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A MODAL THEORY OF ARROWS
ARROW LOGICS I

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A MODAL THEORY OF ARROWS

ARROW LOGICS I

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A MODAL THEORY OF ARROWS. ARROW LOGICS I

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Abstract. The notion of arrow structure /a.s./ is introduced as an algebraic version of the notion of directed multi graph. By means of a special kind of a representation theorem for arrow structures it is shown that the whole information of an a.s. is contained in the set of his arrows equipped with four binary relations describing the four possibilities for two arrows to have a common point. This makes possible to use arrow structures as a semantic base for a special polymodal logic, called in the paper BAL /Basic Arrow Logic/. BAL and various kinds of his extensions are used for expressing in a modal setting different properties of arrow structures. Several kinds of completeness theorems for BAL and some other arrow logics are proved, including completeness with respect to classes of finite models. And the end some open problems and possibilities for further development of the "arrow" approach are formulated.

Introduction

There exist many formal schemes and tools for representing knowledge about different types of data. Sometimes we can better understand this knowledge if it has some graphical representation. In many cases arrows are very suitable visual objects for representing various data structures: different kinds of graphs, binary relations, mappings, categories and so on. An abstract form of this representation scheme is the notion of arrow structure /a.s./, which in this paper is an algebraic version of the notion of directed multi graph. Simply speaking, a.s. is a two sorted algebraic system, consisting of a set of arrows Ar, a set of points Po and two functions 1 and 2 from arrows to points, assigning to each arrow x the point $1(x)$ – the beginning of x, and the point $2(x)$ – the end of x. By means of 1 and 2 we define four relations R_{ij} $i,j=1,2$ such that $xR_{ij}y$ iff $i(x)=j(y)$. These relations define the four possibilities for a two arrows to have a common point. So each a.s. S determine a relational system $W(S)=(Ar, \{R_{ij}/ij=1,2\})$ called arrow frame /a.f./. It is shown that the whole information of an a.s. S is contained in the arrow frame W(S). Arrow frames as relational systems with binary relations are suitable for interpreting polymodal logics, having modal operations, corresponding to each binary relation in the frame. So we introduce a modal language $\mathcal{L}$ with four boxes [ij] with standard Kripke semantics in arrow frames . We show how different properties of arrow frames are modally definable by means of modal formulas of $\mathcal{L}$. The logic of all arrow frames is axiomatized and called BAL – the Basic Arrow Logic. This paper is mainly devoted to study BAL and some of their extensions.

The paper is organized as follows.

Section 1 is devoted to arrow structures and arrow frames.

In section 2 we introduce semantically the notion of arrow logic as the class of all formulas true in a given class of arrow frames. Some definability and undefinability results are proved there. For instance, applying some special techniques, called "copying", we show that the logic of all arrow frames coincides with the logic of all normal arrow frames, which correspond to directed graphs, admitting no more than one arrow between an ordered pair of points.

In section 3 we give axiomatization of the logic of all arrow frames - BAL and prove several completeness theorems.

In section 4, applying the filtration technic from ordinary modal logic we prove that BAL and some other arrow logics possess finite model property and are decidable.

In section 5 we study an extension of BAL with a new connective interpreted by an equivalence relation between arrows, stating that two arrows are equivalent if they have common begins and common ends.

In section 6 we study another extension with a modal constant Loop, which is true in an arrow if it has common begin and end, i.e. if it forms a loop.

Section 7 is devoted to a short survey for possible directions for further development, including extensions with different polyadic modalities, corresponding to some typical relations between arrows as $Path_n$, $Path_\infty$, $Loop_n$, $Trapezium_n$, Triangle and others. There are some natural generalizations of modal logic of binary relations and Lambek Calculus. Finally it is shown a way of many dimensional generalization of arrow structures, which makes possible to consider an n-ary relation in a set as an n-dimensional arrow structure. Among the logics based on n-dimensional arrow frames are some natural generalizations of the so called cylindric modal Logics.

The idea to look for a logic based on two sorted structures having points and arrows, was suggested to me by Johan Van Benthem [BEN 90]. The first results were included in the manuscript [VAK 90] and the many-dimensional generalization in the abstracts [VAK 91a] and [VAK 92]. The terms arrow frame and arrow logic were introduced by Van Benthem [BEN 89] in connection with some generalizations of the modal logic of algebra of relations. Van Bethem's arrow frames consist of a set of objects with composition as a ternary relation, converse as a binary relation and a set of identity arrows. These relational structures are so abstract that there is no any representation theorem stating that the arrows look indeed as arrows, with beginning and end. We adopt Van Benthem's terminology, because it fits very well to the subject of this paper.

1. Arrow structures and arrow frames

By arrow structure (a.s.) we shall mean any system $S=(Ar, Po, 1, 2)$, where

- Ar is a nonempty set, whose elements are called arrows,
- Po is a nonempty set, whose elements are called points. We assume also that $Ar \cap Po = \emptyset$.
- 1 and 2 are total functions from Ar to Po associated to each arrow x the following two points: 1(x) - the first point of x (beginning, source, domain), and 2(x) - the last point of x (end, target, codomain). Graphically:

$$1(x)\bullet \xrightarrow{\qquad x \qquad} \bullet 2(x)$$

If $A=1(x)$ and $B=2(x)$ we say that x connects A with B, or, that (A,B) is a connected pair of points. It is possible for a pair of points (A,B) to be connected by different arrows.

● For some technical reasons we assume the following axiom for arrow structures:

(Ax) For each point A there exists an arrow x such that $A=1(x)$ or $A=2(x)$. In other words, each point is ether the first or the last point of some arrow.

An a.s. S is called normal if it satisfies the following condition of normality

(Nor) If $1(x)=1(y)$ and $2(x)=2(y)$ then $x=y$.

Sometimes, to denote that Ar, Po, 1, 2 are from a given a.s. S, we will write Ar_S, Po_S, 1_S and 2_S.

The main examples of a.s. structures are directed multi-graphs, and for normal a.s. - directed graphs without isolated points. These are notions studied in Graph theory where graphs are visualized, or sometimes defined, by geometrical notions of a point and arrow. In graph intuition arrow is a part of a line with some direction, connected two points. Formally, the notion of an arrow structure coincides with the notion of directed multi-graph without isolated points. We will prefer, however, the term "arrow structure" as more neutral, having models, not only connected with graph intuition, as for example, categories and binary relations.

The example of a.s. constructed from a binary relation can be defined as follows. Let R be a nonempty binary relation in a nonempty set W. Define $Ar=R$, $Po=\{x\in W/(\exists y\in W)(xRy \text{ or } yRx)\}$ and for $(x,y)\in Ar$ define $1((x,y))=x$ and $2((x,y))=y$. Then, obviously $(Ar, Po, 1, 2)$ is a normal a.s. In some sense this example is typical, because each normal a.s. can be represented as an a.s. determined by a non-empty binary relation.

Let S be an a.s. The following binary relation $\rho=\rho_S$ can be defined in the set Po_S. For each $A,B\in Po_S$:

$A\rho B$ iff $(\exists x\in Ar_S)(1(x)=A$ and $2(x)=B)$

According to some properties of ρ_S we can consider different kinds of arrow structures:

● S is serial a.s. if ρ_S is a serial relation (i.e. $\forall A\ \exists B\ A\rho B$),

● S is reflexive a.s. if ρ_S is a reflexive relation,

● S is symmetric a.s. if ρ_S is a symmetric relation,

● S is transitive a.s. if ρ_S is a transitive relation,

● S is total a.s. if ρ_S is a total relation, i.e. $\rho_S=Po_S\times Po_S$.

Let S be a given a.s. The following four relations $R_{ij}=R^S_{ij}$, $i,j\in\{1,2\}$ in Ar_S (called incidence relations in S), will play a fundamental role in this paper:

$xR_{ij}y$ iff $i(x)=j(y)$

The following pictures illustrated the introduced relations:

$xR_{11}y$: $\xleftarrow{\ x\ }\bullet\xrightarrow{\ y\ }$ $\qquad$ $xR_{22}y$ $\xrightarrow{\ x\ }\bullet\xleftarrow{\ y\ }$

$xR_{21}y$: $\xrightarrow{\ x\ }\bullet\xrightarrow{\ y\ }$ $\qquad$ $xR_{12}y$ $\xleftarrow{\ x\ }\bullet\xleftarrow{\ y\ }$

Lemma 1.1. The relations R_{ij} satisfy the following conditions for any $x,y,z\in Ar_S$ and $i,j,k\in\{1,2\}$:

(ρii) $xR^S_{ij}x$,

(σij) If $xR_{ij}y$ then $yR^S_{ji}x$,

(τijk) If $xR_{ij}y$ and $yR_{jk}z$ then $xR_{ik}z$.

Proof. By an easy verification.■

Let $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21})$, $W\neq\emptyset$, be a relational system. $\underline{W}$ will be called arrow frame (a.f.) if it satisfies the axioms (ρii), (σij) and (τijk) for any $i,j,k\in\{1,2\}$and $x,y,z\in W$. The class of all arrow frames will be denoted by ARROW. If S is an a.s. then the a.f. $SAF(S)=(Ar_S, R^S_{11}, R^S_{22}, R^S_{12}, R^S_{21})$ will be called a standard a.f. over S. The class of all standard a.f. will be denoted by (standard)ARROW. One of the main results of this section will be the proof that each a.f. is a standard a.f. over some a.s., i.e. (standard)ARROW=ARROW.

Lemma 1.2. Let S be an a.s. Then the following equivalences are true, where x,y,z range over Ar_S:

(i) S is normal a.s.iff $\forall xy(xR_{11}y \;\&\; xR_{22}y\rightarrow x=y)$,

(ii) S is serial a.s. iff $\mathcal{G}\forall x\exists y\; xR_{21}y$,

(iii) S is reflexive a.s. iff $\forall x\exists y(xR_{11}y \;\&\; yR_{21}x)$ and $\forall x\exists y(xR_{21}y \;\&\; yR_{22}x)$,

(iv) S is symmetric a.s. iff $\forall x\exists y(xR_{12}y \;\&\; yR_{12}x)$,

(v) S is transitive a.s. iff $\forall xy\exists z(xR_{21}y\rightarrow xR_{11}z \;\&\; yR_{22}z)$.

(vi) S is total a.s. iff $\forall ij\in\{1,2\}\forall xy\exists z(xR_{i1}z \;\&\; zR_{2j}y)$

Proof. As an example we shall prove (ii).

($\rightarrow$) Suppose S is serial and let $x\in Ar_S$ and $2(x)=A$. By seriality there exists $B\in Po_S$ such that $A\rho B$. Then for some $y\in Ar_S$ we have $1(y)=A$ and $2(y)=B$, so $2(x)=1(y)$, which yields $xR_{21}y$. Thus $\forall x\exists y\; xR_{21}y$.

($\leftarrow$) Suppose $\forall x\exists y\; xR_{21}y$ and let $A\in Po_S$. Then by (Ax) there exists $x\in Ar_S$ such that $A=1(x)$ or $A=2(x)$.

Case 1: $A=1(x)$. Let $B=2(x)$, then $A\rho B$.

