Institute for Language, Logic and Information

FINITE TYPE STRUCTURES WITHIN COMBINATORY ALGEBRAS

Inge Bethke

ITLI Prepublication Series
for Mathematical Logic and Foundations ML-91-06

University of Amsterdam

The ITLI Prepublication Series

1986
86-01 The Institute of Language, Logic and Information
86-02 Peter van Emde Boas — A Semantical Model for Integration and Modularization of Rules
86-03 Johan van Benthem — Categorial Grammar and Lambda Calculus
86-04 Reinhard Muskens — A Relational Formulation of the Theory of Types
86-05 Kenneth A. Bowen, Dick de Jongh — Some Complete Logics for Branched Time, Part I Well-founded Time, Forward looking Operators
86-06 Johan van Benthem — Logical Syntax

1987
87-01 Jeroen Groenendijk, Martin Stokhof — Type shifting Rules and the Semantics of Interrogatives
87-02 Renate Bartsch — Frame Representations and Discourse Representations
87-03 Jan Willem Klop, Roel de Vrijer — Unique Normal Forms for Lambda Calculus with Surjective Pairing
87-04 Johan van Benthem — Polyadic quantifiers
87-05 Víctor Sánchez Valencia — Traditional Logicians and de Morgan's Example
87-06 Eleonore Oversteegen — Temporal Adverbials in the Two Track Theory of Time
87-07 Johan van Benthem — Categorial Grammar and Type Theory
87-08 Renate Bartsch — The Construction of Properties under Perspectives
87-09 Herman Hendriks — Type Change in Semantics: The Scope of Quantification and Coordination

1988
Logic, Semantics and Philosophy of Language:
LP-88-01 Michiel van Lambalgen — Algorithmic Information Theory
LP-88-02 Yde Venema — Expressiveness and Completeness of an Interval Tense Logic
LP-88-03 — Year Report 1987
LP-88-04 Reinhard Muskens — Going partial in Montague Grammar
LP-88-05 Johan van Benthem — Logical Constants across Varying Types
LP-88-06 Johan van Benthem — Semantic Parallels in Natural Language and Computation
LP-88-07 Renate Bartsch — Tenses, Aspects, and their Scopes in Discourse
LP-88-08 Jeroen Groenendijk, Martin Stokhof — Context and Information in Dynamic Semantics
LP-88-09 Theo M.V. Janssen — A mathematical model for the CAT framework of Eurotra
LP-88-10 Anneke Kleppe — A Blissymbolics Translation Program

Mathematical Logic and Foundations:
ML-88-01 Jaap van Oosten — Lifschitz' Realizabiility
ML-88-02 M.D.G. Swaen — The Arithmetical Fragment of Martin Löf's Type Theories with weak Σ-elimination
ML-88-03 Dick de Jongh, Frank Veltman — Provability Logics for Relative Interpretability
ML-88-04 A.S. Troelstra — On the Early History of Intuitionistic Logic
ML-88-05 A.S. Troelstra — Remarks on Intuitionism and the Philosophy of Mathematics

Computation and Complexity Theory:
CT-88-01 Ming Li, Paul M.B.Vitanyi — Two Decades of Applied Kolmogorov Complexity
CT-88-02 Michiel H.M. Smid — General Lower Bounds for the Partitioning of Range Trees
CT-88-03 Michiel H.M. Smid, Mark H. Overmars, Leen Torenvliet, Peter van Emde Boas — Maintaining Multiple Representations of Dynamic Data Structures
CT-88-04 Dick de Jongh, Lex Hendriks, Gerard R. Renardel de Lavalette — Computations in Fragments of Intuitionistic Propositional Logic
CT-88-05 Peter van Emde Boas — Machine Models and Simulations (revised version)
CT-88-06 Michiel H.M. Smid — A Data Structure for the Union-find Problem having good Single-Operation Complexity
CT-88-07 Johan van Benthem — Time, Logic and Computation
CT-88-08 Michiel H.M. Smid, Mark H. Overmars, Leen Torenvliet, Peter van Emde Boas — Multiple Representations of Dynamic Data Structures
CT-88-09 Theo M.V. Janssen — Towards a Universal Parsing Algorithm for Functional Grammar
CT-88-10 Edith Spaan, Leen Torenvliet, Peter van Emde Boas — Nondeterminism, Fairness and a Fundamental Analogy
CT-88-11 Sieger van Denneheuvel, Peter van Emde Boas — Towards implementing RL

Other prepublications:
X-88-01 Marc Jumelet — On Solovay's Completeness Theorem

1989
Logic, Semantics and Philosophy of Language:
LP-89-01 Johan van Benthem — The Fine-Structure of Categorial Semantics
LP-89-02 Jeroen Groenendijk, Martin Stokhof — Dynamic Predicate Logic, towards a compositional, non-representational semantics of discourse
LP-89-03 Yde Venema — Two-dimensional Modal Logics for Relation Algebras and Temporal Logic of Intervals
LP-89-04 Johan van Benthem — Language in Action
LP-89-05 Johan van Benthem — Modal Logic as a Theory of Information
LP-89-06 Andreja Prijatelj — Intensional Lambek Calculi: Theory and Application
LP-89-07 Heinrich Wansing — The Adequacy Problem for Sequential Propositional Logic
LP-89-08 Víctor Sánchez Valencia — Peirce's Propositional Logic: From Algebra to Graphs
LP-89-09 Zhisheng Huang — Dependency of Belief in Distributed Systems

Mathematical Logic and Foundations:
ML-89-01 Dick de Jongh, Albert Visser — Explicit Fixed Points for Interpretability Logic
ML-89-02 Roel de Vrijer — Extending the Lambda Calculus with Surjective Pairing is conservative
ML-89-03 Dick de Jongh, Franco Montagna — Rosser Orderings and Free Variables
ML-89-04 Dick de Jongh, Marc Jumelet, Franco Montagna — On the Proof of Solovay's Theorem
ML-89-05 Rineke Verbrugge — Σ-completeness and Bounded Arithmetic
ML-89-06 Michiel van Lambalgen — The Axiomatization of Randomness
ML-89-07 Dirk Roorda — Elementary Inductive Definitions in HA: from Strictly Positive towards Monotone
ML-89-08 Dirk Roorda — Investigations into Classical Linear Logic
ML-89-09 Alessandra Carbone — Provable Fixed points in $I\Delta_0+\Omega_1$

Computation and Complexity Theory:
CT-89-01 Michiel H.M. Smid — Dynamic Deferred Data Structures
CT-89-02 Peter van Emde Boas — Machine Models and Simulations
CT-89-03 Ming Li, Herman Neuféglise, Leen Torenvliet, Peter van Emde Boas — On Space Efficient Simulations
CT-89-04 Harry Buhrman, Leen Torenvliet — A Comparison of Reductions on Nondeterministic Space
CT-89-05 Pieter H. Hartel, Michiel H.M. Smid, Leen Torenvliet, Willem G. Vree — A Parallel Functional Implementation of Range Queries
CT-89-06 H.W. Lenstra, Jr. — Finding Isomorphisms between Finite Fields
CT-89-07 Ming Li, Paul M.B. Vitanyi — A Theory of Learning Simple Concepts under Simple Distributions and Average Case Complexity for the Universal Distribution (Prel. Version)
CT-89-08 Harry Buhrman, Steven Homer, Leen Torenvliet — Honest Reductions, Completeness and Nondeterminstic Complexity Classes
CT-89-09 Harry Buhrman, Edith Spaan, Leen Torenvliet — On Adaptive Resource Bounded Computations
CT-89-10 Sieger van Denneheuvel — The Rule Language RL/1
CT-89-11 Zhisheng Huang, Sieger van Denneheuvel, Peter van Emde Boas — Towards Functional Classification of Recursive Query Processing

Other Prepublications:
X-89-01 Marianne Kalsbeek — An Orey Sentence for Predicative Arithmetic
X-89-02 G. Wagemakers — New Foundations: a Survey of Quine's Set Theory
X-89-03 A.S. Troelstra — Index of the Heyting Nachlass
X-89-04 Jeroen Groenendijk, Martin Stokhof — Dynamic Montague Grammar, a first sketch
X-89-05 Maarten de Rijke — The Modal Theory of Inequality
X-89-06 Peter van Emde Boas — Een Relationele Semantiek voor Conceptueel Modelleren: Het RL-project

1990 SEE INSIDE BACK COVER

Instituut voor Taal, Logica en Informatie
Institute for Language, Logic and Information

Faculteit der Wiskunde en Informatica
(Department of Mathematics and Computer Science)
Plantage Muidergracht 24
1018TV Amsterdam

Faculteit der Wijsbegeerte
(Department of Philosophy)
Nieuwe Doelenstraat 15
1012CP Amsterdam

FINITE TYPE STRUCTURES WITHIN COMBINATORY ALGEBRAS

Inge Bethke
Department of Mathematics and Computer Science
University of Amsterdam

ITLI Prepublications
for Mathematical Logic and Foundations
ISSN 0924-2090

Received April 1991

Finite Type Structures within Combinatory Algebras

Inge Bethke
Faculteit Wiskunde en Informatica, Universiteit van Amsterdam
Plantage Muidergracht 24, 1018 TV Amsterdam

Abstract

Inside a combinatory algebra, there are 'internal' versions of the finite type structure over ω, which form models of various systems of finite type arithmetic. This paper compares internal representations of the intensional and extensional functionals. If these classes coincide, the algebra is called ft-extensional. Some criteria for ft-extensionality are given and a number of well-known ca's are shown to be ft-extensional, regardless of the particular choice of representation for ω. In particular, D_A, P_ω, T^ω, H_ω and certain D_∞-models all share the property of ft-extensionality. It is also shown that ft-extensionality is by no means an intrinsic property of ca's, i.e. that there exists a very concrete class of ca's - the class of reflexive coherence spaces - no member of which has this property. This leads to a comparison of ft-extensionality with the well-studied notions of extensionality and weak extensionality. Ft-extensionality turns out to be completely independent.

