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MAXIMAL KRIPKE-TYPE SEMANTICS FOR MODAL AND SUPERINTUITIONISTIC PREDICATE LOGICS

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The semantical approach to logical systems is used widely nowadays. It can be helpful in answering questions about the provability of certain formulas, consistency, decidability, the interpolation property etc. Semantical methods in classical logic are provided by classical model theory. But for non-classical predicate logics, model theory has not yet been elaborated. One of the reasons for this delay is in the variety of semantics for these logics making a general concept of a model too ambiguous.

In the present paper we are concerned only with the semantics for *superintuitionistic logics* (extensions of Heyting's predicate calculus) and *modal logics* (extensions of **S4** quantified). However we hope that the main ideas can be applied to a larger class of logics as well.

Certainly *Kripke semantics* is the most traditional in this area. But incompleteness results prohibit making it universal. Unlike propositional logic where some general theorems on Kripke-completeness can be proved (cf. e.g. [2], [10]) , predicate logics provide many simple examples of Kripke-incompleteness: [9], [5], [7], [12]. In this case general completeness theorems are hardly possible (however cf. [13]), and thus Kripke semantics has to be generalized.

As soon as a *predicate Kripke frame* consists of two parts: of a "propositional" frame and a set of individuals, at least two possible ways for generalizations appear.

If we replace propositional Kripke frames by *algebras* of the corresponding type (i.e. by Heyting algebras, or by **S4**-algebras), we come to *algebraic semantics* (a particular case of which is *topological semantics*).

Another way is to specify some relations between individuals and to retain all truth-definitions in the Kripke-style. In this way, several successive generalizations are known: *Kripke sheaves, Kripke bundles* [12], *functor semantics* [5], [6]. In [12] (cf. also [16]) it is stated that Kripke sheaf semantics is stronger than standard Kripke semantics, and that Kripke bundles are stronger than Kripke sheaves. Also it is true that functor semantics is stronger than Kripke bundle

* We would like to thank Dijk De Jongh for his help in improving English in this paper.

semantics [15]. Moreover, functor semantics is strong enough to provide some general completeness theorems [6].

The question arises, how long this sequence of generalizations might be extended. In other words, does there exist a maximal Kripke-type semantics? To examine this we need a precise definition of "Kripke-type semantics". In the present paper we choose the most general (in our opinion) semantical objects for which truth-definitions are given exclusively in terms of binary relations; so-called *Kripke metaframes* . The main characteristic feature of Kripke metaframe semantics is treating n-tuples of individuals in an abstract way, as "n-dimensional individuals", with possible worlds appearing as "0-dimensional individuals". However a truth definition alone cannot provide a good semantics unless some natural "soundness conditions" are satisfied.

It turns out that for Kripke metaframes, soundness can be described purely in relational terms, without any appeal to non-classical formulas (theorems 2.5.1 and 2.5.5). Thus we come to the notions of a *modal metaframe* and an *intuitionistic metaframe* . These ones generate the "maximal" semantics in question.

Taking the "maximality theorem" into account we can reduce the routine job of proving soundness for *any* other semantics of Kripke-type - that is, for any semantics generated by some class of Kripke metaframes (say, for Kripke bundles or functor semantics) - to the verification of few simple conditions.

Another remarkable feature of modal or intuitionistic metaframe semantics consists in completeness proofs via canonical models. In propositional logic, canonical models are well-known. Their idea (arising to Stone representation theorem) is in treating maximal (or prime) filters of a Lindenbaum algebra as possible worlds. But it is difficult to apply this construction to predicate logics within the standard Kripke semantics.

Indeed, for a "canonical predicate Kripke frame" we need also individuals, and a priori it is unclear, how to construct them from our logic in an algebraic way. Of course, we can merely introduce some individuals from outside, as new constants (akin to Henkin's completeness proof) . But this method works only sometimes (e.g. for quantified **S4** or for Heyting predicate calculus); in some other cases it is suitable only after a non-trivial modification ([8],[14]); and in many cases it cannot work at all, just because completeness fails.

In canonical metaframes the difficulty is overcome by identifying n-dimensional individuals with n-types. (Note that even in the classical case, n-types are certainly non-equivalent to n-tuples of 1-types; this also motivates our notion of a metaframe.) Then

1. Predicate logics and Kripke-type semantics.

1.1. Formulas and logics.

We fix countable sets of *individual variables* $Vr = \{v_1, v_2, ...\}$ and of *m-place predicate letters* $PL^m=\{F_1{}^m, F_2{}^m,...\}$, $m\geq 0$; P^m, Q^m stand for names of elements of PL^m. 0-place predicate letters are also called *propositional letters* . *Atomic formulas without equality* are propositional letters or have the form $P^m(x_1,...,x_m)$, with $P^m \in PL^m$, $x_1,...,x_m \in Vr$; *atomic formulas with equality* can also be of the form $(x=y)$ ($x,y \in Vr$, $=$ is an additional 2-place predicate letter).

Intuitionistic formulas (with or without equality) are built from atomic ones using $\bot$, $\rightarrow$, $\wedge$, $\vee$, $\exists$, $\forall$; *modal formulas* are constructed using $\bot$, $\supset$, $\Diamond$, $\exists$. Other connectives and quantifiers ($\neg$, $\leftrightarrow$ for the intuitionistic case, and $\wedge,\vee$, $\neg$, $\equiv$, $\Box$, $\forall$ for the modal case) are derived as usual. AF, IF, MF denote respectively the sets of atomic, intuitionistic, and modal formulas without equality; the corresponding sets of formulas with equality are denoted by $AF^=$, $IF^=$, $MF^=$.

Free and *bound* occurrences of variables in formulas are defined as usual; FV(A) denotes the set of all parameters (free variables) of a formula A. A formula A is *closed* if $FV(A)=\emptyset$.

For a list of variables $\underline{x}=(x_1,...,x_n)$, the notation $FV(A)\subseteq \underline{x}$ (resp., $FV(A)=\underline{x}$) will be used instead of $FV(A)\subseteq \{x_1,...,x_n\}$ (resp., $FV(A)=\{x_1,...,x_n\}$). Similarly, the notation $y\in \underline{x}$ means: $y\in \{x_1,...,x_n\}$. $P^n(\underline{x})$ will be used as an abbreviation for $P^n(x_1,...,x_n)$ etc. $lh(\underline{x})$ denotes the length of $\underline{x}$.

Let A be a formula, $\underline{x}=(x_1,...,x_n)$, $\underline{y}=(y_1,...,y_n)$ be lists of variables, such that all x_i are distinct. Then $(\underline{y}/\underline{x})A$ denotes the result of simultaneous replacement of free occurrences of each x_i in A by y_i; as usual, bound variables are renamed whenever clashes might appear.

Let $z_1,..., z_n,...$ be the "alphabetic" list (i.e. a sublist of $v_1,..., v_n,...$) of all variables not occurring in A, $\underline{z}=(z_1,...,z_m)$, $m\geq 0$. Then $A^{[m]}$ denotes the result of replacing in A each atomic subformula of the form $P^k(\underline{x})$ by $P^{k+m}(\underline{x}, \underline{z})$; atomic subformulas of the form $(x=y)$ are not touched. It is clear that $FV(A^{[m]})=FV(A)\cup\{z_1,...,z_m\}$ if A contains at least one predicate letter, otherwise $FV(A^{[m]})=FV(A)$.

Recall that a *substitution instance* $(C/P^n(\underline{x}))A$ of a formula A is obtained by replacing every occurrence of $P^n(\underline{y})$ in A by $(\underline{y}/\underline{x})C$, possibly renaming bound variables.

A *superintuitionistic predicate logic* without equality (s.p.l.) or with equality (s.p.l.=) is a set of formulas $\mathbf{L}\subseteq IF^{(=)}$ closed under the rules of detachment (A,A → B/B), generalization (A/∀xA)- for any $x \in Vr$, substitution (i.e. $A \in \mathbf{L}$ only if **L** contains all substitution instances of A) and containing all axioms (and therefore all the theorems) of the intuitionistic predicate calculus (without or with equality, respectively). $\mathbf{QH}^{(=)}$ denotes the least s.p.l (s.p.l.=), that is intuitionistic predicate logic (resp., with equality).

Analogously, a *modal predicate logic* (m.p.l. or m.p.l.=) is a set of formulas $L \subseteq MF^{(=)}$ closed under detachment (A,A ⊃ B/B), generalization, substitution, and necessitation (A/□A), containing all axioms of classical predicate calculus (without or with equality) and axioms of modal system **S4**. The least m.p.l. (=) w.r.t. inclusion is denoted by $\mathbf{QS4}^{(=)}$. (**L**+X) denotes the least m.p.l.(=) containing an m.p.l.(=) **L** and a set $X \subseteq MF^{(=)}$. The same notation will be used also for the intuitionistic case.

Propositional formulas (intuitionistic or modal) are predicate formulas without occurrences of individual variables (it means that they have no occurrences of quantifiers and n-place predicate letters with n>0). A *superintuitionistic propositional logic* is a set of intuitionistic propositional formulas closed under detachment and substitution of propositional formulas for propositional letters and containing all axioms of the intuitionistic propositional calculus; the smallest of these logics is intuitionistic propositional logic (denoted by **H**). Analogously, a *modal propositional logic* is a set of modal propositional formulas closed under propositional substitutions, detachment, and necessitation and containing all axioms of **S4**; the smallest of these logics is **S4**. The notation (**L**+X) will be used in the propositional case as well.

If **L** is a predicate logic (modal or superintuitionistic) then its restriction to propositional formulas obviously is a propositional logic; this is called the *propositional fragment* of **L** and denoted by $\mathbf{L}_p$.

If **L** is a propositional logic (modal or superintuitionistic) then (**QS4**+**L**) (resp., (**QH**+**L**)) is the least m.p.l. (resp., s.p.l.) containing **L**; it is denoted also by **QL**. One can easily show that $(\mathbf{QL})_p=\mathbf{L}$.

Let also $\mathbf{QL}^{=}= \mathbf{QS4}^{=}+\mathbf{L}$ (or $\mathbf{QH}^{=}+\mathbf{L}$).

If D is an arbitrary non-empty set we can extend our basic first order language by adding elements of D as individual constants. Formulas of this extended language are called *D-formulas* (modal or intuitionistic). If A is a D-formula, $x \in Vr$, $a \in D$ then (a/x)A denotes the result of replacing each free occurrence of x in A by a. We also use the notation $(\underline{a}/\underline{x})A$ provided that $\underline{a} \in D^n$, $\underline{x} =(x_1,\dots,x_n)$ is a list of distinct variables.

1.2. Standard Kripke semantics.

Let us recall the main definitions concerning Kripke semantics for propositional and predicate logics [1], [3].

A *propositional Kripke frame* (or merely, a *frame*) is a pair $(F,\leq)$ in which $\leq \subseteq F \times F$ is a pre-order (a reflexive and transitive relation) on a non-empty set F. Sometimes a frame $(F,\leq)$ will be also denoted by F. The pre-order $\leq$ induces an equivalence relation $\approx$ on F:

$$u \approx v \iff u \leq v \ \&\ v \leq u;$$ [\1]

its equivalence classes are called *clusters*. By factorizing the frame F through $\approx$ we obtain a partially ordered (p.o.) set $(F/\approx)$ which is called the *skeleton* of F.

A *p-morphism* of a frame $(F',\leq')$ onto a frame $(F,\leq)$ is an onto map $h: F' \to F$ such that

(i) $\forall u,v \in F' \ (u \leq' v \Rightarrow h(u) \leq h(v))$;
(ii) $\forall u \in F' \forall w \in F \ (h(u) \leq w \Rightarrow \exists v \in F'(u \leq' v \& h(v)=w))$.

A *pre-p-morphism* of F' onto F is an onto map $h: F' \to F$ satisfying (i) and

(ii') $\forall u \in F' \forall w \in F \ (h(u) \leq w \Rightarrow \exists v \in F'(u \leq' v \& h(v) \approx w))$.

It is clear that a pre-p-morphism gives rise to a p-morphism of skeletons; every pre-p-morphism onto a p.o. frame F is a p-morphism.

A *(modal) valuation* in a frame F is a function ξ assigning subsets of F to propositional letters: $\xi(P^0) \subseteq F$ for any $P^0 \in PL^0$. An *intuitionistic valuation* should also satisfy the *monotonicity condition:*

(μ1) $\forall u,v \in F \ (u \leq v \& u \in \xi(P^0) \Rightarrow v \in \xi(P^0))$.

A modal valuation ξ in F gives rise to a *forcing relation* $\xi,u \models A$ between $u \in F$ and a propositional modal formula A; it is defined by the following conditions:

(a) $\xi,u \models P^0 \iff u \in \xi(P^0)$
(b) $\xi,u \not\models \bot$;
(c) $\xi,u \models B \supset C \iff \xi,u \not\models B \vee \xi,u \models C$;

\1 $\&$, $\Rightarrow$, $\iff$, serve as logical metasymbols; the formal symbols $\vee$, $\exists$, $\forall$ have a double usage as metasymbols.

(d) $\xi,u \models \Diamond B \iff \exists v \geq u\ \xi,v \models B$.

We will also write $u \models A$ instead of $\xi,u \models A$ if ξ can be restored from the context.

If ξ is intuitionistic, the forcing relation $\xi,u \models A$ between worlds and intuitionistic formulas is defined by (a), (b) and

(e) $\xi,u \models B \wedge C \iff \xi,u \models B\ \&\ \xi,u \models C$;
(f) $\xi,u \models B \vee C \iff \xi,u \models B \vee \xi,u \models C$;
(g) $\xi,u \models B \rightarrow C \iff \forall v \geq u\ (\xi,v \models B \Rightarrow \xi,v \models C)$.

Then the following *monotonicity condition* is satisfied by any formula A:

(μ2) $u \leq v\ \&\ \xi,u \models A \Rightarrow \xi,v \models A$.

A propositional formula A is *true* w.r.t. the valuation ξ (notation: $\xi \models A$) iff $\forall u \in F\ \xi,u \models A$. A modal (respectively, intuitionistic) formula A is *valid* in F (notation: $F \models A$) iff $\xi \models A$ for any modal (respectively, intuitionistic) valuation ξ. The set of all modal (respectively, intuitionistic) formulas valid in F is denoted by $ML_p(F)$ (resp., $IL_p(F)$) and is called the *modal* (resp., *intuitionistic*) *logic* of the frame F; one can prove that it is indeed a propositional logic of the corresponding type.

We may also observe that $IL_p(F)=IL_p(F/\approx)$ and $IL_p(F') \subseteq IL_p(F)$ if F is a pre-p-morphic image of F'. Similarly, in the modal case $ML_p(F') \subseteq ML_p(F)$ if F is a p-morphic image of F'.

A *predicate Kripke frame* is a triple $\mathbb{F}=(F,\leq,D)$ in which $(F,\leq)$ is a propositional frame (of *possible worlds*) and $D=(D_{(u)})_{u \in F}$ is a family of non-empty sets, such that $D_{(u)} \subseteq D_{(v)}$ whenever $u \leq v$. Sometimes the notation $(F,\leq,D)$ will be also abbreviated to (F,D).

A *(modal) valuation* ξ in $\mathbb{F}$ associates with every predicate letter P^m a family of sets $(\xi_u(P^m))_{u \in F}$ such that $\xi_u(P^m) \subseteq (D_{(u)})^m$; to include the case m=0, we will suppose that $(D_{(u)})^0=\{u\}$. This ξ is called *intuitionistic* if it satisfies the *monotonicity condition*

(μ3) $u \leq v \Rightarrow \xi_u(P^m) \subseteq \xi_v(P^m)$ (if m>0),
$u \leq v\ \&\ u \in \xi_u(P^0) \Rightarrow v \in \xi_v(P^0)$,
for any $u,v \in F$, $P^m \in PL^m$, $P^0 \in PL^0$.

A modal valuation ξ gives rise to a *forcing relation* $\xi,u \models A$ between $u \in F$ and a closed modal $D_{(u)}$-formula A satisfying conditions (b),(c),(d) and

(a') $\xi,u \models P^0 \iff u \in \xi_u(P^0)$;
(a'') $\xi,u \models P^m(\underline{a}) \iff \underline{a} \in \xi_u(P^m)$ (for any m>0, $\underline{a} \in (D_{(u)})^m$);

(h) $\xi,u \models a=b \iff a=b$;
(k) $\xi,u \models \exists xA \iff$ for some $a\in D_{(u)}$ $\xi,u \models (a/x)A$.

An intuitionistic valuation ξ also gives rise to a forcing relation $\xi,u \models A$ between $u\in F$ and a closed intuitionistic $D_{(u)}$-formula A satisfying (b), (e), (f), (g), (a'), (a"), (h), (k) and

(l) $\xi,u \models \forall xA \iff$ for all $v\geq u$, $a\in D_{(v)}$ $\xi,v \models (a/x)A$

The forcing relation satisfies the monotonicity condition (μ2) for any closed intuitionistic $D_{(u)}$-formula A.

