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# COLLAPSING GRAPH MODELS BY PREORDERS

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# Collapsing graph models by preorders

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**Abstract**

We present a strategy for obtaining extensional (partial) combinatory algebras by slightly modifying the well-known construction of graph models for the untyped lambda calculus. Using the notions of semi-functor and semi-adjunction an elegant interpretation of our construction in a category theoretical setting is given.

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## 1 Introduction

A lattice $L$ is called *reflexive* if it contains a copy of its own function space (i.e. the lattice $[L \to L]$ of (Scott-)continuous mappings $L \to L$). To be more precise, a lattice $L$ is reflexive if there exist continuous mappings $F : L \longrightarrow [L \to L]$ and $G : [L \to L] \longrightarrow L$ such that $F \circ G = id_{[L \to L]}$.

For any infinite set $X$, the lattice $(\mathcal{P}(X), \subseteq)$ is reflexive: given some embedding $\langle \cdot, \cdot \rangle : X^{<\omega} \times X \hookrightarrow X$ (where $X^{<\omega}$ denotes the collection of *finite* subsets of $X$) one easily checks that the mappings $F$ and $G$ defined by $F(x)(y) := \{b \mid \exists \beta \subseteq y. \langle \beta, b \rangle \in x\}$ and $G(f) := \{\langle \beta, b \rangle \mid b \in f(\beta)\}$ are continuous and witness reflexivity. Structures $(\mathcal{P}(X), F, G)$ are called *graph models*. Well-known canonical examples are Engeler's $\mathbf{D}_A$ and the Scott/Plotkin-model $\mathcal{P}\omega$ (see also Schellinx(1991)).

Reflexive lattices $L$ are natural models for the untyped lambda-calculus. In particular, through the mapping $F$, they define applicative structures $(L, \bullet)$ that are *combinatory algebras*: just define $a \bullet b$ (*'application of a to b'*) $:= F(a)(b)$ (see e.g. chapter 5 of Barendregt(1984)).

An applicative structure $(L, \bullet)$ is said to be *extensional* if, for all $a, b \in L$, we have that $\forall x. a \bullet x = b \bullet x$ holds if and only if $a = b$: i.e., each element of $L$ *uniquely* represents a mapping $L \to L$. It is easy to show that reflexive lattices $(L, F, G)$ induce extensional combinatory algebras $(L, \bullet)$ iff they are *strict reflexive*; i.e., the mappings $F, G$ additionally have to satisfy $G \circ F = id_L$. Clearly the applicative structure $(\mathcal{P}(X), \bullet)$, obtained from a graph model is *not* extensional: we have $\{\langle \emptyset, b \rangle, \langle \{b\}, b \rangle\} \bullet x = \{\langle \emptyset, b \rangle\} \bullet x$ for all $x \in \mathcal{P}(X)$, while obviously $\{\langle \emptyset, b \rangle, \langle \{b\}, b \rangle\} \neq \{\langle \emptyset, b \rangle\}$. Therefore a graph model $(\mathcal{P}(X), F, G)$, though reflexive, never is *strict*: we can not have $G \circ F = id_{\mathcal{P}(X)}$.

In section 2 we present an "extensionalising strategy" for graph models that finds its origin in Inge Bethke's modification of Engeler's $\mathbf{D}_A$- construction, as described in Bethke(1986). We show how to get *strict* reflexive cpo's, by starting from some infinite *preorder* $(X, \preceq)$ (i.e. $X$ equipped with a reflexive, transitive relation $\preceq$), and defining structures $\mathcal{M}_{\preceq} := (M_{\preceq}, F', G')$, with $M_{\preceq}$ a complete lattice obtained as a quotient from $\mathcal{P}(X)$ by means of $\preceq$, and $F', G'$ continuous mappings defined in analogy to $F$ and $G$: we will formulate conditions on the preorder $(X, \preceq)$ necessary and sufficient for the structure $\mathcal{M}_{\preceq}$ obtained to be *strict reflexive.*

After having looked at a number of examples of this construction in section 3, we turn to a description of the process in terms of *semi-notions* in category theory. It is shown that the extensionalisation procedure by means of preorders boils down to mapping (reflexive) objects in a weak cartesian closed category (the category of graph models **GRA**) to strict reflexive objects in the Karoubi envelope **K(GRA)**, which is cartesian closed (and equivalent to the category of continuous complete lattices).

We end our story by taking a look at a similar procedure for *partial* graph models, which are combinatory algebras with application not everywhere defined. Application of our construction then results in extensional *partial* combinatory algebras.

## 2 Construction

Take some non-empty set $X$ and a mapping $\langle \cdot, \cdot \rangle : X^{<\omega} \times X \longrightarrow X$. Next fix some reflexive, transitive $\preceq$ on $X$, making $(X, \preceq)$ a preorder.

Then

- (i) $\preceq$ induces an extension $\text{E}$ of the membership relation $\in$ between elements of $X$ and subsets of $X$:

$$a \,\text{E}\, x \quad \text{iff} \quad \exists a' \in x.a \preceq a';$$

- (ii) $\text{E}$ induces an extension $\sqsubseteq$ of the set inclusion relation $\subseteq$ on $\mathcal{P}(X)$:

$$x \sqsubseteq y \quad \text{iff} \quad \forall a \,\text{E}\, x.a \,\text{E}\, y \qquad (\text{iff} \quad \forall a \in x.a \,\text{E}\, y);$$

- (iii) $\sqsubseteq$ induces an equivalence relation $\equiv$ on $\mathcal{P}(X)$:

$$x \equiv y \quad \text{iff} \quad x \sqsubseteq y \sqsubseteq x.$$

Now let $[x]$ be the $\equiv$-equivalence class of $x$; let $[x] \leq [y]$ iff $x \sqsubseteq y$; then $[x] = [y]$ iff $x \equiv y$. (Sometimes we will write $a \,\text{E}\, [x]$, meaning $a \,\text{E}\, y$ for some $y \equiv x$; as $y \equiv x$ iff $a \,\text{E}\, y \leftrightarrow a \,\text{E}\, x$, this is harmless and often facilitates notation.)

Define $M_{\preceq} := \left(\mathcal{P}(X)/_{\equiv}, \leq\right)$.

**2.1.** Proposition. *$M_{\preceq}$ is a complete lattice with bottom $[\emptyset]$ and supremum* $\sup A = [\bigcup\{y \mid [y] \in A\}]$ *for all $A \subseteq M_{\preceq}$.*

Proof: Obvious from the definition of $\leq$ in terms of $\sqsubseteq$ and the fact that $\sqsubseteq$ extends $\subseteq$ and has the property that for any $D \subseteq \mathcal{P}(X)$, if $\forall y \in D, y \sqsubseteq z$, then $\cup D \sqsubseteq z$. □

Now, following the construction of the graph model $(\mathcal{P}(X), F, G)$, we define a mapping $G'$ : $[M_{\preceq} \to M_{\preceq}] \longrightarrow M_{\preceq}$ by

$$G'(f) := [\{\langle\gamma, c\rangle \mid c \,\text{E}\, f([\gamma])\}],$$

and $F' : M_{\preceq} \to M_{\preceq}^{M_{\preceq}}$(the set of *all* mappings $M_{\preceq} \to M_{\preceq}$) by

$$F'([x])([y]) := [\{b \mid \exists\beta \sqsubseteq y.\langle\beta, b\rangle \,\text{E}\, x\}].$$

**2.2.** Proposition. *(i)* $F' \in [M_{\preceq} \longrightarrow [M_{\preceq} \to M_{\preceq}]]$;
*(ii)* $G' \in [[M_{\preceq} \to M_{\preceq}] \longrightarrow M_{\preceq}]$.

Proof: (i) Let $Y \subseteq M_{\preceq}$ be directed, $[x] \in M_{\preceq}$. Then

$$F'([x])(\sup Y) =$$

$$= \left[\{b \mid \exists\beta \sqsubseteq \sup Y.\langle\beta, b\rangle \,\text{E}\, x\}\right]$$

$$= \left[\bigcup\{\{b \mid \exists\beta \sqsubseteq z.\langle\beta, b\rangle \,\text{E}\, x\} \quad \mid \quad [z] \in Y\}\right]$$

$$= \sup\{F'([x])([z]) \mid [z] \in Y\},$$

so for all $[x] \in M_{\preceq}$ we have that $F'([x])$ is continuous.
Furthermore

$$F(\sup Y)([x]) =$$

$$= \Big[\{b \mid \exists\beta \sqsubseteq x. \langle\beta, b\rangle \text{ E } \sup Y\}\Big]$$

$$= \Big[\{b \mid \exists\beta \sqsubseteq x. \langle\beta, b\rangle \text{ E } \bigcup\{z \mid [z] \in Y\}\}\Big]$$

$$= \Big[\bigcup\{\{b \mid \exists\beta \sqsubseteq x. \langle\beta, b\rangle \text{ E } z\} \quad \mid \quad [z] \in Y\}\Big]$$

$$= \sup_{[z]\in Y} (F'([z])([x]) = \Big(\sup_{[z]\in Y} F'([z])\Big)([x]),$$

so $F'(\sup Y) = \sup_{[z]\in Y} F'([z])$, i.e. $F'$ is continuous.

(ii) Let $Y \subseteq [M_{\preceq} \to M_{\preceq}]$. Then $\sup Y = \lambda x. \sup\{f(x) \mid f \in Y\}$ and

$$G'(\sup Y) =$$

$$= \Big[\{\langle\gamma, c\rangle | c \text{ E } \sup\{f([\gamma]) \mid f \in Y\}\}\Big]$$

$$= \Big[\{\langle\gamma, c\rangle | c \text{ E } \bigcup_{f\in Y} \{y \mid y = f([\gamma])\}\}\Big]$$

$$= \Big[\bigcup_{f\in Y} \{\langle\gamma, c\rangle \mid c \text{ E } f([\gamma])\}\Big]$$

$$= \sup_{f\in Y} G'(f),$$

so $G'$ is continuous. □

Now the following holds:

**2.3.** THEOREM. $\mathcal{M}_{\preceq} := (M_{\preceq}, F', G')$ *is strict reflexive if and only if the preorder* $(X, \preceq)$ *satisfies*

- *(i)* $\langle\beta, b\rangle \preceq \langle\alpha, a\rangle$ *iff* $\alpha \sqsubseteq \beta$ *and* $b \preceq a$*;*
- *(ii) for all* $d : \exists\beta \in X^{<\omega}\exists b \in X. d \preceq \langle\beta, b\rangle \preceq d$*.*

PROOF: ($\Leftarrow$) *reflexivity:* $F' \circ G' = id_{[M_{\preceq} \to M_{\preceq}]}$.
For, $(F'(G'(f)))([x]) =$

$$=\Big[\{b|\exists\beta\sqsubseteq x.\langle\beta,b\rangle \mathrel{\text{E}} G'(f)\}\Big]$$

$$=\Big[\{b|\exists\beta\sqsubseteq x.\langle\beta,b\rangle \mathrel{\text{E}} \{\langle\gamma,c\rangle\mid c \mathrel{\text{E}} f([\gamma])\}\}\Big]$$

$$=\Big[\{b|\exists\gamma\sqsubseteq x.b \mathrel{\text{E}} f([\gamma])\}\Big] \qquad \text{(by (i)}\rightarrow\text{)}$$

$$=\Big[\bigcup\{\{b\mid b \mathrel{\text{E}} f([\gamma])\}|\gamma\sqsubseteq x\}\Big]$$

$$=\sup_{[\gamma]\leq[x]} f([\gamma])=f([x]) \qquad \text{(by continuity of } f\text{).}$$

*strictness:* $G'\circ F'=id_{M_{\preceq}}$.