1 Introduction

Combinatory logic contains the means for introducing natural numbers, functions on the natural numbers, functions on functions on the natural numbers and so on. Any model of combinatory logic, i.e. any *combinatory algebra*, comes therefore along with an internal representation of the natural numbers and finite type functionals. In particular, it comes along with an *intensional* and an *extensional finite type structure.*

At first sight, there is no immediate reason for expecting that these classes of functionals are closely related or even coincide. After all, extensional functionals are locally determined by the restricted graph of a lower type argument, whereas intensional functionals may also take other data, such as the different ways of representation, of a lower type functional into consideration in order to determine the value. In particular, as we are in an untyped structure, it is to be expected that the value an intensional functional is assigning to a lower type argument may depend on its extended graph, i.e. the applicative behaviour of that argument outside the type structure. However, behind the scenes, this turns out to be not always the case. In quite a number of very well-known models the two type structures do coincide. The purpose of this paper is therefore to compare internal representations of the intensional and extensional finite type structures. It is organized as follows:

In section 2, we collect some well-known notions and facts concerning combinatory algebras. The notion of a ca^+ is taken from Beeson [1985]. The section ends with the introduction of *finite type extensionality*: a combinatory algebra is called ft-extensional if the internal representations of the intensional and extensional finite type structure coincide.

In section 3, a criterion for ft-extensionality is given. Here, we only consider *monotone* combinatory algebras, i.e. combinatory algebras in which application is monotone. Such a combinatory algebra is called *finite type connected*, if every pair of equivalent functionals is connected via a zigzag consisting of functionals of the same type.

In section 4, we exploit the notion of ft-connectedness: D_A, P_ω, D_∞ obtained from a complete lattice, H_ω and T^ω all turn out to be ft-connected and therefore ft-extensional.

Finally in section 5, we show that ft-extensionality is by no means an intrinsic property of combinatory algebras. However, finding combinatory algebras that are not ft-extensional, does not seem to be an easy business. We had to resort to so-called *reflexive coherence spaces*, and it is worth pointing out that this kind of semantics was not conceived as models of combinatory logic or pure λ-calculus, but provided the first denotational model of second order λ-calculus. Section 5 also includes a comparison of ft-extensionality with the well-studied notions of extensionality and weak extensionality. It turns out to be completely independent.

2 Preliminaries

To fix our terminology and notation, we shall collect in this preliminary section a few well-known notions and facts.

Definition 2.1 (i) An *applicative structure* is a structure $(A, *)$ with $*$ a binary operation on A, called *application.*
(ii) A *combinatory algebra* (ca) is a structure $(A, *, K, S)$ with $(A, *)$ an applicative structure and $K, S \in A$ such that for all $a, a', a'' \in A$

(1) $Kaa' = a$,

(2) $Saa'a'' = aa''(a'a'')$.

As in algebra $a * a'$ is usually written as aa' and $(...((a_1a_2)a_3)...a_n)$ will be abbreviated by $a_1a_2...a_n$.
(iii) A *ca*$^+$ is a structure $(A,\ *,\ K,\ S,\ 0,\ S_N,\ P_N,\ D,\ N)$ with $(A,\ *,\ K,\ S)$ a ca, $0, S_N, P_N, D \in A$ and $N \subseteq A$ satisfying

(3) $0 \in N \ \wedge\ \forall a \in N(\, S_N a \in N \ \wedge\ P_N(S_N a) = a \ \wedge\ S_N a \neq 0\,)$,

(4) $\forall a \in N(\, a \neq 0 \rightarrow P_N a \in N \ \wedge\ S_N(P_N a) = a\,)$,

(5) $\forall a, a' \in N \forall b, b' \in A(\, Dbb'aa = b' \ \wedge\ (\, a \neq a' \rightarrow Dbb'aa' = b\,)\,)$.□

A common and important feature of nontrivial ca's , i.e. ca's the cardinality of which exceeds 1, is that in them one can define the additional combinators $0, S_N, P_N$ and D with the aid of the combinators K and S. These are standard tricks in combinatory logic of which we shall now give a flavour (cf. also Barendregt [1984, ch.6,§2]).

Proposition 2.2 Every nontrivial ca $M = (A, *, K, S)$ can be expanded to a ca$^+$.

PROOF. First recall that for any given term t over M one can define a term $\lambda x.t$ over M such that for all $a \in A$ one has that

$$(\dagger) \qquad (\lambda x.t)a = t[x := a]$$

(cf. e.g. Barendregt [1984, ch.5,§1). We now abbreviate

$$I := SKK,\ \top := K,\ \bot := KI,$$

and introduce the combinatory numerals, the successor S_N and predecessor P_N:

$$0 := I,$$

$$\overline{0} := 0,\ \overline{n+1} := \lambda y.y\bot\overline{n},\ N := \{\overline{n} | n \in \omega\}$$

$$S_N := \lambda xy.y\bot x,\ P_N := \lambda x.x\bot.$$

Obviously $0, S_N\overline{n}, P_N\overline{n+1} \in N$, $P_N(S_N\overline{n}) = \overline{n}$ and $S_N(P_N\overline{n+1}) = \overline{n+1}$. It is also readily checked that $S_N\overline{n} \neq 0$. Note, however, that nontriviality is essential for this inequality to hold.

To prove (5) we shall use for $t_1t_2t_3$ the suggestive notation *if* t_1 *then* t_2 *else* t_3, for if $t_1 \equiv \top$ (true) then $t_1t_2t_3 = t_2$, and if $t_1 \equiv \bot$ (false) then $t_1t_2t_3 = t_3$. Now observe that by (†) there exists the fixed point operator $FIX \equiv \lambda x.\chi\chi$ with $\chi \equiv \lambda z.x(zz)$ in M satisfying

$$FIXa = a(FIXa)$$

for all $a \in A$. If we thus put

$$R := \lambda uxyz.if\ z\top\ then\ x\ else\ y(P_N z)(uxy(P_N z))$$

and $REC := FIX(R)$ then REC behaves as a recursor, i.e.

$$
\begin{aligned}
RECaa'\overline{0} &= FIX(R)aa'\overline{0} \\
&= R(FIX(R))aa'\overline{0} \\
&= if\ \overline{0}\top\ then\ a\ else\ a'(P_N\overline{0})(FIX(R)aa'(P_N\overline{0})) \\
&= if\ \top\ then\ a\ else\ a'(P_N\overline{0})(FIX(R)aa'(P_N\overline{0})) = a
\end{aligned}
$$

and

$$
\begin{aligned}
RECaa'\overline{n+1} &= FIX(R)aa'\overline{n+1} \\
&= R(FIX(R))aa'\overline{n+1} \\
&= if\ \overline{n+1}\top\ then\ a\ else\ a'(P_N\overline{n+1})(FIX(R)aa'(P_N\overline{n+1})) \\
&= if\ \bot\ then\ a\ else\ a'\overline{n}(FIX(R)aa'\overline{n}) \\
&= a'\overline{n}(FIX(R)aa'\overline{n}) = a'\overline{n}(RECaa'\overline{n})
\end{aligned}
$$

Hence on the set N of numerals we have explicit definition (via λx) and primitive recursion; $Z := K0$ represents the zero-function and $\Pi_i^n := \lambda x_1 ... x_n.x_i$ a projection. We thus have all primitive recursive functions available and can therefore construct a term t such that $t\overline{n}\overline{m} = \overline{|n-m|}$. The numerical definition-by-cases operator D can then be defined by

$$D := \lambda xyuv.if\ tuv\top\ then\ x\ else\ y.\Box$$

The reason why we define the expansion separately is that we don't want to restrict ourselves in the choice of models by the special relationship between the additional constants and the combinators K, S.

Example 2.3 (Engeler [1981]) Let A be any nonempty set, and let $G(A)$ be the least set containing A and all ordered pairs (B, b) consisting of a finite set $B \subseteq G(A)$ and an element $b \in G(A)$, assuming that elements of A are distinguishable from ordered pairs. On $\mathcal{P}(G(A))$ one can define an application operation $*$ by

$$X * Y = \{b \mid \exists B \subseteq Y((B, b) \in X)\}.$$

$D_A = (\mathcal{P}(G(A)), *)$ is called the *graphmodel.* It can be made into a ca by taking e.g.

$$K := \{(B, (C, b)) \in G(A) \mid b \in B\},$$

$$S := \{(B, (C, (D, b))) \in G(A) \mid b \in BD(CD)\}.$$

D_A is clearly nontrivial and can thus be expanded to a ca^+.

Consider the special case where $A = \omega$. Here instead of appealing to the combinatorial construction in proposition 2.2 one can define N, 0, S_N, P_N and D directly by

$$
\begin{aligned}
N &= \{\{n\} \mid n \in \omega\}, \\
0 &= \{0\}, \\
S_N &= \{(\{n\}, n+1) \mid n \in \omega\}, \\
P_N &= \{(\{n+1\}, n) \mid n \in \omega\},
\end{aligned}
$$

$$D = \{(B,(C,(\{n\},(\{m\},b)))) \in G(\omega) \mid n,m \in \omega \wedge$$
$$((n = m \wedge b \in C) \vee (n \neq m \wedge b \in B))\}.$$

We leave the verification of 2.1 (1)-(5) to the reader. □

Inside a ca$^+$ M there are internal versions of finite type structures over ω, which form models of various systems of finite type arithmetic. In this paper we shall only consider the following *standard* finite type structures of pure types[1]:

Definition 2.4 *Pure types*, denoted by natural numbers, are 0 and with n also $n+1$. □

Intuitively, we think of 0 as the set of natural numbers; the induction step then permits the formation of the collection of functions from the elements of type n into ω. The following gives the two standard interpretations of pure types.

Definition 2.5 Let M be a ca$^+$.
(i) The *intensional type structure over* M, $IT(M)$, is the collection $< IT_n >_{n\in\omega}$ where

$$IT_0 = N,$$

$$IT_{n+1} = \{a \in A \mid \forall a' \in IT_n(\, aa' \in N\,)\}.$$

(ii) The *extensional finite type structure over* M, $ET(M)$, is the collection $< ET_n >_{n\in\omega}$ where

$$ET_0 = N,$$

$$ET_{n+1} = \{a \in A \mid \forall a', a''(\, a' =_n a'' \rightarrow aa' = aa''\,)\},$$

and

$$a =_0 a' \longleftrightarrow a, a' \in N \;\wedge\; a = a',$$

$$a =_{n+1} a' \longleftrightarrow a, a' \in ET_{n+1} \;\wedge\; \forall a'' \in ET_n(\, aa'' = a'a''\,).\square$$

Note that these type structures always coincide at level 0 and 1, i.e. $IT_0 = ET_0$ and $IT_1 = ET_1$, and that therefore, moreover, $ET_2 \subseteq IT_2$. What, however, can be said about the coincidence or divergence of these type structures at levels higher up in the hierarchy of types? Let us concider D_ω, as described in example 2.3:

Example 2.6 In D_ω one has that any type 2 object is extensional, i.e. $IT_2 \subseteq ET_2$, so that the two type structures also coincide at level 2. To understand this coincidence observe the following:
Any function $f : \omega \rightarrow \omega$ can be canonically represented in D_ω by the type

[1] For a treatment of arbitrary finite types, the reader is referred to Bethke [1988].

