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MODAL FRAME CLASSES revisited

Johan van Benthem

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These are notes for a talk given on 1 October 1991 at the Banach Semester in Warsaw on the interplay between algebraic and model-theoretic methods in Modal Logic. (This was to have been the main topic of the di-graph 'Blok & van Benthem 1977' which never left the cradle.) We start with some generalities from the Polish presentation, and then pass on to the main technical contribution offered.

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1 Algebra and Model Theory in Modal Logic

A modal propositional language can be interpreted in modal algebras, and modal logics then describe equational varieties. Initially, this well-known observation only resulted in some predictable 'algebraic completeness theorems'. At a next stage of research, however, natural constructions on algebras turned out to match natural model-theoretic constructions on possible worlds models, via the Stone Representation (going from algebras to models) and induced modal set algebras (going from models to algebras). Benefits for Modal Logic have been manifold, including the discovery of a new model construction ('ultrafilter extension') and proofs of new theorems via techniques from Universal Algebra relying on the theorems of Birkhoff and Jónsson (characterization of modally definable frame classes, classification of all incomplete modal logics). Eventually, this analogy was polished up into a full-fledged categorial duality by various authors (be it one which has been remarkably barren of new insights so far).

The interaction between algebraic methods and model theory in the area is a long and complicated phenomenon, whose history would merit a full-blown dissertation. (For instance, why did it take so long before the fundamental algebraic results of Jónsson & Tarski 1951 were recognized? These contain essentially the Henkin-style completeness theorems for modal S4 and S5 – long before the work of Kripke or Makinson in the sixties – via their associated Lindenbaum algebras and Stone spaces.) An obvious theme here is a 'friendly rivalry' between the two perspectives:

Can 'the algebra' be removed from standard modal results?

There are various possible answers here. First, by the general categorial duality, there must always be some way of doing this: the art is rather to find some *interesting* way, at the level of 'working mathematics'. Here is a 'Global Strategy' to the latter effect: give purely model-theoretic proofs for the algebraic tools *themselves*, so that any 'algebraic' argument involving them becomes 'model-theoretic' automatically. (Thus, van Benthem 1988 gives a purely model-theoretic proof of Jónssons's Theorem.) But one can also attempt a more informative 'Local Strategy', replacing specific algebraic arguments by different model-theoretic ones. Besides ideological purity, this exercise may have concrete benefits: the alternative proof may suggest different generalizations. Van Benthem & Blok 1977 was to have contained a number of open challenges of the local kind, such as the following simple fact: 'Above S4, any immediate predecessor of a tabular modal logic is tabular', whose only existing proof so far is algebraic. In this note, however, we shall eliminate a much better-known case of applied algebra. Even so, the 'constructive tension' between algebra and model theory in Modal Logic has been so fruitful that it might do harm to be too succesful here!

2 The Goldblatt & Thomason Theorem

Goldblatt and Thomason 1974 transformed the Birkhoff characterization of equational varieties into a structural characterization of modally definable frame classes. Their general version is rather 'proof-generated' and artificial, but the result becomes more elegant in an important special case:

Theorem An *elementary* class of frames is modally definable iff it is closed under the formation of generated subframes, disjoint unions, p-morphic images and anti-closed under ultrafilter extensions.

Proof Here is an outline of an algebraic proof, following van Benthem 1979:

- The given closure properties all hold for modal formulas.

- Conversely, one shows that **K** is definable as FRAME (Th_{mod} (**K**)).

The inclusion from left to right is immediate here. For the converse, consider any frame $\mathbb{F}$ such that $\mathbb{F} \models Th_{mod}(\mathbf{K})$. That is, the induced algebra *alg* ($\mathbb{F}$) validates the equational theory of the algebras of the frames in **K** : *alg* ($\mathbb{F}$) $\models Th_{eq}$ (*alg* (**K**)) . Then by Birkhoff's Theorem, *alg* ($\mathbb{F}$) is a homomorphic image of some subalgebra of a direct product of algebras alg ($\mathbb{G}_i$) for a family of frames $\mathbb{G}_i$ from **K** :

alg ($\mathbb{F}$) Hom Sub Prod : Π_i *alg* ($\mathbb{G}_i$) $\mathbb{G}_i \in \mathbf{K}$