Case 2: $A=2(x)$. Take y such that $xR_{21}y$. From here we get $2(x)=1(y)$ and $A=1(y)$, Take $B=2(y)$, then we get $A\rho B$. So in both cases $\forall A\exists B\; A\rho B$.

The remaining conditions can be proved in a similar way. ■

This lemma suggests the following definition concerning arrow frames. Let $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21})$ be an a.f., then $\underline{W}$ is called:

- $\underline{W}$ is normal a.f.iff $\forall xy(xR_{11}y \;\&\; xR_{22}y\rightarrow x=y)$,
- $\underline{W}$ is serial a.f. iff $\mathcal{G}\forall x\exists y\; xR_{21}y$,
- $\underline{W}$ is reflexive a.f. iff $\forall x\exists y(xR_{21}y \;\&\; xR_{22}y)$ and $\forall x\exists y(xR_{12}y \;\&\; xR_{11}y)$,
- $\underline{W}$ is symmetric a.f. iff $\forall x\exists y(xR_{12}y \;\&\; xR_{21}y)$,
- $\underline{W}$ is transitive a.f. iff $\forall xy\exists z(xR_{21}y\rightarrow xR_{11}z \;\&\; yR_{22}z)$,
- $\underline{W}$ is total a.f. iff $\forall ij\in\{1,2\}\forall xy\exists z(xR_{i1}z \;\&\; zR_{2j}y)$, where the variables x,y,z range over the set W.

The class of all normal arrow frames will be denoted by (nor)ARROW. Analogously we introduce the notations (ser)ARROW, (ref)ARROW, (sym)ARROW, (tr)ARROW and (total)ARROW for the classes of all serial a.f., reflexive a.f., symmetric a.f., transitive a.f. and total a.f. respectively. We will use also a notation as (ref)(sym)ARROW denoting the class of all reflexive and symmetric arrow frames.

Obviously, if $\underline{W}$ is total a.f. then $\underline{W}$ is reflexive, symmetric and transitive a.f. An a.f. is called pretotal if it is reflexive, symmetric and transitive. The class of all pretotal a.f. is denoted by (pretotal)ARROW. Using combined notations we have that (pretotal)ARROW=(ref)(sym)(tr)ARROW.

Let $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21})$ be an a.f. and $W'\subseteq W$, $W'\neq\emptyset$ and R'_{ij} are the relations R_{ij} restricted over W'. Then obviously the system $\underline{W}'=(W', R'_{11}, R'_{22}, R'_{12}, R'_{21})$ is an a.f. called subframe of $\underline{W}$. The frame $\underline{W}'$ is called generated subframe of $\underline{W}$ if $\forall i\,j\in\{1,2\}\forall x\in W'\forall y\in W(xR_{ij}y\rightarrow y\in W')$. If $a\in W$ then by $\underline{W}_a$ will be denoted the smallest generated subframe of $\underline{W}$, containig a. $\underline{W}_a$ is called an arrow subframe of $\underline{W}$ generated by a. If $\underline{W}$ is an a.f. and there exists some $a\in W$ such that $\underline{W}=\underline{W}_a$ then $\underline{W}$ is called a generated a.f. (by a). If Σ is a class of arrow frames then by Σ_{gen} we denote the class of all generated frames of Σ.

Lemma 1.3. (i) $((\text{pretotal})\text{ARROW})_{gen}\subseteq(\text{total})\text{ARROW}$,

(ii) $((\text{pretotal})\text{ARROW})_{gen}=(\text{total})\text{ARROW}$

Proof. By an easy verification.■

Let S be an a.s. and for $i\in\{1,2\}$ and $A\in Po_S$ define:

$i(A)=\{x\in Ar_S/i(x)=A\}$, $g(A)=(1(A),2(A))$.

Lemma 1.4. The following is true for each $x,y\in Ar_S$ and $i,j\in\{1,2\}$:

(1) If $x\in i(A)$ and $y\in j(A)$ then $xR^S_{ij}y$,

(2) If $xR^S_{ij}y$ then $x\in i(A)$ iff $y\in j(A)$,

(3) $1(A)\cup 2(A)\neq\emptyset$.

Proof. By an easy verification.■

Lemma 1.4 suggests the following definition. Let $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21})$ be an a.f. and α_1 and α_2 be subsets of W. The pair (α_1,α_2) will be called a generalized point in $\underline{W}$ if it satisfies the following conditions for each $x,y\in W$ and $i,j\in\{1,2\}$:

(1) If $x\in\alpha_i$ and $y\in\alpha_j$ then $xR_{ij}y$,

(2) If $xR_{ij}y$ then $x\in\alpha_i$ iff $y\in\alpha_j$,

(3) $\alpha_1\cup\alpha_2\neq\emptyset$.

The set of generalized points of an a.f. $\underline{W}$ will be denoted by $Po(\underline{W})$. Lemma 1.2 now means that $g(A)=(1(A),2(A))$ is a generalized point in the standard a.f. SAF(S) over S.

For any binary relation R in W and $x\in W$ define $R(x)=\{y\in W/xRy\}$.

Lemma 1.5. Let $\underline{U}=(U, U_{11}, R_{22}, R_{12}, R_{21})$ be an a.f.. Then for any $x,y\in U$ and $i,j\in\{1,2\}$: $xR_{ij}y$ iff $R_{i1}(x)=R_{j1}(y)$ and $R_{i2}(x)=R_{j2}(y)$

Proof. By an easy calculation, using the axioms of a.f.■

Lemma 1.6. Let $\underline{W}$ be an a.f. Then for any $x,y,z\in W$ and $i,j,k\in\{1,2\}$:

(i) The pair $k(z)=(R_{k1}(z), R_{k2}(z))$ is a generalized point in $\underline{W}$.

(ii) For each generalized point (α_1,α_2) there exists $z\in W$ and $k\in\{1,2\}$ such that $k(z)=(\alpha_1,\alpha_2)$.

(iii) $xR_{ij}y$ iff $i(x)=j(y)$.

Proof. (i) Let $i,j\in\{1,2\}$ and $x\in R_{ki}(z)$ and $y\in R_{kj}(z)$. Then we have $zR_{ki}x$ and $zR_{kj}y$. Then by (σki) we obtain $xR_{ik}z$ and by (τikj) we get $xR_{ij}y$. This proves condition (1) from the definition of generalized point. In a similar way one can verify condition (2). By (ρkk) we have $xR_{kk}x$, so $R_{kk}(x)\neq\emptyset$. This shows that $R_{k1}(x)\cup R_{k2}(x)\neq\emptyset$, which proves condition (3).

(ii) Let (α_1,α_2) be a generalized point in $\underline{W}$. Then there exists $z\in W$ such that $z\in\alpha_1\cup\alpha_2$.

Case 1: $z\in\alpha_1$. In this case we will show that $k=1$, i.e. that $(\alpha_1,\alpha_2)=1(z)=(R_{11}(z),R_{12}(z))$ i.e. that $\alpha_1=R_{11}(z)$ and that $\alpha_2=R_{12}(z)$.

Let $x\in\alpha_1$. Since $z\in\alpha_1$, then by (1) of the definition of generalized point we get $xR_{11}z$. So by $(\sigma 11)$ we obtain $zR_{11}x$, which shows that $x\in R_{11}(z)$.

Now let $x\in R_{11}(z)$. Then $zR_{11}x$ and since $z\in\alpha_1$, by (2) of the definition of generalized point we get $x\in\alpha_1$. This proves the equality $\alpha_1=R_{11}$. In a similar way on can prove that $\alpha_2=R_{12}(z)$.

Case 2: $z\in\alpha_2$. In this case $k=2$ and we can proceed as in case 1.

(iii) By lemma 1.5. we have: $xR_{ij}y$ iff $R_{i1}(x)=R_{i2}(y)$ and $R_{j1}(x)=R_{j2}(y)$ iff $(R_{i1}(x),R_{i2}(x))=(R_{j1}(y),{}_{j2}(y))$ iff $i(x)=j(y)$.■

Now we shall give a construction of arrow structures from arrow frames. Let $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21})$. Define a system $S=S(\underline{W})$ as follows: $Ar_S=W$, $Po_S=Po(\underline{W})$ – the set of general points of $\underline{W}$, for $k=1,2$ and $z\in W$ let $k_S(z)=k(z)=(R_{k1}(z),R_{k2}(z))$ as in lemma 1.6. In the next theorem we shall show that $S(\underline{W})$ is an a.s. called the a.s. over $\underline{W}$.

Theorem 1.7. (i) The system $S(\underline{W})$ defined above is an a.s. More over:

(ii) The standard a.f. SAF($S(\underline{W})$) over $S(\underline{W})$ coincides with $\underline{W}$.

(iii) $S(\underline{W})$ is normal (serial, reflexive, symmetric, transitive, total) a.s. iff $\underline{W}$ is normal (serial, reflexive, .. and so on) a.f.

Proof. (i) By lemma 1.6.(i) and (ii) we obtain that the system $S(\underline{W})$ is an a.s.

(ii) By lemma 1.1 and lemma 1.6.(iii) SAF($S(\underline{W})$) is a standard a.f. such that for any $x,y\in W$ and $i,j\in\{1,2\}$: $xR_{ij}y$ iff $i(x)=j(y)$ iff $xR^S_{ij}y$, which shows that SAF($S(\underline{W})$)$=\underline{W}$.

(iii) By lemma 1.2. $S(\underline{W})$ is normal (serial,...) a.s. iff the corresponding standard a.f. SAF($S(\underline{W})$) over $S(\underline{W})$ is normal (serial,...). By (ii) SAF($S(\underline{W})$)$=\underline{W}$, which proves the assertion. ■

Corollary 1.8. (standard)ARROW=ARROW

Proof. From theorem 1.7.■

Let S and S' be two arrow structures. A pair (f,g) of

11-functions $f:Ar_S \rightarrow Ar_{S'}$ and $g:Po_S \rightarrow Po_{S'}$ is called an isomorphism from S onto S' if for any $x \in Ar_S$ and $i=1,2$ we have $g(i_S(x)) = i_{S'}(f(x))$.

Lemma 1.9. Let S be an a.s. $\underline{W}$=SAF(S) be the standard a.f. over S, Po($\underline{W}$) be the set of generalized points of $\underline{W}$, and S'=S($\underline{W}$) be the a.s. over $\underline{W}$. Let for $A \in Po_S$ $g(A)=(1(A),2(A))$ be the function defined before lemma 1.4. and for $x \in Ar_S$ and $i=1,2$ $i_{S'}(x) = (R^{S'}_{i1}(x), R^{S'}_{i2}(x))$ be the function defined as in lemma 1.6.(i). Then:

(i) g is a 11-function from Po_S onto Po($\underline{W}$).

(ii) For any $x \in Ar_S$ and $i=1,2$: $g(i(x))=i_{S'}(x)$.

