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ON THE METAMATHEMATICS OF WEAK THEORIES

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On the metamathematics of weak theories

by

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From the work of Paris-Wilkie [WP87] the theory $I\Delta_0+\Omega_1$ has emerged as one of the weakest theories which is adequate to arithmetize syntax and for which one can prove the "derivability conditions" needed to prove Gödel's incompleteness theorems (provided one uses efficient coding techniques and employs binary numerals). However many questions on the metamathematics of $I\Delta_0+\Omega_1$ remain open, and in particular it is not known whether Solovay's completeness theorem [So76] holds for $I\Delta_0+\Omega_1$. In other words, if we denote by L and PLΩ the provability logics of PA and $I\Delta_0+\Omega_1$ respectively, we have that L is contained in PLΩ while the opposite inclusion is an open problem. Let C be any class of Kripke models such that for all $K \in C$, K does not contain five nodes $a_1, \dots, a_5$ with $a_1 < a_2 < a_3 < a_4$, $a_2 < a_5$ and a_5 incomparable with a_3. We give an upper bound on PLΩ by showing that $L \subseteq PL\Omega \subset Th(C)$ (where the last inclusion is strict). Similar upper bounds, but weaker, were obtained by the second author in her Master's thesis. In subsequent work she showed that PLΩ is contained in the theory of all models of height ≤ 3. The improvement of this paper uses

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analogous techniques but is based on a different definition of the "Solovay constants" due to the first author.

We denote by $\Box$ both the arithmetization of the provability predicate of $I\Delta_0+\Omega_1$ and the corresponding modal operator. $\Diamond A$ is defined as $\neg\Box\neg A$ and $\boxdot A$ is $\Box A \wedge A$. We recall that $PL\Omega$, the provability logic of $I\Delta_0+\Omega_1$, is defined as follows: $PL\Omega \vdash A$ iff for all arithmetical interpretations $*$, $I\Delta_0+\Omega_1 \vdash A^*$.

1. Visser's and Švejdar's principles.

In [Ve89] Verbrugge proved the following result.

1.1. (Švejdar's principle). We have: $I\Delta_0+\Omega_1 \vdash \forall k\, \Box(\Box_k A \to A)$ where $\Box_k A$ is a formalization of the fact that A has a $I\Delta_0+\Omega_1$-proof of Gödel number $\leq k$.

We also need the following theorem of A. Visser [Vi89].

1.2. (Visser's principle). If S and S_i $(i = 1,\ldots,k)$ are Σ_1 sentences, then we have $I\Delta_0+\Omega_1 \vdash \Box[\bigwedge_i(S_i \to \Box S_i) \to S] \to \Box S$.

An easy corollary (see [Vi89]) is:

1.3. If the S_i's are Σ_1-sentences, then $I\Delta_0+\Omega_1 \vdash \Box\bigvee_i S_i \to \Box\bigvee_i \boxdot S_i$.

2. A modified Solovay construction.

We assume knowledge of [So76]. Let (K, $\Vdash$) be a Kripke model based on the frame K (so K is a finite tree). Without loss of generality $K = \{1, \ldots, n\}$ and 1 is the least element (the root) of K. We extend the order of K to the set $K \cup \{0\}$ by putting the new node 0 below every node in K. We will associate to each node $i \in K \cup \{0\}$ an arithmetical sentence L_i. Assuming this has already been done we define the weight w(i) of a node $i \in K$ ($i \neq 0$) as the least p such that $\forall j \leq_K i$ ($j \neq 0$) there is a $I\Delta_0+\Omega_1$-proof of $\neg L_j$ of Gödel number $\leq$ p. If p does not exist $w(i) = \infty$. Note that "$w(i) < \infty$" is an NP-predicate since it is equivalent to the conjunction of $\Box \neg L_c$ over all $c \leq_K i$. We recall that $I\Delta_0+\Omega_1$ is provably complete with respect to NP-formulas, i.e. if A is a suitable arithmetization of an NP-predicate, then $I\Delta_0+\Omega_1 \vdash A \rightarrow \Box A$. The predicate $w(i) < w(j)$ is not guaranteed to be in NP but it is certainly Σ_1. The main difficulty in adapting Solovay's theorem to $I\Delta_0+\Omega_1$ is that it is not known whether $I\Delta_0+\Omega_1$ is provably complete with respect to Σ_1-formulas (this is discussed in detail in [Ve89]). An important (though obvious) property of our weights is that they respect the order of the Kripke frame, that is: $i \leq_K j \rightarrow w(i) \leq w(j)$. Let $i \perp j$ mean that i and j are incomparable in the order $\leq_K$.

