Institute for Logic, Language and Computation

SHAVRUKOV'S THEOREM ON THE SUBALGEBRAS OF DIAGONALIZABLE ALGEBRAS FOR THEORIES CONTAINING $I\Delta_0$+EXP

Domenico Zambella

ILLC Prepublication Series
for Mathematical Logic and Foundations ML-92-05

University of Amsterdam

The ILLC Prepublication Series

1990

Logic, Semantics and Philosophy of Language

LP-90-01 Jaap van der Does A Generalized Quantifier Logic for Naked Infinitives
LP-90-02 Jeroen Groenendijk, Martin Stokhof Dynamic Montague Grammar
LP-90-03 Renate Bartsch Concept Formation and Concept Composition
LP-90-04 Aarne Ranta Intuitionistic Categorial Grammar
LP-90-05 Patrick Blackburn Nominal Tense Logic
LP-90-06 Gennaro Chierchia The Variablity of Impersonal Subjects
LP-90-07 Gennaro Chierchia Anaphora and Dynamic Logic
LP-90-08 Herman Hendriks Flexible Montague Grammar
LP-90-09 Paul Dekker The Scope of Negation in Discourse, towards a Flexible Dynamic Montague grammar
LP-90-10 Theo M.V. Janssen Models for Discourse Markers
LP-90-11 Johan van Benthem General Dynamics
LP-90-12 Serge Lapierre A Functional Partial Semantics for Intensional Logic
LP-90-13 Zhisheng Huang Logics for Belief Dependence
LP-90-14 Jeroen Groenendijk, Martin Stokhof Two Theories of Dynamic Semantics
LP-90-15 Maarten de Rijke The Modal Logic of Inequality
LP-90-16 Zhisheng Huang, Karen Kwast Awareness, Negation and Logical Omniscience
LP-90-17 Paul Dekker Existential Disclosure, Implicit Arguments in Dynamic Semantics

Mathematical Logic and Foundations

ML-90-01 Harold Schellinx Isomorphisms and Non-Isomorphisms of Graph Models
ML-90-02 Jaap van Oosten A Semantical Proof of De Jongh's Theorem
ML-90-03 Yde Venema Relational Games
ML-90-04 Maarten de Rijke Unary Interpretability Logic
ML-90-05 Domenico Zambella Sequences with Simple Initial Segments
ML-90-06 Jaap van Oosten Extension of Lifschitz' Realizability to Higher Order Arithmetic, and a Solution to a Problem of F. Richman
ML-90-07 Maarten de Rijke A Note on the Interpretability Logic of Finitely Axiomatized Theories
ML-90-08 Harold Schellinx Some Syntactical Observations on Linear Logic
ML-90-09 Dick de Jongh, Duccio Pianigiani Solution of a Problem of David Guaspari
ML-90-10 Michiel van Lambalgen Randomness in Set Theory
ML-90-11 Paul C. Gilmore The Consistency of an Extended NaDSet

Computation and Complexity Theory

CT-90-01 John Tromp, Peter van Emde Boas Associative Storage Modification Machines
CT-90-02 Sieger van Denneheuvel, Gerard R. Renardel de Lavalette A Normal Form for PCSJ Expressions
CT-90-03 Ricard Gavaldà, Leen Torenvliet, Osamu Watanabe, José L. Balcázar Generalized Kolmogorov Complexity in Relativized Separations
CT-90-04 Harry Buhrman, Edith Spaan, Leen Torenvliet Bounded Reductions
CT-90-05 Sieger van Denneheuvel, Karen Kwast Efficient Normalization of Database and Constraint Expressions
CT-90-06 Michiel Smid, Peter van Emde Boas Dynamic Data Structures on Multiple Storage Media, a Tutorial
CT-90-07 Kees Doets Greatest Fixed Points of Logic Programs
CT-90-08 Fred de Geus, Ernest Rotterdam, Sieger van Denneheuvel, Peter van Emde Boas Physiological Modelling using RL
CT-90-09 Roel de Vrijer Unique Normal Forms for Combinatory Logic with Parallel Conditional, a case study in conditional rewriting

Other Prepublications

X-90-01 A.S. Troelstra Remarks on Intuitionism and the Philosophy of Mathematics, Revised Version
X-90-02 Maarten de Rijke Some Chapters on Interpretability Logic
X-90-03 L.D. Beklemishev On the Complexity of Arithmetical Interpretations of Modal Formulae
X-90-04 Annual Report 1989
X-90-05 Valentin Shehtman Derived Sets in Euclidean Spaces and Modal Logic
X-90-06 Valentin Goranko, Solomon Passy Using the Universal Modality: Gains and Questions
X-90-07 V.Yu. Shavrukov The Lindenbaum Fixed Point Algebra is Undecidable
X-90-08 L.D. Beklemishev Provability Logics for Natural Turing Progressions of Arithmetical Theories
X-90-09 V.Yu. Shavrukov On Rosser's Provability Predicate
X-90-10 Sieger van Denneheuvel, Peter van Emde Boas An Overview of the Rule Language RL/1
X-90-11 Alessandra Carbone Provable Fixed points in $I\Delta_0+\Omega_1$, revised version
X-90-12 Maarten de Rijke Bi-Unary Interpretability Logic
X-90-13 K.N. Ignatiev Dzhaparidze's Polymodal Logic: Arithmetical Completeness, Fixed Point Property, Craig's Property
X-90-14 L.A. Chagrova Undecidable Problems in Correspondence Theory
X-90-15 A.S. Troelstra Lectures on Linear Logic

1991

Logic, Semantics and Philosophy of Langauge

LP-91-01 Wiebe van der Hoek, Maarten de Rijke Generalized Quantifiers and Modal Logic
LP-91-02 Frank Veltman Defaults in Update Semantics
LP-91-03 Willem Groeneveld Dynamic Semantics and Circular Propositions
LP-91-04 Makoto Kanazawa The Lambek Calculus enriched with Additional Connectives
LP-91-05 Zhisheng Huang, Peter van Emde Boas The Schoenmakers Paradox: Its Solution in a Belief Dependence Framework
LP-91-06 Zhisheng Huang, Peter van Emde Boas Belief Dependence, Revision and Persistence
LP-91-07 Henk Verkuyl, Jaap van der Does The Semantics of Plural Noun Phrases
LP-91-08 Víctor Sánchez Valencia Categorial Grammar and Natural Reasoning
LP-91-09 Arthur Nieuwendijk Semantics and Comparative Logic
LP-91-10 Johan van Benthem Logic and the Flow of Information

Mathematical Logic and Foundations

ML-91-01 Yde Venema Cylindric Modal Logic
ML-91-02 Alessandro Berarducci, Rineke Verbrugge On the Metamathematics of Weak Theories
ML-91-03 Domenico Zambella On the Proofs of Arithmetical Completeness for Interpretability Logic
ML-91-04 Raymond Hoofman, Harold Schellinx Collapsing Graph Models by Preorders

The ITLI Prepublication Series

1986

86-01 The Institute of Language, Logic and Information
86-02 Peter van Emde Boas — A Semantical Model for Integration and Modularization of Rules
86-03 Johan van Benthem — Categorial Grammar and Lambda Calculus
86-04 Reinhard Muskens — A Relational Formulation of the Theory of Types
86-05 Kenneth A. Bowen, Dick de Jongh — Some Complete Logics for Branched Time, Part I Well-founded Time, Forward looking Operators
86-06 Johan van Benthem — Logical Syntax