Proof. Obviously g(A) is a generalized point in $\underline{W}$. Let $g(A)=g(B)$. Then $1(A)=1(B)$ and $2(A)=2(B)$. For A we can find $x \in Ar_S$ such that $1(x)=A$ or $2(x)=A$. Then $x \in 1(A)$ or $x \in 2(A)$. Suppose $x \in 1(A)$. Then $x \in 1(B)$, so $1(x)=B$. From $1(x)=A$ and $1(x)=B$ we get A=B. In the case $x \in 2(A)$ we proceed in the same way and get A=B. This shows that the mapping is injective. To show that it is "onto" suppose that (α_1,α_2) is a generalized point in $\underline{W}$. We shall show that for some $A \in Po_S$ $g(A)=(1(A),2(A))=(\alpha_1,\alpha_2)$. Since $\alpha_1 \cup \alpha_2 \neq \emptyset$ there exists $z \in \alpha_1$ or $z \in \alpha_2$.

Case1: $z \in \alpha_1$. Let $1(z)=A$, so $z \in 1(A)$. We shall show that $1(A)=\alpha_1$ and that $2(A)=\alpha_2$. Suppose $x \in 1(A)$. Then $1(x)=A$, so $1(x)=1(z)$ which yields $xR^S_{11}z$. Since $z \in \alpha_1$, then, by the properties of generalized points we get $x \in \alpha_1$, so $1(A) \subseteq \alpha_1$. Suppose now that $x \in \alpha_1$. Then, since $z \in \alpha_1$, we get $xR_{11}z$, so $1(x)=1(z)=A$. Then $x \in 1(A)$, so $\alpha_1 \subseteq 1(A)$. Consequently $1(A)=\alpha_1$. In a similar way one can show that $2(A)=\alpha_2$. Hence, in this case $g(A)=(\alpha_1,\alpha_2)$.

Case 2: $z \in \alpha_2$. The proof is similar to that of case 1.

(ii) Let $x \in Ar_S$ and $i=1,2$. Since $g(i(x))=(1(i(x)),2(i(x))$ and $i_{S'}(x)=(R^S_{i1}(x),R^S_{i2}(x))$, To show that $g(i(x))=i_{S'}(x)$, we have to prove that $1(i(x))=R^S_{i1}(x)$ and that $2(i(x))=R^S_{i1}(x)$. For that purpose suppose that $y \in 1((i(x))$, so $1(y)=i(x)$. Thus $yR^S_{i1}x$, which yields $y \in R^S_{i1}(x)$. Consequently $1(i(x)) \subseteq R^S_{i1}(x)$. The converse inclusion and the second equality can be proved in a similar way. ■

Theorem 1.10. Let S be an a.s., $\underline{W}$=SAF(S) be the standard a.f. over S and S($\underline{W}$) be the a.s. over $\underline{W}$. Then S is isomorphic with S($\underline{W}$).

Proof. Let for $x \in Ar_S$ $f(x)=x$ and for $A \in Po_S$ $g(A)=(1(A),2(A))$. Then lemma 1.8 shows that the pair (f,g) is the required isomorphism. ■

Theorems 1.10 and 1.7 show that the whole information of an a.s. S is contained in the standard a.f. SAF(S) over S and can be expressed in terms of arrows and the relations R^S_{ij}. An example of such a correspondence is lemma 1.2. As for first order conditions about the relation ρ this correspondence can be defined in an effective way. The intuitive idea of this translation can be explained in the following way. By the axiom (Ax) for each point A there exists $i \in \{1,2\}$ such that $A=i(x)$. So each variable A for a point is translated by a pair (i,x), where x denotes an arrow and i denotes one of the numbers 1 and 2. Suppose now that we have $A\rho B$, $A=i(x)$ and $B=j(y)$. Then by the definition of ρ we have: $(\exists u)(1(u)=i(x)\ \&\ 2(u)=j(y))$ which is equivalent to $(\exists u)(xR_{i1}u\ \&\ uR_{2j}y)$. So if A is translated by (i,x) and B by (j,y), then the corresponding translation of $A\rho B$ will be the formula $\varphi=xS_{ij}y=(\exists u)(xR_{i1}u\ \&\ uR_{2j}y)$. Here obviously $S_{ij}=R_{i1}\circ R_{2j}$. The parameters i and j in φ can be eliminated according to under what kind of quatifiers are A and B. If for example A is under the scope of $(\forall A)$, we change this quantifier by $(\forall i)(\forall x)$ and accordingly for $(\exists A)$. Then quantifiers of the type $(\forall i)$ and $(\exists i)$ can be eliminated in a standard way by conjunctions and disjunctions of formulas, putting on the place of i 1 and 2. Let us take for example the formula $(\forall A)(A\rho A)$. First this formula is translated by $(\forall i)(\forall x)xS_{ii}x$. Eliminating $(\forall i)$ we obtain

$$(\forall x)(xS_{11}x)\ \&\ (\forall x)(xS_{22}x),\text{ which is equivalent to}$$

$$(\forall x)(\exists y)(xR_{11}y\ \&\ yR_{21}x)\ \&\ (\forall x)(\exists y)(xR_{21}y\&\ yR_{22}x)$$

which is exactly the condition of reflexivity of ρ from lemma 1.2. The translation of the formula $(\forall A)(\exists B)(A\rho B)$ is the following:

$$(\forall i)(\forall x)(\exists j)(\exists y)(xS_{ij}y)$$

Eliminating $(\forall i)$ we obtain the conjunction of the following two formulas:

$$\varphi_{1j}=(\forall x)(\exists j)(\exists y)(xS_{1j}y),$$

$$\varphi_{2j}=(\forall x)(\exists j)(\exists y)xS_{2j}y).$$

Eliminating $(\exists j)$ from φ_{1j} and φ_{2j} we obtain the following formulas φ_1 and φ_2:

$$\varphi_1=(\forall x)((\exists y)(xS_{11}y)\vee(\exists y)(xS_{12}y)),$$

$$\varphi_2=(\forall x)((\exists y)(xS_{21}y)\vee(\exists y)(xS_{22}y)).$$ Substituting here S_{ij} we obtain

$$\varphi_1=(\forall x)((\exists y)(\exists z)(xR_{11}z\ \&\ zR_{21}y)\vee(\exists y)(\exists z)(xR_{11}z\ \&\ zR_{22}y)),$$

$$\varphi_2=(\forall x)((\exists y)(\exists z)(xR_{21}z\ \&\ zR_{21}y)\vee(\exists y)(\exists z)(xR_{21}z\ \&\ zR_{22}y)$$

The formula φ_1 is always true in a.s. because in the second disjunct we can put $y=z=x$. It follows logically from φ_2 the following formula $\varphi=(\forall x)(\exists z)(xR_{21}z)$, which is exactly the condition of seriality from lemma 1.2. It is easy to see that φ implies in a.s. the formula φ_2.

The described intuitive idea of translating first order sentences for points in terms of ρ and = in arrow structures into equivalent sentences for arrows in terms of the relations R_{ij} can be given in precise terms, but we will do not that in this paper.

2. Arrow logics - semantic definitions and some definability and nondefinability results

In this section we shall give a semantic definition of modal logics, called arrow logics. For that purpose we introduce the following modal language $\mathcal{L}$. It contains the following symbols:

- VAR – a denumerable set of proposition variables,
- $\neg$, $\wedge$, $\vee$ – classical propositional connectives,
- $[ij]$, $i,j=1,2$ – four modal operations,

● (,) – parentheses.

The definition of the set of all formulas FOR for $\mathscr{L}$ is defined in the usual way.

Abbreviations: $A \Rightarrow B = \neg A \vee B$, $A \Leftrightarrow B = (A \Rightarrow B) \wedge (B \Rightarrow A)$, $1 = A \vee \neg A$, $0 = \neg 1$, $\langle ij \rangle A = \neg [ij] \neg A$.

The general semantics of $\mathscr{L}$ is a Kripke semantics over relational structures of the type $\underline{W} = (W, R_{11}, R_{22}, R_{12}, R_{21})$ with $W \neq \emptyset$, called frames. The standard semantics of $\mathscr{L}$ is over the class ARROW of all arrow frames. Let us remind the basic semantic definitions and notations, which we will use /for more details about Kripke semantics and related notions we refer Segerberg [SEG 71], Hughes & Cresswell [H&C 84] and Van Benthem [Ben 86]/.

Let $\underline{W} = (W, R_{11}, R_{22}, R_{12}, R_{21})$ be a frame. A function $v: VAR \longrightarrow 2^W$ assigning to each variable $p \in VAR$ a subset $v(p)$ of W is called a valuation and the pair $M = (\underline{W}, v)$ is called a model over $\underline{W}$. For $x \in W$ and $A \in FOR$ we define a satisfiability relation $x \Vdash_v A$ in M /to be read "A is true in x at the valuation v"/ by induction on the complexity of the formula A as in the usual Kripke definition:

$x \Vdash_v A$ iff $A \in v(A)$ for $A \in VAR$,

$x \Vdash_v \neg A$ iff $x \nVdash_v A$ /not $x \Vdash_v A$/,

$x \Vdash_v A \wedge B$ iff $x \Vdash_v A$ and $x \Vdash_v B$,

$x \Vdash_v A \vee B$ iff $x \Vdash_v A$ or $x \Vdash_v B$,

$x \Vdash_v [ij]A$ iff $(\forall y \in W)(xR_{ij}y \rightarrow y \Vdash_v A)$.

We say that A is true in the model $M = (\underline{W}, v)$, or that M is a model for A, if for any $x \in W$ we have $x \Vdash_v A$. A is true in the frame $\underline{W}$, or that W is a frame for A, if A is true in any model over $\underline{W}$. A is true in a class Σ of frames if A is true in any member of Σ. A class of formulas L is true in a model M, or M is a model for L, if any member of L is true in M. L is true in a class of frames Σ if any formula from L is true in Σ. L is called the logic of Σ and denoted by $L(\Sigma)$ if it contains all formulas true in Σ. Obviously, this operation of assigning sets of formulas to classes of frames is antymonotonic in the following sense:

If $\Sigma \subseteq \Sigma'$ then $L(\Sigma') \subseteq L(\Sigma)$.

In this paper we will study the logics L((standard)ARROW, L(ARROW), L((nor)ARROW), L((ser)ARROW), L((ref)ARROW, L((sym)ARROW), L((tr)ARROW), L((pretotal)ARROW), L((total)ARROW). The most important logic from this list is L((standard)ARROW). The first result which, can be stated for L(standard)ARROW) and which follows immediately from corollary 1.8. is that

L((standard)ARROW)=L(ARROW).

We say that a condition φ for R_{ij} is modally definable in a class Σ of frames if there exists a formula A such that for any frame $\underline{W} \in \Sigma$: A is true in $\underline{W}$ iff φ holds in $\underline{W}$. If a class of frames is characterized by a condition φ which is modally definable in the class of all frames, then we say that Σ is modally definable class of frames. The following lemma is a standard results in modal definability theory.

Lemma 2.1. /Modal definability of arrow frames/ Let Σ be the class of all frames and $A \in VAR$. Then in the next table the conditions from the left side are modally definable in Σ by the formulas from the right side: $(i,j,k=1,2)$

(ρii) $(\forall x) xR_{ii}x$,	(Pii) $[ii]A \Rightarrow A$,
(σij) $(\forall xy)(xR_{ij}y \rightarrow yR_{ji}x)$	(Σij) $A \vee [ij] \neg [ji] A$,

(τijk) $(\forall xyz)(xR_{ij}y$ and $yR_{jk}z \longrightarrow xR_{ik}z)$ (Tijk) $[ik]A \rightarrow [ij][jk]A$.