For, $(G'(F'([x]))=$

$$=\Big[\{\langle\gamma,c\rangle|c \mathrel{\text{E}} F'([x])([\gamma])\}\Big]$$

$$=\Big[\{\langle\gamma,c\rangle|c \mathrel{\text{E}} \{b\mid\exists\beta\sqsubseteq\gamma.\langle\beta,b\rangle \mathrel{\text{E}} x\}\}\Big]$$

$$=\Big[\{\langle\gamma,c\rangle|\langle\gamma,c\rangle \mathrel{\text{E}} x\}\Big] \qquad \text{(by (i)}\leftarrow\text{)}$$

$$=[x] \qquad \text{(by (ii)).}$$

($\Rightarrow$) Let $\mathcal{M}_{\preceq}$ be strict reflexive. By extensionality of the induced applicative structure we have, for all $x$, $\{\langle\gamma,c\rangle\mid\langle\gamma,c\rangle \mathrel{\text{E}} x\}\equiv x$. Now take any $d$ and put $x=\{d\}$. Then clearly we find $\exists\gamma\exists c.d\preceq\langle\gamma,c\rangle$, and $\langle\gamma,c\rangle \mathrel{\text{E}} x$, so $\langle\gamma,c\rangle\preceq d$. This proofs (ii).

By reflexivity the equality $F'\circ G'=id_{[M_{\preceq}\rightarrow M_{\preceq}]}$ is valid. So, from the transition above marked "(by (i)$\rightarrow$)",

$$\forall x\forall f\forall b\Big(\exists\beta\sqsubseteq x\exists\gamma\exists c(\langle\beta,b\rangle\preceq\langle\gamma,c\rangle\wedge c \mathrel{\text{E}} f([\gamma])) \rightarrow\exists\delta\sqsubseteq x\exists d(d \mathrel{\text{E}} f([\delta])\wedge b\preceq d)\Big). \qquad (\star)$$

In order to prove (i)$\rightarrow$, suppose $\langle\mu,m\rangle\preceq\langle\nu,n\rangle$. We define a mapping $f_n$ by putting $f_n([x])=[\{n\}]$, for all $x$. Obviously $f_n$ is continuous. By taking $x=X$ and $f=f_n$ in $(\star)$ we have

$$\forall b\Big(\exists\beta\exists\gamma\exists c(\langle\beta,b\rangle\preceq\langle\gamma,c\rangle\wedge c\preceq n)\rightarrow\exists d(d\preceq n\wedge b\preceq d)\Big).$$

As $\langle\mu,m\rangle\preceq\langle\nu,n\rangle$ and $n\preceq n$, putting $b=m$, we conclude: $\exists d.d\preceq n\wedge m\preceq d$. So $m\preceq n$ by transitivity of $\preceq$.

To prove that also $\nu\sqsubseteq\mu$, we define a mapping $f_{n\mu}$ as follows:

$$f_{n\mu}([x])=\begin{cases}[\emptyset], & \text{if } x\sqsubseteq\mu;\\ [\{\text{n}\}], & \text{otherwise.}\end{cases}$$

It is easy to check continuity of $f_{n\mu}$. Now take $x=\mu$ and $f=f_{n\mu}$ in $(\star)$. Then

$$\forall b\Big(\exists\beta\sqsubseteq\mu\exists\gamma\exists c(\langle\beta,b\rangle\preceq\langle\gamma,c\rangle\wedge c \mathrel{\text{E}} f_{n\mu}([\gamma]))$$

$$\rightarrow \exists \delta \sqsubseteq \mu \exists d(d \,\mathrm{E}\, f_{n\mu}([\delta]) \wedge b \preceq d)\Big).$$

Suppose $\nu \not\sqsubseteq \mu$. In that case $f_{n\mu}([\nu]) = [\{n\}]$, so $n \,\mathrm{E}\, f_{n\mu}([\nu])$. As $\langle \mu, m\rangle \preceq \langle \nu, n\rangle$ we conclude

$$\exists \delta \sqsubseteq \mu \exists d(d \,\mathrm{E}\, f_{n\mu}([\delta]) \wedge m \preceq d).$$

But $\delta \sqsubseteq \mu$ implies $f_{n\mu}([\delta]) = [\emptyset]$ by definition of $f_{n\mu}$. Contradiction. Therefore $\nu \sqsubseteq \mu$, finishing the proof (i)$\rightarrow$.

Finally, as by strictness the equality $G' \circ F' = id_{M_\preceq}$ is valid, we find (from the transition above marked "(by (i)$\leftarrow$)") that

$$\forall x \forall \gamma \forall c\Big(\exists \beta \exists b(\beta \sqsubseteq \gamma \wedge \langle \beta, b\rangle \,\mathrm{E}\, x \wedge c \preceq b) \rightarrow \langle \gamma, c\rangle \,\mathrm{E}\, x\Big).$$

Taking $x = \{\langle \alpha, a\rangle\}$ we find

$$\forall \gamma \forall c. \alpha \sqsubseteq \gamma \wedge c \preceq a \rightarrow \langle \gamma, c\rangle \preceq \langle \alpha, a\rangle.$$

This proves (i)$\leftarrow$, and ends the proof of our theorem. □

# 3 Some remarks and examples

As we saw in section 2, it is not necessary to start from an *embedding* $\langle \cdot, \cdot \rangle : X^{<\omega} \times X \hookrightarrow X$. Any mapping will do. In fact, from the proof of theorem 2.3 it follows that we also have the following

**3.1.** PROPOSITION. *$\mathcal{M}_\preceq := (M_\preceq, F', G')$ is reflexive iff the preorder $(X, \preceq)$ satisfies*

$$\langle \beta, b\rangle \preceq \langle \alpha, a\rangle \quad \Rightarrow \quad \alpha \sqsubseteq \beta \quad \text{and} \quad b \preceq a. \qquad \Box$$

Taking equality (=) for $\preceq$, this shows us that we recover our 'plain' graph models precisely in case the mapping $\langle \cdot, \cdot \rangle : X^{<\omega} \times X \rightarrow X$ we start from is an embedding.

We should note that, independent from our work, the same result (theorem 2.3) has been obtained by Jean-Louis Krivine (see Krivine(1990)). Krivine moreover observes the following:

**3.2.** PROPOSITION. *Every graph model can be collapsed by a preorder.*

PROOF: Krivine(1990), page 106/107. □

The first example of an extensional combinatory algebra obtained by means of the construction described above was given by Inge Bethke in Bethke(1986), modifying the definition of Engeler's graph model $\mathbf{D}_A$ by using a preorder satisfying the conditions of theorem 2.3. But, in fact, to *any* graph model $\mathcal{M} := (\mathcal{P}(X), \bullet)$ (which is fully determined by the embedding $\langle \cdot, \cdot \rangle : X^{<\omega} \times X \hookrightarrow X$ defining the application-function $\bullet$) we may associate a collection $\mathcal{E}(\mathcal{M})$ of extensional combinatory algrebras (fully determined by the collection of preorders $\preceq$ on $X$ that satisfy the necessary conditions).

We observe that any element of $\mathcal{E}(\mathcal{M})$ can be isomorphically embedded in $\mathcal{M}$: let $\left(\mathcal{P}(X)/_{\equiv}, \star\right) \in \mathcal{M}_e$ and define $\varphi : \left(\mathcal{P}(X)/_{\equiv}, \star\right) \hookrightarrow (\mathcal{P}(X), \bullet)$ by $\varphi([a]) = \bigcup\{b \mid [b] = [a]\}$. One easily checks that $\varphi$ is 1 - 1 and moreover $\varphi([a]) \bullet \varphi([a']) = \varphi([a] \star [a'])$.

The cardinality of the collection $\mathcal{E}(\mathcal{M})$ depends on the graphmodel $\mathcal{M}$ at hand: with a graph model $\mathbf{D}_A$ we can associate many non-isomorphic extensionalisations; on the other hand there are also graph models $\mathcal{M}$ for which $\mathcal{E}(\mathcal{M})$ is a singleton (e.g. the Plotkin/Scott-model $\mathcal{P}\omega$). This will become clear from the examples given below.

First we will turn our attention to variations on Engeler's $\mathcal{D}_A$. In order to do so, let us quickly review the definition.

**3.3.** DEFINITION. Let $A$ be any non-empty set and put:

$$\begin{aligned} G_0(A) &:= A \\ G_{n+1}(A) &:= G_n(A) \cup (G_n(A)^{<\omega} \times G_n(A)) \\ G(A) &:= \bigcup_{n \in \mathbb{N}} G_n(A). \end{aligned}$$

(So $G(A)$ is the smallest set $X \supset A$ such that for all finite $\beta \subseteq X$ and $b \in X$ we have that $(\beta, b) \in X$. The application-defining embedding is just the identity.)

$\mathbf{D}_A$ will denote the graph model $(\mathcal{P}(G(A)), \bullet)$. □

**3.4.** DEFINITION. Let A be any non-empty set, $G(A)$ as in the definition of $\mathbf{D}_A$. Suppose $f : A \to A$ is a 1-1 mapping, $\epsilon : A \to A^{<\omega}$ an arbitrary mapping. Define, for $x, y \in \mathcal{P}(G(A))$,

$$x \sqsubseteq_{f\epsilon} y \quad \textit{iff} \quad \forall a \in x \exists b \in y . a \preceq_{f\epsilon} b,$$

where $a \preceq_{f\epsilon} b$ holds iff either

(1) $a = b$, or
(2) $\exists \beta \exists c \quad (a = (\beta, c) \wedge b \in A \wedge c \preceq_{f\epsilon} f(b) \wedge \epsilon_b \sqsubseteq_{f\epsilon} \beta)$, or
(3) $\exists \alpha \exists c \quad (a \in A \wedge b = (\alpha, c) \wedge f(a) \preceq_{f\epsilon} c \wedge \alpha \sqsubseteq_{f\epsilon} \epsilon_a)$, or
(4) $\exists \alpha \exists \beta \exists c \exists d \quad (a = (\alpha, c) \wedge b = (\beta, d) \wedge \beta \sqsubseteq_{f\epsilon} \alpha \wedge c \preceq_{f\epsilon} d)$. □

**3.5.** REMARK. Given $A, f, \epsilon$ as above, $\preceq_{f\epsilon}$ is well-defined, in the sense that for all $n$ and $a, b \in G_n(A)$, $a \preceq_{f\epsilon} b$ is defined in terms of the restriction of $\preceq_{f\epsilon}$ to $\bigcup_{m<n} G_m(A)$. For this we need that the range of $f$ is contained in $A$ and that the range of $\epsilon$ is contained in $A^{<\omega}$. □

**3.6.** LEMMA. *Given $A, f, \epsilon$ again as above*

- *(i)* $\forall a \in A \quad \forall b. \quad a \preceq_{f\epsilon} b \iff (\epsilon_a, f(a)) \preceq_{f\epsilon} b$;
- *(ii)* $\forall a \in A \quad \forall b. \quad b \preceq_{f\epsilon} a \iff b \preceq_{f\epsilon} (\epsilon_a, f(a))$;
- *(iii)* $\preceq_{f\epsilon}$ *is transitive.*

PROOF: For this we need the injectivity of $f$. The proof is similar to that of proposition 1.3 in Bethke(1986). □

Let $A$ be a non-empty set and $f, \epsilon$ mappings as above. Define $=_{f\epsilon}$ on $\mathcal{P}(G(A))$ by

$$x =_{f\epsilon} y \quad \text{iff} \quad x \sqsubseteq_{f\epsilon} y \quad \text{and} \quad y \sqsubseteq_{f\epsilon} x.$$

By $\mathbf{M}(A, f, \epsilon)$ we will denote the structure $\left(\mathcal{P}(G(A))\big/_{=_{f\epsilon}}, \leq_{f\epsilon}\right)$ as defined in section 2.