1987

87-01 Jeroen Groenendijk, Martin Stokhof — Type shifting Rules and the Semantics of Interrogatives
87-02 Renate Bartsch — Frame Representations and Discourse Representations
87-03 Jan Willem Klop, Roel de Vrijer — Unique Normal Forms for Lambda Calculus with Surjective Pairing
87-04 Johan van Benthem — Polyadic quantifiers
87-05 Víctor Sánchez Valencia — Traditional Logicians and de Morgan's Example
87-06 Eleonore Oversteegen — Temporal Adverbials in the Two Track Theory of Time
87-07 Johan van Benthem — Categorial Grammar and Type Theory
87-08 Renate Bartsch — The Construction of Properties under Perspectives
87-09 Herman Hendriks — Type Change in Semantics: The Scope of Quantification and Coordination

1988

Logic, Semantics and Philosophy of Language:

LP-88-01 Michiel van Lambalgen — Algorithmic Information Theory
LP-88-02 Yde Venema — Expressiveness and Completeness of an Interval Tense Logic
LP-88-03 — Year Report 1987
LP-88-04 Reinhard Muskens — Going partial in Montague Grammar
LP-88-05 Johan van Benthem — Logical Constants across Varying Types
LP-88-06 Johan van Benthem — Semantic Parallels in Natural Language and Computation
LP-88-07 Renate Bartsch — Tenses, Aspects, and their Scopes in Discourse
LP-88-08 Jeroen Groenendijk, Martin Stokhof — Context and Information in Dynamic Semantics
LP-88-09 Theo M.V. Janssen — A mathematical model for the CAT framework of Eurotra
LP-88-10 Anneke Kleppe — A Blissymbolics Translation Program

Mathematical Logic and Foundations:

ML-88-01 Jaap van Oosten — Lifschitz' Realizabiility
ML-88-02 M.D.G. Swaen — The Arithmetical Fragment of Martin Löf's Type Theories with weak Σ-elimination
ML-88-03 Dick de Jongh, Frank Veltman — Provability Logics for Relative Interpretability
ML-88-04 A.S. Troelstra — On the Early History of Intuitionistic Logic
ML-88-05 A.S. Troelstra — Remarks on Intuitionism and the Philosophy of Mathematics

Computation and Complexity Theory:

CT-88-01 Ming Li, Paul M.B.Vitanyi — Two Decades of Applied Kolmogorov Complexity
CT-88-02 Michiel H.M. Smid — General Lower Bounds for the Partitioning of Range Trees
CT-88-03 Michiel H.M. Smid, Mark H. Overmars, Leen Torenvliet, Peter van Emde Boas — Maintaining Multiple Representations of Dynamic Data Structures
CT-88-04 Dick de Jongh, Lex Hendriks, Gerard R. Renardel de Lavalette — Computations in Fragments of Intuitionistic Propositional Logic
CT-88-05 Peter van Emde Boas — Machine Models and Simulations (revised version)
CT-88-06 Michiel H.M. Smid — A Data Structure for the Union-find Problem having good Single-Operation Complexity
CT-88-07 Johan van Benthem — Time, Logic and Computation
CT-88-08 Michiel H.M. Smid, Mark H. Overmars, Leen Torenvliet, Peter van Emde Boas — Multiple Representations of Dynamic Data Structures
CT-88-09 Theo M.V. Janssen — Towards a Universal Parsing Algorithm for Functional Grammar
CT-88-10 Edith Spaan, Leen Torenvliet, Peter van Emde Boas — Nondeterminism, Fairness and a Fundamental Analogy
CT-88-11 Sieger van Denneheuvel, Peter van Emde Boas — Towards implementing RL

Other prepublications:

X-88-01 Marc Jumelet — On Solovay's Completeness Theorem

1989

Logic, Semantics and Philosophy of Language:

LP-89-01 Johan van Benthem — The Fine-Structure of Categorial Semantics
LP-89-02 Jeroen Groenendijk, Martin Stokhof — Dynamic Predicate Logic, towards a compositional, non-representational semantics of discourse
LP-89-03 Yde Venema — Two-dimensional Modal Logics for Relation Algebras and Temporal Logic of Intervals
LP-89-04 Johan van Benthem — Language in Action
LP-89-05 Johan van Benthem — Modal Logic as a Theory of Information
LP-89-06 Andreja Prijatelj — Intensional Lambek Calculi: Theory and Application
LP-89-07 Heinrich Wansing — The Adequacy Problem for Sequential Propositional Logic
LP-89-08 Víctor Sánchez Valencia — Peirce's Propositional Logic: From Algebra to Graphs
LP-89-09 Zhisheng Huang — Dependency of Belief in Distributed Systems

Mathematical Logic and Foundations:

ML-89-01 Dick de Jongh, Albert Visser — Explicit Fixed Points for Interpretability Logic
ML-89-02 Roel de Vrijer — Extending the Lambda Calculus with Surjective Pairing is conservative
ML-89-03 Dick de Jongh, Franco Montagna — Rosser Orderings and Free Variables
ML-89-04 Dick de Jongh, Marc Jumelet, Franco Montagna — On the Proof of Solovay's Theorem
ML-89-05 Rineke Verbrugge — Σ-completeness and Bounded Arithmetic
ML-89-06 Michiel van Lambalgen — The Axiomatization of Randomness
ML-89-07 Dirk Roorda — Elementary Inductive Definitions in HA: from Strictly Positive towards Monotone
ML-89-08 Dirk Roorda — Investigations into Classical Linear Logic
ML-89-09 Alessandra Carbone — Provable Fixed points in $I\Delta_0+\Omega_1$

Computation and Complexity Theory:

CT-89-01 Michiel H.M. Smid — Dynamic Deferred Data Structures
CT-89-02 Peter van Emde Boas — Machine Models and Simulations
CT-89-03 Ming Li, Herman Neuféglise, Leen Torenvliet, Peter van Emde Boas — On Space Efficient Simulations
CT-89-04 Harry Buhrman, Leen Torenvliet — A Comparison of Reductions on Nondeterministic Space
CT-89-05 Pieter H. Hartel, Michiel H.M. Smid, Leen Torenvliet, Willem G. Vree — A Parallel Functional Implementation of Range Queries
CT-89-06 H.W. Lenstra, Jr. — Finding Isomorphisms between Finite Fields
CT-89-07 Ming Li, Paul M.B. Vitanyi — A Theory of Learning Simple Concepts under Simple Distributions and Average Case Complexity for the Universal Distribution (Prel. Version)
CT-89-08 Harry Buhrman, Steven Homer, Leen Torenvliet — Honest Reductions, Completeness and Nondeterminstic Complexity Classes
CT-89-09 Harry Buhrman, Edith Spaan, Leen Torenvliet — On Adaptive Resource Bounded Computations
CT-89-10 Sieger van Denneheuvel — The Rule Language RL/1
CT-89-11 Zhisheng Huang, Sieger van Denneheuvel, Peter van Emde Boas — Towards Functional Classification of Recursive Query Processing

Other Prepublications:

X-89-01 Marianne Kalsbeek — An Orey Sentence for Predicative Arithmetic
X-89-02 G. Wagemakers — New Foundations: a Survey of Quine's Set Theory
X-89-03 A.S. Troelstra — Index of the Heyting Nachlass
X-89-04 Jeroen Groenendijk, Martin Stokhof — Dynamic Montague Grammar, a first sketch
X-89-05 Maarten de Rijke — The Modal Theory of Inequality
X-89-06 Peter van Emde Boas — Een Relationele Semantiek voor Conceptueel Modelleren: Het RL-project

1990 SEE INSIDE BACK COVER

Institute for Logic, Language and Computation
Plantage Muidergracht 24
1018TV Amsterdam
Telephone 020-525.6051, Fax: 020-525.5101

SHAVRUKOV'S THEOREM
ON THE SUBALGEBRAS OF DIAGONALIZABLE ALGEBRAS
FOR THEORIES CONTAINING $I\Delta_0$+EXP

Domenico Zambella
Department of Mathematics and Computer Science
University of Amsterdam

ILLC Prepublications
for Mathematical Logic and Foundations
ISSN 0924-2090
Coordinating editor: Dick de Jongh

received August 1992

Shavrukov's theorem on the subalgebras of diagonalizable algebras for theories containing $I\Delta_0$+exp.

