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CYLINDRIC MODAL LOGIC

Yde Venema

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CYLINDRIC MODAL LOGIC

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Abstract. We study Cylindric Algebras from a perspective of modal logic. A completeness result for this modal logic yields a finite derivation system for the equations valid in the variety of Representable Cylindric Algebras, and a proof calculus for type-free valid formulas.

Contents.

1. Introduction.

In the same manner as Boolean algebras form an interpretation for propositional logic, *cylindric algebras* [HMT] are an algebraic approach towards the predicate calculus. Cylindric algebras (of *dimension* α, α an ordinal) are defined as Boolean algebras with operators (BAO's, cf [JT]) satisfying a specific finite set of equations; there are unary operators c_i corresponding to the existential quantification $\exists v_i$, and nullary operators, i.e. constants, d_{ij}, corresponding to the identities $v_i = v_j$. Such algebras are called *representable* if they are in the variety generated by the so-called *cylindric set-algebras* ; in these algebras the universe V consists of all subsets of a set ${}^{\alpha}U$, the Boolean operators are interpreted as the usual set-theoretical operations, c_i $(0 \leqslant i < \alpha)$ as the *cylindrification* operation C_i given by $C_i(X) = \{u \in {}^{\alpha}U |$ there is a $v \in {}^{\alpha}U$ with $v_j = u_j$ for $j \neq i\}$, and d_{ij} as the *diagonal* set $D_{ij} = \{u \in {}^{\alpha}U | u_i = u_j\}$.
As the cylindrification operators c_i are additive (meaning $c_i(x \vee y) = c_i(x) \vee c_i(y)$), we may treat the subject from a generalized modal perspective. So in our approach cylindric algebras are the *modal algebras* associated with a *modal* logic. A reader unfamiliar with the concept of modal algebras is referred to [Go1]. For the modal language, we will have a unary modal operator $\langle i \rangle$ corresponding to the i-th cylindrification, and a constant δ_{ij} corresponding to the appropriate diagonal constant d_{ij} of the algebraic language. Kripke frames for this language will be relational structures having cylindric-type *complex algebras* [HMT1]. The reader might put forward that the modal language is nothing but the algebraic one in a very thin disguise, and rightly so, as we give a quite straightforward translation between the two. We feel however, that the modal approach is more suitable for our aims, as it enables us to distinguish local truth of a formula (i.e. at a possible world, or element, of a Kripke structure) from global truth (i.e. at every possible world of the Kripke frame).
Now, as usual in this logic-as-algebra framework, completeness theorems are the logical counterpart of the algebraic representation theorems. In this light, we give a new completeness/representation proof for the well-known finite axiomatization result of the two-dimensional representable cylindric algebras. We may also try to use more sophisticated tricks from modal completeness results. The idea to use an axiom system with a special kind of derivation rule in order to obtain completeness results originates with D.M. Gabbay, who used an "irreflexivity rule" to axiomatise the set of ordinary

modal formulas valid in the class of irreflexive Kripke frames (cf. [G]). Similar rules have been applied frequently in the context of more-dimensional modal logics (cf. e.g. [B],[V]). In [V] a system with such a special rule is given which axiomatizes the class of Representable Relation Algebras.
The main result of this paper is a finite derivation system of the identities valid in the class of representable cylindric algebras of finite dimension, obtained by adding one equation to the set axiomatizing cylindric algebras, and a new closure operator to the usual set of derivation rules. For the representable algebras of dimension ω we then prove a similar result. This last result has as its immediate corollary that we can give a proof calculus for the so-called type-free valid formulas which involves only type-free valid formulas. This may indicate a positive solution to Problem 4.16 of [HMT].

This paper sprang from the rather careless written section 3.3 of [V]. We want to thank Hajnal Andréka, Istvan Németi and Ildikó Sain for many corrections and suggestions without which this paper would probably never have been written, let alone be finished.

2. Two-dimensional cylindric modal logic.

As symbols for the language of two-dimensional cylindric modal logic we have an infinite number of propositional variables $p_0, p_1, \ldots, q, r, \ldots$, propositional constants δ_{00}, δ_{01}, δ_{10} and δ_{11}, the Boolean connectives $\wedge$ and $\neg$, and two unary modal operators $\langle 0 \rangle$ and $\langle 1 \rangle$. Formulas are built up in the usual fashion. In this section, however, we will use the following symbols: δ for both δ_{01} and δ_{10}, T for both δ_{00} and δ_{11}, $\diamondsuit$ for $\langle 1 \rangle$ and $\diamondsuit$ for $\langle 0 \rangle$ and the ordinary propositional and modal abbreviations, e.g. $\square\varphi$ for $\neg\diamondsuit\neg\varphi$. (The symbols $\diamondsuit$ and $\diamondsuit$ originate with Segerberg, who treated a similar logic in [S]).

We will define two kinds of semantics for this language: first the more abstract Kripke-frames, and then the intended two-dimensional ones:

A *(Kripke) frame* is a quadruple K = (W,H,V,D) where W is a set of *possible worlds*, H and V are binary *accessibility relations* on W and the *diagonal* D is a subset of W. A *(Kripke) model* is a pair (K,μ) with K a Kripke frame and V a *valuation*, i.e. a map assigning subsets of W to each propositional formula of the language; μ should satisfy $\mu(T) = W$ and $\mu(\delta) = D$. By induction we define a *truth relation* $\models$; we only give the clauses for the modal operators:

$M,w \models \diamondsuit\varphi$ if there is a v with Hwv and $M,v \models \varphi$,

$M,w \models \diamondsuit\varphi$ if there is a v with Vwv and $M,v \models \varphi$.

Concepts like *validity* and *satisfiability* are defined in the usual way.

A *two-dimensional frame based on a set U* is defined as the Kripke frame (W,H,V,D) where

$W = U \times U$,

$H(u_0,u_1)(v_0,v_1)$ iff $u_1 = v_1$,

$V(u_0,u_1)(v_0,v_1)$ iff $u_0 = v_0$ and

$D(u_0,u_1)$ iff $u_0 = u_1$.

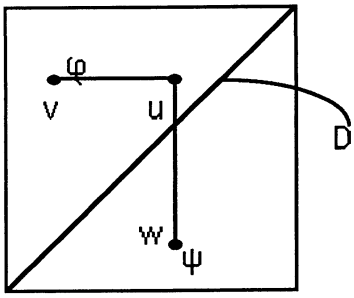
Note that, with this definition, on a two-dimensional model we have

$M,u \models \diamondsuit\varphi$ iff there is a v on the same horizontal line with $M,v \models \varphi$,

$M,u \models \diamondsuit\psi$ iff there is a w on the same vertical line with $M,w \models \psi$

(cf. the figure)

The class of two-dimensional frames is denoted by F_2.



We might now proceed and develop the modal theory of CML_2, by defining concepts like disjoint unions, general frames, etc. As all this can be done in a rather obvious way, we only give notions needed further on: Let K,K' be two Kripke frames, f: $W \mapsto W'$ a map. Consider the following properties:

(1) Huv only if H'f(u)f(v), Vuv only if V'f(u)f(v), Du only if D'f(u);
(2) Du if D'f(u)
(3) If H'f(u)v' then there is a $v \in W$ such that Huv and f(v) = v' (and the same holds for V).

If f has (1) we call it a *homomorphism*, f is a *strong homomorphism* if it also has (2), and if it satisfies the *zigzagcondition* (3) as well, we call it a *zigzagmorphism*. If the zigzagmorphism f: $K \mapsto K'$ is onto, we may call K' a *zigzagmorphic image* of K. It is straightforward to verify that the validity of modal formulas is preserved under taking zigzagmorphic images: If K' is a zigzagmorphic image of K, then $K \models \varphi \Rightarrow K' \models \varphi$.
A frame K is *connected* if $K \models \forall xy \exists z\ xHzVy$. Note that all two-dimensional frames are connected.

Call a frame *nice* if it satisfies the following properties: H and V both are reflexive (i), symmetric (ii), and transitive (iii), H|V = V|H, where R|S = {(x,y)| there is a z with xRz and zSy} (iv), each world has exactly one H-successor (resp. V-successor) in D: (v) for existence and (vi) for uniqueness. Finally, nice frames must validate the following formula (vii): $\forall uvw$ [$(Du \wedge Huv \wedge Vvw \wedge v \neq w) \rightarrow (\exists x(\neg Dx \wedge Vux \wedge Hxw)$].
Let MC1-7 be the following modal formulas:

(MC1) $p \rightarrow \Diamond_H p$
(MC2) $p \rightarrow \boxminus \Diamond_H p$
(MC3) $\Diamond_H \Diamond_H p \rightarrow \Diamond_H p$
(MC4) $\Diamond_H \Diamond_V p \rightarrow \Diamond_V \Diamond_H p$
(MC5) $\Diamond_H \delta$
(MC6) $\Diamond_H(\delta \wedge \varphi) \rightarrow \boxminus(\delta \rightarrow \varphi)$
(MC7) $[\delta \wedge \Diamond_H(p \wedge q \wedge \Diamond_V(p \wedge \neg q))] \rightarrow \Diamond_V(\neg \delta \wedge \Diamond_H(p \wedge \neg q))$

Theorem 2.1.
A frame K is nice iff MC1-7 are valid on K:
In fact the correspondences (i)-(C1) etc. are one by one.

Proof.
For the first six correspondences, the proofs are omitted as they are wellknown, either in their own right or as consequences of the Sahlqvist theorem [SV]. This theorem may also be applied to (vii), but for readers unfamiliar with the theorem we give the proof for $K \models C7 \Rightarrow K \models (vii)$:
Suppose $K \not\models (vii)$. Then there are worlds u,v,w in K with $Du \wedge Huv \wedge Vvw \wedge v \neq w$ (we are a bit sloppy in our notation), for which there is no x satisfying $\neg Dx \wedge Vux \wedge Hxw$ (*). Look at a valuation μ with $\mu(p) = \{v,w\}$, $\mu(q) = \{v\}$: $K,\mu,u \models [\delta \wedge \ominus(p \wedge q \wedge \Diamond\!\!\!\mid(p \wedge \neg q))]$.
Now let x be a world with $\neg Dx$ and Vux. Then by (*) x cannot be an H-predecessor of w, so v is the only H-successor of x where p is true. This gives $x \models \boxminus(p \rightarrow q)$. As x was an arbitrary non-diagonal V-successor of u, this means $u \models \boxbar(\neg\delta \rightarrow \boxminus(p \rightarrow q))$. But then C7 fails to hold in K.

Not every nice frame is two-dimensional, but there does exist a strong connection between the two notions:

Theorem 2.2.
Every connected nice frame is a zigzag-morphic image of a two-dimensional frame.