**3.7. PROPOSITION.** *$\mathbf{M}(A, f, \epsilon)$ is an extensional combinatory algebra.*

PROOF: By definition 3.4 and 3.5, 3.6 the conditions of theorem 2.3 are met. □

The extensional combinatory algebras $M(A)$ defined by Inge Bethke in Bethke(1986) are precisely the models $\mathbf{M}(A, id, \emptyset)$ we get by taking $f$ to be the identity mapping on $A$ and $\epsilon$ the constant mapping $\epsilon_a = \emptyset$, for all $a \in A$.

Whereas the models $\mathbf{M}(A, id, \emptyset)$ have the property that $[G(A)] = [A]$, this is no longer true as soon if we e.g. take $\epsilon_a$ to be non-empty for all $a \in A$. For then $(\emptyset, x) \not\preceq_{f\epsilon} a$, for all $x \in G(A)$, so $G(A) \not\sqsubseteq_{f\epsilon} A$. Therefore taking $\epsilon \not\equiv \emptyset$, one would expect to find extensional models not *isomorphic* (as applicative structures) to $\mathbf{M}(A, id, \emptyset)$.

We will here confine ourselves to showing this to be the case for finite $A$, $f \equiv id$ and $\epsilon_a \neq \emptyset$ for all $a \in A$. For then we have $[\beta] \neq [G(A)]$ for all finite $\beta \subseteq G(A)$: suppose $\beta = \{x_1, \ldots, x_n\}$; let $(\emptyset^m, b)$ denote the element of $G(A)$ given by $(\emptyset, \ldots, \emptyset, b)$, where $b$ is preceded by $m$ occurrences of $\emptyset$. By the pigeonhole-principle $G(A) \sqsubseteq \beta$ would imply, for any $x \in G(A)$, the existence of $1 \leq k \leq n$ such that $(\emptyset^m, b) \preceq_{f\epsilon} x_k$ for infinitely many $m$. Say $x_k = (\gamma_1, \ldots, \gamma_n, a)$, with $a \in A$. Then there is an $m > n$ such that $(\emptyset^m, b) \preceq_{f\epsilon} (\gamma_1, \ldots, \gamma_n, a)$, implying $\epsilon_a \sqsubseteq_{f\epsilon} \emptyset$, which is a contradiction, as $\epsilon_a \neq \emptyset$.

Inspection of the proof of lemma 2.3 in Bethke(1986) now shows us that in $\mathbf{M}(A, id, \epsilon)$ we have

$$\forall [x].(\exists [y], [z].([y] \neq [x] \wedge [z] \bullet [x] = [x] \wedge \forall [x'] \neq [x].[z] \bullet [x'] = [y])) \quad (\star)$$

iff $[x] = [\emptyset]$, whereas in $\mathbf{M}(A, id, \emptyset)$ we have for finite $A$

$$(\star) \quad \text{iff} \quad [x] = [\emptyset] \quad \text{or} \quad [x] = [A].$$

Therefore $\mathbf{M}(A, id, \epsilon)$ and $\mathbf{M}(A, id, \emptyset)$ are not first-order equivalent. Then certainly $\mathbf{M}(A, id, \epsilon) \not\cong \mathbf{M}(A, id, \emptyset)$ (see also Schellinx(1991)).

The relation $\preceq_{f\epsilon}$ from definition 3.4 is such that its restriction to $A$ is just the usual identity-relation. By adding order structure to the atom-set we are able to define even more extensional combinatory algebras. In order to do so, let $\mathcal{R}$ be a reflexive transitive relation on the atom-set $A$, and replace condition (1) in definition 3.4 by

$$(1') \quad a = b \quad \vee \quad (a, b \in A \quad \wedge \quad a\mathcal{R}b).$$

Let $f : A \to A$ be a mapping such that for all $a, b \in A$ we have $a\mathcal{R}b$ iff $f(a)\mathcal{R}f(b)$, and $\epsilon : A \to A^{<\omega}$ arbitrary. We then have a relation $\Subset_{f\epsilon}$ on $\mathcal{P}(G(A))$ defined by

$$x \Subset_{f\epsilon} y \quad \text{iff} \quad \forall a \in x \exists b \in y. a \ll_{f\epsilon} b,$$

where $a \ll_{f\epsilon} b$ holds iff (1′) as above or (2),(3),(4) as in definition 3.4.

One may check that, assuming $\mathcal{R}$ to be decidable, lemmas 3.5 and 3.6 remain valid.

As before we may define a relation $=_{f\epsilon}$ on $\mathcal{P}(G(A))$ by

$$a =_{f\epsilon} b \quad \text{iff} \quad a \Subset_{f\epsilon} b \quad \text{and} \quad b \Subset_{f\epsilon} a.$$

By $\mathbf{M}(\langle A, \mathcal{R}\rangle, f, \epsilon)$ we will denote the structure $\left(\mathcal{P}(G(A))\big/_{=_{f\epsilon}}, \leq_{f\epsilon}\right)$. (Note that $\mathbf{M}(A, f, \epsilon)$ $= \mathbf{M}(\langle A, =\rangle, f, \epsilon)$.)

The following then is obvious.

**3.8.** PROPOSITION. *Let $A$ be a non-empty set, $\mathcal{R}, f, \epsilon$ as above. Then $\mathbf{M}(\langle A, \mathcal{R}\rangle, f, \epsilon)$ is an extensional combinatory algebra.* □

The addition of structure to $A$ indeed gives us different models, as can be seen from the next example, for which we need two lemmas.

**3.9.** LEMMA. *Let $\langle A, \mathcal{R}\rangle$ be as described above. Then in $\mathbf{M}(\langle A, \mathcal{R}\rangle, id, \emptyset)$ we have for all $[x]$,*

$$[z] \bullet [x] = [z] \quad \text{iff} \quad \exists y \subseteq \mathcal{P}(A).[z] = [y].$$

PROOF: (→) Suppose $\forall x.[z] \bullet [x] = [z]$. Define $y := \{a \in A \mid \exists x.[x] = [z] \wedge a \in x\}$. It is left to the reader to show that $[z] = [y]$.
(←) If $[z] = [y]$ for $y \in \mathcal{P}(A)$ then, for all $x$, $[z] \bullet [x] = [y] \bullet [x] = [\{b \mid \exists \beta \Subset x.\{(\beta, b)\} \Subset y\}] = [y] = [z]$. □

**3.10.** LEMMA. *In $\mathbf{M}(A, id, \emptyset)$ we have, for all $y_1, y_2 \subseteq A$, $[y_1] = [y_2]$ iff $y_1 = y_2$.*

PROOF: Easy. □

Let $A = \{a, b\}$ be a two-element set, and $\mathcal{R} = \{(a, a), (b, b), (a, b)\} \in A \times A$. Consider the extensional combinatory algebra $\mathbf{M}(\langle A, \mathcal{R}\rangle, id, \emptyset)$.

**3.11.** PROPOSITION. *For all $A \neq \emptyset$, and $\mathcal{R}$ as above:*

$$\mathbf{M}(\langle\{a, b\}, \mathcal{R}\rangle, id, \emptyset) \not\cong \mathbf{M}(A, id, \emptyset).$$

PROOF: Let $A$ be any non-empty set. Take $\varphi$ to be the $\mathcal{L}$-sentence "*There are precisely 3 different $z$ such that $\forall x.zx = z$*". In $\mathbf{M}(\langle\{a, b\}, \mathcal{R}\rangle, id, \emptyset)$ we have that $[\{b\}] = [\{a, b\}]$, so, by lemma 3.9, $\varphi$ holds. By lemma 3.10 $\varphi$ will not hold in $\mathbf{M}(A, id, \emptyset)$. So the models are not elementary equivalent, therefore they can not be isomorphic. $\square$

Besides $\mathbf{D}_A$, we can obtain other easy definable examples of graph models by using the set $\mathbb{N}$ of natural numbers. We consider embeddings $\langle\cdot,\cdot\rangle : \mathbb{N}^{<\omega} \times \mathbb{N} \hookrightarrow \mathbb{N}$ given by means of an injective (but not necessarily surjective) coding $p : \mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$ of pairs of natural numbers as natural numbers and a bijective coding $e : \mathbb{N} \hookrightarrow \mathbb{N}^{<\omega}$ of finite sets of natural numbers by natural numbers, as follows:

$$\langle \alpha, m\rangle = p(e^{-1}(\alpha), m).$$

Putting $e_n$ for $e(n)$ we then can rewrite the definition of application through an embedding by codings $p$ and $e$ as:

$$x \bullet y = \{m \mid \exists e_n \subseteq y.p(n,m) \in x\}.$$

**3.12.** Definition. A $\mathcal{P}(\mathbb{N})$-*structure* is a graphmodel $[p, e] := (\mathcal{P}(\mathbb{N}), \bullet)$ with application defined by the codings $p$ and $e$ as above.

The Plotkin-Scott model $\mathcal{P}\omega$ is the structure $[p^\star, e^\star]$ defined as follows:

for all $n, m \in \mathbb{N} : p^\star(n,m) = \frac{1}{2}(n+m)(n+m+1)+m$;

for all $n \in \mathbb{N} : e^\star_n = \{k_0, k_1, \ldots, k_{m-1}\}$ iff $n = \sum_{i<m} 2^{k_i} \quad (k_i \neq k_j \text{ if } i \neq j)$;

$e^\star_0 = \emptyset$. $\square$

The following proposition tells us that $\mathcal{E}(\mathcal{P}\omega)$ is a singleton.