Now, the latter product algebra is isomorphic to the algebra of the corresponding disjoint union of frames in **K** : *alg* ($+_i \mathbb{G}_i$) . And then, we apply the Stone Duality to the above sequence of connections – with subalgebras inducing p-morphic images, and homomorphic images (isomorphic embeddings as) generated subframes :

SR (*alg* ($\mathbb{F}$)) g.s. p-morph *SR* (*alg* ($+_i \mathbb{G}_i$))

Note that a frame of the form *SR* (*alg* ($\mathbb{G}$)) is in fact the ultrafilter extension *ue* ($\mathbb{G}$) . Now, the above closure conditions allow the following steps:

$\mathbb{F} \in \mathbf{K}$						$\mathbb{G}_i$'s $\in \mathbf{K}$
↑						↓
↑						$+_i \mathbb{G}_i \in \mathbf{K}$
ue ($\mathbb{F}$) $\in \mathbf{K}$	←	g.s. $\in \mathbf{K}$	←	p-morph	$\in \mathbf{K}$	*

There remains one gap $*$ to be filled in the right-hand corner, on the road toward the desired conclusion that $\mathbb{F} \in \mathbf{K}$:

$$+_i \mathbb{G}_i \in \mathbf{K} \quad \Rightarrow \quad ue\,(+_i \mathbb{G}_i) \in \mathbf{K} \quad ?$$

To close this gap, an additional model-theoretic observation is needed concerning ultrafilter extensions of arbitrary frames $\mathbb{F}$:

Lemma $\quad$ $ue\,(\mathbb{F})$ is a p-morphic image of some elementary equivalent of $\mathbb{F}$.

Then, since $\mathbf{K}$ is elementary and closed under p-morphic images, it must be closed under ultrafilter extensions, and we are done. ♥

The proof of the Lemma uses saturated models as a key tool, and it inspired the following purely model-theoretic analysis.

3 A New Model-Theoretic Proof

We try to make another 'round trip' in the crucial argument given above, but this time without the algebras.

Start with $\mathbb{F} \models Th_{mod}(\mathbf{K})$. We show that each generated subframe $\mathbb{F}_w \in \mathbf{K}$, and then apply a folklore result:

Fact $\quad$ Any frame $\mathbb{F}$ is a p-morphic image of the disjoint union $+_w \mathbb{F}_w$.

First, expand $\mathbb{F}_w$ to a model $(\mathbb{F}, \{\underline{A} \mid A \subseteq W\}, w)$ for a first-order language having new unary predicate letters $\underline{A}$ for each set $A \subseteq W$, as well as a constant $\underline{w}$ to denote the root w .

Now define (with some abuse of notation) the following 'modal description' of $\mathbb{F}_w$ as viewed from the root:

$$DESCR_{\mathbb{F}} \quad = \quad \text{all formulas of the forms } (n = 0, 1, 2, ...)$$
$$\forall y\,(R^n\underline{w}y \to (\underline{-A}y \leftrightarrow \neg\underline{A}y))$$
$$\forall y\,(R^n\underline{w}y \to (\underline{A\cap B}y \leftrightarrow (\underline{A}y \wedge \underline{B}y)))$$
$$\forall y\,(R^n\underline{w}y \to (\underline{m(A)}y \leftrightarrow \exists z\,(Ryz \wedge \underline{A}z))) \;.$$

We can also think of this as a set of genuine modal formulas, with proposition letters p_A for each subset A , and prefixes $\Box^n$ for all $n \in N$. For instance, the second formula would then read as $\Box^n\,(p_{A\cap B} \leftrightarrow (p_A \wedge p_B))$.

Next, set

$$\Delta \quad = \quad \mathrm{DESCR}_{\mathbb{F}} \quad \cup \quad \{\, \underline{A}\,\underline{w} \mid A \text{ contains } w \,\} .$$

<u>Claim</u> Δ is finitely satisfiable in (frames of) **K**.

<u>Proof</u> Suppose it were not. Then, for some finite conjunction δ from $\mathrm{DESCR}_{\mathbb{F}}$ and some atoms $\underline{A1}\,\underline{w}, \ldots, \underline{Ak}\,\underline{w}$, we have $\mathbf{K} \models \neg(\delta \wedge \bigwedge_i \underline{Ai}\,\underline{w})$, and hence the latter modal formula should be true in $\mathbb{F}_w$. But it is not, witness the obvious valuation verifying $\delta \wedge \bigwedge_i \underline{Ai}\,\underline{w}$ there: $V(\underline{A}) = A$. ♥

But then, as in the standard proof of the Compactness Theorem for elementary logic, the whole set Δ is simultaneously satisfiable (under a suitable valuation) in some frame $\mathbb{G}$ which is an <u>ultraproduct</u> of frames verifying the finite subsets of Δ. And because **K** is elementary, the ultraproduct $\mathbb{G} \in \mathbf{K}$.