Proof.
Let K be a connected nice frame. We will define a chain of homomorphisms $(f_\xi)_{\xi<\lambda}$ (where λ is the maximum of $|K|$ and ω), such that the union f_λ of this chain is the desired zigzagmorphism. Every map f_ξ must be seen as an approximation of f_λ.
Look at the set of *possible defects* $P = \lambda x \lambda x W x \{H,V\}$. Call the quadruple $(\beta,\gamma,v,V) \in P$ a *defect* of a homomorphism $f: {}^2\xi \mapsto K$ (where $\xi < \lambda$), if it defies the zigzagcondition, i.e.$f(\beta,\gamma)Vv$ while there is no $\gamma' \in \alpha$ such that $f(\beta,\gamma')=v$; f is called *perfect* if it has no defects. Assume P is well-ordered, then we may speak of the *first* defect min(f) of an imperfect homomorphism $f: {}^2\xi \mapsto K$. By the following lemma such a map has an extension f' lacking the defect min(f).

Lemma.

Let $f: {}^2\alpha \mapsto K$ be a homomorphism, (β,γ,V,v) a defect of f. Then there is an $f' \supseteq f$, $f': {}^2(\alpha+1) \mapsto K$ such that $f(\beta,\alpha) = v$.

Proof.

Without loss of generality we assume that $\beta = 0$ and $\gamma = \alpha - 1$.

We first set $\quad f'(\xi,\eta) = f(\xi,\eta)$ for $\xi,\eta < \alpha$,

$\qquad\qquad\quad f'(0,\alpha) = v$,

viz.

α	v					
	|					
$\alpha-1$	$f(0,\alpha-1)$	$f(1,\alpha-1)$	—	$f(\eta,\alpha-1)$	—	$f(\alpha-1,\alpha-1)$
	|	|		|		|
η	$f(0,\eta)$	$f(1,\eta)$	—	$f(\eta,\eta)$	—	$f(\alpha-1,\eta)$
	|	|		|		|
1	$f(0,1)$	$f(1,1)$	—	$f(\eta,1)$	—	$f(\alpha-1,1)$
0	$f(0,0)$	$f(1,0)$	—	$f(\eta,0)$	—	$f(\alpha-1,0)$
	0	1		η		$\alpha-1$

Then we are concerned with the $f(\eta,\alpha)$, $\eta < \alpha$. By assumption we have $v \neq f(0,\eta)$, and as f is a homomorphism we get the following picture:

$f(0,\alpha)$

|

$|\neq$

$f(0,\eta) \quad --- \quad f(\eta,\eta) \in D \quad ,$

so by the seventh characteristic of being nice, K has a $v_\eta \notin D$ with $Hv_\eta f(0,\alpha)$ and $Vv_\eta f(\eta,\eta)$. We set

$f'(\eta,\alpha) = v_\eta$;

$f'(\alpha,\alpha)$ is defined as the unique diagonal H-successor of any/ all of the $f'(\eta,\alpha)$.

It is straightforward to verify that with this definition the part of f' defined up till now satisfies both the ordinary and the strong homomorphism condition.

For a definition of $f(\alpha,\eta)$ we use the same trick as above to ensure $f(\alpha,\eta) \notin D$: as $f(\alpha,\alpha)$ is in D and $f(\eta,\alpha)$ is not, they cannot be identical. So $f(\eta,\alpha)$ can be defined as any non-diagonal H-successor of $f(\eta,\eta)$ which is a V-predecessor of $f(\alpha,\alpha)$. □

We now define the chain of maps as follows:

$f_0 = \{\langle(0,0),u\rangle\}$ for some u on the diagonal of K.

$f_{\xi+1} = f_\xi$ if f_ξ is perfect
$(f_\xi)'$ otherwise,

$f_\theta = \bigcup_{\xi<\theta} f_\xi$ if $\theta \leqslant \lambda$ is a limit ordinal.

It is now straightforward to verify that f_λ has the desired properties: first it is a strong homomorphism as all the maps in the chain are. Suppose that f is not a zigzagmorphism; then there are quadruples in P witnessing this shortcoming. Let (β,γ,v,V) be the first of these in the well-ordering of P. Then with $\theta = \max(\beta+1,\gamma+1)$, this quadruple is a defect of f_θ, whence (β,γ,v,V) is $\min(f_\theta)$. By its definition then, $f_{\theta+1}$ lacks this defect, as $f_{\theta+1}(\beta,\theta) = v$. As $f_{\theta+1} \subseteq f_\lambda$ the desired contradiction is derived.

Finally, we have to prove that f is onto. This is an easily provable consequence of the connectedness of K. □

We now define our axiom system:

$ACML_2$ is the set of formulas obtained by closing the set of *axioms* consisting of all substitution instances of C0: all propositional tautologies,and (C1) - (C7), under the *inference rules,* necessitation: $\vdash\varphi \Rightarrow \vdash\boxbar\varphi, \boxminus\varphi$, and modus ponens: $\vdash\varphi$ and $\vdash\varphi\rightarrow\psi \Rightarrow \vdash\psi$.

A *deduction* is a finite string of formulas each of which is either an axiom or follows from earlier formulas by a rule of inference. A formula φ is a *thesis* of $ACML_2$ (notation: $ACML_2 \vdash\varphi$ or $\vdash\varphi$ if no confusion arises) if it appears as the last item of a deduction. A formula φ is a *consequence* of a set Γ of formulas, notation $\Gamma\vdash\varphi$, if there are formulas $\gamma_1,\dots,\gamma_n$ in Γ such that $\vdash(\gamma_1\wedge\dots\wedge\gamma_n)\rightarrow\varphi$. A set of formulas Γ is *consistent* if $\bot$ is not a consequence of Γ. For L a set of propositional formulas, an *L-maximal consistent* set (short: *L-MCS*) is a consistent set of L-formulas to which one cannot add L-formulas without violating its consistency.

The *canonical L-model* $M^c(L)$ is defined as $(W^c(L),H^c,V^c,D^c,\mu^c)$, where $W^c(L)$ is the set of L-maximal consistent sets, $H^c(\Gamma,\Sigma)$ if $\{\varphi | \boxminus\varphi \in \Gamma\} \subseteq \Sigma$, $\Gamma \in D^c$ if $\delta \in \Gamma$ and μ^c is given by $\mu^c(p) = \{\Gamma \in W^c(L) | p \in \Gamma\}$. The underlying frame of the canonical L-model is called the *canonical L-frame* and denoted by $F^c(L)$. It is straighforward to prove, by formula-induction, that in $M^c(L)$, $\Gamma \models \varphi$ iff $\varphi \in \Gamma$, for all $\varphi \in \Phi_L$.

Theorem 2.3.
The canonical frame is nice.

Proof.
Again, this theorem can be seen as a direct consequence of the Sahlqvist form of the axioms, cf. [SV] □

Theorem 2.4. (Soundness and Completeness.)
$ACML_2$ is sound and complete with respect to F_2.

Proof.
Soundness is straightforward, and completeness is more or less immediate by the previous theorem and theorem 2.2: Let Σ be a consistent set of formulas in the language L. Then by the usual Lindenbaum construction, Σ has a maximal consistent extension $\Sigma' \in W^c(L)$. By theorem 2.2, $F^c(L)$ is the image of some two-dimensional frame, say based on a set U, under a zigzagmorphism f. Then it is straighforward to verify that, with the valuation μ given by $u \in \mu(p)$ if $p \in f(u)$, $({}^2U,\mu)$ is a two-dimensional model such that for all $u \in {}^2U$, $u \models \varphi$ iff $\varphi \in f(u)$. But then there is a world u in ${}^2U$ with $u \models \sigma$ for all $\sigma \in \Sigma$: $({}^2U,\mu)$ is a model for Σ. □

3. Finite-dimensional cylindric modal logic.

For a finite ordinal α, *α-formulas* are the formulas of the language of α-dimensional cylindric modal logic CML_α. This language has an infinite number of propositional variables $p_0, p_1, \ldots, q, r, \ldots$, propositional constants δ_{ij}, $i,j < \alpha$, the Boolean connectives $\wedge$ and $\neg$, and unary modal operators $\langle i \rangle$ for all $i < \alpha$. For a set of propositional variables L, the set of *L-formulas* Φ_L is the set of α-formulas obtained by allowing only the δ_{ij}'s and the propositional variables in L as atomic formulas. We will use, besides the usual Boolean and modal abbreviations, also the following:

For Γ a subset of α, we define $\langle\Gamma\rangle\varphi$ in the obvious way, e.g.

$\langle\{1,3,5\}\rangle\varphi \equiv \langle 1\rangle\langle 3\rangle\langle 5\rangle\varphi$

$\langle c(i)\rangle\varphi \equiv \langle\alpha\setminus\{i\}\rangle\varphi$

$\langle c(i,j)\rangle\varphi \equiv \langle\alpha\setminus\{i,j\}\rangle\varphi$

$\Diamond\varphi \equiv \langle\alpha\rangle\varphi$.

As in the two-dimensional case we define both the more abstract Kripke semantics, and the intended α-dimensional frames:

A *(Kripke) (α-)frame* is a tuple $K = (W, T_i, E_{ij})_{i,j<\alpha}$ where W is a set of *possible worlds*, each T_i is a binary *accessibility relation* on W and each *diagonal* E_{ij} is a subset of W. A *Kripke model* is a pair (K,μ) with K a Kripke frame and μ a *valuation*, i.e. a map assigning subsets of W to each propositional formula of the language; μ should satisfy $\mu(\delta_{ii}) = W$ and $\mu(\delta_{ij}) = E_{ij}$. By induction we define a *truth relation* $\models$; we only give the clauses for the modal operators:

$M,w \models \langle i\rangle\varphi$ if there is a v with $T_i wv$ and $M,v \models \varphi$.

Concepts like *validity* and *satisfiability* are defined in the usual way.

Now let U be some set. By the *α-frame based on U* we understand the Kripke frame $(W, T_i, E_{ij})_{i,j<\alpha}$ where $W = {}^\alpha U$, $T_i uv$ iff $u_j = v_j$ for all $j \neq i$, and $u \in E_{ij}$ iff $u_i = u_j$. An *α-dimensional frame* is a frame based on some set U; we denote the class of α-dimensional frames by F_α. Note that, with this definition, in an α-dimensional model we have $M,u \models \langle i\rangle\varphi$ iff there is a v in ${}^\alpha U$, at most differing from u in the i-th coordinate, with $M,v \models \varphi$.

For zigzagmorphisms, connected frames, etc. the obvious generalizations of the definitions in the previous section hold.