**3.13.** Proposition. *For the embedding $\langle\cdot,\cdot\rangle : \mathbb{N}^{<\omega} \times \mathbb{N} \hookrightarrow \mathbb{N}$ defined by $[p^\star, e^\star]$ there is a unique reflexive, transitive relation on $\mathbb{N}$ that satisfies the conditions of theorem 2.3.*

Proof: Define, for $x, y \in \mathcal{P}(\mathbb{N})$,

$$x \sqsubseteq y \quad \text{iff} \quad \forall a \in x \exists b \in y . a \preceq b, \tag{1}$$

where $a \preceq b$ iff either

- (i) $a = b$, or
- (ii) $\exists n_1 \exists n_2 \exists m_1 \exists m_2 \quad (a = p^\star(n_1, m_1) \wedge b = p^\star(n_2, m_2) \wedge e^\star_{n_2} \sqsubseteq e^\star_{n_1} \wedge m_1 \preceq m_2)$.

Note that the codings $p^\star, e^\star$ have the properties

$$\forall k . x \in e^\star_k \Rightarrow x < k;$$

$$\forall m, n . m, n \leq p^\star(m,n).$$

(We call codings with these properties *basic codings*.) From this we see, by an easy induction argument, that $\preceq$ on $\mathbb{N}$ is well-defined, in the sense that for all $n$, all $a, b \leq n$, $a \preceq b$ is defined in terms of $\preceq$ restricted to $\{k \mid k < n\}$. Again using induction, one readily shows transitivity of $\preceq$:

$$\forall n \in \mathbb{N} . \forall x_1, x_2, x_3 \leq n : \qquad x_1 \preceq x_2 \wedge x_2 \preceq x_3 \Longrightarrow x_1 \preceq x_3.$$

Now let $\lessdot$ be another reflexive transitive relation on $\mathbb{N}$. Let $\ll$ be the extension of $\lessdot$ to $\mathcal{P}(\mathbb{N})$ as in (1) and suppose $\lessdot$ and $\ll$ satisfy conditions (i) and (ii), i.e.

$$p^\star(n_1, m_1) \lessdot p^\star(n_2, m_2) \quad \text{iff} \quad e^\star_{n_2} \ll e^\star_{n_1} \quad \text{and} \quad m_1 \lessdot m_2.$$

We conclude $\lessdot = \preceq$ from

$$\forall n \in \mathbb{N}. \forall x, y \leq n: \qquad x \preceq y \Longleftrightarrow x \lessdot y,$$

which once more is readily shown by induction on $n$. □

Let $\mathbf{P}\omega$ be the cpo $\left(\mathcal{P}(\mathbb{N})\big/_{\equiv}, \leq\right)$ as in section 2, using the relation $\preceq$ given in the proof of proposition 3.13. Then $\mathbf{P}\omega$ is an extensional combinatory algebra, and $\mathcal{E}(\mathcal{P}\omega) = \{\mathbf{P}\omega\}$.

For $\mathcal{P}(\mathbb{N})$-structures in general, as for $\mathbf{D}_A$, relations satisfying the conditions of 2.3 will not be unique, even if we restrict our attention to the collection of $\mathcal{P}(\mathbb{N})$-structures defined through surjective codings $p$. Proposition 3.13 clearly holds, though, for all $\mathcal{P}(\mathbb{N})$-structures defined by means of basic codings $p, e$ for which moreover $p(n, m) = m$ iff $n = m = 0$.

Obviously isomorphic graph models $\mathcal{G}_1 \cong \mathcal{G}_2$ will give rise to isomorphic extensionalisations $\mathcal{E}(\mathcal{G}_1) \cong \mathcal{E}(\mathcal{G}_2)$ (in the sense that for all $A \in \mathcal{E}(\mathcal{G}_i)$ there exists a $B \in \mathcal{E}(\mathcal{G}_j)$ such that $A \cong B$). Conversely though, we may have $A \in \mathcal{E}(\mathcal{G}), B \in \mathcal{E}(\mathcal{H})$ such that $A \cong B$, but $\mathcal{G} \not\cong \mathcal{H}$. As $\mathbf{D}_{\{a\}} \not\cong \mathcal{P}\omega$ (see Schellinx(1991)), the following proposition gives an example.

**3.14. Proposition.** $\mathbf{P}\omega \cong \mathbf{M}(\{a\}, id, \emptyset)$.

Proof: Let $\preceq_p, \sqsubseteq_p, \leq_p$ denote the relations used in the definition of $\mathbf{P}\omega$; let $\preceq_m, \sqsubseteq_m, \leq_m$ denote the relations used in the definition of $\mathbf{M}(\{a\}, id, \emptyset)$.

Define $\phi : G(\{a\}) \hookrightarrow \mathbb{N}$ by

$\phi(a) = 0,$

$\phi((\beta, b)) = p^\star((e^\star)^{-1}(\{\phi(c) \mid c \in \beta\}), \phi(b)).$

(1) We show by induction on $n$ that $\phi$ is onto:

$$\forall n \forall k \leq n \exists t_k . \phi(t_k) = k. \tag{†}$$

Let $n \geq k = p^\star(a, b)$. If $x \in e^\star_a$ then $x < a < n$, so by inductionhypothesis $\forall x \in e^\star_a \exists t_x . \phi(t_x) = x$. Put $\beta := \{t_x \mid x \in e_a\}$. As $b \leq k$ and (†) by definition is true for $n = 0$, we have by inductionhypothesis that $\phi(t_b) = b$ for some $t_b$. But then $\phi((\beta, t_b)) = p^\star(a, b) = k$.

(2) Now define $\psi : \mathbf{M}(\{a\}, id, \emptyset) \hookrightarrow \mathbf{P}\omega$ by

$$\psi([x]) = [\{\phi(b) \mid b \in x\}].$$

In lemma 3.3 in Bethke(1986) it is proved that

$$\forall n \forall b, b' \in G_n(A). \quad \phi(b) = \phi(b') \Longrightarrow b \preceq_m b' \preceq_m b.$$

Using this one shows by induction on $n$ that

$$\forall n \forall b, b' \in G_n(A). \quad \phi(b) \preceq_p \phi(b') \Longleftrightarrow b \preceq_m b',$$

and then derives that $\psi$ is a well-defined 1-1 mapping. Also it is easy to see that $\psi$ is onto.

(3) Finally let $\bullet$ be the application on $\mathbf{M}(\{a\}, id, \emptyset)$, $*$ the application on $\mathbf{P}\omega$. Then

$$\psi([x] \bullet [y]) = [\{\phi(b) \mid \exists[\beta] \leq_m [y].[\{(\beta, b)\}] \leq_m [x]\}] =$$

$$= [\{c \mid \exists[\gamma] \leq_p \psi([y]).[\{p^*((e^*)^{-1}(\gamma), c)\}] \leq_p \psi([x])\}] = \psi([x]) * \psi([y]),$$

as the reader may verify. So $\psi$ indeed is an applicative isomorphism. $\square$

# 4 Category theoretic preliminaries

We will now review some of the notions that are essential for our description of the extensionalisation procedure in a category theoretical setting. We assume the reader to be familiar with the more basic notions of category theory. (See e.g. MacLane (1971).)

**4.1.** DEFINITION. By a *semi-functor* we mean a mapping $F$ between categories $\mathcal{A}$ and $\mathcal{B}$ having the same properties as a functor, except that $F$ need not preserve identities. So $F$ takes objects/arrows of $\mathcal{A}$ to objects/arrows of $\mathcal{B}$ and, if $f : A \to B$ in $\mathcal{A}$, then $F(f) : F(A) \to F(B)$ in $\mathcal{B}$. Moreover, $F$ is a homomorphism with respect to composition of arrows.

Given a semi-functor $F : \mathcal{A} \to \mathcal{B}$, there is a natural transformation $F(id) : F \to F$ with components $F(id_C) : FC \to FC$. We write $\mathcal{B}(FC, D)_s$ for the set of arrows $f : FC \to D$ in $\mathcal{B}$ that satisfy $f \circ F(id_C) = f$. The set $\mathcal{B}(C, FD)_s$ is defined analogously. A *semi-adjunction* is a tuple $\langle F, G, \mu \rangle$, where $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{A}$ are semi-functors, and $\mu$ is a natural transformation with components

$$\mu_{CD} : \mathcal{B}(FC, D)_s \cong \mathcal{A}(C, GD)_s.$$

(The notions of semi-functor and -adjunction were introduced in Hayashi(1985); however, the definition of semi-adjunction given here is slightly different from Hayashi's original one (see Hoofman(1990/3).) We write $F \vdash_s G$ when $F, G$ are part of a semi-adjunction. If $F, G$ are functors, then $F \vdash_s G$ iff $F \vdash G$. $\square$

Contrary to an 'adjoint of a functor', a 'semi-adjoint of a (semi-)functor' is *not* unique up to natural isomorphism. As an example, consider the category $\mathbf{1}$ having one object $*$ and one arrow $id_*$. For any category $\mathcal{C}$ there is a (unique) functor $T : \mathcal{C} \to \mathbf{1}$ defined by $T(A) = *$ and $T(f) = id_*$ Right semi-adjoints of $T$ correspond to *semi-terminal objects* of $\mathcal{C}$, i.e. objects 1 in $\mathcal{C}$ such that for each $C \in \mathcal{C}$ there is an arrow $t_C : C \to 1$ and if $f : C \to C'$, then $t_{C'} \circ f = t_C$. Semi-terminal objects obviously need not be unique. E.g., in **Set** every object $\neq \emptyset$ is semi-terminal.

On the other hand, semi-adjoints *are* unique up to *semi* natural isomorphism. (See Hoofman(1991))

**4.2.** DEFINITION. A *cartesian closed category* $\mathcal{C}$ is a category $\mathcal{C}$ with three specified right adjoints for the functors

$$\mathcal{C} \longrightarrow \mathbf{1} \qquad \mathcal{C} \longrightarrow \mathcal{C} \times \mathcal{C} \qquad \mathcal{C} \xrightarrow{- \times b} \mathcal{C}$$

$$c \mapsto 0 \qquad c \mapsto \langle c, c \rangle \qquad a \mapsto a \times b$$

Replacing these adjunctions by semi-adjunctions, we obtain the notion of a *semi cartesian closed category* (*semi*-**CCC**). The replacement of adjunctions by semi-adjunctions may be partial, as is witnessed by the notion of *weak cartesian closed category* ($w$**CCC**): a category $\mathcal{C}$ with terminal object $\mathbf{1}$, binary products $A \times B$ for all objects $A, B$ and a semi-adjunction between the *functor* $(\cdot) \times B$ and a *semi-functor* $\cdot \Rightarrow B$.