Corollary 2.2. The class ARROW is modally definable.

Lemma 2.3. Let Σ=ARROW and $A \in$VAR. Then in the next table the conditions from the left side are modally definable in Σ by the formulas from the right side:

seriality of an a.f.	(ser)	$\langle 21 \rangle 1$,
reflexivity of an a.f.	(ref)	$([11][21]A \rightarrow A) \wedge ([21][22]A \rightarrow A)$,
symmetricity of an a.f.	(sym)	$[12][12]A \rightarrow A$,
transitivity of an a.f.	(tr)	$[11][22]A \rightarrow [21]A$.

Proof. As an example we shall show the validity of (tr) in an a.f. $\underline{W}$ implies that $\underline{W}$ is a transitive a.f. For the sake of contradiction, suppose that (tr) is true in $\underline{W}$ and that $\underline{W}$ is not transitive a.f. Then for some $x,y,z \in W$ we have $xR_{21}y$ and not$(\exists z \in W)(xR_{11}z$ & $zR_{22}y)$. Define $v(A)=W \setminus \{y\}$. Then $y \nVdash_v A$ and since $xR_{21}y$ we get $x \nVdash_v [21]A$. We shall show that $x \Vdash_v [11][22]A$. Suppose that this is not true. Then for some $z,t \in W$ we have $xR_{11}z$, $zR_{22}t$ and $t \nVdash_v A$, hence $t=y$. So $(\exists z)(xR_{11}z$ & $zR_{22}y)$, which is a contradiction. ∎

Corollary 2.4. The classes (ser)ARROW, (ref)ARROW, (sym)ARROW, (tr)ARROW and (pretotal)ARROW are modally definable.

We shall show that the condition of normality of an a.f. is not modally definable and consequently that the class (nor)ARROW is not modally definable. We shall show first that the logic L((nor)ARROW) coincides with the logic L(ARROW). For that purpose we shall use a special construction called copying, adapted here for relational structures in the type of arrow frames.

Let $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21})$ and $\underline{W}'=(W', R'_{11}, R'_{22}, R'_{12}, R'_{21})$ be two frames and $M=(\underline{W}, v)$, $M'=(\underline{W}', v')$ be models over $\underline{W}$ and $\underline{W}'$ respectively. Let I be a nonempty set of mappings from W into W'. We say that I is a copying from $\underline{W}$ to $\underline{W}'$ if the following conditions are satisfied for any $i,j \in \{1,2\}$, $x,y \in W$and $f,g \in I$:

(I1) $(\forall y' \in W')(\exists y \in W)(\exists g \in I)g(y)=y'$

(I2) If $f(x)=g(y)$ then $x=y$,

(R_{ij}1) If $xR_{ij}y$ then $(\forall f \in I)(\exists g \in I)f(x)R'_{ij}g(y)$,

(R_{ij}2) If $f(x)R'_{ij}g(y)$ then $xR_{ij}y$.

We say that I is a copying from M to M' if in addition the following condition is satisfied for any $p \in$VAR, $x \in W$ and $f \in I$:

(V) $x \in v(p)$ iff $f(x) \in v'(p)$.

For $x \in W$ and $f \in I$ $f(x)$ is called f-th copy of x and $f(W)=\{f(x)/x \in W\}$ is called f-th copy of W. By (I1) we obtain that $W'=\bigcup\{f(W)/f \in I\}$, so W' is a sum of his copies. If I is one element set $\{f\}$ then f is an isomorphism from $\underline{W}$ onto $\underline{W}'$.

The importance of the copying construction is in the following

Lemma 2.5. (i) (Copying lemma) Let I be a copying from the model M to the model M'. Then for any formula $A \in \mathscr{L}$, $x \in W$ and $f \in I$ the following equivalence holds:

$x \Vdash_v A$ iff $f(x) \Vdash_{v'} A$,

(ii) If I is a copying from the frame $\underline{W}$ to the frame $\underline{W}'$ and v is a valuation, then there exists a valuation v' such that I is a copying from the model $M=(\underline{W}, v)$ to the model $M'=(\underline{W}', v')$.

Proof. (i) The proof is by induction on the complexity of the formula A. For $A \in$VAR the assertion holds by the condition (V) of copying. If A is a Boolean combination of formulas the proof is straightforward. Let $A=[ij]B$ and by the induction hypothesis (i.h.) suppose that the assertion for B holds.

($\rightarrow$) Suppose $x \Vdash_{v} [ij]B$ and $f \in I$. To show that $f(x) \Vdash_{v'} [ij]B$ suppose $f(x)R'_{ij}y'$ and proceed to show that $y' \Vdash_{v} B$. By (I1) $(\exists y \in W)(\exists g \in I)g(y)=y'$, so $f(x)R'_{ij}g(y)$ and by $(R_{ij}2)$ we get $xR_{ij}y$. From $xR_{ij}y$ and $x \Vdash_{v} [ij]B$ we get $y \Vdash_{v} B$. Then by the i.h. we get $g(y) \Vdash_{v'} B$, so $y' \Vdash_{v'} B$.

($\leftarrow$) Suppose $f(x) \Vdash_{v'} [ij]B$. To show that $x \Vdash_{v} [ij]B$ suppose $xR_{ij}y$ and proceed to show that $y \Vdash_{v} B$. From $xR_{ij}y$ we obtain by $(R_{ij}1)$ that there exists $g \in W$ such that $f(x)R'_{ij}g(y)$. Then, since $f(x) \Vdash_{v'} [ij]B$, we get $g(y) \Vdash_{v'} B$ and by the i.h. $y \Vdash_{v} B$.

(ii) Define for $p \in VAR$:

$v'(p)=\{x' \in W' / (\exists x \in W)(\exists f \in I) f(x)=x' \text{ and } x \in v(p)\}$

We shall show that the condition (V) of copying is fulfilled. Let $x \in W$ and $f \in I$ and suppose $x \in v(p)$. Then by the definition of v' we have $f(x) \in v'(p)$. For the converse implication suppose $f(x) \in v'(p)$. Then there exists $y \in W$ and $g \in I$ such that $f(x)=g(y)$ and $y \in v(p)$. By (I2) we get $x=y$, so $x \in v(p)$.■

Lemma 2.6. Let $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21})$ be an arrow frame. Then there exists a normal arrow frame $\underline{W}'=(W', R'_{11}, R'_{22}, R'_{12}, R'_{21})$ and a copying I from $\underline{W}$ to $\underline{W}'$ and if $\underline{W}$ is a finite a.f. the same is $\underline{W}'$.

Proof. Let $\underline{B}(W)=(B(W), 0, 1, +, .)$ be the Boolean ring over the set W, namely B(W) is the set of all subsets of W, $0=\emptyset$, $1=W$, $A+B=(A\setminus B)\cup(B\setminus A)$ and $A.B=AB=A\cap B$. Note that in Boolean rings $a-b=a+b$.

We put $W'=W \times B(W)$, $I=B(W)$ and for $f \in I$ and $x \in W$ we define $f(x)=(x,f)$. Obviously the conditions (I1) and (I2) from the definition of copying are fulfilled and each element of W' is in the form of $f(x)$ for some $f \in I$ and $x \in W$.

For the relations R'_{ij} we have the following definition:

$f(x)R'_{ij}g(y)$ iff $xR_{ij}y$ & $(f+i.\{x\}=g+j.\{y\})$. Here the indices i, $j \in \{1,2\}$ are considered as elements of B(W): 1 is the unit of B(W) and $2=1+1=1-1=0$.

To verify the condition $(R_{ij}1)$ suppose $xR_{ij}y$ and $f \in I$. Put $g=f+i.\{x\}-j.\{y\}$. Then $f+i.\{x\}=g+j.\{y\}$, which implies $f(x)R'_{ij}g(y)$. Condition $(R_{ij}2)$ follows directly from the definition of R'_{ij}. So I is a copying.

The proof that W' with the relations R'_{ij} is an arrow frame is straightforward. For the condition of normality suppose $f(x)R'_{11}g(y)$ and $f(x)R'_{22}g(y)$. Then we obtain $xR_{11}y$ & $(f+1.\{x\}=g+1.\{y\})$ and $xR_{22}y$ & $(f+2.\{x\}=g+2.\{y\})$. From here, since $2=0$, we get $f=g$ and $f+\{x\}=g+\{y\}$, which implies $\{x\}=\{y\}$, hence $x=y$ and $f(x)=g(x)$. Thus W' is a normal a.f.

Suppose now that $\underline{W}$ is a finite a.f. Then the Boolean ring over W is finite too and hence $\underline{W}'$ is a finite a.f.■

If Σ is a class of a.f. then the class of all finite a.f. from Σ is denoted by Σ_{fin}.

Theorem 2.7. (i) L((nor)ARROW)=L(ARROW).

(ii) $L(((nor)ARROW)_{fin})=L(ARROW_{fin})$.

Proof. (i) Since (nor)ARROW$\subseteq$ARROW we get L(ARROW)$\subseteq$ L((nor)ARROW). To prove that L((nor)ARROW)$\subseteq$L(ARROW) suppose $A\notin L(ARROW)$. Then there exists an a.s. $\underline{W}$, $x\in W$ and a valuation v such that $x \Vdash\!\!\!\!/_{v} A$. By lemma 2.6. there exists a normal a.s. $\underline{W}'$ and a copying I from $\underline{W}$ to W'. By lemma 2.5.(ii) there exists a valuation v' in W' such that I is a copying from the model $(\underline{W}, v)$ to the model $(\underline{W}', v')$. Then by the copying lemma we get for any $f\in I$ that $f(x) \Vdash\!\!\!\!/_{v'} A$. So A is not true in $\underline{W}'$and hence $A\notin L((nor)ARROW)$. So $L((nor)ARROW)\subseteq L(ARROW)$.

(ii) The proof is the same as the proof of (i), using the fact that lemma 2.6 guaranties that $\underline{W}'$ is a finite a.f. ∎

Corollary 2.8. The condition of normality of an a.f. is not modally definable.

Proof. Suppose that there exists a formula φ such that for any a.f. $\underline{W}$: φ is true in $\underline{W}$ iff $\underline{W}$ is normal. So $\varphi\in L((nor)ARROW)$. Let $\underline{W}_0$ be an a.f. which is not normal. Then φ is not true in $\underline{W}_0$, so $\varphi\notin L(ARROW)$, hence by theorem 2.7 $\varphi\notin L((nor)ARROW)$, which is a contradiction.∎

Another example of modally undefinable condition is totality. First we need the following standard result from modal logic.

Lemma 2.9. Let Σ be a nonempty class of a.f. closed under subframes and let Σ_{gen} be the class of generated frames of Σ. Then $L(\Sigma)=L(\Sigma_{gen})$.

Corollary 2.10. (i) $L((pretotal)ARROW)=L(((pretotal)ARROW)_{gen})=L((total)ARROW$.

(ii) $L((pretotal)ARROW)_{fin})=L((((pretotal)ARROW)_{gen})_{fin})=L(((total)ARROW)_{fin})$.

Proof. (i) The first equality follows from lemma 2.9 and the second – from lemma 1.3.

(ii) Use the fact that generated frame of a finite frame is a finite frame too. ∎

Corollary 2.11. The condition of totality of an a.f. is not modally definable.