1 object $F \equiv \{(\{n\}, f(n)) | n \in \omega\}$, since by the definition of application $F * \{n\} = \{f(n)\}$. This representation, however, is by no means unique. If, for example, f is constant, it is also represented by $\{(\emptyset, f(0))\}$, and, in general, it is also irrelevant what pairs of the form (B, b), where B is neither empty nor a singleton, are contained in any representation of f. This means that every type 1 function f has a whole range of representations in D_ω. But note that if X_1 and X_2 both represent the same function, i.e. $X_1 =_1 X_2$, then $X_1 \cup X_2$ also represents that function, since application is additive in the first argument, i.e. $(X_1 \cup X_2) * Y = (X_1 * Y) \cup (X_2 * Y)$. Hence $X_1 =_1 X_1 \cup X_2 =_1 X_2$. Since application is monotone in the second argument, it then follows that any type 2 object must be extensional, i.e. if $Z \in IT_2$, and $X_1 =_1 X_2$, then $Z * X_1 = Z * X_2$; for $Z * X_1 \subseteq Z * (X_1 \cup X_2)$, $Z * X_2 \subseteq Z * (X_1 \cup X_2)$ and as the results of these application are all singletons, we have in fact that $Z * X_1 = Z * (X_1 \cup X_2) = Z * X_2$. $\square$

We shall see in the next section that this particular coincidence at level 2 extends in D_ω to all pure finite types and that D_ω is therefore *extensional on finite types.*

Definition 2.7 A ca$^+$ M is called *ft-extensional* (extensional on finite types) iff

$$IT(M) = ET(M). \square$$

In the next section we shall present sufficient conditions on ca$^+$'s in order to be ft-extensional.

3 FT-Connected ca$^+$'s

The crux of the proof that every type 2 object in D_ω is extensional is threefold: firstly, every pair of equivalent type 1 objects is bounded above by another type 1 object, namely it's union; secondly, application is monotone; thirdly, the numerals are *consistent,* i.e.

$$\forall X, Y \in N(X \subseteq Y \longrightarrow X = Y).$$

The latter property, however, is independent of the special choice of N in D_ω and is shared by all monotone ca$^+$'s, i.e. ca$^+$'s that are monotone as applicative structures.

Definition 3.1 A *monotone* applicative structure is a structure $(A, *, \sqsubseteq)$ where $(A, \sqsubseteq)$ is a poset satisfying for all $a, a', a'' \in A$

$$a \sqsubseteq a' \longrightarrow aa'' \sqsubseteq a'a'' \;\wedge\; a''a \sqsubseteq a''a'. \;\square$$

Lemma 3.2 Let $(M, \sqsubseteq)$ be a monotone ca$^+$. Then M satisfies the following *consistency property*

$$\forall a, a' \in N(a \sqsubseteq a' \longrightarrow a = a').$$

PROOF. Let $a, a' \in N$ be such that $a \sqsubseteq a'$. Assume $a \neq a'$. Then it follows from the monotonicity of M that $a' = Daa'aa \sqsubseteq Daa'aa' = a$. Hence $a' \sqsubseteq a$ and therefore $a = a'$. Contradiction. □

So monotonicity guarantees consistency, or to put it in another way, the incomparability of the numerals. Monotonicity, however, does not guarantee that, as in the case of D_ω, IT_2, or in general IT_n, is closed under the joins of equivalent objects. This property is an important ingredient of the proof that D_ω is ft-extensional. It can, however, in a more general setting be weakened to the notion of *finite type connectedness.*

Definition 3.3 Let $(M, \sqsubseteq)$ be a monotone ca$^+$. Then
(i) $a, a' \in IT_n$ are called *n-connected* iff there exists a sequence $a_0, ..., a_{m+1}$ in IT_n such that $a = a_0, a_{m+1} = a'$ and $a_i \sqsubseteq a_{i+1}$ or $a_i \sqsupseteq a_{i+1}$, for all $0 \leq i \leq m$. If a, a' are n-connected, we shall write this as $a \leftrightsquigarrow_n a'$.
(ii) M is called *ft-connected* iff for all $n \in \omega$ and all $a, a' \in IT_n$

$$IT_n = ET_n \ \wedge \ a =_n a' \rightarrow a \leftrightsquigarrow_n a'. \ \square$$

As the numerals are incomparable, one then has

Theorem 3.4 Let $(M, \sqsubseteq)$ be a ft-connected ca$^+$. Then M is ft-extensional.

PROOF. As $IT_0 = ET_0$, it is sufficient to prove that

$$IT_n = ET_n \longrightarrow IT_{n+1} = ET_{n+1}.$$

Assume $IT_n = ET_n$. Then $ET_{n+1} \subseteq IT_{n+1}$. For the converse let $a \in IT_{n+1}$ and $b, b' \in ET_n$ be such that $b =_n b'$. Since M is ft-connected there is a sequence $b_0, ..., b_{m+1} \in IT_n$ such that $b = b_0, b' = b_{m+1}$ and $b_i \sqsubseteq b_{i+1}$ or $b_{i+1} \sqsupseteq b_i$, for all $0 \leq i \leq m$. Then $ab_0, ..., ab_{m+1} \in N$ and, since application is monotone, $ab_i \sqsubseteq ab_{i+1}$ or $ab_{i+1} \sqsupseteq ab_i$, for all $0 \leq i \leq m$. Hence $ab = ab_0 = ... = ab_{m+1} = ab'$, by lemma 3.2. Thus $a \in ET_{n+1}$. □

Having seen that ft-connectedness is a sufficient condition on ca$^+$'s in order to be ft-extensional, we can also seek for sufficient conditions for ft-connectedness. The one we shall give below is again inspired by the algebraic structure of and the behaviour of application in D_ω.

Definition 3.5 A monotone applicative structure $M = (A, *, \sqsubseteq)$ is called *finitely additive in the first argument (fafa)* iff for all $a, a', a'' \in A$

(i) $a \sqcup a'$ exists in $(A, \sqsubseteq)$,

(ii) $(a \sqcup a')a'' = aa'' \sqcup a'a''$.□

Proposition 3.6 Let M be fafa and M' be a ca$^+$-expansion of M. Then M' is ft-connected.

PROOF. One proves by induction on n that for all $a, a' \in IT_n$

$$IT_n = ET_n \ \wedge \ a =_n a' \longrightarrow a \sqcup a' \in IT_n.$$

For the induction step let $a, a' \in IT_{n+1}$ be equivalent and assume that $IT_{n+1} = ET_{n+1}$. In order to prove that $a \sqcup a' \in IT_{n+1}$, it is sufficient to prove that $a \sqcup a' \in ET_{n+1}$, and for this it suffices to show that $(a \sqcup a')b = ab$, for all $b \in ET_n$. Thus let $b \in ET_n$. Then, as a and a' are equivalent, it follows that $ab = a'b$. Whence $(a \sqcup a')b = ab \sqcup a'b = ab$. □

Corollary 3.7 Let M be fafa and M' be a ca$^+$-expansion of M. Then M' is ft-extensional. □

4 Examples.

In this section we shall discuss several examples of ft-extensional ca$^+$'s such as D_A, P_ω, certain D_∞-models, H_ω and T^ω.

The Graphmodels D_A. Every graphmodel D_A is clearly fafa and thus ft-extensional. □

The Graphmodels P_ω.(Plotkin [1972], Scott [1975]) P_ω is a coded version of D_A and has been extensively studied in the context of models for the λ-calculus. Its universe is $\mathcal{P}(\omega)$ and application is defined by

$$X * Y = \{m \mid \exists e_n \subseteq Y(\,(n, m) \in X\,)\}$$

where $(.,.)$ is some bijective coding of pairs of natural numbers and $\{e_n \mid n \in \omega\}$ is some enumeration of the finite subsets of ω. The structure of these models, as has been shown by Baeten and Boerboom [1979], depends heavily on the specific coding used in the construction. Although P_ω-models and D_A-models are never isomorphic as ca's (see Longo [1983]), they enjoy the same sufficient properties in order to be ft-extensional: again P_ω is closed under unions and application satisfies $(X \cup Y)Z = XZ \cup YZ$. □

Additive Reflexive Complete Lattices.(Scott [1969]) The first structures used as a mathematical foundation for the semantics of the untyped λ-calculus were reflexive complete lattices. Let us briefly recall the key concepts.

Let A be a complete lattice. A subset $A' \subseteq A$ is *directed* if, for every finite set $A'' \subseteq A'$, there is an upper bound $a \in A'$ for A''. Given complete lattices A, B, a function $f : A \to B$ is said to be *Scott-continuous* if f is monotone and preserves joins of directed subsets of A. $[A \to B]$, the set of Scott-continuous functions between A and B, partially ordered pointwise is then itself a complete lattice with $\bigsqcup \mathcal{F} = \lambda a. \bigsqcup\{f(a) | f \in \mathcal{F}\}$.

A *reflexive* complete lattice is a triple $< A, F, G >$ with A a complete lattice such that the set of Scott-continuous self-maps, $[A \to A]$, is a retract of A via

F, G, i.e. $F : A \to [A \to A]$, $G : [A \to A] \to A$ are Scott-continuous maps such that $F \circ G = id_{[A \to A]}$. These structures define in a natural way λ-models where the application operation $*$ is given by

$$a * a' = F(a)(a').$$

As $*$ is continuous (and hence monotone), $(A, *)$ can be made into a monotone ca by defining the combinators in terms of λ-abstraction as follows:

$$S = \lambda xyz.xz(yz) \qquad K = \lambda xy.x$$

One can restrict the class of reflexive complete lattices by the additional requirement $id_A \sqsubseteq G \circ F$. This yields the class of so-called *additive* reflexive complete lattices the members of which, as we shall show below, are fafa and therefore enjoy the property of ft-extensionality.

Lemma 4.1 Let $< A, F, G >$ be an additive reflexive complete lattice. Then F preserves $\sqcup$, i.e.

$$\forall a, a' \in A \; F(a \sqcup a') = F(a) \sqcup F(a').$$

PROOF. Let $a, a' \in A$. Since F is monotone, we have $F(a), F(a') \sqsubseteq F(a \sqcup a')$. Now suppose that $f \in [A \to A]$ is an arbitrary upper bound of $\{F(a), F(a')\}$, i.e. $F(a) \sqsubseteq f$ and $F(a') \sqsubseteq f$. Thus

$$a \sqsubseteq G(F(a)) \sqsubseteq G(f)$$

and

$$a' \sqsubseteq G(F(a')) \sqsubseteq G(f),$$

since G is monotone and $< A, F, G >$ is additive. Therefore $a \sqcup a' \sqsubseteq G(f)$, whence $F(a \sqcup a') \sqsubseteq F(G(f)) = f$. This shows that $F(a \sqcup a')$ is indeed the least upper bound of $\{F(a), F(a')\}$. $\square$

Now let the ca M be obtained in the canonical way from an additive reflexive complete lattice $< A, F, G >$. Then

Theorem 4.2 Let M' be a ca$^+$-expansion of M. Then M' is ft-extensional.