Abstract. Recently Volodya Shavrukov [1] pioneered the study of subalgebras of diagonalizable algebras of theories of arithmetic. We intend to show that his results extend to weaker theories (namely to theories containing $I\Delta_0$+exp).

§0 Introduction. A diagonalizable algebra is a Boolean algebra $(\mathcal{D},\rightarrow,\bot)$ with an additional operator $\Box$ which satisfies the axioms:

$\forall x,y \;\; \Box(x\rightarrow y)\rightarrow(\Box x\rightarrow\Box y) = \top$ and $\forall x \;\; \Box(\Box x\rightarrow x)\rightarrow\Box x = \top$.

Let T be a sufficiently strong axiomatized theory in the language of arithmetic. The predicate of provability of T generates in a natural way an operator on the Lindenbaum algebra of T. The resulting diagonalizable algebra $\mathcal{D}_T$ is called the *diagonalizable algebra of T*. The subalgebras of $\mathcal{D}_T$ have been studed in [1], in particular the general problem of when a diagonalizable algebra $\mathcal{D}$ is embeddable in $\mathcal{D}_T$.

We will translate this question into a problem of provability logic and for this we need some notation. Let $\mathcal{L}$ be the set of modal formulas generated by the language $(\rightarrow,\Box,\bot,\{p_i\}_{i\in\omega})$. We write $B\vDash A$ if A can be derived using modus ponens and necesitation from B and Löb's axioms (hence $\vDash A$ means that A is a theorem of Löb's logic and $B\vDash A$ means $\vDash \boxdot B\rightarrow A$, where $\boxdot B$ is $B\wedge\Box B$), we write $B\Vdash A$ iff $\vDash B\rightarrow A$. When $\mathcal{A}$ is a set of modal formulas in the language $\mathcal{L}$, $\mathcal{A}\vDash A$ and $\mathcal{A}\Vdash A$ are defined analogously. Given a set $\mathcal{A}$, consider the equivalence relation on $\mathcal{L}$: $A\approx_{\mathcal{A}} B$ iff $\mathcal{A}\vDash A\leftrightarrow B$, and let $\mathcal{L}/\mathcal{A}$ be the sets of $\approx_{\mathcal{A}}$-equivalence classes. The operator which maps the equivalence class of A to that of $\Box A$ is a well defined operator on $\mathcal{L}/\mathcal{A}$ which turns it into a diagonalizable algebra. For every (denumerable) diagonalizable algebra $\mathcal{D}$ there is a set $\mathcal{A}$ such that $\mathcal{D}$ is isomorfic to $\mathcal{L}/\mathcal{A}$.

Let T be an axiomatized theory in the language of the arithmetic and let Thm(.) be the provability predicate of T. A *T-interpretation* is a map ι which maps formulas of $\mathcal{L}$ to sentences of the language of arithmetic such that T proves:

(i) $\iota(\Box A)\leftrightarrow \mathrm{Thm}[\iota(A)]$; (ii) $\neg\iota(\bot)$; (iii) $\iota(A\rightarrow B)\leftrightarrow(\iota(A)\rightarrow\iota(B))$.

(In the following we shall simply say an *interpretation* since the theory T will be fixed.) If for every formula A in $\mathcal{L}$, $\mathcal{A}\vDash A$ iff $T\vdash \iota(A)$ we say that ι *interprets* $\mathcal{A}$ in T. We say that $\mathcal{A}$ is *interpretable* in T if there exists an interpretation which interprets $\mathcal{A}$ in T.

Given an interpretation of $\mathcal{A}$ in T there exists a natural embedding of $\mathcal{L}/\mathcal{A}$ in $\mathcal{D}_T$ and vice versa. So, for any given theory T, the problem the problem of classifying diagonalizable algebras in $\mathcal{D}_T$, reduces to classifying the sets of modal formulas $\mathcal{A}$ which are interpretable in T.

We write as usual $\Box^0\bot$ for $\bot$ and $\Box^{n+1}\bot$ for $\Box\Box^n\bot$; the minimal n such that $\mathcal{A}\models\Box^n\bot$ is called the *height* of $\mathcal{A}$. If such an n does not exist, we say that $\mathcal{A}$ has *infinite height*. We say that $\mathcal{A}$ has the *strong disjunction property (s.d.p.)* or, equivalently, that $\mathcal{A}$ is *strongly disjunctive (s.d)* iff $\mathcal{A}$ is consistent and for all formulas A and B if $\mathcal{A}\models\Box A\vee\Box B$ then either $\mathcal{A}\models A$ or $\mathcal{A}\models B$. The same classification is, mutata mutandis, applied to diagonalizable algebras. In the following T will a fixed axiomatized theory (i.e. the theory is given along with its primitive recursive axiomatization). The language of T contains the language of the arithmetic and -only for the sake of convenience- a symbol for exponentiation. Thm(.) is the provability predicate of T. We write $Thm^0(\bot)$ for the sentence $0\neq0$ and $Thm^{n+1}(\bot)$ for $Thm(Thm^n(\bot))$ (we shall always omit the Gödel-number symbols $\ulcorner\ \urcorner$). The minimal n such that $T\vdash Thm^n(\bot)$ is called the *height* of T. If such an n does not exist we say that T has *infinite height*. The height of T is in fact the height of its diagonalizable algebra $\mathcal{D}_T$. If all Σ_1-sentences provable in T are true in the standard model, then T is *Σ_1-sound*, otherwise T is *Σ_1-ill*. Shavrukov proved that every r.e. set of modal formulas is interpretable in the diagonalizable algebra of a Σ_1-ill theory containing Σ_1-induction provided it has the the same height as the theory. Moreover every r.e. and s.d. set of modal formulas is interpretable in the diagonalizable algebra of a Σ_1-sound theory containing Σ_1-induction. In particular, Σ_1-sound theories containing Σ_1-induction have the same set of subalgebras and the same holds for Σ_1-ill theories of any fixed height. (Recall that the Gödel numbering of arithmetical sentences gives a natural recursive enumeration of a set $\mathcal{A}$ such that $\mathcal{L}/\mathcal{A}$ is isomorphic to $\mathcal{D}_T$.)

The construction makes use of a Solovay function which ranges over a Kripke model. In the case of infinite height theories the models used are of infinite height so Σ_1-induction is needed to guarantee the existence of the limit. In the case of finite height theories this model has standard height, so the proof in [1] goes through for $I\Delta_0$+exp. Thus in the present exposition we concentrate only on theories of infinite height.

I whish to thank Volodya Shavrukov for numerous suggestions and corrections. I owe very much also to the stimulating criticisms and friendly encouragements of Lev Beklemishev.