Proof. Suppose that there exists a formula φ such that for any a.f. $\underline{W}$: φ is true in $\underline{W}$ iff $\underline{W}$ is total a.f. Then $\varphi\in L((total)ARROW)$ and by corollary 2.10 $\varphi\in L((pretotal)ARROW)$. Let $\underline{W}_0$ be a pretotal a.f. which is not total (such frames obviously exist). Then φ is not true in $\underline{W}_0$, so $\varphi\notin L((pretotal)ARROW)$ – a contradiction.∎

3. Axiomatization of some arrow logics

In this section we introduce a syntactical definition of arrow logic as sets of formulas containing some formulas as axioms and closed under some rules. The minimal set of axioms which we shall use, contains those from the minimal modal logic for each modality [ij] and the formulas, which modally define arrow frames. The formal system, obtained in this way is denoted by BAL and called Basic Arrow Logic.

Axioms and rules for BAL.

(Bool) All or enough Boolean tautologies,

(K[ij]) $[ij](A\Rightarrow B)\Rightarrow([ij]A\Rightarrow[ij]B)$,

(Pii) $[ii]A\Rightarrow A$,

(Σij) $A\vee[ij]\neg[ji]A$,

(Tijk) $[ik]A\Rightarrow[ij][jk]A$,

(MP) $\dfrac{A, A\Rightarrow B}{B}$, (N[ij]) $\dfrac{A}{[ij]A}$, i,j are anymembers of{1,2} and

A and B are arbitrary formulas.

We identify BAL with the set of its theorems.

By an arrow logic (a.l.) we mean any set L of formulas containing BAL and closed under the rules (MP), (N[ij]) and the rule of substitution of propositional variables. So BAL is the smallest arrow logic. We adopt the following notation. If X is a finite sequence of formulas , (taken as a new axiom) then by BAL+X we denote the smallest arrow logic containing all formulas from X. We shall us the following formulas as additional axioms:

(ser) $\langle 21\rangle 1$,

(ref) $([11][21]A\to A)\wedge([21][22]A\to A)$,

(sym) $[12][12]A\to A$,

(tr) $[11][22]A\to[21]A$.

Let $X\subseteq\{ser, ref, sym, tr\}$ and let for instance $X=\{ser, tr\}$. Then BAL+X= BAL+ser+tr. We will use also the notation (X)ARROW and for that concrete X (X)ARROW= (ser)(tr)ARROW.

Let L be an a.l. and Σ be a class of arrow frames. We say that L is sound in Σ if $L\subseteq L(\Sigma)$, L is complete in Σ if $L(\Sigma)\subseteq L$, and that L is characterized by Σ, or that $L(\Sigma)$ is axiomatized by L, if L is sound and complete in Σ, i.e. if $L=L(\Sigma)$.

In the completeness proofs we shall use the standard method of canonical models. We shall give a brief description of the method. For more details and some definitions we refer Segerberg [SEG 71] or Hughes & Cresswell [H&C 84].

Let L be an a.l. The frame $\underline{W}_L=(W_L, R^L_{11}, R^L_{22}, R^L_{12}, R^L_{21})$ will be called canonical frame for the logic L if W_L is the set of all maximal consistent sets in L and the relations R^L_{ij} are defined in W_L as follows: $xR^L_{ij}y$ iff $\{A\in FOR/[ij]A\in x\}\subseteq y$. For $p\in VAR$ the function $v_L(p)=\{x\in W_L/p\in x\}$ is called canonical valuation and the pair $M_L=(\underline{W}_L, v_L)$ is called the canonical model for L. The following is a standard result from modal logic.

Lemma 3.1. (i) Truth lemma for the canonical model for L. The following is true for any formula A and $x\in W_L$: $x\Vdash_{v_L} A$ iff $A\in x$.

(ii) If $A\notin L$ then there exists $x\in W_L$ such that $A\notin x$.

Lemma 3.2. Let L be an a.l. Then the canonical frame $\underline{W}_L$ of L is an a.f.

Proof. It is well known fact from the standard modal logic that the axiom (Pii) yields the condition (ρii) for the canonical frame. In the same way the axioms (Σij) and (Tijk) yield the conditions (σij) and (τijk) for $\underline{W}_L$. Thus $\underline{W}_L$ is an a.f.■

Theorem 3.3. BAL is sound and complete in the class of all arrow frames.

Proof. Soundness follows by lemma 2.1 and the completeness can be proved by the method of canonical models. Let L=BAL. By lemma 3.2 the canonical frame for L is an a.f. To show that $L(ARROW)\subseteq L$ suppose that $A\notin L$. Then by lemma 3.1.(ii) there exists $x\in W_L$ such that $A\notin x$. Then by the truth lemma we have $x\not\Vdash_{v_L} A$, so A is not true in the a.f. $\underline{W}_L$. Then $A\notin L(ARROW)$, which proves the theorem.■

Corollary 3.4. BAL=L(ARROW)=L((nor)ARROW).

Proof - from theorem 3.3 and theorem 2.7. ■

Lemma 3.5. Let L be an a.l. Then the following conditions are true:

(i) (ser)$\in$L iff $\underline{W}_L$ is a serial a.f.,

(ii) (ref)$\in$L iff $\underline{W}_L$ is a reflexive a.f.,

(iii) (sym)$\in$L iff $\underline{W}_L$ is a symmetric a.f.,

(iv) (tr)$\in$L iff $\underline{W}_L$ is a transitive a.f.

Proof. As an example we shall show (iv)($\longrightarrow$). Suppose (tr)= $[11][22]A\rightarrow[21]A\in L$. and proceed to show the condition of transitivity of $\underline{W}_L$: $(\forall xy\in W_L)(\exists z\in W_L)(xR^L_{21}y\longrightarrow xR^L_{11}z\ \&\ zR^L_{22}y)$.

Let $M_1=\{A/[11]A\in x\}$, $M_2=\{A/(\exists B\in y)(\neg A\rightarrow[22]\neg B\in L)\}$ and $M=M_1\cup M_2$. Then the following assertion is true:

Assertion. (i) If $A_1,\ldots,A_n\in M_i$ then $A_1\wedge\ldots\wedge A_n\in M_i$, $i=1,2$,

(ii) If $A\in M_i$ and $A\rightarrow B\in L$ then $B\in M_i$, $i=1,2$,

(iii) $M_1\cup M_2$ is L-inconsistent set iff $\exists A\in FOR$: $A\in M_1$ and $\neg A\in M_2$,

(iv) If $xR^L_{21}y$ then M is L-consistent set of formulas.

(v) Let z be a maximal consistent set. Then $M_2\subseteq z$ implies $zR^L_{22}y$, and $M_1\subseteq z$ implies $xR^L_{11}z$.

(vi) If xRy then $(\exists z\in W_L)(xR^L_{11}z\ \&\ zR^L_{22}y)$.

Proof. The proof of (i) and (ii) is straightforward and (iii) follows from (i) and (ii).

Let us proof (iv). Suppose $xR^L_{21}y$ and that M is not L-consistent. Then by (iii) there exists a formula A such that $A\in M_1$ and $\neg A\in M_2$. Then $[11]A\in x$ and $\exists B\in y$ such that $\neg\neg A\rightarrow[22]\neg B\in L$, hence $A\rightarrow[22]\neg B\in L$. Then by the rule (N[11]) we get $[11](A\rightarrow[22]\neg B)\in L$ and by axiom (K[11]) and (MP) we obtain that $[11]A\rightarrow[11][22]\neg B\in L$. But $[11]A\in x$, so $[11][22]\neg B\in x$, Then by the axiom (tr): $[11][22]\neg B\rightarrow[21]\neg B$ and (MP) we get $[21]\neg B\in x$ and since $xR^L_{21}y$ we get $\neg B\in y$. Since $B\in x$ we obtain a contradiction.

(v) Suppose $M_2\subseteq z\in W_L$. Suppose that not $zR^L_{22}y$. Then for some formula A we have: $[22]B\in z$ and $B\notin y$, so $\neg B\in y$. Since $\neg\neg[22]B\rightarrow[22]\neg\neg B\in L$, then by the definition of M_2 we get that $\neg[22]B\in M_2$, hence $\neg[22]B\in z$ – a contradiction. The second part of (v) follows by the definition of R^L_{11}.

(vi) Suppose $xR^L_{21}y$. Then by (iv) M is an L-consistent set. Then there exists a maximal consistent set z such that $M\subseteq z$ and by (v) we have $xR^L_{11}z$ and $zR^L_{22}y$. Now the proof of (iv)($\longrightarrow$) follows directly from assertion (vi). ■

Theorem 3.6. Let X$\subseteq$\{ser, ref, sym, tr\}. Then BAL+X=L((X)ARROW).

Proof. The consistency part of the theorem follows from lemma 2.3 and the completeness part can be obtained from lemma 3.5. as in the proof of theorem 3.3. ■

Corollary 3.7. (i) BAL+ref+sym+tr=L((pretotal)ARROW),

(ii) BAL+ref+sym+tr=L((total)ARROW).

Proof. (i) is a direct consequence of theorem 3.6 and (ii) follows from corollary 2.10. ■

4. Filtration and finite model property

In this section, applying the techniques of filtration coming from classical modal logic, we shall show that BAL and some of its

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extensions posses finite model property and are decidable. We adopt the Segerberg's definition of filtration, adapted for the language $\mathcal{L}$ of arrow logics (see [SEG 71]).

Let $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21})$ be an a.f. and $M=(\underline{W}, v)$ be a model over $\underline{W}$. Let Ψ be a finite set of formulas, closed under subformulas. For $x,y\in W$ define:

$x\sim y$ iff 9$(\forall A\in\Psi)(x\Vdash_v A$ iff $y\Vdash_v A)$, $|x|=\{y\in W/x\sim y\}$, $W'=\{|x|/x\in W\}$, for $p\in VAR$ $v'(p)=\{|x|/x\in v(p)\}$.

Let R'_{ij} $i,j=1,2$ be any binary relations in W' such that $\underline{W}'=(W', R'_{11}, R'_{22}, R'_{12}, R'_{21})$ be an a.f. We say that the model $M'=(W', v')$ is a filtration of the model M trough Ψ if following conditions are satisfied for any $i,j=1,2$ and $x,y\in W$:

($FR_{ij}1$) If $xR_{ij}y$ then $|x|R'_{ij}|y|$,

($FR_{ij}2$) If $|x|R|y|$ then $(\forall[ij]A\in\Psi)(x\Vdash_v [ij]A \rightarrow y\Vdash_v A)$.

The following lemma is a standard result in filtration theory.

Lemma 4.1. ([Seg 71])(i)/ Filtration lemma/ For any formula $A\in\Psi$ and $x\in W$ the following is true: $x\Vdash_v A$ iff $|x|\Vdash_{v'} A$.

(ii) $CardW'\leq 2^n$, where $n=Card\Psi$.

Let L be an a.l. We say that L admits a filtration if for any frame $\underline{W}$ for L and a model $M=(\underline{W},v)$ over $\underline{W}$ and for any formula A there exist a finite set of formulas Ψ containing A and closed under subformulas and a filtration $M'=(\underline{W}',v')$ of M trough Ψ, such that $\underline{W}'$ is a frame for L.

