Institute for Language, Logic and Information

COMPARING THE THEORY OF REPRESENTATIONS AND CONSTRUCTIVE MATHEMATICS

A.S. Troelstra

ITLI Prepublication Series
for Mathematical Logic and Foundations ML-92-01

University of Amsterdam

The ITLI Prepublication Series

1986 86-01 The Institute of Language, Logic and Information
86-02 Peter van Emde Boas A Semantical Model for Integration and Modularization of Rules
86-03 Johan van Benthem Categorial Grammar and Lambda Calculus
86-04 Reinhard Muskens A Relational Formulation of the Theory of Types
86-05 Kenneth A. Bowen, Dick de Jongh Some Complete Logics for Branched Time, Part I Well-founded Time, Forward looking Operators
86-06 Johan van Benthem Logical Syntax
1987 87-01 Jeroen Groenendijk, Martin Stokhof Type shifting Rules and the Semantics of Interrogatives
87-02 Renate Bartsch Frame Representations and Discourse Representations
87-03 Jan Willem Klop, Roel de Vrijer Unique Normal Forms for Lambda Calculus with Surjective Pairing
87-04 Johan van Benthem Polyadic quantifiers
87-05 Víctor Sánchez Valencia Traditional Logicians and de Morgan's Example
87-06 Eleonore Oversteegen Temporal Adverbials in the Two Track Theory of Time
87-07 Johan van Benthem Categorial Grammar and Type Theory
87-08 Renate Bartsch The Construction of Properties under Perspectives
87-09 Herman Hendriks Type Change in Semantics: The Scope of Quantification and Coordination
1988 LP-88-01 Michiel van Lambalgen *Logic, Semantics and Philosophy of Language:* Algorithmic Information Theory
LP-88-02 Yde Venema Expressiveness and Completeness of an Interval Tense Logic
LP-88-03 Year Report 1987
LP-88-04 Reinhard Muskens Going partial in Montague Grammar
LP-88-05 Johan van Benthem Logical Constants across Varying Types
LP-88-06 Johan van Benthem Semantic Parallels in Natural Language and Computation
LP-88-07 Renate Bartsch Tenses, Aspects, and their Scopes in Discourse
LP-88-08 Jeroen Groenendijk, Martin Stokhof Context and Information in Dynamic Semantics
LP-88-09 Theo M.V. Janssen A mathematical model for the CAT framework of Eurotra
LP-88-10 Anneke Kleppe A Blissymbolics Translation Program
ML-88-01 Jaap van Oosten *Mathematical Logic and Foundations:* Lifschitz' Realizabiility
ML-88-02 M.D.G. Swaen The Arithmetical Fragment of Martin Löf's Type Theories with weak Σ-elimination
ML-88-03 Dick de Jongh, Frank Veltman Provability Logics for Relative Interpretability
ML-88-04 A.S. Troelstra On the Early History of Intuitionistic Logic
ML-88-05 A.S. Troelstra Remarks on Intuitionism and the Philosophy of Mathematics
CT-88-01 Ming Li, Paul M.B.Vitanyi *Computation and Complexity Theory:* Two Decades of Applied Kolmogorov Complexity
CT-88-02 Michiel H.M. Smid General Lower Bounds for the Partitioning of Range Trees
CT-88-03 Michiel H.M. Smid, Mark H. Overmars, Leen Torenvliet, Peter van Emde Boas
Maintaining Multiple Representations of Dynamic Data Structures
CT-88-04 Dick de Jongh, Lex Hendriks, Gerard R. Renardel de Lavalette Computations in Fragments of Intuitionistic Propositional Logic
CT-88-05 Peter van Emde Boas Machine Models and Simulations (revised version)
CT-88-06 Michiel H.M. Smid A Data Structure for the Union-find Problem having good Single-Operation Complexity
CT-88-07 Johan van Benthem Time, Logic and Computation
CT-88-08 Michiel H.M. Smid, Mark H. Overmars, Leen Torenvliet, Peter van Emde Boas Multiple Representations of Dynamic Data Structures
CT-88-09 Theo M.V. Janssen Towards a Universal Parsing Algorithm for Functional Grammar
CT-88-10 Edith Spaan, Leen Torenvliet, Peter van Emde Boas Nondeterminism, Fairness and a Fundamental Analogy
CT-88-11 Sieger van Denneheuvel, Peter van Emde Boas Towards implementing RL
X-88-01 Marc Jumelet *Other prepublications:* On Solovay's Completeness Theorem
1989 LP-89-01 Johan van Benthem *Logic, Semantics and Philosophy of Language:* The Fine-Structure of Categorial Semantics
LP-89-02 Jeroen Groenendijk, Martin Stokhof Dynamic Predicate Logic, towards a compositional, non-representational semantics of discourse
LP-89-03 Yde Venema Two-dimensional Modal Logics for Relation Algebras and Temporal Logic of Intervals
LP-89-04 Johan van Benthem Language in Action
LP-89-05 Johan van Benthem Modal Logic as a Theory of Information
LP-89-06 Andreja Prijatelj Intensional Lambek Calculi: Theory and Application
LP-89-07 Heinrich Wansing The Adequacy Problem for Sequential Propositional Logic
LP-89-08 Víctor Sánchez Valencia Peirce's Propositional Logic: From Algebra to Graphs
LP-89-09 Zhisheng Huang Dependency of Belief in Distributed Systems
ML-89-01 Dick de Jongh, Albert Visser *Mathematical Logic and Foundations:* Explicit Fixed Points for Interpretability Logic
ML-89-02 Roel de Vrijer Extending the Lambda Calculus with Surjective Pairing is conservative
ML-89-03 Dick de Jongh, Franco Montagna Rosser Orderings and Free Variables
ML-89-04 Dick de Jongh, Marc Jumelet, Franco Montagna On the Proof of Solovay's Theorem
ML-89-05 Rineke Verbrugge Σ-completeness and Bounded Arithmetic
ML-89-06 Michiel van Lambalgen The Axiomatization of Randomness
ML-89-07 Dirk Roorda Elementary Inductive Definitions in HA: from Strictly Positive towards Monotone
ML-89-08 Dirk Roorda Investigations into Classical Linear Logic
ML-89-09 Alessandra Carbone Provable Fixed points in $I\Delta_0+\Omega_1$
CT-89-01 Michiel H.M. Smid *Computation and Complexity Theory:* Dynamic Deferred Data Structures
CT-89-02 Peter van Emde Boas Machine Models and Simulations
CT-89-03 Ming Li, Herman Neuféglise, Leen Torenvliet, Peter van Emde Boas On Space Efficient Simulations
CT-89-04 Harry Buhrman, Leen Torenvliet A Comparison of Reductions on Nondeterministic Space
CT-89-05 Pieter H. Hartel, Michiel H.M. Smid, Leen Torenvliet, Willem G. Vree A Parallel Functional Implementation of Range Queries
CT-89-06 H.W. Lenstra, Jr. Finding Isomorphisms between Finite Fields
CT-89-07 Ming Li, Paul M.B. Vitanyi A Theory of Learning Simple Concepts under Simple Distributions and
Average Case Complexity for the Universal Distribution (Prel. Version)
CT-89-08 Harry Buhrman, Steven Homer, Leen Torenvliet Honest Reductions, Completeness and Nondeterminstic Complexity Classes
CT-89-09 Harry Buhrman, Edith Spaan, Leen Torenvliet On Adaptive Resource Bounded Computations
CT-89-10 Sieger van Denneheuvel The Rule Language RL/1
CT-89-11 Zhisheng Huang, Sieger van Denneheuvel, Peter van Emde Boas Towards Functional Classification of Recursive Query Processing
X-89-01 Marianne Kalsbeek *Other Prepublications:* An Orey Sentence for Predicative Arithmetic
X-89-02 G. Wagemakers New Foundations: a Survey of Quine's Set Theory
X-89-03 A.S. Troelstra Index of the Heyting Nachlass
X-89-04 Jeroen Groenendijk, Martin Stokhof Dynamic Montague Grammar, a first sketch
X-89-05 Maarten de Rijke The Modal Theory of Inequality
X-89-06 Peter van Emde Boas Een Relationele Semantiek voor Conceptueel Modelleren: Het RL-project
1990 *Logic, Semantics and Philosophy of Language*
LP-90-01 Jaap van der Does A Generalized Quantifier Logic for Naked Infinitives
LP-90-02 Jeroen Groenendijk, Martin Stokhof Dynamic Montague Grammar
LP-90-03 Renate Bartsch Concept Formation and Concept Composition
LP-90-04 Aarne Ranta Intuitionistic Categorial Grammar
LP-90-05 Patrick Blackburn Nominal Tense Logic
LP-90-06 Gennaro Chierchia The Variablity of Impersonal Subjects
LP-90-07 Gennaro Chierchia Anaphora and Dynamic Logic
LP-90-08 Herman Hendriks Flexible Montague Grammar
LP-90-09 Paul Dekker The Scope of Negation in Discourse, towards a flexible dynamic Montague grammar
LP-90-10 Theo M.V. Janssen Models for Discourse Markers
LP-90-11 Johan van Benthem General Dynamics
LP-90-12 Serge Lapierre A Functional Partial Semantics for Intensional Logic
LP-90-13 Zhisheng Huang Logics for Belief Dependence
LP-90-14 Jeroen Groenendijk, Martin Stokhof Two Theories of Dynamic Semantics
LP-90-15 Maarten de Rijke The Modal Logic of Inequality
LP-90-16 Zhisheng Huang, Karen Kwast Awareness, Negation and Logical Omniscience
LP-90-17 Paul Dekker Existential Disclosure, Implicit Arguments in Dynamic Semantics