The following is an equivalent algebraic formulation of $w$**CCC**:

A *weak cartesian closed category* is a category $\mathcal{C}$ with a terminal object $\mathbf{1}$ and binary products $A \times B$, and with the following data:

- For each pair of objects $A, B \in \mathcal{C}$ an object $A \Rightarrow B \in \mathcal{C}$, and an arrow $\mathbf{ev}_{A,B} \in \mathcal{C}((A \Rightarrow B) \times A, B)$. Furthermore, for each arrow $f \in \mathcal{C}(D \times A, B)$ an arrow $\mathbf{cur}(f) \in \mathcal{C}(D, A \Rightarrow B)$.

satisfying the following equations (omitting subscripts):

1. $\mathbf{ev} \circ (\mathbf{cur}(f) \times id) = f$
2. $\mathbf{cur}(f \circ (g \times id)) = \mathbf{cur}(f) \circ g$

(From the algebraic definition we obtain a semi-(bi-)functor $(\cdot \Rightarrow \cdot)$ defined on arrows $f : X \to Y, g : U \to V$ by $(f \Rightarrow g) := \mathbf{cur}(g \circ \mathbf{ev} \circ (id \times f)) : (Y \Rightarrow U) \to (X \Rightarrow V)$.)

A cartesian closed category then is a $w$**CCC** in which also $\mathbf{cur}(\mathbf{ev}) \equiv id$ holds. □

Observe that, like 'being cartesian closed', the properties 'being semi/weak cartesian closed' are *essentially categorical* properties, i.e. they are preserved under equivalence of categories.

**4.3.** DEFINITION. A pair $(f, g)$ of mappings $f : X \to Y, g : Y \to X$ is called a *retraction* iff $f \circ g \equiv id_Y$. We say that $Y$ is a *retract* of $X$. □

E.g., by definition a lattice $L$ is reflexive just if the lattice $[L \to L]$ of continuous mappings $L \to L$ is a retract of $L$.

**4.4.** DEFINITION. A *strict reflexive object* $A$ in a $w$**CCC** is an object $A$ such that $A \cong (A \Rightarrow A)$. □

**4.5.** PROPOSITION. *A strict reflexive object $A$ in a weak cartesian closed (and locally small) category $\mathcal{C}$ induces a retraction $\mathcal{C}(\mathbf{1}, A) \rightleftarrows \mathcal{C}(A, A)$ in* **Set**. *If $\mathcal{C}$ is cartesian closed, the retraction in fact is an isomorphism.*

PROOF: Let $A$ be a reflexive object in $\mathcal{C}$ and let $\phi : A \cong (A \Rightarrow A)$ be an isomorphism. Let $k$ be the canonical isomorphism $A \cong \mathbf{1} \times A$. Then $F := \lambda x.\mathbf{ev}(\phi x \times id)k$ is a mapping $\mathcal{C}(\mathbf{1}, A) \to \mathcal{C}(A, A)$, and $G := \lambda f.\phi^{-1}\mathbf{cur}(fk^{-1})$ is a mapping $\mathcal{C}(A, A) \to \mathcal{C}(\mathbf{1}, A)$. One easily checks that $F \circ G \equiv id_{\mathcal{C}(A,A)}$ and in case $\mathbf{cur}(\mathbf{ev}) \equiv id$, also $G \circ F \equiv id_{\mathcal{C}(\mathbf{1},A)}$. □

If we take for $\mathcal{C}$ the category **CPO** of complete partial orders and continuous functions, we even obtain a retraction *in* **CPO**: $\mathcal{C}(\mathbf{1}, A) \cong A, \mathcal{C}(A \to A) \cong [A \to A]$ and $F, G$ are continuous.

4.6. DEFINITION. Given a category $\mathcal{A}$, the *Karoubi envelope* of $\mathcal{A}$ is the category $\mathbf{K}(\mathcal{A})$ having as objects pairs $(A, f)$, where $A$ is an object from $\mathcal{A}$ and $f \in \mathcal{A}(A, A)$ an idempotent $\mathcal{A}$-arrow, i.e. $f \circ f = f$. Morphisms $(A, f) \to (B, g)$ in $\mathbf{K}(\mathcal{A})$ are arrows $h : A \to B$ in $\mathcal{A}$ such that $g \circ h \circ f = h$ (or equivalently $g \circ h = h$ and $h \circ f = h$). Composition of arrows is composition in $\mathcal{A}$; $id_{(A,f)} = f$.

As any idempotent arrow $f$ uniquely determines its target and source $A$, we will often identify an object $(A, f)$ in $\mathbf{K}(\mathcal{A})$ with its 'arrow part' $f$. □

Let $F : \mathcal{A} \to \mathcal{B}$ be a (semi-)functor. Then define $\mathbf{K}(F) : \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ by

$$\mathbf{K}(F)(A, f) := (F(A), F(f)), \quad \mathbf{K}(F)(g) := F(g).$$

The proof of the following proposition is an amusing exercise.

4.7. PROPOSITION. *The Karoubi envelope $\mathbf{K}(F)$ of a semi-functor $F$ is a functor.* □

One easily checks that if $F \vdash_s G$, then $\mathbf{K}(F) \vdash \mathbf{K}(G)$. The following then is immediate:

4.8. THEOREM. *If $\mathcal{A}$ is a semi/weak cartesian closed category, then its Karoubi envelope $\mathbf{K}(\mathcal{A})$ is cartesian closed.* □

Now the next fairly trivial observation will appear to be at the heart of the extensionalisation procedure given in section 2: let $A$ be any object in a weak cartesian closed category $\mathcal{A}$, and let $f : A \to A$ be an idempotent arrow in $\mathcal{A}$. Let also mappings $i : (A \Rightarrow A) \to A$ and $j : A \to (A \Rightarrow A)$ in $\mathcal{A}$ be given. Then we have:

4.9. PROPOSITION. *The arrows $(f \Rightarrow f)jf$ and $fi(f \Rightarrow f)$ determine an isomorphism $f \cong (f \Rightarrow f)$ in $\mathbf{K}(\mathcal{A})$ if and only if*

- *(i) $fi(f \Rightarrow f)jf = f$;*
- *(ii) $(f \Rightarrow f)jfi(f \Rightarrow f) = (f \Rightarrow f)$.*

PROOF: Easy. □

# 5 The category GRA of graph models

By the category of graph models we mean the category **GRA**, having as objects all *powersets*, i.e. all sets of the form $\mathcal{P}(X)$, where $X$ is a set; morphisms are all functions continuous with respect to the Scott-topology on the lattices $(\mathcal{P}(X), \subseteq)$.

- $\mathcal{P}(\emptyset) = \{\emptyset\}$ is terminal object in **GRA**.

- Writing $X \uplus Y$ for the disjoint union of the sets $X$ and $Y$, one easily checks that **GRA** has binary products $\mathcal{P}(X) \times \mathcal{P}(Y) := \mathcal{P}(X \uplus Y)$.
- For the semi-exponents, we put $(\mathcal{P}(X) \Rightarrow \mathcal{P}(Y)) := \mathcal{P}(X^{<\omega} \times Y)$.

Let $F \in \mathcal{P}(X^{<\omega} \times Y)$. We define an evaluation function $\mathbf{ev} : (\mathcal{P}(X) \Rightarrow \mathcal{P}(Y)) \times \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$ by

$$\mathbf{ev}(F, a) := \{c \mid \exists \gamma \subseteq a.(\gamma, c) \in F\}.$$

For $f : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(Z)$ define $\mathbf{cur}(f) : \mathcal{P}(X) \to \mathcal{P}(Y^{<\omega} \times Z)$ by

$$\mathbf{cur}(f)(b) := \{(\gamma, c) \mid c \in f(b, \gamma)\}.$$

Clearly **ev** and $\mathbf{cur}(f)$ are continuous. We leave it to the reader to check that conditions 1 and 2 of definition 4.2 are satisfied. Therefore **GRA** is a weak cartesian closed category.

Any concrete embedding $\langle \cdot, \cdot \rangle : X^{<\omega} \times X \hookrightarrow X$ induces (as in the proof of proposition 4.5), through the mappings **cur** and **ev** a retraction $(F, G)$ between $\mathcal{P}(X)$ and the lattice $\mathbf{GRA}(\mathcal{P}(X), \mathcal{P}(X)) = [\mathcal{P}(X) \to \mathcal{P}(X)]$ of continuous functions from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ known from the definition of graph models. In particular, also strict reflexive objects in **GRA** (given by *surjective* embeddings) induce these retractions, which never are isomorphisms. The result always is a *non-extensional* lambda model.
We will show that the construction of *extensional* combinatory algebras as described in section 2 in fact boils down to the *construction of strict reflexive objects* in the Karoubi envelope **K(GRA)** of the category of graph models. As **K(GRA)** is cartesian closed, these strict reflexive objects induce an *isomorphism* between the object and its hom-set, and therefore give rise to extensional lambda models.

Let $f$ be some continuous idempotent mapping $\mathcal{P}(X) \to \mathcal{P}(X)$. Then $f$ is an object of **K(GRA)**, with exponent $(f \Rightarrow f) := \mathbf{cur}(f \circ \mathbf{ev} \circ (id \times f))$.
Suppose $\langle \cdot, \cdot \rangle$ is some mapping $X^{<\omega} \times X \to X$. Then the mappings $j : \mathcal{P}(X^{<\omega} \times X) \to \mathcal{P}(X)$ given by $x \mapsto \{a \mid a = \langle \beta, b \rangle \& (\beta, b) \in x\}$, and $i : \mathcal{P}(X) \to \mathcal{P}(X^{<\omega} \times X)$ given by $y \mapsto \{(\beta, b) \mid \langle \beta, b \rangle = a \in y\}$ are continuous, so they are arrows in **GRA** and proposition 4.9 tells us under what conditions $f$ is a *strict reflexive object* in **K(GRA)**.
Returning to the construction described in section 2, we observe the following: take $\preceq$ to be a preorder on some set $X$, then, for $x \in \mathcal{P}(X)$, define $[x] := \{b \mid \exists b' \in x.b \preceq b'\}$. Now it's easy to see that $p := \lambda x.\{b \mid \exists \beta \subseteq x.b \in [\beta]\}$ is an idempotent arrow in **GRA**. Note that $p(x) \equiv [x]$. Furthermore $(p \Rightarrow p)(x) = \mathbf{cur}(p \circ \mathbf{ev} \circ (id \times p))(x) = \{(\gamma, c) \mid c \in [\{c' \mid \exists \gamma' \subseteq [\gamma]\ (\gamma', c') \in x\}]\}$. Using proposition 4.9 we then can prove the following

**5.1.** THEOREM. *The arrows $(p \Rightarrow p)ip$ and $pj(p \Rightarrow p)$ determine an isomorphism $p \cong (p \Rightarrow p)$ in **K(GRA)** if and only if the preorder $(X, \preceq)$ satisfies*

- *(i) $\langle \beta, b \rangle \preceq \langle \alpha, a \rangle$ iff $\alpha \sqsubseteq \beta$ and $b \preceq a$;*
- *(ii) for all $d : \exists \beta \in X^{<\omega} \exists b \in X.d \preceq \langle \beta, b \rangle \preceq d$.* $\square$

In fact, we will not give the proof here, but leave the calculations to the zealous reader. We shall however provide the (quite similar) details in the next section, while working in a more basic category, equivalent to **GRA**.