PROOF. It suffices to prove that M is fafa. Clearly, $(A, *, \sqsubseteq)$ is monotone and, as A is a complete lattice, $a \sqcup a' \in A$, for all $a, a' \in A$. Moreover, by the preceding lemma we have that

$$\begin{aligned}(a \sqcup a')a'' &= F(a \sqcup a')(a'') \\ &= (F(a) \sqcup F(a'))(a'') \\ &= F(a)(a'') \sqcup F(a')(a'') \\ &= aa'' \sqcup a'a''. \; \square\end{aligned}$$

Note that this also covers Scott's famous inverse limit spaces D_∞ where the initial space D_0 is a complete lattice. $\Box$

All the examples discussed so far are complete lattices and ft-extensional by virtue of corollary 3.7. The situation is slightly more complicated with respect to the last two examples, the hypergraphmodel H_ω and the model T^ω.

The Hypergraphmodel H_ω.(Sanchis [1979]) Sanchis' H_ω is the monotone ca $(\mathcal{P}(\omega), *, \subseteq)$ where application is defined by

$$X * Y := \{m \,|\, \forall f \exists p \exists e_n \subseteq Y(<\overline{f}(p), n, m>\in X)\}.$$

Here, $< .,.,. >$ is some bijective coding of triples of natural numbers, $\{e_n \,|\, n \in \omega\}$ is some enumeration of the finite subsets of ω and, if f is a function from ω to ω, then $\overline{f}(p)$ is some code for the sequence $f(0), ..., f(p-1)$. H_ω is a complete lattice but not fafa.

Proposition 4.3 H_ω is not fafa.

PROOF. Put (assuming $e_0 = \emptyset$)

$$X := \{<\overline{f}(1), 0, 0> \,|\, f : \omega \to \omega \ \wedge \ f(0) > 0\} \,,$$

$$Y := \{<\overline{\lambda x.0}(1), 0, 0>\}$$

and let $Z \subseteq \omega$. Then
(i) $(X \cup Y)Z = \{0\}$, since, if $f(0) = 0$ then $<\overline{f}(1), 0, 0>\in Y \subseteq X \cup Y$, and, if $f(0) > 0$ then $<\overline{f}(1), 0, 0>\in X \subseteq X \cup Y$,
(ii) $XZ = \emptyset = YZ$, since $<\overline{\lambda x.0}(p), n, m>\notin X$, and $<\overline{\lambda x.1}(p), n, m>\notin Y$, for every $e_n \subseteq Z$, $p, m \in \omega$.
Whence $(X \cup Y)Z \neq XZ \cup YZ$. $\Box$

There exists, however, a closure function $\gamma : H_\omega \to H_\omega$ which associates with each subset of ω its closure under 'extensions' of triples while preserving its applicative behaviour.

Definition 4.4 If $X \in H_\omega$, define

$$\gamma(X) := \{<\alpha, n, m> \,|\, \exists \beta \preceq \alpha \exists e_k \subseteq e_n(<\beta, k, m>\in X)\},$$

where we let α, β, γ range over codes of finite sequences and write $\alpha \preceq \beta$ if α codes a sequence that is an initial segment of the sequence coded by β. $\Box$

γ as defined above has the following properties:

Proposition 4.5 For any $X, Y, Z \in \mathcal{P}(\omega)$:

(i) $X \subseteq \gamma(X)$

(ii) $\gamma(X)Z = XZ$,

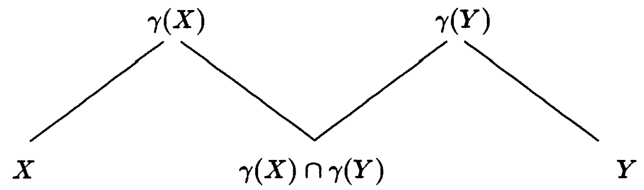
(iii) $XZ = YZ \rightarrow (\gamma(X) \cap \gamma(Y))Z = XZ$.

PROOF. We leave (i) and (ii) to the reader. For (iii) observe that, since $\gamma(X) \cap \gamma(Y) \subseteq \gamma(X)$, it follows that $(\gamma(X) \cap \gamma(Y))Z \subseteq \gamma(X)Z = XZ$, by monotonicity and (ii). For the converse, let $m \in XZ$ and f be any function from ω to ω. As $m \in YZ$, there are $p, q \in \omega$ and $e_n, e_l \subseteq Z$ such that $< \overline{f}(p), n, m > \in X$ and $< \overline{f}(q), l, m > \in Y$. Hence $< \overline{f}(r), k, m > \in \gamma(X) \cap \gamma(Y)$, for $r = max\{p, q\}$ and $e_k = e_n \cup e_l$. Whence $m \in (\gamma(X) \cap \gamma(Y))Z$. □

From proposition 4.5 it now follows that every pair of equivalent functionals is connected by a sequence of functionals of the same type.

Theorem 4.6 Let M be a ca$^+$-expansion of H_ω . Then M is ft-extensional.

PROOF. We shall prove that M is ft-connected. Equivalent type-0-objects are trivially 0-connected. Assume that $IT_{n+1} = ET_{n+1}$ and let $X, Y \in IT_{n+1}$ be equivalent. Then $\gamma(X), \gamma(Y) \in IT_{n+1}$, by 4.5(ii). Moreover, as $X =_{n+1} Y$, it follows from 4.5(iii) that $(\gamma(X) \cap \gamma(Y))Z = XZ$, for all $Z \in ET_n$. So $\gamma(X) \cap \gamma(Y) \in ET_{n+1} = IT_{n+1}$. Hence



constitutes an $n+1$-connection by 4.5(i). Therefore $X \leftrightsquigarrow_{n+1} Y$. □

The Model T^ω.(Plotkin [1978]) T^ω was first introduced by Plotkin. However, here we refer to the description given by Barendregt and Longo in [1980].

T^ω is a subset of $\mathcal{P}(\omega)^2$ equipped with a very special application operation. The importance of this model lies in the effectiveness properties of its semantics and the way its natural order matches the partial order on B, the λ-model of Böhm-like trees. We shall neither use nor comment on these properties. The only reason for including this model in our list of examples is that it is, as opposed to the preceding examples, not a complete lattice. For a thorough investigation of T^ω we refer the reader to Barendregt and Longo [1980].

The universe of T^ω is $\{< A, B > \mid A, B \in \mathcal{P}(\omega) \wedge A \cap B = \emptyset\}$. If $a \in T^\omega$ we write $a =< a_-, a_+ >$ and call $a \in T^\omega$ *finite* if $a_- \cup a_+$ is so. We let $\{e_n \mid n \in \omega\}$ be some enumeration of the finite elements of T^ω and $(.,.)$ be some bijective coding of pairs of natural numbers.

On T^ω one can define a partial order by

$$a \sqsubseteq b \longleftrightarrow a_- \subseteq b_- \text{ and } a_+ \subseteq b_+.$$

It is readily checked that $(T^\omega, \sqsubseteq)$ forms a complete partial order with bottom $< \emptyset, \emptyset >$ and $\bigsqcup D =< \cup\{d_- \mid d \in D\}, \cup\{d_+ \mid d \in D\} >$, for directed $D \subseteq T^\omega$.

T^ω belongs to the class of reflexive complete partial orders and defines - in the same way as reflexive complete lattices - a λ-model where application is continuous, and therefore a monotone ca. $F : T^\omega \to [T^\omega \to T^\omega]$ and $G : [T^\omega \to T^\omega] \to T^\omega$ are defined as follows:

Definition 4.7 For $n, m \in \omega$, $a, b \in T^\omega$ and $f \in [T^\omega \to T^\omega]$, define

(i) $n \uparrow m \longleftrightarrow \exists a \in T^\omega(e_n \sqsubseteq a \ \wedge \ e_m \sqsubseteq a)$,

(ii) $D_{(n,2m+1)} := \{(n', 2m) \mid n' \uparrow n \ \wedge \ (n', 2m) \leq (n, 2m+1)\}$,
$D_{(n,2m)} := \{(n', 2m+1) \mid n' \uparrow n \ \wedge \ (n', 2m+1) \leq (n, 2m)\}$,

(iii) $(F(a)(b))_- := \{m \mid \exists e_n \sqsubseteq b((n, 2m) \in a_- \ \wedge \ D_{(n,2m)} \subseteq a_+)\}$,
$(F(a)(b))_+ := \{m \mid \exists e_n \sqsubseteq b((n, 2m+1) \in a_- \ \wedge \ D_{(n,2m+1)} \subseteq a_+)\}$,

(iv) $(G(f))_- := \{(n, 2m) \mid m \in (f(e_n))_-\} \cup \{(n, 2m+1) \mid m \in (f(e_n))_+\}$,
$(G(f))_+ := \{(n, 2m) \mid \exists(e_n \sqsubseteq e_l \ \wedge \ m \in (f(e_l))_+)\}$
$\cup\{(n, 2m+1) \mid \exists(e_n \sqsubseteq e_l \ \wedge \ m \in (f(e_l))_-)\}$. □

To prevent any misgivings as to the relationship between the sets D_n and the numerical definition-by-cases operator D, let us stress that there is none. We just keep close to the notations introduced in Barendregt and Longo [1980].

T^ω is not fafa. First of all, T^ω is not closed under finite joins: e.g. $< \emptyset, \{0\} >, < \{0\}, \emptyset > \in T^\omega$, but if $< \emptyset, \{0\} >, < \{0\}, \emptyset > \sqsubseteq a$, then $0 \in a_- \cap a_+$. But even if $a \sqcup b$ does exist it does not necessarily satisfy $(a \sqcup b)c = ab \sqcup bc$. Observe, however, that T^ω is closed under finite meets, i.e. for all $a, b \in T^\omega$

$$a \sqcap b =< a_- \cap b_-, a_+ \cap b_+ > \in T^\omega.$$

But $\sqcap$ does not in general satisfy $(a \sqcap b)c = ac \sqcap bc$ either, so that the whole enterprise is not merely a matter of reversing the order. Application does, however, preserve meets of certain elements which we, for the purpose of this paper, shall call *sober* and *saturated*.