An α-frame is called *nice* if it satisfies the properties N1 - N8:

($N1_i$) T_i is reflexive,
($N2_i$) T_i is symmetric,
($N3_i$) T_i is transitive,
($N4_{ij}$) $T_i|T_j = T_j|T_i$,
($N5_i$) $E_{ii} = W$,
($N6_{ij}$) If $i \neq j$, each world has at most one T_i-successor in E_{ij}
($N7_{ijk}$) If $k \notin \{i,j\}$, $\forall u\ (E_{ij}u \leftrightarrow \exists v\ (T_k uv \wedge E_{ik}v \wedge E_{jk}v)$
($N8_{ij}$) If $i \neq j$,
$\forall uvw\ [\ (E_{ij}u \wedge T_i uv \wedge T_j vw \wedge v \neq w) \rightarrow \exists x\ (\neg E_{ij}x \wedge T_j ux \wedge T_i xw)\ ]$

These properties all correspond to modal formulas, just like in theorem 2.6. Consider the following α-formulas:

($MC1_i$) $p \rightarrow \langle i \rangle p$
($MC2_i$) $p \rightarrow [i]\langle i \rangle p$
($MC3_i$) $\langle i \rangle\langle i \rangle p \rightarrow \langle i \rangle p$
($MC4_{ij}$) $\langle i \rangle\langle j \rangle p \rightarrow \langle j \rangle\langle i \rangle p$
($MC5_i$) δ_{ii}
($MC6_{ij}$) $\langle i \rangle(\delta_{ij} \wedge \varphi) \rightarrow [i](\delta_{ij} \rightarrow \varphi)$ $(i \neq j)$
($MC7_{ijk}$) $\delta_{ij} \leftrightarrow \langle k \rangle(\delta_{ik} \wedge \delta_{jk})$ $(k \notin \{i,j\})$
($MC8_{ij}$) $[\delta_{ij} \wedge \langle i \rangle(\ p \wedge q \wedge \langle j \rangle(p \wedge \neg q))] \rightarrow \langle j \rangle(\ \neg\delta_{ij} \wedge \langle i \rangle(p \wedge \neg q))$ $(i \neq j)$

Proposition 3.1.
For α-frames K, we have $K \models MC1_i$ iff $K \models N1_i$, etc.

The proof of this proposition is as in the previous section. □

So a frame is nice iff MC1-MC8 hold in it.
One can easily prove that in a nice frame, any composition of T-relations is again an equivalence relation. Let P_i be the equivalence relation composed of all accessibility relations T_j except T_i, then P_i is the accesibility relation of the operator $\langle c(i) \rangle$. In α-dimensional frames P_i of course denotes the relation of lying in the same "i-hyperplane": x and y are in the same i-hyperplane if they have the same i-th coordinate.

Definition 3.1.
Consider the following derivation system A_α:
The *axioms* are
(MC0): all propositional tautologies, and
(MC1)-(MC8),
and as *derivation rules* we have

Necessitation (Nec): $\vdash\varphi \Rightarrow \vdash[i]\varphi$ and
Modus Ponens (MP): $\vdash\varphi$ and $\vdash\varphi\rightarrow\psi \Rightarrow \vdash\psi$.

The following concepts are defined as in 2.9: deduction,thesis, consequence, consistent, L-maximal consistent set.
As in the previous section, we can easily prove the following

Theorem 3.2.:
A_α is sound and complete with respect to the nice frames.

Unfortunately, we do not have an immediate analogon of 2.7: not every connected nice frame is a zigzagmorphic image of an α-dimensional one, if $\alpha > 2$. So A_α is not complete with respect to F_α. In the sequel we will show that by *adding a derivation rule* to A_α, we can attain completeness. But first, let's have a look at where the problem arises.

The general line of the completeness proof will be, just like for most completeness proofs, to give a model for a consistent set of formulas Γ. Here such a model is built up in a countable number of stages; in every stage of the proof we are dealing with a finite approximation of the model. Such an approximation will be called a *matrix* and has the form of a homomorphism Λ_n: ${}^\alpha n \mapsto F$ where F is some canonical frame; if we can show that the union of these maps is a *quasi*-zigzagmorphism (to be defined later on), the desired model rolls out immediately. Now a finite approximation may have a shortcoming with respect to the *quasi*-zigzag-condition and the aim of the construction is to remove these shortcomings, one in each stage of the construction. Removing a shortcoming will mean extending the homomorphism Λ_n by assigning an MCS to one element of ${}^\alpha n+1 \setminus {}^\alpha n$. But then *all* new elements of ${}^\alpha n+1$ have to be assigned MCSs, and in general we can only be sure that this continuation can be done "respecting homomorphy", if the approximation satisfies some special conditions: all information concerning the MCSs of the extension should already "be present in" the old MCSs. Now in general this is not the case, so we have to interrupt the process by *adding* the necessary information to the homomorphism. The new derivation rule, called the Consistency Rule (CR), is devised just to make this possible. However, adding new information to an L-*maximal* set means extending the language; therefore each Λ_{n+1} does not only extend Λ_n with respect to the domain, but also w.r.t. the codomain: $W^c(L_{n+1})$ instead of $W^c(L_n)$, for some $L_{n+1} \supseteq L_n$. The union Λ' of the chain of homomorphisms will then be a quasi-zigzagmorphism into $F^c(\bigcup_{n\in\omega}L_n)$ (Λ' need not be surjective).

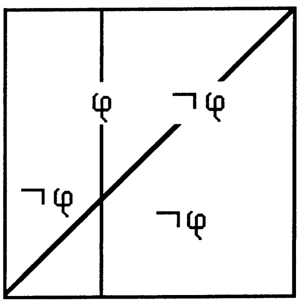
Before giving the formal definition of the new derivation system, matrices and their defects, we give some derived theses of the old system and facts needed later on. The more technical proofs are skipped here and can be found in the appendix.

To start with, we would like to have an operator $THIS_i$ such that $[i](THIS_i\varphi \rightarrow \psi)$ is provably equivalent to $\langle i\rangle(THIS_i\varphi \wedge \psi)$. For then, any MCS Γ containing $\langle i\rangle(THIS_i\varphi \wedge \psi)$, has *exactly one* T_i-successor Δ in the canonical frame in which $THIS_i\varphi \wedge \psi$ can be found. (For, suppose both $\Delta \neq \Delta'$ would be T_i-successors of Γ. Then some ψ is in $\Delta\backslash\Delta'$, whence $\langle i\rangle(THIS_i\varphi \wedge \psi) \wedge \langle i\rangle(THIS_i\varphi \wedge \neg\psi) \in \Gamma$, contradicting the assumption about $THIS_i$.) It is in this sense that we speak of one MCS (Γ) containing all information on another (Δ).

Before defining $THIS_i\varphi$, we consider the following abbreviations:

$$[\![i|j]\!]\varphi \equiv [j](\delta_{ij} \rightarrow [i](\neg\delta_{ij} \rightarrow [j]\varphi))$$
$$[\![i,j]\!]\varphi \equiv [j]\varphi \wedge [\![i|j]\!]\neg\varphi.$$

First consider the case $\alpha=2$: in a two-dimensional frame we have $K,u_0,u_1,\mu \models [\![0,1]\!]\varphi$ iff φ holds on the vertical line through (u_0,u_1) and nowhere else, cf.



And indeed, we can show that

$$\vdash \langle 0\rangle([\![0,1]\!]\varphi \wedge \psi) \rightarrow [0]([\![0,1]\!]\varphi \rightarrow \psi) \qquad \text{(T1)}$$

For the general case, set

$$THIS_i\varphi \equiv [c(i)]\varphi \wedge \bigwedge_{j\neq i}[\![i|j]\!][c(i)]\neg\varphi$$

Note that for $K \in F_\alpha$:

$K,\mu,(u_0,...,u_i,...,u_j,...,u_{\alpha-1}) \models [\![i|j]\!]\varphi$ iff for all u_i',u_j' such that $u_i'\neq u_i$, $K,\mu,(u_0,...,u_i',...,u_j',...,u_{\alpha-1}) \models \varphi$,

whence

$K,\mu,u \models THIS_i\varphi \Leftrightarrow$ for all v: $K,\mu,v \models \varphi$ iff $v_i = u_i$, i.e. φ holds exactly at all points in the hyperspace through u.

In the appendix it is proved that

$\vdash \mathrm{THIS}_i\varphi \leftrightarrow [c(i)]\mathrm{THIS}_i\varphi$ (T2)

$\vdash \mathrm{THIS}_i\varphi \rightarrow \Box(\varphi \leftrightarrow \mathrm{THIS}_i\varphi)$ (T3)

$\vdash \langle i\rangle(\mathrm{THIS}_i\varphi \wedge \psi) \leftrightarrow [i](\mathrm{THIS}_i\varphi \rightarrow \psi)$ (T4)

In the case $\alpha=2$, again consider the formula $[\![0,1]\!]p$. If this formula is true somewhere in a two-dimensional model $M = (K,\mu)$, there is *exactly* one point u on the diagonal where p holds. This gives: $\langle i\rangle(p \wedge \delta_{01})$ is true exactly on the points on the horizontal line through u. So, $M,u \vDash [\![1,0]\!]\langle 1\rangle(p \wedge \delta_{01})$.
This implies: having a THIS_0-formula gives a THIS_1-one. We may even prove:

$\vdash ([\![0,1]\!]p \wedge \delta_{01}) \rightarrow [\![1,0]\!]\langle 1\rangle(\delta_{01} \wedge p).$ (T5)

In the general case, something similar is going on: in an α-dimensional frame $K = {}^{\alpha}U$, if $\mathrm{THIS}_i p$ characterizes the i-hyperplane $\{x \in {}^{\alpha}U \mid x_i=a\}$, then $\mathrm{THIS}_j(\langle i\rangle(\delta_{ij} \wedge p))$ does so with $\{x \in {}^{\alpha}U \mid x_j=a\}$. And again, we may prove that

$\vdash (\delta_{ij} \wedge \mathrm{THIS}_i p) \rightarrow \mathrm{THIS}j(\langle i\rangle(\delta_{ij} \wedge p)).$ (T6)

In the same way as THIS_i which is an operator characterizing hyperspaces, we may define an operator OH (<u>O</u>nly <u>H</u>ere) pointing out single possible worlds in connected nice frames. Set

$\mathrm{OH}\varphi \equiv \varphi \wedge \bigwedge\!\!\!\bigwedge_{j\neq i}[\![i|j]\!][c(i)]\neg\varphi.$

and for connected, nice K we have $K,\mu,u \vDash \mathrm{OH}\varphi$ iff u is the only world in K with $u \vDash \varphi$, as we can prove:

$\vdash \Diamond(\mathrm{OH}\varphi \wedge \psi) \rightarrow \Box(\mathrm{OH}\varphi \rightarrow \psi)$ (T7)

$\vdash (\Diamond(\mathrm{OH}\varphi \wedge \psi) \wedge \Diamond(\mathrm{OH}\xi \wedge \langle i\rangle\mathrm{OH}\varphi)) \rightarrow \Diamond(\mathrm{OH}\xi \wedge \langle i\rangle\psi)$ (T8)

<u>Definition 3.2.</u>
A^+_α is the deduction system A_α, with one extra derivation rule. Using the notation $\vdash^+_\alpha \varphi$ for "φ is derivable in A^+_α", we give this rule by the following:

CR Let $j,i \notin \Gamma \subseteq \alpha$, φ a formula such that p does not occur in φ.
If $\vdash^+_\alpha [\Gamma][\![i,j]\!]p \rightarrow \varphi$, then $\vdash^+_\alpha \varphi$.

Notion like deduction, etc. are defined just as for A_α. When no confusion arises, we may drop the subscript α in $\vdash^+_\alpha$.
To call CR a derivation rule is slightly misleading; it is in fact a schema of derivation rules, CR_Γ, Γ a subset of α. We would like to stress the fact, however, that there are only *finitely many* rules CR_Γ in A^+_α, as α is finite.
Note that it may be necessary to apply CR more than once in a derivation: a deduction of φ could look like

$(1)\ \vdash [\Gamma_1][\![i_1,j_1]\!]p_1 \rightarrow \psi$ (...)
$(2)\ \vdash^+ \psi$ (1,CR)
$(3)\ \vdash^+ \psi \rightarrow [\Gamma_0][\![i_0,j_0]\!]p_0 \rightarrow \varphi$ (...)
$(4)\ \vdash^+ [\Gamma_0][\![i_0,j_0]\!]p_0 \rightarrow \varphi$ (2,3,MP)
$(5)\ \vdash^+ \varphi$ (4,CR)

So for φ a formula in a language L with finitely many propositional variables, it may be impossible to deduce φ using only formulas in L. In an extended version of this paper we will give an example of a derivation necessarily using CR.