Corollary 4.2. (i) Let Σ be a class of arrow frames, let Σ_{fin} be the class of all finite arrow frames from Σ and let $L(\Sigma)$ admits a filtration. Then $L(\Sigma)=L(\Sigma_{fin})$.

(ii) If $L(\Sigma)$ is finitely axiomatizable then it is decidable.

Lemma 4.3. Let $\underline{W}$ be an a.f., $M=(\underline{W}, v)$ be a model over $\underline{W}$ and $M'=(\underline{W}',v,)$ be a filtration of M trough Ψ. Then:

(i) If $\underline{W}$ is a serial a.f. then $\underline{W}'$is a serial a.f.,

(ii) If $\underline{W}$ is a reflexive a.f. then $\underline{W}'$ is a reflexive a.f.,

(iii) If $\underline{W}$ is a symmetric a.f. the $\underline{W}'$ is a symmetric a.f.,

(iv) If $\underline{W}$ is a total a.f. then $\underline{W}'$is a total a.f.

Proof. As an example we shall prove (iii). We have to show that $(\forall|x|\in W')(\exists|y|\in W')(|x|R'_{12}|y|$ & $|y|R'_{12}|x|)$. Suppose $|x|\in W'$. Then there exists $y\in W$ such that $xR_{12}y$ & $yR_{12}x$. Then by the condition ($FR_{12}1$) of the filtration we obtain $|x|R'_{12}|y|$ & $|y|R'_{12}|x|$. ∎

Theorem 4.4. The logic L(ARROW) admits a filtration.

Proof. Let A be a formula and let Ψ be the smallest set of formulas containing A, closed under subformulas and satisfying the following condition

(*) If for some $i,j=1,2$ $[ij]A\in\Psi$ then for any $ij=1,2$ $[ij]A\in\Psi$.

It is easy to see that Ψ is finite and if n is the number of subformulas of A then $Card\Psi\leq 2^{4n}$. Then define W' and v' as in the definition of filtration. We define the relations R'_{ij} in W' as follows:

$|x|R'_{ij}|y|$ iff $(\forall[ij]A\in\Psi)(\forall k\in\{1,2\})(x\Vdash_v [ik]A\longleftrightarrow y\Vdash_v [jk]A)$.

First we shall show that the frame $\underline{W}'$is an a.f. The conditions (ρii) and (σij) follow directly from the definition of R'_{ij}. For the condition (τijk) suppose $|x|R'_{ij}|y|$ and $|y|R'_{jk}|z|$. To prove $|x|R'_{ik}|z|$ suppose $[ik]A\in\Psi$, $l\in\{1,2\}$ and for the direction ($\rightarrow$) suppose $x \Vdash_v [il]A$ and proceed to show that $z \Vdash_v [kl]A$. From $[ik]A \in \Psi$ we get $[ij]A, [jl]A \in \Psi$. Then $|x|R'_{ij}|y|$, $[ij]A \in \Psi$ and $x \Vdash_v [il]A$ imply $y \Vdash_v [jl]A$. This and $[jl]A \in \Psi$ and $|y|R'_{jk}|z|$ imply $z \Vdash_v [kl]A$.

The converse direction ($\leftarrow$) can be proved in a similar way.

It remains to show that the conditions of filtration ($FR_{ij}1$) and ($FR_{ij}2$) are satisfied.

For the condition ($FR_{ij}1$) suppose $xR_{ij}y$, $[ij]A \in \Psi$, $k \in \{1,2\}$ and for the direction ($\rightarrow$) suppose $x \Vdash_v [ik]A$, $yR_{jk}z$ and proceed to show that $z \Vdash_v A$. From $xR_{ij}y$ and $yR_{jk}z$ we get $xR_{ik}z$ and since $x \Vdash_v [ik]A$ we get $z \Vdash_v A$. For the direction ($\leftarrow$) suppose $y \Vdash_v [jk]A$, $xR_{ik}z$ and proceed to show that $z \Vdash_v A$. From $xR_{ij}y$ we get $yR_{ji}x$ and by $xR_{ik}z$ we get $yR_{jk}z$. From here and $y \Vdash_v [jk]A$ we obtain $z \Vdash_v A$. This ends the proof of ($FR_{ij}1$).

For the condition ($FR_{ij}2$) suppose $|x|R'_{ij}|y|$, $[ij]A \in \Psi$ and $x \Vdash_v [ij]A$. From here we obtain $y \Vdash_v [jj]A$ and since $yR_{jj}y$ we get $y \Vdash_v A$. This completes the proof of the theorem.

Corollary 4.5.

(i) $BAL = L(ARROW) = L(ARROW_{fin}) = L(((nor)ARROW)_{fin})$.

(ii) BAL is a decidable logic.

Proof. (i) The first two equalities follow from corollary 3.4 and theorem 4.4. The last equality follows from theorem 2.7.

(ii) is a consequence of corollary 4.2 and corollary 3.4. ■

Theorem 4.6. Let $X \subseteq \{ser, ref, sym\}$. Then the logic $L((X)ARROW)$ admits a filtration.

Proof. Use the same filtration as in theorem 4.4 and apply lemma 4.3.■

Corollary 4.7. Let $X \subseteq \{ser, ref, sym\}$. Then:

(i) $B+X = L((X)ARROW) = L(((X)ARROW)_{fin})$

(ii) B+X is a decidable logic.

Theorem 4.8. The logic $L((total)ARROW)$ admits a filtration.

Proof. Use the same filtration as in theorem 4.4 and apply lemma 4.3.■

Corollary 4.9.

(i) $B+ref+sym+tr = L((pretotal)ARROW) = L((total)ARROW = L(((total)ARROW)_{fin}) = L(((pretotal)ARROW)_{fin})$

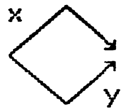
(ii) B+ref+sym+tr is a decidable logic.

5. An extension of BAL with a modality for equivalent arrows

We saw that the condition of a normality is not modally definable. This means that the language $\mathcal{L}$ is not strong enough to tell us the difference between normal and non normal a.f. In this section we shall show that there exists a natural extension of the language $\mathcal{L}$ in which normality become modally definable.

Let $\underline{W}$ be an a.f. and for $x,y \in W$ define

($\equiv$) $x \equiv y$ iff $xR_{11}y$ & $xR_{22}y$.

Graphically $x \equiv y$: 

The relation $\equiv$ is called an equivalence of two arrows.

By means of $\equiv$ the normality condition is equivalent to the following one:

(Nor') $(\forall x y \in W)(x \equiv y \rightarrow x = y)$.

If we extend our language $\mathcal{L}$ with a new modality $[\equiv]$, interpreted in a.f. with the relation $\equiv$, then (Nor') is modally definable by the formula $p \rightarrow [\equiv]p$.

Let the extension of $\mathcal{L}$ with $[\equiv]$ be denoted with $\mathcal{L}([\equiv])$. The general semantics of $\mathcal{L}([\equiv])$ is defined in the class of all relational structures /called also frames/ of the form $\underline{W} = (W, R_{11}, R_{22}, R_{12}, R_{21}, \equiv)$. The standard semantics of $\mathcal{L}([\equiv])$ is defined in the class of arrow frames with the relation $\equiv$ defined by $(\equiv)$.

We shall show that the condition $(\equiv)$ is not modally definable. For that purpose we introduce the following nonstandard semantics of $\mathcal{L}([\equiv])$.

By a nonstandard $\equiv$-arrow frame ($\equiv$-a.f.) we mean any system $\underline{W} = (W, R_{11}, R_{22}, R_{12}, R_{21}, \equiv)$ satisfying the following conditions for any $x,y,z \in W$ and $i,j,k \in \{1,2\}$:

(ρii), (σij) and (τijk),

$(\equiv\rho)$ $\quad x \equiv x$,

$(\equiv\sigma)$ $\quad x \equiv y \rightarrow y \equiv x$,

$(\equiv\tau)$ $\quad x \equiv y \ \& \ y \equiv z \rightarrow x \equiv z$,

$(\equiv \subseteq R_{ii})$ $\quad x \equiv y \rightarrow x R_{ii} y$

The class of all nonstandard $\equiv$-arrow frames is denoted by Nonstandard-$\equiv$-ARROW.

It can be easily seen that if a nonstandard $\equiv$-a.f. satisfies the following conditions

If a nonstandard $\equiv$-a.f. satisfies the condition

$(R_{11} \cap R_{22} \subseteq \equiv)$ $\quad x R_{11} y \ \& \ x R_{22} y \rightarrow x \equiv y$,

then it is called a standard $\equiv$-a.f. It is easily seen that in any standard $\equiv$-a.f. we have

$x \equiv y \longleftrightarrow x R_{11} y \ \& \ x R_{22} y$.

The class of all nonstandard and standard $\equiv$-arrow frames are denoted respectively by Nonstandard-$\equiv$-ARROW and Standard-$\equiv$-ARROW.

All conditions from the definition of nonstandard $\equiv$-a.f. are modally definable by the following formulas respectively:

$(\equiv P)$ $\quad [\equiv]A \rightarrow A$,

$(\equiv \Sigma)$ $\quad A \vee [\equiv]\neg[\equiv]A$,

$(\equiv T)$ $\quad [\equiv]A \rightarrow [\equiv][\equiv]A$,

$(\subseteq_{ii})$ $\quad [ii]A \rightarrow [\equiv]A$, $i = 1,2$.

We shall show that the condition $(R_{ii} \subseteq \equiv)$ is not modally definable. For that purpose we shall proof first that L(Standard-$\equiv$-ARROW)=L(Nonstandard-$\equiv$-ARROW).

Lemma 5.1. Let $\underline{W} = (W, R_{11}, R_{22}, R_{12}, R_{21}, \equiv)$ be a nonstandard $\equiv$-a.f. Then there exists a standard $\equiv$-a.f. $\underline{W}' = (W', R'_{11}, R'_{22}, R'_{12}, R'_{21}, \equiv')$ and a copying from $\underline{W}$ to $\underline{W}'$ and if $\underline{W}$ is a finite a.f. then $\underline{W}'$ is a finite a.f. too.

Proof. Use the same construction as in the lemma 2.6 with the following modification. Let $\equiv(x) = \{y \in W / x \equiv y\}$. Since $\equiv$ is an equivalence relation then $\equiv(x) = \equiv(y)$ implies $x \equiv y$. The definitions of R'_{ij}

and $\equiv'$ are the following:

$f(x)R'_{ij}g(y)$ iff $xR_{ij}y$ & $(f+i.\equiv(x) = g+j.\equiv(y))$

$f(x)\equiv' g(y)$ iff $x\equiv y$ & $f=g$.

The details that this will do are left to the reader. ∎

Corollary 5.2. L(Standard-$\equiv$-ARROW)=L(Nonstandard-$\equiv$-ARROW).

Now the axiomatization of L(Nonstandard-$\equiv$-ARROW) is easy.

Denote by $[\equiv]$BAL the following axiomatic system:

Axioms and rules for $[\equiv]$BAL

(I) All axioms and rules of BAL,

(II) The following new axioms:

($\equiv$P) $[\equiv]A\rightarrow A$,

($\equiv\Sigma$) $A\vee[\equiv]\neg[\equiv]A$,

($\equiv$T) $[\equiv]A\rightarrow[\equiv][\equiv]A$,

($\leq$ii) $[ii]A\rightarrow[\equiv]A$, i=1,2.