Instituut voor Taal, Logica en Informatie
Institute for Language, Logic and Information

Faculteit der Wiskunde en Informatica
(Department of Mathematics and Computer Science)
Plantage Muidergracht 24
1018TV Amsterdam

Faculteit der Wijsbegeerte
(Department of Philosophy)
Nieuwe Doelenstraat 15
1012CP Amsterdam

COMPARING THE THEORY OF REPRESENTATIONS AND CONSTRUCTIVE MATHEMATICS

A.S. Troelstra
Department of Mathematics and Computer Science
University of Amsterdam

ITLI Prepublications
for Mathematical Logic and Foundations
ISSN 0924-2090

Received January 1992

To be published in
the Proceedings of the conference
Computer Science Logic'91
Springer Lecture Notes on Compter Science

Comparing the theory of representations and constructive mathematics

A.S.Troelstra*

Faculteit Wiskunde en Informatica
Universiteit van Amsterdam
Plantage Muidergracht 24
1018TV AMSTERDAM (NL)
email: anne@fwi.uva.nl

Abstract

The paper explores the analogy between reducibility statements of Weihrauch's theory of representations and theorems of constructive mathematics which can be reformulated as inclusions between sets. Kleene's function-realizability is the key to understanding of the analogy, and suggests an alternative way of looking at the theory of reducibilities.

1 Introduction

In a series of interesting papers Kreitz and Weihrauch ([KW1, KW2, KW3, W1, W2, WK]) have developed a theory of representations (henceforth "TR" for short), lifting notions of "effectiveness" on Baire space $\mathbb{B}$ to other structures, of cardinality not greater than $\mathbb{B}$, via the notion of "representation" which is analogous to Ershov's notion of numeration for countable structures. TR has been proposed as an alternative to the various forms of constructive mathematics and has been applied in complexity theory (see e.g.[M]). Just as classical recursive mathematics, TR introduces constructivity considerations in a classical setting.

Looking at examples of representations and the reducibility relations between them, one is struck by an obvious informal parallel or analogy between these results and certain well-known facts from constructive mathematics ("CM" for short) in the style of Bishop, Brouwer and Markov. (For general background on CM, and

*Paper presented at the conference Computer Science Logic '91, october 1991, Berne, Switzerland. I am indebted to K.Weihrauch for discussions and helpful comments on an earlier version of this paper.

references, see [TD].) In this paper we discuss and explain to some extent this parallelism, and show by means of some examples how results from constructive mathematics may be converted into results on the reducibility of representations, and conversely, how many reducibility results in the theory of representations might actually have been obtained by "translating" a theorem of constructive mathematics.

Although we set in this paper some steps on the road towards a more explicit formulation of the analogies, we are far from having demonstrated "equivalence" between TR and some form of CM. In order to make the idea of "equivalence" precise, one would first have to agree on an appropriate and natural formalism for CM and on a suitable formal setting for TR. Even under the assumption that such an agreement can be reached, I do not think it likely that full equivalence results. However, even the exploration of partial and limited analogies between TR and CM can be worthwhile, e.g. because known results from CM may show us what to expect and look for in the case of TR.

In the papers by Kreitz and Weihrauch we encounter many different representations for (classically) the same set of objects. This is reminiscent of early work in intuitionistic mathematics and Markov's constructive mathematics, where many different constructive analogues of the same classical notions are studied. There later developments showed that only very few of the many alternatives were well-behaved mathematically. (Bishop's constructive mathematics seems to have skipped this "proliferation-of-notions" stage.)

Similarly, it is to be expected that on further systematic development of TR, only a few of the many possible representententations for the same set will turn out to have good properties. Here too analogies might be exploited to advantage.

NOTATION. Let α, β, γ, possibly sub-or superscripted, be used for elements of $\mathbb{B}$. We assume that our standard coding of finite sequences of $\mathbb{N}$ is *onto* $\mathbb{N}$. $\mathrm{j}_1, \mathrm{j}_2$ are the inverses to a surjective pairing $(\ ,\ ) : \mathbb{N}^2 \to \mathbb{N}$; pairing and its inverses are lifted to $\mathbb{B}$ by $(\alpha, \beta) := \lambda n.(\alpha n, \beta n)$ etc. We write $\bar{\beta}n$ for the (code of) the sequence $\langle \beta 0, \beta 1, \ldots, \beta(n-1)\rangle$, $\bar{\beta}0 \equiv \langle\ \rangle$ (empty sequence). We write $\hat{n}$ as short for $\langle n \rangle$. $*$ denotes concatenation of codes.

$$\begin{array}{lcl} \alpha(\beta) = x & := & \alpha(\bar{\beta}(\min_z[\alpha(\bar{\beta}z) > 0]) = x + 1, \\ \alpha|\beta = \gamma & := & \forall x(\alpha(\hat{x} * \beta) = \gamma x). \end{array}$$

$\Phi_\alpha : \mathbb{B} \rightharpoonup \mathbb{N}$ and $\Psi_\alpha : \mathbb{B} \rightharpoonup \mathbb{B}$ are partial continuous functionals given by

$$\begin{array}{l} \Phi_\alpha(\beta) = n := \alpha(\beta) = n, \\ \Psi_\alpha(\beta) = \gamma := \alpha|\beta = \gamma. \end{array}$$

Let $\langle I_n \rangle_n$ be a standard enumeration of intervals with dyadic rational endpoints, such that $I_{(j,m)} := (\nu_{\mathrm{D}}(j) - 2^{-m}, \nu_{\mathrm{D}}(j) + 2^{-m})$ and ν_{D} is a standard enumeration of the dyadic rationals $k2^{-n}, k \in \mathbb{Z}, n \in \mathbb{N}$. Let $\langle r_n \rangle_n$ be a standard enumeration of $\mathbb{Q}$

2 The theory of representations: definitions and examples

DEFINITION. A *representation* of a set M is a partial surjective mapping $\delta : \mathbb{B} \rightharpoonup M$. We use $\delta, \delta', \delta''$ for arbitrary representations.

If δ, δ' are two representations of M, we say that δ is *reducible* to δ' iff there is a partial continuous $\Gamma : \mathbb{B} \rightharpoonup \mathbb{B}$ such that

$$\forall \alpha \in \mathrm{dom}(\delta)(\delta(\alpha) = \delta'(\Gamma\alpha))$$

If Γ is recursive, we say that δ is *computably reducible* to δ' (c-*reducible*). We write $\delta \preceq \delta'$, $\delta \preceq_{\mathrm{c}} \delta'$ for reducibility and c-reducibility respectively.

δ and δ' are *equivalent* (*c-equivalent*) iff $\delta \preceq \delta'$ and $\delta' \preceq \delta$ ($\delta \preceq_{\mathrm{c}} \delta'$ and $\delta' \preceq_{\mathrm{c}} \delta$). Notation: $\delta \equiv \delta'$ ($\delta \equiv_{\mathrm{c}} \delta'$).

Examples of representations

(A) $\alpha \mapsto \Phi_\alpha$, $\alpha \mapsto \Psi_\alpha$ are representations of certain classes $[\mathbb{B} \rightharpoonup \mathbb{N}]$, $[\mathbb{B} \rightharpoonup \mathbb{B}]$ of partial continuous functions.

(B) Let $\mathrm{En}(\alpha) := \{n : \exists m(\alpha m = n+1)\}$, then En is a representation of $\mathrm{P}(\mathbb{N})$, the powerset of $\mathbb{N}$, by enumerations; $\mathrm{En}^{\mathrm{c}}(\alpha) := \{n : \forall m(\alpha m \neq n+1)\}$.

(C) Let $\mathrm{Cf}(\alpha) := \{n : \alpha n = 0\}$, then Cf is a representation of $\mathrm{P}(\mathbb{N})$ by characteristic functions.

(D) We define a representation $\rho_<$ of $\mathbb{R}$ as follows.

$$\begin{aligned} \alpha \in \mathrm{dom}(\rho_<) &:= \exists m \in \mathbb{N} \forall n \in \mathrm{En}(\alpha)(r_n < m) \\ \rho_<(\alpha) &:= \mathrm{l.u.b.}\{r_n : n \in \mathrm{En}(\alpha)\} \end{aligned}$$

(E) Similarly for $\rho_>$, with (least) upper bound replaced by (greatest) lower bound.