A predicate formula A is *true* w.r.t. ξ (notation: $\xi \models A$) iff $\xi,u \models [\forall]A$ for any $u\in F$ ($[\forall]A$ denotes the universal closure of A). In fact if $\underline{x} =(x_1,\dots,x_n)$ is a list of distinct variables containing FV(A) then

$\xi \models A$ iff for each $u\in F$, $\underline{a}\in (D_{(u)})^n$, $\xi,u \models (\underline{a}/\underline{x})A$.

This refers also to the case n=0; then $(\underline{a}/\underline{x})A=A$.

A is *valid* in $\mathbb{F}$ (notation: $\mathbb{F} \models A$) iff $\xi \models A$ for any valuation ξ (of the corresponding type). The set of all formulas valid in $\mathbb{F}$ (modal or intuitionistic, without or with equality) is called the (*modal, intuitionistic* etc.) *logic* of the frame $\mathbb{F}$ and denoted by $ML^{(=)}(\mathbb{F})$ (resp., $IL^{(=)}(\mathbb{F})$).

A m.p.l.(=) L is *Kripke-complete* if $L=\bigcap_{i\in I} ML^{(=)}(\mathbb{F}_i)$ for some family $(\mathbb{F}_i)_{i\in I}$ of predicate Kripke frames; an analogous definition is given for the intuitionistic case.

QH, **QS4** and some of their extensions are Kripke-complete, but there are also very simple examples of Kripke-incomplete logics (e.g. $\mathbf{QH}^=$, and $\mathbf{QS4}^=$; cf. also [7], [12]). Therefore some generalizations of standard Kripke semantics were proposed, and now we consider two of them.

1.3 Kripke bundles and functor semantics.

A *Kripke bundle* [12] is a triple $\mathbb{F}=(F,D,\pi)$ in which π is a p-morphism of a frame (D, $\leq'$) onto a frame (F,$\leq$). Then D is split into "individual domains" (or "fibres") $D_{(u)} =\pi^{-1}(u)$ ($u\in F$).

Functor semantics [5] deals with $\mathcal{SET}$-valued functors. Namely let $\mathcal{C}$ be a (small) category, F be its *frame representation*. It means that $F=\mathrm{Ob}\,\mathcal{C}$ is the set of $\mathcal{C}$-objects, and $u\leq v$ iff $\mathcal{C}(u,v)\neq\emptyset$ (i.e. there exist $\mathcal{C}$-morphisms from u to v). A *$\mathcal{C}$-set* (or a *$\mathcal{SET}$-valued functor over* $\mathcal{C}$) can be defined as a triple $\mathbb{F}=(F,\underline{D},E)$ in which $\underline{D}=(D_{(u)})_{u\in F}$ is a family of non-empty disjoint sets, $E=(E_\mu)_{\mu\in \mathrm{Mor}\,\mathcal{C}}$ is a family of functions parametrized by $\mathcal{C}$-morphisms, and $E_\mu : D_{(u)} \to D_{(v)}$ whenever $\mu \in \mathcal{C}(u,v)$. As usual, it is required that $E_{\mu\circ\mu'} = E_\mu \circ E_{\mu'}$, $E_{1_u} = 1_{D(u)}$.

In both cases we introduce auxiliary relations in $D_n = \bigcup_{u\in F}(D_{(u)})^n$

- $\underline{a}\leq_{n,u,v} \underline{b} \Leftrightarrow \underline{a}\in (D_{(u)})^n \;\&\; \underline{b}\in (D_{(v)})^n \;\& u\leq v \;\&$ $(\forall i\; a_i\leq' b_i) \& \forall i\neq j\; (a_i = a_j \Rightarrow b_i = b_j)$ if $\mathbb{F}$ is a Kripke bundle;
- $\underline{a}\leq_{n,u,v} \underline{b} \Leftrightarrow \underline{a}\in (D_{(u)})^n \;\&\; \underline{b}\in (D_{(v)})^n \;\& \exists\mu \in \mathcal{C}(u,v) \forall i\; E_\mu(a_i)=b_i$ if $\mathbb{F}$ is a $\mathcal{C}$-set;
- $\underline{a}\leq_n \underline{b} \Leftrightarrow \exists\; u,v\in F\; (\underline{a}\leq_{n,u,v} \underline{b})$.

These definitions refer to the case n=0 as well. As before, we set: $(D_{(u)})^0=\{u\}$, $D_0=F$; then $\underline{a}\leq_0 \underline{b} \Leftrightarrow \exists\; u,v\in F\; (\underline{a}=u \;\&\; \underline{b}=v \& u\leq v)$.

A *(modal) valuation* in $\mathbb{F}$ is a function sending every m-place predicate letter P^m to a set $\xi(P^m)\subseteq D_m$. Like in standard Kripke semantics, we can consider also $\xi_u(P^m)=\xi(P^m)\cap (D_{(u)})^m$ for each $u\in F$. Then the *forcing relation* $\xi,u \models A$ between $u\in F$ and a closed $D_{(u)}$-formu-la A is defined by conditions (b),(c),(a'),(a''),(h),(k); and (d) is replaced by

(d') $\xi,u \models (\underline{a}/\underline{x})\Diamond B \Leftrightarrow \exists v \exists \underline{b}\; (\underline{a}\leq_{n,u,v} \underline{b} \;\&\; \xi,v \models (\underline{b}/\underline{x})B)$.

(Here B is a modal formula, $\underline{x}$=FV(B), lh($\underline{x}$)=n.)

In particular, for the case n=0 (i.e. if B is closed) we have:

$\xi,u \models \Diamond B \Leftrightarrow \exists v\geq u\; \xi,v \models B$.

An *intuitionistic valuation* in $\mathbb{F}$ is a modal valuation satisfying the monotonicity condition:

(μ4) $\underline{a}\leq_n\underline{b} \;\&\; \underline{a}\in \xi(P^n) \Rightarrow \underline{b}\in \xi(P^n)$.

In this case the forcing relation between u and closed $D_{(u)}$-formulas is defined by the conditions (a'), (a''), (b), (h), (k), (e), (f) and

(g') $\xi,u \models (\underline{a}/\underline{x})(B \to C) \Leftrightarrow$ for all $\underline{b}$, $v\geq u$ $(\underline{a}\leq_{n,u,v} \underline{b} \;\&\; \xi,v \models (\underline{b}/\underline{x})B$ $\Rightarrow \xi,v \models (\underline{b}/\underline{x})C)$

(here we suppose FV(B → C)=$\underline{x}$, lh($\underline{x}$)=n);

(l') $\xi,u \models \forall y\; (\underline{a}/\underline{x})B \Leftrightarrow$ for all $\underline{b}$, $v\geq u$, $c\in D_{(v)}$ $(\underline{a}\leq_{n,u,v} \underline{b} \Rightarrow$ $\xi,v\models (\underline{b},c/\underline{x},y)B)$

(here we suppose that FV(B)=$\underline{x}$,y or FV(B)=$\underline{x}$, and that lh($\underline{x}$)=n).

The truth of a predicate formula w.r.t. a valuation ξ, and validity in a Kripke bundle or in a $\mathcal{C}$-set are now defined as in standard Kripke semantics. However the set of valid formulas $ML^{\sim(=)}(\mathbb{F})$ (or $IL^{\sim(=)}(\mathbb{F})$) is not always substitution closed (and hence it is not always a logic in the sense of Sec.1.1). E.g. (cf. [12]) for a Kripke bundle $\mathbb{F}_0=(F_0,D_0,\pi_0) = (\{u\},\{a_1,a_2\},\pi_0)$, with $a_1<'a_2$, $\pi_0(a_1) = \pi_0(a_2)=u$, we have:

$$\Diamond P^0 \supset P^0 \in ML^{\sim}(\mathbb{F}_0),\ (P^0 \vee \neg P^0) \in IL^{\sim}(\mathbb{F}_0),$$

but

$$\Diamond P^1(x) \supset P^1(x) \notin ML^{\sim}(\mathbb{F}_0),\ (P^1(x) \vee \neg P^1(x)) \notin IL^{\sim}(\mathbb{F}_0).$$

Therefore the *predicate logic* $ML^{(=)}(\mathbb{F})$ (or $IL^{(=)}(\mathbb{F})$) of a Kripke bundle or a $\mathcal{C}$-set $\mathbb{F}$ is defined as the set of formulas whose substitution instances are always valid in $\mathbb{F}$; it is rather easily proved that

$$ML^{(=)}(\mathbb{F}) = \{A \in MF^{(=)} \mid \forall m \geq 0\ \mathbb{F} \models A^{[m]}\},$$
$$IL^{(=)}(\mathbb{F}) = \{A \in IF^{(=)} \mid \forall m \geq 0\ \mathbb{F} \models A^{[m]}\}.$$

The previous examples reveal another difference between Kripke bundles and predicate Kripke frames. Viz., for a predicate Kripke frame $\mathbb{F}=(F,\leq,D)$ we have:

$$ML(\mathbb{F})_p=ML_p(F) \text{ and } IL(\mathbb{F})_p=IL_p(F).$$

On the other hand, for Kripke bundle $\mathbb{F}_0$ we have:

$$\Diamond P^0 \supset P^0 \in ML_p(F_0)\backslash ML(\mathbb{F}_0)_p,\ (P^0 \vee \neg P^0) \in IL_p(F_0)\backslash IL(\mathbb{F}_0)_p.$$

The standard Kripke semantics is a particular case of Kripke bundle semantics. More precisely, a predicate Kripke frame $\mathbb{F}=(F,\leq,D)$ corresponds to a Kripke bundle $\mathbb{F}'=(F,D',\pi)$, with $D'=\{(u,a) \mid u \in F, a \in D_{(u)}\}$, $\pi(u,a)=a$. (Thus $D'_{(u)} = \{u\} \times D_{(u)}$).) It is easily proved that $ML^{(=)}(\mathbb{F}')=ML^{(=)}(\mathbb{F})$, $IL^{(=)}(\mathbb{F}')=IL^{(=)}(\mathbb{F})$.

Furthermore, Kripke bundle semantics is a particular case of functor semantics. Namely, a Kripke bundle $\mathbb{F}=(F,D,\pi)$ corresponds to a category $\mathcal{C}$ whose frame representation is F, such that $\mathcal{C}(u,v)=\{f \mid f$ is a function from $D_{(u)}$ to $D_{(v)}$, $a \leq' f(a)$ for any $a \in D_{(u)}\}$, and to a $\mathcal{C}$-set

$$\mathbb{F}'=(F,\underline{D},E), \text{ with } \underline{D}=(D_{(u)})_{u \in F},\ E=(E_\mu)_{\mu \in Mor\,\mathcal{C}},\ E_\mu = \mu \text{ for any } \mu.$$

In this case the corresponding relations $\leq_n$ in $\mathbb{F}$ and in $\mathbb{F}'$ coincide. To show this, assume that $\underline{a}\leq_{n,u,v}\underline{b}$ in $\mathbb{F}$. Then the condition $a_i = a_j \Rightarrow b_i = b_j$ provides functionality of the relation $\{(a_i,b_i) \mid 1\leq i\leq n\}$; hence there exists a function $\mu: D_{(u)} \rightarrow D_{(v)}$ such that $\mu(a_i)=b_i$, and thus $a\leq'\mu(a)$ since $\underline{a}\leq_{n,u,v}\underline{b}$ in $\mathbb{F}$. Consequently $\underline{a}\leq_n\underline{b}$ in $\mathbb{F}'$. The converse implication is proved similarly.

Therefore, the forcing relations in $\mathbb{F}$ and in $\mathbb{F}'$ also coincide, and $ML^{(=)}(\mathbb{F}')=ML^{(=)}(\mathbb{F})$, $IL^{(=)}(\mathbb{F}')=IL^{(=)}(\mathbb{F})$.

One can observe that the definitions of forcing in Kripke bundles and in $\mathcal{C}$-sets look very similar to that in Kripke semantics, especially if we use the notation $\xi,\underline{a} \models A\,[\underline{x}]$ instead of $\xi,u \models (\underline{a}/\underline{x})A$ (u can be dropped because it is determined by $\underline{a}$). Then (d') can be rewritten as

$$\xi,\underline{a} \models \Diamond B\,[\underline{x}] \Leftrightarrow \exists \underline{b}\ (\underline{a}\leq_n \underline{b}\ \&\ \xi,\underline{b} \models B\,[\underline{x}]).$$

This observation motivates a further step from functor and Kripke bundle semantics to the metaframe semantics described in the next section.

2. Metaframes.

2.1. Preliminaries.

Let Σ_0 be the restriction of the category $\mathcal{SET}$ of sets and maps to the set ω. Thus objects in Σ_0 are finite ordinals: $0=\emptyset$, $1=\{0\}$,..., $n+1=\{0,1,...,n\}$,..., and $\Sigma_0(m,n)$ is the set of all maps from m to n.[2] For the sake of convenience, we use an isomorphic version Σ of Σ_0 whose objects are: $I_0=\emptyset$, $I_1=\{1\}$,..., $I_n=\{1,...,n\}$, and whose morphisms are set-theoretic maps. $\Sigma(m,n)$ abbreviates $\Sigma(I_m,I_n)$, and $\Sigma^{\sim}(m,n)$ denotes the set of all injective maps from I_m to I_n. Then

$$\Sigma(m,n)\neq\emptyset \Leftrightarrow n>0 \vee m=0,$$

$$\Sigma^{\sim}(m,n)\neq\emptyset \Leftrightarrow 0\leq m\leq n.$$

We have also: $\Sigma(0,n) = \Sigma^{\sim}(0,n) = \{\Lambda_n\}$, Λ_n being an empty map from $\emptyset$ to I_n.

Every map $\sigma\in\Sigma(m,n)$ can be extended to a map $\sigma^+\in\Sigma(m+1,n+1)$, by setting $\sigma^+(m+1)=n+1$; σ^+ is called the *minimal extension* of σ.

\2 Consequently, Σ_0 is equivalent to the category $\mathcal{FINSET}$ of finite sets and maps.

As usual, $\circ$ denotes composition of maps. Some maps have special notations. $id(m,n) \in \Sigma^{\sim}(m,n)$ is such that $id(m,n)=i$ for each $i \in I_m$ (thus $id(n,n)$ is the identity map, $id(0,n)=\Lambda_n$). Let also $j_n=id(n-1,n)$, $id(n)=id(n,n)$. Also let $pr_{n,i} \in \Sigma^{\sim}(1,n)$ be such that $pr_{n,i}(1)=i$ (and thus, $pr_{n,1}=id(1,n)$).

Every n-tuple $\underline{x} = (x_1,\dots,x_n)$ of variables can be identified with a map from I_n to Vr. If $\sigma \in \Sigma(m,n)$ we can construct a composition $\underline{x} \circ \sigma$ corresponding to the n-tuple $\underline{x}_\sigma = (x_{\sigma(1)},\dots,x_{\sigma(m)})$; if m=0 and $\sigma=\Lambda_n$ then $\underline{x}_\sigma$ is empty. The notation $\underline{a}_\sigma$ will be used also for arbitrary n-tuples (not necessarily from Vr^n). The map of n-tuples sending $\underline{a}$ to $\underline{a}_\sigma$ is called a *σ-transformation* . Usually we will make no difference between $\underline{a}_\sigma$ and $\underline{a} \circ \sigma$.

If $z \in Vr$, $\underline{x} \in Vr^n$, let $(\underline{x}-z)$ be the result of deleting all occurrences of z from $\underline{x}$ (thus, if $\underline{x} = (x_1,\dots,x_n)$ consists of distinct variables then $lh(\underline{x}-x_i)=n-1$, and $\underline{x}-z=\underline{x}$ if $z \notin \underline{x}$). Let $\underline{x}+z=\underline{x}-z,z$ be the result of putting z behind $\underline{x}$-z. Let $n'= lh(\underline{x}+z)$, $\sigma_{\underline{x}-z} \in \Sigma^{\sim}(n'-1,n)$, so that $\underline{x}-z=\underline{x}\circ\sigma_{\underline{x}-z}=(\underline{x}+z)\circ j_{n'}$.

Thus if $z=x_i$ we have: $n'=n$, $\sigma_{\underline{x}-z}(j)= j$ if $1 \le j < i$, $\sigma_{\underline{x}-z}(j)= j+1$ if $i \le j \le n-1$. If $z \notin \underline{x}$ we have: $n'=n+1$, and $\sigma_{\underline{x}-z}=id(n)$.

2.2. Forcing in metaframes.

A *predicate Kripke metaframe* (or briefly, a *metaframe*) is a diagram morphism (cf. [4]) of the dual category Σ^0 to the category of frames and maps. Thus a metaframe can be defined as a pair $\mathcal{F} = (\mathbb{D}, \mathbb{\Pi})$ in which $\mathbb{D} = (D_n, \le_n)_{n\in\omega}$ is a sequence of frames, $\mathbb{\Pi} = (\pi_\sigma)_{\sigma\in Mor\Sigma}$ is a family of maps parametrized by Σ-morphisms, and $\pi_\sigma: D_n \to D_m$ whenever $\sigma \in \Sigma(m,n)$. We say that $\mathcal{F}$ is a metaframe *over a frame* F if $F=(D_0, \le_0)$. Non-formally F can be treated as a frame of "possible worlds"; D_n as an "n-dimensional" individual domain, $\le_n$ as an accessibility relation between "abstract n-tuples", the π_σ as "abstract transformations of n-tuples". For this reason, $\pi_\sigma(\underline{a})$ will sometimes be denoted as $(\underline{a}\cdot\sigma)$ if π is clear from the context.