§1 A lemma. In this section we prove a lemma which will be used to characterize the r.e. sets of modal formulas interpretable in a theory $T\supseteq I\Delta_0$+exp. We assume the reader to be familiar with the techniques introduced in [2].

A finite tree-like Kripke model k (in the sequel simply a *model*) is a triple $(W,R,\Vdash)$ where (W,R) is a finite tree with nodes W and order relation R, $a\in\omega$ and $\Vdash$ is a finite subset of $W\times\omega$. We call W the *universe of k* and (W,R) the *frame of k*. We write $w\Vdash p_i$ if $(w,i)\in\Vdash$. The relation $w\Vdash A$

(w *forces* A) is then expanded to all the formulas of $\mathcal{L}$ in the usual way. We say that k' is a generated submodel (in the sequel simply a *submodel*) of k if the frame of k' is a subtree of the frame of k and the forcing relation of k' equals the intersection of the forcing relation of k with the universe of k'. We write $k \Vdash A$ (*k forces A*) iff the formula A is forced in the root of the model coded by k, we write $k \vDash A$ (k *is a model of A*) if every node of k forces A. Then we have that k is a model A iff k forces $\Box A$. If $\mathcal{A}$ is a finite set of formulas we write $k \Vdash \mathcal{A}$ (resp. $k \vDash \mathcal{A}$) if for every $A \in \mathcal{A}$, $k \Vdash A$ (resp. $k \vDash A$). Then it easy to chek that if $\mathcal{A}$ is finite then $\mathcal{A} \vDash A$ iff every model of $\mathcal{A}$ is a model of A, and $\mathcal{A} \Vdash A$ iff every model which forces $\mathcal{A}$ forces A (if $\mathcal{A}$ is infinite this may not be the case since we will deal only with finite models).

In a first-order formula an occurence of a quantifier is said to be bounded if it is of the form $\forall x<t$ or $\exists x<t$ where t is a term of the language of T. The Δ_0-formulas of T are the formulas provably equivalent to formulas with only bounded quantifiers (having assumed exponentiation as a primitive function of the language we should properly write $\Delta_0(\exp)$ but in the present paper there will be no risk of confusion). The Σ_1 formulas are those equivalent to a Δ_0-formula preceeded by an existential quantifier. The theory whose axioms are those of Robinson arithmetic plus the characteristic axioms for exponentiation and the induction schema for Δ_0-formulas is called $I\Delta_0+\exp$; the theory which contains also the schema of Σ_1-induction is called $I\Sigma_1$. We refer the reder to [3] for more details on these theories.

We fix a natural coding of modal formulas and of models in arithmetic; we shall use the same symbol both for a formula (resp. model) and its code. We require that the coding assigns to proper submodels of k a smaller code then to k itself. Having exponentiation as a primitive function, we may require without loss of generality that $k \Vdash A$ and $k \vDash A$ translate into Δ_0-formulas. Given an r.e. set $\mathcal{A}$ of modal formulas we may find, formalizing in the language of arithmetic the algorithm enumerating $\mathcal{A}$, a Δ_0-formula "$A \in \mathcal{A}_x$" (here A and x are the free variables of the formula) such that for every $A \in \mathcal{L}$, $A \in \mathcal{A}$ iff $\exists n \in \omega$, $T \vdash A \in \mathcal{A}_n$. We also require that, provably in T, for every x there are only finitely many A such that $A \in \mathcal{A}_x$. We call such a formula a *description* of $\mathcal{A}$ (in T). We may formalize in T also the notion of Löb's derivability so that we can use the expression $\mathcal{A}_n \vDash A$ both when arguing in the real world and in the theory. Formalizing the proof of the completeness theorem for Löb's logic in $I\Delta_0+\exp$ one can find a Δ_0-formula describing the relation $\mathcal{A}_n \vDash A$. We shall also use the expression "$\mathcal{A} \vDash A$" when reasoning in T; this stands for for $\exists x\,(\mathcal{A}_x \vDash A)$.

Once we fix a description of $\mathcal{A}$, it makes perfectly sense to say *"T proves that $\mathcal{A}$ is s.d."* this simply means:

$$T \vdash \neg(\mathcal{A} \vDash \bot) \wedge \forall A,B\,(\mathcal{A} \vDash \Box A \vee \Box B) \rightarrow (\mathcal{A} \vDash A \vee \mathcal{A} \vDash B).$$

Obviously, an r.e. set of formulas $\mathcal{A}$ may have different descriptions and for one description the theory T may prove that $\mathcal{A}$ is s.d. and for an other description it may not, possibly the "opinion" of T may be incorrect. We shall deal with these phenomena in the next section; for the moment we keep the description fixed and assume T proves that $\mathcal{A}$ is s.d..

Lemma 1. Let T be an axiomatized theory of infinite height containing $I\Delta_0$+exp and $\mathcal{A}$ an r.e. set of modal formulas. If there is a description of $\mathcal{A}$ in T such that T proves that $\mathcal{A}$ is s.d. then $\mathcal{A}$ is interpretable in T.

Proof. Let T be an axiomatized theory and $A \in \mathcal{A}_n$ be a description of an r.e. set of modal formulas as in the hypothesis of the lemma. We shall define a Solovay function h(n) whose value is either 0 or the code of a model of $\mathcal{A}_{,m}$ for some $m \leq n$. We agree that $0 \Vdash A$ is some fixed provably false sentence (e.g. $0 \neq 0$), so the expression $h(n) \Vdash A$ will always have a meaning. The Solovay function is defined, simultaneously with the sentences λ_0 and λ_A, by an arithmetical fixed point. The definition is the following.

Let λ_0 be the sentence $\forall n\, h(n)=0$. We order the modal formulas by increasing code and let A_i be the i-th formula in this order (this enumeration of formulas is redundant, since here formulas are actually codes, but we introduce it for better readability). For every i and every string $\sigma \in 2^i$ define a formula:

$$A_\sigma := \bigwedge\{A_n \mid n<i \text{ and } \sigma(n)=1\} \wedge \bigwedge\{\neg A_n \mid n<i \text{ and } \sigma(n)=0\}$$

for each modal formula $A=A_i$ define the sentence:

$$\lambda_A := \exists \sigma \in 2^{i+1}[\sigma(i)=1 \wedge \exists^\infty n\, h(n) \Vdash A_\sigma \wedge \forall \tau \in 2^{i+1} (\tau<\sigma \rightarrow \forall^\infty n\, h(n) \nVdash A_\tau)],$$

where $\tau<\sigma$ has to be read as τ precedes σ in the lexicographic order. $\exists^\infty n$ is an abbreviation of $\forall m \exists n>m$ and $\forall^\infty n$ of $\neg\exists^\infty n \neg$.

Let $h(0)=0$. For $n>0$ if n codes a proof of $\lambda_0 \vee \lambda_A$ for some formula A, then:

(a) if $h(n)=0$ and $\mathcal{A}_{,n} \nVdash A$, then choose the minimal model k of $\mathcal{A}_n$ which forces $\neg A$ and define $h(n+1)=k$.

(b) if $h(n)=h\neq 0$ and the root of some submodel of h forces $\neg A$ then let k be the minimal such submodel and define $h(n+1)=k$.

In all other cases let $h(n+1)=h(n)$.