## 6 A category of relations equivalent to GRA

We take a look at the following category: objects are sets, arrows $f : A \to B$ are relations $R \subseteq A^{<\omega} \times B$ satisfying the following monotonicity condition:

$$\beta \subseteq \beta' \ \&\ \beta R b \Rightarrow \beta' R b.$$

For the identity on an object $A$ we take the arrow $id_A$ defined by $\alpha(id_A)a \Leftrightarrow a \in \alpha$. Composition $S \star R$ of arrows $R : A \to B, S : B \to C$ is defined by

$$\alpha(S \star R)c \Leftrightarrow \exists\beta(\alpha R\beta \wedge \beta Sc),$$

where $\alpha R\beta$ is an abbreviation for $\forall b \in \beta.\alpha Rb$. Similarly we will write $aRb$ for $\{a\}Rb$.
We will denote the category thus defined by *Kl*(**REL**) as it is in fact known as the semi-Kleisli category of the category of relations (see Hoofman, 1990/2).

**6.1.** PROPOSITION. *Kl(**REL**) is equivalent to the category of graph models, Kl(**REL**)* $\equiv$ **GRA**.

PROOF: We will define appropriate functors, and leave the details of verification to the reader.

- The functor $F : Kl(\mathbf{REL}) \to \mathbf{GRA}$ is defined on objects by $X \mapsto \mathcal{P}(X)$, on arrows by $R \mapsto \lambda x.\{b \mid \exists\beta \subseteq x.\beta Rb\}$;
- The functor $G : \mathbf{GRA} \to Kl(\mathbf{REL})$ is defined on objects by $\mathcal{P}(X) \mapsto X$, on arrows by $f \mapsto \{(\beta, b) \mid b \in f(\beta)\}$. □

**6.2.** DEFINITION. Let $R : A \to B$ be an arrow in *Kl*(**REL**). We say that $R$ is *linear* iff it satisfies $\alpha Rb \Leftrightarrow \exists a \in \alpha.aRb$. □

So a linear arrow is fully determined by its values on singletons.
As can be readily verified (e.g. using proposition 6.1) *Kl*(**REL**) is a *w***CCC** and hence we now have a (bi-)functor $\cdot \Rightarrow \cdot : Kl(\mathbf{REL})^{op} \times Kl(\mathbf{REL}) \to Kl(\mathbf{REL})$ acting on objects $A, B$ by $(A \Rightarrow B) := A^{<\omega} \times B$ and on arrows $R : A \to A', S : B' \to B$ by $\Xi(R \Rightarrow S)(\beta, b) \Leftrightarrow \{a \mid \exists(\alpha, a) \in \Xi.\beta R\alpha\}Sb$.

**6.3.** LEMMA. *If $S : A \to B$ and $T : B \to C$ are linear, then so is $T \star S : A \to C$. If $R : A \to A$ is linear, then so is $(R \Rightarrow R)$; furthermore we have $(\beta, b)(R \Rightarrow R)(\alpha, a)$ iff $\alpha R\beta$ and $bRa$.*
PROOF: Easy. □

Now take an idempotent arrow $R : A \to A$ in *Kl*(**REL**), so $R \star R = R$. Let $I : (A \Rightarrow A) \to A, J : A \to (A \Rightarrow A)$ also be arrows in *Kl*(**REL**). Again, using proposition 4.9, $R$ is a strict reflexive object in $\mathbf{K}(Kl(\mathbf{REL}))$ if and only if

(i) $(R \Rightarrow R) = (R \Rightarrow R)JRI(R \Rightarrow R)$;
(ii) $R = RI(R \Rightarrow R)JR$.

In order to obtain in $Kl(\mathbf{REL})$ the analogue of theorem 5.1 we have to impose some further restrictions on the arrows $R, I, J$: if we take $R, I$ and $J$ to be *linear* arrows we can, using the definition of composition and lemma 6.3, rewrite these conditions as:
(i) $\alpha R\beta \wedge bRa$ *iff*

$$\exists\gamma,\gamma',c,c',d,d'.\gamma R\beta \wedge bRc \wedge (\gamma,c)Id \wedge dRd' \wedge d'J(\gamma',c') \wedge \alpha R\gamma' \wedge c'R\alpha.$$

(ii) $bRa$ *iff*

$$\exists\gamma,\gamma',c,c',d,D'.bRd' \wedge d'J(\gamma',c') \wedge \gamma R\gamma' \wedge c'Rc \wedge (\gamma,c)Id \wedge dRa.$$

In order to increase legibility, let us write $<$ for $R$. Hence $\alpha < \beta$ stands for $\forall b \in \beta(\alpha < b)$, and by linearity of $<$ this becomes $\forall b \in \beta\exists a \in \alpha(a < b)$. Furthermore, we write $(\alpha, a) < (\beta, b)$ for $\beta < \alpha \wedge a < b$. Now the two conditions can be written as:

1. $(\beta,b) < (\alpha,a)$ *iff* $\exists(\gamma,c),(\gamma',c'),d,d'.(\beta,b) < (\gamma,c)Id < d'J(\gamma',c') < (\alpha,a)$;
2. $b < a$ *iff* $\exists(\gamma,c),(\gamma',c'),d,d'.b < d'J(\gamma',c') < (\gamma,c)Id < a$.

Note that by idempotency of the relation $<$ we have for all $\alpha, \beta, a, b$
(i) $a < b \Leftrightarrow \exists c.a < c < b$;
(ii) $\alpha < \beta \Leftrightarrow \exists\gamma.\alpha < \gamma < \beta$;
(iii) $(\beta,b) < (\alpha,a) \Leftrightarrow \exists(\gamma,c).(\beta,b) < (\gamma,c) < (\alpha,a)$.

By restricting the relations $I, J$ we obtain various specific instances of the conditions for $<$ to be a reflexive object in $\mathbf{K}(Kl(\mathbf{REL}))$. For example, let $I$ be a function $\langle\cdot,\cdot\rangle : A^{<\omega} \times A \to A$, and define $J$ by $aJ(\beta,b) \Leftrightarrow a = \langle\beta,b\rangle$. Our two conditions may then be written as:

1' $(\beta,b) < (\alpha,a)$ *iff* $\exists\gamma,\gamma',c,c'.(\beta,b) < (\gamma,c) \wedge \langle\gamma,c\rangle < \langle\gamma',c'\rangle \wedge (\gamma',c') < (\alpha,a)$

2' $b < a$ *iff* $\exists\gamma,\gamma',c,c'.b < \langle\gamma',c'\rangle \wedge (\gamma',c') < (\gamma,c) \wedge \langle\gamma,c\rangle < a$.

Adding one more restriction, namely reflexivity of the relation, now once more leads us to the conditions we encountered in the extensionalisation procedure of section 2:

**6.4. Theorem.** *Let $<, I, J$ be as defined above, and $<$ reflexive. Then $(< \Rightarrow <)$ and $<$ are isomorphic in the Karoubi envelope if and only if*

1. $(\beta,b) < (\alpha,a)$ iff $\langle\beta,b\rangle < \langle\alpha,a\rangle$
2. $\forall a\exists\beta,b.\ a < \langle\beta,b\rangle < a$.

Proof: We show that 1, 2 are equivalent to 1', 2' above. First suppose 1, 2 hold, then 1' holds:

- If $(\beta,b) < (\alpha,a)$, then $(\beta,b) < (\beta,b) \wedge \langle\beta,b\rangle < \langle\beta,b\rangle \wedge (\beta,b) < (\alpha,a)$ by reflexivity of $<$.
- If $\exists\gamma,\gamma',c,c'.(\beta,b) < (\gamma,c) \wedge \langle\gamma,c\rangle < \langle\gamma',c'\rangle \wedge (\gamma',c') < (\alpha,a))$, then $\exists\gamma,\gamma',c,c'.(\beta,b) < (\gamma,c) < (\gamma',c') < (\alpha,a))$ by 1. Hence $(\beta,b) < (\alpha,a)$ by transitivity of $<$.

and 2' holds:

- If $b < a$, then by 2 there exists $(\gamma, c)$ such that $b < \langle\gamma, c\rangle < b < a$. Hence $b < \langle\gamma, c\rangle \wedge (\gamma, c) < (\gamma, c) \wedge \langle\gamma, c\rangle < a$.

- If $\exists\gamma, \gamma', c, c'.b < \langle\gamma', c'\rangle \wedge (\gamma', c') < (\gamma, c) \wedge \langle\gamma, c\rangle < a$ then $\exists\gamma, \gamma', c, x'.\, b < \langle\gamma', c'\rangle < \langle\gamma, c\rangle < a)$ by 1. Hence $b < a$ by transitivity of $<$.

The other way round, suppose 1', 2' hold, then 1 holds:

- If $(\beta, b) < (\alpha, a)$, then $\langle\beta, b\rangle < \langle\beta, b\rangle \wedge (\beta, b) < (\alpha, a) \wedge \langle\alpha, a\rangle < \langle\alpha, a\rangle$ by reflexivity of $<$. Hence $\langle\beta, b\rangle < \langle\alpha, a\rangle$ by 2'.

- If $\langle\beta, b\rangle < \langle\alpha, a\rangle$, then $(\beta, b) < (\beta, b) \wedge \langle\beta, b\rangle < \langle\alpha, a\rangle \wedge (\alpha, a) < (\alpha, a)$ by reflexivity of $<$. Hence $(\beta, b) < (\alpha, a)$ by 1'.

and 2 holds:

- We have $a < a$ by reflexivity of $<$, and hence $\exists\gamma, \gamma', c, c'.a < \langle\gamma', c'\rangle \wedge (\gamma', c') < (\gamma, c) \wedge \langle\gamma, c\rangle < a)$ by 2'. From $(\gamma', c') < (\gamma, c)$ it follows that $\langle\gamma', c'\rangle < \langle\gamma, c\rangle$ by 1 (which we have already established). Hence $a < \langle\gamma, c\rangle < a$ by transitivity of $<$. □

Also interesting is the case in which $I$ is defined as before, but $J$ is defined independently of $I$ by $aJ(\beta, b) \Leftrightarrow a = [\beta, b]$, where $[-,-]$ is a function $A^{<\omega} \times A \to A$. The conditions for isomorphism in the Karoubi envelope now reduce to

1" $(\beta, b) < (\alpha, a)$ *iff* $\exists\gamma, \gamma', c, c'.(\beta, b) < (\gamma, c) \wedge \langle\gamma, c\rangle < [\gamma', c'] \wedge (\gamma', c') < (\alpha, a)$;

2" $b < a$ *iff* $\exists\gamma, \gamma', c, c'.b < [\gamma', c'] \wedge (\gamma', c') < (\gamma, c) \wedge \langle\gamma, c\rangle < a$.

Again, if $<$ is reflexive we can give more simple sufficient, but this time not necessary, requirements.