An "abstract projection" π_{Λ_n} gives rise to a splitting $D_n = \bigcup_{u\in F} D_{n,u}$ in which $D_{n,u} = (\pi_{\Lambda_n})^{-1}(u)$ (some of these $D_{n,u}$ can be empty). $D_{n,u}$ is an "n-dimensional domain in the world u". We have equivalence relations in D_n : $\underline{a} \approx_n \underline{b} \iff \underline{a} \le_n \underline{b}$ & $\underline{b} \le_n \underline{a}$; let also $\underline{a} <_n \underline{b} \iff \underline{a} \le_n \underline{b}$ & not $\underline{b} \le_n \underline{a}$.

Every $\mathcal{C}$-set $\mathbb{F}=(F,D,E)$ corresponds to a metaframe $\mathcal{F}[\mathbb{F}]$, with

D_n, $\leq_n$ defined in 1.3, and $\pi_\sigma(\underline{a})=\underline{a}_\sigma$ for any $\underline{a} \in D_n$, $\sigma \in \Sigma(m,n)$. So in this case abstract n-tuples and transformations are real.

An *$\mathcal{F}$-formula* (intuitionistic or modal, with or without equality) of level n is a triple $(\underline{a}, \underline{x}, A)$ in which A is a formula (of the corresponding type), $\underline{a} \in D_n$, $\underline{x}$ is a list of distinct variables of length n, $FV(A) \subseteq \underline{x}$. (Thus if n=0, A must be closed, $\underline{a} \in D_0=F$.) A *(modal) valuation* in a metaframe $\mathcal{F}$ is a function ξ sending every m-place predicate letter P^m to a set $\xi(P^m) \subseteq D_m$; an *intuitionistic valuation* should also satisfy the *monotonicity condition* :

(μ) $\underline{a} \leq_m \underline{b}$ & $\underline{a} \in \xi(P^m) \Rightarrow \underline{b} \in \xi(P^m)$.

The pair $\mathcal{M} = (\mathcal{F}, \xi)$ is called a *metaframe model* . The *truth* of an $\mathcal{F}$-formula with equality $(\underline{a}, \underline{x}, A)$ *in* $\mathcal{M}$ is denoted by $\mathcal{M}, \underline{a} \models A\,[\underline{x}]$ or by $\xi, \underline{a} \models A\,[\underline{x}]$ and defined inductively (we assume here that $\underline{a} \in D_n$, $\underline{x} = (x_1,...,x_n)$, $n'=lh(\underline{x}+z)$):

- $\xi, \underline{a} \models P^m(\underline{x}_\sigma)\,[\underline{x}] \Leftrightarrow \pi_\sigma(\underline{a}) \in \xi(P^m)$;
- $\xi, \underline{a} \not\models \bot\,[\underline{x}]$;
- $\xi, \underline{a} \models x_i=x_j \Leftrightarrow \pi_{pr_{n,i}}(\underline{a})=\pi_{pr_{n,j}}(\underline{a})$;

for modal formulas:

- $\xi, \underline{a} \models (B \supset C)\,[\underline{x}] \Leftrightarrow \xi, \underline{a} \not\models B\,[\underline{x}]$ or $\xi, \underline{a} \models C\,[\underline{x}]$;
- $\xi, \underline{a} \models \Diamond B\,[\underline{x}] \Leftrightarrow \exists \underline{b}\,(\underline{a} \leq_n \underline{b}\ \&\ \xi, \underline{b} \models B\,[\underline{x}])$;
- $\xi, \underline{a} \models \exists z B\,[\underline{x}] \Leftrightarrow \exists \underline{b} \in D_{n'}\,(\pi_{j_{n'}}(\underline{b})=\pi_{\sigma_{\underline{x}-z}}(\underline{a})\ \&\ \xi, \underline{b} \models B\,[\underline{x}+z])$;

for intuitionistic formulas:

- $\xi, \underline{a} \models (B \wedge C)\,[\underline{x}] \Leftrightarrow \xi, \underline{a} \models B\,[\underline{x}]$ and $\xi, \underline{a} \models C\,[\underline{x}]$;
- $\xi, \underline{a} \models (B \vee C)\,[\underline{x}] \Leftrightarrow \xi, \underline{a} \models B\,[\underline{x}]$ or $\xi, \underline{a} \models C\,[\underline{x}]$;
- $\xi, \underline{a} \models (B \rightarrow C)\,[\underline{x}] \Leftrightarrow \forall \underline{b}\,(\underline{a} \leq_n \underline{b}\ \&\ \xi, \underline{b} \models B\,[\underline{x}] \Rightarrow \xi, \underline{b} \models C\,[\underline{x}])$;
- $\xi, \underline{a} \models \exists z B\,[\underline{x}] \Leftrightarrow \exists \underline{b} \in D_{n'}\,(\pi_{j_{n'}}(\underline{b}) \leq_{n'-1} \pi_{\sigma_{\underline{x}-z}}(\underline{a})\ \&\ \xi, \underline{b} \models B\,[\underline{x}+z])$;
- $\xi, \underline{a} \models \forall z B\,[\underline{x}] \Leftrightarrow \forall \underline{b} \in D_{n'}\,(\pi_{\sigma_{\underline{x}-z}}(\underline{a}) \leq_{n'-1} \pi_{j_{n'}}(\underline{b}) \Rightarrow \xi, \underline{b} \models B[\underline{x}+z])$.

As in the functor or Kripke bundle semantics, $\models$ can be called "forcing relation" between $(\underline{a}, \underline{x})$ and A.

As we have already mentioned, forcing in metaframes is equivalent to forcing in $\mathcal{C}$-sets in the following sense. If a metaframe $\mathcal{F}[\mathbb{F}]$ corresponds to a $\mathcal{C}$-set $\mathbb{F}$ then

$$\xi, \underline{a} \models A\,[\underline{x}]\ (\text{in } \mathcal{F}[\mathbb{F}]) \Leftrightarrow \xi, u \models (\underline{a}/\underline{x})A\ (\text{in } \mathbb{F})$$

provided that $u \in F$, $\underline{x} \in Vr^n$, $\underline{a} \in (D_{(u)})^n$, A is a modal (or intuitionistic) formula, ξ is a modal (or intuitionistic) valuation in $\mathbb{F}$, $FV(A) \subseteq \underline{x}$. This can be shown by an easy inductive proof, and it is useful to have this equivalence in mind for a better motivation of the previous definition.

E.g. for the case $A=\exists zB$ (modal) our definition corresponds to the following two equivalences in $\mathcal{C}$-sets:

$$u \models (\underline{a}/\underline{x})\exists x_i B \Leftrightarrow \exists \underline{b} \in D_n \,((b_1,\dots,b_{n-1})=(a_1,\dots,a_{i-1},a_{i+1},\dots,a_n) \;\&\; u \models (b_1,\dots,b_n/x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)B),$$

$$u \models (\underline{a}/\underline{x})\exists zB \Leftrightarrow \exists \underline{b} \in D_{n+1} \,((b_1,\dots,b_n)=\underline{a} \;\&\; u \models (\underline{b}/\underline{x}, z)B\,) \text{ (if } z \notin \underline{x}).$$

(Indeed, if $z=x_i$ we have:

$n'=n$, $\pi_{\sigma_{\underline{x}-z}}(\underline{a})=\underline{a}\circ\sigma_{\underline{x}-z} = (a_1,\dots,a_{i-1},a_{i+1},\dots,a_n), \pi_{j_{n'}}(\underline{b})=\underline{b}\circ j_n$;

if $z \notin \underline{x}$ we have:

$n'=n+1, \pi_{\sigma_{\underline{x}-z}}(\underline{a})=\underline{a}\circ id(n)=\underline{a}$, $\pi_{j_{n'}}(\underline{b})=\underline{b}\circ j_{n+1}$.)

For the intuitionistic case, analogous transcriptions can be made; the only peculiarity (the use $\leq_{n'-1}$ of instead of "=") appears in the definition of forcing for $\exists$-formulas, but in fact this is not essential for logics without equality (cf. remarks after 2.5.1 and 2.6.2).

We will often drop ξ in the notation "$\xi, \underline{a} \models A\,[\underline{x}]$".

Let $\underline{x}$ be a list of distinct variables of length n, $FV(A) \subseteq \underline{x}$. The formula A is called *true under* ξ (or *in* $\mathcal{M}$) w.r.t. $\underline{x}$ (notation: $\xi \models A\,[\underline{x}]$) if $\xi, \underline{a} \models A\,[\underline{x}]$ for any $\underline{a} \in D_n$. A modal (intuitionistic) formula A is *valid in* $\mathcal{F}$ (notation: $\mathcal{F} \models A$) iff $\xi \models A\,[\underline{x}]$ for any $\underline{x} \supseteq FV(A)$ and for any modal (intuitionistic) valuation ξ in $\mathcal{F}$.

We set

$$ML^{(=)\sim}(\mathcal{F}) = \{A \in MF^{(=)} \mid \mathcal{F} \models A\},\; IL^{(=)\sim}(\mathcal{F}) = \{A \in IF^{(=)} \mid \mathcal{F} \models A\},$$

$$ML^{(=)}(\mathcal{F}) = \{A \in MF^{(=)} \mid \forall n \geq 0\; \mathcal{F} \models A^{[n]}\} \text{ (} A^{[n]} \text{ was defined in 1.1)},$$

$$IL^{(=)}(\mathcal{F}) = \{A \in IF^{(=)} \mid \forall n \geq 0\; \mathcal{F} \models A^{[n]}\}.$$

It is clear that $ML^{(=)}(\mathcal{F}) \subseteq ML^{(=)\sim}(\mathcal{F})$, and $ML^{(=)\sim}(\mathcal{F}) \subseteq ML^{(=)\sim}(\mathcal{F}') \Rightarrow ML^{(=)}(\mathcal{F}) \subseteq ML^{(=)}(\mathcal{F}')$. The same is true for the intuitionistic case.

2.3. Soundness.

The sets $ML^{(=)\sim}(\mathcal{F})$, $IL^{(=)\sim}(\mathcal{F})$ are not necessarily predicate logics in the sense of Sec.1.2. Moreover, there exist metaframes $\mathcal{F}$ such that these sets are neither substitution closed nor extensions of **QS4** (or **QH**).

A metaframe $\mathcal{F}$ is called *modally sound* , without "=" or with "=" respectively (briefly, *m-sound* or *m*$^{=}$*-sound*) if it satisfies the following requirements:

$(I)^{m(=)}$ *(logical soundness)* . $ML^{(=)}(\mathcal{F})$ is a m.p.l. (=).

$(II)^{m(=)}$ *(truth invariance).* For any $A \in MF^{(=)}$ and for any modal valuation ξ in $\mathcal{F}$, the truth-value of $\xi \models A\,[\underline{x}]$ does not depend on $\underline{x}$ (provided that $FV(A) \subseteq \underline{x}$).

$(III)^{m(=)}$ *(variable renaming invariance).* For any $A \in MF^{(=)}$, $\sigma \in \Sigma(m,n)$ and any lists of distinct variables: $\underline{x} = (x_1,..,x_n)$, $\underline{y} = (y_1,..,y_m)$ such that $FV(A) \subseteq \underline{y}$, for any valuation ξ in $\mathcal{F}$, and any $\underline{a} \in D_n$:

$\xi, \underline{a} \models (\underline{x}_\sigma/\underline{y})A\,[\underline{x}] \iff \xi, \pi_\sigma(\underline{a}) \models A\,[\underline{y}]$.

A particular case of (III) is the following "invariance of forcing $\xi, \underline{a} \models A\,[\underline{x}]$ w.r.t. the list $\underline{x} \supseteq FV(A)$ ":

$(III')^{m(=)}$. For any $A \in MF^{(=)}$, $\sigma \in \Sigma^{\sim}(m,n)$ (sic!) and any list of distinct variables $\underline{x} = (x_1,..,x_n)$ such that $FV(A) \subseteq \underline{x}_\sigma$, for any valuation ξ in $\mathcal{F}$, and any $\underline{a} \in D_n$:

$\xi, \underline{a} \models A\,[\underline{x}] \iff \xi, \pi_\sigma(\underline{a}) \models A\,[\underline{x}_\sigma]$.

Intuitionistic soundness without "=" or with "=" (briefly, *i-soundness* or *i*$^{=}$*-soundness*) is characterized by analogous properties $(I)^{i(=)}$, $(II)^{i(=)}$, $(III)^{i(=)}$ and also by

$(IV)^{i(=)}$*(monotonicity)* For any $A \in IF^{(=)}$, any list of distinct variables $\underline{x} = (x_1,..,x_n)$ containing FV(A), and any valuation ξ in $\mathcal{F}$:

$\underline{a} \leq_n \underline{b}\ \&\ \xi, \underline{a} \models A\,[\underline{x}] \Rightarrow \xi, \underline{b} \models A\,[\underline{x}]$.

Condition (I) is necessary for constructing a semantics for modal and superintuitionistic logics. Condition (II) allows us to define the truth of a formula under a valuation. The conditions (III) and (IV) are closely related to (I); however we do not know if (III) (or (IV)) follows from the conjunction (I)&(II).

We will consider only $m^{(=)}$-sound and $i^{(=)}$-sound metaframes. In other words, we are interested only in metaframe semantics deriving from some classes of sound metaframes (of the corresponding type). Thus a maximal metaframe semantics (for each of four types of predicate logics) is generated by all sound metaframes.

Our goal now is to give a non-logical description of soundness.

2.4. Modal and intuitionistic metaframes.

A metaframe $\mathcal{F} = (\mathbb{D}, \mathbb{\Pi})$ is called *modal* (briefly, *m-metaframe*) if it satisfies the following conditions:

(0) $\pi_{\Lambda_1}(D_1)=F$;

(1) $\sigma \in \Sigma(m,n)$, $\underline{a}, \underline{b} \in D_n$, $\underline{a} \leq_n \underline{b} \Rightarrow \pi_\sigma(\underline{a}) \leq_m \pi_\sigma(\underline{b})$;

(2) $\sigma \in \Sigma(m,n)$, $\underline{a} \in D_n$, $\underline{c} \in D_m$, $\pi_\sigma(\underline{a}) \leq_m \underline{c} \Rightarrow \exists \underline{b} \in D_n(\underline{a} \leq_n \underline{b}\ \&\ \pi_\sigma(\underline{b})=\underline{c})$;

(3) ("lift property") if $\sigma \in \Sigma(m,n)$, $\underline{a} \in D_n$, $\underline{b} \in D_{m+1}$, $\pi_\sigma(\underline{a})=\pi_{j_{m+1}}(\underline{b})=\underline{d} \in D_m$, then there exists $\underline{c} \in D_{n+1}$ such that $\pi_{\sigma^+}(\underline{c})=\underline{b}$, $\pi_{j_{n+1}}(\underline{c})=\underline{a}$ i.e. the following square commutes:

$$\begin{array}{lcc} \pi_{j_{n+1}}: \underline{c} & \mapsto & \underline{a} \\ \quad\ \downarrow \pi_{\sigma^+} & & \downarrow \pi_\sigma \\ \pi_{j_{m+1}}: \underline{b} & \mapsto & \underline{d} \end{array}$$

(4) if $\sigma \in \Sigma(m,n)$, $\sigma' \in \Sigma(k,m)$, then $\pi_{\sigma \circ \sigma'} = \pi_{\sigma'} \circ \pi_\sigma$;

(5) $\pi_{id(n)} = \mathbf{1}_{D_n}$ (the identity map).

Conditions (1) and (2) mean that π_σ is a p-morphism of $(D_n, \leq_n)$ onto a generated subframe of $(D_m, \leq_m)$. The conditions (1), (4), (5) mean that $\mathcal{F}$ is a cofunctor from Σ to the category *QO* of quasi-ordered sets and monotonic maps.

Let us observe also that (4) implies $\pi_{\Lambda_n} = \pi_{\Lambda_m} \circ \pi_\sigma$ for any $\sigma \in \Sigma(m,n)$; and (4)&(0) implies $\pi_{\Lambda_n}(D_n)=F$. In fact, if (4),(5) are satisfied, (0) is equivalent to

(0') $\bigcup_{n>0} \pi_{\Lambda_n}(D_n) = F$,

and also to

(6) if $\sigma \in \Sigma^{\sim}(m,n)$ then $\pi_\sigma(D_n)=D_m$.

A modal metaframe $\mathcal{F}$ is called a *modal metaframe with equality (m=-metaframe)* if it satisfies

($0^=$) for any $\underline{a} \in D_{n+1}$, $\underline{a} \cdot pr_{n+1,n} = \underline{a} \cdot pr_{n+1,n+1} \Rightarrow \underline{a} \cdot j_{n+1} = \underline{a} \cdot j_{n+}$.