2.1. Definition. Given $i \perp j$ let i^j be the least element $\leq_K$ i and incomparable with j, and j^i be the least element $\leq_K$ j and incomparable with i. If at least one of $w(i^j)$ and $w(j^i)$ is $< \infty$, then we say that P(i, j) holds iff either $w(i^j) < w(j^i)$ or the weights of i^j and j^i

are equal but i^j has smaller index than j^i (in a fixed enumeration of K). If P(i, j) holds we say that i has priority over j.

Note that if $a \leq i$, $b \leq j$ and $a \perp b$, then $P(a, b) \leftrightarrow P(i, j)$. By the diagonal lemma we can assume that the formulas L_i are chosen so that they satisfy the following system of "fixed-point equations":

2.2. Definition. If $i \neq 0$ define L_i by: $I\Delta_0+\Omega_1 \vdash L_i \leftrightarrow [\ w(i) < \infty \wedge \forall j >_K i\ w(j) = \infty \ \wedge\ \forall\ j \perp i\ P(i, j)]$. Define $L_0 \leftrightarrow \neg L_1 \wedge \ldots \wedge \neg L_n$.

The above definition can be described informally as follows: consider all the weights associated to the nodes of the tree K. If they are all ∞ then L_0 holds (and all the other L_i's fail). Otherwise look at the set M of the maximal nodes (in the tree order) among the nodes with weigth $< \infty$. The elements of M are totally ordered by the partial order P(x, y). Let j be the least element of M with respect to P(x, y). Then L_j holds (and the other L_i's fail).

3. The conditions

First some terminology: we will often write $<$ and $\leq$ without subscripts to refer to the order relation on $K\cup\{0\}$. We often write $\vdash \varphi$ to mean $I\Delta_0+\Omega_1 \vdash \varphi$. We next state some properties of the constants L_i (i ranges over $K\cup\{0\}$). We omit the proofs of the easy lemmas.

L1. $I\Delta_0+\Omega_1 \vdash \exists!\, i\in K\cup\{0\}\ L_i$.

L2. $i < j \Rightarrow \ \vdash L_i \rightarrow \Diamond L_j$.

Proof. The assertion to be proved follows from the particular case when j is an immediate successor of i (since if a consistent theory proves the consistency of another theory, then the latter is consistent). So without loss of generality we assume that j is an immediate successor of i. If $i = 0$, then $j = 1$ and the thesis follows from the definition of L_0 plus the observation that $\vdash \Box \neg L_1 \rightarrow w(1) < \infty$. If $i \neq 0$ we have $\vdash w(i) < \infty \wedge \Box \neg L_j \rightarrow w(j) < \infty$. On the other hand, by definition of L_i, we have $\vdash L_i \rightarrow w(j) = \infty \wedge w(i) < \infty$. The thesis follows. QED.

L3. $j \geq i \Rightarrow \ \vdash L_j \rightarrow \Box \neg L_i$ unless $i = j = 0$.

Proof. If $j > i$ we have the chain of implications $L_j \rightarrow \Box \neg L_j \rightarrow \Box\Box \neg L_j \rightarrow \Box \neg L_i$, where the last implication follows from (L2). If $i = j > 0$, the thesis follows immediately from the definition of L_i. QED.

L3.1. $\mathbb{N} \models L_0$.

Proof. An application of L3 with $i = j \neq 0$, gives $\mathbb{N} \not\models L_i$. Since one of the L_i's must hold, the only possibility left is $\mathbb{N} \models L_0$. QED.

L4. $\vdash L_1 \rightarrow \Box \bigvee_{i>1} L_i$.

Proof. By the previous lemmas, $\vdash L_1 \rightarrow \Box \neg L_0 \wedge \Box \neg L_1$. Since one of the L_i's must hold $\vdash L_1 \rightarrow \Box \bigvee_{i>1} L_i$. QED.

