the set $\mathcal{Q}$ of axioms for Q is complete for both interpretations. However, since definable sets need not be measurable, or have the Baire property, Friedman considers completeness with respect to a class of nonstandard models, so called *Borel structures.*
Let $\mathcal{M}$ be a model for a first order language L, with domain a subset of 2^ω of positive Lebesgue measure. On the measure theoretic interpretation, the satisfaction clause for Q is:

$$\mathcal{M} \models Qx\phi(x,a_1,...,a_n) \text{ iff } \lambda\{ x \mid \mathcal{M} \not\models \phi(x,a_1,....,a_n)\} = 0, \text{ where } \lambda = (\tfrac{1}{2},\tfrac{1}{2})^\omega$$

is Lebesgue measure on 2^ω.
The other logical constants have standard interpretation. A structure whose domain is a subset of 2^ω and all of whose relations and functions are Borel is called a *Borel structure* (or model). A Borel structure for $L \cup \{Q\}$ is called *totally Borel* if any relation defined using a formula from $L \cup \{Q\}$ (possibly with parameters) is Borel. Evidently the axioms for Q are valid on any totally Borel model.
Similarly, when Q is interpreted in terms of category, we put

$$\mathcal{M} \models Qx\phi(x,a_1,...,a_n) \text{ iff } \{ x \mid \mathcal{M} \models \phi(x,a_1,....,a_n)\} \text{ is comeagre;}$$

and we have definitions of Borel structure etc. analogous to the ones given before.
The main theorems in this subject are:

1.1 THEOREM (See Steinhorn [1985a,b]) Let T be a consistent theory in L with an infinite model. Then T has a totally Borel model with domain 2^ω.
1.2 THEOREM (See Steinhorn [1985a]) Let T be a theory in $L \cup \{Q\}$. Then T has a totally Borel model whose domain is a full subset of 2^ω iff T is consistent in $\mathcal{Q}$. Mutatis mutandis, the same theorem holds for category.

We now study the effect of adding $\mathcal{Q}$ to ZF. We shall first present a few quick consistency results using a theorem due to Shelah and Stern, in order not to overburden the exposition with the details of a forcing construction. In sections 2 and 3 we give a finer analysis.

1.3 THEOREM Assume there is a standard model of ZF. (I) Then there is a standard model of ZFC with the following properties:
(1) Any set of reals definable from ordinal and real parameters has the property of Baire (Shelah [1984], theorem 7.17).
(2) There is a sequence of $\aleph_1$ reals (Stern [1985], theorem 2(iii)).
(3) Any ordinal definable set of reals is Lebesgue measurable (Stern [1985], theorem 2(v)).
(II) As a consequence, there is a standard model of ZF + DC in which all sets of reals have the Baire property (Shelah [1984], theorem 7.17).
The proofs show moreover that subsets of $(2^{\omega})^n$, $n > 1$, have corresponding regularity properties.

We first show how *not* to add $\mathcal{Q}$ to ZF. Generally, we have

1.4 LEMMA Let T be any first order theory. Then T + $\mathcal{Q}$ is conservative over T.
PROOFSKETCH This sketch is only intended to give the reader moral certainty; a more elementary proof can be found in van Lambalgen [1990] (theorem 2.1.3). Let $\mathcal{M}$ be a model of ZFC such that $T \in \mathcal{M}$. Let $\mathcal{N}$ be an extension of $\mathcal{M}$ in which every set of reals has the Baire property (theorem 1.3 (II)).
Let $\mathcal{A}$ be a totally Borel model of T in $\mathcal{M}$, and let $\mathcal{A}^{\#}$ be $\mathcal{A}$ as seen in $\mathcal{N}$. By absoluteness, $\mathcal{A}^{\#}$ verifies T. Since in $\mathcal{N}$ all sets definable using $\forall$ and Q have the Baire property, we can expand $\mathcal{A}^{\#}$ to a model $\langle\mathcal{A}^{\#},Q\rangle$ of the Friedman axioms. (Use the fact that Q6 corresponds to the Kuratowski - Ulam theorem.) ⊠

In particular, ZFC + $\mathcal{Q}$ is conservative over ZFC, even when the variables bound by Q run over *all* sets. However, the theory ZFC + $\mathcal{Q}$ is unnatural, since we do not allow Q to occur in the schemata of ZF. This unnaturalness is reflected semantically: since no Borel relation can be a wellordering, the Borel models constructed in the above lemma are not transitive. We therefore consider only extensions of set theory in which Q does occur in the separation and substitution schemata. It is useful to distinguish two extensions of this sort: ZF$\mathcal{Q}$ and ZF$\mathcal{Q}^0$. In ZF$\mathcal{Q}$, Q formulas may contain parameters for arbitrary sets; in ZF$\mathcal{Q}^0$, only real and ordinal parameters are allowed in Q formulas.

Hence in ZFQ we allow formulas "$Qx(x \in A)$" for *arbitrary* sets A in 2^ω; in the weaker theory ZFQ^0 we still have formulas "$Qx(x \in B)$", with B a *Borel* subset of 2^ω.
We emphasize that systems of this type (unlike the ones studied in section 2) are fairly weak in one respect: one cannot even express that random sequences are closed under nontrivial operations. (For a weaker notion of closure, see section 3.) Nevertheless, the axioms pose strong constraints on the set theoretic universe. We first give a simple result which shows that it is really the Fubini property Q6 that is responsible for the striking features of ZFQ that will be discussed later.

1.5 LEMMA ZFQ minus Q6 is conservative over ZF.
PROOF Define $Qx\phi$ explicitly as $\exists y \forall x(x \notin y \rightarrow \phi)$, then Q1 - 5 are satisfied. ⊠

1.6 LEMMA If ZF is consistent, so is ZFQ + DC.
PROOF This follows from 1.3 (II). ⊠

1.7 LEMMA If ZF is consistent, so is ZFQ^0 + AC.
PROOF Use theorem 1.3 (I) (1). For an alternative proof, see lemma 3.4 below. ⊠

The main technical tool is the following additivity property of Q.

1.8 THEOREM (ZFQ) Let the family $(A_\alpha)_{\alpha<\kappa}$ be given, where κ is an initial ordinal. $\forall \alpha < \kappa Qx(x \notin A_\alpha) \rightarrow Qx \forall \alpha < \kappa(x \notin A_\alpha)$.
PROOF (The argument is adapted from van Benthem [1990]). We use induction on initial ordinals κ. Note that we need Q in the separation axiom to conclude from this that the result holds for *all* families $(A_\alpha)_{\alpha<\kappa}$. We may suppose the A_α are pairwise disjoint.
The case $\kappa = 1$ is trivial. So suppose for all $\lambda < \kappa$ and all families $(B_\alpha)_{\alpha<\lambda}$, $\forall \alpha < \lambda Qx(x \notin B_\alpha) \rightarrow Qx \forall \alpha < \lambda(x \notin B_\alpha)$. Define a relation S on 2^ω by $S(x,y) \Leftrightarrow \forall \alpha < \kappa[x \in A_\alpha \rightarrow (\exists \beta > \alpha(y \in A_\beta) \vee \forall \beta < \kappa(y \notin A_\beta))] \vee [\forall \alpha < \kappa(x \notin A_\alpha) \wedge y \in 2^\omega]$.
Put $B := \bigcup_{\alpha<\kappa} A_\alpha$.
We then have $\forall x Qy S(x,y)$. For if $x \notin B$, then $\{y \mid S(x,y)\} = 2^\omega$; and if $x \in B$, then $\{y \mid S(x,y)\} = 2^\omega - C$, where C is a union of less than κ A_α's; now apply the induction hypothesis. Hence also $QxQyS(x,y)$, whence by Q6, $QyQxS(x,y)$. Consider the set $\{x \mid S(x,y)\}$. Obviously, if $y \in B$, then $\{x \mid S(x,y)\} = B^c \cup C$, where C is a union of less than κ A_α's; and if $y \notin B$, then $\{x \mid S(x,y)\} = 2^\omega$, which is in Q. It follows from this observation that $\{ y \mid QxS(x,y)\} \supseteq B^c$. If we have equality, then $QxQyS(x,y)$ implies

$Qx(x \notin B)$. If not, then for some $y \in B$, $\{x \mid S(x,y)\} = B^c \cup C$, where C is a union of less than κ A_α's, and we can apply the induction hypothesis to obtain $Qx(x \notin C)$. But $Qx(x \in B^c \vee x \in C) \wedge Qx(x \notin C)$ implies $Qx(x \in B^c)$. ⊠

In particular, we see that the continuum is not the union of κ Q - nullsets, for *any* initial ordinal κ.

1.9 CORLLARY (ZF*Q*) Suppose $B \subseteq 2^\omega$ has a wellordering. Then $Qx(x \notin B)$.

PROOF Suppose $B = \{ x_\alpha \mid \alpha < \kappa\}$. By Q2, $Qx(x \notin \{x_\alpha\})$. Now apply the preceding theorem. ⊠

In other words, if a set can be "counted" at all, it must be small. Note that it is consistent with ZF*Q* that every wellorderable subset of 2^ω is countable; this holds in Solovay's model where every set of reals is Lebesgue measurable. But, as we have seen in 1.3 (I) (2), the existence of a sequence of $\aleph_1$ reals (hence of a nonmeasurable set) is also consistent with ZF*Q*. Hence, if CH' denotes the aleph - free version of the continuum hypothesis, ZF*Q* is consistent with both CH' and $\neg$CH'.

1.10 COROLLARY (ZF*Q*) $\neg$AC.

PROOF If not, 2^ω would have a wellordering, whence by the preceding corollary, $Qx(x \notin 2^\omega)$; but this contradicts Q1. ⊠

We thus see that it is the Fubini axiom Q6 that is responsible for eliminating some sets from models of ZFC.

By keeping track of definability, we obtain

1.11 COROLLARY (ZFQ^0) Let $\phi(x,\alpha)$ be a formula which may contain additional ordinal and real parameters (but no other parameters). Then $\forall\alpha < \kappa Qx\phi(x,\alpha) \rightarrow Qx\forall\alpha < \kappa\phi(x,\alpha)$. Moreover, if B = Field($\leq$) for some definable wellordering $\leq$, then $Qx(x \notin B)$.

For the constructible hierarchy this means the following: if L(x) is the formula "x is constructible", we get $Qx\neg L(x)$.

2 AN EXTENSIONAL STUDY OF RANDOMNESS We shall see presently how one can introduce a randomness predicate R and axioms governing R, such that ZFQ^0 is interpretable in the resulting theory. The moral of this construction is, that we may think of a formula "$Qx\phi(x)$" as meaning: "for all random x, $\phi(x)$"; in other words, that we can extensionalize the notion of randomness implicit in *Q*.