In fact for any m-metaframe this condition is equivalent to

($0^\#$) if $\underline{a}, \underline{b} \in D_n$, then $(\forall i\ \underline{a} \cdot pr_{n,i} = \underline{b} \cdot pr_{n,i}) \Rightarrow \underline{a} = \underline{b}$.

Corresponding notions in the intuitionistic case are somewhat more complicated; the main difference consists in replacing most of "=" by "$\leq_n$" or by "$\approx_n$". Viz., $\mathcal{F} = (\mathbb{D}, \mathbb{H})$ is an *intuitionistic metaframe* (briefly, *i-metaframe*) if it satisfies the following conditions:

(0'), (1) as before;

(2') $\sigma \in \Sigma(m,n)$, $\underline{a} \in D_n$, $\underline{c} \in D_m$, $\pi_\sigma(\underline{a}) \leq_m \underline{c} \Rightarrow \exists \underline{b} \in D_n(\underline{a} \leq_n \underline{b}$ & $\pi_\sigma(\underline{b}) \approx_m \underline{c})$;

(3') for any $\sigma \in \Sigma(m,n)$, $\underline{a} \in D_n$, $\underline{b} \in D_{m+1}$:

(i) $\pi_{j_{m+1}}(\underline{b}) \leq_m \pi_\sigma(\underline{a}) \Rightarrow \exists \underline{c} \in D_{n+1}(\pi_{j_{n+1}}(\underline{c}) \leq_n \underline{a}$ & $\underline{b} \leq_{m+1} \pi_{\sigma^+}(\underline{c}))$,

(ii) $\pi_\sigma(\underline{a}) \leq_m \pi_{j_{m+1}}(\underline{b}) \Rightarrow \exists \underline{c} \in D_{n+1}(\underline{a} \leq_n \pi_{j_{n+1}}(\underline{c})$ & $\pi_{\sigma^+}(\underline{c}) \leq_{m+1} \underline{b})$;

(4') if $\sigma \in \Sigma(m,n)$, $\sigma' \in \Sigma(k,m)$, $\underline{a} \in D_n$, then $\underline{a} \cdot (\sigma \circ \sigma') \approx_k (\underline{a} \cdot \sigma) \cdot \sigma'$;

(5) as before.

As in the modal case, we have then

(6') for any $\sigma \in \Sigma^{\sim}(m,n)$,
$\forall \underline{b} \in D_m \exists \underline{a} \in D_n \, \pi_\sigma(\underline{a}) \approx_m \underline{b}$,

and in a particular case m=0:

$\forall u \in F \, \exists \underline{a} \in D_n \, \pi_{\Lambda_n}(\underline{a}) \approx_0 u$.

The conditions (1), (4'), (5) mean that $\mathcal{F}$ corresponds to (so to say) "a cofunctor from Σ to $\mathcal{QO}$ up to clusters". In the presence of (1) and (2'), $\leq_n$ in (3') (i) can be replaced by $\approx_n$, and $\leq_{m+1}$ in (3') (ii) can be replaced by $\approx_{m+1}$ as well.

An i-metaframe $\mathcal{F}$ is called an *intuitionistic metaframe with equality (i$^=$-metaframe)* if it also satisfies conditions

$(0^=)'$ $\underline{a} \leq_n \underline{b}$ & $\underline{a} \cdot pr_{n,i} = \underline{a} \cdot pr_{n,j} \Rightarrow \underline{b} \cdot pr_{n,i} = \underline{b} \cdot pr_{n,j}$;

$(0^=)''$ for any $\underline{a} \in D_{n+1}$,
$\underline{a} \cdot pr_{n+1,n} = \underline{a} \cdot pr_{n+1,n+1} \Rightarrow \underline{a} \cdot j_{n+1} \approx_n \underline{a} \cdot j_n^+$;

$(0^=)'''$ for any m,n>0, $\sigma \in \Sigma(m,n)$, $\underline{a} \in D_n$,
$\underline{a} \cdot (\sigma \circ pr_{m,i}) = \underline{a} \cdot (\sigma \circ pr_{m,j}) \Leftrightarrow (\underline{a} \cdot \sigma) \cdot pr_{m,i} = (\underline{a} \cdot \sigma) \cdot pr_{m,j}$.

In some situations it is convenient to use a narrower class of intuitionistic metaframes. A *strong intuitionistic metaframe (i$^+$-metaframe)* is a metaframe satisfying conditions (0), (1), (2), (4), (5), and (3'). In this situation (3') can be replaced by

$(3^\#)$ for any $\sigma \in \Sigma(m,n)$, $\underline{a} \in D_n$, $\underline{b} \in D_{m+1}$ such that $\pi_\sigma(\underline{a}) = \pi_{j_{m+1}}(\underline{b})$:

(i) $\exists \underline{c} \in D_{n+1}(\pi_{j_{n+1}}(\underline{c}) = \underline{a}$ & $\underline{b} \leq_{m+1} \pi_{\sigma^+}(\underline{c}))$,

(ii) $\exists \underline{c} \in D_{n+1}(\underline{a} \leq_n \pi_{j_{n+1}}(\underline{c})$ & $\pi_{\sigma^+}(\underline{c}) = \underline{b})$.

A *strong intuitionistic metaframe with equality* (*$i^{+=}$-metaframe)* is an i^+-metaframe satisfying also $(0^=)'$, $(0^=)''$.

It is clear that $(0^=)$ implies $(0^=)''$, (4) implies $(0^=)'''$, (n) implies (n') for each n=0,2,3,4, and (0') follows from (0#)&(1)&(2)&(4). Hence we obtain the following implications between the notions introduced:

$$\begin{array}{ccccc} m^=\text{-metaframe} & \Rightarrow & i^{+=}\text{-metaframe} & \Rightarrow & i^=\text{-metaframe} \\ \Downarrow & & \Downarrow & & \Downarrow \\ m\text{-metaframe} & \Rightarrow & i^+\text{-metaframe} & \Rightarrow & i\text{-metaframe} . \end{array}$$

Also it is clear that $i^{+=}$-metaframe $\Leftrightarrow$ i^+-metaframe & $i^=$-metaframe.

2.5. Characterizations of soundness.

Lemma 2.5.0 Let $\mathcal{F} = (\mathbb{D}, \mathbb{H})$ be an i- (or m-) metaframe satisfying $(II)^i$ (or $(II)^m$ respectively), A be an intuitionistic (respectively, modal) propositional formula, n≥0. Then

$\mathcal{F} \models A^{[n]} \Leftrightarrow A \in IL_p(D_n, \leq_n)$ (respectively, $A \in ML_p(D_n, \leq_n)$).

Remark. Once Theorem 2.5.1 is proved, the reference to (II) in this lemma becomes redundant. But 2.5.0 will be used for proving 2.5.1.

Proof. This is an immediate consequence of truth definitions in 2.2 and 1.2 and of conditions (II), (5). ⊠

Theorem 2.5.1. ('Soundness Theorem")

/1/ Every $i^{(=)}$-metaframe is $i^{(=)}$-sound.

/2/ Every $m^{(=)}$-metaframe is $m^{(=)}$-sound.

Proof. (Sketch) At first we prove (III') by an induction on the construction of A (using (1), (2), (4), (5), and also $(0^=)'''$ for the case $A=(x_1=x_2)$, and (3) for the quantifier case). The condition (IV) is also proved inductively (applying $(0^=)'$ for the case $A=(x_i=x_j)$ and (1) for the cases $A=P(\underline{x}_\sigma)$, and $A=\forall zB$, $A=\exists zB$). Then using (III') (and (IV) in the intuitionistic case) we prove (III) and the following claim:

(III") $\xi, \underline{a} \models A\ [\underline{x}]$ does not depend on renaming of bound variables in A.

Condition (II) follows immediately from (III) and (6). Now by 2.5.0, we prove validity of the propositional axioms of **H** (or **S4**). Verification of the predicate axioms is straightforward using (III) (and (IV) for the intuitionistic case). The equality axioms are checked using $(0^=)$ (and (IV)). (III) (together with (IV), in the intuitionistic case) implies that $ML^{\sim(=)}(\mathcal{F})$ and $IL^{\sim(=)}(\mathcal{F})$ are closed under substitutions for individual variables and also under *exact* substitutions for predicate letters; we call a substitution $(C/P^n(\underline{x}))$ (cf. 1.1.) "exact" if $FV(C) \subseteq \underline{x}$. To prove the last statement we show that

$$\xi, \underline{a} \models (C/P^n(\underline{x}))A\ [\underline{x}] \iff \xi', \underline{a} \models A\ [\underline{x}]$$

provided that $\xi'(P^n) = \{ \underline{b} \in D_n \mid \xi, \underline{b} \models C\ [\underline{x}]\}$. Then we observe that each substitution in A is an exact substitution in some $A^{[n]}$ and therefore $ML^{\sim(=)}(\mathcal{F})$, $IL^{\sim(=)}(\mathcal{F})$ are substitution closed, and the proof of (I) is completed.⊠

Remark Due to Theorem 2.5.1, for i-metaframes, in the truth-definition for $\exists$-formulas, $\leq_{n'-1}$ can be replaced by $\approx_{n'-1}$ (to show this we need (2') and (IV)). For i^+-metaframes (and in particular, for m-metaframes) condition (2) allows us to replace $\leq_{n'-1}$ by =, i.e. the definition is the same as in the modal case. Moreover, if $z \notin \underline{x}$, in i^+-metaframes we have:

$$\xi, \underline{a} \models \exists zB\ [\underline{x}] \iff \exists \underline{b} \in D_{n+1}\ (\pi_{j_{n+1}}(\underline{b}) = \underline{a}\ \&\ \xi, \underline{b} \models B\ [\underline{x}+z]\ ).$$

Theorem 2.5.2. /1/ Every $i^=$-sound i-metaframe is an $i^=$-metaframe (and thus, every $i^=$-sound i^+-metaframe is an $i^{+=}$-metaframe).

/2/ If $\mathcal{F}$ is an m-metaframe and $\mathbf{QH}^= \subseteq IL^=(\mathcal{F})$, then $\mathcal{F}$ is an $m^=$-metaframe (and thus, every $m^=$-sound m-metaframe is an $m^=$-metaframe).

Proof. (Sketch) Conditions $(0^=)'$ and $(0^=)'''$ are (IV) and (III) for $A=(x_i=x_j)$; condition $(0^=)''$ follows from the axiom

$$\forall \underline{x}, \underline{y}, z\ (y=z \rightarrow (P^{n+1}(\underline{x},y) \leftrightarrow P^{n+1}(\underline{x},z))$$

in the intuitionistic case, and $(0^=)$ follows from an analogous axiom in the modal case. ⊠

Call a metaframe $\mathcal{F} = (\mathbb{D}, \mathbb{R})$ an *i*-metaframe* if it satisfies (1), (4'), and

(2')* for any $\sigma \in \Sigma(m,n)$, $\underline{a} \in D_n$, $\underline{b} \in D_m$:

$$\underline{a} \cdot \sigma \leq_m \underline{c} \Rightarrow \exists \underline{b}\ (\ \underline{a} \leq_n \underline{b}\ \&\ (\underline{b} \cdot \sigma) \approx_m \underline{c} \cdot id(m));$$

(3')* for any $\sigma \in \Sigma(m,n)$, $\underline{a} \in D_n$, $\underline{b} \in D_{m+1}$:

(i) $\underline{b} \cdot j_{m+1} \leq_m \underline{a} \cdot \sigma \Rightarrow \exists \underline{c} \in D_{n+1}(\underline{c} \cdot j_{n+1} \leq_n \underline{a} \cdot id(n)\ \&$
$\underline{b} \cdot id(m+1) \leq_{m+1} (\underline{c} \cdot \sigma^+))$;

(ii) $\underline{a} \cdot \sigma \leq_m \underline{b} \cdot j_{m+1} \Rightarrow \exists \underline{c} \in D_{n+1}(\underline{a} \cdot id(n) \leq_n \underline{c} \cdot j_{n+1}\ \&$
$(\underline{c} \cdot \sigma^+) \leq_{m+1} \underline{b} \cdot id(m+1))$;

(6')* for any $\sigma \in \Sigma^{\sim}(m,n)$,

$$\forall \underline{b} \in D_m \exists \underline{a} \in D_n\ (\underline{a} \cdot \sigma) \leq_m \underline{b} \cdot id(m)\ .$$

$\mathcal{F}$ is an *$i^{*=}$-metaframe* if it also satisfies $(0^=)'$, $(0^=)''$,$(0^=)'''$. $\mathcal{F}$ is an *m^*-metaframe* if it satisfies (4) and

(1*) for any $\sigma \in \Sigma(m,n)$, $\underline{a},\underline{b} \in D_n$:
$\underline{a} \le_m \underline{b} \Rightarrow \exists \underline{c} \in D_m(\underline{a}\cdot\sigma \le_m \underline{c}$ & $\underline{b}\cdot\sigma = \underline{c}\cdot id(m))$;

(2*) for any $\sigma \in \Sigma(m,n)$, $\underline{a} \in D_n$, $\underline{b} \in D_m$:
$\underline{a}\cdot\sigma \le_m \underline{c} \Rightarrow \exists \underline{b} \in D_n(\underline{a} \le_n \underline{b}$ & $(\underline{b}\cdot\sigma) = \underline{c}\cdot id(m))$

(3*) for any $\sigma \in \Sigma(m,n)$, $\underline{a} \in D_n$, $\underline{b} \in D_{m+1}$:
$\underline{b}\cdot j_{m+1} = \underline{a}\cdot\sigma \Rightarrow \exists \underline{c} \in D_{n+1}(\underline{c}\cdot j_{n+1} = \underline{a}\cdot id(n)$ & $\underline{b}\cdot id(m+1)=(\underline{c}\cdot\sigma^+))$;

(6*) if $\sigma \in \Sigma^{\sim}(m,n)$ then $\pi_\sigma(D_n)=\pi_{id(m)}(D_m)$.

Thus an i-metaframe is an i*-metaframe satisfying (5) and (0) (analogously, for $i^=$-, and for m-metaframes); omitting (5) makes all other conditions in the previous definitions rather awkward. However these definitions are involved in the following important statement.

<u>Proposition 2.5.3</u> ("Soundness criteria")
/1/ A metaframe is $i^{(=)}$-sound iff it is an $i^{*(=)}$-metaframe.
/2/ A metaframe is m-sound iff it is an m*-metaframe.

<u>Proof.</u> (Sketch) The "if " part is proved as in 2.5.1. To prove "only if" in /1/, we notice that (1) follows from (IV) applied to $A=P^m(\underline{x}_\sigma)$, the conditions (2')*, (3')*, (4') follow from (III) applied to $A=(P^m(\underline{y})\to Q^m(\underline{y}))$, $\exists zP^{m+1}(\underline{y},z)$, $\forall zP^{m+1}(\underline{y},z)$, $A=P^k(\underline{y}_{\sigma'})$, and the condition (6')* follows from (II) applied to $A=P^m(\underline{x}_\sigma)$. The conditions $(0^=)'$, $(0^=)''$,$(0^=)'''$ are verified as in Theorem 2.5.2.

For "only if " in /2/, the proof is analogous. The conditions (1*), (2*) follow from (III) applied to $A=\Diamond P^m(\underline{y})$, the conditions (3*) and (4) follow from (III) applied to $A=\exists zP^{m+1}(\underline{y},z)$ and $A=P^k(\underline{y}_\sigma)$; and for (6*) we apply (II) to $A=P^m(\underline{x}_\sigma)$. ⊠

<u>Lemma 2.5.4</u> /1/ For any $i^{*(=)}$-metaframe $\mathcal{F}$ there exists an $i^{(=)}$-metaframe $\mathcal{F}'$ such that $IL^{(=)}(\mathcal{F}) = IL^{(=)}(\mathcal{F}')$.

/2/ For any m*-metaframe $\mathcal{F}$ there exists an m-metaframe $\mathcal{F}'$ such that $ML^=(\mathcal{F}) = ML^=(\mathcal{F}')$.

<u>Proof.</u> (Sketch) /1/ Let $\mathcal{F} = (\mathbb{D}, \Pi)$ and consider $\mathcal{F}'' = (\mathbb{D}'', \Pi'')$, with $D_n'' = \{\underline{a} \in D_n \mid \underline{a}\cdot id(n) \approx_n \underline{a}\}$ ($=\{\underline{b} \in D_n \mid \exists \underline{a} \in D_n\ \underline{a}\cdot id(n) \approx_n \underline{b}\}$, by (1), (4')), and with $\le_n$ and π_σ coming as restrictions from $\mathcal{F}$ (this is correct, since by (4'), π_σ maps D_n into D_m'' for any $\sigma \in \Sigma(m,n)$). Properties of $i^{*(=)}$-metaframes for $\mathcal{F}$ imply all properties of $i^{(=)}$-metaframes for $\mathcal{F}''$, except (5). Instead of (5), $\mathcal{F}''$ satisfies

(5') $\underline{a}\cdot id(n) \approx_n \underline{a}$ (if $\underline{a}\in D_n''$).