If the theory T is strong enough one can use for λ_A simply the sentence $\exists m \forall n>m\, h(n) \Vdash A$. Then $\lambda_0 \vee \lambda_A$ simply means that the limit of h is a model which forces the formula A, in particular, if h moved to h(n+1) because n codes a proof of $\lambda_0 \vee \lambda_A$, there will be a proof that h(n+1) is not the limit of the function (in fact h(n+1) is choosen so that $h(n+1) \Vdash \neg A$). But in $I\Delta_0$+exp it is not possible to prove that the limit of the Solovay function exists (one needs Σ_1-induction), in particular it cannot be excluded that for some formula A both $h(n) \Vdash A$ and $h(n) \Vdash \neg A$ occurs for infinitely many n; thus one would not have as desired, $\lambda_{\neg A} \leftrightarrow \neg\lambda_A$. To help the reader's intuition we present the following semi-formal description of λ_A which should clarify the definition above. To each formula A we attach an infinite set C(A) such that either $\forall n \in C(A)\ h(n) \Vdash A$ or $\forall n \in C(A)\ h(n) \Vdash \neg A$. The set C(A) is defined in the following way. Let $C(A_0) = \{n \mid h(n) \Vdash \neg A_0\}$ if this is infinite, $C(A_0)=\{n \mid h(n) \Vdash A_0\}$ otherwise. Let $C(A_{i+1}) = \{n \in C(A_i) \mid h(n) \Vdash \neg A_{i+1}\}$ if this is infinite, $C(A_{i+1}) = \{n \in C(A_i) \mid h(n) \Vdash A_{i+1}\}$ otherwise. Finally, let λ_A be the sentence $\forall n \in C(A)\ h(n) \Vdash A$.

Claim 1. T proves $\forall n\ [\ h(n) \neq 0 \rightarrow \mathrm{Thm}[\exists m\, h(m) \text{ is a proper submodel of } h(\dot{n})]\ ]$.

Proof. In fact if $h(n)\neq 0$ then at some stage s<n for some formula A, s codes a proof $\lambda_0 \vee \lambda_A$ and $h(s+1)=h(n) \Vdash \neg A$. By provable Σ_1 completeness $\mathrm{Thm}[\neg\lambda_0]$ this together with $\mathrm{Thm}[\lambda_0 \vee \lambda_A]$ yields $\mathrm{Thm}[\lambda_A]$ and in particular $\mathrm{Thm}[\exists^\infty n\ h(n) \Vdash A]$. From $h(n) \Vdash \neg A$ we get $\mathrm{Thm}[h(\dot{n}) \Vdash \neg A]$ by provable Σ_1 completeness, thus the claim follows.

Claim 2. $\forall n \in \omega\ \exists m \in \omega$ such that T proves $h(n)\neq 0 \rightarrow \mathrm{Thm}^m(\bot)$. (So, since T has infinite height, for every standard n, h(n)=0.)

Proof. This is an easy corollary of the previous claim.

To define $\iota(A)$ we need to assign "ad hoc" a model to 0. Following Shavrukov we shall construct a formula $\mathcal{T}$ in such a way that for all standard formulas A and B the following properties are provable in T.

(1) $\neg\mathcal{T}(\bot)$

(2) $\mathcal{T}(A \rightarrow B) \leftrightarrow (\mathcal{T}(A) \rightarrow \mathcal{T}(B))$

(3) $\mathcal{A} \models A \rightarrow \mathcal{T}(A)$.

(4) $\mathcal{T}(\Box A) \rightarrow \mathcal{A} \models A$.

(Roughly speaking the formula $\mathcal{T}(A)$ says that A belongs to some a maximal consistent set $\mathcal{T}$ containing $\mathcal{A} \cup \{\neg\Box A \mid \mathcal{A} \not\models \Box A\}$. Such a set $\mathcal{T}$ exists (for T) since otherwise for some $A_0,\ldots,A_n$ such that $\mathcal{A} \not\models \Box A_0, \ldots, \mathcal{A} \not\models \Box A_n$ we would have $\mathcal{A} \models \Box A_0 \vee \ldots \vee \Box A_n$. This contradicts the s.d.p. of $\mathcal{A}$.) For the proof of the lemma only (1)-(4) are needed, so we prefer to postpone the definition of $\mathcal{T}$ and the proof of (1)-(4) after the proof of the lemma.

We define τ_A to be the sentence $\lambda_0 \wedge \mathcal{T}(A)$, and finally define: $\iota(A) := \lambda_A \vee \tau_A$, i.e. $\lambda_A \vee [\lambda_0 \wedge \mathcal{T}(A)]$. We shall prove that ι is an interpretation (claim 5) and that ι interprets $\mathcal{A}$ in T (claim 6).

Claim 3. For every $A \in \mathcal{L}$, T proves $\forall^\infty n\ h(n) \Vdash A \rightarrow \lambda_A$.

Proof. Since A is standard we can replace in the definition of λ_A the quantifications over strings by finite conjunctions and disjunctions. So the claim is trivial.

Claim 4. For every $A\in\mathcal{L}$, T proves $\forall n\, [h(n)=0 \wedge \mathcal{A}_{,n}\models A \rightarrow \iota(A)]$.

Proof. Assume h(n)=0 and $\mathcal{A}_{,n}\models A$. Reasoning in T we want to show $\lambda_A\vee\tau_A$. Since h(n)=0 and $\mathcal{A}_{,n}\models A$, the function can leave 0 only to a model of A and eventually move to some submodel of it. So $\neg\lambda_0$ implies $\forall^\infty n\, h(n)\models A$. By the previous claim, this implies λ_A. On the other hand, by (3), we have $\mathcal{T}(A)$, so, λ_0 implies τ_A.

Claim 5. The function ι is an interpretation. (i.e. properties (i)-(iii) are provable in T.)

Proof. We have to prove that for every standard formula A properties (i)-(iii) are provable in T, i.e. $\iota(\Box A)\leftrightarrow \mathrm{Thm}[\iota(A)]$, $\neg\,\iota(\bot)$ and $\iota(A\rightarrow B)\leftrightarrow (\iota(A)\rightarrow\iota(B))$. The proof is more readable if we derive them both from T+λ_0 and from T+$\neg\lambda_0$. In fact under the hypothesis λ_0 the sentence $\iota(A)$ is equivalent to $\mathcal{T}(A)$ (by our convention that $0\nVdash A$), while, under the hypothesis $\neg\lambda_0$, $\iota(A)$ is equivalent to λ_A.

T+λ_0 $\vdash$ $\iota(\Box A)\rightarrow \mathrm{Thm}[\iota(A)]$. Assume $\iota(\Box A)$ and λ_0 and reason in T. As we just remarked, under the assumption λ_0, $\iota(\Box A)$ reduces to $\mathcal{T}(\Box A)$. By (4) we obtain $\mathcal{A}\models A$, so, for some n, $\mathcal{A}_{,n}\models A$. Since we assumed λ_0, h(n)=0. Both $\mathcal{A}_{,n}\models A$ and h(n)=0 are Σ_1 sentence so by provable Σ_1-completeness we have $\mathrm{Thm}[\mathcal{A}_{\dot n}\models A]$ and $\mathrm{Thm}[h(\dot n)=0]$. By claim 4 we have $\mathrm{Thm}[\iota(A)]$.

T+λ_0 $\vdash$ $\mathrm{Thm}[\iota(A)]\rightarrow\iota(\Box A)$. Assume $\mathrm{Thm}[\lambda_A\vee\tau_A]$ and λ_0. It sufficies to show, reasoning in T, that $\mathcal{T}(\Box A)$. Since $\mathrm{Thm}[\lambda_A\vee\tau_A]$, a fortiori $\mathrm{Thm}[\lambda_0\vee\lambda_A]$. Let n the code of the minimal proof of $\lambda_0\vee\lambda_A$, Since we assumed λ_0, h(n)=0. Then $\mathcal{A}_{,n}\models A$, otherwise the function would leave 0 at stage n+1, contradicting λ_0. Then $\mathcal{A}\models \Box A$ and so, by (3), $\mathcal{T}(\Box A)$.