Our procedure is to eliminate Q in terms of the R predicate and $\forall$. In the first approximation, this elimination is a translation from Q–formulas to R–formulas which is the identity on formulas not containing Q, commutes with $\neg$, $\wedge$, $\forall$ and transforms $Qx\phi$ into $\forall x(R(x) \rightarrow \phi)$. However, we quickly run into trouble if ϕ is of the form $x \neq y$, say; for then $Qx(x \neq y)$ implies that R is empty. Instead of R(x) we therefore use a relation $R(x,\vec{y})$ with the intended interpretation " x is independent of $\vec{y}$" or "$\vec{y}$ has no information about x". Here, $\vec{y}$ denotes a vector, *of unspecified length*, of variables; hence R is a relation of indefinite arity. We always assume that the variable x in $R(x,\vec{y})$ runs over 2^{ω}; the $\vec{y}$, however, may be interpreted by arbitrary sets. $\vec{y}$ may be empty, in which case we write $R(x,\emptyset) =: R(x)$. We may think of x with R(x) as random sequences. We are now in a position to define our translation formally; afterwards we shall specify the axioms which will turn the translation into an embedding.

2.1 DEFINITION Let L be an arbitrary countable first-order language. We define a translation * of L(Q) into L(R) as follows:

* is the identity on L formulas
* commutes with $\wedge$, $\neg$, $\rightarrow$, $\forall$
$(Qx\phi(x,\vec{y}))^* := \forall x(R(x,\vec{y}) \rightarrow \phi(x,\vec{y})^*)$

We now give the axioms for the independence relation. Properties 1 - 5 are a slight weakening of the usual axioms for algebraic or linear independence. Observe that R5 corresponds to the Steinitz exchange principle.

R0. Axioms and inference rules for classical predicate logic
R1. $\exists x R(x)$, $\forall \vec{y} \exists x R(x,\vec{y})$
R2. $R(x,\vec{y}\vec{z}) \rightarrow R(x,\vec{z})$
R3. (a) $R(x,\vec{y}) \rightarrow R(x,\pi\vec{y})$ for any permutation π; (b) $R(x,y\vec{z}) \rightarrow R(x,yy\vec{z})$
R4. $\neg R(x,x)$
R5. $R(y,\vec{z}) \wedge R(x,y\vec{z}) \rightarrow R(y,x\vec{z})$.

The next axiom looks odd and we shall comment on it after the proof of the embedding theorem.

R6. Suppose $\phi(x,\vec{y})$ is in L(R), and z does not occur free in ϕ. Then
$\forall x(R(x,z\,\vec{y}\,) \rightarrow \phi(x,\vec{y})) \rightarrow \forall x(R(x,\vec{y}) \rightarrow \phi(x,\vec{y}))$;
The system consisting of axioms R0 - 6 will be denoted $\mathcal{R}$.

Our results in this section will rely upon the possibility of interpreting R by means of Solovay forcing. This was spelt out in van Lambalgen [1990], but since in the present paper we have opted for a stronger version of R6, we have to verify this anew.

Let $\mathcal{M}$ be a countable transitive model of ZF + V=L. In $\mathcal{M}$, consider $(2^{\omega})^{\kappa}$, where $\kappa \geq \omega_1$. We equip $(2^{\omega})^{\kappa}$ with the product topology and the product measure λ^{κ} defined on the Borel σ - algebra $B((2^{\omega})^{\kappa})$. Let I denote the σ - ideal of λ^{κ} nullsets, then the quotient algebra $\mathcal{B} := B((2^{\omega})^{\kappa})/I$ is a complete Boolean algebra. Let $\mathcal{G} \subseteq \mathcal{B}$ be a generic ultrafilter; construct the generic extension $\mathcal{M}[\mathcal{G}]$. We shall refer to this extension as "generically adding κ random reals".

For any sequence $\vec{y}$ of elements in $2^{\omega} \cap \mathcal{M}[\mathcal{G}]$, $\mathcal{M}[\vec{y}]$ is welldefined (via relative constructibility).

In $\mathcal{M}[\mathcal{G}]$, interpret $R(x,\vec{y})$ as

$R(x,\vec{y})$ iff for all Borel sets B with code in $\mathcal{M}[\vec{y}]$, if $\lambda B = 1$, then $x \in B$.

If $R(x,\vec{y})$, we say that x is (Solovay) random over $\mathcal{M}[\vec{y}]$.

2.2 LEMMA $R(x,\vec{y})$, interpreted as Solovay randomness, satisfies $\mathcal{R}$.

PROOF R1 holds because the set of reals random over $\mathcal{M}[\vec{y}]$ has outer measure 1 in $\mathcal{M}[\mathcal{G}]$. R2 – 4 are trivial and R5 was verified in van Lambalgen [1990], theorem 2.2.1. To prove R6, we first observe that R is in fact definable; hence it suffices to show that for ϕ in the language of ZF, if z does not occur free in ϕ,

$$\mathcal{M}[\mathcal{G}] \models \forall x(R(x,z\vec{y}) \rightarrow \phi(x,\vec{y})) \rightarrow \forall x(R(x,\vec{y}) \rightarrow \phi(x,\vec{y})).$$

Suppose $\mathcal{M}[\mathcal{G}] \models \forall x(R(x,z\vec{y}) \rightarrow \phi(x,\vec{y}))$. If x is an element of $\mathcal{M}[\mathcal{G}]$, let $\underline{x}$ denote its name. We claim that there exists a formula ψ such that $\mathcal{M}[\mathcal{G}] \models \phi(x,\vec{y}) \Leftrightarrow \mathcal{M}[\vec{y},x] \models \psi(x,\vec{y})$. By the homogeneity of $\mathcal{B}$, we can write $\mathcal{M}[\mathcal{G}] = \mathcal{M}[\vec{y},x][\mathcal{G}']$, where $\mathcal{G}' \subseteq \mathcal{B}$ is a generic ultrafilter. If π is any automorphism on $\mathcal{B}$ such that the induced automorphism on $\mathcal{M}[\mathcal{G}]$ fixes $\vec{y}$ and x, then $\mathcal{M}[\vec{y},x][\pi\mathcal{G}'] \models \phi(x,\vec{y})$. It follows that $\mathbf{1}_{\mathcal{B}} \Vdash \phi(\underline{x},\underline{\vec{y}})$ and since forcing is expressible in the groundmodel, in this case $\mathcal{M}[\vec{y},x]$, we have the required formula $\psi(x,\vec{y})$.

By the properties of Solovay forcing, there exists a Borel set B, with code in $\mathcal{M}[\vec{y}]$, such that for all x random over $\mathcal{M}[\vec{y}]$, $x \in B \Leftrightarrow \mathcal{M}[\vec{y},x] \models \psi(x,\vec{y})$.

By hypothesis, for all x random over $\mathcal{M}[\vec{y},z]$, $\mathcal{M}[\vec{y},x] \models \psi(x,\vec{y})$. Since this set has outer measure 1, we must have $\lambda B = 1$. But then, because B has code in $\mathcal{M}[\vec{y}]$, also for all x random over $\mathcal{M}[\vec{y}]$, $\mathcal{M}[\vec{y},x] \models \psi(x,\vec{y})$. This shows $\mathcal{M}[\mathcal{G}] \models \forall x(R(x,\vec{y}) \rightarrow \phi(x,\vec{y}))$. ⊠

2.3 THEOREM * is a faithful relative interpretation of L(Q) into L(R), i.e. for all ϕ in L(Q): $\mathcal{Q} \vdash \phi$ iff $\mathcal{R} \vdash \phi^*$.

PROOFSKETCH For a weaker system $\mathcal{R}$ this was proved in van Lambalgen [1990]. We do the $\Rightarrow$ part again to show the reader which properties of R correspond to quantifier properties. R1 ensures that * is correctly defined as a relative interpretation. The

remainder of the argument proceeds by a routine induction on the length of proofs in $\mathcal{Q}$.
That the axioms for Q are derivable in $\mathcal{R}$ can be seen as follows:

- $(\neg Qx\ x\neq x)^* = \neg\forall x(R(x) \to x\neq x) \leftrightarrow \exists xR(x)$, hence Q1 corresponds to R1 under *
- $(\forall xQy\ x\neq y)^* = \forall x\forall y(R(y,x) \to x\neq y) \leftrightarrow \forall x\neg R(x,x)$ hence Q2 corresponds to R4 under *
- Q3 holds trivially
- $(Qx\phi(x,\vec{y}) \wedge \forall x(\phi(x,\vec{y}) \to \psi(x,\vec{z})) \to Qx\psi(x,\vec{z}))^* =$
$\forall x(R(x,\vec{y}) \to \phi(x,\vec{y})^*) \wedge \forall x(\phi(x,\vec{y})^* \to \psi(x,\vec{z})^*) \to \forall x(R(x,\vec{z}) \to \psi(x,\vec{z})^*)$;
the antecedent implies that $\forall x(R(x,\vec{y}\vec{z}) \to \psi(x,\vec{z})^*)$, hence by R6 also $\forall x(R(x,\vec{z}) \to \psi(x,\vec{z})^*)$
- $(Qx\phi(x,\vec{y}) \wedge Qx\psi(x,\vec{z}) \to Qx(\phi(x,\vec{y})\wedge\psi(x,\vec{z})))^* =$
$\forall x(R(x,\vec{y}) \to \phi(x,\vec{y})^*) \wedge \forall x(R(x,\vec{z}) \to \psi(x,\vec{z})^*) \to$
$\forall x(R(x,\vec{y}\vec{z}) \to \phi(x,\vec{y})^*\wedge\psi(x,\vec{z})^*)$, hence (Q5)* can be derived in $\mathcal{R}$ using R2
- $(QxQy\phi(x,y,\vec{z}) \leftrightarrow QyQx\phi(x,y,\vec{z}))^* =$
$\forall x\forall y(R(x,\vec{z})\wedge R(y,x\vec{z}) \to \phi(x,y,\vec{z})^*) \leftrightarrow \forall y\forall x(R(y,\vec{z})\wedge R(x,y\vec{z}) \to \phi(x,y,\vec{z})^*)$,
hence (Q6)* can be derived in $\mathcal{R}$ using R5 and R2.