(F) Let $\mathbb{Q}_n = \{k2^{-n} : k \in \mathbb{Z}\}$, $n \in \mathbb{N}$, and put

$$\begin{aligned} \alpha \in \mathrm{dom}(\rho_{\mathbb{R}}) &:= \forall n(r_{\alpha n} \in \mathbb{Q}_n \wedge |r_{\alpha n} - r_{\alpha(n+1)}| \leq 2^{-(n+1)}), \\ \rho_{\mathbb{R}}(\alpha) &:= \lim \langle r_{\alpha n} \rangle_n. \end{aligned}$$

(G) A complete, separable metric space $\mathcal{M}$ is a triple $(M, d, \langle p_n \rangle_n)$, where d is a distance function on the set M, and $\langle p_n \rangle_n$ is dense in M. $\mathcal{M}$ is in fact completely determined by the distance function on $\langle p_n \rangle_n$, that is to say $\mathcal{M}$ is determined by a function α defined on (coded) triples of natural numbers, such that

$$\forall k(|d(p_i, p_j) - r_{\alpha(i,j,k)}| < 2^{-k})$$

A representation δ_{NC} (Cf. [KW2, 4.12]) may be given for the points of $\mathcal{M}$ by

$$\begin{aligned} \mathrm{dom}(\delta_{\mathrm{NC}}) &:= \{\beta : \forall m(d(p_{\beta(m+1)}, p_{\beta m}) \leq 2^{-m}\} \\ &= \{\beta : \forall m, k(r_{\alpha(\beta(m+1), \beta m, k)} < 2^{-m} + 2^{-k})\} \\ \delta_{\mathrm{NC}}(\beta) &:= \lim \langle p_{\delta n} \rangle_n \end{aligned}$$

Less well-behaved is the representation δ_C, where the Cauchy modulus is left implicit:

$$\begin{array}{ll} \mathrm{dom}(\delta_C) & := \{\beta : \langle p_{\beta n}\rangle_n \text{ is a Cauchy sequence}\} \\ \delta_C(\beta) & := \lim\langle p_{\beta n}\rangle_n. \end{array}$$

(H) A slightly different representation $\delta_{\mathcal{M}}$ has as domain all of $\mathbb{B}$: let $\langle p_{\gamma(k,m,n)}\rangle_n$ be a standard enumeration (possibly with repetitions) of $\{p_i : d(p_i, p_m) < 2^{-k}\}$, and let

$$\delta_{\mathcal{M}}(\beta) := \lim\langle p_{\delta n}\rangle_n$$

where $\delta 0 = \beta 0$, $\delta(n+1) = \gamma(n, \delta n, \beta(n+1))$. (Special case: $\delta_{\mathbb{R}}$ with $\langle p_n\rangle_n$ a standard enumeration of $\mathbb{Q}$.)

(I) The (listable) open sets of $\mathbb{R}$ may be represented by ω with

$$\mathrm{dom}(\omega) := \mathbb{B},\ \omega(\alpha) := \bigcup\{I_n : n \in \mathrm{En}(\alpha)\}.$$

This suggests a complementary representation of closed sets as $\delta_{\mathrm{cl}}(\alpha) := \mathbb{R} \setminus \omega(\alpha)$.

It is not difficult to show that

(1) $\qquad \mathrm{Cf} \preceq \mathrm{En},\ \neg\mathrm{En} \preceq \mathrm{Cf},$

(2) $\qquad \rho_{\mathbb{R}} \preceq \rho_<,\ \rho_{\mathbb{R}} \preceq \rho_>,\ \neg\rho_< \preceq \rho_{\mathbb{R}},\ \neg\rho_> \preceq \rho_{\mathbb{R}}, \delta_{\mathbb{R}} \equiv \rho_{\mathbb{R}}.$

It is also easy to see that given two representations δ, δ' of a set, we can define a greatest lower bound $\delta \sqcap \delta'$ by

$$\begin{array}{l} \alpha \in \mathrm{dom}(\delta \sqcap \delta') := \mathrm{j}_1\alpha \in \mathrm{dom}(\delta) \wedge \mathrm{j}_2\alpha \in \mathrm{dom}(\delta') \wedge \delta(\mathrm{j}_1\alpha) = \delta'(\mathrm{j}_2\alpha), \\ (\delta \sqcap \delta')(\alpha) := \delta(\mathrm{j}_1\alpha)\ (= \delta'(\mathrm{j}_2\alpha)). \end{array}$$

Then

(3) $\qquad \mathrm{En} \sqcap \mathrm{En}^c \equiv \mathrm{Cf},\ \rho_< \sqcap \rho_> \equiv \rho_{\mathbb{R}}$

Remarks on CM versus TR. Results such as (1) and (2) invite comparison between CM and TR. In the classical setting of TR differences between representations δ, δ' of the same set X correspond to differences in the amount of information acessible (obtainable) from the representing functions (i.e. the amount of information encoded in the representing functions) concerning the elements of the set. In CM, where all objects are supposed to be given to us (are defined) by the presentation of certain bits of information, the δ, δ' correspond to different sets. Thus, in TR, Cf and En reflect different information concerning elements of $\mathrm{P}(\mathbb{N})$; in CM these correspond to decidable and enumerable sets respectively.

Similarly $\rho_{\mathbb{R}}$, $\rho_<$, $\rho_>$ correspond in CM to respectively the reals (in the sense of one of the standard definitions), "limits" of monotone increasing sequences in $\mathbb{Q}$ with upper bound, and "limits" of monotone decreasing sequences in $\mathbb{Q}$ with lower bound (see [TD]).

The analogue of (1) in CM is : decidable implies enumerable, but not conversely; the analogue of (2) is (approximately): every real is approximated by a bounded monotone increasing sequence in $\mathbb{Q}$ with upper bound, and by a bounded monotone decreasing sequence in $\mathbb{Q}$ with lower bound, but not conversely.

Results such as (3) also have their counterpart in CM: $\mathrm{En} \sqcap \mathrm{En}^c \equiv \mathrm{Cf}$ expresses that a set which is enumerable with enumerable complement is decidable (with an appeal to Markov's principle $\forall x(A \vee \neg A) \wedge \neg\neg\exists x A \to \exists x A$), and $\rho_< \sqcap \rho_> \equiv \rho_{\mathbb{R}}$ corresponds to the fact that if we have monotone increasing $\langle r_n \rangle_n \subset \mathbb{Q}$, monotone decreasing $\langle s_n \rangle_n \subset \mathbb{Q}$, $\forall nm(r_n < s_m)$, $\forall k \exists n(s_n - r_n < 2^k)$, then $\lim \langle r_n \rangle_n = \lim \langle s_n \rangle_n$ is a real.

As to the representations δ_{NC}, δ_{C}, it is not hard to see that for the reals $\delta_{\mathrm{NC}} \preceq \delta_{\mathrm{C}}$, but $\neg\delta_{\mathrm{C}} \preceq \delta_{\mathrm{NC}}$, while $\delta_{\mathrm{NC}} \equiv \delta_{\mathbb{R}}$. $\neg\delta_{\mathrm{C}} \preceq \delta_{\mathrm{NC}}$ expresses that the Cauchy modulus of a fundamental sequence of basis points cannot be computed continuously from the sequence itself; this reflects the distinction between F-numbers and FR-numbers in Markov's constructive mathematics (see [Sh]).

3 From CM to TR

The obvious parallels between certain facts from CM on the one hand, and certain reducibilities in TR on the other hand, raises the question whether we can perhaps systematically translate (certain classes of) statements of CM into reducibility statements of TR.

What the examples of analogies suggest, is the following. Let us assume that we have settled on a common formal language for (parts of) CM and TR, containing at least variables ranging over $\mathbb{N}$ and $\mathbb{B}$.

On the side of CM, we deal with sets $\mathcal{X}$, the elements of which are regarded as given by information encodable by elements of $\mathbb{B}$. That is to say, $\mathcal{X}$ may be thought of as a pair $(X, \delta : X \twoheadrightarrow X^*)$ with $X \subset \mathbb{B}$, $\delta(\alpha)$ the element described by α, δ surjective; this is nothing but a representation of X^*. Modulo an isomorphism, we can always take δ to be a map assigning to α its equivalence class $\alpha_\sim$, i.e. $\alpha \sim \beta \leftrightarrow \delta\alpha = \delta\beta$. (Examples: Cf can be presented as $(\mathbb{B}, \sim_0)$, where $\alpha \sim_0 \beta := \forall n(\alpha n = 0 \leftrightarrow \beta n = 0)$, and $\rho_{\mathbb{R}}$ is $(\mathrm{dom}(\rho_{\mathbb{R}}), \sim_1)$ where $\alpha \sim_1 \beta := \forall n(|r_{\alpha k} - r_{\beta k}| \leq 2^{-k+1}$.)