Definition 4.8 For $a \in T^\omega$, define

(i) a is *sober* if, for all $m \in \omega$

$$m \in a_- \to D_m \subseteq a_+,$$

(ii) a is *saturated* if, for all $n, m, l \in \omega$

$$(n, m) \in a_- \ \wedge \ e_n \sqsubseteq e_l \to (l, m) \in a_-. \ \square$$

a is sober, if every $m \in a_-$ actually contributes to a's applicative behaviour, and it is saturated, if it is - in a certain way - upwards closed. These two properties are, in particular, shared by all canonically chosen representatives of continuous self-maps, i.e.

Lemma 4.9 Let $f \in [T^\omega \to T^\omega]$. Then $G(f)$ is sober and saturated.

PROOF. To prove that $G(f)$ is sober, let $m \in (G(f))_-$. Say $m = (n, 2l)$ (the case $m = (n, 2l+1)$ is proved similarly). Then $l \in (f(e_n))_-$. Now suppose that $(n', 2l+1)$ is an arbitrary element of $D_{(n,2l)}$. Then e_n and $e_{n'}$ have an upper bound in T^ω, and hence $e_n \sqcup e_{n'} \in T^\omega$. Say $e_n \sqcup e_{n'} = e_k$. Then $e_{n'} \sqsubseteq e_k$ and, since f is monotone, $l \in (f(e_n))_- \subseteq (f(e_k))_-$. Whence $(n', 2l+1) \in (G(f))_+$. The fact that $G(f)$ is saturated follows from the monotonicity of f. □

Lemma 4.10 For all $a, b, c \in T^\omega$:

(i) If a and b are both sober and saturated, then $(a \sqcap b)c = ac \sqcap bc$.

(ii) $(\,G(F(a)) \sqcap G(F(b))\,)c = ac \sqcap bc$

(iii) If a is sober and saturated, then

$$\forall c' \sqsubseteq c \; ac' = bc' \to (a \sqcap b)c = ac$$

(iv) $(\,G(F(a)) \sqcap a\,)b = ab$

PROOF. (i) Clearly $(a \sqcap b)c \sqsubseteq ac \sqcap bc$, since application is monotone. For the converse, let $m \in (ac \sqcap bc)_- = (ac)_- \cap (bc)_-$. Then there are $e_n, e_l \sqsubseteq c$ such that $(n, 2m) \in a_-, (l, 2m) \in b_-, D_{(n,2m)} \subseteq a_+$ and $D_{(l,2m)} \subseteq b_+$. Now put $e_k := e_n \sqcup e_l$. Then, as a and b are both saturated, it follows that $(k, 2m) \in a_-$ and $(k, 2m) \in b_-$. Moreover, by sobricity, we have that $D_{(k,2m)} \subseteq a_+$ and $D_{(k,2m)} \subseteq a_+$. Whence $(k, 2m) \in a_- \cap b_- = (a \sqcap b)_-$, and $D_{(k,2m)} \subseteq a_+ \cap b_+ = (a \sqcap b)_+$. Therefore $m \in ((a \sqcap b)c)_-$, since $e_k \sqsubseteq c$. Thus $(ac \sqcap bc)_- \subseteq ((a \sqcap b)c)_-$. $(ac \sqcap bc)_+ \subseteq ((a \sqcap b)c)_+$ is proved similarly.
For (ii) combine the preceding lemma with (i) and the fact that T^ω is reflexive, i.e. $F \circ G = id_{[T^\omega \to T^\omega]}$.
(iii) $(a \sqcap b)c \sqsubseteq ac$ follows by monotonicity. For the converse, let $m \in (ac)_-$. Then $m \in (bc)_-$ and therefore $(n, 2m) \in b_-$ and $D_{(n,2m)} \subseteq b_+$, for some $e_n \sqsubseteq c$. Then also $m \in be_n = ae_n$. Thus $(l, 2m) \in a_-$ and $D_{(l,2m)} \subseteq a_+$, for some $e_l \sqsubseteq e_n$. Hence $(n, 2m) \in a_-$ and $D_{(n,2m)} \subseteq a_+$, since a is sober and saturated. This proves that $(n, 2m) \in (a \sqcap b)_-$ and $D_{(n,2m)} \subseteq (a \sqcap b)_+$, for certain $e_n \sqsubseteq c$. Whence $m \in ((a \sqcap b)c)_-$. So $(ac)_- \subseteq ((a \sqcap b)c)_-$. $(ac)_+ \subseteq ((a \sqcap b)c)_+$ is proved similarly.
(iv) follows again from (iii). □

In the same way as in the case of H_ω, we can apply this lemma in order to show that every pair of equivalent functionals is connected by a sequence of functionals of the same type.

Theorem 4.11 Let M be a ca$^+$-expansion of T^ω. Then M is ft-extensional.

PROOF. Let $a, b \in IT_{n+1}$ be equivalent. Then

$$\begin{tikzcd}[column sep=small]
a \arrow[dr, dash] & & G(F(a)) \arrow[dl, dash] \arrow[dr, dash] & & G(F(b)) \arrow[dl, dash] \arrow[dr, dash] & & b \arrow[dl, dash] \\
& G(F(a)) \sqcap a & & G(F(a)) \sqcap G(F(b)) & & G(F(b)) \sqcap b &
\end{tikzcd}$$

constitutes a $n+1$-connection by lemma 4.10. Hence $a \leftrightsquigarrow_{n+1} b$. $\square$

In the next section we shall show that ft-extensionality is by no means an intrinsic property of ca's, i.e. that there exists a very concrete class of cas no member of which has this property.

5 Coherence Spaces

We have so far seen that quite a number of well-known ca's exhibit the property of ft-extensionality. The question then is whether this is necessarily so, i.e. whether ft-extensionality is an intrinsic property of ca's, or whether this is due to the particular choice of examples. In this section we shall show that the latter is the case, that, as a matter of fact, there is a whole class of ca's - the class of reflexive coherence spaces - each member of which is not ft-extensional.

There is a second question which we wish to address in this section, namely the question of the interdependencies between certain degrees of extensionality within the hierarchy of ca's.
In ca's, every algebraic function is representable. In so-called λ-algebras, this representation of algebraic functions can be given uniformly by the interpretation of λ-terms. In λ-models - or, equivalently, *weakly extensional* λ-algebras - there is even a canonical representation for every representable function, the association of which to any representable function being representable itself. In the ultimate structures of this hierarchy, *extensional* ca's, every representable function has a unique representative. We thus end up with three degrees of extensionality: ft-extensionality, weak extensionality and (global) extensionality. There is the well-known fact that extensionality implies weak extensionality. This however - and showing this is the second aim of this section - is the only dependency. That is,

$$
\begin{tikzcd}
\text{ft-extensionality} \arrow[drr, "/\!/" description, "(2)"'] & & \text{extensionality} \arrow[ll, "/\!/" description, "(3)"] \arrow[d, shift right] \\
& & \text{weak extensionality} \arrow[u, shift right, "/\!/" description, "(1)"']
\end{tikzcd}
$$

(1) is known from the literature: e.g. P_ω is weakly extensional without being extensional (cf. Barendregt [1984, ch.18,§1]), and in the previous section we have already encountered an example for (2): H_ω is ft-extensional but not weakly extensional (cf. Koymans [1984]). (3) will follow from the remainder of this section.

Let us first recall some of the definitions concerning coherence spaces and briefly review the theory of λ-structures obtained from them. Our exposition is based in part on Girard [1986] and Girard, Taylor and Lafont [1989].

Definition 5.1 A *coherence space* is a set (of sets) $\mathcal{A}$ which satisfies:

i) *Down-closure*: if $X \in \mathcal{A}$ and $X' \subseteq X$, then $X' \in \mathcal{A}$,

ii) *Binary completeness*: if $\mathcal{A}' \subseteq \mathcal{A}$ and if for all $X, X' \in \mathcal{A}' \, (X \cup X' \in \mathcal{A})$, then $\bigcup \mathcal{A}' \in \mathcal{A}$. □

In particular, we have the ***undefined object***, $\emptyset \in \mathcal{A}$. One may therefore consider $\mathcal{A}$ as a cpo (partially ordered by inclusion), and as such it is ***algebraic***, i.e. any set is the directed union of its finite subsets. So coherence spaces are a very special sort of cpos. However, they are better regarded as undirected graphs.

Elements of the set $\bigcup \mathcal{A}$ are called ***atoms***. This set will also be denoted by $|\mathcal{A}|$. The *compatibility relation* between atoms is defined by

$$a \stackrel{\frown}{\smile} a' \ (\mathrm{mod}\ \mathcal{A}) \text{ iff } \{a, a'\} \in \mathcal{A}.$$

This constitutes a reflexive symmetric relation on $|\mathcal{A}|$, so $(|\mathcal{A}|, \stackrel{\frown}{\smile})$ is a graph, called the *web* of $\mathcal{A}$.

The construction of the web of a coherence space is a bijection between coherence spaces and (reflexive symmetric) graphs. From the web one can recover the coherence space by

$$X \in \mathcal{A} \longleftrightarrow X \subseteq |\mathcal{A}| \ \wedge\ \forall a, a' \in X (a \stackrel{\frown}{\smile} a').$$

So a coherence space $\mathcal{A}$ is the set of all coherent subsets of $|\mathcal{A}|$.

Whereas in Scott-style domain theory the functions between domains are exactly those which preserve directed joins, this is no longer the case here.

Definition 5.2 Given two coherence space $\mathcal{A}$ and $\mathcal{B}$, a function f from $\mathcal{A}$ to $\mathcal{B}$ is *stable* if

i) if $X \subseteq X' \in \mathcal{A}$, then $f(X) \subseteq f(X')$ (monotonicity)

ii) if $\mathcal{A}'$ is a directed subset of $\mathcal{A}$, then $f(\bigcup \mathcal{A}') = \bigcup f(\mathcal{A}')$ (directed union)

iii) if $X, X', X \cup X' \in \mathcal{A}$, then $f(X \cap X') = f(X) \cap f(X')$ (stability). □

Whereas the first two conditions are entirely familiar from the topological setting, the third - the stability property itself - does not have any obvious topological significance. However, if the ordered sets $\mathcal{A}$ and $\mathcal{B}$ are considered as categories, then i) states that f is a functor, ii) that it preserves directed joins and iii) that it also preserves pullbacks.