The induction step is almost trivial, since both in the case of $\mathcal{Q}$ and $\mathcal{R}$ the inference rules are those of classical predicate logic. To check the validity of the identity axioms, we have to verify that * commutes with substitution, i.e. that $(Qx\phi(x,y\vec{z}))^*[y = t(\vec{v})] \leftrightarrow (Qx\phi(x,y\vec{z})[y = t(\vec{v})])^*$, but this can be shown using R2 and R6.
In van Lambalgen [1990] (theorem 2.2.1) we proved the converse direction by extending a totally Borel model (cf. theorem 1.2) for $\mathcal{Q}\cup\{\neg\phi\}$ using forcing; this argument has to be changed slightly. Suppose $\mathcal{Q}\cup\{\neg\phi\}$ is consistent. Let $\mathcal{M}$ be a model of ZF + V=L, and let $\mathcal{A} \in \mathcal{M}$ be a totally Borel model for $\mathcal{Q}\cup\{\neg\phi\}$. Let $\mathcal{M}[\mathcal{G}]$ be the extension for generically adding ω_1 random reals. Now consider $\mathcal{A}$ in $\mathcal{M}[\mathcal{G}]$ (by Borel absoluteness) and expand $\mathcal{A}$ to $<\mathcal{A},R>$, where R is interpreted as Solovay forcing. That $<\mathcal{A},R> \models \mathcal{R}$ was verified in lemma 2.2. By induction one shows

$$\mathcal{A}\models \psi(\vec{y}) \Leftrightarrow <\mathcal{A},R> \models \psi^*(\vec{y}),$$

which proves $<\mathcal{A},R> \models \neg\phi^*$. Hence if $\mathcal{Q}\cup\{\neg\phi\}$ is consistent, so is $\mathcal{R}\cup\{\neg\phi^*\}$. ⊠

The above proof makes clear that the odd - looking R6 corresponds to that seemingly most innocent of all quantifier properties, monotonicity. To see what it means, let us first consider the contraposition:

$$\exists x(R(x,\vec{y}) \wedge \phi(x,\vec{y})) \to \exists x(R(x,z\,\vec{y}\,) \wedge \phi(x,\vec{y})).$$

In conjunction with the irreflexivity of R, this statement shows that if ϕ is satisfied by some random sequence, it is satisfied by infinitely many of them. Again, we see that random sequences have some indistinguishability properties. Moreover, in van Lambalgen [1990] it is shown that, as a consequence of R6, random sequences are

transcendental in the modeltheoretic sense: $R(a,\vec{b})$ implies that a is not in the algebraic closure of $\vec{b}$. As easy consequences we get
- $\forall x\ (R(x) \rightarrow x \neq a)$, for constants a in L; and more generally
- $\forall x\ (R(x,\vec{y}) \rightarrow x \neq t(\vec{y}))$, where t is a term in L. In other words, if x is independent of $\vec{y}$, then x cannot be obtained by applying a definable function to $\vec{y}$. Hence the R - analogue of monotonicity for Q is a strong statement; perhaps this is connected to the reservations on monotonicity expressed in section 1. Further clarification of R6, in terms of its relation to forcing, will be given after the proof of lemma 3.4.

We now study the effect of adding $\mathcal{R}$ to ZFC. As in the case of $\mathcal{Q}$, there are two possibilities. The system ZF$\mathcal{R}$ results from ZF by adding $\mathcal{R}$, where the parameters $\vec{y}$ can be interpreted by arbitrary sets and where we allow R to occur in the schemata of ZF. The system ZF$\mathcal{R}^0$ + AC results from ZFC by adding $\mathcal{R}$, with the following provisos

(1) R may occur in the schemata
(2) the variables in R are assumed to range over 2^ω
(3) the L(R) formulas occurring in R6 may contain ordinal parameters.

Irt is easy to see that in this case the fundamental axiom is R6:

2.4 LEMMA ZF$\mathcal{R}$ minus R6 is conservative over ZF; similarly for ZF$\mathcal{R}^0$.
PROOF If $\vec{y}$ denotes the vector $<y_1,..,y_n>$, $\#(x,\vec{y})$ is defined as: $x \neq y_1 \wedge .... \wedge x \neq y_n$. Now interpret R as: $R(x,\vec{y}) \Leftrightarrow \#(x,\vec{y})$. ⊠

Hence, for a reason that is not yet clear to me, the role of the Fubini axiom Q6 in $\mathcal{Q}$ is taken over by the transcendentality axiom R6 in $\mathcal{R}$.
The fundamental consistency result is given by:

2.5 LEMMA ZF$\mathcal{R}^0$ + AC can be interpreted in the model obtained by generically adding κ random reals, for $\kappa \geq \omega_1$.
PROOF Interpret R as Solovay randomness and apply 2.2. Since R is definable, the schemata with the new predicate R are also satisfied. ⊠

COROLLARY 2.6 (a) If ZF is consistent, so is ZF$\mathcal{Q}^0$ + AC. (b) If ZF is consistent, so is ZF$\mathcal{Q}$ + DC.
PROOF (a) By theorem 2.3. (b) Consider again $\mathcal{M}[\mathcal{G}]$; ZF$\mathcal{Q}$ + DC holds in the inner model of $\mathcal{M}[\mathcal{G}]$ of sets which are hereditarily definable from ordinal and real parameters. ⊠

We still have to prove one more consistency result, namely, if ZF is consistent, so is ZF$\mathcal{R}$ + DC. Before we turn to this question, and to the question posed in the introduction, to wit, what is the relation between the intensional, and the extensional approach to randomness?, let us first draw an easy consequence from ZF$\mathcal{R}^0$ + AC. We already know the next result (it follows from 1.11 and 2.5), but we give another proof here because it shows that R in the schemata of ZF makes life a little easier.

THEOREM 2.7 (ZF$\mathcal{R}^0$+ AC) There is no definable wellordering (not even with real and ordinal parameters).

PROOF Suppose $x < y$ is a formula that defines a wellordering. Suppose for the present that $x < y$ does not contain additional real parameters (ordinal parameters are allowed). Define $S(x,y) := R(y,x) \wedge \forall z(R(z,x) \rightarrow y \leq z)$. We claim that $\forall x \exists ! y S(x,y)$. Uniqueness is obvious. If for some x, $\forall y(R(y,x) \rightarrow \exists z(R(z,x) \wedge z<y)$, then by using AC and recursion one can construct an infinite descending < - chain (here, it is essential that R occurs in the schemata); this proves the claim.

But since S defines a function, we also have $\forall x \exists ! y S(x,y) \rightarrow \forall xy(R(y,x) \rightarrow \neg S(x,y)))$. For by R4, $\forall xz(S(x,z) \rightarrow \forall y(R(y,zx) \rightarrow \neg S(x,y)))$, whence by R6, $\forall xz(S(x,z) \rightarrow \forall y(R(y,x) \rightarrow \neg S(x,y)))$. $\forall x \exists y S(x,y)$ now implies $\forall xy(R(y,x) \rightarrow \neg S(x,y))$, but this is a contradiction. (Observe that we haven't used R5 in the proof.) If there are other parameters in $x < y$, simply add these to R in the definition of S. ⊠

We now examine the relation between ZF$\mathcal{R}$ and ZF$\mathcal{Q}$ and their ^{0} - versions. These results are in part due to Domenico Zambella.

2.8 THEOREM (i) ZF$\mathcal{R}$ has an inner model satisfying ZF$\mathcal{Q}$. (ii) ZF$\mathcal{Q}$ has an inner model satisfying ZF$\mathcal{R}$ (as a consequence ZF$\mathcal{R}$ is consistent).

PROOF (i) follows from theorem 2.3 *. In the other direction, it suffices to find an explicit definition of an R, satisfying R1–5 and the weak form of R6, in ZF$\mathcal{Q}$. Suppose first that we have a formula $\theta(x,\alpha,\vec{y})$ enumerating (in α) all sets of reals which are definable (in L(Q)!) from $\vec{y}$ and ordinal parameters. Define

$$R(x,\vec{y}) \Leftrightarrow \forall\alpha(Qx\theta(x,\alpha,\vec{y}) \rightarrow \theta(x,\alpha,\vec{y})).$$

Observe that we must have $\forall \vec{y} Qx R(x,\vec{y})$; for this statement is equivalent to $\forall \vec{y} Qx \forall\alpha(Qx\theta(x,\alpha,\vec{y}) \rightarrow \theta(x,\alpha,\vec{y}))$, which follows from the tautology $\forall \vec{y} \forall\alpha(Qx\theta(x,\alpha,\vec{y}) \rightarrow Qx\theta(x,\alpha,\vec{y}))$ by theorem 1.8. Hence we have established R1. R2 and R3 are trivial. For R4 observe that $R(x,y)$ implies $Qx(x \neq y) \rightarrow x \neq y$, which implies $x \neq y$ by Q2. The verification of R5 is somewhat more involved:

We have to show $R(y,\vec{z}) \wedge R(x,y\vec{z}) \to R(y,x\vec{z})$. Suppose $\neg R(y,x\vec{z})$. Then for some $\phi(y,x\vec{z})$, $\neg(Qy\phi(y,x\vec{z}) \to \phi(y,x\vec{z}))$. By $R(x,y\vec{z})$, $\neg Qx\neg(Qy\phi(y,x\vec{z}) \to \phi(y,x\vec{z}))$, and by $R(y,\vec{z})$, $\neg Qy\neg Qx\neg(Qy\phi(y,x\vec{z}) \to \phi(y,x\vec{z}))$. We apply Q6 and obtain $\neg Qx\neg Qy\neg(Qy\phi(y,x\vec{z}) \to \phi(y,x\vec{z}))$, whence also $\neg Qx\neg(Qy\phi(y,x\vec{z}) \wedge Qy\neg \phi(y,x\vec{z}))$. From this we get $\neg Qx(x=x)$, but this conflicts with Q2 and Q4. Lastly, R6 holds trivially:

For suppose $\forall x(R(x,z\vec{y}) \to \phi(x,\vec{y}))$, then by definition $\forall x[\forall\alpha(Qx\theta(x,\alpha,z\vec{y}) \to \theta(x,\alpha,z\vec{y})) \to \phi(x,\vec{y})]$. We have to show $\forall x[\forall\alpha(Qx\theta(x,\alpha,\vec{y}) \to \theta(x,\alpha,\vec{y})) \to \phi(x,\vec{y})]$. Suppose for some x and α, $Qx\theta(x,\alpha,\vec{y}) \to \theta(x,\alpha,\vec{y})$, then also $Qx(\theta(x,\alpha,\vec{y}) \wedge z = z) \to (\theta(x,\alpha,\vec{y}) \wedge z = z)$, hence $\phi(x,\vec{y})$. This proves $\forall x(R(x,\vec{y}) \to \phi(x,\vec{y}))$.

It remains to construct the formula $\theta(x,\alpha,\vec{y})$. The obvious way to do this, is to copy the construction of the formula defining ordinal definability. To this end, we must extend the reflection theorem to formulas in the language L(Q), i.e. we have to show:

if ϕ is a formula in L(Q), then $\forall\alpha\exists\beta>\alpha(\phi \leftrightarrow \phi^{V_\alpha})$.

We show how to amend Kunen's proof ([1980], 137). We first observe that the *quantifier* Q is represented as a *set* in ZF$\mathcal{Q}$; for by the separation axiom we have

$$\exists B\forall A(A \in B \leftrightarrow A \subseteq 2^\omega \wedge Qx(x \in A)).$$

We shall call this B Q. We can therefore rewrite a formula $Qx\phi(x)$ as $\exists u \in Q\forall x(x \in u \leftrightarrow \phi(x))$, so we have a reduction to the case of first order formulas. ⊠

2.9 COROLLARY (i) ZF$\mathcal{R}^0$ has an inner model satisfying ZF$\mathcal{Q}^0$. (ii) ZF$\mathcal{Q}^0$ has an inner model satisfying ZF$\mathcal{R}^0$.