Also $IL^{\sim(=)}(\mathcal{F}) = IL^{\sim(=)}(\mathcal{F}'')$, since inductive reasoning shows that

$$\xi, \underline{a} \models A[\underline{x}] \iff \xi'', \underline{a}\cdot id(n) \models A[\underline{x}]$$

holds for any valuation ξ in $\mathcal{F}$, its restriction ξ'' to $\mathcal{F}''$, for any $A\in IF^{(=)}$, with $FV(A)\subseteq \underline{x}$, $\underline{a}\in D_n$.

Finally we set $\mathcal{F}' = (\mathbb{D}', \mathbb{\Pi}')$, with

$D_0' = D_0'' \setminus \{ u\in D_0'' \mid \forall n>0\ D_{n,u}'' = \emptyset\}$; $D_n' = D_n''$ if $n>0$,
$\pi_\sigma'(\underline{a}) = \underline{a}$ if $\sigma=id(n)$, $n\geq 0$; $\pi_\sigma'(\underline{a}) = \pi_\sigma(\underline{a})$ otherwise.

Then $\mathcal{F}'$ is an $i^{(=)}$-metaframe (the conditions (0') and (5) follow from the definition of D_0' and $\pi_{id(n)}'$), and $IL^{\sim(=)}(\mathcal{F}') = IL^{\sim(=)}(\mathcal{F}'')$ (the difference between $\pi_{id(n)}$ and $\pi_{id(n)}'$ does not matter here since $\mathcal{F}''$ satisfies (5') and (IV); and D_0'' can be replaced by D_0' due to (6')).

For /2/, the proof is analogous. We set

$\mathcal{F}' = (\mathbb{D}', \mathbb{\Pi}')$, $D_n' = \{ \underline{a}\in D_n \mid \underline{a}\cdot id(n)=\underline{a} \}$ (= $\pi_{id(n)}(D_n)$, by (4)), $\pi_\sigma' = (\pi_\sigma \mid D_n)$, $\underline{a}\leq_n \underline{b} \iff \exists \underline{c}\ (\underline{a}\leq_n \underline{c}\ \&\ \underline{c}\cdot id(n)=\underline{b})$.

Then $\mathcal{F}'$ is a metaframe required. ⊠

Now from 2.5.3 and 2.5.4 we deduce

<u>Theorem 2.5.5.</u> ("Maximality Theorem")

/1/ For any $i^{(=)}$-sound metaframe $\mathcal{F}$ there exists an $i^{(=)}$-metaframe $\mathcal{F}'$ such that $IL^{(=)}(\mathcal{F}) = IL^{(=)}(\mathcal{F}')$.

/2/ For any $m^{(=)}$-sound metaframe $\mathcal{F}$ there exists an $m^{(=)}$-metaframe $\mathcal{F}'$ such that $ML^{(=)}(\mathcal{F}) = ML^{(=)}(\mathcal{F}')$.

This theorem says that $m^{(=)}$- ($i^{(=)}$-) sound metaframes are logically equivalent to $m^{(=)}$- ($i^{(=)}$-) metaframes , and therefore maximal metaframe semantics are provided by $m^{(=)}$-metaframes or by $i^{(=)}$-metaframes.

<u>2.6. Remarks on strong intuitionistic metaframes.</u>

Let $\mathcal{F} = (\mathbb{D}, \mathbb{\Pi})$ be a metaframe satisfying (1). A metaframe $(\mathbb{D}', \mathbb{\Pi}')$ is called a *skeleton* of $\mathcal{F}$ and denoted by $(\mathcal{F}/\approx)$ if $(D_n', \leq_n') = (D_n/\approx_n)$ (cf. 1.2), and $\pi_\sigma'(|\underline{a}|) = |\pi_\sigma(\underline{a})|$ (provided that $\underline{a}\in D_n$, $\sigma\in\Sigma(m,n)$); $|\underline{b}|$ denotes the cluster including $\underline{b}$.

A metaframe ($\mathbb{D}''$, $\mathbb{H}''$) is called a *0-skeleton* of $\mathcal{F}$ and denoted by ($\mathcal{F}/\approx_0$) if $(D_0'', \leq_0'') = (D_0/\approx_0)$, $(D_n'', \leq_n'') = (D_n, \leq_n)$ for n>0; $\pi_\sigma''=\pi_\sigma$ for $\sigma \in \Sigma(m,n)$, m,n>0, and $\pi_{\wedge_n}''(\underline{a}) = |\pi_{\wedge_n}(\underline{a})|$ (for n>0, $\underline{a} \in D_n$).

(1) guarantees the correctness of these definitions.

If $\mathcal{F}$ is an i-metaframe then ($\mathcal{F}/\approx$) is an i^+-metaframe. Also if $\mathcal{F}$ is an $i^{(+)(=)}$-metaframe then ($\mathcal{F}/\approx_0$) is an $i^{(+)(=)}$-metaframe as well, but this is not necessarily true for ($\mathcal{F}/\approx$) because $(0^=)'$, $(0^=)''$ may be violated in $(D_1/\approx_1)$.

Lemma 2.6.1 /1/ $IL^{(=)}(\mathcal{F}/\approx_0)=IL^{(=)}(\mathcal{F})$ for any $i^{(=)}$-metaframe $\mathcal{F}$.

/2/ $IL(\mathcal{F}/\approx)=IL(\mathcal{F})$ for any i-metaframe $\mathcal{F}$.

Proof. For any valuation ξ in $\mathcal{F}$ and for the corresponding "factorized" valuation ξ' in $\mathcal{F}' = (\mathcal{F}/\approx_{(0)})$, for any $\underline{a} \in D_n$, $\underline{x} \supseteq FV(A)$ we have:

$\xi, \underline{a} \models A[\underline{x}] \Leftrightarrow \xi', \underline{b} \models A[\underline{x}]$,

with $\underline{b}=|\underline{a}|$ in the proof of /2/, and in the proof of /1/: $\underline{b}=\underline{a}$ if n>0, $\underline{b}=|\underline{a}|$ if n=0. ⊠

Remark. /2/ is not true for logics with equality.

Proposition 2.6.2. For any i-metaframe $\mathcal{F}$ there exists an i^+-metaframe $\mathcal{F}'$ such that $IL(\mathcal{F}')=IL(\mathcal{F})$.

Proof. We can observe that ($\mathcal{F}/\approx$) is an i^+-metaframe and apply Lemma 2.6.1. ⊠

Therefore for logics without equality, i^+-metaframes generate the same semantics as i-metaframes (i.e. the maximal one, due to Theorem 2.5.3).

We do not know if this proposition can be extended to superintuitionistic logics with equality; we are able to show only that in this case each $i^=$-metaframe is logically equivalent to some $i^=$-metaframe satisfying (2) and (6) (but not necessarily (4)).

3. Completeness in metaframe semantics.

3.1. Canonical metaframe models for modal logics.

A predicate formula A is called an *n-formula* if $FV(A) \subseteq \{v_1,\dots,v_n\}$. A set of n-formulas is called an *n-set* . Let **L** be a m.p.l. As in classical logic, an n-set $\underline{a} \subseteq MF$ is called an ***L-n-type*** if

A set of n-formulas is called an *n-set* . Let **L** be a m.p.l. As in classical logic, an n-set $\underline{a} \subseteq MF$ is called an ***L-n-type*** if

- $\underline{a}$ is ***L-consistent*** (i.e. for any $A_1,\dots,A_m \in \underline{a}$, $\neg(\bigwedge_{1\le i\le m} A_i) \notin \mathbf{L}$)[\3] ;
- $\underline{a}$ is *maximal* i.e. for any modal n-formula A, either $A \in \mathbf{L}$ or $\neg A \in \mathbf{L}$.

The following Lemma 3.1.1 is standard, and 3.1.2 is trivial.

Lemma 3.1.1. (Lindenbaum)
Each **L**-consistent n-set of modal formulas is contained in some **L**-n-type.

Lemma 3.1.2. If $\underline{a}$ is an **L**-n-type, $\sigma \in \Sigma(m,n)$, then the m-set $\{A \mid A$ is a modal m-formula, $(v_{\sigma(1)},\dots,v_{\sigma(m)}/v_1,\dots,v_m)A \in \underline{a}\}$ (denoted by $(\underline{a}\cdot\sigma)$) is an **L**-m-type.

Let $D_{L,n}$ be the set of all **L**-n-types; and consider the following relation in $D_{L,n}$:

$$\underline{a} \le_{L,n} \underline{b} \Leftrightarrow \forall A\ (\Box A \in \underline{a} \Rightarrow A \in \underline{b}).$$

A standard proof shows that $\le_{L,n}$ is reflexive and transitive.

The *canonical metaframe* of the logic **L** is defined as $\mathcal{F}_L = (\mathbb{D}_L, \mathbb{\Pi}_L)$, with $\mathbb{D}_L = (D_{L,n}, \le_{L,n})_{n\in\omega}$, $\mathbb{\Pi}_L = (\pi_{L,\sigma})_{\sigma\in Mor\Sigma}$, $\pi_{L,\sigma}(\underline{a}) = (\underline{a}\cdot\sigma)$. The *canonical metaframe model* is defined as $\mathcal{M}_L = (\mathcal{F}_L , \xi_L)$, with

$$\xi_L(P^n) = \{\underline{a} \in D_{L,n} \mid P^n(v_1,\dots,v_n) \in \underline{a}\} .$$

Proposition 3.1.3. $\mathcal{F}_L$ is a modal metaframe.

Proof. Verification of (1), (4) and (0) is rather easy; (5) is obvious.

To prove (2), we assume that $\underline{a} \in D_{L,n}$, $\underline{a}\cdot\sigma \le_{L,n} \underline{c}$ i.e. that for any modal m-formula A, $\Box A_\sigma \in \underline{a}$ implies $A \in \underline{c}$ (here A_σ denotes $(v_{\sigma(1)},\dots,v_{\sigma(m)}/v_1,\dots,v_m)A$); we have to find $\underline{a} \in D_{L,n}$ such that $\underline{a} \le_{L,n} \underline{b}$, $\underline{b}\cdot\sigma = \underline{c}$. Or equivalently, $\underline{b}$ should contain the set

$$\beta = \{A \mid \Box A \in \underline{a}\} \cup \{C_\sigma \mid C \in \underline{c}\},$$

and by 3.1.1 we have to show **L**-consistency of β. Suppose the contrary. It is easily seen that both $\{A \mid \Box A \in \underline{a}\}$ and $\{C_\sigma \mid C \in \underline{c}\}$ are closed under conjunctions; so inconsistency of β implies $\neg(A \wedge C_\sigma) \in \mathbf{L}$ for some $C \in \underline{c}$, $\Box A \in \underline{a}$. But then $(A \supset \neg C_\sigma) \in \mathbf{L}$, $(\Box A \supset \Box\neg C_\sigma) \in \mathbf{L}$, and thus $\Box\neg C_\sigma \in \underline{a}$. Hence $\neg C \in \underline{c}$ by our assumption, and this contradicts the consistency of $\underline{c}$.

\3 Everywhere we suppose that the case m=0 is also included; in this case the conjunction is empty and equals to $(\bot \supset \bot)$.

that $\underline{c}$ should contain the set $\gamma = \underline{a} \cup \{B_{\sigma^+} \mid B \in \underline{b}\}$, and we have to show **L**-consistency of γ. Assuming the contrary, we obtain since $\underline{a}$ and $\{B_{\sigma^+} \mid B \in \underline{b}\}$ are $\wedge$-closed: $\neg(A \wedge B_{\sigma^+}) \in \mathbf{L}$ for some $B \in \underline{b}$, $A \in \underline{a}$, and hence $(A \supset \neg B_{\sigma^+}) \in \mathbf{L}$. Remembering that

$B_{\sigma^+} = (v_{\sigma(1)}, \ldots, v_{\sigma(m)}, v_{n+1}/v_1, \ldots, v_m, v_{m+1})B$ and $v_{n+1} \notin FV(A)$,

we have: $(A \supset \forall v_{n+1} \neg B_{\sigma^+}) \in \mathbf{L}$, $(A \supset \neg \exists v_{n+1} B_{\sigma^+}) \in \mathbf{L}$, and thus $\neg \exists v_{n+1} B_{\sigma^+} \in \underline{a}$. On the other hand $\exists v_{n+1} B_{\sigma^+} \in \underline{a}$ since $\exists v_{n+1} B_{\sigma^+}$ is equivalent in **L** to $(\exists v_{m+1} B)_\sigma$, and $B \in \underline{b}$ implies $\exists v_{m+1} B \in \underline{b}$, $\exists v_{m+1} B \in \underline{b} \cdot j_{m+1} = \underline{a} \cdot \sigma$, $(\exists v_{m+1} B)_\sigma \in \underline{a}$. This contradiction yields the **L**-consistency of γ. ⊠

<u>Theorem 3.1.4.</u> ("Fundamental Theorem")
For any **L**-n-type $\underline{a}$ and modal n-formula A,
$\mathcal{M}_L, \underline{a} \vDash A\ [v_1, \ldots, v_n] \iff A \in \underline{a}$.

<u>Proof.</u> (Sketch) As in modal propositional logic, we apply induction on the number of logical symbols in A. Here we consider two cases.

If $A = P^m(\underline{v}_\sigma)$, $\underline{v} = (v_1, \ldots, v_n)$, $\sigma \in \Sigma(m,n)$, then $\underline{a} \vDash A\ [\underline{v}] \iff (\underline{a} \cdot \sigma) \in \xi_L(P^m) \iff P^m(v_1, \ldots, v_m) \in (\underline{a} \cdot \sigma) \iff A \in \underline{a}$ (by definitions of $\vDash$, ξ_L and $(\underline{a} \cdot \sigma)$).

If $A = \exists z B$ then we can suppose $z = v_{n+1}$. Indeed, re-naming of bound variables in A leads to a formula A' such that $(A \equiv A') \in \mathbf{L}$; and $A \in \underline{a} \iff A' \in \underline{a}$ since $\underline{a}$ is a type; on the other hand $\mathcal{F}_L$ is m-sound (by 3.1.3 and 2.5.1), and thus $\underline{a} \vDash (A \equiv A')$ i.e. $\underline{a} \vDash A$ iff $\underline{a} \vDash A'$.

Now if $A = \exists v_{n+1} B$, by the inductive hypothesis and the definition of forcing we have:

$$\underline{a} \vDash A \iff \exists \underline{b} \in D_{L,n+1}(\underline{b} \cdot j_{n+1} = \underline{a}\ \&\ B \in \underline{b}).$$

Recall that $\underline{b} \cdot j_{n+1} = \underline{a}$ means: $C \in \underline{a} \iff C \in \underline{b}$ for any n-formula C. If such $\underline{b}$ exists we have $\exists v_{n+1} B\ (=A) \in \underline{b}$ (since $(B \supset \exists v_{n+1} B) \in \mathbf{L}$), and thus $A \in \underline{a}$. Conversely, if $A \in \underline{a}$ then we can obtain $\underline{b}$ by extending the **L**-consistent set $\underline{a} \cup \{B\}$. ⊠

<u>Corollary 3.1.5.</u> $\mathbf{L} \supseteq ML(\mathcal{F}_L)$ for any m.p.l. **L**.

<u>Proof.</u> If $A \notin \mathbf{L}$ for some n-formula A then $\{\neg A\}$ is **L**-consistent, and thus $\neg A \in \underline{a}$ for some **L**-n-type $\underline{a}$ (by 3.1.1), $\mathcal{M}_L, \underline{a} \vDash \neg A\ [v_1, \ldots, v_n]$ by 3.1.4, and hence $\mathcal{F}_L \nvDash A$. ⊠

<u>3.2. Canonical metaframe models for superintuitionistic logics.</u>

Now we will modify the previous construction for the intuitionistic case.

Let **L** be a s.p.l. An n-set $\underline{a} \subseteq IF$ is called an ***L-n-type*** if

- $\underline{a}$ is *L-closed* : if $A_1,\dots,A_m \in \underline{a}$ and $((\bigwedge_{1\le i\le m} A_i) \to B) \in \mathbf{L}$ then $B \in \underline{a}$;
- $\underline{a}$ is *disjunctive* : if $(B \vee C) \in \underline{a}$ then $B \in \underline{a}$ or $C \in \underline{a}$;
- $\bot \notin \underline{a}$.

Lemma 3.2.1. Let (S,T) be an *L-consistent pair* of n-sets i.e. for any $A_1,\dots,A_n \in S$, $B_1,\dots,B_m \in T$, $((\bigwedge_i A_i) \to (\bigvee_j) B_j) \notin \mathbf{L}$.[4] Then there exists an $\mathbf{L}$-n-type $\underline{a}$ such that $S \subseteq \underline{a}$, $T \cap \underline{a} = \emptyset$.

The proof is standard.

Lemma 3.2.2. If $\underline{a}$ is an $\mathbf{L}$-n-type, $\sigma \in \Sigma(m,n)$, then $(\underline{a}\cdot\sigma)$ is an $\mathbf{L}$-m-type.