Let $\mathcal{Y} = (Y, \delta' : Y \twoheadrightarrow Y^*)$. $\mathcal{X} \subset \mathcal{Y}$ is expressible in codes as $\forall \alpha \in X \exists \beta \in Y(\delta\alpha = \delta'\beta)$. Within a classical setting, a more constructive reading of this can be enforced by requiring the existence of a continuous Γ such that $\forall \alpha \in X(\delta\alpha = \delta'(\Gamma\alpha))$ (i.e. $\delta \preceq \delta'$. So $\delta \preceq \delta'$ appears as the TR-analogue of the constructive reading of $\forall \alpha \in X \exists \beta \in Y(\delta\alpha = \delta'\beta)$.

It is a wellknown fact from the metamathematics of constructive systems, that there is a wide class of sets X and constructive formal systems S such that $\mathrm{S} \vdash \forall \alpha \in X \exists \beta \in Y A(\alpha, \beta) \Rightarrow \mathrm{S} \vdash \exists \Gamma \forall \alpha \in X\, A(\alpha, \Gamma\alpha)$ (Γ continuous) holds (more on this in the next section).

Thus, as a rule, a *constructive* proof of an inclusion will automatically generate a (constructively proved) reducibility statement. Of course, in the setting of TR reducibility statements may also be proved by classical means. Many mathematical theorems can be reworded as inclusions (for an example, see section 5).

4 Realizability

We shall restrict attention to statements of CM and TR formalizable in the language of elementary analysis **EL**, with variables for natural numbers (x, y, z, n, m) and variables ranging over $\mathbb{B}$ (α, β, γ). For a precise description see e.g. [TD, page 144].

Although the results below do have generalizations, there are several reasons for this restriction. First of all, we want to avoid undue complications. We are in this section not aiming at maximal generality, but are satisfied with an illustration of the basic ideas; secondly, if we want to formulate analogues in TR for statements in CM, it is pretty obvious what to do with $\mathbb{N}, \mathbb{B}, \mathbb{R}$ etc., since the elements of these sets are obviously encodable by elements of $\mathbb{B}$. But for the interpretation of $P(\mathbb{N})$ (and higher powersets) in the context of TR, we must choose (what from the viewpoint of CM is) a subclass encodable by a subset of $\mathbb{B}$ — such as the decidable or the enumerable sets. There is a certain arbitrariness in this choice, and it seems better to regard En, Cf as the TR-analogues of enumerable and decidable sets respectively, and not as analogues of $P(\mathbb{N})$.

DEFINITION. The class of *almost negative formulas* is the least class containing formulas $t = s$, $\exists x(t = s)$, $\exists \alpha(t = s)$ and closed under $\wedge, \rightarrow$, $\forall x$, $\forall \alpha$, $\neg$. □
N.B. $\neg A$ may actually be rendered as $A \rightarrow 1 = 0$, $A \vee B$ is definable as $\exists x((x = 0 \rightarrow A) \wedge (x \neq 0 \rightarrow B))$.

Almost negative formulas do not contain "essential" existential quantifiers. That is to say, $\exists x(t = s)$ is innocent in the sense that if true, the x can be found as $\min_x[t = s]$; $\exists \alpha(t = s]$ is reduced to the preceding case since it is equivalent to $\exists x(t[\alpha/x * \lambda x.0] = s[\alpha/x * \lambda x.0])$ (all terms of **EL** are continuous in their function parameters and $\{x * \lambda x.0 : x \in \mathbb{N}\}$ ranges over a dense subset of $\mathbb{B}$). In the presence of Markov's principle $\forall x(A \vee \neg A) \wedge \neg\neg\exists x A \rightarrow \exists x A$, almost negative formulas are equivalent to negative ones not containing any existential quantifier or disjunction.

For notions of constructive mathematics definable in **EL** we can easily give a recipe for finding a corresponding representation with almost negative domain, namely Kleene's function realizability, formalized in detail in [K].

Function realizability associates to each formula A an almost negative formula with an extra function variable α (α not free in A), written $\alpha\,\mathbf{r}\,A$ ("α realizes A").

If the elements of a set X are presented as a subset of $\mathbb{B}^l \times \mathbb{N}^k$ definable by a formula $A(\vec{\beta}, \vec{x})$ in **EL**, with an appropriate equivalence relation $\sim$, the obvious choice of representation is δ_A with

$$\begin{aligned}\mathrm{dom}(\delta_A) &:= \{(\alpha, \vec{\beta}, \vec{x}) : \alpha\,\mathbf{r}\,A(\vec{\beta}, \vec{x})\} \\ \delta_A(\alpha, \vec{\beta}, \vec{x}) &:= (\vec{\beta}, \vec{x})/\sim\end{aligned}$$

We recall the definition of $\alpha\,\mathbf{r}\,A$, presented by induction on the complexity of A:

DEFINITION. ($A \vee B$ treated as defined)

$$\begin{aligned}
\alpha\,\mathbf{r}\,(t = s) &:= t = s \\
\alpha\,\mathbf{r}\,(A \wedge B) &:= (\mathrm{j}_1\alpha\,\mathbf{r}\,A) \wedge (\mathrm{j}_2\alpha\,\mathbf{r}\,B) \\
\alpha\,\mathbf{r}\,(A \to B) &:= \forall\beta(\beta\,\mathbf{r}\,A \to \alpha|\beta\,\mathbf{r}\,B) \\
\alpha\,\mathbf{r}\,\forall\beta A &:= \forall\beta(\alpha|\beta\,\mathbf{r}\,A) \\
\alpha\,\mathbf{r}\,\forall x\,A &:= \forall\beta(\alpha|\beta\,\mathbf{r}\,A[x/\beta 0] \\
\alpha\,\mathbf{r}\,\exists\beta A &:= (\mathrm{j}_2\alpha)\,\mathbf{r}\,A[\beta/\mathrm{j}_1\alpha] \\
\alpha\,\mathbf{r}\,\exists x\,A &:= (\mathrm{j}_2\alpha)\,\mathbf{r}\,A[x/(\mathrm{j}_1\alpha)0]
\end{aligned}$$

(Alternative clauses for $\forall x, \exists x$ leading to an equivalent notion are: $\alpha\,\mathbf{r}\,\forall x\,A := \forall x(\lambda y.\alpha(x,y)\,\mathbf{r}\,A)$, $\alpha\,\mathbf{r}\,\exists x\,A := \lambda y.\alpha(y+1)\,\mathbf{r}\,A[x/\alpha 0]$). $\Box$

It is easy to see that $\alpha\,\mathbf{r}\,A$ is logically equivalent to an almost negative formula. For almost negative formulas B realizability is equivalent to truth: $B \leftrightarrow \exists\alpha(\alpha\,\mathbf{r}\,B)$. But in general $\exists\alpha(\alpha\,\mathbf{r}\,A)$ is not provably equivalent to A.

If we apply this recipe to the concept underlying $\rho_<$, i.e. "least-upper-bounds" of enumerable sets of rationals with an upper bound, we have α's encoding such sets and satisfying

$$A(\alpha) := \exists n \in \mathbb{Z} \forall m \in \mathrm{En}(\alpha)(r_m < n).$$

$\forall m \in \mathrm{En}(\alpha)(r_m < n)$ is almost negative, and the representation suggested by the recipe is not $\rho_<$ but the modified $\rho^*_<$ in which the existential quantifier has been made explicit:

$$\begin{aligned}\mathrm{dom}(\rho^*_<) &:= \{\alpha : \forall m \in \mathrm{En}(\lambda y.\alpha(y+1))(r_m < \alpha 0)\} \\ \rho^*_< &:= \mathrm{l.u.b.}\{r_m : m \in \mathrm{En}(\alpha)\}.\end{aligned}$$

It is easy to see that $\rho^*_< \preceq \rho_<$, but not $\rho_< \preceq \rho^*_<$. In many other examples the predicates defining the domain of a representation δ are in fact almost negative and can be straightforwardly read as deriving from a definition of a corresponding notion in CM. For an interesting example, see also δ_{loc} in the next section.

In the case of sets in CM with a complicated definition, replacing $A(\beta)$ by $\exists\alpha(\alpha\,\mathbf{r}\,A(\beta))$ as the notion considered (which is what our proposal for a "privileged" representation amounts to) may actually involve a *change* in the notion studied, since in general $A \leftrightarrow \exists\alpha(\alpha\,\mathbf{r}\,A)$ is not provable.

On the other hand, the logical schema $A \leftrightarrow \exists\alpha(\alpha\,\mathbf{r}\,A)$ is equivalent to a generalized continuity schema:

GC $\quad \forall\alpha[A\alpha \to \exists\beta\, B(\alpha,\beta)] \to \exists\gamma\forall\alpha[A\alpha \to B(\alpha, \gamma|\alpha]$

where A is almost negative, and the soundness theorem for function realizability establishes the consistency of GC relative to many constructive systems, in particular relative to $\mathbf{EL} + \mathrm{BI} + \mathrm{M}$ (containing FAN, AC_{01}, CONT_1).