PROOFSKETCH Only the argument for (ii) needs modification. We can no longer conclude immediately from separation that Q is represented as a set in ZF$\mathcal{Q}^0$. We therefore use "bootstrapping": let $\theta_0(x,\alpha,\vec{y})$ denote the formula representing all sets ordinal definable (in $\{\in, =\}$) from $\vec{y}$. Using θ_0, we can define (in ZF$\mathcal{Q}^0$) a set Q^1 by

$$A \in Q^1 \leftrightarrow \exists\alpha(\forall x(x \in A \leftrightarrow \theta_0(x,\alpha,\vec{y})) \wedge Qz\theta_0(z,\alpha,\vec{y})).$$

As in 2.8, we can use Q^1 to show that the reflection theorem holds for L(Q) formulas involving one Q only . This absoluteness can then be used to construct a formula enumerating all sets (of reals) which are definable from ordinals and reals using formulas which may contain one Q; etc. ⊠

Hence there is not much to choose between the extensional approach and the approach which speaks about random sequences only implicitly. Of course, this holds true only provided one accepts $\mathcal{Q}$; and as we have seen, one may argue that both monotonicity

(Q4) and classical logic not justified on a more intensional way of looking at randomness. In any case it is clear that the Fubini property, in the guise of theorem 1.8, plays a major role in the equivalence.

3 DATA AXIOMS So far our axioms have made explicit only very general properties of randomness; since the axioms also allow an interpretation in terms of category, essential features of randomness are missing. Even worse, the axioms do not force simple properties like density of random sequences:

3.1 LEMMA ZF$\mathcal{Q}$ has models in which the random sequences are not dense.
PROOF Let [0] be the cylinder set of all sequences starting with 0. We may suppose $\neg Qx(x \notin [0])$, otherwise we are done. Define a new quantifier $Q^{[0]}$ by $Q^{[0]}\phi \leftrightarrow Qx(x \in [0] \rightarrow \phi)$. Then ZF$\mathcal{Q}$ is also satisfied when we interpret Q as $Q^{[0]}$. Hence in the resulting model there are no random sequences in [1]. ⊠

To formulate additonal axioms, we take our cue from the intuitionistic theory of lawless sequences. A lawless sequence (cf. also the next paragraph) is a process of choosing infinitely many 0's and 1's such that at any stage only finitely many values are known and no restrictions are imposed upon future choices. Two characteristic axioms are:

> Density: every finite binary word is the initial segment of a lawless sequence

and

> Open Data: if we know $A(\alpha)$ for lawless α, this can only be due to our knowledge of an initial segment $\alpha(n)$ of α; hence for all β, if $\beta(n) = \alpha(n)$, then $A(\beta)$.

We will now investigate the analogues of these axioms for random sequences. By now we have four formal systems, and the formulation of the data and density axioms differs for each of them. To spare the reader, we look only at the two most important cases: ZF$\mathcal{Q}$ and ZF$\mathcal{R}^0$.
We first consider ZF$\mathcal{Q}$. As a density axiom we propose

> Density: let A be a Borel set such that $\lambda A > 0$, then $\neg Qx\neg(x \in A)$.

In other words, a set of positive probability contains a random sequence. As a kind of converse, we have

> Inner Data: $Qx(\phi(x) \rightarrow \exists A(A \text{ Borel} \wedge \lambda A > 0 \wedge x \in A \wedge Qz(z \in A \rightarrow \phi(z,\vec{y})))$.

The justification of this principle runs as follows. We think of random sequences as incomplete objects, i.e. objects about which we have only partial information; typically their mode of generation and some initial segment. (In this respect, random sequences are analogous to lawless sequences.) Suppose we know that $\phi(x)$ holds for some random x. Since we cannot know x in its entirety, our knowledge of the truth of $\phi(x)$ must be based on our knowledge of the *process* generating x. But it seems that such a process must be of the following type: a randomness preserving (Borel) function f: $2^\omega \to 2^\omega$ applied to a sequence generated by coin tossing (or some isomorphic process). Now f preserves randomness if $\lambda f^{-1} << \lambda$, i.e. if λf^{-1} is absolutely continuous with respect to λ. Hence we have justified

$$Qx(\phi(x) \to \exists f(f: 2^\omega \to 2^\omega \text{ Borel} \wedge \lambda f^{-1} << \lambda \wedge x \in \text{ran}(f) \wedge Qx\phi(f(x))),$$

but this is equivalent to Inner Data by general nonsense.

It is clear that, by resorting to a specimen of intuitionistic reasoning in the above justification, we have given a more or less constructive meaning to the dual quantifier $\neg Q \neg$. For a thoroughgoing classical analysis of randomness, see the remarks on Myhill's work in 6.1.

We next consider $ZF\mathcal{R}^0$. One could obtain a R version of Inner Data by restricting the formulas $\phi(x)$ to have only real and ordinal parameters, and applying * from definition 2.1. However, this would not yield a statement that is true for our preferred interpretation of R, Solovay randomness. The true R - analogue is obtained by explicitly taking parameters into consideration. Suppose $\phi(x) = \phi(x,\vec{y})$, where all real parameters are exhibited. Observe that, by applying Q2, Q4 and Q5, ID is equivalent to

$$Qx(\phi(x,\vec{y}) \to \exists A(A \text{ Borel} \wedge \lambda A > 0 \wedge x \in A \wedge Qz(z \in A \wedge \#(z,\vec{y}) \to \phi(z,\vec{y}))).$$

If we translate this version of Inner Data in the R language, using *, we get

$$\forall x(R(x,\vec{y}) \wedge \phi(x,\vec{y}) \to \exists A(\lambda A > 0 \wedge x \in A \wedge \forall z(R(z,\vec{y}) \wedge \#(z,\vec{y}) \wedge z \in A \to \phi(z,\vec{y})))),$$

but by R4 this is equivalent to

$$\forall x(R(x,\vec{y}) \wedge \phi(x,\vec{y}) \to \exists A(\lambda A > 0 \wedge x \in A \wedge \forall z(R(z,\vec{y}) \wedge \#(z,\vec{y}) \wedge z \in A \to \phi(z,\vec{y})))).$$

As we shall see in the proof of 3.2, this statement expresses a truth about Solovay forcing. In fact, I do not know of any other way of verifying Inner Data than by way of the parametrized version. The reader may also wish to compare this formulation of Inner Data with the parametrized form of the axiom of Open Data, given in 4.

Henceforth we abbreviate Inner Data to ID, and Density to D; we use these abbreviations indiscriminately for the Q and R versions. When we add both D and ID to a theory we indicate this by IDD.

3.2 LEMMA $ZF\mathcal{R}^0$ + AC + IDD holds in the model obtained by generically adding $\kappa \geq \omega_1$ random reals to a groundmodel satisfying V = L.

PROOF Let $\mathcal{M}[\mathcal{G}]$ be the generic extension constructed in theorem 2.6. It suffices to show that Density and ID hold in $\mathcal{M}[\mathcal{G}]$. Density follows from the definition of Solovay randomness. Moreover, ID was already implicitly verified in the proof of R6. For we showed that, given a formula $\phi(x,\vec{y})$, there exists a Borel set A with code in $\mathcal{M}[\vec{y}]$, such that for all x random over $\mathcal{M}[\vec{y}]$, $x \in A$ iff $\mathcal{M}[\mathcal{G}] \models \phi(x,\vec{y})$. If $\phi(x,\vec{y})$ is verified for some random x, $\lambda A > 0$. Hence we have shown

$$\forall x(R(x,\vec{y}) \wedge \phi(x,\vec{y}) \rightarrow \exists A(\lambda A > 0 \wedge x \in A \wedge \forall z(R(z,\vec{y}) \wedge z \in A \rightarrow \phi(z,\vec{y})))). \boxtimes$$

Exactly as in section 2 we obtain

3.3 COROLLARY $ZF\mathcal{Q}$+ DC + IDD is consistent.

A consequence of ID that is useful in applications is given in

3.4 LEMMA ($ZF\mathcal{Q}$+ DC + ID) For any formula $\phi(x)$ there is a Borel set A such that $Qx(x \in A \leftrightarrow \phi(x))$.

PROOF We may assume $\neg Qx\neg\phi(x)$, for otherwise we can take $A = \emptyset$. Consider the set $\{A \mid \lambda A > 0 \wedge Qx(x \in A \rightarrow \phi(x))\}$ which exists in $ZF\mathcal{Q}$. Let A_0 be an element of maximal measure. A_0 exists because $s := \sup\{\lambda A \mid \lambda A > 0 \wedge Qx(x \in A \rightarrow \phi(x))\} \leq 1$; choose (DC!) $A_1, ..., A_n,...$ such that $\lambda A_n \rightarrow s$, and put $A_0 := \bigcup A_n$. Then we must have $Qx(x \in A_0 \leftrightarrow \phi(x))$, for otherwise $\neg Qx\neg(\phi(x) \wedge x \notin A_0)$ and ID shows that A_0 wasn't maximal after all. $\boxtimes$

The reader will have noticed that the proofs of R6 and ID in the random real extension are very similar. It may therefore be instructive to observe that ID can be derived from two weaker axioms, with the help of R6. The axioms are

(1) Let B be a Borel set with Borel code u, such that $\lambda B = 1$. Then $\forall x(R(x,u) \rightarrow x \in B)$.

(2) for any formula $\phi(x,\vec{y})$, there exists a Borel set B such that $\forall x(R(x,\vec{y}) \rightarrow (\phi(x,\vec{y}) \leftrightarrow x \in B))$.

Suppose $R(x,\vec{y}) \wedge \phi(x,\vec{y})$. To prove ID, it suffices to show that for the B given by (2), $\lambda B > 0$. Suppose $\lambda B = 0$. Let B have Borel code u. By (1), $\forall x(R(x,u) \rightarrow x \notin B)$, whence by R2, $\forall x(R(x,u\vec{y}) \rightarrow x \notin B)$. By (2), $\forall x(R(x,u\vec{y}) \rightarrow \neg\phi(x,\vec{y}))$. We may now apply R6 to obtain $\forall x(R(x,\vec{y}) \rightarrow \neg\phi(x,\vec{y}))$, a contradiction.