Theorem 5.3. $[\equiv]$BAL is sound and complete in the class Nonstandard-$\equiv$-ARROW.

Proof - by the canonical construction. ∎

Corollary 5.4. $[\equiv]$BAL=L(Nonstandard-$\equiv$-ARROW)= L(Standard-$[\equiv]$-ARROW).

Theorem 5.5. (i) $[\equiv]$BAL= L(Nonstandard-$\equiv$-ARROW)= L((Nonstandard-$\equiv$-ARROW)$_{fin}$)

(ii) $[\equiv]$BAL is a decidable logic.

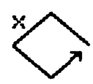
Proof. Apply the filtration technic with the following modification: the definition of the relations R'_{ij} is the same as in theorem 4.4., the definition of $\equiv'$ is the following

$$|x|\equiv'|y| \text{ iff } (\forall[\equiv]A\in\Psi)(x\Vdash_v[\equiv]A \longleftrightarrow y\Vdash_v[\equiv]A) \ \& \ |x|R'_{11}|y| \ \& \ |x|R'_{22}|y|.$$

The details that this definition of filtration will do is left to the reader. ∎

6. Extensions of BAL with propositional constant Loop

We say that an arrow x forms a loop if $xR_{12}x$. Graphically



Let $\underline{W}$ be an a.f. We let $Loop_{\underline{W}}=\{x\in W/xR_{12}x\}$.

Lemma 6.1. In the language $\mathcal{L}$ Loop is not expressible in a sense that there is no a formula A in $\mathcal{L}$ such that for any a.f. $\underline{W}$, valuation v and $x\in W$: $x\Vdash_v A$ iff $x\in Loop_{\underline{W}}$.

Proof. Let W={a, b, c}, $R_{11}=R_{22}=\{(a,a), (b,b), (c,c)\}$, $R_{12}=R_{21}=\{(a,a), (b,c), (c,b)\}$. It is east to see that W with the relations R_{ij} is an a.f. Let v be a valuation in W such that for any p$\in$VAR v(p)=$\emptyset$. Then by induction on the complexity of a formula one can see that for any formula A the set $v(A)=\{x\in W/x\Vdash_v A\}$ is ether W or $\emptyset$. Suppose now that there exists a formula A such that $x\Vdash_v A$ iff $x\in$Loop. Then v(A)={a} which contradicts the previous result. ∎

Let $\mathcal{L}$(Loop) be an extension of the language $\mathcal{L}$ by a new propositional constant Loop with the following standard semantics: for any a.f. $\underline{W}$, valuation v and $x\in W$: $x\Vdash_v$ loop iff $x\in Loop_{\underline{W}}$. Loop has also a nonstandard semantics which can be defined in the following way. By a nonstandard Loop arrow frame we mean any system $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21}, \delta)$ such that $(W, R_{11}, R_{22}, R_{12}, R_{21})$ is an a.f. and δ /sometimes denoted by $\delta_{\underline{W}}$/ is a subset of W. Then the interpretation of Loop in a nonstandard Loop a.f. $\underline{W}$ is: $x \Vdash_{V}$ Loop iff $x\in\delta_{\underline{W}}$. A nonstandard Loop a.f. $\underline{W}$ is called a standard one if the following two conditions are satisfied stating together that $\mathrm{Loop}_{\underline{W}}=\delta_{\underline{W}}$:

(Loop 1) $(\forall x\in W)(x\in\delta_{\underline{W}} \rightarrow x\in\mathrm{Loop}_{\underline{W}})$,

(Loop 2) $(\forall x\in W)(x\in\mathrm{Loop}_{\underline{W}} \rightarrow x\in\delta_{\underline{W}})$.

The class of all nonstandard Loop arrow frames is denoted by NonstandardLoopARROW. Accordingly the class of all standard Loop a.f. is denoted by StandardLoopARROW. It can be easily shown that the condition (Loop 1) is modally definable in NonstandardLoopARROW by the following formula

(Loop) $\mathrm{Loop}\rightarrow([12]A\rightarrow A)$.

If a nonstandard Loop a.f. satisfies (Loop 1) we call it a general Loop a.f. The class of general Loop arrow frames is denoted by GeneralLoopARROW. We shall show that condition (Loop 2) is not modally definable in GeneralLoopARROW. For that purpose we shall use the copying construction, which for frames with δ contains an additional condition

(δ) For any $x\in W$ and $f\in I$: $x\in\delta$ iff $f(x)\in\delta'$.

The copying lemma for this version of copying is also true.

Lemma 6.2. Let $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21}, \delta)$ be a general Loop a.f. Then there exists a standard Loop a.f. $\underline{W}'=(W', R'_{11}, R'_{22}, R'_{12}, R'_{21}, \delta')$ and a copying I from $\underline{W}$ to $\underline{W}'$and if $\underline{W}$ is a finite then $\underline{W}'$ is a finite frame too.

Proof. The construction of I and W' is the same as in the proof of lemma 2.6. To define R'_{ij} we first define the function

$$\delta(x)=\begin{cases}0 & \text{if } x\in\delta\\ 1 & \text{if } x\notin\delta\end{cases}$$

where 0 and 1 are considered as zero and unit of the Boolean ring. Then:

$f(x)R'_{ij}g(y)$ iff $xR_{ij}y$ & $(f+i.\delta(x)=g+j.\delta(y))$.

$\delta'=\{x'/\exists x\in\delta\ \exists f\in I\ f(x)=x'\}$

The proof that this is a copying and that $\underline{W}$ is an a.f. is the same as in lemma 2.6. Let us show that $\underline{W}'$ is a standard Loop a.f.

For the condition (Loop 1) suppose $x'\in\delta'$. Then $x'=f(x)$ for some $x\in\delta$ and $f\in I$. So we have $xR_{12}x$, $\delta(x)=0$ and hence $f+1.\delta(x)=f+2.\delta(x)$. This shows that $f(x)R'_{12}f(x)$, hence $x'R'_{12}x'$.

For the condition (Loop 2) suppose $x'R'_{12}x'$. Then for some $x\in W$ and $f\in I$ we have $f(x)=x'$ and $f(x)R'_{12}f(x)$. Then $xR_{12}x$ and $f+\delta(x)=f$, so $\delta(x)=0$, which yields that $x\in\delta$. Thus $x'\in\delta'$. ■

Lemma 6.2 implies the following

Theorem 6.3. L(LoopARROW)=L(GeneralLoopARROW).

Corollary 6.4. Condition (Loop 2) is not modally definable.

Now the axiomatization of L(LoopARROW) is easy: we axiomatize L(GeneralLoopARROW) adding to the axioms of BAL the axiom

(Loop) $\mathrm{Loop}\rightarrow([12]A\rightarrow A)$.

The obtained system is denoted by LoopBAL. Using the canonical construction one can prove the following

Theorem 6.5. LoopBAL is sound and complete in GeneralLoopARROW.

Corollary 6.6. LoopBAL=L(GeneralLoopARROW)=L(LoopARROW).

The constant Loop makes possible to distinguish the logics L(LoopARROW) and L((nor)LoopARROW). Namely we have

Lemma 6.7. Let $\varphi=A\wedge Loop\to[12](Loop\to A)$. Then:

(i) $\varphi\notin$L(LoopARROW),

(ii) $\varphi\in$L((nor)LoopARROW),

(iii) L(LoopARROW)$\neq$L((nor)LoopARROW).

Proof - straightforward by the completeness theorem.■

The formula φ from lemma 6.7 modally defines in GeneralLoopARROW the following condition

(nor_0) $(\forall xy)(xR_{11}y$ & $x\in\delta$ & $y\in\delta \longrightarrow x=y)$

Let $\underline{W}$ be a general Loop a.f. We call $\underline{W}$ quasi-normal if it satisfies the condition Nor_0.

Lemma 6.8. Let $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21}, \delta)$ be a quasi-normal general Loop a.f. Then there exists a normal Loop a.f. $\underline{W}'=(W, R'_{11}, R'_{22}, R'_{12}, R'_{21}, \delta')$ and a copying I from $\underline{W}$ to $\underline{W}'$ and if $\underline{W}$ is finite then $\underline{W}'$ is finite too.

Proof. The construction of I, W' and R'_{ij} is the same as in lemma 6.2 with the following modification of the function $\delta(x)$:

$$\delta(x)=\begin{cases} 0 & \text{if } x\in\delta \\ \{x\} & \text{if } x\notin\delta \end{cases}$$

The proof that $\underline{W}'$ is a standard Loop a.f. is the same as in lemma 6.2. Let us show the condition of normality. For, suppose $f(x)R'_{11}g(y)$ and $f(x)R'_{22}g(y)$. Then we have $xR_{11}y$ & $(f+\delta(x)=g+\delta(y))$ and $xR_{22}y$ & $(f=g)$. From here we get $\delta(x)=\delta(y)$.

Case 1: $\delta(x)=\emptyset$. Then $\delta(y)=\emptyset$ and hence $x,y\in\delta_{\underline{W}}$. By $xR_{11}y$ and $x,y\in\delta_{\underline{W}}$ we get by (nor_0) $x=y$ and by $f=g$ we obtain $f(x)=f(y)$.

Case 2: $\delta(x)\neq\emptyset$. Then $\delta(y)\neq\emptyset$ and hence $\{x\}=\{y\}$, so $x=y$ and consequently $f(x)=g(y)$. This proves the condition of normality. ■

From lemma 6.8 we obtain the following

Theorem 6.9. L((nor_0)GeneralLoopARROW)=L((nor)LoopARROW).

Let NorLoopBAL=LoopBAL+$A\wedge Loop\to[12](Loop\to A)$. Using the canonical method we can easily prove the following

Theorem 6.10. NorLoopBAL is sound and complete in the class (nor_0)GeneralLoopARROW.

Corollary 6.11. NorLoopBAL=L((nor)LoopARROW).

Lemma 6.12. The logics L(GeneralLoopARROW) and L((nor_0)GeneralLoopARROW) admit a filtration and are decidable.

Proof. For the L(generalLoopARROW) use the same filtration as for the logic L(ARROW) with the following definition for δ:

$\delta'=\{|x|/x\in\delta\}$.

We have to show that the filtered frame satisfies the condition (Loop 1). Suppose $|x|\in\delta'$. Then $x\in\delta$ and by (Loop 1) we get $xR_{12}x$. Then by the properties of filtration we get $|x|R'_{12}|y|$.

For the logic L((nor_0)GeneralLoopARROW) we modify the definition of R'_{ij} as follows:

$|x|R'_{ij}|y|$ iff $(\forall[ij]A\in\Psi)(\forall k\in\{1,2\})(x\Vdash_{\overline{V}}[ik]A \longleftrightarrow x\Vdash_{\overline{V}}[jk]A)$ &
$(x,y\in\delta' \longrightarrow |x|=|y|)$.

The proof that this definition works is left to the reader. ∎

Corollary 6.13. The logics LoopBAL and NorLoopBAL possess finite model property and are decidable.