T+λ_0 $\vdash$ $\neg\iota(\bot)$. Immediately from (1).

T+λ_0 $\vdash$ $\iota(A\rightarrow B)\leftrightarrow (\iota(A)\rightarrow\iota(B))$. Immediately from (2).

T+$\neg\lambda_0$ $\vdash$ $\iota(\Box A)\rightarrow \mathrm{Thm}[\iota(A)]$. Assume $\iota(\Box A)$ and $\neg\lambda_0$. It sufficies to prove $\mathrm{Thm}[\lambda_A]$ in T. By our assumption $\lambda_{\Box A}$ holds, in particular for some n, $h(n)\Vdash\Box A$. The latter is a Σ_1 sentence so $\mathrm{Thm}[h(\dot n)\Vdash\Box A]$. Since $h(n)\neq 0$, by claim 1 we have Thm["$\exists m$ h(m) is a submodel of $h(\dot n)$"], thus $\mathrm{Thm}[\forall^\infty n\, h(n)\Vdash A]$. By claim 3, $\mathrm{Thm}[\lambda_A]$ follows.

T+$\neg\lambda_0$$\vdash$ $\mathrm{Thm}[\iota(A)]\rightarrow\iota(\Box A)$. Assume $\mathrm{Thm}[\lambda_A\vee\tau_A]$ and $\neg\lambda_0$. It sufficies to derive $\lambda_{\Box A}$ reasonong in T. Since $\mathrm{Thm}[\lambda_A\vee\tau_A]$, a fortiori $\mathrm{Thm}[\lambda_0\vee\lambda_A]$. Let n a code of a proof of $\lambda_0\vee\lambda_A$ which is large enough to have $h(n)\neq 0$. (Such an n exists since we assumed $\neg\lambda_0$ and any provable sentence has arbitrary large proofs.) Then, directly from the definition of h and from the fact that the code of a model is larger than the code of its proper submodels, we can conclude that $\forall^{\infty}n\ h(n)\Vdash\Box A$. Thus $\lambda_{\Box A}$ follows by claim 5.

$T+\neg\lambda_0\vdash\neg\iota(\bot)$. Immediate.

$T+\neg\lambda_0\vdash\iota(A\rightarrow B)\leftrightarrow(\iota(A)\rightarrow\iota(B))$. Is left to the reader.

Claim 6. For every $A\in\mathcal{L}$, $\mathcal{A}\vDash A$ iff $T\vdash\iota(A)$.

Proof. ($\Longrightarrow$) Assume $\mathcal{A}\vDash A$ then for some $\mathcal{A}_{,n}\vDash A$. Since n is standard $h(n)=0$ and, by Σ_1-completeness, $T\vdash h(n)=0\wedge\mathcal{A}_{,n}\vDash A$. So $\iota(A)$ by claim 6. Vice versa, ($\Longleftarrow$), if $T\vdash\iota(A)$ we have in particular that $T\vdash\lambda_0\vee\lambda_A$. Assume for a contradiction that $\mathcal{A}\nvDash A$ and let n be the code of the proof of $\lambda_0\vee\lambda_A$. In particular we have that $\mathcal{A}_{,n}\nvDash A$ then $h(n+1)\neq 0$. This n is a standard number so this contradicts the fact that h will spend all its standard live in 0.

The proof of the lemma is completed but for the definition of the predicate $\mathcal{T}$. We introduce the formula $V(\sigma)$ which roughly says: A_σ is $\Box$-conservative over $\mathcal{A}$, namely

$$V(\sigma):=\forall A\,[\,(\mathcal{A}\vDash A_\sigma\rightarrow\Box A)\rightarrow(\mathcal{A}\vDash\Box A)\,].$$

Assume strings have been coded into numbers in some natural way, (e.g. choose $\Sigma_{\sigma(i)|=1}\,2^i$ as code of σ) so that on strings of equal lenght the relation "<" coincides with the relation "preceeds lexicografically" or, when strings are thought as nodes of a binary tree, "is on the left of". Let $U(\sigma)$ be the sentence which sais that σ is the leftmost string satisfying $V(\sigma)$,

$$U(\sigma):=V(\sigma)\wedge\forall\tau\in 2^{i+1}(\tau<\sigma\rightarrow\neg\ V(\tau))].$$

If $A=A_i$ let $\mathcal{T}(A)$ hold if there is $\sigma\in 2^{i+1}$ such that $U(\sigma)$ and $\sigma(i)=1$. Note that if $\sigma,\tau\in 2^j$ for some $i>j$ and both $U(\sigma)$ and $U(\tau)$ then $\tau(i)=\sigma(i)$. We have to show that for every standard formula properties (1) to (4) of $\mathcal{T}$ are provable in T. As first thing let us remark that for all standard i, T proves $\exists\sigma\in 2^{i+1}\,U(\sigma)$, i.e. i.e. there exists the leftmost string σ satisfying $V(\sigma)$. Reason in T. A string satisfying $V(\sigma)$ must exists otherwise for every $\sigma\in 2^{i+1}$ there would be a modal formula C_σ such that $\mathcal{A}\vDash A_\sigma\rightarrow\Box C_\sigma$ and $\mathcal{A}\nvDash\Box C_\sigma$. Since $\bigvee_{\sigma\in 2^{i+1}}A_\sigma$ is a tautology, one would have $\mathcal{A}\vDash\bigvee_{\sigma\in 2^{i+1}}\Box C_\sigma$. By the s.d.p. of $\mathcal{A}$ (provable in T) $\mathcal{A}\vDash\Box C_\sigma$ for some σ, a contradiction. Now, once we know that one string σ exists satisfying $V(\sigma)$, the existence of the minimal one is again a consequence of the standardness of i since the quantifiers over strings in 2^{i+1} may be transformed in finite conjunctions and disjunctions. This proves our remark, now we check in turn that properties (1) to (4) which we required for $\mathcal{T}$ are provable in T.

(1) $\neg\mathcal{T}(\bot)$

(2) $\mathcal{T}(A\to B) \leftrightarrow (\mathcal{T}(A) \to \mathcal{T}(B))$

(3) $\mathcal{A}\models A \to \mathcal{T}(A)$.

(4) $\mathcal{T}(\Box A) \to \mathcal{A}\models A$.

We reason in T. It is obvuious that no string σ such that $V(\sigma)$, $\sigma(\bot)=1$, so (1) holds. (We write $\sigma(A)$ for $\sigma(i)$ where $A=A_i$.) To prove (2) assume first that $\mathcal{T}(A\to B)$ and $\mathcal{T}(A)$. Let σ be a sufficiently long string such that $U(\sigma)$ and $\sigma(A\to B)=\sigma(A)=1$. Then $\sigma(B)=1$ otherwise $A_\sigma \leftrightarrow \bot$ and surely could not satisfy $V(\sigma)$. The converse is similar. Property (3) is also a diret consequence of the existence of an arbitrary (standard) long string satisfying $U(\sigma)$. For such a string we must have $\sigma(A)=1$ otherwise $\mathcal{A}\models A_\sigma \to \bot$ and, by the definition of $V(\sigma)$, that $\mathcal{A}\models \bot$. Last to prove (4) assume that $\mathcal{T}(\Box A)$. Let σ be a sufficiently long string such that $U(\sigma)$ and $\sigma(\Box A)=1$. Then $\mathcal{A}\models A_\sigma\to \Box A$ so, by the definition of $V(\sigma)$, $\mathcal{A}\models \Box A$. By the s.d.p. of $\mathcal{A}$ we get $\mathcal{A}\models A$.