From now on all notions like derivability, consistency, etc. are understood to be defined with respect to A^+_α.
For L a set of propositional formulas, the *canonical L-model* $M^c(L)$ is defined as $(W^c(L),\equiv_i,E_{ij},\mu^c)$, where $W^c(L)$ is the set of L-maximal consistent sets, $\Gamma\equiv_i\Sigma$ if $\{\varphi|\ [i]\varphi \in \Gamma\} \subseteq \Sigma$, $\Gamma \in E_{ij}$ if $\delta_{ij} \in \Gamma$ and μ^c is given by $\mu^c(p) = \{\Gamma \in W^c(L)|\ p \in \Gamma\}$. The underlying frame of the canonical L-model is called the canonical L-frame. As in the previous section, in $M^c(L)$ we have $\varphi\in\Gamma \iff \Gamma\vDash\varphi$ for L-formulas φ, and the canonical frame is nice.

Let $f: K \mapsto F^c(L')$ be a strong homomorphism, $L\subseteq L'$. Then f is called a *quasi-L-zigzagmorphism* if it satisfies the following *quasi-zigzagcondition* :

(3') for all u in K and L-formulas φ, if $\langle i\rangle\varphi \in f(u)$ then there is a v in K with T_iuv and $\varphi \in f(u)$.

We need quasi-zigzagmorphisms in our completeness proof because of the following proposition, and its corollary:

Proposition 3.3.
Let $f: K \mapsto F^c(L')$ be a quasi-L-zigzagmorphism and let μ be the valuation on K given by : $u \in \mu(p)$ iff $p \in f(u)$. Then for all φ in Φ_L:
$K,\mu,u \vDash \varphi \iff \varphi \in f(u)$.

Proof.
The proof is by formula-induction: we only give the step $\varphi\equiv \langle i\rangle\psi$:
$K,\mu,u \vDash \langle i\rangle\psi \iff$ there is a v in K with T_iuv and $K,\mu,v \vDash \varphi \iff$
there is a v in K with T_iuv and $\varphi \in f(u) \iff \langle i\rangle\varphi \in f(u)$. □

Corollary 3.4.
Let $\Sigma \in W^c(L)$, and $f: {}^\alpha m \mapsto F^c(L')$ a quasi-L-zigzagmorphism such that $\Sigma \subseteq f(u)$ for some $u \in {}^\alpha m$. Then there is a model for Σ on ${}^\alpha m$.

The reason why, at a first glance, the rule CR looks more peculiar than necessary, is that it is not stated in the way it is used:

Proposition 3.5.
If $\Delta \subseteq \Phi_L$ is consistent, $p \notin L$, and $i,j, \notin \Gamma \subseteq \alpha$, then $\Delta \cup [\Gamma][\![i,j]\!]p$ is consistent.

Proof.
Suppose otherwise, then $\vdash^+ [\Gamma][\![i,j]\!]p \rightarrow \neg\varphi$ for some $\varphi \in \Delta$. But as p does not occur in φ, this means $\vdash^+ \neg\varphi$, contradicting the consistency of Δ. □

Corollary 3.6.
If $\Delta \subseteq \Phi_L$ is consistent, $p \notin L$ and $\langle i\rangle\varphi \in \Delta$, then $\Delta \cup \{\langle i\rangle(\varphi \wedge THIS_i p)\}$ is consistent.

The proof of this corollary is given in the appendix. □

Perhaps the meaning of the rule CR is made most clear when we look at its soundness:

Theorem 3.7 (SOUNDNESS)
$\vdash^+ \varphi \quad \Rightarrow \quad F_\alpha \models \varphi$.

Proof.
We leave it to the reader to verify that the axioms of $A^+{}_\alpha$ are valid in F_α and that the set of F_α-valid formulas closed under MP and Nec. For CR, we must show that if $F_\alpha \models [\Gamma][\![i,j]\!]p \rightarrow \varphi$ then $F_\alpha \models \varphi$ (where Γ,i,j,p,φ are as in the wording of CR).
Suppose, for contraposition, that $F_\alpha \not\models \varphi$, i.e. there is a model $({}^\alpha U,\mu)$ with a world u such that ${}^\alpha U,\mu,u \models \neg\varphi$. Let μ' be the following valuation on ${}^\alpha U$:

$\mu'(q) = \mu(q)$ if $q \neq p$,
$\mu'(p) = \{v \in {}^\alpha U | \ v_i = u_i\}$.

Then clearly ${}^\alpha U,\mu',u \models [\Gamma][\![i,j]\!]p \wedge \varphi$. So $F_\alpha \not\models [\Gamma][\![i,j]\!]p \rightarrow \varphi$. □

Let L be a set of propositional formulas. An *L-matrix of size n* is a strong homomorphism $\Lambda: {}^\alpha n \mapsto F^c(L)$. The size of Λ is denoted by $|\Lambda|$. An L-matrix is called *distinguishing* if there is, for all sequences $a \in {}^\alpha n$ and $i,j < \alpha$, an L-formula $\varphi(a)$ with $OH\varphi(a) \in \Lambda(a)$: $\varphi(a)$ must be seen as the formula characterizing a or $\Lambda(a)$. Not only do distinguishing matrices give formulas characterizing points, but they can also characterize hyperplanes, lines, etc. We give two alternative characterizations of distinguishing matrices:

Proposition 3.8.

For a matrix Λ of size n, the following are equivalent:

(1) Λ is distinguishing.

(2) For all $i,j < \alpha$, $a \in {}^{\alpha}n$, there is a formula $\varphi(a,i,j)$ such that $[c(i,j)][\![i,j]\!]\varphi(a,i,j) \in \Lambda(a)$

(3) For all $i < \alpha$, $a \in {}^{\alpha}n$, there is a formula $\varphi(i,a)$ such that $THIS_i\varphi(i,a) \in \Lambda(a)$.

Proof.

The proof of this proposition is given in the appendix. □

Matrices need not be perfect; an *L-defect* of an L'-matrix Λ is a triple (a,i,φ) with $a \in {}^{\alpha}n$, $i < \alpha$ and φ an L-formula such that $\langle i\rangle\varphi \in \Lambda(a)$ while there is no b in ${}^{\alpha}n$ such that $b_j = a_j$ for $j \neq i$ and $\varphi \in \Lambda(b)$ (i.e. (a,i,φ) is a witness of the fact that Λ does not satisfy the quasi-zigzagcondition.)

The next two lemmas are the cornerstone of the completeness proof. The first lemma sais that if in a distinguishing matrix formulas in a new language are added to one MCS, this news may spread to the other ones (like an ink spot). The content of the second lemma is that if a matrix is distinghuishing, each of its defects can be repaired.

Ink Spot Lemma 3.9.

Let Λ be a distinguishing L-matrix of size n; suppose $\Lambda(a) \subseteq \Gamma \in W^c(L')$. Then there is a distinguishing L'-matrix Λ' such that $\Lambda'(a) = \Gamma$ and for all $x \in {}^{\alpha}n$: $\Lambda(x) = \Lambda'(x) \cap \Phi_L$.

Proof.

By definition of a distinguishing matrix there is, for all $x \in {}^{\alpha}n$, a formula φ_x such that $OH\varphi_x \in \Lambda(x)$; as Λ is a homomorphism this means $\Box(OH\varphi_x \rightarrow \psi) \in \Gamma$ for all $x \in {}^{\alpha}n$, $\psi \in \Lambda(x)$. Set

$\Lambda'(x) = \{\varphi \mid \Box(OH\varphi_x \rightarrow \varphi) \in \Gamma\}$.

Then $\Gamma = \Lambda'(a)$ and $\Lambda'(x)$ is consistent. $\Lambda'(x)$ is maximal by T7.

To verify that Λ' is a strong homomorphism, first suppose $\psi \in \Lambda'(x)$, $x,y \in {}^{\alpha}n$ such that they only may differ in the i-th coordinate; then

$\langle i\rangle(OH\varphi_x) \in \Lambda(y) \Rightarrow$

$\Diamond(\psi \wedge \varphi_x) \wedge \Diamond(OH\varphi_y \wedge \langle i\rangle OH\varphi_x) \in \Gamma \Rightarrow$ (by T8)

$\Diamond(OH\varphi_y \wedge \langle i\rangle\psi) \in \Gamma \Rightarrow$
$\Box(OH\varphi_y \rightarrow \langle i\rangle\psi \in \Gamma \Rightarrow \langle i\rangle\psi \in \Lambda'(y)$.
The proof for the other conditions is straightforward. □

Repair lemma 3.10.
Let $p \notin L$. Any distinguishing L-matrix with a defect has an $L\cup\{p\}$-extension Λ' of size $|\Lambda|+1$, lacking this defect.

Proof.
Without loss of generality we may assume that the defect has the form $(a,0,\varphi)$. Corollary 3.6 gives that $X = \Lambda(a) \cup \{\langle 0\rangle(\varphi \wedge THIS_0 p)\}$ is consistent, so by the Ink Spot Lemma there is an L'-matrix Ω of size n (set $n = |\Lambda|$) extending Λ, such that X is contained in $\Omega(a)$. The desired matrix Λ' will be such that for $x \in {}^{\alpha}n$, $\Lambda'(x) = \Omega(x)$. So we have to provide Λ' with L'-MCSs for those α-sequences x having one or more coordinates equal to n. We will do this step by step, in α stages.

In the first stage, we give $\Lambda'(x)$ for those x having n as their zero-th coordinate. This stage is divided into two steps:

step1:
We set out to define $\Lambda'(x)$ for those x with *only* $x_0 = n$.
As $\vdash [c(0)]THIS_0 p \leftrightarrow THIS_0 p$,
$\langle 0\rangle[c(0)]THIS_0 p$ is in $\Omega(a_0,a_1,\dots,a_{\alpha-1}) \Rightarrow$ so is $[c(0)]\langle 0\rangle THIS_0 p$.
As Ω is a homomorphism, $\langle 0\rangle THIS_0 p$ is in every $\Omega(a_0,x_1,\dots,x_{\alpha-1})$.
Set

$$\Lambda'(x) = \{\psi \mid [0](THIS_0 p \rightarrow \psi) \in \Omega(a_0,x_1,\dots,x_{\alpha-1})\},$$

then clearly $\Lambda'(x)$ is consistent. By T4 it is maximal too.