The language $\mathscr{L}([\equiv],\mathrm{Loop})$ is an extension of the language $\mathscr{L}([\equiv])$ with the constant Loop. The standard semantics of this language is a combination of the standard semantics of $\mathscr{L}([\equiv])$ and $\mathscr{L}(\mathrm{Loop})$. This semantics is also modally undefinable. To axiomatize it we introduce a general semantics for $\mathscr{L}([\equiv],\mathrm{Loop})$ as follows.

A frame $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21}, \equiv, \delta)$ is called a general Loop-≡ arrow frame if $(W, R_{11}, R_{22}, R_{12}, R_{21})$ is an a.f. and ≡ and δ satisfy the conditions on the left side in the next table:

$(\rho\equiv)$	$x\equiv x$,	$(\equiv P)$	$[\equiv]A\rightarrow A$,
$(\sigma\equiv)$	$x\equiv y\longrightarrow y\equiv x$,	$(\equiv\Sigma)$	$A\vee[\equiv]\neg[\equiv]A$,
$(\tau\equiv)$	$x\equiv y\,\&\,y\equiv z\longrightarrow x\equiv z$,	$(\equiv T)$	$[\equiv]A\rightarrow[\equiv][\equiv]A$,
$(\equiv\subseteq R_{ii})$	$x\equiv y\longrightarrow xR_{ii}y$, $i=1,2$,	$(\subseteq ii)$	$[ii]A\rightarrow[\equiv]A$, $i=1,2$,
(Loop 1)	$xR_{12}x\longrightarrow x\in\delta$,	(Loop)	$\mathrm{Loop}\rightarrow([\equiv]A\rightarrow A)$,
$(\equiv\delta)$	$x\equiv y\ \&\ x\in\delta\longrightarrow y\in\delta$,	(≡Loop)	$\mathrm{Loop}\rightarrow[\equiv]\mathrm{Loop}$,
$(\equiv R_{11}\delta)$	$xR_{11}y\ \&\ x\in\delta\ \&\ y\in\delta\longrightarrow x\equiv y$	(≡11Loop)	$\mathrm{Loop}\wedge[\equiv]A\rightarrow[11](\mathrm{Loop}\rightarrow A)$.

If in addition $\underline{W}$ satisfies the condition $(R_{11}\cap R_{22}\subseteq\equiv)$ and (Loop2) it is called standard Loop-≡ arrow frame. The classes of all general Loop-≡ arrow frames and standard Loop-≡-arrow frames are denoted by GeneralLoop-≡-ARROW and StandardLoop-≡-ARROW respectively.

All conditions from the left side of the above table are modally definable by the corresponding formulas from the right side.

We axiomatize the logic $\mathscr{L}([\equiv],\mathrm{Loop})$ by adding all these formulas as axioms to the logic BAL The obtained system is /denoted by [≡]LoopBAL.

Theorem 6.14. The logic [≡]LoopBAL is sound and complete in the class GeneralLoop-≡-ARROW.

Proof – by the canonical construction. ∎

Lemma 6.15. Let $\underline{W}=(W, R_{11}, R_{22}, R_{12}, R_{21}, \equiv, \delta)$ be a general Loop-≡ a.f. Then there exist a standard Loop-≡ a.f. $\underline{W}'=(W', R'_{11}, R'_{22}, R'_{12}, R'_{21}, \equiv', \delta')$ and a copying I from $\underline{W}$ to $\underline{W}'$.

Proof. The set W', I, δ' and R'_{ij} are defined as in lemma 6.8 with the following modification of the function $\delta(x)$:

$$\delta(x)=\begin{cases}0 & \text{if } x\in\delta\\ \equiv(x) & \text{if } x\notin\delta\end{cases}$$

The relation ≡' is defined as in lemma 5.1. The proof that this construction works is left to the reader. ∎

Corollary 6.16. [≡]LoopBAL=L(GeneralLoop-≡-ARROW)= L(StandardLoop-≡-ARROW).

7. Further perspectives

A. Extensions of BAL with additional connectives.

Sections 5 and 6 can be considered as examples of possible extensions of the language $\mathscr{L}$ with operators having their standard semantics in terms of arrow frames. There are many possibilities of such extensions, depending of what kind of relations between arrows we want to describe in a modal setting. The main scheme is the following: to each n+1-ary relation $R(x_0,x_1,\dots,x_n)$ to introduce an n-place modal box operation $[R](A_1,\dots,A_n)$ with the following semantics, coming from the representation theory of Boolean algebras with operators ([J&T 51], see also [VAK 91]):

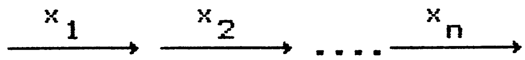
$x_0 \Vdash_v [R](A_1,\dots,A_n)$ iff

$(\forall x_1,\dots,x_n \in W)(R(x_0,x_1,\dots,x_n) \to x_1 \Vdash_v A_1$ or...or $x \Vdash_v A_n)$

The dual operator $\langle R\rangle(A_1,\dots,A_n)$ is defined by $\neg[R](\neg A_1,\dots,\neg A_n)$.

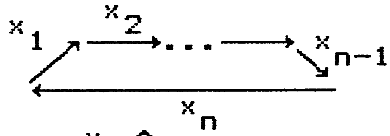
In the following we list some natural relations between arrows, which are candidates for a modal study:

$Path_n(x_1,\dots,x_n)$ iff $x_1R_{21}x_2$ & $x_2R_{21}x_3$ & &$x_{n-1}R_{21}x_n$, $n\geq 2$

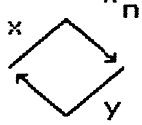


$Path_\infty(x_1,x_2,x_3,\dots.)$ iff $(\forall n)Path(x_1,\dots,x_n)$

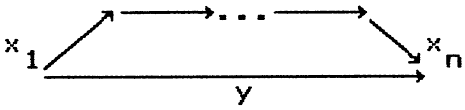
$Loop_n(x_1,x_2,\dots,x_n)$ iff $Path_{n+1}(x_1,\dots,x_n,x_1)$



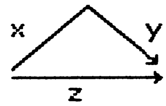
Converse: xSy iff $Loop_2(x,y)$



$Trapezium_n(x_1,\dots,x_n,y)$ iff $Path_n(x_1,\dots,x_n)$ & $x_1R_{11}y$ & $x_nR_{22}y$



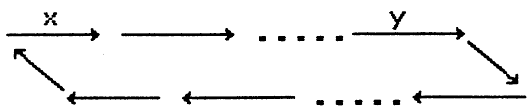
$Triangle(x,y,z)$ iff $Trapezium_2(x,y,z)$



Connectedness: $Con(x,y)$ iff

$\exists n\geq 2\exists x_1\dots x_n$: $x=x_1$ & $x_n=y$ & $Path_n(x_1,\dots,x_n)$

Double side connectedness: $Dcon(x,y)$ iff $Con(x,y)$ & $Con(y,x)$



The relations $Path_n$, $Path_\infty$, $Loop_n$ can be used to define also semantics for suitable propositional constants:

$Path_n$, $Path_\infty$, $Loop_n$, Loop as follows.

$x_1 \Vdash_v$ **$Path_n$** iff $(\exists x_2,\dots x_n \in W)Path_n(x_1,x_2,\dots,x_n)$,

$x_1 \Vdash_v$ **$Path_\infty$** iff $(\exists x_2,x_3,\dots)Path_\infty(x_1,x_2,x_3,\dots)$,

$x_1 \Vdash_v$ **$Loop_n$** iff $(\exists x_2,\dots,x_n)Loop_n(x_1,x_2,\dots,x_n)$

$x \Vdash_v$ **Loop** iff $\exists n$ $x \Vdash_v$ **$Loop_n$**.

These considerations motivate the following general problem: develop a modal theory /axiomatization, definability, (un)decidability/ of some extensions of BAL with modal operations corresponding to the above defined relations in arrow structures.

For example, the extension of BAL with the modal operations $A \bullet B = \langle Triangle\rangle(A,B)$, $A^{-1} = [Converse]A$ and the propositional constant Id=Loop is a natural generalization of the modal logic of binary relations ([BEN 89], [VEN 89], [VEN 91]). This logic has a closed connection with various versions of representable relativized relational algebras ([KRA 89], [MA 82], [NÉM 91]).

B. Arrow semantics of Lambek Calculus and its generalizations.
Let A/B and A\B are "duals" of $A \bullet B$ with the following semantics:

$x \Vdash_v B/A$ iff $(\forall yz \in W)(Triangle(x,y,z)\ \&\ y \Vdash_v A \rightarrow z \Vdash_v B)$,

$y \Vdash_v A\backslash B$ iff $(\forall xz \in W)(Triangle(x,y,z)\ \&\ x \Vdash_v A \rightarrow z \Vdash_v B)$

The modal connectives $A \bullet B$, A\B, and A/B can be considered as the operations in the Lambek Calculus. Mikulás [Mik 92] proves a completeness theorem for the Lambek Calculus with respect to a relational semantics of the above type over transitive normal arrow frames /this is an equivalent reformulation of Mikulás result in "arrow" terminology/. Roorda [Roor 91] and [Roor 91a] studies a modal version of Lambek Calculus extended with classical Boolean operations. So it is natural to study an extension of BAL with the above dyadic modal operations, which will give another intuition for the operations in the Lambek Calculus.

C. Arrow logics and point logics over arrow systems.
With each arrow structure S=(Ar, Po, 1, 2) we can associate thefollowing two relational systems: the arrow frame $(Ar, R_{11}, R_{22}, R_{12}, R_{21})$ and the point frame (Po, ρ). The first system is used as a semantic base of the logic BAL and the later can be used as a semantic base of an ordinary modal language with a modal operator $\Box$. So each class Σ of arrow systems determines a class of arrow frames $Ar(\Sigma)$ and a class $Po(\Sigma)$ of point frames. A general question, which arises is the problem of comparative study of the corresponding logics $L(Ar(\Sigma))$ and $L(Po(\Sigma))$. A kind of a correspondence between first order properties of $Po(\Sigma)$ and $Ar(\Sigma)$ was shown in section 1.

D. Many-dimensional generalizations of arrow systems.
The introduced in this paper arrow structures can be generalized to the so called n-dimensional arrow structures ([VAK 91a]) in the following way. Let S=(Ar, Po, 1,...n) be a two sorted algebraic system. S is called an n-dimensional arrow structure if for any i=1,...,n, i is a function from Ar to Po satisfying the axiom:

$(\forall A \in Po)(\exists i\ 1 \leq i \leq n)(\exists x \in Ar)(i(x)=A)$

The arrows in an n-dimensional a.s. looks like as follows:

$$\begin{array}{cccc} 1(x) & 2(x) & 3(x)\ldots\ldots\ldots & n(x) \\ \bullet\!\!-\!\!-\!\!-\!\! & \bullet\!\!-\!\!-\!\!-\!\! & \bullet\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\rightarrow & \bullet \end{array}$$

A natural example of n-dimensional a.s. is the set of all n-tuples of a given n-ary relation. Among the logics based on n-dimensional arrow frames are some generalizations of the so called cylindric modal Logics ([VEN 89], [VEN 91]). These logics have also a very closed connection with some versions of representable relativized cylindric algebras [NÉM 91].

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