Conversely, in $ZF\mathcal{R}^0$(minus R6) + D, ID implies (1), (2) and R6. (2) follows from 3.4. To prove (1), let B be a Borel set with code u such that $\lambda B =1$ and suppose that

for some x, $R(x,\vec{y}) \wedge x \notin B$. By ID, there exists A such $\lambda A > 0$ and $\forall z(R(z,\vec{y}) \wedge z \in A \rightarrow z \notin B)$. We can rewrite this as $\forall z(R(z,\vec{y}) \rightarrow (z \notin B \vee z \notin A))$; but since $\lambda(B^c \cup A^c) < 1$, we have a contradiction with D. Lastly, we prove R6. Suppose $\forall x(R(x,z\vec{y}) \rightarrow \phi(x,\vec{y}))$ and $\exists x(R(x,\vec{y}) \wedge \neg\phi(x,\vec{y}))$. ID gives us a Borel set A such that $\lambda A > 0$ and $\forall x(R(x,\vec{y}) \wedge x \in A \rightarrow \neg\phi(x,\vec{y}))$. By hypothesis, $\forall x(R(x,\vec{y}) \wedge x \in A \rightarrow \neg R(x,z\vec{y}))$, which is equivalent to $\forall x(R(x,\vec{y}) \wedge R(x,z\vec{y}) \rightarrow x \notin A)$. By R2 we obtain $\forall x(R(x,z\vec{y}) \rightarrow x \notin A)$, which conflicts with D.
The presence of (1) of course explains why we had to use Solovay forcing.

We have already noted that the Q language is not strong enough to express the closure of the universe of random sequences under suitable operations. E.g. if f is measure preserving, there is no way to express "for all random x, f(x) is random". A weaker type of closure is given by

3.5 LEMMA (ZFQ+IDD) Let $f:2^\omega \rightarrow 2^\omega$ be a bijective measure preserving Borel function. Then $Qx\phi(x) \leftrightarrow Qx\phi(f(x))$.
PROOF Suppose $Qx\phi(x) \wedge \neg Qx\phi(f(x))$. By Inner Data, we obtain a Borel set A such that $\lambda A > 0$ and $Qx(x \in A \leftrightarrow \neg\phi(f(x)))$. Then f[A], the image of A under f, is Borel and we have $Qx(x \in f[A] \leftrightarrow \neg\phi(x))$; but since $Qx\phi(x)$, we get $Qx(x \notin f[A])$, whence $\lambda f[A] = 0$; a contradiction. For the other direction, consider f^{-1} (which is also Borel and measure preserving). ⊠

We are now ready for the main result of this section.
3.6 THEOREM (ZFQ+ DC + IDD) There exists a translation invariant extension μ of Lebesgue measure to all of $\wp(2^\omega)$. This μ can be computed as follows: every $B \in \wp(2^\omega)$ can be written as $A\Delta N$, where A is Borel and N is a μ – nullset, and $\mu A\Delta N = \lambda A$.
PROOF The idea that Fubini's theorem is connected to extensions of Lebesgue measure is suggested by Simms [1989]. Let $\mathcal{A}$ denote the Borel σ – algebra on 2^ω. Let $\mathcal{N}$ denote $\{B \in \wp(2^\omega) \mid Qx(x \notin B)\}$. By theorem 1.8, $\mathcal{N}$ is closed under countable unions. By Q4, $\mathcal{N}$ is closed under subsets. Hence $\mathcal{N}$ is a σ– ideal. Let $\mathcal{B} = \{A\Delta N \mid A \in \mathcal{A}, N \in \mathcal{N}\}$. It is easy to see that $\mathcal{B}$ is a σ – algebra. We would like to define μ on $\mathcal{B}$ such that $\mu A\Delta N = \lambda A$. To make μ welldefined, we should have $A_1\Delta N_1 = A_2\Delta N_2$ implies $\lambda A_1 = \lambda A_2$. But $A_1\Delta N_1 = A_2\Delta N_2$ implies $A_1\Delta A_2 = N_1\Delta N_2$ and since $N_1\Delta N_2 \in \mathcal{N}$, we get $Qx(x \notin A_1\Delta A_2)$. By Inner Density, $\lambda A_1\Delta A_2 = 0$. Hence μ is welldefined and is easily shown to be σ – additive. Moreover, $N \in \mathcal{N}$ implies $\mu N = 0$.

We now show that $\mathcal{B} = \wp(2^{\omega})$. Choose $B \in \wp(2^{\omega})$. Apply lemma 3.4 to get a Borel A such that $Qx(x \in A \leftrightarrow x \in B)$, i.e. $Qx(x \notin A\Delta B)$. Hence $A\Delta B \in \mathcal{N}$. We may now write $B = A\Delta(A\Delta B)$, which shows that $B \in \mathcal{B}$.
That μ is translation invariant follows from lemma 3.5. ⊠

3.7 LEMMA (ZF$\mathcal{Q}$+ DC + IDD) μ is κ–additive, for any ordinal κ.
PROOF Suppose $\{A_{\alpha}\}_{\alpha<\kappa}$ are pairwise disjoint subsets of 2^{ω}. At most countably many of them can have positive μ measure; say these are $\{A_n\}_{n<\omega}$. Then $\sum_{\alpha<\kappa}\mu A_{\alpha} = \sum_{n<\omega}\mu A_n + \mu\bigcup\{A_{\alpha}|\ \omega \leq \alpha < \kappa\}$; by the additivity property 1.8, the last term equals 0. ⊠

The meaning of these results can perhaps best be explained by referring to two rivalling traditions in probability theory. In the approach universally used today (associated with the name of Kolmogorov), a probability is taken to be a σ–additive measure on a σ–algebra. Other notions that are of interest to a probabilist, foremost among them independence, are defined in terms of measure. But there is an older tradition, now nearly extinct, in which rather randomness and independence are taken as primitive. This approach goes back to von Mises, who tried to capture the independence inherent in random sequences by means of axioms about subsequence selection. Although these axioms themselves are fairly unwieldy, it still seems to me that the general idea of treating fundamental probabilistic notions as primitives has some potential. For instance, we have seen just now that the whole business about σ–algebras actually becomes superfluous. This may not be immediately apparent, because we formulated D and ID in terms of Borel sets and Lebesgue measure; but a glance at the axioms will show that we could have used closed sets instead, and *their* Lebesgue measure is easily computed.
It may also be of interest to note that von Mises' axioms in their orignal form are derivable in ZF$\mathcal{Q}$+ DC +IDD. First some notation:
Put $LLN(x) := \forall\varepsilon\exists n_0\forall n{\geq}n_0 \left|\frac{1}{n}\sum_{k=1}^{n} x_k - \frac{1}{2}\right| < \varepsilon$ and define a partial operation $/: 2^{\omega}\times 2^{\omega} \rightarrow 2^{\omega}$ by: $({}^{x}/{}_{y})_n = x_m$ if m is the index of the n^{th} 1 in y and undefined if there is no such index. (The subsequence ${}^{x}/{}_{y}$ of x is infinite if y has infinitely many 1's.)
In van Lambalgen [1990] we used the systems $\mathcal{Q}$ and $\mathcal{R}$ to formalize von Mises' notion of randomness. We added for instance the following axioms:

$QxLLN(x)$ ("a random sequence satisfies the law of large numbers")

and

QxQyLLN($^x/y$) ("If one selects a subsequence from a random sequence x by means of a random sequence y which is independent of x, that subsequence satisfies the law of large numbers").

It is easy to see that these properties are derivable in ZF$\mathcal{Q}$+ DC +IDD. QxLLN(x) follows from ID and the fact that $\lambda\{x \mid LLN(x)\} = 1$. To prove QxQyLLN($^x/y$), we first observe that /y: $2^\omega \to 2^\omega$ is measure preserving. Hence by 3.5, $\forall$yQxLLN($^x/y$). By applying monotonicity and Fubini we obtain QxQyLLN($^x/y$).

The next consequence of ZF$\mathcal{Q}$+ DC + IDD is also related to AC:

3.8 THEOREM (ZF$\mathcal{Q}$+ DC + IDD) There are no ultrafilters on $\wp(\omega)$.

PROOF Let U be an ultrafilter on $\wp(\omega)$. We identify U with a subset of 2^ω. By theorem 3.6, U must be μ - measurable. Hence U must be of the form A Δ N, where A is Borel and N $\in \mathcal{N}$. We claim that $\lambda A = 0$ or 1. This follows if we can show that A is a tailset. So let τ be a transformation on 2^ω that changes a fixed finite number of coordinates of $x \in 2^\omega$. It suffices to show that $x \in A$ implies $\tau x \in A$. Observe that, by lemma 3.3, each $M \in \mathcal{N}$ is a tailset. If $x \in A \cap U$, then $x \notin N$ and hence $\tau x \notin N$; but since for all y, $y \in U$ implies $\tau y \in U$, we must have $\tau x \in A$. Suppose now that $x \notin A \cap U^c$. Again, generally $y \in U^c$ implies $\tau y \in U^c$. If $x \notin N$, $x \in A \cup N$, $x \notin A \cap N$, whence $x \in U$. So we must have $x \in N$ and hence $\tau x \in N$. If $\tau x \notin A$, then $\tau x \in A \cup N$, $\tau x \notin A \cap N$ and it follows that $\tau x \in U$, a contradiction. This shows that $\tau x \in A$, i.e. that A is a tailset. By the 0 – 1 law, A has measure 0 or 1. By definition of μ, μU equals 0 or 1. But U^c is obtained from U by the measure preserving transformation of exchanging 0's and 1's, and we get a contradiction from lemma 3.5.⊠

4 EXCURSION INTO INTUITIONISM: LAWLESS SEQUENCES We indicate here, as a consequence of the preceding results, why the addition of the axioms LS for lawless sequences to intuitionistic set theory ZF^I is not conservative. ZF^I (called T in Powell [1975]) is the following set of axioms, formulated in intuitionistic logic:

1. (Extensionality) $x = y \wedge x \in w \to y \in w$
2. ($\in$ - induction) $\forall x(\forall u \in x\phi(u) \to \phi(x)) \to \forall x\phi(x)$
3. (Separation) $\exists y\forall z(z \in y \leftrightarrow z \in x \wedge \phi)$
4. (Pairs) $\exists y(u \in y \wedge v \in y)$
5. (Union) $\exists y\forall z(\exists u \in x(z \in u) \to z \in y)$
6. (Replacement) $\exists y\forall v(\exists u \in x(\phi(v) \wedge \forall w(\phi(w) \to v = w)) \to v \in y)$
7. (Power set) $\exists y\forall z(\forall u(u \in z \to u \in x) \to z \in y)$
8. (Double complement) $\exists y\forall z(\neg\neg z \in x \to z \in y)$

9. (Infinity) $\exists y(\exists z(z \in y) \wedge \forall x \in y \exists z \in y(x \in z))$

Here, $\neg\phi$ is defined as $\phi \rightarrow \perp$, where $\perp$ is some absurdity; e.g. $\perp = \forall xy(x \in y)$. Furthermore, equality is defined as $x = y \Leftrightarrow \forall z(z \in x \leftrightarrow z \in y)$.

We now turn to lawless sequences. Lawless sequences are processes of assigning (in our case) 0 or 1 to the arguments 1,2,3, .., such that (1) at any stage only finitely many values are known and (2) at no stage the possibility for choosing values is restricted.