The proof is trivial.

Let $D_{\mathbf{L},n}$ be the set of all $\mathbf{L}$-n-types; it is partially ordered by the relation :

$$\underline{a} \le_{\mathbf{L},n} \underline{b} \iff \underline{a} \subseteq \underline{b}.$$

The *canonical metaframe* of the logic $\mathbf{L}$ is defined as $\mathcal{F}_\mathbf{L} = (\mathbb{D}_\mathbf{L}, \Pi_\mathbf{L})$, with $\mathbb{D}_\mathbf{L} = (D_{\mathbf{L},n}, \le_{\mathbf{L},n})_{n\in\omega}$, $\Pi_\mathbf{L} = (\pi_{\mathbf{L},\sigma})_{\sigma\in \mathrm{Mor}\Sigma}$, $\pi_{\mathbf{L},\sigma}(\underline{a}) = (\underline{a}\cdot\sigma)$. The *canonical metaframe model* is defined as $\mathcal{M}_\mathbf{L} = (\mathcal{F}_\mathbf{L} , \xi_\mathbf{L})$, with

$$\xi_\mathbf{L}(P^n) = \{\underline{a} \in D_{\mathbf{L},n} \mid P^n(v_1,\dots,v_n) \in \underline{a}\} ;$$

obviously $\xi_\mathbf{L}$ is an intuitionistic valuation.

Proposition 3.2.3. $\mathcal{F}_\mathbf{L}$ is an i^+-metaframe.

Proof. Verification of (1), (4) and (5) is trivial; (0) is easy.

For (2), assume that $\underline{a}\cdot\sigma \subseteq \underline{c}$, $\underline{a} \in D_{\mathbf{L},n}$, $\underline{c} \in D_{\mathbf{L},m}$, $\sigma \in \Sigma(m,n)$. Due to Lemma 3.2.1, to find $\underline{b} \supseteq \underline{a}$ such that $\underline{b}\cdot\sigma = \underline{c}$ it is enough to show $\mathbf{L}$-consistency of the pair $(\underline{a} \cup (\underline{c}|\sigma), ((-\underline{c})|\sigma))$ (here $(-\underline{c})$ denotes $\{D \mid D$ is an m-formula, $D \notin \underline{c}\}$, and for any m-set S, $(S|\sigma) = \{X_\sigma \mid X \in S\}$). Assuming the contrary we have $(A \wedge C_\sigma \to D_\sigma) \in \mathbf{L}$ for some $A \in \underline{a}$, $C \in \underline{c}$, $D \in (-\underline{c})$ (since $\underline{a}$, $\underline{c}$ are $\wedge$-closed, $(-\underline{c})$ is $\vee$-closed). But then $(A \to (C_\sigma \to D_\sigma)) \in \mathbf{L}$, $(C_\sigma \to D_\sigma)$ $(=(C\to D)_\sigma) \in \underline{a}$, $(C \to D) \in \underline{c}$, and this contradicts to $C \in \underline{c}$, $D \in (-\underline{c})$.

For (3#), assume that $\underline{a} \in D_{\mathbf{L},n}$, $\underline{b} \in D_{\mathbf{L},m+1}$, $\sigma \in \Sigma(m,n)$, $\underline{a}\cdot\sigma = \underline{b}\cdot j_{m+1}$ i.e. $B_\sigma \in \underline{a} \iff B \in \underline{b}$ for any m-formula B. By Lemma 3.2.1, the required statements (i), (ii) follow from $\mathbf{L}$-consistency of the pairs $(\underline{a} \cup (\underline{b}|\sigma^+), -\underline{a})$ and $(\underline{a} \cup (\underline{b}|\sigma^+), ((-\underline{b})|\sigma^+))$. The first one is consistent because otherwise $(A \wedge B_{\sigma^+} \to A') \in \mathbf{L}$ for some $A \in \underline{a}$, $B \in \underline{b}$, $A' \in (-\underline{a})$, and then

[4] The empty disjunction is defined to be $\bot$.

(since A,A' are n-formulas) $(A \wedge \exists v_{n+1} B_{\sigma^+} \to A') \in \mathbf{L}$, and the latter formula is equivalent in **L** to $(A \wedge (\exists v_{m+1} B)_\sigma \to A')$. But $B \in \underline{b}$ implies $\exists v_{m+1} B \in \underline{b}$, and $(\exists v_{m+1} B)_\sigma \in \underline{a}$ by our assumption, hence $A' \in \underline{a}$, and the contradiction follows. The second pair is **L**-consistent because otherwise $(A \wedge B_{\sigma^+} \to (B')_{\sigma^+}) \in \mathbf{L}$ for some $A \in \underline{a}$, $B \in \underline{b}$, $B' \in (-\underline{b})$, and then $(A \to \forall v_{n+1}(B_{\sigma^+} \to (B')_{\sigma^+}) \in \mathbf{L}$, $(A \to (\forall v_{m+1}(B \to B'))_\sigma) \in \mathbf{L}$, $(\forall v_{m+1}(B \to B'))_\sigma) \in \underline{a}$, $\forall v_{m+1}(B \to B') \in \underline{b}$,$(B \to B') \in \underline{b}$, $B' \in \underline{b}$, and we come to a contradiction. ⊠

Theorem 3.2.4 ("Fundamental Theorem")
For any **L**-n-type $\underline{a}$ and intuitionistic n-formula A,
$\mathcal{M}_\mathbf{L}, \underline{a} \models A [v_1,\dots,v_n] \Leftrightarrow A \in \underline{a}$.

Proof. It goes along the same lines as in 3.1.4. Here we consider the case $A = \forall v_{n+1} B$ only. By the inductive hypothesis and the definition of forcing we have:

$$\underline{a} \models A \Leftrightarrow \forall \underline{b} \in D_{\mathbf{L},n+1}(\underline{b} \cdot j_{n+1} \supseteq \underline{a} \Rightarrow B \in \underline{b}).$$

Thus $A \in \underline{a}$ only if $\underline{a} \models A$ (for, $A \in \underline{a}$ and $\underline{b} \cdot j_{n+1} \supseteq \underline{a}$ imply $A \in \underline{b}$, and also $B \in \underline{b}$ since $(A \to B) \in \mathbf{L}$). Conversely, if $A \notin \underline{a}$ then we can construct $\underline{b} \in D_{\mathbf{L},n+1}$ such that ($\underline{b} \cdot j_{n+1} \supseteq \underline{a}$ & $B \notin \underline{b}$) since the pair $(\underline{a}, \{B\})$ is **L**-consistent . ⊠

Corollary 3.2.5. $\mathbf{L} \supseteq IL(\mathcal{F}_\mathbf{L})$ for any s.p.l. **L**.

Proof. If $A \notin \mathbf{L}$ for some n-formula A then $(\emptyset, \{A\})$ is **L**-consistent, and thus $A \notin \underline{a}$ for some **L**-n-type $\underline{a}$ (by 3.1.1), $\mathcal{M}_\mathbf{L}, \underline{a} \not\models A [v_1,\dots,v_n]$ by 3.2.4, and hence $\mathcal{F}_\mathbf{L} \not\models A$. ⊠

3.3. Completeness theorems (modal case).

Recall that a *general (propositional Kripke) frame* (cf. [1]) is a triple $(F, \leq, \mathcal{X})$ consisting of a frame $(F,\leq)$ and a non-empty family $\mathcal{X}$ of subsets of F which is closed under Boolean operations and under the operation $\Box S = \{x \mid \forall y (x \leq y \Rightarrow y \in S)\}$. A general frame $\Phi = (F, \leq, \mathcal{X})$ is called *descriptive* if

- Φ is *tight* : $\forall x,y \in F(\forall S \in \mathcal{X}(x \in \Box S \Rightarrow y \in S) \Rightarrow x \leq y)$;
- Φ is *compact* : if $\mathcal{X}' \subseteq \mathcal{X}$ and $\mathcal{X}'$ is *finitely centered* (i.e. $\bigcap \mathcal{Y} \neq \emptyset$ for any finite non-empty $\mathcal{Y} \subseteq \mathcal{X}$) then $\bigcap \mathcal{X}' \neq \emptyset$.\5

\5 For non-reflexive frames (which are not considered here) Φ should also be "distinguished".

A *valuation in* Φ is a valuation ξ in F such that $\xi(P^0) \in \mathcal{X}$ for any $P^0 \in PL^0$. A propositional formula A is called *valid in* Φ (notation: $\Phi \models A$) if $\xi \models A$ for any valuation ξ in Φ. For a logic **L**, $\Phi \models \mathbf{L}$ abbreviates $\forall A \in \mathbf{L}\ \Phi \models A$; $F \models \mathbf{L}$ has an analogous meaning.

A propositional modal logic **L** is called *canonical* [1] if $\Phi \models \mathbf{L}$ implies $F \models \mathbf{L}$ for any descriptive general frame $\Phi = (F, \leq, \mathcal{X})$.

Quite a lot of well-known modal logics are canonical (**S4**, **S4.1**, **S4.2**, **S4.3** are among them). All "Sahlqvist logics" [10] are canonical; this follows from [11].

Every canonical propositional logic is complete in Kripke semantics; this is a consequence of the propositional "Fundamental Theorem".

Now let **L** be a m.p.l.; for any modal n-formula A ($n \geq 0$) consider the set

$|A|_{L,n} = \{ \underline{a} \in D_{L,n} \mid A \in \underline{a} \}$ (= $\{ \underline{a} \in D_{L,n} \mid \mathcal{M}_L, \underline{a} \models A [v_1, \ldots, v_n] \}$).

Let

$\mathcal{X}_{L,n} = \{ |A|_{L,n} \mid A$ is an n-formula$\}$, $\Phi_{L,n} = (D_{L,n}, \leq_{L,n}, \mathcal{X}_{L,n})$.

Lemma 3.3.1. $\Phi_{L,n}$ is a descriptive general frame.

Proof. To show that $\Phi_{L,n}$ is a general frame, we observe that $|A \wedge B|_{L,n} = |A|_{L,n} \cap |B|_{L,n}$, $|\Box A|_{L,n} = \Box |A|_{L,n}$ etc. $\Phi_{L,n}$ is tight according to the definition of $\leq_{L,n}$. To show compactness, suppose $\mathcal{X}' \subseteq \mathcal{X}_{L,n}$ is finitely centered. Then the set $\alpha = \{A \mid |A|_{L,n} \in \mathcal{X}'\}$ is **L**-consistent; indeed consider $A_1, \ldots, A_m \in \alpha$; take any $\underline{b} \in |A_1|_{L,n} \cap \ldots \cap |A_m|_{L,n}$; then $A_1, \ldots, A_m \in \underline{b}$, and thus $\neg(A_1 \wedge \ldots \wedge A_m) \notin \mathbf{L}$. Now if $\alpha \subseteq \underline{a}$, $\underline{a} \in D_{L,n}$ (3.1.1) then $\underline{a} \in \bigcap \mathcal{X}'$. ⊠

We skip an easy inductive proof of the following

Lemma 3.3.2. Let η be a valuation in $\Phi_{L,n}$, B_i ($i \geq 0$) be n-formulas such that $\eta(F_i^0) = |B_i|_{L,n}$. For any propositional modal formula A whose propositional letters are among $F_1^0, \ldots, F_k^0$ let

$A^\eta = (B_1, \ldots, B_k / F_1^0, \ldots, F_k^0)A$.

Then for any $\underline{a} \in D_{L,n}$

$\eta, \underline{a} \models A \Leftrightarrow \mathcal{M}_L, \underline{a} \models A^\eta [v_1, \ldots, v_n]$.

(Recall that F_i^0 is the i-th propositional letter, vide 1.1.)

Lemma 3.3.3. $\Phi_{L,n} \models \mathbf{L}_p$.

Proof. Let $A \in \mathbf{L}$ be a propositional formula A and consider a valuation η in $\Phi_{L,n}$. Then for any $\underline{a} \in D_{L,n}$, $A^\eta \in \underline{a}$. Hence $\eta, \underline{a} \models A$ by 3.1.4 and 3.3.2. Therefore $\Phi_{L,n} \models A$. ⊠

We call an m.p.l. $\mathbf{L}$ *metaframe-canonical* if $\mathbf{L} \subseteq ML(\mathcal{F}_{\mathbf{L}})$. In this case $\mathbf{L} = ML(\mathcal{F}_{\mathbf{L}})$ by 3.1.5, and thus every metaframe-canonical logic is complete w.r.t. modal metaframes.

Theorem 3.3.4. Let **L** be a canonical propositional modal logic, then **QL** is metaframe-canonical.

Proof. By 3.3.3, for any $n \geq 0$ we have $\Phi_{\mathbf{QL},n} \models \mathbf{L}$, and thus $(D_{\mathbf{QL},n}, \leq_{\mathbf{QL},n}) \models \mathbf{L}$ since $\Phi_{\mathbf{QL},n}$ is descriptive(3.3.1) and $\mathbf{L}$ is canonical. Hence $\mathcal{F}_{\mathbf{QL}} \models A^{[n]}$ for any $A \in \mathbf{L}$ (by 2.5.0), i.e. $\mathbf{L} \subseteq ML(\mathcal{F}_{\mathbf{QL}})$. But $ML(\mathcal{F}_{\mathbf{QL}})$ is an m.p.l. since $\mathcal{F}_{\mathbf{QL}}$ is an m-metaframe (3.1.3) and by the Soundness Theorem (2.5.1). Consequently, $\mathbf{QL} \subseteq ML(\mathcal{F}_{\mathbf{QL}})$. ⊠

Recall that the *Barcan formula* is $Ba = \Diamond \exists x P^1(x) \supset \exists x \Diamond P^1(x)$.

Lemma 3.3.5. Let **L** be an m.p.l. containing Ba, then $Ba \in ML(\mathcal{F}_{\mathbf{L}})$.
Proof. Let us show that the metaframe $\mathcal{F}_{\mathbf{L}}$ satisfies the following condition:

($\boxdot$) for any $\underline{a}, \underline{b} \in D_{\mathbf{L},n}$, $\underline{c} \in D_{\mathbf{L},n+1}$,

$$\underline{a} \leq_{\mathbf{L},n} \underline{b} = \underline{c} \cdot j_{n+1} \Rightarrow \exists \underline{d} \in D_{\mathbf{L},n+1} \, (\underline{d} \leq_{\mathbf{L},n} \underline{c} \;\&\; \underline{d} \cdot j_{n+1} = \underline{a}).$$

Assume that $\underline{a} \leq_{\mathbf{L},n} \underline{b} = \underline{c} \cdot j_{n+1}$. To find $\underline{d}$ it is sufficient to prove $\mathbf{L}$-consistency of the (n+1)-set $\alpha = \underline{a} \cup \{\Diamond C \mid C \in \underline{c}\}$. But α is consistent because $C \in \underline{c}$, $A \in \underline{a}$, $(\Diamond C \supset \neg A) \in \mathbf{L}$ imply $(\exists v_{n+1} \Diamond C \supset \neg A) \in \mathbf{L}$, and hence $(\Diamond \exists v_{n+1} C \supset \neg A) \in \mathbf{L}$ (due to Ba). But $C \in \underline{c}$ implies $\exists v_{n+1} C \in \underline{c}$, $\exists v_{n+1} C \in \underline{b}$, $\Diamond \exists v_{n+1} C \in \underline{a}$ (since $\underline{a} \leq_{\mathbf{L},n} \underline{b}$), and this yields $\mathbf{L}$-inconsistency of $\underline{a}$.

The condition ($\boxdot$) guarantees that $Ba \in ML(\mathcal{F}_{\mathbf{L}})$ i.e. that $\mathcal{F}_{\mathbf{L}} \models Ba^{[n]}$ for any $n \geq 0$. To show this consider an equivalent of $Ba^{[n]}$: $\Diamond \exists v_{n+1} P^{n+1}(\underline{v}, v_{n+1}) \supset \exists v_{n+1} \Diamond P^{n+1}(\underline{v}, v_{n+1})$ (with $\underline{v} = (v_1, \ldots, v_n)$). By $(II)^m$ it is enough to establish that $\xi, \underline{a} \models Ba^{[n]} [\underline{v}]$ for any valuation ξ in $\mathcal{F}_{\mathbf{L}}$ and for any $\underline{a} \in D_{\mathbf{L},n}$. But this follows easily from ($\boxdot$). ⊠

Theorem 3.3.6. Let **L** be a canonical propositional modal logic. Then **QLB**=**QL**+Ba is metaframe-canonical.

3.4. Completeness theorems (intuitionistic case).

Now let us briefly show how to transfer the previous considerations to superintuitionistic logics.