So, looking at examples of concrete representations as found in the papers by Kreitz and Weihrauch, we see that in many cases the domains are definable by almost negative formulas, and hence the representations may be regarded as being derived from a CM-notion by our recipe.

But what about representations with a domain not defined by an almost negative formula, can we also think of these as derived from a suitable CM-notion? The principal difficulties we meet in the examples are of two types: (a) the definition of $\mathrm{dom}(\delta)$ contains set-quantifiers, and (b) the definition of $\mathrm{dom}(\delta)$ contains (essential) quantifiers $\exists\alpha$ or $\exists x$.

As long as we are working in the context of **EL**, the first difficulty can only be circumvented by finding an equivalent definition in the language of **EL**.

In the second case, where a definition of $\mathrm{dom}(\delta)$ in **EL** contains quantifiers $\exists\alpha$ or $\exists n$, there is a quite general solution: we may replace these quantifiers by $\neg\neg\exists\alpha, \neg\neg\exists n$, or equivalently by $\neg\forall\alpha\neg, \neg\forall n\neg$. In the classical setting of TR this is actually the same representation, but now the definition is (almost) negative. (Under this transformation, δ_{C} corresponds to the quasi-numbers of constructive recursive mathematics, and $\rho_<$ to monotone increasing sequences which are not unbounded.)

An example of the elimination of a set quantifier is the following. In [KW3, page 29] the following representation δ'_{cl} of a class of closed subsets of $\mathbb{R}$ is considered:

$$\begin{array}{ll} \alpha \in \mathrm{dom}(\delta'_{\mathrm{cl}}) & := \exists X \subset \mathbb{R}(X \text{ closed } \wedge \mathrm{En}(\alpha) = \{k : I_k \cap X \neq \emptyset\}), \\ \delta'_{\mathrm{cl}}(\alpha) & := \bigcap_m \bigcup_n \{I_{(n,m)} : (n,m) \in \mathrm{En}(\alpha)\}. \end{array}$$

We can replace the quantifier $\exists X \subset \mathbb{R}$ by an extra condition on the enumeration expressing that whenever a finite set $I_{n_1}, \ldots, I_{n_p}$ covers an I_m with $m \in \mathrm{En}(\alpha)$ then at least one $n_i \in \mathrm{En}(\alpha)$. One can then prove that $\delta'_{\mathrm{cl}}(\alpha)$ as defined above is indeed closed, and that α indeed enumerates all I_k with nonempty intersection with δ'_{cl}. In passing from an inclusion statement $\forall\alpha \in \mathrm{dom}(\delta)\exists\beta \in \mathrm{dom}(\delta')(\delta(\alpha) = \delta'(\beta))$ in CM to $\delta \preceq \delta'$ the statement is strengthened by requiring continuous dependence of β from α; but this is usually an automatic consequence of a constructive proof of the conclusion, by the following

THEOREM. *For suitable formal theories* S *in the language of* **EL**, *the following derived rule holds:*

$$\mathbf{S} \vdash \forall\alpha[A\alpha \to \exists\beta B(\alpha,\beta)] \Rightarrow \mathbf{S} \vdash \exists\gamma\forall\alpha[A(\alpha) \to \gamma|\alpha\downarrow \wedge B(\alpha,\gamma|\alpha)]$$

(So γ codes a partial continuous function in $[\mathbb{B} \rightharpoonup \mathbb{B}]$). Moreover, if $\forall\alpha\exists\beta A(\alpha,\beta)$ is closed, then γ may be taken to be recursive. For S *we can take* **EL**, *with some of the following schemata added:*

(1) bar induction BI*;*

(2) the fan theorem FAN, *a consequence of* BI:
$\forall\alpha \leq \beta\exists x A(\bar{\alpha}x) \to \exists z \forall\alpha \leq \beta\exists x \leq z A(\bar{\alpha}x)$

(3) Markov's principle (cf. end of section 2)

(4) the countable axiom of choice AC_{01}*:* $\forall x\exists\alpha A(x,\alpha) \to \exists\beta\forall x A(x, \lambda y.\beta(x,y))$;

(5) the continuity principle $CONT_1$*:* $\forall\alpha\exists\beta A(\alpha,\beta) \to \exists\gamma\forall\alpha A(\alpha, \gamma|\alpha)$*, or some weaker version.*

PROOF. By **q**-realizability for functions; see [T, 3.3, 3.7.9]. □

The theorem may be applied as follows. Suppose that we can express in **EL**, for representations δ, δ', δ with an almost negative domain,

$$\forall\alpha \in \text{dom}(\delta)\exists\beta(\delta(\alpha) = \delta'(\beta))$$

and this is provable in a suitable system **S** (e.g. $\mathbf{EL} + \text{FAN} + \text{MP} + \text{AC}_{01}$), which is a subsystem of classical analysis, then by the theorem

$$\mathbf{S} \vdash \exists\gamma\forall\alpha \in \text{dom}(\delta)(\gamma|\alpha\downarrow \wedge \delta(\alpha) = \delta'(\gamma|\alpha)).$$

That is to say, in cases where δ, δ' have been chosen to correspond to the constructive definitions of sets X, Y, the constructive proof of $\forall x \in X\exists y \in Y(x = y)$ automatically yields the stronger $\delta \preceq \delta'$ (established constructively as well as classically).

For an extension to stronger systems, see e.g. [Fr], where an extension of **q**-realizability to intuitionistic second-order arithmetic is given. For further generalizations to higher-order logic, see e.g. [vO].

5 An example

How translation of CM-statements, and the use of the metatheorem works out in practice, we can see from the following more or less representative example. In [Bi, page 177] we find the following

THEOREM. *Let X be an inhabited closed located set in a complete separable metric space $\mathcal{M} \equiv (M, d, \langle p_n\rangle_n)$, $x \in M$ such that $\forall y \in X(d(y,x) > 0)$, then $d(X,x) > 0$.* □

PROOF. For a constructive proof, see [Bi]. A classical proof is even easier.

We give a reformulation in TR. We first define a representation of inhabited closed located sets.

DEFINITION.

$$\begin{aligned} \alpha \in \text{dom}(\delta_{\text{loc}}) \;&:= \exists X \subset M(\exists x(x \in X) \wedge X \text{ located} \wedge \\ &\qquad \forall\beta \in \text{dom}(\delta_{\mathcal{M}})(d(X, \delta_{\mathcal{M}}(\beta)) = \delta_{\mathbb{R}}(\alpha \mid \beta))) \\ \delta_{\text{loc}}(\alpha) \;&:= \{\delta_{\mathcal{M}}(\beta) : \delta_{\mathbb{R}}(\alpha \mid \beta) = 0\}. \;\square \end{aligned}$$

So $\lambda\beta.(\alpha|\beta)$ describes the continuous distance function $d(X, \delta_{\mathcal{M}}(\beta))$ ($\delta_{\mathcal{M}}$ as defined in section 2, (H)).

DEFINITION. For pairs (X,x), X a closed located set of $\mathcal{M}$, $x \in M$, $d(X,x) > 0$, we choose a representation ρ^* as follows. If $\alpha = (\beta, \hat{n} * \gamma)$, and $X = \delta_{\mathrm{loc}}(\beta)$, $x = \delta_{\mathcal{M}}(\gamma)$ and $d(X,x) > 2^{-n}$, then $\alpha \in \mathrm{dom}(\rho^*)$, and $\rho^*(\beta, \hat{n} * \gamma) = (\delta_{\mathrm{loc}}(\beta), \delta_{\mathcal{M}}(\gamma))$.

For pairs (X,x), X closed located, inhabited, $x \in M$ such that $\forall y \in X(d(y,x) > 0)$, we define ρ^{**} as follows. If $\alpha = (\beta, \beta', \beta'')$, $\delta_{\mathrm{loc}}(\beta) = X$, $\delta_{\mathcal{M}}(\beta') = x$, and β'' such that

$$\delta_{\mathcal{M}}(\gamma) \in X \rightarrow \beta''(\gamma)\downarrow \wedge\, d(\delta_{\mathcal{M}}(\gamma), \delta_{\mathcal{M}}(\beta')) > 2^{-\beta''(\gamma)},$$

then $\alpha \in \mathrm{dom}(\rho^{**})$, and clearly

$$\rho^{**}(\beta, \beta', \beta'') = (\delta_{\mathrm{loc}}(\beta), \delta_{\mathcal{M}}(\beta')).$$

It is not difficult to strengthen the proof of the theorem to a constructive proof of the equivalence $\rho^* \equiv \rho^{**}$. (Even if this reformulation of the theorem looks a bit complicated at first sight, it is nevertheless a straightforward spelling out of the required explicit information.)

If we want to obtain $\rho^* \equiv \rho^{**}$ via an appeal to the derived rule in the preceding section, we must put in some extra work.