We will now show that the part of the matrix defined up till now is a strong homomorphism:

First, let $x,y \in {}^{\alpha}n+1$ (with $x_j \neq n$ and $y_j \neq n$ if $j \neq 0$) only differ in their i-th coordinate. We must show that $\Lambda'(x) \equiv_i \Lambda'(y)$.
For $i=0$, this is immediate (by the fact that Ω is a strong homomorphism and/or by definition of Λ').
If $i \neq 0$, x and y are either both old or both new points.
In the first case, there is nothing (new) to prove.
In the second case, let x' and y' be the projections of x and y in the 0-hyperplane through a, i.e. let $x' = (a_0,x_1,\dots,x_{\alpha-1})$ and $y' = (a_0,y_1,\dots,y_{\alpha-1})$. Now suppose $[i]\psi \in \Lambda'(x)$, then

$[i](\psi \wedge THIS_0 p) \in \Lambda'(x)$ (as $[i]THIS_i p \in \Lambda'(x)$)
$\Rightarrow \langle 0\rangle[i](\psi \wedge THIS_0 p) \in \Omega(x')$ (by def. of $\Lambda'(x)$)
$\Rightarrow [i]\langle 0\rangle(\psi \wedge THIS_0 p) \in \Omega(x')$ (by modal logic)
$\Rightarrow \langle 0\rangle(\psi \wedge THIS_0 p) \in \Omega(y')$ (as Ω is a homomorphism)
$\Rightarrow [0](THIS_0 p \rightarrow \psi) \in \Omega(y')$ (by (6b))
$\Rightarrow \psi \in \Lambda'(y)$ by its definition. (by def. of $\Lambda'(y)$)

Concerning the diagonal, we only check that the new Λ'-images are not on the E_{0i}-diagonal. Take $i=1$ and suppose $\delta_{01} \in \Lambda'(n,x_1,\dots,x_{\alpha-1})$, then

$\langle 0\rangle(\delta_{01} \wedge THIS_0 p) \in \Lambda'(n,x_1,\dots,x_{\alpha-1})$
$\Rightarrow [0](\delta_{01} \rightarrow THIS_0 p) \in \Lambda'(n,x_1,\dots,x_{\alpha-1})$
$\Rightarrow THIS_0 p \in \Lambda'(x_1,x_1,x_2,\dots,x_{\alpha-1})$
$\Rightarrow THIS_0 p \in \Lambda'(x_1,a_1,a_2,\dots,a_{\alpha-1})$, which is impossible by $x_1 \neq n$.

step2:
We can now treat all x's with, besides x_0, other coordinates identical to n. Consider such an x: let $\Gamma = \{i \in \alpha \mid x_i = n,\ i \neq 0\}$ and let x' be the sequence obtained by replacing all Γ-coordinates in x with 0 (then x' is such that $\Lambda'(x')$ was defined in step 1). Set

$\Lambda'(x) = \{\psi \mid \langle\Gamma\rangle(\psi \wedge \bigwedge_{i\in\Gamma} \delta_{0i}) \in \Lambda'(x')\}$.

Now clearly each of these sets is maximal and consistent, and in this step we cannot have destroyed the strong homomorphism constraint.

Note that for all x in the hyperplane with $x_0=n$, we have $THIS_0 p$ in $\Lambda'(x)$. The point is, that using this, we also have a defining formula for every hyperplane of points having their *i-th* coordinate equal to n: by T6 we may take $\langle c(i)\rangle(\delta_{0i} \wedge p)$.

We are now ready for the last $\alpha-1$ stages of the construction: Let, for $0 < i < \alpha$, X_i be the set of ${}^{\alpha}n+1$-sequences for which the i=th coordinate is the first one equalling n. In the i-th stage from now we define Λ'-images for sequences in X_i. Each stage has two steps.

For the first half of the i-th stage, consider the sequences x such that x_i is the only coordinate equal to n. Let x' be the sequence $(x_0,\dots,x_{i-1},a_i,x_{i+1},\dots,x_{\alpha-1})$. Then $x' \in {}^{\alpha}n$. Set

$\Lambda'(x) = \{\psi \mid [i](THIS_i(\langle c(i)\rangle(\delta_{0j} \wedge p)) \rightarrow \psi) \in \Omega(x')$.

Again by T4, this (consistent!) set is maximal, as $\langle i\rangle THIS_i\langle c(i)\rangle(\delta_{0j} \wedge p)$ is in $\Omega(x')$.

For the <u>second step</u>, use a similar procedure as in the last step of the first stage.

We leave it to the reader to verify that, with this definition, the new matrix is a strong homomorphism from ${}^{\alpha}n+1$ into $F^c(L\cup\{p\})$. Λ' is distinguishing by proposition 3.8, and of course Λ' does not have the defect which started this business, as it was built to have $\varphi \in \Lambda'(n,a_1,\ldots,a_{\alpha-1})$. □

<u>Theorem 3.10</u>. (COMPLETENESS)
Let Σ be a consistent set of CML_{α}-formulas. Then Σ has an α-dimensional model.

<u>Proof.</u>
Without loss of generality we may assume that Σ is maximal. For simplicity we assume that $\delta_{ij} \in \Sigma$ for all $i,j < \alpha$. Let $p_1,p_2,\ldots$ form a set of mutual distinct propositional constants not in L. Set $L_0 = L$, $L_{n+1} = L_n \cup \{p_{n+1}\}$, $L' = \bigcup_{n\in\omega} L_n$. We will construct a chain of distinguishing L_n-matrices Λ_n, $n<\omega$, such that the union Λ' of this chain is a quasi-zigzagmorphism: ${}^{\alpha}m \mapsto F^c(L')$, (where $m = \sup\{|\Lambda_n| \mid n\in\omega\}$), such that $\Sigma\subseteq\Lambda'(0,0,\ldots,0)$.
Look at the set of *possible defects* $P = {}^{\alpha}\omega \times \alpha \times \Phi_L$. Assume we have an enumeration of P, so that we may speak of the *first* defect $\min(\Lambda)$ of a (distinguishing) imperfect matrix Λ. By the repair lemma such a map Λ has an extension, a distinguishing L_{n+1}-matrix Λ', lacking the defect $\min(\Lambda)$. Now define the chain of maps as follows:

$\Lambda_0 = \{\langle(0,0,\ldots,0),\Sigma\rangle\}$

$\Lambda_{n+1} = \begin{cases} \Lambda_n & \text{if } \Lambda_n \text{ is perfect} \\ (\Lambda_n)' & \text{otherwise,} \end{cases}$

$\Lambda' = \bigcup_{n\in\omega} \Lambda_n$.

It is then straightforward to verify that Λ' has the desired properties, so by corollary 3.4 we have found an α-dimensional model for Σ. □

Note that in fact we have proved strong completeness and soundness:

$$\Sigma \models_{\alpha} \varphi \iff \Sigma \vdash^{+}_{\alpha} \varphi$$

where $\Sigma \models_{\alpha} \varphi$ means: for every α-dimensional model (U,μ) and world $u \in {}^{\alpha}U$, if $(U,\mu),u \models \sigma$ for all $\sigma \in \Sigma$, then $(U,\mu),u \models \varphi$, and $\Sigma \vdash^{+}_{\alpha} \varphi$ means: there is a finite $\Sigma_0 \subseteq \Sigma$ such that $\vdash^{+}_{\alpha} (\bigwedge\!\!\!\bigwedge \Sigma_0) \to \varphi$.

4. An axiomatization of Representable Cylindric Algebras.

In this section we give our main result, viz. a finite schema of equational axioms and rules generating all the equations valid in the class of representable cylindric algebras of dimension $\alpha \leqslant \omega$. (The case $\alpha > \omega$ is an easy generalization). For definitions concerning cylindric algebras the reader is referred to [HMT]. Some notation: $(R)CA_\alpha$ is short for (representable) cylindric algebra of dimension α, L_α is the algebraic language of CA_α's. For a class A of CA_α-type structures, let EQ_A be the set of equations valid in A.
$EQ_{RCA\alpha}$ is recursively enumerable, and some derivation systems are known. Monk proved in [M] that no finite schema of equational axioms can generate $EQ_{RCA\alpha}$ if one allows only the ordinary algebraic derivation rules; in the same article he gave a system with infinitely many axioms. In [AN2], Andréka and Németi showed a finite schema of axioms and rules generating $EQ_{RCA\alpha}$, but they need an axiom which is not in equational form. Our system has finitely many equational axioms and rules, but one of the latter (CR) is somewhat unorthodox.

As was mentioned in the introduction, in our approach CA's appear as the *modal* algebras of our logic. We will use, however, the terminology of [HMT]: we define, for a frame $K = (W, T_i, E_{ij})_{i,j<\alpha}$, its *complex algebra* as $CmK = (Pow(W), \cup, \cap, -_W, \emptyset, W, T_i^*, E_{ij})_{i,j<\alpha}$, where $T_i^*(X) = \{y \in W \mid \text{for some } x \in X, xT_iy\}$. Obviously CmK has the type of cylindric algebras. For a class A of frames, CmA denotes the class of all complex algebras of frames in A. The reader is referred to [Go2] for a general treatment of the theory of complex algebras.

The modal language CML_α and the algebraic L_α are in fact very close in talking about frames. Let $P = \{p_0, p_1, \ldots\}$ be a set of propositional variables and $X = \{x_0, x_1, \ldots\}$ a set of L_α-variables. Define a map ϱ from CML_α-formulas to L_α-terms in X:

$\varrho(p_i) = x_i$,
$\varrho(\delta_{ij}) = d_{ij}$,
$\varrho(\neg\varphi) = -\varrho(\varphi)$
$\varrho(\varphi \wedge \psi) = \varrho(\varphi) \wedge \varrho(\psi)$
$\varrho(\langle i \rangle \varphi) = c_i \varrho(\varphi)$.

It is easily seen that ϱ is a bijection; let σ be the inverse of ϱ. Now using ϱ, we give a translation ϱ' from CML_α-formulas to L_α-identities: $\varrho'(\varphi)$ is defined as $\varrho(\varphi) = 1$.

The bad news about this translation ϱ' is that it is not a bijection, as it has only images of the form $t=1$. The good news is that, given the underlying Boolean algebraic setting, every equation has an equivalent in the range of ϱ', viz. $s=t$ has $(s\wedge t) \vee (-s\wedge -t) = 1$. So in the sequel we may and will conveniently think all L_α-equations having the form $t=1$. To increase readability, we write $s \leqslant t$ for $s\wedge -t=0$.
The strong similarity between logic and algebra is expressed by the following

<u>Proposition 4.1.</u> For any α-frame K and CML_α-formula φ
$K \models \varphi \iff CmK \models \varrho'(\varphi)$.

<u>Proof.</u>
A straightforward induction. □

By definition, a CA is representable if it belongs to the variety generated by the cylindric set algebras [HMT 3.1.1], which are in our phrasing just the complex algebras of α-dimensional frames. So by Birkhoff's theorem we get

<u>Proposition 4.2.</u> $EQ_{RCA\alpha} = EQ_{CmF\alpha}$.

The above two propositions, together with the completeness result of the previous section, are enough to give a finite axiomatization of $EQ_{RCA\alpha}$ for $3 \leqslant \alpha < \omega$.