L5. Suppose that $a \perp b$ and that $\forall c\ (c < a \to c < b)$. Then $\vdash L_b \to \Box\neg L_a$.

Proof. Our assumption implies that $\vdash P(a, b) \to w(a) < w(b)$, hence $\vdash P(a, b) \to \Box_{w(b)}\neg L_a$. So by Švejdar's principle $\vdash w(b) < \infty \to \Box(P(a, b) \to \neg L_a)$. But $\vdash L_a \to P(a, b)$. So $\vdash w(b) < \infty \to \Box\neg L_a$. Now the thesis follows by observing that $\vdash L_b \to w(b) < \infty$. QED.

As an immediate corollary we have:

L5.1. Suppose that $a \perp b$ and that $\forall c\ (c < a \to c < b)$. Then $\vdash L_b \to w(a) < \infty$. Moreover, since $w(a) < \infty$ is a NP-predicate, $\vdash L_b \to \Box(w(a) < \infty)$.

L6. If a is an immediate successor of 1, then $\vdash L_1 \to \Box\Box\neg L_a$.

Proof. By L4 $\vdash L_1 \to \Box \bigvee_{i>1} L_i$. Every $i > 1$ is either $\geq a$ or incomparable with a. In either case $\vdash L_i \to \Box\neg L_a$ by L3 and L5 respectively. So $\vdash \Box(\bigvee_{i>1} L_i \to \Box\neg L_a)$ and the thesis follows. QED.

L6.1. If a is an immediate successor of 1, then $\vdash L_1 \to \Box(w(a) < \infty)$.

L7. Let $a \perp b$. Then $\vdash \Box(w(a^b) < \infty) \to \Box(L_a \to \Box\neg L_b)$. Moreover, since $\vdash w(a^b) \leq w(a)$, we have, a fortiori, $\vdash \Box(w(a) < \infty) \to \Box(L_a \to \Box\neg L_b)$.

Proof. First note that $\vdash w(a^b) < \infty \to P(a, b) \vee P(b, a)$. So $\vdash \Box(w(a^b) < \infty) \to \Box(P(a, b) \vee P(b, a))$. Hence by Visser's principle (1.3) $\vdash \Box(w(a^b) < \infty) \to \Box(\boxdot P(a, b) \vee \boxdot P(b, a))$. We need to prove $\vdash \Box(w(a^b) < \infty) \to \Box(L_a \to \Box\neg L_b)$. So work inside $\vdash$ and assume $\Box(w(a^b) < \infty)$. Then we have seen that $\Box(\boxdot P(a, b) \vee \boxdot P(b, a))$ must hold and we must prove $\Box(L_a \to \Box\neg L_b)$. So move inside $\Box$ and assume L_a. In the presence of $\boxdot P(a, b) \vee \boxdot P(b, a)$ we have that L_a implies $\boxdot P(a, b)$, which in turn implies $\Box\neg L_b$. QED.

L7.1. Let $a \perp b$. Then $\vdash \Box(w(a^b) < \infty) \to \Box(L_a \to w(b) < \infty)$.

Proof. $w(b) < \infty$ is equivalent to the conjunction of $\Box\neg L_c$ over all $c \leq b$. Any such c is either incomparable with a or $\leq a$. If $c \perp a$, then $a^b = a^c$ and, by L7, $\vdash \Box(w(a^b) < \infty) \to \Box(L_a \to \Box\neg L_c)$. If $c \leq a$, then $\vdash L_a \to \Box\neg L_c$, so again $\vdash \Box(w(a^b) < \infty) \to \Box(L_a \to \Box\neg L_c)$. QED.

L7.2. Let $a \perp b$. Then $\vdash \Box(w(a^b) < \infty) \to \Box(L_a \to \boxdot(w(b) < \infty))$.

Proof. By L7.1 and the fact that $w(b) < \infty$ is an NP-predicate. QED.

3.2. Definition. Given $a \in K$ with $a \neq 1$, let $a\# \leq a$ be the least node of K such that $a^b \leq a\#$ for all $b \perp a$. In other words $a\#$ is the least node with $a\# \leq a$ such that there are no bifurcation nodes in the half-open interval [a#, a). In particular a# = a iff the father of a is a bifurcation node.

L8. $\vdash \Box(w(a\#) < \infty) \rightarrow \Box(L_a \rightarrow \Box\bigvee_{i>a} L_i)$.