Now, consider the <u>generated subframe</u> $\mathbb{G}_w$. It is still in **K**, and moreover, it satisfies all of Δ (by the modal form of the latter's principles). In fact, being generated from a root w (we use the same notation for roots throughout) it even satisfies the set Δ^+, which is Δ with the unrestricted universal form of the above descriptive principles, without the prefixes '$(R^n wy \rightarrow$'.

Next, take an *ω-saturated* elementary extension $\mathbb{H}$ of $\mathbb{G}_w$ (the latter still contains the world w, even though it need no longer be generated by it). As **K** is elementary, $\mathbb{H}$ must belong to it. What we now get is the following picture:

$\mathbb{F} \models \mathrm{Th}_{mod}(\mathbf{K})$		frames in **K**	
$\uparrow$	p-morphic image	$\downarrow$	ultraproduct
$+_w\, \mathbb{F}_w$		$\mathbb{G} \in \mathbf{K}$	
	disjoint union	$\downarrow$	generated subframe
		$\mathbb{G}_w$	
		$\downarrow$	elementary equivalent
		$\mathbb{H}$	

Here we need a 'bridge' from right to left. So, define the following map on $\mathbb{H}$:

$$U(x) \quad := \quad \{A \subseteq W \mid H \models \underline{A}\,[x]\} .$$

Fact U is a surjective p-morphism from $(\mathbb{H}, w)$ onto *ue* $(\mathbb{F}_w)$:

(1) U(x) is an ultrafilter

(2) U(w) is the principal ultrafilter $w^\sim$ on $\mathbb{F}_w$ generated by w

(3) $x R y \Rightarrow U(x) R^\sim U(y)$

(4) $U(x) R^\sim V \Rightarrow \exists y (Rxy \,\&\, U(y) = V)$

(5) U is onto.

Proof

(1) The construction of Δ^+ gives U(x) the characteristic properties of ultrafilters:

$$A \in U(x) \,\&\, B \in U(x) \text{ iff } A \cap B \in U(x)$$
$$A \notin U(x) \text{ iff } -A \in U(x).$$

(2) This assertion is evident by definition.

(3) If $A \in U(y)$, then $\mathbb{H} \models \underline{A}\,[y]$, and since $\mathbb{H} \models \underline{m(A)}x \leftrightarrow \exists z\,(Rxz \wedge \underline{A}z)$, we have $\mathbb{H} \models \underline{m(A)}\,[x]$ and hence also $m(A) \in U(x)$.

(4) Consider $\{Rxy\} \cup \{\underline{A}y \mid A \in V\}$. This set is finitely satisfiable in $\mathbb{H}$:

Since $A_1, \ldots, A_k \in V$, $\bigcap_i A_i \in V$, and therefore, $m(\bigcap_i A_i) \in U(x)$. So, $\mathbb{H} \models \underline{m(\bigcap_i A_i)}\,[x]$, and hence an R-successor y as required exists in $\mathbb{H}$, by the truth of Δ^+ in the latter model.

Therefore, by Saturation, a suitable y with $U(y) = V$ exists in $\mathbb{H}$ as well.

(5) U is surjective.

This may be seen by another instance of Saturation in $\mathbb{H}$. For each ultrafilter V in *ue* $(\mathbb{F}, w)$, the set $\{\underline{A}x \mid A \in V\}$ is finitely satisfiable:

Each finite intersection $\bigcap_i A_i$ is non-empty and hence it is verified by some R^n–successor of w in $\mathbb{F}_w$. But then, $A = m^n(\bigcap_i A_i)$ belongs to $w^\sim$, and hence $\underline{A}$ holds at w in $\mathbb{G}_w$ and hence in $\mathbb{H}$. By the truth of Δ^+, this yields a verifying point in $\mathbb{H}$.