We think of ZF^I with variables for lawless sequences as a two - sorted theory: one sort for sets, one sort for lawless sequences $\alpha, \beta, ... \in 2^\omega$. We first introduce some notational conventions. We use w as a variable over finite binary sequences. [w] is the set of infinite binary sequences which have initial segment w.

$\#(\alpha,\beta_1,....,\beta_n)$ is defined as $\alpha \neq \beta_1 \wedge .... \wedge \alpha \neq \beta_n$; and the quantifier $\underline{\forall}\gamma B(\gamma,\beta_1,....,\beta_n)$ is defined as $\forall\gamma(\#(\gamma,\beta_1,....,\beta_n) \rightarrow B(\gamma,\beta_1,....,\beta_n))$.

To show nonconservativity we need only axioms which are an immediate consequence of the intuitive explanation of lawlessness given above (and not the more problematic continuity principles relying on the bar theorem). The axioms are as follows:

LS1 $\forall w \exists \alpha(\alpha \in [w])$

LS2 $\alpha = \beta \vee \alpha \neq \beta$

LS3 $A(\alpha,\beta_1,....,\beta_n) \wedge \#(\alpha,\beta_1,....,\beta_n) \rightarrow \exists w(\alpha \in [w] \wedge \underline{\forall}\gamma \in [w]A(\gamma,\beta_1,....,\beta_n)$

We add a few words of explanation; for the full motivation the reader is referred to Troelstra [1977] or Troelstra and van Dalen [1988].

LS1 says that we are allowed to specify an initial segment in advance. LS2 is based on the fact that extensional and intensional identity coincide for lawless sequences. Let $\equiv$ denote intensional identity: $\alpha \equiv \beta$ means that α and β are given to us as the same process. Then obviously $\alpha \equiv \beta \vee \neg\alpha \equiv \beta$ and it is easy to show that for extensional identity $=$, $\alpha = \beta \leftrightarrow \alpha \equiv \beta$.

LS3 is clearly inherent in the meaning of lawlessness, especially if we consider the parameterfree version: $A(\alpha) \rightarrow \exists w(\alpha \in [w] \wedge \forall\gamma \in [w]A(\gamma))$. For if we have a proof of $A(\alpha)$, this can only be on the basis of a finite segment of α. In the general case of formulas with parameters, we have to add some provisos (like the condition $\#(\alpha,\beta_1,....,\beta_n)$) to avoid inconsistency. The formula $A(\alpha,\beta_1,....,\beta_n)$ may contain quantifiers over arbitrary sets, but the parameters $\beta_1,....,\beta_n$ are assumed to be lawless. (Troelstra remarks: "LS3 seems also to be justified for predicates containing ... parameters for non-lawlike objects not constructed from lawless sequences ... , provided extensionality holds w.r.t. all function and set parameters." (Troelstra [1977], p. 29) In other words, one would need an independence relation more general than $\neq$ to formulate LS3.)

Call the resulting two sorted system $ZF^I\mathcal{LS}$. We then have

4.1 THEOREM $ZF^I\mathcal{LS}$ is not conservative over ZF^I.

PROOF We first indicate how to embed a slight variant of $ZF\mathcal{Q}^0$ into $ZF^I\mathcal{LS}$. For reasons that will become clear below we have to consider the Q - closures of the axioms Q1 – 6, instead of the universal closures. Call this system weak - $ZF\mathcal{Q}^0$. Inspection of the proof of corollary 1.11 shows that it takes place inside weak - $ZF\mathcal{Q}^0$.

Powell [1975] constructs an interpretation * of ZF into ZF^I as follows. Put $\sim\sim z := \{x \mid \neg\neg x \in z\}$ and define by recursion a function s which satisfies $s(x) = \sim\sim\{s(u) \mid u \in x\}$. We can now define * by recursion:

$$(x \in y)^* = s(x) \in s(y)$$
$$(\neg\phi)^* = \neg\phi^* := \phi^* \to \bot$$
$$(\phi \vee \psi)^* = \neg(\neg\phi^* \wedge \neg\psi^*)$$
$$(\phi \wedge \psi)^* = \phi^* \wedge \psi^*$$
$$(\phi \to \psi)^* = \phi^* \to \psi^*$$
$$(\forall x\phi)^* = \forall xu(u = s(x) \to \phi^*(u))$$
$$(\exists x\phi)^* = \neg(\forall x\neg\phi)^*.$$

Powell shows $ZF \vdash \phi \Leftrightarrow ZF^I \vdash \phi^*$.

We extend this interpretation with the following clause for Q: $(Qx\phi(x,y_1,\ldots,y_n))^* := \forall\alpha(\#(x,y_1,\ldots,y_n) \to \phi^*(\alpha,s(y_1),\ldots,s(y_n)))$. If one of the parameters $y_1,\ldots,y_n$ is in fact a lawless sequence β, we put $s(\beta) = \beta$.

All Q axioms except Q4 are trivially true under this interpretation. To verify Q4, one uses the axioms of density and open data, and it is here that our convention on weak – $ZF\mathcal{Q}^0$ becomes important, for now we may assume that all parameters are lawless. So suppose we have $\forall\alpha(\alpha \neq \beta \to \phi^*(\alpha,\beta))$ and $\forall\alpha(\phi^*(\alpha,\beta) \to \psi^*(\alpha,\gamma))$. Then also $\forall\alpha(\alpha \neq \beta,\gamma \to \psi^*(\alpha,\gamma))$ and we have to show $\forall\alpha(\alpha \neq \gamma \to \psi^*(\alpha,\gamma))$.

If $\beta = \gamma$ we are done. So (LS1) suppose $\alpha \neq \gamma$, $\beta \neq \gamma$. By applying LS3 to β in $\forall\alpha(\alpha \neq \beta,\gamma \to \psi^*(\alpha,\gamma))$ we get

$$\exists w(\beta \in [w] \wedge \forall\beta' \in [w](\beta' \neq \gamma \to \forall\alpha(\alpha \neq \beta',\gamma \to \psi^*(\alpha,\gamma))).$$

By LS2 we can take $\beta' \in [w]$ such that $\alpha \neq \beta'$ and $\gamma \neq \beta'$ and $\psi^*(\alpha,\gamma)$. This completes the proof of the interpretation of Q4.

Suppose $ZF^I\mathcal{LS}$ were conservative over ZF^I. Hence if weak - $ZF\mathcal{Q}^0 \vdash \phi$, then $ZF^I\mathcal{LS} \vdash \phi^*$. By assumption $ZF^I \vdash \phi^*$ and since * is an embedding, $ZF \vdash \phi$. This would show that weak - $ZF\mathcal{Q}^0$ is conservative over ZF, quod non. ⊠

4.2 REMARK We could have interpreted full $ZF\mathcal{Q}^0$ in $ZF^I\mathcal{LS}$ if in the latter system LS3 were formulated with an abstract independence relation instead of $\neq$ (cf. Troelstra's remark above).

This result shows that, in contrast to the situation in second order intuitionistic arithmetic, when added to set theory lawless sequences can no longer be considered a "figure of speech" (Troelstra and van Dalen [1988], p. 644). In other words, quantifiers over lawless sequences can no longer be interpreted by means of expressions not containing those quantifiers.

5 EPILOGUE Gödel writes in a letter to Tarski, commenting on the failure of his "square axioms" for the continuum:

> My confidence that $2^{\aleph_0} = \aleph_2$ has of course somewhat been shaken. But it still seems plausible to me. One reason is that I don't believe in any kind of irrationality such as, e.g. random sequences in any absolute sense. (Gödel [1990], 175)

The preceding considerations seem to show that randomness is related to structure or complexity rather than cardinality. It is true that Freiling [1986] has claimed that suitable randomness axioms can force $2^{\aleph_0} \geq \aleph_{n+1}$, for any n. But an examination of his proofs (see the next section) makes clear that what he really shows is that the continuum cannot have a wellordering of length ω_n, for any n. We would rather interpret his results as indicating that the continuum cannot be wellordered at all. This is what ZF$\mathcal{Q}$ implies and, as can be seen below, ZF$\mathcal{Q}$ is a natural extrapolation of Freiling's axioms. Of course, what we mean is really a conditional assertion: *if* one believes in the reality of randomness and *if* one believes that ZF$\mathcal{Q}$ is a correct description of this reality, then the axiom of choice is false. For different notions of set AC may still be true.
To us, the interesting feature of the randomness axioms is rather their strong connection with forcing. Fairly straightforward hypotheses on random sequences turn out to describe essential properties of a particular type of forcing extension; as such they are a "poor man's Martin's axiom". But, unlike Martin's axiom, the axioms can also lay claim to some intuitive justification.

6 APPENDIX: RELATED APPROACHES Here we discuss two other attempts to add axioms for randomness to set theory, those of Myhill and Freiling.
6.1 MYHILL'S AXIOMS These were alluded to in the quotation from Kreisel in section 0. They are reported in paragraph 10 of Kruse [1967].
Again, let 2^ω be the space of infinite binary sequences and let λ denote Lebesgue measure on this space, i.e. $(\frac{1}{2},\frac{1}{2})^\omega$. Let R(x) be a predicate which should intuitively be interpreted as "x is random". Myhill tries to formalize the intuition that random sequences should satisfy "all" properties of probability 1, and no other properties. His axioms (M) are

(M1) $\lambda\{x \in 2^{\omega} \mid R(x)\} = 1$

(M2) If $\phi(x)$ is a formula in one free variable, then $\lambda\{x \in 2^{\omega} \mid \phi(x)\} = 1$ implies $\forall x(R(x) \rightarrow \phi(x))$.

The restriction that ϕ contain no parameter beside x is obviously necessary, since otherwise we could take the formula $x \neq y$. The formulation of (M2) contains a deliberate ambiguity, however: is R allowed to occur in ϕ or not? If not, then M is obviously conservative over ZFC. On the other hand, if we do allow R in ϕ, then R need no longer be explicitly definable. Myhill comments:

> [This] would accord with a prejudice of mine which I derived from Feller, i.e. that randomness is an intensional notion, not definable in the usual mathematical terms. The "circularity" of the schema above with R allowed to appear in ϕ is quite justified if we are convinced that R belongs to a new order of ideas, entirely outside the set theoretic order.

An immediate consequence is that M is no longer conservative over ZFC, for it implies the by now familiar consequence that there is no definable wellordering of the continuum: if there were such a wellordering, then we could define the least random element, in contradiction with (M2). It is clear that this proof works only when R is also allowed to occur in the schemata of ZF. Note that, on the liberal interpretation of (M2), the consistency of ZFC + M (with R allowed in the schemata of ZF) is not immediate.