Reformulation of the representation δ_{loc}

Let X be inhabited, located in a complete, separable metric space $\mathcal{M}$, and let $f(x) = d(X,x)$. the $X = \{x : f(x) = 0\}$. Our definition of δ_{loc} given above is not satisfactory, inasmuch its definition is not in the language of **EL**. We can correct this as follows. It is not difficult to see that X itself is a complete, separable metric space, and that we can construct explicitly a sequence $\langle q_n \rangle_n$ of points in X, dense in X. The distance function $\lambda x.d(X,x)$ must satisfy

$$\begin{array}{l}\forall x, y \in M(d(X,y) = 0 \rightarrow d(x,y) \geq d(x,X)),\\ \forall x, k \exists y(d(X,y) = 0 \wedge d(x,y) < d(x,X) + 2^{-k}).\end{array}$$

Now an arbitrary continuous $f : M \longrightarrow \mathbb{R}$ represents the distance function of $X_f \equiv \{x : f(x) = 0\}$, i.e. $d(X_f, x) = f(x)$, if the following conditions are satisfied:

$$(*) \qquad \begin{cases}\forall x, y(f(y) = 0 \rightarrow d(x,y) \geq f(x)),\ \forall n(f(q_n) = 0)\\ \forall x, k \exists n(d(x,q_n) < f(x) + 2^{-k})\end{cases}$$

If f is represented by a γ such that $\delta_{\mathbb{R}}(\gamma|\alpha) = f(\delta_{\mathcal{M}}(\alpha))$, and d by a γ' such that $\delta_{\mathbb{R}}(\gamma'|(\alpha,\beta)) = d(\delta_{\mathcal{M}}(\alpha), \delta_{\mathcal{M}}(\beta))$, and the sequence $\langle q_n \rangle_n$ by a γ'' such that $q_n = \delta_{\mathcal{M}}(\lambda y.\gamma''(n,y))$, it is not hard to verify that $(*)$ can be expressed by an almost negative formula.

We can take $(*)$ as the condition (on γ and γ'') determining the domain of an appropriate representation of the located, closed, inhabited sets. Now the theorem on the derived rule applies. The "balance of work" in the case of a constructive proof is more or less neutral: compared with a direct constructive proof of $\rho^* \equiv \rho^{**}$, we had to put in a bit extra work in order to formulate things in **EL**, while we saved a little by an appeal to the derived rule. In this particular case we can give a much shorter proof by reasoning classically: from the classically proven $\forall y d(\delta_{\rm loc}(\alpha), y) > 0 \to \exists k\, d(\delta_{\rm loc}(\alpha), x) > 2^{-k}$ we can find the k continuously in α.

REMARK. In adopting a coding as an appropriate rendering of a concept of constructive mathematics, one often uses a lemma: if f is a continuous map from $\mathcal{M}$ to $\mathcal{M}'$, $\mathcal{M}$ and $\mathcal{M}'$ complete separable metric spaces, then (classically, or if we assume intuitionistic continuity axioms) there is a continuous $\Gamma : \mathbb{B} \longrightarrow \mathbb{B}$ such that $f(\delta_{\mathcal{M}}(\alpha)) = \delta_{\mathcal{M}'}(\Gamma\alpha)$ (proof left to the reader).

6 Compactness of bounded closed subsets

As a second illustrative example we show how conversely a reducibility result from [KW] might also have been obtained from a proof of CM with application of the derived rule.

DEFINITION. $\delta_{\rm cl}$ represents closed sets as countable intersections of complements of basis intervals:

$$\delta_{\rm cl} := \bigcap\{\mathbb{R} \setminus I_k : k \in {\rm En}(\alpha)\} = \mathbb{R} \setminus \bigcup\{I_k : k \in {\rm En}(\alpha)\}.$$

For the corresponding notion of bounded sets we define $\delta_{\rm bcl}$:

$$\hat{n} * \alpha \in {\rm dom}(\delta_{\rm bcl}) := \delta_{\rm cl}(\alpha) \subset [-n, n]\ (n \in \mathbb{N}),$$
$$\delta_{\rm bcl}(\hat{n} * \alpha) := \delta_{\rm cl}(\alpha) \text{ whenever } \hat{n} * \alpha \in {\rm dom}(\delta_{\rm bcl}). \ \square$$

${\rm dom}(\delta_{\rm bcl})$ is easily seen to be definable by an almost negative formula (e.g. $\hat{n} * \alpha \in {\rm dom}(\delta_{\rm bcl}) \leftrightarrow \forall r \in \mathbb{Q}(r < -n \vee n < r \to \exists m(\alpha m > 0 \wedge r \in I_{\alpha m-1})))$. We want to compare $\delta_{\rm bcl}$ with $\delta_{\rm whb}$, a representation of the compact subsets of $\mathbb{R}$.

DEFINITION. Put

$$C_\alpha := \{I_j : j \in {\rm En}(\alpha)\},$$
$$C_{\alpha,n} := \{I_j : \exists i \in D_n(\alpha i = j + 1)\}$$

where $\langle D_n \rangle_n$ is some standard enumeration of finite sets, e.g. $D_n = \{m_0 < \ldots < m_{p-1}\} \Leftrightarrow n = \sum_{i<p} 2^{m_i}$.

A function $\beta \in \mathbb{B}$ *witnesses the compactness* of $X \subset \mathbb{R}$ (*is a witness for compactness* of X) iff

$$\forall\alpha[(X \subset \bigcup C_\alpha) \leftrightarrow (\alpha \in {\rm dom}(\Phi_\beta))] \text{ and}$$
$$\forall\alpha[(X \subset \bigcup C_\alpha) \to X \subset \bigcup C_{\alpha,\beta(\alpha)}].$$

The *weak Heine-Borel representation* is then given by

$$\alpha \in {\rm dom}(\delta_{\rm whb}) := \exists X \subset \mathbb{R}(\alpha \text{ witnesses compactness of X})$$
$$\delta_{\rm whb}(\alpha) := \bigcap\{\bigcup C_{\beta,\alpha(\beta)} : \alpha(\beta){\downarrow} \wedge \beta \in \mathbb{B}\}$$

THEOREM. $\delta_{\text{bcl}} \preceq \delta_{\text{whb}}$.
PROOF. We show constructively $\forall\alpha \in \text{dom}(\delta_{\text{bcl}})\exists\gamma \in \text{dom}(\delta_{\text{whb}})(\delta_{\text{bcl}}(\alpha) = \delta_{\text{whb}}(\gamma))$. Let $\hat{n}*\alpha \in \text{dom}(\delta_{\text{bcl}})$, then $\delta_{\text{bcl}}(\hat{n}*\alpha) = [-n,n]\setminus\bigcup\{I_j : j \in \text{En}(\alpha)\} = [-n,n]\setminus\bigcup C_\alpha$. *Assuming the compactness of* $[-n,n]$ (which is a consequence of FAN) we have

$$\delta_{\text{bcl}}(\hat{n} * \alpha) \subset \bigcup C_\beta \Leftrightarrow \exists m([-n,n] \subset \bigcup C_{\alpha,m} \cup \bigcup C_{\beta,m})$$

and

$$[-n,n] \subset \bigcup C_{\alpha,n} \cup \bigcup C_{\beta,m} \Rightarrow \delta_{\text{bcl}}(\hat{n} * \alpha) \subset \bigcup C_{\beta,m}.$$

Since $[-n,n] \subset \bigcup C_{\alpha,n} \cup \bigcup C_{\beta,m}$ is decidable in n, m, α, β, we can compute m for any β covering $\delta_{\text{bcl}}(\hat{n} * \alpha)$. □

This proof is constructive (assuming FAN), so, with an appeal to our derived rule M may be found continuously from β, by Φ_γ say. Then γ is a witness for the compactness of $\delta_{\text{bcl}}(\hat{n} * \alpha)$, and may be found (again appealing to the derived rule) continuously from α.

REMARK. If we compare the proof with the similar argument in [KW], we see that we have saved little work by an appeal to the derived rule, since the continuity of the dependencies is not difficult to see. But the proof shows at least that this result fits into our "metamathematical schema".

More interesting is a proof of the converse, $\delta_{\text{whb}} \preceq \delta_{\text{bcl}}$. The proof in [KW3] looks (nearly) constructive, but does not bring the statement under our schema, since (a) constructively we also need to show that $\delta_{\text{whb}}(\alpha)$ defines a closed set (in CM a witness of compactness does not uniquely determine the set being witnessed: $[1,2] \cup [2,3]$ and $[1,3]$ have the same witnesses of compactness, but $[1,2] \cup [2,3]$ is not closed), and (b) the domain of δ_{whb} has not been defined in **EL**. So we have to prove a

LEMMA. $\alpha \in \text{dom}(\delta_{\text{whb}})$ *is expressible by an almost negative predicate.*
PROOF.

$$(1) \qquad \forall\gamma[(\bigcap_\beta\{\bigcup C_{\beta,\alpha(\beta)} : \alpha(\beta)\downarrow\} \subset \bigcup C_\gamma) \to \alpha(\gamma)\downarrow]$$

expresses that α witnesses compactness of $\bigcap_\beta\{\bigcup C_{\beta,\alpha(\beta)} : \alpha(\beta)\downarrow\}$. We can rewrite (1) as

$$(2) \qquad \forall\gamma[\forall\gamma'(\forall\beta(\alpha(\beta)\downarrow \to \delta_{\mathbb{R}}(\gamma') \in \bigcup C_{\beta,\alpha(\beta)}) \to \delta_{\mathbb{R}}(\gamma') \in \bigcup C_\gamma) \to \alpha(\gamma)\downarrow]$$

and this is easily verified to be equivalent to an almost negative statement.