Therefore, by Saturation, the whole above set must be satisfiable in $\mathbb{H}$ at a point x, which gives the required U-original for V. ♥

Finally, $\mathbb{F}_w$ must be in **K**, by following all closure steps in the above reasoning. ♥

Comments

- Each of the closure conditions in the Theorem is *necessary* for its proof. For instance, the non-modal class of all frames consisting of isolated points and reflexive points (some of each) demonstrates the necessity of having generated subframes: all other closure conditions are satisfied.
- The use of saturated models here goes back to Fine 1974 and van Benthem 1979, 1980 (see also Kracht 1991).

• Evidently, ultrafilter extensions are themselves like saturated models, but it is not so easy to spell out the exact sense in which this is true. (By an earlier Lemma, they are always p-morphic images of saturated frames. Is there some elegant converse?) Section 8 below has some further suggestions.

• The Theorem admits of various reformulations. E.g., its clauses for closure under generated subframes and p-morphic images would collapse into one if we were to allow *partial p-morphisms*, having an R-closed domain only. Moreover, the construction of the above frames $\mathbb{G}$ and $\mathbb{H}$ only requires an *ultraproduct* and an *ultrapower*, respectively. So, adding closure under ultraproducts for **K** characterizes those frame classes which are definable by means of 'elementary modal' formulas.

4 Extension to Richer Modal Formalisms

One virtue of the above model-theoretic argument is that it extends to richer modal formalisms (which need not have an obvious algebraic version). For instance, consider the modal language with an added *existential modality* "in at least one world".

Theorem The new modally definable elementary classes of frames are the ones closed under the formation of p-morphic images and anti-closed under ultrafilter extensions.

Proof In the above argument, one can now satisfy the full set $DESCR_{\mathbb{F}}$ at the start, without the restriction to iterated R–successors of some fixed w, while also satisfying the complete set descriptions of all worlds w, in some frame $\mathbb{G}$ of **K**. Hence, one can start with the frame $\mathbb{F}$ as it is. Then, the frame $\mathbb{H}$ can be taken to be a saturated elementary extension of $\mathbb{G}$, and the map U sends it p-morphically onto *ue* ($\mathbb{F}$). ♥

The next task would be to give a similar analysis for the still richer modal language with a *difference modality* "in at least one different world", which will give us more power of discrimination for the above p-morphism U (without turning it into an isomorphism though). Here, we only pose one obvious

Question What would be an appropriate modal formalism matching the elementary frame classes satisfying only anti-closure under ultrafilter extensions?

Finally, one way of extending this kind of reasoning to full monadic second-order logic may be found in van Benthem 1985, chapters XVII and XVIII.

5 Specialization to Finite Frames

On simpler frame classes, the above arguments may 'collapse'. In particular, ultrafilter extensions of *finite* frames are just those frames themselves. Thus, Rodenburg 1986 has a finite version of the Goldblatt-Thomason Theorem for an intuitionistic propositional language. This inspired the following results in van Benthem 1989:

Theorem On *finite transitive* frames, closure under disjoint unions, generated subframes and p-morphic images suffices for modal definability.

The relevant technique is the use of so-called 'Jankov formulas' completely describing a finite frame: which amounts to a finite version of the above $DESCR_{\mathbb{F}}$ construction. Some form of transitivity is essential here, as these three closure conditions do not suffice in general. (E.g., the one-point reflexive frame verifies the modal theory of the class of finite strict linear orders, without being obtainable from them via our three constructions.) For the general case, the above argument only yields a more proof-generated notion of 'local p-morphic image': being those frames which can be obtained as images of frames in the class **K** under 'n-step cut-off p-morphisms' for arbitrary n (details of the required definition may be read off in the above argument):

Theorem A class of finite frames is modally definable iff it is closed under local p-morphic images, generated subframes and finite disjoint unions.

Marcus Kracht has pointed out that these results already follow from the work on modal splitting algebras in Rautenberg 1980, Kracht 1990.

6 Generalization to General Frames

One reason why the general version of the Goldblatt & Thomason Theorem is less attractive is that arbitrary modal *frame* classes do not seem to be the most appropriate semantic realm for the modal language. The characterization of modally definable classes of *models* via 'invariance for bisimulation' (van Benthem 1977, Theorem 1.9) seems the truly fundamental result:

Theorem A class of models is modally definable iff it is closed under ultraproducts and bisimulations.