This completes the proof of lemma 1.□

§2. **The theorems.** We shall use lemma 1 to prove the two theorems announced in the introduction. They characterize the r.e. sets interpretable in a theory of infinite height.

Theorem 1. Let $\mathcal{A}$ is an r.e. set of modal formulas and T is a Σ_1 sound theory containing $I\Delta_0$+exp, then $\mathcal{A}$ is interpretable in T iff $\mathcal{A}$ is s.d..

Theorem 2. Let $\mathcal{A}$ is an r.e. set of modal formulas and T is a Σ_1 ill theory of infinite height containing $I\Delta_0$+exp, then $\mathcal{A}$ is interpretable in T iff $\mathcal{A}$ has infinite height.

The "only if" part of both theorems is trivial. To prove the first theorem we show that, if $\mathcal{A}$ is an r.e. set with the s.d.p. and T is a Σ_1-sound theory, then we can find a description of $\mathcal{A}$ in T such that T proves the s.d.p. of $\mathcal{A}$. Analogously for the second theorem. For the sake of readability we shall give these proofs in an informal style, namely we shall merely describe algorithms and give for granted their formalization in the language of T.

Suppose $\mathcal{A}$ is an r.e. set of modal formulas and let $A\in \mathcal{A}_{,s}$ be any description of $\mathcal{A}$. To this description we associate in a natural way the algorithm $\{\mathcal{A}_{,s}\}_{s\in\omega}$ enumerating $\mathcal{A}$, i.e. an increasing recursive sequence of finite sets $\{\mathcal{A}_{,s}\}_{s\in\omega}$ such that $\mathcal{A}=\bigcup_{s\in\omega}\mathcal{A}_{,s}$. We shall construct a new algorithm $\{\mathcal{V}_{,s}\}_{s\in\omega}$ enumerating the same set $\mathcal{A}$ such that the canonical translation of $\{\mathcal{V}_{,s}\}_{s\in\omega}$ in the language of the arithmetic yields a description with the desired properties.

The proofs of theorems 1 and 2 needs two modal lemmas, respectively lemma 2 and 3. These are the adaptations of some lemmas of [1]. We shall present them in a form which is easly formalized and proved in $I\Delta_0$+exp. Their proofs are moved to the end of this section.

A finite set $\mathcal{C}$ of formulas is said to be *adequate* if: (i) $\bot \in \mathcal{C}$, (ii) all subformulas of any $B \in \mathcal{C}$ are in $\mathcal{C}$ (iii) for every $B,C \in \mathcal{C}$ there exists $D \in \mathcal{C}$ such that $\Vdash D \leftrightarrow (B \rightarrow C)$.

Lemma 2. Let $\mathcal{C}$ be a finite adequate set containing $\mathcal{A}$. The following are equivalent:

(a) $\mathcal{A}$ is s.d. (b) $\mathcal{A} \nvDash \bot$ and $\forall B,C \in \mathcal{C}$ $\mathcal{A} \vDash \Box B \vee \Box C \Rightarrow \mathcal{A} \vDash B$ or $\mathcal{A} \vDash C$.□

Proof of theorem 1. We are now ready to present the algorithm required to prove theorem 1. We may code (with infinitely many repetitions) finite sets of formulas with natural numbers. The property "s codes an adequate set" is Δ_0. With the same notation of the example given above consider the following algorithm $\{\mathcal{V}_{,s}\}_{s \in \omega}$.

(Stage 0) $\mathcal{V}_{,0} = \emptyset$.

(Stage s+1) If s codes an adequate set $\mathcal{C}$, if condition (b) of lemma 2 holds for for the set $\mathcal{A}_{,s} \cap \mathcal{C}$ and $\mathcal{A}_{,s} \cap \mathcal{C} \Vdash \mathcal{V}_{,s}$, let $\mathcal{V}_{,s+1} = \mathcal{V}_{,s} \cup (\mathcal{A}_{,s} \cap \mathcal{C})$, otherwise let $\mathcal{V}_{,s+1} = \mathcal{V}_{,s}$.

From lemma 2 follows that that $\mathcal{A} = \bigcup_{s \in \omega} \mathcal{V}_{,s}$ so $\{\mathcal{V}_{,s}\}_{s \in \omega}$ yields a description of $\mathcal{A}$. Formalizing lemma 2 in $I\Delta_0$+exp, we have that T proves the s.d.p. of $\mathcal{A}$.□

Lemma 3. Let $\mathcal{C}$ be a finite adequate set containing $\mathcal{A}$. The following are equivalent:

(1) $\mathcal{A}$ has infinite height (2) there exists $B \in \mathcal{C}$ such that B is s.d. and $B \vDash \bigwedge \mathcal{A}$.□

Proof of theorem 2. Given a Σ_1 ill theory T choose a Δ_0-formula $\sigma(x)$ such that $T \vdash \exists x \sigma(x)$ and $\omega \vDash \forall x \neg \sigma(x)$. In every model of T there is a Δ_0 definable number n, namely the minimal witness of $\exists x\, \sigma(x)$. The idea of the proof is the following: given any algorithm $\mathcal{A}_{,s}$ enumerating $\mathcal{A}$ we construct a new algorithm which simulates $\mathcal{A}_{,s}$ until the nonstandard stage n, but once this stage is reached we stop the simulation and enumerates some arbitrary s.d. set containing $\mathcal{A}_{,n}$. In the real world this stage n is never reached, so this new algorithm enumerates the same set as the old one. But in any model of T this algorithm enumerates a finite s.d. set. Lemma 3 is used to guarantee that some s.d. formula $B \vDash \mathcal{A}_{,s}$ always exists.

(Stage 0) $\mathcal{V}_{,0} = \emptyset$.

(Stage s+1) Let $\mathcal{C}$ be the finite set of formulas coded by s. If $\mathcal{C}$ is adequate, $\mathcal{A}_{,s} \cap \mathcal{C} \supseteq \mathcal{V}_{,s}$ and there is a s.d. formula A in $\mathcal{C}$ such that $A \vDash \mathcal{A}_{,s} \cap \mathcal{C}$, (i.e. condition (b) of lemma 2 holds).Then:

Case 1: if $\forall x \leq s\ \neg\sigma(x)$ let $\mathcal{V}_{,s+1} = \mathcal{V}_{,s} \cup (\mathcal{A}_{,s} \cap \mathcal{C})$.

Case 2: if $\exists x < s\ \sigma(x)$ let $\mathcal{V}_{,s+1} = \mathcal{V}_{,s} \cup \{A\}$.

Otherwise let $\mathcal{V}_{,s+1} = \mathcal{V}_{,s}$.