Proof. Immediate from L7 and L3 and the fact that for all $b \perp a$, $a^b \leq a\#$ (so $w(a^b) \leq w(a\#)$). QED.

L9. Let b be an immediate successor of a. Then $\vdash \Box(w(a\#) < \infty) \rightarrow \Box(L_a \rightarrow \Box\Box\neg L_b)$. A fortiori, $\vdash \Box(w(a) < \infty) \rightarrow \Box(L_a \rightarrow \Box\Box\neg L_b)$.

Proof. Every $i \geq a$ is either incomparable with b, in which case by L5 $\vdash L_i \rightarrow \Box\neg L_b$, or it is $\geq b$, in which case again $\vdash L_i \rightarrow \Box\neg L_b$ by L3. So $\vdash \bigvee_{i>a} L_i \rightarrow \Box\neg L_b$. Now the thesis follows from L8: $\vdash \Box(w(a\#) < \infty) \rightarrow \Box(L_a \rightarrow \Box\bigvee_{i>a} L_i)$. Note that we used both Švejdar's (L5) and Visser's principle (L8). QED.

L9.1. Let b be an immediate successor of a. Then $\vdash \Box(w(a\#) < \infty) \rightarrow \Box(L_a \rightarrow \Box(w(b) < \infty))$.

4. Upper and lower bounds on PLΩ.

A lower bound on PLΩ is given by the containment $L \subseteq PL\Omega$. In theorem 4.4 we prove a non-trivial opposite inclusion. Without loss of continuity the reader can skip the next paragraphs and go to definition 4.3.

The proof of theorem 4.4. suggests the introduction of the following terminology: let s be an increasing sequence of elements $a_1 < a_2 < \ldots < a_n$ from K such that $a_1 = 1$ (the root) and for each $i < n$ there are no bifurcations strictly between a_i and a_{i+1}, i.e. $\neg\exists$ c: $a_i < c < a_{i+1}\#$. If $1 \leq i \leq n$, we write $a_i \Vdash_s \varphi$ as an abbreviation for $I\Delta_0+\Omega_1 \vdash L_{a_1} \to \Box(L_{a_2} \to \ldots \to \Box(L_{a_i} \to \varphi))\ldots))$. If $i = 1$ this reduces to $I\Delta_0+\Omega_1 \vdash L_{a_1} \to \varphi$. Whenever we use the notation $a \Vdash_s \varphi$ we implicitly assume that s is a sequence $a_1 < a_2 < \ldots < a_n$ as above and a is one of the a_i's. We define $a \Vdash_{all} \varphi$ as $a \Vdash_s \varphi$ where s is the sequence of all the elements $\leq$ a.

4.1. Lemma. $\vdash \Box(w(a_i\#) < \infty) \to \Box(L_{a_i} \to \Box(w(a_{i+1}\#) < \infty))$.

Proof. Since $\neg\exists c$: $a_i < c < a_{i+1}\#$, either $a_{i+1}\#$ is an immediate successor of a_i in which case we apply L9.1, or $a_{i+1}\# \leq a_i$ and we can apply L3. QED.

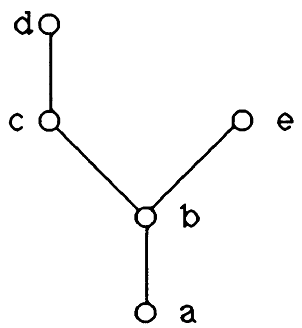
4.2. Theorem. $a_n \Vdash_s \Box(\bigvee_{i>a_n} L_i)$. In particular for every $a \in K$, $a \Vdash_{all} \Box(\bigvee_{i>a} L_i)$.

Proof. If $n = 1$ we apply L4. If $n > 1$, $a_2\#$ is either a_1 (= 1) or an immediate successor of a_1. In any case, by L3 or L6.1, $\vdash L_{a_1} \to \Box(w(a_2\#) < \infty)$. Combining this with a repeated use of lemma 4.1 yields: $a_{n-1} \Vdash_s \Box(w(a_n\#) < \infty)$. On the other hand, by L8, $\vdash \Box(w(a_n\#) < \infty) \to \Box(L_{a_n} \to \Box(\bigvee_{i>a_n} L_i))$. We conclude, $a_{n-1} \Vdash_s \Box(L_{a_n} \to \Box(\bigvee_{i>a_n} L_i))$, i.e. $a_n \Vdash_s \Box(\bigvee_{i>a_n} L_i)$. QED.