Example 5.3 Clearly, every stable function is Scott-continuous. A typical example of a function which is continuous but not stable is the following: Let **2** be the coherence space obtained from the set of atoms $\{0, 1\}$ and the universal compatibility relation. **2** can be represented pictorially by

$$\begin{tikzcd} & \{0,1\} \arrow[dl, dash] \arrow[dr, dash] & \\ \{0\} \arrow[dr, dash] & & \{1\} \arrow[dl, dash] \\ & \emptyset & \end{tikzcd}$$

Let furthermore $\mathcal{B}$ be an arbitrary but not atomless coherence space and pick $b \in |\mathcal{B}|$. Define $f : \mathcal{A} \to \mathcal{B}$ by

$$f(X) = \begin{cases} \{b\} & \text{if } X \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

Then f is Scott-continuous, as it is monotonic and preserves directed unions. However, f does not meet the stability condition:

$$f(\{0\} \cap \{1\}) = f(\emptyset) = \emptyset \neq \{b\} = f(\{0\}) \cap f(\{1\}). \ \square$$

As such, the collection of stable functions from $\mathcal{A}$ to $\mathcal{B}$ is not presented as a coherence space. However, it can be considered as belonging to this very special class of spaces. Here the crucial observation is that for a given stable function, a fixed argument and a finite portion of its value there is a finite least part of that argument which suffices to give that value portion. Or loosely speaking, if one has some information on the output, one knows which part of the input was used to get it.

Lemma 5.4 (Normalisation Lemma) If f is a stable function from $\mathcal{A}$ to $\mathcal{B}$, $X \in \mathcal{A}$ and $b \in f(X)$, then there is a finite $Z \subseteq X$ such that

$$b \in f(Z) \ \wedge \ \forall Y \subseteq X(b \in f(Y) \longrightarrow Z \subseteq Y).$$

PROOF. See e.g. Girard [1986]. □

Since a stable function f from $\mathcal{A}$ to $\mathcal{B}$ is determined by its values on least finite sets, f has a unique graph representation, called *trace*. This gives a bijection $\mathcal{T}$ between the set of stable functions and the coherence space of traces with an obvious inverse $\mathcal{F}$ which maps traces onto stable functions.

Theorem 5.5 (Representation Theorem) Let $\mathcal{A}$ and $\mathcal{B}$ be coherence spaces and $\mathcal{A}_{fin}$ be the set of finite sets in $\mathcal{A}$.

i) Define a compatibility relation on $\mathcal{A}_{fin} \times |\mathcal{B}|$ by $(Z, b) \frown (Z', b')$ iff

   1. $Z \cup Z' \in \mathcal{A} \longrightarrow b \frown b'$,
   2. $Z \cup Z' \in \mathcal{A} \ \wedge b = b' \longrightarrow Z = Z'$.

   Moreover, let $[\mathcal{A} \rightarrow_s \mathcal{B}]$ be the set defined by

   $$X \in [\mathcal{A} \rightarrow_s \mathcal{B}] \longleftrightarrow X \subseteq \mathcal{A}_{fin} \times |\mathcal{B}| \ \wedge \ \forall x, x' \in X(x \frown x').$$

   Then $[\mathcal{A} \rightarrow_s \mathcal{B}]$ is a coherence space.

ii) Let f be a stable function from $\mathcal{A}$ to $\mathcal{B}$. Define the *trace* of f, $\mathcal{T}(f)$, by

   $$\mathcal{T}(f) = \{(Z, b) \in \mathcal{A}_{fin} \times |\mathcal{B}| \mid b \in f(Z) \wedge \forall Z' \subseteq Z(b \in f(Z') \longrightarrow Z' = Z)\}.$$

   Then $\mathcal{T}(f) \in [\mathcal{A} \rightarrow_s \mathcal{B}]$.

iii) Let $X \in [\mathcal{A} \rightarrow_s \mathcal{B}]$. For $Y \in \mathcal{A}$, define $\mathcal{F}(X)(Y)$ by

   $$\mathcal{F}(X)(Y) = \{b \in |\mathcal{B}| \mid \exists Y' \subseteq Y ((Y', b) \in X)\}.$$

   Then $\mathcal{F}(X)$ is a stable function from $\mathcal{A}$ to $\mathcal{B}$.

iv) $\mathcal{T}$ and $\mathcal{F}$ are mutually inverse constructions, i.e. for all stable functions f from $\mathcal{A}$ to $\mathcal{B}$ and all $X \in [\mathcal{A} \rightarrow_s \mathcal{B}]$ one has $f = \mathcal{F}(\mathcal{T}(f))$ and $X = \mathcal{T}(\mathcal{F}(X))$.

PROOF. See e.g. Girard [1986]. □

Being a coherence space, $[\mathcal{A} \rightarrow_s \mathcal{B}]$ is naturally ordered by inclusion. The bijection between $[\mathcal{A} \rightarrow_s \mathcal{B}]$ and the stable functions from $\mathcal{A}$ to $\mathcal{B}$ then induces an order relation on the set of stable functions by

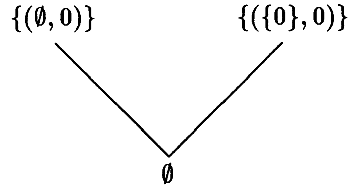
$$f \sqsubseteq g \longleftrightarrow \mathcal{T}(f) \subseteq \mathcal{T}(g).$$

$\sqsubseteq$ is strictly coarser than the pointwise ordering. Note that $f \sqsubseteq g$ implies $f(X) \subseteq g(X)$ for all X. However, the reverse is false:

Example 5.6 Let $\mathbf{1}$ be the coherence space consisting of $\emptyset$ and $\{0\}$. Then there are three stable functions from $\mathbf{1}$ to itself:

$$f_1(\emptyset) = f_1(\{0\}) = \emptyset$$
$$f_2(\emptyset) = f_2(\{0\}) = \{0\}$$
$$f_3(\emptyset) = \emptyset \;\; f_3(\{0\}) = \{0\}.$$

Their respective traces are $\mathcal{T}(f_1) = \emptyset$, $\mathcal{T}(f_2) = \{(\emptyset, 0)\}$ and $\mathcal{T}(f_3) = \{(\{0\}, 0)\}$. Typically, $f_3 \sqsubseteq f_2$ fails, while $f_3(X) \subseteq f_2(X)$ for $X \in \mathbf{1}$. Hence $[\mathbf{1} \to_s \mathbf{1}]$ can be represented by



Roughly speaking, a function which is $\sqsubseteq$-bigger is just 'wider' (in that more elements have at least a given value), whereas a pointwise-bigger function can be 'higher' as well.

Coherence spaces can be used to give a semantics to the untyped λ-calculus. Here one can proceed in the same way as in the case of reflexive complete lattices or reflexive complete partial orders, that is

Definition 5.7 Let $\mathcal{A}$ be a coherence space.

i) $\mathcal{A}$ is *reflexive* if $[\mathcal{A} \to_s \mathcal{A}]$ is a retract of $\mathcal{A}$, i.e. there are stable functions

$$F : \mathcal{A} \to [\mathcal{A} \to_s \mathcal{A}]$$
$$G : [\mathcal{A} \to_s \mathcal{A}] \to \mathcal{A}$$

such that $F \circ G = id_{[\mathcal{A} \to_s \mathcal{A}]}$.

ii) Let $\mathcal{A}$ be reflexive via the maps F and G.

1. For $X, Y \in \mathcal{A}$, define

$$X * Y = \{a \in |\mathcal{A}| \mid \exists Y' \subseteq Y(\,(Y', a) \in F(X)\,)\}.$$

2. Let ρ be a valuation in $\mathcal{A}$. Define the interpretation $[\![\;]\!]_\rho : \Lambda \to \mathcal{A}$ by induction as follows

$$[\![x]\!]_\rho = \rho(x)$$
$$[\![MN]\!]_\rho = [\![M]\!]_\rho * [\![N]\!]_\rho$$
$$[\![\lambda x.M]\!]_\rho = G(\mathcal{T}(\lambda X \in \mathcal{A}.[\![M]\!]_{\rho(x:=X)})).\ \square$$

Checking that $[\![\;]\!]_\rho$ is well-defined is a boring but straightforward exercise. For this and the following theorem we refer the reader to e.g. Girard [1986].

Theorem 5.8 Let $\mathcal{A}$ be a reflexive coherence space via F, G and let $\mathcal{M} = (\mathcal{A}, *, [\![\;]\!])$. Then

i) $\mathcal{M}$ is a λ-model.

ii) $\mathcal{M}$ is extensional iff $G \circ F = id_{\mathcal{A}}$. □

Whereas $*$ is continuous in both its arguments in the case of e.g. complete lattices, it is stable here.

Proposition 5.9 Let $\mathcal{A}$ be a reflexive coherence space via F, G. Then $*$ is stable in both its arguments. In particular, one has that for all $X, X', Y, Y' \in \mathcal{A}$, if $X \cup X', Y \cup Y' \in \mathcal{A}$, then

$$(X \cap X') * (Y \cap Y') = X * Y \cap X' * Y'.$$

PROOF. Since $X * Y = \mathcal{F}(F(X))(Y)$ and $\mathcal{F}(F(X))$ is stable, it follows that $*$ is stable in its second argument, and, since F is monotone and preserves directed unions, it follows that $*$ is monotone and directed union preserving in its first argument. In order to prove the stability condition for the first argument, it is sufficient to prove the second claim.
Thus let $X, X' \in \mathcal{A}$ and $Y, Y' \in \mathcal{A}$ be bounded above. Then

$$(X \cap X') * (Y \cap Y') \subseteq X * Y \cap X' * Y'$$

follows by monotonicity. Conversely, if $a \in X * Y \cap X' * Y'$, this means that $(Y_1, a) \in F(X)$, $(Y_2, a) \in F(X')$, for some $Y_1 \subseteq Y$ and $Y_2 \subseteq Y'$. Then $(Y_1, a) \supset\!\subset (Y_2, a)$, since $(Y_1, a), (Y_2, a) \in F(X \cup X')$. Whence $Y_1 = Y_2$, since $Y_1 \cup Y_2 \subseteq Y \cup Y' \in \mathcal{A}$. It follows that $Y_1 \subseteq Y \cap Y'$, and, as F is stable, also $(Y_1, a) \in F(X) \cap F(X') = F(X \cap X')$. Therefore $a \in (X \cap X') * (Y \cap Y')$. □

Coherence spaces and rigid embeddings form a category. Girard [1986] showed that the stable function space constructor $[\,.\, \to_s .\,]$, the cartesian product constructor $\times$ and the coalesced sum constructor $+$ are functorial in this category. Moreover, these functors are well-behaved in the sense that recursive equations in this category written using them can be solved using standard limit constructions. In particular, the equation $\mathcal{A} = [\mathcal{A} \to_s \mathcal{A}]$ has nontrivial solutions which, according to theorem 5.8, provide us with nontrivial extensional λ-models or equivalently, nontrivial extensional ca's.