<u>Definition 4.1.</u>
Let Σ_α be the smallest set of equations containing
(C0) - (C7), the equations governing CA_α, cf. [HMT1, p 162].
(C8) $c_i(x\wedge y\wedge c_j(x\wedge -y)) \leqslant c_j(c_i x\wedge -d_{ij})$
which is closed under ordinary algebraic deduction and under the following rule:
(CR) Let $y, x_0, \ldots, x_{n-1}$ be L_α-variables such that y does not occur among the x_i. Let $i,j \in \alpha$ and $\Gamma \subseteq \alpha$ be such that $i,j \notin \Gamma$.
If $-c_\Gamma (c_j y \vee c_j(d_{ij} \wedge c_i(-d_{ij} \wedge c_j y))) \leqslant t(x_0, \ldots, x_{n-1})$ is in Σ_α then so is $t(x_0, \ldots, x_{n-1}) = 1$.

Here c_Γ denotes the generalized cylindrification operator defined in [HMT1, p 203]; it is the algebraic counterpart of the modal operator $\langle\Gamma\rangle$ used in the previous section.
Clearly Σ_α is the algebraic version of the set of $A^+{}_\alpha$-axioms, as we can easily prove the following

Proposition 3. For all CML_α-formulas φ

$$\vdash^{+}_{\alpha} \varphi \iff \varrho'(\varphi) \in \Sigma_\alpha.$$

Proof.
We leave this proof to the appendix. □

Compared to the axioms and rules recursively enumerating $EQ_{CA\alpha}$, Σ_α has one extra axiom, and the new derivation rule, CR.
We are now ready to state and prove the fundamental theorem of this paper:

Theorem 4.4. For $3 \leqslant \alpha < \omega$, $\Sigma_\alpha = EQ_{RCA\alpha}$.

Proof.
First, let $t=1 \in \Sigma_\alpha$. By the above proposition, $\vdash^{+}_{\alpha} \sigma(t)$, so by $\vdash^{+}_{\alpha}$-soundness, $F_\alpha \models \sigma(t)$. Proposition 4.1 gives $CmF_\alpha \models t=1$, so by proposition 4.2, $t=1 \in EQ_{RCA\alpha}$.
For the other direction: let $t=1$, or identically, $\varrho(\sigma(t))=1$ be in $EQ_{RCA\alpha}$. Proposition 4.2 gives $CmF_\alpha \models \varrho(\sigma(t))=1$, so proposition 4.1 yields $F_\alpha \models \sigma(t)$. Then by completeness, $\vdash^{+}_{\alpha} \sigma(t)$. Finally, by proposition 4.3, $t=1 \in \Sigma_\alpha$. □

As RCA_α is known not to be finitely axiomatizable in the ordinary sense there must be equations in $\Sigma_\alpha = EQ_{RCA\alpha}$ which need one or more applications of CR in their derivations. In an extended version of this paper we give an example of (the modal counterpart of) a $\vdash^{+}$-derivation of an equation ε in $EQ_{RCA\alpha} \setminus EQ_{CA\alpha}$.

For $\alpha = 2$, the situation is much simpler, as we can even dispose of the derivation rule CR. Let Σ_2 be the smallest set of L_2-equations containing (C0) - (C8) which is closed under ordinary algebraic deduction.

Theorem 4.4 (ii)

$$\Sigma_2 = EQ_{RCA2}$$

Proof.
By the same route as the above proof, here using the completeness theorem 2.4 for CML_2. □

It is now straightforward to turn the above results into a recursive enumeration of the equations holding in RCA_α for α an arbitrary infinite ordinal. We only treat the case $\alpha=\omega$. First, we need the corollaries 4.1.15,16 of [HMT] to obtain the following

Proposition 4.5. If $\alpha<\omega$ and ε is an equation in L_α, then
$RCA_\alpha \models \varepsilon \iff RCA_\omega \models \varepsilon$.

We define the set Σ_ω of L_ω-equations in the same way as the Σ_α's with $\alpha<\omega$, the only difference being the rule CR_ω which may only be applied for *finite* Γ. (Of course, otherwise c_Γ would not be defined.) Clearly then $\Sigma_\omega = \bigcup_{\alpha<\omega}\Sigma_\alpha$.

Theorem 4.6. $\Sigma_\omega = EQ_{RCA\omega}$.

Proof.
Let $\varepsilon \in \Sigma_\omega$. It is straightforward to show that there must be an $\alpha<\omega$ such that $\varepsilon \in \Sigma_\alpha$. By the previous theorem, $RCA_\alpha \models \varepsilon$ so by the above proposition 4.5 we have $RCA_\omega \models \varepsilon$.
For the other direction, let $RCA_\omega \models \varepsilon$ and let $\alpha<\omega$ be such that $\varepsilon \in L_\alpha$. Then $RCA_\alpha \models \varepsilon$ by proposition 4.5, so $\varepsilon \in \Sigma_\alpha$ by Theorem 4.4. Then clearly $\varepsilon \in \Sigma_\omega$. □

Note that the scheme CR_ω now consists of *infinitely* many rules CR_Γ, as the set of finite subsets of ω is (countably) infinite. On the other hand, we call Σ_ω an "almost finite" derivation system because of the following observation:
Let ε be an L_ω-equation for which we want to give a derivation in Σ_ω. As ε contains only finitely many symbols, there is an $\alpha < \omega$ with $\varepsilon \in L_\alpha$. By proposition 4.5 it then suffices to use the finite subsystem Σ_α of Σ_ω to search for a derivation of ε.

We finish this section with a short remark concerning a connection between Theorem 3.2.13 of [HMT] and our results. We would like to thank H. Andréka, I. Németi and I. Sain for bringing this theorem to our notice.
We conjecture that our result can be seen as a way to turn the mentioned (representation) theorem into a derivation system, but we feel that this matter should be investigated from a more general perspective and will not go deeper into it here.

5. Type-free valid formulas.

As CA's form an algebraic approach towards the predicate calculus, we may see whether our axiomatization result has any consequences for the latter subject. Interestingly, it turns out that we can give a proof calculus for the so-called type-free valid formulas which involves only these type-free valid formulas, thus indicating a positive solution to Problem 4.16 of [HMT]. (We would like thank H. Andréka, I. Németi and I. Sain for pointing out this corollary of Theorem 4.4.)

For a detailed treatment of the connection between the predicate calculus and CA's the reader is referred to section 4.3 of [HMT], and to [N]. Here we will only define the notions needed to state our result.

A *very ordinary language* for the predicate calculus is a pair $\Lambda = (R,\varrho)$ such that R and ϱ are functions with domain ω. Every R_i is a *relation symbol* of *rank* ϱ_i; there are supposed to be infinitely many relation symbols. *Formulas* are defined in the usual way, using variables $v_0,v_1,\ldots$ *Restricted formulas* are those in which every relational atomic subformula has the form $R_i(v_0,\ldots,v_{\varrho i-1})$. Note that every formula has a restricted equivalent. Note too that, given a language $\Lambda = (R,\varrho)$, we may write R_i in stead of $R_i(v_0,\ldots,v_{\varrho i-1})$ without loss of information.

Now let f be a permutation of ω. We denote by f^+ the permutation of first order formulas, induced by f as follows: for any φ, $f^+\varphi$ is the formula obtained from φ by replacing each atomic subformula R_i (or $R_i(v_0,\ldots,v_{\varrho i-1})$) of φ by R_{fi} (or $R_{fi}(v_0\ldots v_{\varrho fi-1})$). A restricted formula is *type-free valid* if $\models f^+\varphi$ for every permutation f of ω, where $\models$ denotes ordinary first order validity.

Andréka and Németi gave a proof theory for these formulas (cf [AGN], [AN1], [AN2]), but this calculus involves a roundabout through the ordinary proof calculus of first-order formulas.

The notion of type-free valid formulas (for short: tfvf's) arises naturally in the light of the connections between CA's and first order predicate calculus. To see this we first define a translation from L_α-terms to predicate formulas:

Let Λu be a very ordinary language. For t a term of L_ω, $\xi'(t)$ is defined as follows:

$\xi'(x_i) = R_i$

$\xi'(d_{ij}) = v_i = v_j$

$\xi'(s\wedge t) = \xi'(s) \wedge \xi'(t)$

$\xi'(s \vee t) = \xi'(s) \vee \xi'(t)$
$\xi'(-t) = \neg\xi'(t)$
$\xi'(0) = \bot$
$\xi'(1) = \top$
$\xi'(c_i t) = \exists v_i\, \xi'(t)$.

Clearly ξ' is a bijection onto the set of restricted Λu-formulas; let $\tau\mu'$ be its inverse.
A fundamental result concerning tfvf's is given by Theorem 4.3.64 of [HMT], here partly stated as

Proposition 5.1.
For every formula φ the following are equivalent:
(i) $RCA_\omega \models \tau\mu'(\varphi) = 1$
(ii) φ is type-free valid.

Now, having an axiomatization of all RCA_ω-valid equations, and the bijection ξ', we can immediately give the desired proof calculus for type-free valid formulas.
To state the special derivation rule needed here, we let, for a finite sequence $Q = \langle q_0,\dots q_n\rangle$ of finite ordinals, $\forall_Q$ denote the quantifier sequence $\forall q_0 \dots \forall q_n$.

Now let $\Lambda = (R,\varrho)$ be a very ordinary language.
$T\lambda^\Lambda$ is defined as the smallest set Φ of Λ-formulas satisfying

(I) Φ contains the $T\lambda^\Lambda$-*axioms*, these being all restricted instances of one of the following schema's:

$(T\lambda 0)$	φ, φ a propositional tautology	
$(T\lambda 1)$	$\neg\exists v_i \bot$	
$(T\lambda 2)$	$\varphi \to \exists v_i \varphi$	
$(T\lambda 3)$	$\exists v_i(\varphi \wedge \exists v_i \psi) \leftrightarrow (\exists v_i \varphi \wedge \exists v_i \psi)$	
$(T\lambda 4)$	$\exists v_i \exists v_j\, \varphi \to \exists v_j \exists v_i\, \varphi$	
$(T\lambda 5)$	$v_i = v_i$	
$(T\lambda 6)$	$v_i = v_j \to \exists v_k\, (v_i = v_k \wedge v_k = v_j)$	if $k \neq i,j$
$(T\lambda 7)$	$\exists v_i\, (v_i = v_j \wedge \varphi) \to \forall v_i\, (v_i = v_j \to \varphi)$	if $i \neq j$
$(T\lambda 8)$	$v_i = v_j \wedge \exists v_i(\varphi \wedge \psi \wedge \exists v_j(\varphi \wedge \neg\psi)) \to \exists v_j(v_i \neq v_j \wedge \exists v_i \varphi)$	if $i \neq j$

(II) Φ is closed under the following rules:
(MP) $\psi \in \Phi$ whenever $\varphi,\ \varphi \to \psi \in \Phi$.
(UG) $\forall v_i \varphi \in \Phi$ whenever $\varphi \in \Phi$
(CR) $\varphi \in \Phi$
whenever R_k is not a relational atomic subformula of φ, and

$\forall_Q (\forall v_j R_k \wedge \forall v_j(v_i = v_j \rightarrow \forall v_i(\neg v_i = v_j \rightarrow \forall v_j \neg R_k))) \rightarrow \varphi$ is in Φ for some finite sequence Q of finite ordinals in which i and j do not occur.