6.1.1 LEMMA ZFC + M is consistent.

PROOFSKETCH Use the construction of theorem 1.3 (I), starting from a model $\mathcal{M}$ of ZF + V = L. Interpret R(x) as: x is (Solovay) random over $\mathcal{M}$. In the generic extension of $\mathcal{M}$ verifying 1.3 (I), the random reals have measure 1. Furthermore, R is definable, hence to verify (M2) it suffices to show that a definable set of reals with measure 1 contains all random reals. This is established in Stern [1985], section 4.7.⊠

In order to facilitate comparison between M and the axioms introduced in the previous sections, let us reformulate M in terms of axioms of density and data. (M1) is equivalent to the axiom of Outer Density:

If $\{x \mid \phi(x)\}$ has positive outer measure, then $\exists x(R(x) \wedge \phi(x))$;

and (M2) is equivalent to Outer Data:

If $\exists x(R(x) \wedge \phi(x))$, then $\{x \mid \phi(x)\}$ has positive outer measure.

It then becomes clear that Myhill's approach is eminently classical. If $\{x \mid \phi(x)\}$ has positive outer measure, we do not have enough information to construct a random x which almost certainly verifies ϕ. By the same token, knowledge of the truth of $\exists x(R(x) \wedge \phi(x))$ does not allow us to infer that we must have been able to construct (with practical certainty) a random x such that $\phi(x)$; hence we obtain only the weaker conclusion that $\{x \mid \phi(x)\}$ has positive outer measure.

One final remark: in view of the extensional theory of randomness outlined in section 2, the following modification of Myhill's axioms seems reasonable: we have a relation $R(x,\vec{y})$ such that

(M0) $R(x,\vec{y})$ satisfies $\mathcal{R}$

(M1) $\forall \vec{y}\ \lambda\{x \in 2^{\omega} \mid R(x,\vec{y})\} = 1$

(M2) If $\lambda\{x \in 2^{\omega} \mid \phi(x,\vec{y})\} = 1$, then $\forall x(R(x,\vec{y}) \rightarrow \phi(x,\vec{y}))$.

Is it possible to prove the consistency of this theory short of using an inaccessible?

6.2 FREILING AND THE CONTINUUM HYPOTHESIS We will next investigate the relation of Freiling's axioms of symmetry [1986] to the axioms introduced here. Freiling writes:

> Suppose we were to throw a random dart at the real number line and ask whether the dart landed on a rational number. The outcome is, of course, predictable. We could say in advance that the dart will (with probability one) land on an irrational number. Furthermore, let us agree that the reason does not depend on any particular property of the set of real numbers except that it is countable and its members are determined before we make our throw.
> Now suppose we were to throw two darts and ask whether the second dart was a rational multiple of the first one. The answer would likewise be no, since by the time we throw the second dart there are only countably many points which it has to miss, and membership in this countable set is predetermined by the first dart.
> Suppose then that we have a function f: $\mathbb{R} \rightarrow \mathbb{R}_{\aleph_0}$ (i.e. f assigns to each real a countable set of reals). The second dart will not be in the countable set assigned to the first dart. Now by the symmetry of the situation (the real line does not know which dart was thrown first or second), we could also say that the first dart will not be in the set assigned to the second. This leads us to the following natural proposition:
> $A_{\aleph_0}$ $\quad \forall f: \mathbb{R} \rightarrow \mathbb{R}_{\aleph_0} \exists xy(x \notin f(y) \wedge y \notin f(x))$,
> the intuition being that x and y could be found by independently throwing two random darts.

He then proceeds to prove that $A_{\aleph_0}$ is equivalent to $\neg$CH. We intend to show here that Freiling's intuitive motivation for $A_{\aleph_0}$ is entirely captured by $\mathcal{Q}$ in the following sense: a suitable fragment of ZF$\mathcal{Q}$ (basically one allows two iterations of Q only) suffices to derive $A_{\aleph_0}$ and, conversely, that fragment can be interpreted in ZFC + $A_{\aleph_0}$.

Although the resulting fragment is admittedly adhoc, we shall try to motivate it by referring back to the proof of 1.8. It will be observed that, in that proof, we needed only statements of the form $Qx(\langle x,y\rangle \in U)$ or $QxQy(\langle x,y\rangle \in V)$. Accordingly, we can define a class of *elementary statements* as follows.

6.2.1 DEFINITION The class of n – elementary statements is defined by

(1) $<x_1, \ldots, x_n> \in U$, where $U \subseteq (2^\omega)^n$, is n – elementary

(2) n – elementary statements are closed under $\rightarrow, \neg, \wedge, \vee, \forall, \exists$

(3) if ϕ is elementary and not of the form $\forall x\psi$ or $\exists x\psi$, then $Qx\phi$ is n – elementary.

$ZF\mathcal{E}_n$ is obtained from $ZF\mathcal{Q}$ by applying the following restrictions

(1) the Q axioms are formulated for n – elementary statements only

(2) we allow only n iterations of Q

(3) in n – elementary statements, we allow only those $U \subseteq (2^\omega)^n$ for which $\{ x \mid <x_1, \ldots, x_{n-1}, x> \in U\}$ is countable for all $<x_1, \ldots, x_{n-1}>$.

We apologize for the lack of elegance, both of the preceding definitions and of the proof of the main result, 6.2.7. We include this material only because it shows that Freiling's intuitions fit squarely in the framework of the preceding sections. Furthermore, it may be of interest to see that Q can also have a cardinality interpretation and that the size of the continuum is related to the number of iterations of Q.

6.2.2 THEOREM $(ZF\mathcal{E}_2)$ $A_{\aleph_0}$

PROOF Suppose we are given $f: 2^\omega \rightarrow (2^\omega)_{\aleph_0}$. It suffices to show $QxQy(x \notin f(y) \wedge y \notin f(x))$, for generally $Qx\phi$ implies $\exists x\phi$. By Q6 and Q5, we only have to prove $QxQy(y \notin f(x))$. However, this follows from $\forall xQy(y \notin f(x))$, which is a consequence of 1.8.

⊠

Clearly, Freiling's symmetry principle ("the real line does not know the order of the darts") corresponds to Q6.

6.2.3 COROLLARY $(ZF\mathcal{E}_2)$ $\neg CH$

PROOF If CH, there is a wellordering $\leq$ of 2^ω of length $\aleph_1$. Now consider $f(y) := \{x \mid x \leq y\}$. Then $\forall x\forall y(x \in f(y) \vee y \in f(x))$; but since each $f(y)$ is countable we get a contradiction from $A_{\aleph_0}$. ⊠

Actually something stronger holds. Let $(2^\omega)_n$ denote the set of n - element subsets of x. Since $(2^\omega)_n$ can be coded into 2^ω, we can also consider functions $f: (2^\omega)_n \rightarrow (2^\omega)_{\aleph_0}$. This means that we are able to formulate and prove a generalization of $A_{\aleph_0}$ in our set up. First a

6.2.4 DEFINITION Let $f: (2^\omega)^n \rightarrow (2^\omega)_{\aleph_0}$. A set $X \subseteq 2^\omega$ is called f - incomparable if for any n distinct elements $x_1, \ldots, x_n$ of X, $x_n \notin f(\{x_1, \ldots, x_{n-1}\})$.

The generalization of $A_{\aleph_0}$ can then be formulated as follows:

$A^n_{\aleph_0}$ $\forall f: (2^\omega)_{n-1} \to (2^\omega)_{\aleph_0}\ \exists$ f - incomparable set of size n.

6.2.5 THEOREM ($ZF\mathcal{E}_n$) For all n, $A^n_{\aleph_0}$.

PROOF Along the same lines as for n = 2. To show how Q6 is applied we do the case n = 3. Choose f. By 1.8 $\forall x\forall yQz(z \notin f(\{x,y\})$, hence by monotonicity $QxQyQz(z \notin f(\{x,y\})$. By Q6, we get $QzQyQx(z \notin f(\{x,y\})$, whence $QxQyQz(x \notin f(\{z,y\})$ by Q3; and similarly $QxQyQz(y \notin f(\{x,z\})$. Now apply Q5. ⊠

Freiling proves

6.2.6 THEOREM (ZFC) $A^n_{\aleph_0}$ is equivalent to $2^{\aleph_0} \geq \aleph_n$.

We will now establish an equivalence between Freiling's approach and ours.

6.2.7 THEOREM $ZF\mathcal{E}_2$+ AC is interpretable in ZFC + $A_{\aleph_0}$.

PROOF The obvious interpretation of Q would be to put A in Q if A is co-countable, but this runs afoul of Q6. We therefore have to give a contextual definition of Q, depending on an additional parameter. Indeed, it seems impossible to give a *uniform* interpretation of Q in the absence of strong hypotheses like, e.g., "all sets of reals have the Baire property".

Let $\mathcal{A}$ be the σ - algebra of countable (i.e. finite or countably infinite) and co-countable sets. Define $\nu : \mathcal{A} \to \{0,1\}$ by $\nu A = 1$ iff A is co-countable. ν is σ–additive. We show how to extend $\nu \times \nu$ on $2^\omega \times 2^\omega$ to a measure μ on an extension of $\mathcal{A}\times\mathcal{A}$ that will serve to interpret Q. Expressions like "full", "null" or "almost all" refer to ν.

We first need a lemma which shows that Sierpinski's counterexample to a Fubini theorem without joint measurability condition does not exist under $A_{\aleph_0}$.

6.2.8 LEMMA (ZFC + $A_{\aleph_0}$) There is no set that is null for almost all horizontal sections and full for almost all vertical sections.

PROOF See Freiling [1986], p.197. ⊠

Let $\mathcal{N}_x$ denote $\{ A \subseteq 2^\omega \mid \forall x \in 2^\omega\ A_x$ is null $\}$ and similarly $\mathcal{N}^y := \{ A \subseteq 2^\omega \mid \forall y \in 2^\omega\ A^y$ is null $\}$. Obviously $\mathcal{N}_x$ and $\mathcal{N}^y$ are σ–ideals.

6.2.9 LEMMA (ZFC + $A_{\aleph_0}$) Let A be $\nu \times \nu$ measurable and suppose $A \subseteq \bigcup B_i$, where $\{B_i\}$ is a countable family contained in $\mathcal{N}_x \cup \mathcal{N}^y$. Then $\nu \times \nu A = 0$.