Example 5.10 Let $\mathcal{A}_0$=**1** and $\mathcal{A}_{n+1} = [\mathcal{A}_n \to_s \mathcal{A}_n]$. One can define two stable functions ϕ_S, ϕ_P between $\mathcal{A}_0$ and $\mathcal{A}_1$ by $\phi_S(\emptyset) = \phi_P(\emptyset) = \emptyset$, $\phi_S(\{0\}) = \{(\emptyset, 0)\}$ and $\phi_P(\{0\}) = \{(\{0\}, 0)\}$. One then obtains two solutions $\mathcal{A}_S, \mathcal{A}_P$ to the equation $\mathcal{A} = [\mathcal{A} \to_s \mathcal{A}]$ by taking the inverse limit $\lim_{\leftarrow}(\mathcal{A}_n, \phi_n)$ with initial projection ϕ_S and ϕ_P, respectively. The subscripts S and P are reminiscent of the two analogue solutions which were given respectively by Scott [1972] and Park [1976] to the equation $D = [D \to D]$ in the category of Scott domains.

In the remainder of this section we shall now assume that $\mathcal{M}$ is a nontrivial (not necessarily extensional) ca which is obtained in the canonical way from a reflexive coherence space. That is, $\mathcal{M} = (\mathcal{A}, *)$, $|\mathcal{A}| \neq \emptyset$, $\mathcal{A}$ is a reflexive coherence space (via F,G) and $*$ is as in definition 5.7. We shall prove that $\mathcal{M}$ is not ft-extensional.

The first thing to note about $\mathcal{M}$ is that $\mathcal{A}$ is not closed under arbitrary unions or, to put it in another way, the web of $\mathcal{A}$ is not the universal one.

Lemma 5.11 There are $a, a' \in |\mathcal{A}|$ such that $\neg(\, a \supset\!\!\!\!\smile a'\,)$.

PROOF. If the web of $\mathcal{A}$ would be the universal one, then $\mathcal{A}$ would be a complete lattice, and, as $\mathcal{A}$ is reflexive, $[\mathcal{A} \to_s \mathcal{A}]$ would be a complete lattice too. It is therefore sufficient to observe that $[\mathcal{A} \to_s \mathcal{A}]$ is not closed under arbitrary unions.

Let $a \in |\mathcal{A}|$. Then $\{a\} \in \mathcal{A}$. So $(\emptyset, a), (\{a\}, a) \in \mathcal{A}_{fin} \times |\mathcal{A}|$ and therefore $\{(\emptyset, a)\}, \{(\{a\}, a)\} \in [\mathcal{A} \to_s \mathcal{A}]$. But as $(\emptyset, a)$ and $(\{a\}, a)$ are incompatible by 5.5.i)2., it follows that $\{(\emptyset, a), (\{a\}, a)\} \notin [\mathcal{A} \to_s \mathcal{A}]$. □

Let us now expand $\mathcal{M}$ to a ca$^+$, say $\mathcal{M}'$. Then, the next thing to note about such an expansion is that our natural numbers are not bounded above.

Proposition 5.12 $\forall X, X' \in N(\, X \cup X' \in \mathcal{A} \longrightarrow X = X'\,)$.

PROOF. Let $a, a' \in |\mathcal{A}|$ be incompatible and $X, X' \in N$ be bounded above. Assume $X \neq X' \in \mathcal{A}$ and put $Y = D * \{a\} * \{a'\} * X$. Then

$$\{a\} = Y * X' \subseteq Y * (X \cup X'),$$

$$\{a'\} = Y * X \subseteq Y * (X \cup X'),$$

since application is monotone. Thus a, a' are compatible. Contradiction. □

As a matter of fact, this already holds for finite approximants of the natural numbers. That is, we can carefully choose finite subsets of the natural numbers such that no pair in the resulting collection has an upper bound. This construction makes again essentially use of the fact that we have at our disposal incompatible atoms in $|\mathcal{A}|$ together with the D operator.

Definition 5.13 Choose incompatible $a, a' \in |\mathcal{A}|$ and put

$$Y_\Gamma = D * \{a\} * \{a'\} * 0.$$

Define $\Gamma \subseteq \mathcal{A}_{fin}$ by

$$Z \in \Gamma \longleftrightarrow (Z, a) \in F(Y_\Gamma) \;\vee\; (Z, a') \in F(Y_\Gamma). \;\; \square$$

Lemma 5.14
i) $\forall X \in N \exists Z \subseteq X(\, Z \in \Gamma\,)$
ii) $\forall Z, Z' \in \Gamma(\, Z \cup Z' \in \mathcal{A} \longrightarrow Z = Z'\,)$
iii) $\emptyset \notin \Gamma$

PROOF.(i) Depending on whether $X = 0$ or $X \neq 0$, one has that $Y_\Gamma * X = \{a\}$ or $Y_\Gamma * X = \{a'\}$. Thus $(Z, a) \in F(Y_\Gamma)$ or $(Z, a') \in F(Y_\Gamma)$, for some $Z \subseteq X$. Hence $Z \in \Gamma$, for some $Z \subseteq X$.
(ii) Let $(Z, a_1), (Z', a_2) \in F(Y_\Gamma)$ for $a_1, a_2 \in \{a, a'\}$. As $F(Y_\Gamma)$ is a set of compatible atoms, it follows that $(Z, a_1) \asymp (Z', a_2)$. Assume $Z \cup Z' \in \mathcal{A}$. Then $a_1 \asymp a_2$ and hence $a_1 = a_2$, since a and a' are incompatible. So $Z = Z'$.
(iii) If $(\emptyset, a) \in F(Y_\Gamma)$, then $a \in Y_\Gamma * \emptyset \subseteq Y_\Gamma * 0 = \{a'\}$, and if $(\emptyset, a') \in F(Y_\Gamma)$, then $a' \in Y_\Gamma * \emptyset \subseteq Y_\Gamma * 1 = \{a\}$, i.e. $a = a'$. Hence $\emptyset \notin \Gamma$. □

The existence of this set of finite approximants allows us to construct two $\sqsubseteq$-incomparable but equivalent type-1 functions. They will be both constant 0 on N, however, whereas one will be constant 0 on the whole of $\mathcal{A}$, the other will exhibit this behaviour only for those arguments which are approximated by members of Γ.

Proposition 5.15 There are equivalent $X, X' \in IT_1$ such that $X * \emptyset = 0$ and $X' * \emptyset = \emptyset$.

PROOF. Put $X := G(\mathcal{T}(\lambda Y.0))$ and define $f : \mathcal{A} \to \mathcal{A}$ by

$$f(Y) = \begin{cases} 0 & \text{if } \exists Z \in \Gamma\ Z \subseteq Y \\ \emptyset & \text{otherwise} \end{cases}$$

Observe that f is stable: for the stability condition apply lemma 5.14(ii). Thus $X' := G(\mathcal{T}(f)) \in \mathcal{A}$. Moreover, $X' * Y = 0$ for all $Y \in N$, by lemma 5.14(i). Thus $X =_1 X'$. But $X' * \emptyset = \emptyset$ by lemma 5.14(iii). □

Finally, we shall show that the finite type structures do not coincide in $\mathcal{M}'$. The reason for this disagreement is that we have in $\mathcal{M}'$ a type-2 functional which distinguishes type-1 functions according to how much input is needed to compute the value at 0, i.e. a functional that takes the trace of a type-1 function into consideration rather than its applicative behaviour on N.

Definition 5.16 Define $h : \mathcal{A} \to \mathcal{A}$ by

$$h(X) = \begin{cases} 0 & \text{if } \exists Z \in \Gamma(Z \subseteq X * \emptyset) \\ 1 & \text{if } \exists Z \in \Gamma(Z \not\subseteq X * \emptyset\ \wedge\ Z \subseteq X * 0) \\ \emptyset & \text{otherwise. } \square \end{cases}$$

Proposition 5.17 h is stable.

PROOF. First observe that, if $X, X' \in \mathcal{A}$ are bounded above, $Y, Y' \in \{\emptyset, 0\}$ and $Z, Z' \in \Gamma$, then

$$Z \subseteq X * Y \ \wedge \ Z' \subseteq X' * Y' \longrightarrow Z' \subseteq (X \cap X') * (Y \cap Y'). \quad (\dagger)$$

For, if $Z \subseteq X * Y$ and $Z' \subseteq X' * Y$", then $Z, Z' \subseteq (X \cup X') * (Y \cup Y')$, and hence, $Z = Z'$, by lemma 5.14(ii). Thus

$$Z' \subseteq X * Y \cap X' * Y' = (X \cap X') * (Y \cap Y')$$

by proposition 5.9.

It follows that h is well-defined (in ($\dagger$), take $X = X'$, $Y = \emptyset$ and $Y' = 0$).

Monotonicity: Observe that the only nontrivial case, where $X' \subseteq X$ and $h(X') = 1$, is again covered by ($\dagger$).

Directed union: Let $\mathcal{A}' \subseteq \mathcal{A}$ be directed. Then $\bigcup h(\mathcal{A}') \subseteq h(\bigcup \mathcal{A}')$, since h is monotone. The converse inclusion follows from the fact that $Z \in \Gamma$ is finite and the preservation of directed unions by $*$.

Stability: Let $X, X' \in \mathcal{A}$ be such that $X \cup X' \in \mathcal{A}$. Clearly, $h(X) \cap h(X') = h(X \cap X')$ if $h(X) = \emptyset$ or $h(X') = \emptyset$. So assume that $h(X), h(X') \in \{0, 1\}$. Then there are $Z, Z' \in \Gamma$ such that $Z \subseteq X * 0$ and $Z' \subseteq X' * 0$. Hence $Z' \subseteq (X \cap X') * 0$ by ($\dagger$). Therefore $h(X \cap X') \in \{0, 1\}$. Thus, as $0 \not\subseteq 1$, $1 \not\subseteq 0$ and $h(X \cap X') \subseteq h(X), h(X')$, it follows that $h(X) = h(X \cap X') = h(X')$. Whence $h(X \cap X') = h(X) \cap h(X')$. $\square$

We have now arrived at the position where we have all the necessary ingredients at our disposal in order to prove that

Theorem 5.18 $IT(\mathcal{M}') \neq ET(\mathcal{M}')$

PROOF. $G(\mathcal{T}(h)) \in \mathcal{A}$, since h is stable. Moreover, $G(\mathcal{T}(h)) \in IT_2$: let $X \in IT_1$. Then $X * 0 \in N$. Hence $Z \subseteq X * 0$ for some $Z \in \Gamma$. Thus

$$G(\mathcal{T}(h)) * X \in \{0, 1\}.$$

Now let $X, X' \in IT_1$ be as in proposition 5.15. Then $h(X) = 0$ and $h(X') = 1$. Thus

$$G(\mathcal{T}(h)) * X \neq G(\mathcal{T}(h)) * X',$$

and therefore $IT_2 \neq ET_2$. $\square$

What we have shown is that the class of λ-models obtained in the canonical way from reflexive coherence spaces lacks the property of ft-extensionality. And as this class is rich enough to include extensional models, we may conclude that

Corollary 5.19 Extensionality does not imply ft-extensionality, i.e. there is an extensional ca$^+$ $\mathcal{M}$ such that $IT(\mathcal{M}) \neq ET(\mathcal{M})$. $\square$

References

J. Baeten and B. Boerboom, *Ω can be anything it shouldn't be*, **Indagationes Mathematicae**, vol. 41 (1979), pp. 111-120.