The above theorem implies that the relation $a \Vdash_S \varphi$ shares most of the usual properties of the forcing relation $\Vdash$, except that we are not able to prove (or disprove) the implication $\forall b > a\ b \Vdash_S \varphi \Rightarrow a \Vdash_S \Box\varphi$. If this implication were true Solovay's theorem for $I\Delta_0+\Omega_1$ would readily follow. For the moment being we must content ourselves with a weaker result (theorem 4.4 below).

4.3. Definition. Let K and T be Kripke frames. We say that K omits T if K does not contain an homomorphic copy of T which preserves the order relation and the incomparability relation.

4.4. Theorem. Let A be a modal formula such that $\neg A$ is true at the root of some Kripke model $(K, \Vdash)$ whose frame K omits the tree K_0 depicted below. Then $PL\Omega \nvdash A$. In other words $PL\Omega \subseteq Th(Omit(K_0))$, where $Omit(K_0)$ is the class of trees that omit K_0.



The tree K_0

To prove the theorem we assume, without loss of generality, that the nodes of K are 1, ... , n where 1 is the root. As above we extend the order of K to $K \cup \{0\}$ by adjoining 0 as a new root. We make the frame $K \cup \{0\}$ into a Kripke model by stipulating that for every atomic modal formula A, $0 \Vdash A$ iff $1 \Vdash A$. For each atomic modal formula A, we define an arithmetical formula A* given by the disjunction $\bigvee_{i \Vdash A} L_i$ (note that the definition of L_i depends on the tree K, but not on the forcing relation $\Vdash$). We extend the map * to an arithmetical interpretation of all the modal formulas by requiring that * preserves the boolean connectives and maps the modal operator $\Box$ into the provability predicate of $I\Delta_0+\Omega_1$.

4.5. Lemma. If K omits K_0, then for every modal formula A we have:

i) $1 \Vdash A \Rightarrow I\Delta_0+\Omega_1 \vdash L_1 \rightarrow A^*$

ii) $1 \Vdash \neg A \Rightarrow I\Delta_0+\Omega_1 \vdash L_1 \rightarrow \neg A^*$

Moreover for each $i >_K 1$:

iii) $i \Vdash A \Rightarrow I\Delta_0+\Omega_1 \vdash L_1 \rightarrow \Box(L_i \rightarrow A^*)$

iv) $i \Vdash \neg A \Rightarrow I\Delta_0+\Omega_1 \vdash L_1 \rightarrow \Box(L_i \rightarrow \neg A^*)$

We prove the lemma by induction on the complexity of A. The atomic case is immediate from the definition of A*. Boolean connectives pose no problem because conjunctions commute with provability and the inductive step for negations has been built in in the inductive hypothesis. So it is enough to consider the case $A \equiv \Box B$. We prove (i) to (iv).

(i): If $1 \Vdash \Box B$, then $\forall\ i > 1$, $i \Vdash B$, so by the induction hypothesis $I\Delta_0+\Omega_1 \vdash L_1 \rightarrow \Box(L_i \rightarrow B^*)$ for all $i > 1$. Now the thesis follows by observing that $I\Delta_0+\Omega_1 \vdash L_1 \rightarrow \Box(\bigvee_{i>1} L_i)$.

(ii): If $1 \Vdash \neg\Box B$, then for some $i > 1$, $i \Vdash \neg B$, so by the induction hypothesis $I\Delta_0+\Omega_1 \vdash L_1 \to \Box(L_i \to \neg B^*)$. Now the thesis follows from $I\Delta_0+\Omega_1 \vdash L_1 \to \Diamond L_i$.