A subset S of a frame $(F, \leq)$ is called *conic* if $\Box S = S$. A *general (propositional Kripke) intuitionistic frame* is a triple $\Phi = (F, \leq, \mathcal{X})$ consisting of a frame $(F, \leq)$ and of a family $\mathcal{X}$ of its conic subsets

A subset S of a frame (F,≤) is called *conic* if $\Box S=S$. A ***general (propositional Kripke) intuitionistic frame*** is a triple $\Phi = (F,\leq, \mathcal{X})$ consisting of a frame (F,≤) and of a family $\mathcal{X}$ of its conic subsets containing $\emptyset$ and closed under finite unions and intersections and under the operation $S\mapsto T = \bigcap \{U \mid U\subseteq F, S\cap U\subseteq T\}$.

Φ is called *descriptive* if it is *tight* (as in 3.3) and *compact* :

if $(\mathcal{X}', \mathcal{X}'')$ is a *finitely centered pair* of subsets of $\mathcal{X}$ (i.e. if for any finite $\mathcal{Y}'\subseteq \mathcal{X}'$, $\mathcal{Y}''\subseteq \mathcal{X}''$, $(\bigcap \mathcal{Y}')\setminus(\bigcup \mathcal{Y}'')\neq\emptyset$) then $(\bigcap \mathcal{X}')\setminus(\bigcup \mathcal{X}'')\neq\emptyset$.[6]

Further definitions (of valuations, validity, canonicity, $|A|_{\mathbf{L},n}$ etc.) are analogous to 3.3.

Lemma 3.4.1. $\Phi_{\mathbf{L},n}$ is descriptive.

Proof. Each of the sets $|A|_{\mathbf{L},n}$ is obviously conic. Also we have: $|A\wedge B|_{\mathbf{L},n} = |A|_{\mathbf{L},n} \cap |B|_{\mathbf{L},n}$, $|A\vee B|_{\mathbf{L},n} = |A|_{\mathbf{L},n} \cup |B|_{\mathbf{L},n}$,$| A\rightarrow B|_{\mathbf{L},n} = |A|_{\mathbf{L},n} \mapsto |B|_{\mathbf{L},n}$, $|\bot|_{\mathbf{L},n} = \emptyset$. Thus $\Phi_{\mathbf{L},n}$ is a general intuitionistic frame. Its tightness is clear; compactness follows immediately from 3.2.1. ⊠

The following two lemmas are analogues of 3.3.2 and 3.3.3.

Lemma 3.4.2. Let η be a valuation in $\Phi_{\mathbf{L},n}$, B_i $(i\geq 0)$ be n-formulas such that $\eta(F_i^0) = |B_i|_{\mathbf{L},n}$. For any propositional intuitionistic formula A whose propositional letters are among $F_1^0,\ldots, F_k^0$ let

$A^\eta = (B_1,\ldots, B_k/F_1^0,\ldots, F_k^0)A.$

Then for any $\underline{a}\in D_{\mathbf{L},n}$

$\eta, \underline{a} \models A \Leftrightarrow \mathcal{M}_{\mathbf{L}} , \underline{a} \models A^\eta [v_1,\ldots, v_n].$

Lemma 3.4.3. $\Phi_{\mathbf{L},n} \models \mathbf{L}_p$.

We call a s.p.l. **L** *metaframe-canonical* if $\mathbf{L}\subseteq IL(\mathcal{F}_{\mathbf{L}})$. In this case $\mathbf{L} = IL(\mathcal{F}_{\mathbf{L}})$ by 3.2.5, and thus every metaframe-canonical logic is complete w.r.t. strong intuitionistic metaframes.

Theorem 3.4.4. Let **L** be a canonical propositional superintuitionistic logic, then **QL** is metaframe-complete.

The proof is analogous to 3.2.4.

Remark. In fact Theorem 3.3.4 can be strengthened: $\mathbf{QL}^=$ (and **QL** as well) is complete in functor semantics. This stronger version was proved by S.Ghilardi [6]. However it is not clear if his methods can be extended to the modal case.

The following predicate formula:

$$D = \forall x(P^1(x)\vee q)\rightarrow (\forall xP^1(x)\vee q)$$

---

[6] Here we suppose that $\bigcap \emptyset = F$, $\bigcup\emptyset = \emptyset$.

is called the *constant domain principle.*

Lemma 3.4.5. Let **L** be a s.p.l. containing D, then $D \in IL(\mathcal{F}_L)$.

Proof. We show that $\mathcal{F}_L$ satisfies the condition (□) from the proof of 3.3.5. Let $\underline{a} \leq_{L,n} \underline{b} = \underline{c} \cdot j_{n+1}$. To find $\underline{d}$, it is sufficient to prove L-consistency of the pair $(\underline{a}, -\underline{a} \cup -\underline{c})$. Assume the contrary. Then $(A \to A' \vee C') \in \mathbf{L}$ for some $A \in \underline{a}$, $A' \in (-\underline{a})$, $C' \in (-\underline{c})$. Applying rules of **QH** and the formula D we have: $(A \to A' \vee \forall v_{n+1} C') \in \mathbf{L}$ and hence we obtain subsequently: $\forall v_{n+1} C' \in \underline{a}$, $\forall v_{n+1} C' \in \underline{b}$, $\forall v_{n+1} C' \in \underline{c}$, $C' \in \underline{c}$, and this is a contradiction.

Now $D \in IL(\mathcal{F}_L)$ follows from (□) rather easily. ⊠

Finally we come to the following

Theorem 3.4.6. Let **L** be a canonical propositional superintuitionistic logic. Then **QLD=QL**+D is metaframe-canonical.

4. Representation theorems.

4.1. Pre-bundles.

A *(Kripke) pre-bundle* over a frame F is a triple $\mathbb{F}=(F,D,\pi)$ in which π is a pre-p-morphism of a frame $(D, \leq')$ onto a frame $(F,\leq)$. Thus every pre-bundle over a p.o. set is a Kripke bundle.

As with Kripke bundles, for a pre-bundle there exists a splitting $D = \bigcup_{u \in F}(D_{(u)})$; the $D_{(u)} = \pi^{-1}(u)$ are non-empty and disjoint; and there is a metaframe $\mathcal{F}^*[\mathbb{F}] = (\mathbb{D}^*, \Pi^*)$ such that $\mathbb{D}^* = (D_n^*, \leq_n^*)_{n \geq 0}$, $(D_0^*, \leq_0^*) = (F, \leq_0)$, $D_n^* = \bigcup_{u \in F}(D_{(u)})^n$, $\leq_n^*$ is the same as $\leq_n$ in 1.3 for Kripke bundles, $\Pi^* = (\pi_\sigma^*)_{\sigma \in Mor\Sigma}$, $\pi_\sigma^*(\underline{a}) = \underline{a}_\sigma$.

In general, $\mathcal{F}^*[\mathbb{F}]$ need not be an i-metaframe. For example, if $F=\{u_0,u_1,u_2\}$ and $u_0 < u_1 \approx u_2$, $D_{(u_0)}=\{a_1,a_2\}$, $D_{(u_1)}=\{b_1\}$, $D_{(u_2)}=\{b_2\}$, $a_i <' b_i$ $(i=1,2)$, $b_1 \approx' b_2$ then (2') fails in $\mathcal{F}^*[\mathbb{F}]$.

Therefore, the "semantics of pre-bundles" is incorrect.

4.2. Quasi-Cartesian and Cartesian metaframes.

A metaframe $\mathcal{F} = (\mathbb{D}, \Pi)$ is called *quasi-Cartesian* (briefly, *qC-metaframe*) if

- $\pi_{\wedge 1}(D_1)=F$, i.e. $\mathbb{F}=(F,D,\pi) = ((D_0, \leq_0),(D_1, \leq_1),\pi_{\wedge 1})$ is a pre-bundle;
- for any n, $D_n \subseteq D_n^*$ (D_n^* comes from $\mathbb{F}$);
- for any $\sigma \in \Sigma(m,n)$, $\pi_\sigma = \pi_\sigma^* | D_n$.

$\mathcal{F}$ is called *Cartesian* (briefly, *C-metaframe*) if also $D_n = D_n^*$ (and thus, $\pi_\sigma = \pi_\sigma^*$ for any $\sigma \in \Sigma(m,n)$, $D_{n,u} = (D_{(u)})^n$).

In these cases we say that $\mathcal{F}$ is a (q)C-metaframe *over* the pre-bundle $\mathbb{F}$.

We will also use such abbreviations as "Cm-metaframe", "qCi-metaframe" etc.

In qC-metaframes (as well as in those derived from $\mathcal{C}$-sets) abstract n-tuples and transformations are real; but the equality $\leq_n^* = \leq_n$ can be false for some n>1. This allows us to use the notation $(\mathbb{F}, \mathbb{D})$ for arbitrary qC-metaframes over $\mathbb{F}$.

For qC-metaframes, definitions of forcing from 2.2 can be somewhat simplified and made more like standard Kripke semantics.

Namely, we can define $\xi, u \vDash (\underline{a}/\underline{x})A$ for $u \in F$, $\underline{a} \in D_{n,u}$ and $D_{(u)}$-formula $(\underline{a}/\underline{x})A$ (which will be abbreviated to $A(\underline{a})$). Each $\mathcal{F}$-formula $(\underline{a}, \underline{x}, A)$ corresponds to the unique $D_{(u)}$-formula $A(\underline{a})$, but this correspondence is not one-to-one. (E.g. $P^2(a,b)$ corresponds to $((a,b),(x,y), P^2(x,y))$ and also to $(a, x, P^2(x,x))$.) Nevertheless, the definition given below is correct because, due to property (III) from 2.2, $\xi, \underline{a} \vDash A[\underline{x}]$ must be the same for all $\mathcal{F}$-formulas corresponding to $A(\underline{a})$.

Intuitionistic case.

- $\xi, u \vDash P^m(\underline{a}) \iff \underline{a} \in \xi(P^m)$;
- $\xi, u \nvDash \bot$;
- $\xi, u \vDash a{=}b \iff a{=}b$;
- $\xi, u \vDash (B \wedge C)(\underline{a}) \iff \xi, u \vDash B(\underline{a})$ and $\xi, u \vDash C(\underline{a})$;
- $\xi, u \vDash (B \vee C)(\underline{a}) \iff \xi, u \vDash B(\underline{a})$ or $\xi, u \vDash C(\underline{a})$;
- $\xi, u \vDash (B \to C)(\underline{a}) \iff \forall v \forall \underline{b}$ $(u \leq_0 v$ & $\underline{b} \in D_{n,v}$ & $\underline{a} \leq_n \underline{b}$ & $\xi, v \vDash B(\underline{b}) \Rightarrow \xi, v \vDash C(\underline{b})$);
- $\xi, u \vDash \forall z B(\underline{a}, z) \iff \forall v \forall \underline{b} \forall c$ $(u \leq_0 v$ & $(\underline{b}, c) \in D_{n+1,v}$ & $\underline{a} \leq_n \underline{b} \Rightarrow \xi, v \vDash B(\underline{b}, c))$;
- $\xi, \underline{a} \vDash \exists z B(\underline{a}, z) \iff \exists v \exists \underline{b} \exists c$ $(u \approx_0 v$ & $(\underline{b}, c) \in D_{n+1,v}$ & $\underline{a} \approx_n \underline{b}$ & $\xi, v \vDash B(\underline{b}, c)$) in qCi-metaframes,
  or $\iff \exists c$ $((\underline{a}, c) \in D_{n+1,v}$ & $\xi, u \vDash B(\underline{a}, c))$ in qCi$^+$-metaframes.

For the modal case (i.e. for Cartesian m$^{(=)}$-metaframes) the definition is analogous; the first three items are the same; for $\exists$-formulas it is the same as in qCi$^+$-metaframes, and

- $\xi, u \vDash (B \supset C)(\underline{a}) \iff \xi, u \nvDash B(\underline{a})$ or $\xi, u \vDash C(\underline{a})$;
- $\xi, u \vDash \Diamond B(\underline{a}) \iff \exists v \exists \underline{b}$ $(u \leq_0 v$& $\underline{b} \in D_{n,v}$ & $\underline{a} \leq_n \underline{b}$ & $\xi, v \vDash B(\underline{b}))$.

A formula A (with $FV(A)=\{x_1,...,x_n\}$) is true w.r.t ξ iff $\xi, u \models A(\underline{a})$ for any $u \in F$, $\underline{a} \in D_{n,u}$.

4.3. Representation of $m^=$- and $i^=$-metaframes.

Theorem 4.3.1. (i) Let $\mathcal{F} = (\mathbb{D}, \mathbb{\Pi})$ be a qC-metaframe. Then the following conditions are equivalent:

- $\mathcal{F}$ is an m-metaframe,
- $\mathcal{F}$ is an $m^=$-metaframe,
- $\mathcal{F}$ is a Cartesian i^+-metaframe,
- $\mathcal{F}$ is a Cartesian metaframe satisfying conditions (1) and (2).

(ii) Every $m^=$-metaframe is isomorphic to a Cartesian $m^=$-metaframe over a Kripke bundle.

Proof. (Sketch).

(i) If $\mathcal{F}$ is a qC-metaframe, (3) can be rewritten as:
"for any $\sigma \in \Sigma(m,n)$, $u \in F$, if $\underline{a}=(a_1,...,a_n) \in D_{n,u}$, $\underline{b}=(a_{\sigma(1)},...,a_{\sigma(m)},b_{m+1}) \in D_{m+1,u}$, then $\underline{c}=(a_1,...,a_n,b_{m+1}) \in D_{n+1,u}$".
Hence it is clear that $\mathcal{F}$ satisfies (3) iff it is Cartesian.

(ii) Let $\mathcal{F} = (\mathbb{D}, \mathbb{\Pi})$ be an $m^=$-metaframe. Consider a Kripke bundle $\mathbb{F}=(F,D,\pi) = ((D_0, \leq_0),(D_1, \leq_1),\pi_{\wedge 1})$. For $\underline{a} \in D_{n,u}$, $n>0$, set $k(\underline{a})=(\underline{a}\cdot pr_{n,1},...,\underline{a}\cdot pr_{n,n}) \in (D_{(u)})^n$. By (3) and ($0^{\#}$),

$\forall u \in F\ \forall n>0\ \forall \underline{b} \in (D_{(u)})^n\ \exists!\ \underline{a} \in D_{n,u}\ k(\underline{a})=\underline{b}$.

Consequently, k is an isomorphism between $\mathcal{F}$ and the Cartesian $m^=$-metaframe $\mathcal{F}' = (\mathbb{F}, \mathbb{D}')$ in which $\mathbb{D}' = (D_n^*, \leq_n')_{n\geq 0}$. ⊠

Therefore the maximal metaframe semantics for m.p.l.$^=$'s is given by Cartesian $m^=$-metaframes. The next theorem proves an analogous fact for s.p.l.$^{(=)}$'s; however Theorem 4.3.1 itself cannot be transferred to the intuitionistic case.

Theorem 4.3.2. (i) For any $i^{(=)}$-metaframe $\mathcal{F} = (\mathbb{D}, \mathbb{\Pi})$ there exists a quasi-Cartesian $i^{(=)}$-metaframe $\mathcal{F}' = (\mathbb{F}', \mathbb{D}')$ over a p.o. frame F' (and thus, $\mathbb{F}'$ is a Kripke bundle) such that $IL^{(=)}(\mathcal{F}')=IL^{(=)}(\mathcal{F})$.

(ii) For any $i^{(=)}$-metaframe $\mathcal{F} = (\mathbb{D}, \mathbb{\Pi})$ there exists a Cartesian $i^{(=)}$-metaframe $\mathcal{F}' = (\mathbb{F}', \mathbb{D}')$ with finite domains (i.e. all $D'_{(u)}$ are finite) such that $IL^{(=)}(\mathcal{F}')=IL^{(=)}(\mathcal{F})$.

Proof. (Sketch) Case I (without equality)

(i) Due to 2.6.1, we can suppose here that $\mathcal{F}$ is an i-metaframe over a p.o. frame F. Let $\mathcal{F}' = (\mathbb{F}', \mathbb{D}')$, with $\mathbb{F}'=(F,D',\pi')$,

$\mathbb{D}' = (D_n', \leq_n')_{n\geq 0}$, $D' = \bigcup_{k>0} (D_k \times I_k)$;

$\pi'(\underline{a},j)=\pi\,\Lambda_k(\underline{a})$ (if $(\underline{a},j)\in(D_k\times I_k)$),

$D_n'=\{(\underline{a}^*)_\sigma \mid \underline{a}\in D_k,\ \sigma\in\Sigma(n,k),\ k>0\}$ ($\underline{a}^*$ denotes $((\underline{a},1),\ldots,(\underline{a},k))$ if $\underline{a}\in D_k$);

$(\underline{a}^*)_\sigma \le_n' (\underline{b}^*)_\tau \iff (\underline{a}\cdot\sigma)\le_n(\underline{b}\cdot\tau)$ (in $\mathcal{F}$).

A straightforward check shows that $\mathcal{F}'$ is a qC-metaframe and an i-metaframe. By the construction, $(\mathcal{F}'/\approx')=(\mathcal{F}/\approx)$, and hence $IL(\mathcal{F}')=IL(\mathcal{F})$ (2.6.1).

(ii) Suppose $\mathcal{F}=(\mathbb{F},\mathbb{D})$ is a qCi-metaframe.