H. P. Barendregt, **The Lambda Calculus. Its Syntax and Semantics**, 2^{nd} edition, North-Holland, Amsterdam, 1984.

H. P. Barendregt and G. Longo, *Equality of λ-terms in the model T^ω*, **To H. B. Curry: Essays on Combinatory Logik, Lambda-Calculus and Formalism**, Academic Press, New York and London, 1980, pp. 303-339.

M. J. Beeson, **Foundations of Constructive Mathematics**, Springer-Verlag, 1985.

I. Bethke, **Notes on Partial Combinatory Algebras**, Dissertation, Amsterdam, 1988.

E. Engeler, *Algebras and combinators*, **Algebra Universalis**, Vol. 13 (1981), pp. 389-392.

J-Y. Girard, *The system F of variable types, fifteen years later*, **Theoretical Computer Science**, vol. 45 (1986), pp. 159-192.

J-Y. Girard, P. Taylor, Y. Lafont, **Proofs and Types**, Cambridge University Press, Cambridge, 1989.

K. Koymans, **Models of the Lambda Calculus**, Dissertation, Mathematical Center Tracts, Amsterdam, 1984.

G. Longo, *Set-theoretical models of λ-calculus: theories, expansions, isomorphisms*, **Annals of Pure and Applied Logic**, vol. 24 (1983), pp. 153-188.

D. Park, **The Y-combinator in Scott's lambda calculus models** (revised version), Theory of Computation Report No. 13, University of Warwick, Dept. of Computer Science.

G. Plotkin, **A set-theoretical definition of application**, Memo MIP-R-95, School of AI, Edinburgh, 1972, 32 pp.

G. Plotkin, *T^ω as a universal domain*, **J. Comput. Syst. Sci**, vol. 17 (1978), pp. 209-236.

L. E. Sanchis, *Reducibilities in two models for combinatory logik*, **Journal of Symbolic Logic**, vol. 44 (1979), pp. 221-234.

D. S. Scott, **Models for the λ-calculus**, Manuscript, 1969, 53 pp.

D. S. Scott, *Continuous lattices*, in: F. W. Lawvere (ed.), **Toposes, Algebraic Geometry and Logic**, Lecture Notes in Mathematics 274, Springer-Verlag, 1972, pp.97-136.

D. S. Scott, *Lambda calculus and recursion theory*, **Proceedings of the 3^{rd} Scandinavian Logic Symposium**, North-Holland, Amsterdam, 1975, pp. 154-193.

The ITLI Prepublication Series

1990

Logic, Semantics and Philosophy of Language

LP-90-01 Jaap van der Does — A Generalized Quantifier Logic for Naked Infinitives
LP-90-02 Jeroen Groenendijk, Martin Stokhof — Dynamic Montague Grammar
LP-90-03 Renate Bartsch — Concept Formation and Concept Composition
LP-90-04 Aarne Ranta — Intuitionistic Categorial Grammar
LP-90-05 Patrick Blackburn — Nominal Tense Logic
LP-90-06 Gennaro Chierchia — The Variablity of Impersonal Subjects
LP-90-07 Gennaro Chierchia — Anaphora and Dynamic Logic
LP-90-08 Herman Hendriks — Flexible Montague Grammar
LP-90-09 Paul Dekker — The Scope of Negation in Discourse, towards a flexible dynamic Montague grammar
LP-90-10 Theo M.V. Janssen — Models for Discourse Markers
LP-90-11 Johan van Benthem — General Dynamics
LP-90-12 Serge Lapierre — A Functional Partial Semantics for Intensional Logic
LP-90-13 Zhisheng Huang — Logics for Belief Dependence
LP-90-14 Jeroen Groenendijk, Martin Stokhof — Two Theories of Dynamic Semantics
LP-90-15 Maarten de Rijke — The Modal Logic of Inequality
LP-90-16 Zhisheng Huang, Karen Kwast — Awareness, Negation and Logical Omniscience
LP-90-17 Paul Dekker — Existential Disclosure, Implicit Arguments in Dynamic Semantics

Mathematical Logic and Foundations

ML-90-01 Harold Schellinx — Isomorphisms and Non-Isomorphisms of Graph Models
ML-90-02 Jaap van Oosten — A Semantical Proof of De Jongh's Theorem
ML-90-03 Yde Venema — Relational Games
ML-90-04 Maarten de Rijke — Unary Interpretability Logic
ML-90-05 Domenico Zambella — Sequences with Simple Initial Segments
ML-90-06 Jaap van Oosten — Extension of Lifschitz' Realizability to Higher Order Arithmetic, and a Solution to a Problem of F. Richman
ML-90-07 Maarten de Rijke — A Note on the Interpretability Logic of Finitely Axiomatized Theories
ML-90-08 Harold Schellinx — Some Syntactical Observations on Linear Logic
ML-90-09 Dick de Jongh, Duccio Pianigiani — Solution of a Problem of David Guaspari
ML-90-10 Michiel van Lambalgen — Randomness in Set Theory
ML-90-11 Paul C. Gilmore — The Consistency of an Extended NaDSet

Computation and Complexity Theory

CT-90-01 John Tromp, Peter van Emde Boas — Associative Storage Modification Machines
CT-90-02 Sieger van Denneheuvel, Gerard R. Renardel de Lavalette — A Normal Form for PCSJ Expressions
CT-90-03 Ricard Gavaldà, Leen Torenvliet, Osamu Watanabe, José L. Balcázar — Generalized Kolmogorov Complexity in Relativized Separations
CT-90-04 Harry Buhrman, Edith Spaan, Leen Torenvliet — Bounded Reductions
CT-90-05 Sieger van Denneheuvel, Karen Kwast — Efficient Normalization of Database and Constraint Expressions
CT-90-06 Michiel Smid, Peter van Emde Boas — Dynamic Data Structures on Multiple Storage Media, a Tutorial
CT-90-07 Kees Doets — Greatest Fixed Points of Logic Programs
CT-90-08 Fred de Geus, Ernest Rotterdam, Sieger van Denneheuvel, Peter van Emde Boas — Physiological Modelling using RL
CT-90-09 Roel de Vrijer — Unique Normal Forms for Combinatory Logic with Parallel Conditional, a case study in conditional rewriting

Other Prepublications

X-90-01 A.S. Troelstra — Remarks on Intuitionism and the Philosophy of Mathematics, Revised Version
X-90-02 Maarten de Rijke — Some Chapters on Interpretability Logic
X-90-03 L.D. Beklemishev — On the Complexity of Arithmetical Interpretations of Modal Formulae
X-90-04 — Annual Report 1989
X-90-05 Valentin Shehtman — Derived Sets in Euclidean Spaces and Modal Logic
X-90-06 Valentin Goranko, Solomon Passy — Using the Universal Modality: Gains and Questions
X-90-07 V.Yu. Shavrukov — The Lindenbaum Fixed Point Algebra is Undecidable
X-90-08 L.D. Beklemishev — Provability Logics for Natural Turing Progressions of Arithmetical Theories
X-90-09 V.Yu. Shavrukov — On Rosser's Provability Predicate
X-90-10 Sieger van Denneheuvel, Peter van Emde Boas — An Overview of the Rule Language RL/1
X-90-11 Alessandra Carbone — Provable Fixed points in $I\Delta_0+\Omega_1$, revised version
X-90-12 Maarten de Rijke — Bi-Unary Interpretability Logic
X-90-13 K.N. Ignatiev — Dzhaparidze's Polymodal Logic: Arithmetical Completeness, Fixed Point Property, Craig's Property
X-90-14 L.A. Chagrova — Undecidable Problems in Correspondence Theory
X-90-15 A.S. Troelstra — Lectures on Linear Logic

1991

Mathematical Logic and Foundations

ML-91-01 Yde Venema — Cylindric Modal Logic
ML-91-02 Alessandro Berarducci, Rineke Verbrugge — On the Metamathematics of Weak Theories
ML-91-03 Domenico Zambella — On the Proofs of Arithmetical Completeness for Interpretability Logic
ML-91-04 Raymond Hoofman, Harold Schellinx — Collapsing Graph Models by Preorders
ML-91-05 A.S. Troelstra — History of Constructivism in the Twentieth Century
ML-91-06 Inge Bethke — Finite Type Structures within Combinatory Algebras

Computation and Complexity Theory

CT-91-01 Ming Li, Paul M.B. Vitanyi — Kolmogorov Complexity Arguments in Combinatorics
CT-91-02 Ming Li, John Tromp, Paul M.B. Vitanyi — How to Share Concurrent Wait-Free Variables
CT-91-03 Ming Li, Paul M.B. Vitanyi — Average Case Complexity under the Universal Distribution Equals Worst Case Complexity
CT-91-04 Sieger van Denneheuvel, Karen Kwast — Weak Equivalence
CT-91-05 Sieger van Denneheuvel, Karen Kwast — Weak Equivalence for Constraint Sets
CT-91-06 Edith Spaan — Census Techniques on Relativized Space Classes

Other Prepublications

X-91-01 Alexander Chagrov, Michael Zakharyaschev — The Disjunction Property of Intermediate Propositional Logics
X-91-02 Alexander Chagrov, Michael Zakharyaschev — On the Undecidability of the Disjunction Property of Intermediate Propositional Logics
X-91-03 V. Yu. Shavrukov — Subalgebras of Diagonizable Algebras of Theories containing Arithmetic
X-91-04 K.N. Ignatiev — Partial Conservativity and Modal Logics
X-91-05 Johan van Benthem — Temporal Logic
X-91-06 — Annual Report 1990
X-91-07 A.S. Troelstra — Lectures on Linear Logic, Errata and Supplement