<u>Theorem 5.2.</u>
For Λ a very ordinary language and φ a Λ-formula, the following are equivalent:
(i) $\varphi \in T\lambda^{\Lambda}$
(ii) φ is type-free valid.

<u>Proof.</u>
By a straighforward (proof) induction one can show that $\varphi \in T\lambda^{\Lambda} \Longleftrightarrow \tau\mu'(\varphi) \in \Sigma_{\omega}$. The result is then immediate by theorem 4.6 and proposition 5.1. □

Appendix.

Here we give the (sketchy) proofs and derivations which we left out in the previous sections.

Deductions of the theorems in section 3.

In the derivations we do not give every step; e.g. we immediately conclude $\vdash \psi \rightarrow \bot$ from $\vdash (\delta_{ij} \wedge \langle i \rangle \psi) \rightarrow \bot$. In this and similar cases we use the following abbreviations which give an indication of the type of step left out:

- PL Propositional logic, e.g. $\vdash p \rightarrow (q \rightarrow r) \Rightarrow \vdash p \rightarrow (\neg q \rightarrow \neg r)$
- ML Modal. Logic, e.g. $\vdash \varphi \rightarrow \psi \Rightarrow \vdash [c(i)]\varphi \rightarrow [c(i)]\psi$.
- CML Cylindric Modal Logic, e.g. the above example, or the fact that every $[i]$ is an S5 modality.
- df definition, e.g. $\vdash [\![i,j]\!]\varphi \rightarrow \varphi$ by CML and definition of $[\![i,j]\!]$.

(a) T1 ≡ 7.

0 $\vdash (\delta_{10} \wedge \langle 1 \rangle ([\![0,1]\!]\varphi \wedge \psi \wedge \langle 0 \rangle ([\![0,1]\!]\varphi \wedge \neg\psi))) \rightarrow$
$\langle 0 \rangle (\neg\delta_{10} \wedge \langle 1 \rangle [\![0,1]\!]\varphi)$ (C8)

1 $\vdash (\delta_{10} \wedge \langle 1 \rangle ([\![0,1]\!]\varphi \wedge \psi \wedge \langle 0 \rangle ([\![0,1]\!]\varphi \wedge \neg\psi))) \rightarrow$ (0,ML)
$\langle 0 \rangle (\neg\delta_{10} \wedge \langle 1 \rangle \varphi)$

2 $\vdash (\delta_{10} \wedge \langle 1 \rangle ([\![0,1]\!]\varphi \wedge \psi \wedge \langle 0 \rangle ([\![0,1]\!]\varphi \wedge \neg\psi))) \rightarrow [\![0,1]\!]\varphi$ (CML)

3 $\vdash (\delta_{10} \wedge \langle 1 \rangle ([\![0,1]\!]\varphi \wedge \psi \wedge \langle 0 \rangle ([\![0,1]\!]\varphi \wedge \neg\psi))) \rightarrow$
$[0](\neg\delta_{10} \rightarrow [1]\neg\varphi)$ (2,def.$[\![0,1]\!]$

4 $\vdash (\delta_{10} \wedge \langle 1 \rangle ([\![0,1]\!]\varphi \wedge \psi \wedge \langle 0 \rangle ([\![0,1]\!]\varphi \wedge \neg\psi))) \rightarrow \bot$ (1,3,ML)

5 $\vdash ([\![0,1]\!]\varphi \wedge \psi \wedge \langle 0 \rangle ([\![0,1]\!]\varphi \wedge \neg\psi)) \rightarrow \bot$ (4,CML)

6 $\vdash ([\![0,1]\!]\varphi \wedge \psi) \rightarrow [0]([\![0,1]\!]\varphi \rightarrow \psi)$ (5,ML)

7 $\vdash \langle 0 \rangle [\![0,1]\!]\varphi \wedge \psi) \rightarrow [0]([\![0,1]\!]\varphi \rightarrow \psi)$ (6,CML)

(b) T2 ≡ 17, T3 ≡ 29, T3 ≡ 33:

8 $\vdash \delta_{ij} \leftrightarrow \langle k \rangle \delta_{ij}$ (CML)

9 $\vdash \neg\delta_{ij} \leftrightarrow \langle k \rangle \neg\delta_{ij}$ (CML)

10 $\vdash [k](\neg\delta_{ij} \rightarrow \varphi) \rightarrow (\neg\delta_{ij} \rightarrow [k]\varphi)$ (ML,8)

11 $\vdash \langle k \rangle (\neg\delta_{ij} \wedge \neg\varphi) \rightarrow (\neg\delta_{ij} \wedge \langle k \rangle \neg\varphi)$ (ML,9)

12 $\vdash (\neg\delta_{ij} \rightarrow [k]\varphi) \rightarrow [k](\neg\delta_{ij} \rightarrow \varphi)$ (11,PL)

13 $\vdash (\neg\delta_{ij} \rightarrow [k]\varphi) \leftrightarrow [k](\neg\delta_{ij} \rightarrow \varphi)$ (10,12,PL)

14 $\vdash [\![i|j]\!][k]\varphi \leftrightarrow [k][\![i|j]\!]\varphi$ (13,CML)

15 $\vdash [\![i|j]\!]\varphi \leftrightarrow [j][\![i|j]\!]\varphi \leftrightarrow [\![i|j]\!][j]\varphi$ (CML)

16 $\vdash [\![i|j]\!][c(i)]\varphi \leftrightarrow [c(i)][\![i|j]\!]\varphi$ (14,15)

17 $\vdash \mathrm{THIS}_i\varphi \rightarrow [c(i)]\mathrm{THIS}_i\varphi$ (16,ML)

18 $\vdash (\delta_{ij} \wedge \langle j\rangle(\varphi \wedge \psi) \wedge \langle i\rangle(\varphi \wedge \neg\psi)) \to \langle i\rangle(\neg\delta_{ij} \wedge \langle j\rangle\varphi)$ (C8)

19 $\vdash (\varphi \wedge \psi) \to (\langle i\rangle(\varphi \wedge \neg\psi) \to \langle j\rangle(\delta_{ij} \wedge \langle i\rangle(\neg\delta_{ij} \wedge \langle j\rangle\varphi)))$ (18,CML)

20 $\vdash (\varphi \wedge \psi) \to ([j](\delta_{ij} \to [i](\neg\delta_{ij} \to [j]\neg\varphi)) \to [i](\varphi \to \psi))$ (19,PL)

21 $\vdash (\psi \wedge \llbracket i,j \rrbracket\varphi) \to [i](\varphi \to \psi)$ (20,PL)

22 $\vdash (\llbracket i,j \rrbracket\varphi \wedge \llbracket i,j \rrbracket\varphi) \to [i](\varphi \to \llbracket i,j \rrbracket\varphi)$ (21)

23 $\vdash [j]\llbracket i,j \rrbracket\varphi \to [j][i](\varphi \to \llbracket i,j \rrbracket\varphi)$ (22,ML)

24 $\vdash \llbracket i,j \rrbracket\varphi \to [j][i](\varphi \to \llbracket i,j \rrbracket\varphi)$ (23,15)

25 $\vdash [c(i,j)]\llbracket i,j \rrbracket\varphi \to \Box(\varphi \to \llbracket i,j \rrbracket\varphi)$ (24,ML)

26 $\vdash (\varphi \wedge THIS_i\varphi) \to (\llbracket i|j \rrbracket\neg\varphi \to [i](\varphi \to THIS_i\varphi))$ (20)

27 $\vdash [c(i)]THIS_i\varphi \to ([c(i)]\llbracket i|j \rrbracket\neg\varphi \to \Box(\varphi \to THIS_i\varphi))$ (26,ML)

28 $\vdash THIS_i\varphi \to ([c(i)]\llbracket i|j \rrbracket\neg\varphi \to \Box(\varphi \to THIS_i\varphi))$ (17,27,MP)

29 $\vdash THIS_i\varphi \to \Box(\varphi \to THIS_i\varphi)$ (25,def. THIS)

30 $\vdash (\delta_{ij} \wedge THIS_i\varphi \wedge \langle i\rangle(\neg\delta_{ij} \wedge THIS_i\varphi)) \to \bot$ (def $THIS_i$)

31 $\vdash (\delta_{ij} \wedge \langle j\rangle(THIS_i\varphi \wedge \psi \wedge \langle i\rangle(THIS_i\varphi \wedge \neg\psi))) \to \bot$ (30,C8)

32 $\vdash (THIS_i\varphi \wedge \psi \wedge \langle i\rangle(THIS_i\varphi \wedge \neg\psi)) \to \bot$ (31,CML)

33 $\vdash \langle i\rangle(THIS_i\varphi \wedge \psi) \to [i](THIS_i\varphi \to \psi)$ (32,ML)

(c) T5 ≡ 42, T6 ≡ 59:

34 $\vdash \delta_{ij} \to (\langle j\rangle(\neg\delta_{ij} \wedge \langle i\rangle(\delta_{ij} \wedge \varphi)) \to \langle i\rangle(\neg\delta_{ij} \wedge \langle j\rangle\varphi))$ (C8)

35 $\vdash \delta_{ij} \to ([i](\neg\delta_{ij} \to [j]\neg\varphi) \to [j](\neg\delta_{ij} \to \neg\langle i\rangle(\delta_{ij} \wedge \varphi)))$ (34,PL)

36 $\vdash (\delta_{ij} \wedge [j]\varphi \wedge [i](\neg\delta_{ij} \to [j]\neg\varphi)) \to [j](\neg\delta_{ij} \to \neg\langle i\rangle(\delta_{ij} \wedge \varphi))$(35,PL)

37 $\vdash (\delta_{ij} \wedge \llbracket i,j \rrbracket\varphi) \to [j](\neg\delta_{ij} \to \neg\langle i\rangle(\delta_{ij} \wedge \varphi))$ (36,df.$\llbracket i,j \rrbracket$)

38 $\vdash (\delta_{ij} \wedge \llbracket i,j \rrbracket\varphi) \to \langle i\rangle(\delta_{ij} \wedge [j](\neg\delta_{ij} \to \neg\langle i\rangle(\delta_{ij} \wedge \varphi))$ (37,CML)

39 $\vdash (\delta_{ij} \wedge \llbracket i,j \rrbracket\varphi) \to \langle i\rangle(\delta_{ij} \wedge [j](\neg\delta_{ij} \to [i]\neg\langle i\rangle(\delta_{ij} \wedge \varphi))$ (38,CML)

40 $\vdash (\delta_{ij} \wedge \llbracket i,j \rrbracket\varphi) \to (\delta_{ij} \wedge \varphi)$ (df $\llbracket i,j \rrbracket$)

41 $\vdash (\delta_{ij} \wedge \llbracket i,j \rrbracket\varphi) \to [i]\langle i\rangle(\delta_{ij} \wedge \varphi)$ (40,CML)

42 $\vdash (\delta_{ij} \wedge \llbracket i,j \rrbracket\varphi) \to \llbracket j,i \rrbracket\langle i\rangle(\delta_{ij} \wedge \varphi)$ (39,41,PL)

We now give the derivation of T6 with i=0, j=0; in 43-59, $k \notin \{0,1\}$. We abbreviate $\Delta \equiv \delta_{01} \wedge THIS_0\varphi$, $\psi \equiv \langle 0\rangle(\delta_{01} \wedge \varphi)$, i.e. we want to prove $\Delta \vdash THIS_1\psi$.