**6.5.** THEOREM. *Let $<, I, J$ be as defined above, and $<$ reflexive. If*

1. $(\beta, b) < (\alpha, a)$ iff $\langle\beta, b\rangle < \langle\alpha, a\rangle$,
2. $\forall a \exists\beta, b.\ a < \langle\beta, b\rangle < a$,
3. $\forall\alpha, a.\ [\alpha, a] < \langle\alpha, a\rangle < [\alpha, a]$,

*then $(< \Rightarrow <)$ and $<$ are isomorphic in the Karoubi envelope.*

PROOF: It is an easy exercise to prove that the requirements of the theorem imply the requirements 1",2" above. □

To complete the picture, let us study the way in which theorems 2.3, 5.1 and 6.4 are related. For this the following lemma will be of use.

**6.6.** LEMMA. *Let $\mathcal{A},\mathcal{B}$ be categories. If $\mathcal{A}\equiv\mathcal{B}$, then $\mathbf{K}(\mathcal{A})\equiv\mathbf{K}(\mathcal{B})$.*

PROOF: Suppose $\mathcal{A}\equiv\mathcal{B}$, as witnessed by functors $F:\mathcal{A}\to\mathcal{B}, G:\mathcal{B}\to\mathcal{A}$, and natural isomorphisms $\mu: FG\to id_{\mathcal{A}}, \nu: GF\to id_{\mathcal{B}}$. Then $(\mathbf{K}(F),\mathbf{K}(G),\mathbf{K}(\mu),\mathbf{K}(\nu))$ determines an equivalence $\mathbf{K}(\mathcal{A})\equiv\mathbf{K}(\mathcal{B})$. □

Now using the functor $G$ from proposition 6.1, from the idempotent arrow $p$ in **GRA** (as in 5.1), we obtain a reflexive idempotent linear relation $G(p)$ in $Kl(\mathbf{REL})$.

Conversely, given a reflexive idempotent linear relation $<$ in $Kl(\mathbf{REL})$, we obtain an idempotent arrow $F(<)$ in **GRA** defined by $F(<):=\lambda x.\{b\mid\exists\beta\subseteq x.\beta<b\}$. Then

$$a\preceq b \quad \textit{iff} \quad \{b\}<a$$

defines a preorder on $X$ and $F(<)=\lambda x.\{b\mid\exists\beta\subseteq x.b\in[\beta]\}$.

Therefore, as $\mathbf{GRA}\equiv Kl(\mathbf{REL})$, in fact theorems 5.1 and 6.4 are the same statements, expressed within different, but equivalent, categories.

Finally, to establish the relation with theorem 2.3, we note the following

**6.7.** PROPOSITION. *$\mathbf{K}(Kl(\mathbf{REL}))\equiv\mathbf{CCLat}$, where* **CCLat** *denotes the category of complete continuous lattices.*

PROOF: For $(X,R)$ in $\mathbf{K}(Kl(\mathbf{REL}))$ we define $\mathtt{Pt}(X,R):=(\{[A]\mid A\in\mathcal{P}(X)\},\subseteq)$, with $[A]:=\{a\mid\exists\beta\subseteq A.\beta Ra\}$. Then $\mathtt{Pt}(X,R)$ is a complete continuous lattice. For arrow $T:(A,R)\to(B,S)$ we define $\mathtt{Pt}(T):=\lambda x.\{b\mid\exists\beta\subseteq x.\beta Tb\}$. This defines a functor $\mathtt{Pt}:\mathbf{K}(Kl(\mathbf{REL}))\to\mathbf{CCLat}$.

Conversely, let $(D,\leq)$ be a complete continuous lattice with basis $\mathsf{B}_D$. Define $R:\mathsf{B}_D^{<\omega}\to\mathsf{B}_D$ by

$$\beta Rb \quad \textit{iff} \quad b\ll\bigvee\beta,$$

(where $x\ll y$ (" $x$ *way below* $y$") iff for each directed subset $S$ of $\mathsf{B}_D$ we have that $y\leq\bigvee S$ implies that there is an $y'\in S$ such that $x\leq y'$.)

Now one can check that $\mathtt{Rep}(D,\leq):=(\mathsf{B}_D,R)$ is in $\mathbf{K}(Kl(\mathbf{REL}))$.

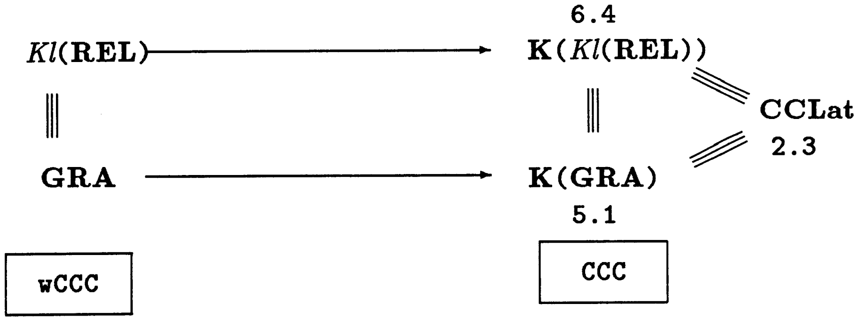
For continuous $f:(D,\leq)\to(D',\leq')$, define $\mathtt{Rep}(f):\mathtt{Rep}(D,\leq)\to\mathtt{Rep}(D',\leq')$ by

$$\beta\mathtt{Rep}(f)b \quad \textit{iff} \quad b\ll f(\bigvee\beta)$$

This defines an arrow in $\mathbf{K}(Kl(\mathbf{REL}))$, and $\mathtt{Rep}$ is a functor $\mathbf{CCLat}\to\mathbf{K}(Kl(\mathbf{REL}))$. In fact the functors $\mathtt{Pt}$, $\mathtt{Rep}$ establishes an equivalence of categories. For details we refer to Hoofman (1990/1). □

Now, by applying the equivalence, we find that the isomorphism in $\mathbf{K}(Kl(\mathbf{REL}))$ under the conditions of theorem 6.4 induces precisely the isomorphism between continuous lattices given by the construction of section 2.

The following diagram summarizes our observations:




# 7 Getting partial

In this final section we will briefly describe an interesting modification of the construction given in section 2. Recall that the concept of combinatory algebra can be extended to applicative structures on which the application is not everywhere defined (see Bethke(1987)). We will refer to such structures as *partial* combinatory algebras (*pca*). Concrete examples of pca's can be constructed as follows: take some infinite set $X$, and $\langle\cdot,\cdot\rangle$ a *non-surjective* embedding $(X^{<\omega}\backslash\emptyset)\times X \hookrightarrow X$. By non-surjectivity we may fix some $p$ not in the range of $\langle\cdot,\cdot\rangle$.

Also recall that given some cpo $M$ with bottom $\perp$, a continuous function $f$ from $M$ to $M$ is called *strict* if $f(\perp)=\perp$. We write $[M \xrightarrow{s} M]$ for the cpo of *strict continuous* functions from $M$ to $M$. Then, for $x, y \in \mathcal{P}(X)$, define $\mathtt{F}(x)(y) := \{b \mid \exists\beta \subseteq y.\langle\beta, b\rangle \in x\}$. For $f \in [\mathcal{P}(X) \xrightarrow{s} \mathcal{P}(X)]$ define $\mathtt{G}(f) := \{\langle\gamma, c\rangle \mid c \in f(\gamma)\} \cup \{p\}$.

Next a partial application $*$ on $\mathcal{P}(X)\backslash\emptyset$ is given by

$$x * y := \begin{cases} \mathtt{F}(x)(y), & \text{if } \mathtt{F}(x)(y) \neq \emptyset; \\ \mathit{undefined}, & \text{otherwise.} \end{cases}$$

Let $\mathcal{G} := (\mathcal{P}(X)\backslash\emptyset, *)$.

We will call the partial applicative structures thus obtained *partial graph models.*

**7.1.** LEMMA. *(i)* $\mathtt{F} \in \left[\mathcal{P}(X) \longrightarrow [\mathcal{P}(X) \xrightarrow{s} \mathcal{P}(X)]\right]$;
*(ii)* $\mathtt{G} \in \left[[\mathcal{P}(X) \xrightarrow{s} \mathcal{P}(X)] \longrightarrow \mathcal{P}(X)\right]$ *and* $\mathit{Range}(\mathtt{G}) \subseteq \mathcal{P}(X)\backslash\emptyset$;
*(iii)* $\mathtt{F} \circ \mathtt{G} = id_{[\mathcal{P}(X) \xrightarrow{s} \mathcal{P}(X)]}$.

PROOF: Straightforward. □

**7.2.** DEFINITION. A cpo $\mathcal{M}$ for which there exists mappings $\mathtt{F} : \mathcal{M} \longrightarrow [\mathcal{M} \xrightarrow{s} \mathcal{M}]$ and $\mathtt{G} : [\mathcal{M} \xrightarrow{s} \mathcal{M}] \longrightarrow \mathcal{M}$ satisfying the conditions of lemma 7.1 is called *p-reflexive.* □

**7.3.** DEFINITION. As usual, we write $x \simeq y$ for "*if either $x$ or $y$ is defined, then both are defined and equal*". We say that a partial applicative structure $(M, \bullet)$ is *extensional* iff for all $x, y \in M$ we have that $(\forall z \in M. x \bullet z \simeq y \bullet z)$ implies that $x = y$. □

**7.4.** PROPOSITION. *Partial graph models are non-extensional partial combinatory algebras.*

PROOF: From 7.1 it follows that partial graph models are p-reflexive cpo's via the mappings F and G. Therefore they are partial combinatory algebras (for details see Bethke(1987), theorem 2.8). To see that partial graph models never are extensional, just observe that for all $x, y \in \mathcal{P}(X)\backslash\emptyset$ we have $(x \cup \{p\}) * y \simeq x * y$, while clearly $x \cup \{p\} \neq x$, whenever $p \notin x$. □

In Bethke(1987) it is shown that for a $p$-reflexive cpo $(\mathcal{A}, F, G)$ to determine an extensional pca it is necessary and sufficient that the cpo is also *p-strict*, i.e. $G \circ F = id_{\mathcal{A}\backslash\{\perp\}}$. So by 7.4 we have that a $p$-reflexive cpo $(\mathcal{P}(X), \mathtt{F}, \mathtt{G})$ as defined above can never be $p$-strict.

## Constructing extensional partial combinatory algebras

Take a non-empty set $X$ and let us fix some mapping $\langle\cdot,\cdot\rangle : (X^{<\omega}\backslash\emptyset) \times X \to X$ as well as a preorder $\preceq$ on $X$. Define $\mathcal{M} := \left(\mathcal{P}(X)/_{\equiv}, \leq\right)$ as in section 2. By Proposition 2.1 $\mathcal{M}$ is a complete lattice with bottom $[\emptyset]$ and supremum $\sup A = [\bigcup\{y \mid [y] \in A\}]$, for all $A \subseteq \mathcal{M}$. We now fix some element $p \in X$ and following the construction of the partial graph model $\mathcal{G}$, define mappings $\mathtt{G}'$ by

$$\mathtt{G}'(f) := \left[\{\langle\gamma, c\rangle \mid c \mathrel{\mathrm{E}} f([\gamma])\} \cup \{p\}\right]$$

and $\mathtt{F}'$ by

$$\mathtt{F}'([x])([y]) := \left[\{b \mid \exists\beta \sqsubseteq y.\langle\beta, b\rangle \mathrel{\mathrm{E}} x\}\right].$$

**7.5.** LEMMA. *(i)* $\mathtt{F}' \in \left[\mathcal{M} \longrightarrow [\mathcal{M} \xrightarrow{s} \mathcal{M}]\right]$;

*(ii)* $\mathtt{G}' \in \left[[\mathcal{M} \xrightarrow{s} \mathcal{M}] \longrightarrow \mathcal{M}\right]$;

*(iii)* $Range(\mathtt{G}') \subseteq \mathcal{M}\backslash[\emptyset]$.