PROOF (Cf. Simms [1989], lemma 2) Suppose not, then we would have $\nu \times \nu A = 1$. We may suppose $A \subseteq X \cup Y$, where $X \in \mathcal{N}_x$ and $Y \in \mathcal{N}^y$. Hence for all x, $A_x \subseteq X_x \cup Y_x$. By Fubini's theorem applied to $\nu \times \nu$, for almost all x, A_x is full.
Since for all x, X_x is null, it follows that Y_x must be full for almost all x, which is impossible by the previous lemma. ⊠

We can now define the extension μ of $\nu \times \nu$. Let $\mathcal{B}$ be the σ - algebra generated by $\mathcal{A} \cup (\mathcal{N}_x \cup \mathcal{N}^y)$. It is easy to show that $\mathcal{B} = \{ B \mid B = A\Delta N, A \in \mathcal{A}, N \in \mathcal{N}_x \cup \mathcal{N}^y\}$. Define μ on $\mathcal{B}$ by $\mu(A\Delta N) = \nu \times \nu A$. μ is a welldefined σ - additive measure because $A_1\Delta N_1 = A_2\Delta N_2$ implies $A_1\Delta A_2 = N_1\Delta N_2$; now apply the previous lemma. Obviously μ satisfies $\mu N = 0$ for all $N \in \mathcal{N}_x \cup \mathcal{N}^y$. We can explicitly define a disintegration $\{ \mu_x \mid x \in 2^\omega\}$ of μ as follows: put $\mu_x(A_x\Delta N_x) := \nu A_x$. Then μ_x is a σ - additive measure that extends ν, for fixed $B \in \mathcal{B}$ the function $x \to \mu_x B_x$ is ν - measurable and we have

(*) $\mu(A\Delta N) = \nu \times \nu A = \int \nu A_x d\nu(x) = \int \mu_x A_x d\nu(x) = \int \mu_x(A_x\Delta N_x)d\nu(x)$.

This property will serve to validate Q6. We interpret Q in ZFC + $A_{\aleph_0}$ as follows:

(1) $Qx\psi(x) \Leftrightarrow \nu\{x \mid \psi(x)\} = 1$

(2) $Qx\phi(x,y) \Leftrightarrow \mu_x\{y \mid \phi(x,y)\} = 1$

for 2 – elementary ϕ and ψ. This interpretation is correct because, if ϕ is 2 – elementary, either $\forall x$ $\{y \mid \phi(x,y)\}$ is null or $\forall x$ $\{y \mid \neg\phi(x,y)\}$ is null; whence by construction of μ, $\{\langle x,y\rangle \mid \phi(x,y)\}$ is μ - measurable.

We now show that the Q axioms are valid under this interpretation. Q1 – 3,5 hold trivially.

We prove Q6 by means of (*):

$QxQy\phi(x,y) \Leftrightarrow \nu\{x \mid \mu_x\{y \mid \phi(x,y)\} = 1\} = 1 \Leftrightarrow \int \mu_x\{y \mid \phi(x,y)\}d\nu(x) = 1 \Leftrightarrow \mu\{\langle x,y\rangle \mid \phi(x,y)\} = 1 \Leftrightarrow \int \mu_y\{x \mid \phi(x,y)\}d\nu(y) = 1 \Leftrightarrow \nu\{y \mid \mu_y\{y \mid \phi(x,y)\}=1\}=1 \Leftrightarrow QyQx\phi(x,y)$.

The validity of Q4 is connected to some simple symmetry properties. Suppose $Qx\phi(x,y)$ and $\forall x(\phi(x,y) \to y(x,z))$. Then $\mu_y\{x \mid \phi(x,y)\} =1$, hence $\mu_y\{x \mid \psi(x,z)\} =1$. Let $\pi: 2^\omega \to 2^\omega$ be an automorphism that maps y to z and let $\tau = id \times \pi$ be the induced automorphism on 2^ω. Obviously μ is invariant under τ. Hence if we put $C := \{\langle x,z\rangle \mid \psi(x,z)\}$ we have $\mu C = (\mu\tau^{-1})C$ which implies $\mu_z C_z = (\mu\tau^{-1})_z C_z = \mu_y C_z = 1$. ⊠

6.2.10 COROLLARY $ZF\mathcal{E}_2$ + AC is interpretable in ZFC + $2^{\aleph_0} = \aleph_2$.
Analogous results hold for n iterations of Q, $n \geq 3$.

The main philosophical difference between Freiling's approach and ours, is that $ZF\mathcal{Q}$ is a purely qualitative theory. Unlike Freiling, we remain agnostic about which sets are small (hence will almost certainly be missed by a random dart), we formulate properties of small sets *per se*. (It is only after adding ID that we commit ourselves.) What Freiling proves in each case is that the continuum cannot have a wellordering of length ω_n; only by applying AC can we conclude from this that $2^{\aleph_0} \geq \aleph_{n+1}$. It seems to us that his arguments may be taken as well as evidence that AC is false for a universe containing random sequences. Indeed, ZFQ, which contains Q6, the natural extrapolation of Freiling's symmetry principle, forces AC to be false. But we have seen that, although the continuum has no wellordering at all, the aleph - free version of CH can still be true in the presence of ZFQ (even with ID added). We conclude from this that randomness is related to the structure, rather than to the cardinality of the continuum.

REFERENCES

van Benthem , J.F.A.K. [1990], Inference, Methodology and Semantics, in: P. Bystrov (ed.), Festschrift for V.A. Smirnov, Ak. Nauk USSR, Moscow.

Borel, E. [1990], Les probabilités dénombrables et leurs applications arithmétiques, reprinted in Leçons sur la théorie des fonctions, Gauthiers - Villars.

Freiling, C. [1986], Axioms of symmetry: throwing darts at the real number line, *J. Symb. Logic* **51**, 190 - 200.

Gödel, K. [1990], Collected Works Vol. II, S. Feferman et al. (eds.), Oxford University Press.

Kakuda, Y. [1989], The role of a filter quantifier in set theory, in J. Shinoda, T.A. Slaman, T. Tugué (eds.), Mathematical logic and applications, Lectures Notes in Mathematics 1388, Springer Verlag.

Kaufmann, M. [1983], Set theory with a filter quantifier, *J. Symb. Logic* **48**, 263 - 287.

Kunen, K. [1980], Set theory, North–Holland.

Kreisel, G. [1969], Two notes on the foundations of set theory, *Dialectica* **23**, 93 - 114.

Kruse, A.H. [1967], Some notions of random sequence and their set theoretic foundations, *Z. Logik u. Grundl. d. Math.* **13**, 299 - 322.

van Lambalgen, M. [1990], The axiomatization of randomness, *J. Symb. Logic* **55**,

Oxtoby, J.C. [1980], Measure and Category, Springer Verlag.

Powell, W. [1975], Extending Gödel's negative interpretation to ZF, *J. Symb. Logic* **40**, 221 - 229.

Simms, J. [1989], Traditional Cavalieri principles applied to the modern notion of area, *J. Phil. Logic* **18**, No. 3, 275 – 314.

Shelah, S. [1984], Can you take Solovay's inaccessible away?, *Isr. J. Math.* **48**, No. 1, 1 - 47.

Solovay, R.M. [1970], A model of set theory in which every set of reals is Lebesgue measurable, *Ann. of Math.* **92**, 1 - 56.

Stern, J. [1985], Regularity properties of definable sets of reals, *Ann. Math. Logic* **29**, 289 – 324.

Troelstra, A.S. [1977], Choice sequences, Oxford University Press.

Troelstra, A.S. and D. van Dalen [1988], Constructivism in mathematics, Vol. II, North Holland.

The ITLI Prepublication Series

1990

Logic, Semantics and Philosophy of Language

LP-90-01 Jaap van der Does	A Generalized Quantifier Logic for Naked Infinitives
LP-90-02 Jeroen Groenendijk, Martin Stokhof	Dynamic Montague Grammar
LP-90-03 Renate Bartsch	Concept Formation and Concept Composition
LP-90-04 Aarne Ranta	Intuitionistic Categorial Grammar
LP-90-05 Patrick Blackburn	Nominal Tense Logic
LP-90-06 Gennaro Chierchia	The Variablity of Impersonal Subjects
LP-90-07 Gennaro Chierchia	Anaphora and Dynamic Logic
LP-90-08 Herman Hendriks	Flexible Montague Grammar
LP-90-09 Paul Dekker	The Scope of Negation in Discourse, towards a flexible dynamic Montague grammar
LP-90-10 Theo M.V. Janssen	Models for Discourse Markers
LP-90-11 Johan van Benthem	General Dynamics
LP-90-12 Serge Lapierre	A Functional Partial Semantics for Intensional Logic
LP-90-13 Zisheng Huang	Logics for Belief Dependence
Mathematical Logic and Foundations	
ML-90-01 Harold Schellinx	Isomorphisms and Non-Isomorphisms of Graph Models
ML-90-02 Jaap van Oosten	A Semantical Proof of De Jongh's Theorem
ML-90-03 Yde Venema	Relational Games
ML-90-04 Maarten de Rijke	Unary Interpretability Logic
ML-90-05 Domenico Zambella	Sequences with Simple Initial Segments
ML-90-06 Jaap van Oosten	Extension of Lifschitz' Realizability to Higher Order Arithmetic, and a Solution to a Problem of F. Richman
ML-90-07 Maarten de Rijke	A Note on the Interpretability Logic of Finitely Axiomatized Theories
ML-90-08 Harold Schellinx	Some Syntactical Observations on Linear Logic
ML-90-09 Dick de Jongh, Duccio Pianigiani	Solution of a Problem of David Guaspari
ML-90-10 Michiel van Lambalgen	Randomness in Set Theory
Computation and Complexity Theory	
CT-90-01 John Tromp, Peter van Emde Boas	Associative Storage Modification Machines
CT-90-02 Sieger van Denneheuvel, Gerard R. Renardel de Lavalette	A Normal Form for PCSJ Expressions
CT-90-03 Ricard Gavaldà, Leen Torenvliet, Osamu Watanabe, José L. Balcázar	Generalized Kolmogorov Complexity in Relativized Separations
CT-90-04 Harry Buhrman, Leen Torenvliet	Bounded Reductions
CT-90-05 Sieger van Denneheuvel, Karen Kwast	Efficient Normalization of Database and Constraint Expressions
CT-90-06 Michiel Smid, Peter van Emde Boas	Dynamic Data Structures on Multiple Storage Media, a Tutorial
Other Prepublications	
X-90-01 A.S. Troelstra	Remarks on Intuitionism and the Philosophy of Mathematics, Revised Version
X-90-02 Maarten de Rijke	Some Chapters on Interpretability Logic
X-90-03 L.D. Beklemishev	On the Complexity of Arithmetical Interpretations of Modal Formulae
X-90-04	Annual Report 1989
X-90-05 Valentin Shehtman	Derived Sets in Euclidean Spaces and Modal Logic
X-90-06 Valentin Goranko, Solomon Passy	Using the Universal Modality: Gains and Questions
X-90-07 V.Yu. Shavrukov	The Lindenbaum Fixed Point Algebra is Undecidable
X-90-08 L.D. Beklemishev	Provability Logics for Natural Turing Progressions of Arithmetical Theories
X-90-09 V.Yu. Shavrukov	On Rosser's Provability Predicate
X-90-10 Sieger van Denneheuvel, Peter van Emde Boas	An Overview of the Rule Language RL/1
X-90-11 Alessandra Carbone	Provable Fixed points in $I\Delta_0+\Omega_1$, revised version
X-90-12 Maarten de Rijke	Bi-Unary Interpretability Logic