Let $\mathcal{F}'=(\mathbb{F}',\mathbb{D}')$, with

$\mathbb{F}'=((D_0',\le_0'),(D',\le_1'),\pi')$,

$D_0'=\bigcup_{k>0}D_k$;

$D'=\bigcup_{k>0}(D_k\times I_k)$;

$\pi'(\underline{a},j)=\underline{a}$ if $(\underline{a},j)\in D'$ (and thus $D'_{(\underline{a})}=\{\underline{a}\}\times I_k$ whenever $\underline{a}\in I_k$; each $D'_{(\underline{a})}$ is finite);

$\underline{a}\le_0'\underline{b}\iff \underline{a}\cdot\Lambda_m\le_0\underline{b}\cdot\Lambda_n$ in $\mathcal{F}$ (if $\underline{a}\in D_m$, $\underline{b}\in D_n$);

$\mathbb{D}'=(D_n',\le_n')_{n\ge0}$,

$D_n'=((D')^*)_n$ coming from $\mathbb{F}'$ (hence $D_n'=\bigcup_{k>0}(\{\underline{a}\}\times I_k)^n=\bigcup_{k>0}\{(\underline{a}^*)_\sigma \mid \underline{a}\in D_k,\ \sigma\in\Sigma(n,k)\}$);

$(\underline{a}^*)_\sigma\le_n'(\underline{b}^*)_\tau\iff(\underline{a}\cdot\sigma)\le_n(\underline{b}\cdot\tau)$ in $\mathcal{F}$ (if $n>0$).

Then one can prove that $\mathcal{F}'$ is a C-metaframe and an i-metaframe and that $(\mathcal{F}'/\approx')=(\mathcal{F}/\approx)$. Therefore $IL(\mathcal{F}')=IL(\mathcal{F})$.

Case II (with equality).

(i) For this case we need an auxiliary definition.

A *(quasi-) Cartesian equipped intuitionistic metaframe ((q)Cei-metaframe)* is a triple $\mathcal{F}=(\mathbb{F},\mathbb{D},=_1)$ such that $(\mathbb{F},\mathbb{D})$ is (quasi-) Cartesian i-metaframe ($\mathbb{F}=(F,D,\pi)$, $\mathbb{D}'=(D_n',\le_n')_{n\ge0}$) and $=_1$ is an equivalence relation on D satisfying three conditions (cf. $(0^=)'$,$(0=)''$ in 2.4):

(•) $a=_1b\Rightarrow a\approx_1b,\ \pi(a)=\pi(b)$;

(••) $(a_1,\ldots,a_n)\le_n(b_1,\ldots,b_n),\ a_i=_1a_j\Rightarrow b_i=_1b_j$ $(n>0)$;

(•••) $(a_1,\ldots,a_n,a_{n+1})\in D_{n+1},\ a_n=_1a_{n+1}\Rightarrow$
$(a_1,\ldots,a_{n-1},a_n)\approx_n(a_1,\ldots,a_{n-1},a_{n+1})$ $(n>0)$.

Note that a (quasi-) Cartesian i-metaframe is a (q)Cei- metaframe in which $(=_1)$ is an equality relation.

A (q)Cei-metaframe $\mathcal{F}=(\mathbb{F},\mathbb{D},=_1)$ gives rise to an $i^=$-metaframe $(\mathcal{F}/\approx_1)=(\mathbb{D}',\mathbb{H}^*)$, in which $\mathbb{D}'=(D_n/=_n,\le_n')_{n\ge0}$,

$(a_1,...,a_n) =_n (b_1,...,b_n) \iff (\forall i\ a_i = b_i) \lor (a_1 = ... = a_n =_1 b_1 = ... = b_n)$ if $n>0$;

$u =_0 v \iff u=v$ (for $u,v \in F$);

$\leq_n'$ and $\pi_\sigma{}^*$ come from $\leq_n$ and π_σ ("via representatives").

Then we can define the s.p.l.$^{(=)}$ of $\mathcal{F}$ as $IL^{(=)}(\mathcal{F}) = IL^{(=)}(\mathcal{F}/{=_1})$, or equivalently, we can reformulate the definition in 2.2, with only the change: $\xi, \underline{a} \models x_i = x_j \iff \pi_{pr_{n,i}}(\underline{a}) =_1 \pi_{pr_{n,j}}(\underline{a})$.

Lemma 4.3.3. 1) For any i$^=$-metaframe $\mathcal{F}$ there exists a qCei-metaframe $\mathcal{F}'$ (over the same frame) such that $IL^=(\mathcal{F}) = IL^=(\mathcal{F}')$.

2) For any qCei-metaframe $\mathcal{F}$ there exists a Cei-metaframe $\mathcal{F}'$ with finite domains such that $IL^=(\mathcal{F}) = IL^=(\mathcal{F}')$; moreover, if $\mathcal{F}$ is a qCi$^=$-metaframe then $\mathcal{F}'$ is a Ci$^=$-metaframe.

A proof of 4.3.3 repeats that of 4.3.2 for the case I. We set:

$(\underline{a},j) =_1' (\underline{b},k) \iff \underline{a}=\underline{b}\ \&\ \pi_{pr_{n,j}}(\underline{a}) = \pi_{pr_{n,k}}(\underline{a})$ to get 1), and

$(\underline{a},j) =_1' (\underline{b},k) \iff \underline{a}=\underline{b}\ \&\ a_j =_1 b_k$ to get 2).

To show logical equivalence of $\mathcal{F}$ and $\mathcal{F}'$, instead of 2.6.1 we use the following claim:

if ξ, ξ' are valuations in $\mathcal{F}$ and $\mathcal{F}'$ respectively, and $\xi'(P^m) = \{(\underline{a}^*)_\sigma \mid (\underline{a}\cdot\sigma) \in \xi(P^m)\}$ for any $P^m \in PL^m, m \geq 0$, then

$\xi, \underline{a}\cdot\sigma \models A[\underline{x}] \iff \xi', (\underline{a}^*)_\sigma \models A[\underline{x}]$.

Lemma 4.3.4. 1) For any Cei-metaframe $\mathcal{F}$ there exists a qCi-metaframe $\mathcal{F}'$ (over the same pre-bundle) such that $IL^=(\mathcal{F}) = IL^=(\mathcal{F}')$.

2) The 0-skeleton $(\mathcal{F}/{\approx_0})$ of a qCi$^{(=)}$-metaframe $\mathcal{F}$ is also a qCi$^{(=)}$-metaframe.

Remark. In general, $(\mathcal{F}/{\approx_0})$ need not be Cartesian if $\mathcal{F}$ is a Cartesian i$^=$-metaframe.

Proof of 4.3.4. 1) If a qCei-metaframe $\mathcal{F} = (\mathbb{F}, \mathbb{D}, =_1)$ is Cartesian, the condition (•••) is equivalent to:

$\forall u \in F\ \forall a_1,...,a_n, b_1,...,b_n \in D_{(u)}\ ((\forall i\ a_i =_1 b_i) \Rightarrow (a_1,...,a_n) \approx_n (b_1,...,b_n))$.

Setting

$\mathcal{F}' = (\mathbb{F}, \mathbb{D}')$, $\mathbb{D}' = (D_n', \leq_n')_{n\geq 0}$, $D_n' = \{\underline{a} \in D_n \mid \forall i,j\ (a_i =_1 a_j \Rightarrow a_i = a_j)\}$,

$\leq_n' = \leq_n' | D_n$,

we have for any $n>0$:

$\leq_n{}' = \leq_n{}' | D_n$,

we have for any n>0:

$\forall \underline{a} \in D_n \ \exists \underline{b} \in D_n{}' \ \forall i \ a_i =_1 b_i$.

Hence, if ξ is a valuation in $\mathcal{F}$ and ξ' is its restriction to $\mathcal{F}'$ then for any $A \in IF^=$, $u \in F$, $\underline{a} \in D_{n,u}{}'$,

$\xi, \underline{a} \models A[\underline{x}] \Longleftrightarrow \xi', \underline{a} \models A[\underline{x}]$

(this is proved by an induction on the length of A). ⊠

Now the case II of 4.3.2 is clear. Given an $i^=$-metaframe $\mathcal{F}$, we construct its equivalents: a qCei-metaframe $\mathcal{F}_1$ (Lemma 4.3.3, 1)), a Cei-metaframe $\mathcal{F}_2$ (Lemma 4.3.3, 2)), a $qCi^=$-metaframe $\mathcal{F}_3$ (Lemma 4.3.4, 1)), and a $qCi^=$-metaframe $\mathcal{F}_4$ over a p.o. frame (4.3.4, 2)). This yields (i).

The statement (ii) follows from (i), by 4.3.3, 2). ⊠

4.4. Cartesian metaframes and $\mathcal{C}$-sets.

Theorem 4.4.1. 1) Every Cartesian m-metaframe with (at most) countable domains can be presented as $\mathcal{F}[\mathbb{F}]$ for some $\mathcal{C}$-set $\mathbb{F}$.

2) There exists a counterexample to 1) having only one uncountable domain.

Proof. (Sketch) 1) Suppose $\mathcal{F} = (\mathbb{D}, \mathbb{H})$ to be a cm-metaframe over a frame F. Let $\mathcal{C}$ be a subcategory of $\mathcal{SET}$ such that

$Ob\,\mathcal{C} = \{D_{(u)} \mid u \in F\}$,

$\mathcal{C}(u,v) = \{ f: D_{(u)} \to D_{(v)} \mid \forall n>0 \ \forall a_1, \ldots, a_n \in D_{(u)} \ (a_1, \ldots, a_n) \leq_n (f(a_1), \ldots, f(a_n)) \}$.

Let $\mathbb{F}$ be an inclusion functor $\mathcal{C} \to \mathcal{SET}$. Then $\mathcal{F}[\mathbb{F}] = \mathcal{F}$ since

$\underline{a} \leq_n \underline{b} \Longleftrightarrow \exists f \in \mathcal{C}(u,v) \ \forall i \ b_i = f(a_i)$ (whenever $u \leq_0 v$, $\underline{a} \in D_{n,u}$, $\underline{b} \in D_{n,v}$)

To prove ($\Rightarrow$), we enumerate $D_{(u)}$: $D_{(u)} = \{a_1, \ldots, a_n, \ldots\}$, and using the property (2) from 2.4, we get the sequence $b_1, \ldots, b_n, \ldots$ satisfying $(a_1,\ldots,a_k) \leq_k (b_1,\ldots,b_k)$ for all k; then the function f sending each a_k to b_k is in $\mathcal{C}(u,v)$.

2) Let $\mathcal{F} = (\mathbb{F}, \mathbb{D})$ be a Cartesian metaframe over the two-element chain $F = (\{u_0, v_0\}, \leq_0)$, with $D_{(u_0)} = \omega$, $D_{(v_0)} = 2^\omega$,

there exists a function $f=\mathbb{F}^{\sim}(\mu): D_{(u_0)} \to D_{(v_0)}$, and thus $a_1 \neq a_2$, $f(a_1)=f(a_2)=b$ for some $a_1, a_2 \in D_{(u_0)}$, $b \in D_{(v_0)}$. Hence $(a_1,a_2) \leq_2 (b,b)$ is not the case in $\mathcal{F}$, and this is a contradiction. ⊠

Therefore all Cm=-metaframes with countable domains are nothing but $\mathcal{C}$-sets. However we do not know if arbitrary cm=-metaframes are equivalent to $\mathcal{C}$-sets.

The following theorem shows that for modal logics without equality the situation is different; moreover, for this case m-metaframes are stronger than m=-metaframes (and therefore, than $\mathcal{C}$-sets).

Theorem 4.4.2. Let $C_1'= P(x) \supset P(y) \vee Q(x) \vee Q(y)$, $C_2'= P(x,y) \supset P(x,x) \vee Q(x) \vee Q(y)$, $C_i = Q(z) \vee \Box(\Box Q(z) \supset \forall x \forall y C_i')$, $i=1,2$.
Then

1) $\mathcal{F} \models C_1 \Rightarrow \mathcal{F} \models C_2$, for any m=-metaframe $\mathcal{F}$;
2) $\mathcal{F}^{\sim} \models C_1$, $\mathcal{F}^{\sim} \not\models C_2$, for some m-metaframe $\mathcal{F}^{\sim}$.

Proof. 1) By 4.3.1, we can assume that $\mathcal{F}$ is Cartesian. So let $\mathcal{F} = (\mathbb{F}, \mathbb{D})$, and suppose , $\xi, u_0 \not\models C_2^{[m]}(\underline{a},a)$ for some valuation ξ in $\mathcal{F}$, $(\underline{a},a) \in (D_{(u_0)})^{m+1}$. Then we have (for some $u_1, d_0, d_1, \underline{b}, b$):

$u_0 \leq_0 u_1$, $(\underline{a},a) \leq_{m+1} (\underline{b},b)$, $(\underline{b},b) \in (D_{(u_1)})^{m+1}$, $d_0, d_1 \in D_{(u_1)}$;
$\xi, u_0 \not\models Q(\underline{a},a)$;
$\xi, u_1 \models \Box Q(\underline{b},b) \wedge P(\underline{b},d_0,d_1) \wedge \neg P(\underline{b},d_0,d_0) \wedge \neg Q(\underline{b},d_0) \wedge \neg Q(\underline{b},d_1)$.

Hence $d_0 \neq d_1$. Taking $\xi'(Q)=\xi(Q)$, $\xi'(P)=\{(\underline{b},d_0)\}$ we have :

$\xi', u_0 \not\models Q(\underline{a},a)$;
$\xi', u_1 \models \Box Q(\underline{b},b) \wedge P(\underline{b},d_0) \wedge \neg P(\underline{b},d_1) \wedge \neg Q(\underline{b},d_0) \wedge \neg Q(\underline{b},d_1)$,

and therefore $\xi', u_0 \not\models C_1^{[m]}(\underline{a},a)$.

2) Let $D_1 = \{c_1, d_0, d_1\}$; for $n>0$, let $\approx_n$, $\leq_n$ be the following relations in D_1^n :

$\underline{a} \approx_n \underline{b} \Leftrightarrow \underline{a}=\underline{b} \vee \forall i\,(a_i=b_i=c_1 \vee \{a_i,b_i\}=\{d_0,d_1\})$;
$\underline{a} \leq_n \underline{b} \Leftrightarrow \underline{a} \approx_n \underline{b} \vee \forall i\; b_i=c_1$.

Then

$\underline{a} \approx_n \underline{b} \Leftrightarrow \underline{a} \leq_n \underline{b}\ \&\ \underline{b} \leq_n \underline{a}$.

Let $(D_n^{\sim}, \leq_n^{\sim})$ be the skeleton of $(D_1^n, \leq_n)$ for $n>0$ and

$\underline{a} \approx_n \underline{b} \iff \underline{a} \leq_n \underline{b}$ & $\underline{b} \leq_n \underline{a}$.

Let $(D_n{}^{\sim}, \leq_n{}^{\sim})$ be the skeleton of $(D_1{}^{n}, \leq_n)$ for n>0 and $(D_0{}^{\sim}, \leq_0{}^{\sim}) = (\{u\}, =)$ be a reflexive singleton. $\underline{a}^{\sim}$ denotes the class $\underline{a}$ (mod$\approx_n$) (if $\underline{a} \in D_n$).

Let $\mathcal{F}^{\sim} = (\mathbb{D}^{\sim}, \mathbb{\Pi})$ be a metaframe in which $\mathbb{\Pi} = (\pi_\sigma)_{\sigma \in \mathrm{Mor}\Sigma}$, π_σ is a constant map if $\sigma = \Lambda_n$, $\pi_\sigma(\underline{a}^{\sim}) = (\underline{a}_\sigma)^{\sim}$ otherwise.

Let ξ be a valuation in $\mathcal{F}^{\sim}$ such that $\xi(Q) = \{c_1{}^{\sim}\}$, $\xi(P) = \{(d_0, d_1)^{\sim}\}$. Then $\xi, d_0{}^{\sim} \not\models C_2\ [z]$.

On the other hand, $\mathcal{F}^{\sim} \models C_1{}^{[m]}$. For, consider any model in $\mathcal{F}^{\sim}$, and suppose $(\underline{a}, a)^{\sim} \not\models Q(\underline{z}, z)\ [\underline{z}, z]$, $(\underline{b}, b)^{\sim} \not\models \Box Q(\underline{z}, z) \supset (\forall x \forall y C_1{}')^{[m]}\ [\underline{z}, z]$, $(\underline{a}, a) \leq_{m+1} (\underline{b}, b)$. Then $(\underline{a}, a) \approx_{m+1} (\underline{b}, b)$ is not true, and hence $b = c_1$, $\underline{b} = (c_1, \ldots, c_1)$. But it is easily checked that $(\underline{b}, b)^{\sim} \models Q(\underline{z}, z)$ implies $(\underline{b}, b)^{\sim} \models (\forall x \forall y C_1{}')^{[m]}\ [\underline{z}, z]$ (note that $(\underline{b}, d_0)^{\sim} = (\underline{b}, d_1)^{\sim}$), and the contradiction follows. ⊠

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