PROOF: Left to the reader. □

Again we can define a partial application $\star$ on $(\mathcal{M}\backslash[\emptyset])$ by

$$[x] \star [y] := \begin{cases} \mathtt{F}'([x])([y]), & \text{if } \mathtt{F}'([x])([y]) \neq [\emptyset]; \\ undefined, & \text{otherwise.} \end{cases}$$

Thus we obtain a partial applicative structure $\mathcal{G}_{\preceq} := (\mathcal{M}\backslash[\emptyset], \star)$.

The next theorem tells us which conditions on the preorder $(X, \preceq)$ are sufficient and necessary for $\mathcal{G}_{\preceq}$ to be an *extensional* pca.

**7.6.** THEOREM. *$\mathcal{G}_{\preceq} := (\mathcal{M}, \mathtt{F}', \mathtt{G}')$ is p-strict p-reflexive if and only if the preorder $(X, \preceq)$ satisfies*

- *(i)* for all $\beta, b : \langle\beta, b\rangle \not\preceq p$;
- *(ii)* for all $d : p \preceq d$;
- *(iii)* for all $d : d \not\preceq p$ iff $\exists\beta, b. d \preceq \langle\beta, b\rangle \preceq d$;
- *(iv)* for all $\alpha, \beta \neq \emptyset : \langle\beta, b\rangle \preceq \langle\alpha, a\rangle$ iff $\alpha \sqsubseteq \beta$ and $b \preceq a$.

PROOF: ($\Leftarrow$) *p-reflexivity:* $\mathtt{F}' \circ \mathtt{G}' = id_{[\mathcal{M} \xrightarrow{s} \mathcal{M}]}$.

For, $(\mathtt{F}'(\mathtt{G}'(f)))([x]) =$

$$= \left[\{b | \exists \beta \sqsubseteq x . \langle \beta, b\rangle \mathrel{\text{E}} \mathtt{G}'(f)\}\right]$$

$$= \left[\left\{b | \exists \beta \sqsubseteq x . \langle \beta, b\rangle \mathrel{\text{E}} \{\langle \gamma, c\rangle \mid c \mathrel{\text{E}} f([\gamma])\} \cup \{p\}\right\}\right]$$

$$= \left[\left\{b | \exists \beta \sqsubseteq x . \langle \beta, b\rangle \mathrel{\text{E}} \{\langle \gamma, c\rangle \mid c \mathrel{\text{E}} f([\gamma])\}\right\}\right] \qquad \text{(by (i))}$$

$$= \left[\{b | \exists \gamma \sqsubseteq x . \gamma \neq \emptyset \wedge b \mathrel{\text{E}} f([\gamma])\}\right] \qquad \text{(by (iv)}\rightarrow\text{)}$$

$$= f([x]) \qquad \text{(by strictness and continuity of } f\text{)}.$$

*p-strictness:* $\mathtt{G}' \circ \mathtt{F}' = id_{\mathcal{M} \setminus [\emptyset]}$.

For, let $[x] \in \mathcal{M}, x \neq \emptyset$.

Then, $(\mathtt{G}'(\mathtt{F}'([x])) =$

$$= \left[\{\langle \gamma, c\rangle | c \mathrel{\text{E}} \mathtt{F}'([x])([\gamma])\} \cup \{p\}\right]$$

$$= \left[\left\{\langle \gamma, c\rangle | c \mathrel{\text{E}} \{b \mid \exists \beta \sqsubseteq \gamma . \langle \beta, b\rangle \mathrel{\text{E}} x\}\right\} \cup \{p\}\right]$$

$$= \left[\{\langle \gamma, c\rangle | \langle \gamma, c\rangle \mathrel{\text{E}} x\} \cup \{p\}\right] \qquad \text{(by (iv)}\leftarrow\text{)}$$

$$= [x] \qquad \text{(by (ii) and (iii))}.$$

($\Rightarrow$) Suppose $(\mathcal{M}, \mathtt{F}', \mathtt{G}')$ is a $p$-strict $p$-reflexive cpo. By $p$-reflexivity the equality $\mathtt{F}' \circ \mathtt{G}' = id_{[\mathcal{M} \xrightarrow{s} \mathcal{M}]}$ is valid. Now let $\langle \beta, b\rangle \preceq p$. Then define

$$f([x]) := \begin{cases} [\emptyset], & \text{if } x \sqsubseteq \beta; \\ [\beta], & \text{otherwise.} \end{cases}$$

Clearly $f$ is strict and continuous, and $f([\beta]) = [\emptyset]$. But $b \mathrel{\text{E}} \mathtt{F}'(\mathtt{G}'(f))([\beta])$, contradicting $p$-reflexivity. This proves (i).

By extensionality of the induced partial applicative structure and (i) we have for $x \neq \emptyset$, that $[\{\langle \gamma, c\rangle \mid \langle \gamma, c\rangle \mathrel{\text{E}} x\} \cup \{p\}] \equiv [x]$. Then $p \preceq d$, for any $d$. So (ii) is clear. Also by extensionality, if $\langle \gamma, c\rangle \preceq d$, then $\{\langle \gamma, c\rangle \mid \langle \gamma, c\rangle \mathrel{\text{E}} \{d\}\} \equiv \{d\}$, so $\exists \beta \exists b . d \preceq \langle \beta, b\rangle \preceq d$. Therefore, if for no $\langle \beta, b\rangle$ we have $d \preceq \langle \beta, b\rangle \preceq d$, we cannot have $\langle \gamma, c\rangle \preceq d$. But then, by extensionality, $\{d, p\} \equiv \{p\}$, so $d \preceq p$. Conversely, if $\langle \beta, b\rangle \preceq d$ and $d \preceq p$ we have $\langle \beta, b\rangle \preceq p$ by transitivity, contradicting (i). So $d \not\preceq p$, and we have proved also (iii).

The proof of (iv) is similar to the proof of (i) of theorem 2.3. We leave the details to the reader. □

As a corollary of the proof of the theorem we obtain:

**7.7.** PROPOSITION. *$\mathcal{G}_{\preceq} := (\mathcal{M}, \mathtt{F}', \mathtt{G}')$ is p-reflexive if and only if the preorder $(X, \preceq)$ satisfies*

- *(1) for all $\beta, b : \langle\beta, b\rangle \not\preceq p$;*
- *(2) for all $\alpha, \beta \neq \emptyset : \langle\beta, b\rangle \preceq \langle\alpha, a\rangle \quad \Rightarrow \quad \alpha \sqsubseteq \beta$ and $b \preceq a$.* □.

This shows (by taking equality (=) for $\preceq$) that we recover our 'plain' partial graph models precisely in case the mapping $\langle\cdot,\cdot\rangle : X^{<\omega}\backslash\emptyset \times X \to X$ we start from is an embedding, which moreover is non-surjective (as $p$ may not be in its range).

For an example of a structure satisfying the conditions of theorem 7.6 we refer the reader to Bethke(1987) (definitions 2.9, 2.10).

Like for the construction in section 2, we may alternatively give a description within the category **GRA** and its Karoubi-envelope **K(GRA)**: when $f$ is the idempotent mapping in **GRA** induced by a preorder $\preceq$ (see section 5), then $(f \Rightarrow f) := \mathbf{cur}(f \circ \mathbf{ev} \circ (id \times f))$ is an idempotent mapping $\mathcal{P}(X^{<\omega} \times X) \to \mathcal{P}(X^{<\omega} \times X)$, but *also* an idempotent mapping $\mathcal{P}(X^{<\omega}\backslash\emptyset \times X) \to \mathcal{P}(X^{<\omega}\backslash\emptyset \times X)$. Take a mapping $\langle\cdot,\cdot\rangle : X^{<\omega}\backslash\emptyset \times X \to X$ and fix $p \in X$. Then define $i : \mathcal{P}(X^{<\omega}\backslash\emptyset \times X) \to \mathcal{P}(X)$ by $x \mapsto \{a \mid a = \langle\beta, b\rangle \& (\beta, b) \in x\} \cup \{p\}$ and $j : \mathcal{P}(X) \to \mathcal{P}(X^{<\omega}\backslash\emptyset \times X)$ by $y \mapsto \{(\beta, b) \mid \langle\beta, b\rangle = a \in y\}$.

In **K(GRA)** we need to satisfy the following:

- (1) $(f \Rightarrow f)jfi(f \Rightarrow f) = (f \Rightarrow f)$;
- (2) for $x \neq \emptyset : fi(f \Rightarrow f)jf = f$.

It is not difficult to check that (1) and (2) hold if and only if the mapping $\langle\cdot,\cdot\rangle$ and the preorder $\preceq$ fulfill the conditions (i) - (iv) of theorem 7.6.

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The considerations in this report provide a full answer to a question of prof. Anne Troelstra, namely whether, and if so, in which sense, the constructions described in Bethke(1986,1987) can be considered as being canonical ways of obtaining extensional (partial) combinatory algebras. We'd like to thank him and Inge Bethke for their patient attention and useful comments.

## References

H.P. BARENDREGT[1984]
*The Lambda Calculus. Its Syntax and Semantics.* Nort-Holland.

I. BETHKE[1986]
How to construct extensional combinatory algebras. *Indagationes Mathematicae 89*: 243-257.

I. BETHKE[1987]
On the existence of extensional partial combinatory algebras. *Journal of Symbolic Logic 52-3.*

S. HAYASHI[1985]
Adjunction of semi-functors: categorical structures in non-extensional lambda calculus. *Theoretical Computer Science 45*: 95-104.

R. HOOFMAN[1990/1]
Continuous information systems. *Department of Computer Science, Utrecht University, preprint RUU-CS-90-25*

R. HOOFMAN[1990/2]
Linear logic, domain theory and semi-functors. *Department of Computer Science, Utrecht University, preprint RUU-CS-90-34.*

R. HOOFMAN[1990/3]
A note on semi-adjunctions. *Department of Computer Science, Utrecht University, preprint RUU-CS-90-41*

R. HOOFMAN[1991]
Ph.D.-thesis. *Department of Computer Science, Utrecht University.*

J.L. KRIVINE[1990]
*Lambda-calcul, Types et Modèles.* Masson.

S. MACLANE[1971]
*Categories for the Working Mathematician.* Springer Verlag, New York.

H. SCHELLINX[1991]
Isomorphisms and non-isomorphisms of graph models. *Journal of Symbolic Logic 56.*

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