(iii): If $i > 1$ and $i \Vdash \Box B$, then for all $j > i$, $j \Vdash B$, so for all such j's (if any), by induction hypothesis, $I\Delta_0+\Omega_1 \vdash L_1 \to \Box(L_j \to B^*)$. Now we would like to prove $I\Delta_0+\Omega_1 \vdash L_1 \to \Box(L_i \to \Box\bigvee_{j>i} L_j)$ in order to conclude $I\Delta_0+\Omega_1 \vdash L_1 \to \Box(L_i \to \Box B^*)$. For this aim it is enough to prove $I\Delta_0+\Omega_1 \vdash L_1 \to \Box(L_i \to \Box\neg L_j)$ for all j incomparable with i (for the other nodes, i.e. those with $j \leq i$, this is automatically true by lemma L3). So suppose $j \perp i$. There are two cases:

Case 1. If $\forall c\ (c < j \to c < i)$, then by an application of Švejdar's principle (L5), $I\Delta_0+\Omega_1 \vdash L_i \to \Box\neg L_j$, and a fortiori $I\Delta_0+\Omega_1 \vdash L_1 \to \Box(L_i \to \Box\neg L_j)$.

Case 2. $\exists c < j: c \perp i$. It follows that the only element below both i and j must be the root otherwise K_0 would not be omitted. Therefore ij is an immediate successor of 1. By L6.1 we have $I\Delta_0+\Omega_1 \vdash L_1 \to \Box(w(ij) < \infty)$. Now by lemma L7, $I\Delta_0+\Omega_1 \vdash \Box(w(ij) < \infty) \to \Box(L_i \to \Box\neg L_j)$. Combining these two, we have $I\Delta_0+\Omega_1 \vdash L_1 \to \Box(L_i \to \Box\neg L_j)$ as desired.

The last case to consider is (iv): Assume $i > 1$ and $i \Vdash \neg\Box B$. Then for some $j > i$ we must have $j \Vdash \neg B$, so by induction hypothesis $\vdash L_1 \to \Box(L_j) \to \neg B^*)$. By L2 $\vdash L_i \to \Diamond L_j$, so $\vdash L_1 \to \Box(L_i \to \neg\Box B^*)$ as desired. QED.

Proof of 4.4. If K omits K_0 and $1 \Vdash \neg A$, where 1 is the root of K, then, by the lemma, $I\Delta_0+\Omega_1 \vdash L_1 \to \neg A^*$, where * is the arithmetical interpretation induced by K. By L3 $\mathbb{N} \models L_0$. So by L2 the theory

$I\Delta_0+\Omega_1+L_1$ is consistent. Hence $I\Delta_0+\Omega_1 \nvdash A^*$ and we conclude $PL\Omega \nvdash A$ as desired. QED.

5. Bifurcation depth

In the above sections, we have tried to obtain the best possible upper bound on $PL\Omega$. In particular, we have proved that $PL\Omega \subseteq Th(Omit(K_0))$. In this section, we will prove that the inclusion is strict, using the fact that the 'disjunction property' holds for $PL\Omega$, as proved by Montagna.

5.1. (Montagna's disjunction property). If $PL\Omega \vdash \Box A \vee \Box B$, then $PL\Omega \vdash A$ or $PL\Omega \vdash B$.

Proof. Suppose that for some interpretations $^\circ$ and $^{\circ\circ}$, $I\Delta_0+\Omega_1 \nvdash A(\vec{p}^{\,\circ})$ and $I\Delta_0+\Omega_1 \nvdash B(\vec{p}^{\,\circ\circ})$, where $\vec{p}$ contains all propositional variables occurring in the modal formulas A and B. We have to prove that there is an interpretation $*$ such that $I\Delta_0+\Omega_1 \nvdash (\Box A \vee \Box B)^*$. By multiple diagonalization, define for all $p_i \in \vec{p}$ an arithmetical formula p_i^* such that $I\Delta_0+\Omega_1 \vdash p_i^* \leftrightarrow [(\Box A(\vec{p}^{\,*}) \preccurlyeq \Box B(\vec{p}^{\,*}) \wedge p_i^\circ) \vee (\Box B(\vec{p}^{\,*}) \prec \Box A(\vec{p}^{\,*}) \wedge p_i^{\circ\circ})]$. Here $\Box A \preccurlyeq \Box B$, $\Box A \prec \Box B$ means that there is a proof of A and there is no proof of B with smaller, resp. smaller or equal, Gödel number.

We will show that $I\Delta_0+\Omega_1 \nvdash (\Box A \vee \Box B)^*$. So suppose, to derive a contradiction, that $I\Delta_0+\Omega_1 \vdash \Box A(\vec{p}^{\,*}) \vee \Box B(\vec{p}^{\,*})$.