In order to meet objection (a) above, we need a

LEMMA. *In* **EL** + MP *we can prove that whenever*

(1) $$\forall\gamma[(\bigcap_{\beta}\{\bigcup C_{\beta,\alpha(\beta)} : \alpha(\beta)\downarrow\} \subset \bigcup C_\gamma) \to \alpha(\gamma)\downarrow]$$

then $\delta_{\mathrm{whb}}(\alpha) \equiv X_\alpha$ *is closed.*

PROOF. Let $x \notin X_\alpha$, $X_\alpha \subset [-n, n]$. (We may assume this without loss of generality, since $C_{\lambda n.n}$ certainly covers, so $\alpha(\lambda n.n)\downarrow$, and $\bigcup C_{\lambda n.n,\alpha(\lambda n.n)}$ provides us with a bound.) Consider the following subset of $\langle I_n \rangle_n$ consisting of intervals of the forms

$$\{(r, r') : r \leq -n - 1 \wedge r' < x\},\ \{(r, r') : x < r \wedge n + 1 \leq r'\}.$$

This set covers X_α, for if $x' \in X_\alpha$, $x \neq x$, then by Markov's principle $x' \,\sharp\, x$, i.e. $x' < x$ or $x < x'$, hence $x' \in (-n - 1, r)$ for some $r < x$ or $x' \in (r', n + 1)$ for some $r' > x$. Since X_α is compact, there is a finite collection

$$\{(r'_1, r_1), \ldots, (r'_p, r_p)\} \cup \{(s_1, s'_1), \ldots, (s_q, s'_q)\}$$

with $r'_i \leq -n - 1, r_i < x, x < s_j, n + 1 \leq s'_j$ covering X_α. Now each $x' \in X_\alpha$ has a distance to x of at least $\inf\{x - \sup\{r_1, \ldots, r_p\}, \inf\{s_1, \ldots, s_q\} - x\} > 0$. Therefore, if x is in the closure of X_α, also $\neg\neg x \in X_\alpha$. If $x \in X_\alpha$, then

$$\forall\beta(\alpha(\beta)\downarrow \ \to \exists j(x \in I_j \in C_{\beta,\alpha(\beta)})),$$

hence if $\neg\neg x \in X_\alpha$, then

$$\forall\beta(\alpha(\beta)\downarrow \to \neg\neg\exists j(x \in I_j \in C_{\beta,\alpha(\beta)})).$$

By Markov's principle, we can drop $\neg\neg$, so $x \in X$. □

By means of the preceding two lemma's, the proof in [KW] is now easily adapted to obtain

PROPOSITION. $\mathrm{dom}(\delta_{\mathrm{whb}}) \subset \mathrm{dom}(\delta_{\mathrm{bcl}})$ *is constructively provable.* □

This brings the reducibility $\delta_{\mathrm{whb}} \preceq \delta_{\mathrm{bcl}}$ under our general schema. But note that in this case the balance of work is even *negative*: in order to make the proof of the proposition constructive we needed to put in extra work.

7 Concluding remarks

(1) Our discussion covers most, but not all specific representations and reducibilities discussed in the work of Weihrauch and Kreitz. Thus in [W2] representations for *sets* of continuous maps between separable metric spaces in which a pointset X appears as a parameter (the domain of definition of the function). To bring this under the schema, we need to extend the results of section 4 to **EL** extended with a (purely schematic) set variable. We have checked that this is possible, but we are not really satisfied with our treatment.

The formulation of an TR-analogue to a CM-inclusion or -equality statement may become awkward if the CM-statement is expressed in a language with set variables. As an example, consider the following simple theorem taken from [BB, page 37]:

THEOREM. *For inhabited sets $X \subset \mathbb{R}$ with an upper bound, X has an l.u.b. iff* (*) $\forall x, y \in \mathbb{R}(x < y \rightarrow y \geq X \vee \exists x' \in X(x < x'))$, *where* $y \geq X := \forall x \in X(y \geq x)$. □

This theorem may be recast as $\delta_{1,X} \equiv \delta_{2,X}$ where $\delta_{1,X}$ and $\delta_{2,X}$ are representations depending on a parameter X. A code in dom($\delta_{1,X}$) should specify an $x_0 \in X$, an upper bound for X, and a decision function applicable to pairs x, y with $x < y$ for (*), plus a function yielding the $x' \in X$ if the second alternative in (*) holds. $\delta_{1,X}$ assigns to this code simply l.u.b.(X). A code for $\delta_{2,X}$ should specify an element of X, an upper bound x_1 of X, and a sequence $\langle y_n \rangle_n \subset X$ such that $y_n + 2^{-n} > x_1$.

As the example shows, the fact that P($\mathbb{R}$) is not encodable by $\mathbb{B}$ may be circumvented by parametric representations. We suspect that this can be done quite generally (cf. the representations of the continuous partial functions between metric spaces mentioned above), but this aspect calls for further investigation.

(2) For reducibilities between representations expressible in **EL**, function realizability provides a key to understanding the analogies between CM and TR. However, in general there is no saving of labour in deriving reducibilities from CM-results. But the CM-results suggest reducibilities, and the discussion shows that attention to the logical form of the definitions of domains of representations may help to explain why certain representations are better behaved than others — a type of explanation different in spirit from the topological criteria in [KW2], hence adding a little bit of insight.

(3) We believe that comparison between CM and TR may be useful in selecting the mathematically best-behaved representations.

References

[B] M.J.Beeson, *Foundations of Constructive Mathematics*, Springer-Verlag, Berlin 1985.

[BB] E. Bishop, D.S. Bridges, *Constructive Analysis*, Springer-Verlag, Berlin 1985.

[Bi] E.A. Bishop, *Foundations of Constructive Analysis* (1967), McGrawHill, New York.

[Fr] H.M. Friedman, On the derivability of instantiation properties, *The Journal of Symbolic Logic 42* (1977), 506–514.

[K] S.C. Kleene, Formalized recursive functionals and formalized realizability, *memoirs of the American mathematical Society 89* (1969).

[KW1] C. Kreitz, K. Weihrauch, A unified approach to constructive and recursive analysis, in: M.M. Richter et al. (eds.), *Computation and Proof theory*, Springer-Verlag, Berlin 1984, 259-278.

[KW2] C. Kreitz, K. Weihrauch, Theory of representations, *Theoretical Computer Science 38* (1985), 35–53.

[KW3] C. Kreitz, K. Weihrauch, Compactness in constructive analysis revisited, *Annals of Pure and Applied Logic 36* (1987), 29–38.

[M] N. Th. Müller, Computational complexity of real functions and real numbers. Informatik Berichte 59 (1986), Fern-Universität Hagen, BRD.

[ML] P. Martin-Löf, *Intuitionistic Type Theory*, Bibliopolis, Napoli.

[Sh] N.A. Shanin, *Constructive Real Numbers and Function Spaces*, American Mathematical Society, providence (RI), 1968 (translation of the russian original).

[T] A.S. Troelstra (ed.), *Metamathematical investigation of Intuitionistic Arithmetic and Analysis*, Springer verlag, berlin 1973.

[TD] A.S. Troelstra, D. van Dalen, *Constructivism in Mathematics* (1988), North-Holland Publ. Co., Amsterdam

[vO] J. van Oosten, Exercises in Realizability. Ph.D. thesis, Universiteit van Amsterdam, 1990.

[W1] K. Weihrauch, Type 2 recursion theory, *Theoretical Computer Science 38* (1985), 17–33.

[W2] K. Weihrauch, Computability on computable metric spaces, *Theoretical Computer Science*, to appear.

[WK] K. Weihrauch, C. Kreitz, Representations of the real numbers and of the open subsets of the set of real numbers, *Annals of Pure and Applied Logic 35* (1987), 247–260.