43 $\Delta \vdash [0](\neg\delta_{01} \to [1][c(0,1)]\neg\varphi)$ (df THIS)

44 $\Delta \vdash [0](\neg\delta_{10} \to [c(0,1)][1](\delta_{10} \to \neg\varphi))$ (43,CML)

45 $\Delta \vdash [0](\neg\delta_{10} \to [c(0,1)][j][1](\delta_{10} \to \neg\varphi)$ (44,CML)

46 $\Delta \vdash [c(1)](\neg\delta_{10} \to [1][j](\delta_{10} \to \neg\varphi)$ (45,13)

47 $\vdash \xi \to (\delta_{1j} \to \xi)$ (PL)

48 $\Delta \vdash [c(1)]((\neg\delta_{10} \wedge \delta_{1j}) \to [1][j](\delta_{10} \to \neg\varphi))$ (46,47,ML)

49 $\Delta \vdash [c(1)]((\delta_{1j} \wedge \neg\delta_{0j}) \to [1][j](\delta_{10} \to \neg\varphi))$ (48,CML)

50 $\Delta \vdash [c(1)](\delta_{1j} \to [1](\neg\delta_{0j} \to (\delta_{10} \to [j]\neg\varphi)))$ (49,13)

51 $\Delta \vdash [c(1)](\delta_{1j} \to [1](\neg\delta_{1j} \to (\delta_{10} \to [j]\neg\varphi)))$ (50,CML)

52 $\Delta \vdash [j](\delta_{1j} \rightarrow [1](\neg\delta_{1j} \rightarrow [c(1)](\delta_{10} \rightarrow \neg\varphi)))$ (51,13)
53 $\Delta \vdash [j](\delta_{1j} \rightarrow [1](\neg\delta_{1j} \rightarrow [j][c(1)][0](\delta_{01} \rightarrow \neg\varphi)))$ (52,CML)
54 $\Delta \vdash [j](\delta_{1j} \rightarrow [1](\neg\delta_{1j} \rightarrow [j][c(1)]\neg\psi))$ (df ψ)
55 $\Delta \vdash \llbracket 1|j \rrbracket [c(1)]\neg\psi$ (df $\llbracket 1|j \rrbracket$

56 $\Delta \vdash [c(0,1)](\delta_{01} \wedge \varphi)$ (df THIS)
57 $\Delta \vdash [c(0,1)][0]\langle 0\rangle(\delta_{01} \wedge \varphi)$ (56,CML)
58 $\Delta \vdash [c(1)]\psi$ (57,dfψ)

59 $\Delta \vdash \mathrm{THIS}_1\psi$ (55,58)

(d) T7 ≡ 72, T8 ≡ 74.
60 $\vdash OH\varphi \rightarrow (\langle c(i)\rangle\varphi \wedge \bigwedge_{j\neq i}\llbracket i|j \rrbracket[c(i)]\neg\langle c(i)\rangle\varphi$ (def.OH)
61 $\vdash OH\varphi \rightarrow \mathrm{THIS}_i\langle c(i)\rangle\varphi$ (60,CML))
62 $\vdash OH\varphi \rightarrow \Box(\langle c(i)\rangle\varphi \rightarrow \mathrm{THIS}_i\langle c(i)\rangle\varphi)$ (29,61)
63 $\vdash OH\varphi \rightarrow \Box(\varphi \rightarrow \bigwedge_i \mathrm{THIS}_i\langle c(i)\rangle\varphi)$ (62,ML)
64 $\vdash OH\varphi \rightarrow \Box(\varphi \rightarrow \bigwedge_i \bigwedge_{j\neq i}\llbracket i|j \rrbracket[c(i)]\neg\langle c(i)\rangle\varphi)$ (df THIS_i)
65 $\vdash OH\varphi \rightarrow \Box(\varphi \rightarrow OH\varphi)$ (64,CML,df OH)

Now let $\Gamma \subseteq \alpha$ be such that $i \notin \Gamma$. Then
66 $\vdash OH\varphi \rightarrow \llbracket i|j \rrbracket[c(i)]\neg\varphi$ (62,df THIS)
67 $\vdash OH\varphi \rightarrow [\Gamma]\llbracket i|j \rrbracket\neg\varphi$ (66,16)
68 $\vdash (\, OH\varphi \wedge \langle\Gamma \cup \{i\}\rangle(\neg\psi \wedge OH\varphi)\,) \rightarrow$
$\langle\Gamma\rangle(\, \langle i\rangle(\neg\psi \wedge OH\varphi) \wedge \llbracket i|j \rrbracket\neg\varphi\,)$ (67,ML)
69 $\vdash (\, OH\varphi \wedge \langle\Gamma \cup \{i\}\rangle(\neg\psi \wedge OH\varphi)\,) \rightarrow$
$\langle\Gamma\rangle(\, \langle i\rangle(\neg\psi \wedge OH\varphi) \wedge \llbracket i|j \rrbracket\neg OH\varphi\,)$ (68,65)
70 $\vdash (\, OH\varphi \wedge \langle\Gamma \cup \{i\}\rangle(\neg\psi \wedge OH\varphi)\,) \rightarrow \langle\Gamma\rangle(\neg\psi \wedge OH\varphi)$ (69,CML)
71 $\vdash (\, OH\varphi \wedge \Diamond(\neg\psi \wedge OH\varphi)\,) \rightarrow (\neg\psi \wedge OH\varphi)$ (71,repeatedly)
72 $\vdash \Diamond(\psi \wedge OH\varphi) \rightarrow \Box(OH\varphi \rightarrow \psi)$ (71,CML)

73 $\vdash (\Diamond(OH\varphi \wedge \psi) \wedge \Diamond(OH\xi \wedge \langle i\rangle OH\varphi)) \rightarrow$
$\Diamond(OH\xi \wedge \langle i\rangle OH\varphi \wedge \Box(OH\varphi \rightarrow \psi)\,)$ (72,CML)
74 $\vdash (\Diamond(OH\varphi \wedge \psi) \wedge \Diamond(OH\xi \wedge \langle i\rangle OH\varphi)) \rightarrow \Diamond(OH\xi \wedge \langle i\rangle\psi)$ (73,ML)

<u>Proof of proposition 3.8.</u>
(1)⇒(3):
Take $\varphi(i,a) \equiv \langle c(i)\rangle\varphi_a$, and use thesis 61.
(3)⇒(2):
Take $\varphi(a,i,j) \equiv \varphi(i,a)$, and use thesis 16.
(2)⇒(1):
Take $\varphi(a) \equiv \bigwedge_{j\neq i}\varphi(a,i,j)$, and use thesis 16.

Proof of corollary 3.6.

Without loss of generality we may assume that Δ is maximal and $i=0$. It is easy to verify that there is an L-MCS H with $\Delta \equiv_0 H$ and $\varphi \in H$. Let $q_1,\ldots,q_{\alpha-1}$ be distinct propositional formulas not in $L\cup\{p\}$; define $L' = L \cup \{p\} \cup \{q_i | 0 < i < \alpha\}$.

By proposition 3.5, $H \cup \{ \bigwedge\!\!\!\bigwedge_{j\neq 0}[c(0,i)][\![0,i]\!]q_i\}$ is consistent, so it has a maximal consistent extension H'. As the set $\Delta \cup \{\varphi | [0]\varphi \in H'\}$ is consistent, Δ has a L'-maximal consistent extension $\Delta' \equiv_0 H'$.

Let ψ be the formula $\langle c(0)\rangle \bigwedge\!\!\!\bigwedge_{j\neq 0} q_i$. As $\bigwedge\!\!\!\bigwedge_{j\neq 0}[c(0,i)][\![0,i]\!]q_i$ is provably equivalent to $THIS_0\psi$, we have:

$\varphi \wedge THIS_i\psi \in H' \Rightarrow$

$\langle i\rangle(\varphi \wedge THIS_i\psi) \in \Delta' \Rightarrow$

$\Delta \nvdash^+ [i](THIS_i\psi \rightarrow \neg\varphi) \Rightarrow$

$\Delta \nvdash^+ [i](THIS_i p \rightarrow \neg\varphi) \Rightarrow$

$\Delta \cup \{\langle i\rangle(\varphi \wedge THIS_i p)\}$ is consistent. □

Proof of proposition 4.3.

The proof of the left inclusion is by induction on the length of the proof for φ.

First, let φ be an axiom of A'_α. If φ is an instance of MC0, MC1, or one of MC4,...,MC8, it is immediately clear that $\varrho'(\varphi) \in \Sigma_\alpha$.

If φ is an instance of MC2, i.e. $\varphi \equiv \psi \rightarrow [i]\langle i\rangle\psi$ then

$\varrho'(\varphi) \equiv -\varrho(\psi) \vee -c_i{-}c_i\varrho(\psi) = 1$,

so $\varrho'(\varphi)$ is equivalent to $\varrho(\psi) \leqslant -c_i{-}c_i\varrho(\psi)$.

Then $\varrho'(\varphi) \in \Sigma_\alpha$, as $\varrho(\psi) \leqslant c_i\varrho(\psi) \in \Sigma_\alpha$ by C2, and

$c_i\varrho(\psi) = -c_i{-}c_i\varrho(\psi)$

by Theorem 1.2.11 of [HMT].

For φ an instance of CM3, the proof goes likewise, using Theorem 1.2.3 of [HMT]..

Now suppose we obtained φ as a $\vdash^+_\alpha$-theorem after using a consistency rule. We only treat the following case:

$\vdash^+_\alpha \varphi$ because of $\vdash^+_\alpha [\Gamma][\![i,j]\!]p_i \rightarrow \varphi$, where the usual restrictions hold. By the induction hypothesis,

$c_\Gamma\, (c_j x_i \vee c_j(d_{ij} \wedge c_i(-d_{ij} \wedge c_j x_i))) \vee \varrho(\varphi) = 1$

is in Σ_α, and as x_i can not appear among the variables of $\varrho(\varphi)$, this immediately gives $\varrho'(\varphi) \in \Sigma_\alpha$.

For the other direction, we only treat the case in which $\varphi'(\varphi)$ is an instance of C1 or C3.

C1: By Necessitation we have $\vdash^+_\alpha [i]T$, so $\varrho'([i]T) \equiv -c_i 0 = 1 \in \Sigma_\alpha$.

C3: For any modal S5-operator $\Box$, $\Diamond(\varphi \wedge \Diamond\psi) \leftrightarrow \Diamond\varphi \wedge \Diamond\psi$ is a theorem, so $\vdash^{+}_{\alpha}\langle i\rangle(\varphi \wedge \langle i\rangle\psi) \leftrightarrow \langle i\rangle\varphi \wedge \langle i\rangle\psi$. After some Boolean manipulations, this gives $c_i(\varrho(\varphi) \wedge c_i\varrho(\psi)) = c_i\varrho(\varphi) \wedge c_i\varrho(\psi) \in \Sigma_\alpha$.

$\Box$

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