Then $I\Delta_0+\Omega_1 \vdash \Box A(\vec{p}^{\,*}) \preccurlyeq \Box B(\vec{p}^{\,*}) \vee \Box B(\vec{p}^{\,*}) \prec \Box A(\vec{p}^{\,*})$, so, because $I\Delta_0+\Omega_1$ is a true theory, either

1) $\Box A(\vec{p}^*) \preccurlyeq \Box B(\vec{p}^*)$ and $I\Delta_0+\Omega_1 \vdash p_i^* \leftrightarrow p_i^\circ$ for all i (by definition of $\vec{p}^*$), or

2) $\Box B(\vec{p}^*) \prec \Box A(\vec{p}^*)$ and $I\Delta_0+\Omega_1 \vdash p_i^* \leftrightarrow p_i^{\circ\circ}$ for all i.

In case 1), we have $I\Delta_0+\Omega_1 \vdash A(\vec{p}^*)$, so $I\Delta_0+\Omega_1 \vdash A(\vec{p}^\circ)$, contradicting our assumption. Similarly, case 2) contradicts the assumption $I\Delta_0+\Omega_1 \nvdash B(\vec{p}^{\circ\circ})$. QED.

5.2. Definition. We define T_n by induction.

$T_1(p) = \Diamond p \wedge \Diamond \neg p$

$T_{i+1}(\vec{p},\vec{q},r) = \Diamond(T_i(\vec{p}) \wedge \Box r) \wedge \Diamond(T_i(\vec{q}) \wedge \Box \neg r)$, where all propositional variables in $\vec{p},\vec{q},r$ are different, and $\vec{p}$ and $\vec{q}$ are of length 2^i-1.

The idea behind T_n is that if the root of a tree K forces T_n, then K does not omit the full binary tree of bifurcation depth n. This will be proved in theorem 5.4. below.

5.3. Theorem. For any n, $PL\Omega + T_n$ is consistent.

Proof. By induction.

Basic case: Suppose, in order to derive a contradiction, that $PL\Omega \vdash \neg T_1(p)$, i.e. $PL\Omega \vdash \Box\neg p \vee \Box p$. Then by Montagna's disjunction property, either $PL\Omega \vdash \neg p$ or $PL\Omega \vdash p$. Both are clearly false: In the first case, we can take $p^*=\top$, and in the second case $p^*=\bot$.

Induction step: Suppose as induction hypothesis that for any $\vec{p}$ consisting of 2^i-1 different propositional variables, $PL\Omega + T_i(\vec{p})$ is consistent.

To derive a contradiction, suppose that $PL\Omega \vdash \neg T_{i+1}(\vec{p},\vec{q},r)$, that is $PL\Omega \vdash \Box(\neg T_i(\vec{p}) \vee \neg\Box r) \vee \Box(\neg T_i(\vec{q}) \vee \neg\Box\neg r)$. Then by Montagna's disjunction property, either 1) $PL\Omega \vdash \neg T_i(\vec{p}) \vee \neg\Box r$ or 2) $PL\Omega \vdash \neg T_i(\vec{q}) \vee \neg\Box\neg r$. We show that 1) cannot hold. By the induction hypothesis, $PL\Omega \nvdash \neg T_i(\vec{p})$. Since r does not appear in $T_i(\vec{p})$, we can take $r=\top$. But then $PL\Omega \vdash \Box r$, so $PL\Omega \nvdash \neg T_i(\vec{p}) \vee \neg\Box r$. By an analogous proof, we can show that 2) cannot hold, which gives the desired contradiction. QED.

5.4. Theorem. T_n is not consistent with $Th(C_n)$, where C_n is the class of trees that omit the full binary tree T_n of bifurcation depth n.

Proof. We will show by induction that if K is a finite tree with root k such that $k \Vdash_K T_n$, then we can embed the full binary tree T_n (with nodes numbered 1 to 2^{n+1}-1 in breadth first order) into K. The embedding will preserve $\leq$ and $\perp$.

Basic case: suppose $k \Vdash_K T_1(p)$, then there are nodes i,j such that $k \leq_K i$, $k \leq_K j$, $i \Vdash_K p$ and $j \Vdash_K \neg p$, so $i \neq j$. Therefore, the full binary tree with one level of bifurcation can be homomorphically embedded into K.