The ITLI Prepublication Series

ML-90-01 Harold Schellinx *Mathematical Logic and Foundations* Isomorphisms and Non-Isomorphisms of Graph Models
ML-90-02 Jaap van Oosten A Semantical Proof of De Jongh's Theorem
ML-90-03 Yde Venema Relational Games
ML-90-04 Maarten de Rijke Unary Interpretability Logic
ML-90-05 Domenico Zambella Sequences with Simple Initial Segments
ML-90-06 Jaap van Oosten Extension of Lifschitz' Realizability to Higher Order Arithmetic, and a Solution to a Problem of F. Richman
ML-90-07 Maarten de Rijke A Note on the Interpretability Logic of Finitely Axiomatized Theories
ML-90-08 Harold Schellinx Some Syntactical Observations on Linear Logic
ML-90-09 Dick de Jongh, Duccio Pianigiani Solution of a Problem of David Guaspari
ML-90-10 Michiel van Lambalgen Randomness in Set Theory
ML-90-11 Paul C. Gilmore The Consistency of an Extended NaDSet
CT-90-01 John Tromp, Peter van Emde Boas *Computation and Complexity Theory* Associative Storage Modification Machines
CT-90-02 Sieger van Denneheuvel, Gerard R. Renardel de Lavalette A Normal Form for PCSJ Expressions
CT-90-03 Ricard Gavaldà, Leen Torenvliet, Osamu Watanabe, José L. Balcázar Generalized Kolmogorov Complexity in Relativized Separations
CT-90-04 Harry Buhrman, Edith Spaan, Leen Torenvliet Bounded Reductions
CT-90-05 Sieger van Denneheuvel, Karen Kwast Efficient Normalization of Database and Constraint Expressions
CT-90-06 Michiel Smid, Peter van Emde Boas Dynamic Data Structures on Multiple Storage Media, a Tutorial
CT-90-07 Kees Doets Greatest Fixed Points of Logic Programs
CT-90-08 Fred de Geus, Ernest Rotterdam, Sieger van Denneheuvel, Peter van Emde Boas Physiological Modelling using RL
CT-90-09 Roel de Vrijer Unique Normal Forms for Combinatory Logic with Parallel Conditional, a case study in conditional rewriting
X-90-01 A.S. Troelstra *Other Prepublications* Remarks on Intuitionism and the Philosophy of Mathematics, Revised Version
X-90-02 Maarten de Rijke Some Chapters on Interpretability Logic
X-90-03 L.D. Beklemishev On the Complexity of Arithmetical Interpretations of Modal Formulae
X-90-04 Annual Report 1989
X-90-05 Valentin Shehtman Derived Sets in Euclidean Spaces and Modal Logic
X-90-06 Valentin Goranko, Solomon Passy Using the Universal Modality: Gains and Questions
X-90-07 V.Yu. Shavrukov The Lindenbaum Fixed Point Algebra is Undecidable
X-90-08 L.D. Beklemishev Provability Logics for Natural Turing Progressions of Arithmetical Theories
X-90-09 V.Yu. Shavrukov On Rosser's Provability Predicate
X-90-10 Sieger van Denneheuvel, Peter van Emde Boas An Overview of the Rule Language RL/1
X-90-11 Alessandra Carbone Provable Fixed points in $I\Delta_0+\Omega_1$, revised version
X-90-12 Maarten de Rijke Bi-Unary Interpretability Logic
X-90-13 K.N. Ignatiev Dzhaparidze's Polymodal Logic: Arithmetical Completeness, Fixed Point Property, Craig's Property
X-90-14 L.A. Chagrova Undecidable Problems in Correspondence Theory
X-90-15 A.S. Troelstra Lectures on Linear Logic
1991 LP-91-01 Wiebe van der Hoek, Maarten de Rijke *Logic, Semantics and Philosophy of Langauge* Generalized Quantifiers and Modal Logic
LP-91-02 Frank Veltman Defaults in Update Semantics
LP-91-03 Willem Groeneveld Dynamic Semantics and Circular Propositions
LP-91-04 Makoto Kanazawa The Lambek Calculus enriched with additional Connectives
LP-91-05 Zhisheng Huang, Peter van Emde Boas The Schoenmakers Paradox: Its Solution in a Belief Dependence Framework
LP-91-06 Zhisheng Huang, Peter van Emde Boas Belief Dependence, Revision and Persistence
LP-91-07 Henk Verkuyl, Jaap van der Does The Semantics of Plural Noun Phrases
LP-91-08 Víctor Sánchez Valencia Categorial Grammar and Natural Reasoning
LP-91-09 Arthur Nieuwendijk Semantics and Comparative Logic
LP-91-10 Johan van Benthem Logic and the Flow of Information
ML-91-01 Yde Venema *Mathematical Logic and Foundations* Cylindric Modal Logic
ML-91-02 Alessandro Berarducci, Rineke Verbrugge On the Metamathematics of Weak Theories
ML-91-03 Domenico Zambella On the Proofs of Arithmetical Completeness for Interpretability Logic
ML-91-04 Raymond Hoofman, Harold Schellinx Collapsing Graph Models by Preorders
ML-91-05 A.S. Troelstra History of Constructivism in the Twentieth Century
ML-91-06 Inge Bethke Finite Type Structures within Combinatory Algebras
ML-91-07 Yde Venema Modal Derivation Rules
ML-91-08 Inge Bethke Going Stable in Graph Models
ML-91-09 V.Yu. Shavrukov A Note on the Diagonalizable Algebras of PA and ZF
ML-91-10 Maarten de Rijke, Yde Venema Sahlqvist's Theorem for Boolean Algebras with Operators
ML-91-11 Rineke Verbrugge Feasible Interpretability
ML-91-12 Johan van Benthem Modal Frame Classes, revisited
CT-91-01 Ming Li, Paul M.B. Vitányi *Computation and Complexity Theory* Kolmogorov Complexity Arguments in Combinatorics
CT-91-02 Ming Li, John Tromp, Paul M.B. Vitányi How to Share Concurrent Wait-Free Variables
CT-91-03 Ming Li, Paul M.B. Vitányi Average Case Complexity under the Universal Distribution Equals Worst Case Complexity
CT-91-04 Sieger van Denneheuvel, Karen Kwast Weak Equivalence
CT-91-05 Sieger van Denneheuvel, Karen Kwast Weak Equivalence for Constraint Sets
CT-91-06 Edith Spaan Census Techniques on Relativized Space Classes
CT-91-07 Karen L. Kwast The Incomplete Database
CT-91-08 Kees Doets Levationis Laus
CT-91-09 Ming Li, Paul M.B. Vitányi Combinatorial Properties of Finite Sequences with high Kolmogorov Complexity
CT-91-10 John Tromp, Paul Vitányi A Randomized Algorithm for Two-Process Wait-Free Test-and-Set
CT-91-11 Lane A. Hemachandra, Edith Spaan Quasi-Injective Reductions
CT-91-12 Krzysztof R. Apt, Dino Pedreschi Reasoning about Termination of Prolog Programs
CL-91-01 J.C. Scholtes *Computational Linguistics* Kohonen Feature Maps in Natural Language Processing
CL-91-02 J.C. Scholtes Neural Nets and their Relevance for Information Retrieval
CL-91-03 Hub Prüst, Remko Scha, Martin van den Berg A Formal Discourse Grammar tackling Verb Phrase Anaphora
X-91-01 Alexander Chagrov, Michael Zakharyaschev *Other Prepublications* The Disjunction Property of Intermediate Propositional Logics
X-91-02 Alexander Chagrov, Michael Zakharyaschev On the Undecidability of the Disjunction Property of Intermediate Propositional Logics
X-91-03 V. Yu. Shavrukov Subalgebras of Diagonalizable Algebras of Theories containing Arithmetic
X-91-04 K.N. Ignatiev Partial Conservativity and Modal Logics
X-91-05 Johan van Benthem Temporal Logic
X-91-06 Annual Report 1990
X-91-07 A.S. Troelstra Lectures on Linear Logic, Errata and Supplement
X-91-08 Giorgie Dzhaparidze Logic of Tolerance
X-91-09 L.D. Beklemishev On Bimodal Provability Logics for Π_1-axiomatized Extensions of Arithmetical Theories
X-91-10 Michiel van Lambalgen Independence, Randomness and the Axiom of Choice
X-91-11 Michael Zakharyaschev Canonical Formulas for K4. Part I: Basic Results
X-91-12 Herman Hendriks Flexibele Categoriale Syntaxis en Semantiek: de proefschriften van Frans Zwarts en Michael Moortgat
X-91-13 Max I. Kanovich The Multiplicative Fragment of Linear Logic is NP-Complete
X-91-14 Max I. Kanovich The Horn Fragment of Linear Logic is NP-Complete
X-91-15 V. Yu. Shavrukov Subalgebras of Diagonalizable Algebras of Theories containing Arithmetic, revised version
X-91-16 V.G. Kanovei Undecidable Hypotheses in Edward Nelson's Internal Set Theory
X-91-17 Michiel van Lambalgen Independence, Randomness and the Axiom of Choice, Revised Version
X-91-18 Giovanna Cepparello New Semantics for Predicate Modal Logic: an Analysis from a standard point of view
X-91-19 Papers presented at the Provability Interpretability Arithmetic Conference, 24-31 Aug. 1991, Dept. of Phil., Utrecht University
1992 LP-92-01 Víctor Sánchez Valencia Lambek Grammar: an Information-based Categorial Grammar
ML-92-01 A.S. Troelstra Comparing the theory of Representations and Constructive Mathematics