But even if the latter view is thought contentious, there may be a preference for characterizing modal classes of *general* frames, rather than full ones, being the more natural class of modal structures. The relevant result here may be found in Goldblatt 1975 (Theorem 12.11), using the appropriate sense of the frame constructions:

Theorem A class of general frames is modally definable iff it is closed under generated subframes, p-morphic images, disjoint unions as well as both closed and anti-closed under ultrafilter extensions.

Here all these notions are to be taken in the appropriate sense for general frames. Analyzing this same situation via the main argument in Section 3 above, one arrives at the following modifications. The description of the general frames $\mathbb{F}_w$ will only refer to sets in their admissible range. Then, to obtain the frame $\mathbb{G}_w$, one needs a special 'two-sorted' ultraproduct for general frames (cf. van Benthem 1985), defined in such a way that validity of universal monadic second-order sentences in all components is passed on to the whole ultraproduct. Next, the required saturated extension $\mathbb{H}$ can be obtained through an ultrapower of the same sort. The other steps in the argument remain essentially the same. What this means is that we get the above Theorem with the closure condition under ultrafilter extensions replaced by one for (two-sorted) ultraproducts. This may be understood as follows:

- ultrafilter extensions of general frames may be obtained as p-morphic images of general frame ultrapowers, via the argument in van Benthem 1979.
- two-sorted ultraproducts $\Pi_U \mathbb{F}_i$ may be related to the ultrafilter extension *ue* $(+_i \mathbb{F}_i)$, via the embedding used in the Main Lemma of van Benthem 1989. (The precise connection here must be left to further investigation.)

7 An Afterthought on Ultrafilter Extensions

In a way, ultrafilter extensions seem a very natural 'completion' of frames. The underlying general idea seems to be this. Fix some suitable formal language on frames. Each individual type (that is, each finitely satisfiable set of first-order formulas with one free variable) becomes a new individual. This will include all old individuals via their 'records' (i.e., the types realized by them), but also formerly merely 'potential' objects. The richness of the new individual domain will depend on the language, of course: ultrafilter extensions amount to a case where each set of former individuals has a predicate for its name, but one can usually do with less.

Remark For instance, in the Stone Representation for Boolean algebras, one can keep the cardinality of the underlying point set closer to that of the algebra being represented by taking a suitably ***saturated elementary submodel*** of the full ultrafilter model. (The price of this parsimony seems to be non-canonicity for the smaller model though.)

The art is then to find a good lifted definition for the old relations between individuals in the extended model. For instance, in the modal case, we set

RUV iff for each ϕ in V, $m(\phi)$ is in U,

where m is the usual set-theoretic projection sending X to $\{ y \in W \mid \exists x \in X: Ryx \}$. This coincides with the original relation R on the former individuals. An extension of this idea to predicates of an arbitrary arity is immediate:

$RUV_1 \ldots V_n$ iff for all $\phi_1 \in V_1, \ldots, \phi_n \in V_n$, $m^n(\phi_1, \ldots, \phi_n)$ is in U.

Here, m^n is some suitable n-place set operation in the original model $\mathbb{M}$ generalizing the usual modal m.

Of course, this definition need not make the extended model $\mathbb{M}^+$ elementarily equivalent to the old one. For instance, on the natural numbers $\mathbb{N}$ with the first-order language over $\{<, =\}$, one new reflexive point gets added at the end, so that the ordering is no longer irreflexive. Nevertheless, one may consider special ***fragments*** of the full first-order language guaranteeing the following 'harmony' for their unary formulas $\phi(x)$ with respect to unary types T:

$\mathbb{M}^+ \models \phi\,[T]$ iff $\phi \in T$.

Here is where 'modal fragments' come in. What may be allowed in constructing ϕ is:

- unary atoms
- Boolean operations
- restricted quantifiers of the forms
  $\exists y\,(R^2xy \wedge \psi(y))$, $\exists yz\,(R^3xyz \wedge \psi(y) \wedge \chi(z))$, etcetera.

Thus, modal fragments of first-order languages are precisely suitable for this kind of model-theoretic completion.

Eventually, it may also be worth exploring representations via arbitrary finitary types $T(x_1, \ldots, x_n)$. These are reminiscent of 'multi-dimensional' modal frames, where suitable definitions are to be found for the earlier relations in an arbitrary n-ary setting. (Cf. Shehtman & Skvortsov 1988, van Lambalgen 1991 on such finitary approaches.)

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