Induction step: suppose that $k \Vdash_K T_{i+1}(\vec{p},\vec{q},r)$, i.e.
$k \Vdash_K \Diamond(T_i(\vec{p}) \wedge \Box r) \wedge \Diamond(T_i(\vec{q}) \wedge \Box\neg r)$. Then there are nodes k_1,k_2 such that $k \leq_K k_1$, $k \leq_K k_2$, $k_1 \Vdash_K T_i(\vec{p}) \wedge \Box r$ and $k_2 \Vdash_K T_i(\vec{q}) \wedge \Box\neg r$. By induction hypothesis, we can homomorphically embed a copy of the full binary tree T_i of bifurcation depth i into the subtree of K of points $\geq_K k_1$. Analogously, we can homomorphically embed a copy of T_i into the subtree of K of points $\geq_K k_2$. Because $k_1 \Vdash_K \Box r$ and $k_2 \Vdash_K$

$\Box\neg r$, $k_1 \perp k_2$ and the two images are disjoint. Therefore, we can combine both homomorphic embeddings into one, and map the root of T_{i+1} into k, giving an homomorphic embedding of T_{i+1} into K. QED.

5.5. Corollary. $Th(C_n)$ does not have the disjunction property.

Proof. Suppose n>1. From the proof of theorem 5.4, it follows that $Th(C_n) \vdash \neg T_n(\vec{p},\vec{q},r)$, i.e.

$Th(C_n) \vdash \Box(\neg T_{n-1}(\vec{p}) \vee \neg\Box r) \vee \Box(\neg T_{n-1}(\vec{q}) \vee \neg\Box\neg r)$.

However, it is easy to see that both $T_{n-1}(\vec{p}) \wedge \Box r$ and $T_{n-1}(\vec{q}) \wedge \Box\neg r$ have models on T_{n-1}, and $T_{n-1} \in C_n$. Thus $Th(C_n) \nvdash \neg T_{n-1}(\vec{p}) \vee \neg\Box r$ and $Th(C_n) \nvdash \neg T_{n-1}(\vec{q}) \vee \neg\Box\neg r$. The case for n=1 is proved in an analogous way. QED.

5.6. Corollary. Let C be a class of trees included in C_n. Then $Th(C) \neq PL\Omega$.

Proof. We have $Th(C) \supseteq Th(C_n)$, and $Th(C_n) \not\subseteq PL\Omega$. QED.

In particular, $PL\Omega$ is different from $Th(Omit(K_0))$.

6. Concluding remarks.

The above section gives some support to the conjecture that $PL\Omega$ is equal to L, provided $PL\Omega$ is equal to the theory of some class of frames. It can be shown that a necessary and sufficient condition for

$PL\Omega$ to be equal to L is that for every finite Kripke tree K, $PL\Omega \nvdash \neg A_K$ where A_K is the conjunction of:

L_1

$\Box(L_i \rightarrow \Diamond L_j)$ for $i < j$

$\Box(L_i \rightarrow \Box \neg L_j)$ for $i \not< j$

$\Box(L_i \rightarrow \neg L_j)$ for $i \neq j$

$\Box(L_1 \vee \ldots \vee L_n)$.

In fact we have:

6.1 Proposition. $PL\Omega \subseteq Th(\{K\}) \Leftrightarrow PL\Omega \nvdash \neg A_K$.

Proof. ($\Leftarrow$) If $I\Delta_0 + \Omega_1 \nvdash \neg A_K^*$, let $M \models I\Delta_0 + \Omega_1$, $M \models A_K^*$. If we define for all propositional letters p, $p^\circ = \bigvee_{i \Vdash p} L_i^*$, we can carry out Solovay's proof inside the model M by proving for all modal formulas B, $i \Vdash B \Rightarrow M \models \Box(L_i^* \rightarrow B^\circ)$; so in particular, if $1 \Vdash B$, then $M \models B^\circ$.

($\Rightarrow$) We have $1 \Vdash A_K$, so $\neg A_K \notin Th(\{K\})$. If $PL\Omega \subseteq Th(\{K\})$, then $PL\Omega \nvdash \neg A_K$. QED.

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