By lemma 3 the formula A required in case 1 always exists. In the real world case 2 never obtains, so $\mathcal{A} = \bigcup_{s\in\omega} \mathcal{V}_{,s}$. For the theory T case 2 obtains eventually, say at stage n, so $\bigcup_{s\in\omega} \mathcal{V}_{,s}$ is s.d.. This completes the proof of theorem 2.□

Proof of lemma 2. The direction (a) $\Longrightarrow$ (b) is trivial. For the proof of (b) $\Longrightarrow$ (a) assume that $\mathcal{A} \nvDash \bot$ and for all $B, C \in \mathcal{C}$ if $\mathcal{A} \vDash \Box B \vee \Box C$ then $\mathcal{A} \vDash B$ or $\mathcal{A} \vDash C$. Fix a set $\mathcal{A}t \subseteq \mathcal{C}$ such that:

$$\mathcal{A}t := \{G \in \mathcal{C} \mid \forall C \in \mathcal{C}\ G \Vdash C \text{ or } G \Vdash \neg C\}.$$

The elements of $\mathcal{A}t$ are called *atoms*. Let $\gamma = \{G \in \mathcal{A}t \mid A \nvDash \neg G\}$. By the adequateness of $\mathcal{C}$, $\vDash \bigvee \mathcal{A}t \leftrightarrow \top$ so, since $\mathcal{A} \nvDash \bot$, $\gamma \neq \emptyset$. We claim that there is a model of $\mathcal{A} \cup \{\Diamond G \mid G \in \gamma\}$. In fact, if not then $\mathcal{A} \vDash \bigvee_{G\in\gamma} \neg\Box G$. By (b), there is $G \in \gamma$ such that $\mathcal{A} \vDash \neg G$ quod non. This proves the claim.

Suppose now that for some formulas B_1, B_2 both $\mathcal{A} \nvDash B_1$ and $\mathcal{A} \nvDash B_2$, so we may assume that there are two models k_1 and k_2 of $\mathcal{A}$ forcing respectively $\neg B_1$ and $\neg B_2$. We shall show that $\mathcal{A} \nvDash \Box B_1 \vee \Box B_2$ by constructing a model k' of $\mathcal{A}$ which contains k_1 and k_2 as proper submodels. The s.d.p. of $\mathcal{A}$ will follow.

Let k be a model of $\mathcal{A} \cup \{\Diamond G \mid G \in \gamma\}$. Let r, r_1 and r_2 be the roots of respectively k,k_1 and k_2. Let R,R_1 and R_2 be the respective accessibility relations. Let k' be the model obtained grafting k_1 and k_2 above the root of k. More precisely, the universe of k' is the disjoint union of the universes of k,k_1 and k_2 and the accessibility relation of k' is the transitive closure of the relation $R \cup R_1 \cup R_2 \cup \{(r,r_1),(r,r_2)\}$. The forcing relation of k' is the union of the forcing relations of k,k_1 and k_2.

We claim that k' is a model of $\mathcal{A}$ and $k' \Vdash \neg\Box B_1 \wedge \neg\Box B_2$. Obviously k' forces $\neg\Box B_1 \wedge \neg\Box B_2$ because k_1 and k_2 are submodels of k' forcing respectively B_1 and B_2. To show that k' is a model of $\mathcal{A}$, we prove by induction on the complexity of subformulas $C \in \mathcal{C}$ that $k' \Vdash C$ iff $k \Vdash C$. The basis step is trivial as well as the induction for Boolean connectives. We prove the induction step for $\Box$. Assume $k' \Vdash \neg\Box C$. Then for some proper submodel w' of k', $w' \Vdash \neg C$. The model w' is a submodel of k_1 or of k_2 or is a proper submodel of k. If w' is a proper submodel of k, then $k \Vdash \neg\Box C$ follows. Otherwise, let G be the atom forced in w'; since $C \in \mathcal{C}$, by the definition of atom: either $G \Vdash C$ or $G \Vdash \neg C$. But $G \Vdash C$ leads immediately to contradiction so, $G \Vdash \neg C$. Since both k_1 and k_2 are models of $\mathcal{A}$, $G \in \gamma$. By our choice of k, $k \Vdash \bigwedge_{G\in\gamma} \Diamond G$, so there is a proper submodel w of k which forces G. Hence $w \Vdash \neg C$ and $k \Vdash \neg\Box C$. Vice versa if $k \Vdash \neg\Box C$ then for some proper submodel w of k,

$w \Vdash \neg C$. Since w is also a proper submodel of k', $k' \Vdash \neg\Box C$ follows. This completes the proof of the lemma.□

Proof of lemma 3. ($\Leftarrow$) Is immediate. ($\Rightarrow$) Let us first observe that if $\mathcal{A}$ has infinite height and $\mathcal{A} \models \Box C \vee \Box D$ then either $\mathcal{A} \cup \{C\}$ or $\mathcal{A} \cup \{D\}$ has infinite height. In fact, if $\mathcal{A} \models \Box C \vee \Box D$ and for some n both $\mathcal{A} \cup \{C\} \models \Box^n \bot$ and $\mathcal{A} \cup \{D\} \models \Box^n \bot$ then $\mathcal{A} \models \Box C \to \Box^{n+1} \bot$ and $\mathcal{A} \models \Box D \to \Box^{n+1} \bot$. Thus $\mathcal{A} \models \Box^{n+1} \bot$ and A has finite height. Now, list the formulas of $\mathcal{C} = \{C_1, ..., C_n\}$ define $\mathcal{A}_0 := \mathcal{A}$ and for all $i<n$ let $\mathcal{A}_{i+1} := \mathcal{A}_i \cup \{C\}$ for the first C in $\mathcal{C}$ such that for some D in $\mathcal{C}$, $\mathcal{A}_i \models \Box C \vee \Box D$, $\mathcal{A}_i \wedge C$ has infinite height and $\mathcal{A}_i \not\models C$, $\mathcal{A}_{i+1} := \mathcal{A}_i$ otherwise. Finally choose in $\mathcal{C}$ a formula B equivalent to $\bigwedge \mathcal{A}_n$. By lemma 2 and the previous observation, B satisfies the required properties.□

References.

[1] V.Yu. Shavrukov, Subalgebras of diagonalizable algebras of theories containing arithmetic, *ITLI Prepublication Series*, **X-91-03**, University of Amsterdam (1991).

[2] R.Solovay, Provability interpretations of modal logic. *Israel Journal of Mathematics* **25**, 287-304, (1976).

[3] P.Hajeck and P.Pudlak, *Methamathematics of first order arithmetic* . Springer Verlag, in print.

[4] D. de Jongh, M.Jumelet and F.Montagna, On the proof of Solovay's theorem. *Studia Logica* **L,1** 51-69 (1991).

The ILLC Prepublication Series

ML-91-05 A.S. Troelstra — History of Constructivism in the Twentieth Century
ML-91-06 Inge Bethke — Finite Type Structures within Combinatory Algebras
ML-91-07 Yde Venema — Modal Derivation Rules
ML-91-08 Inge Bethke — Going Stable in Graph Models
ML-91-09 V.Yu. Shavrukov — A Note on the Diagonalizable Algebras of PA and ZF
ML-91-10 Maarten de Rijke, Yde Venema — Sahlqvist's Theorem for Boolean Algebras with Operators
ML-91-11 Rineke Verbrugge — Feasible Interpretability
ML-91-12 Johan van Benthem — Modal Frame Classes, revisited

Computation and Complexity Theory

CT-91-01 Ming Li, Paul M.B. Vitányi — Kolmogorov Complexity Arguments in Combinatorics
CT-91-02 Ming Li, John Tromp, Paul M.B. Vitányi — How to Share Concurrent Wait-Free Variables
CT-91-03 Ming Li, Paul M.B. Vitányi — Average Case Complexity under the Universal Distribution Equals Worst Case Complexity
CT-91-04 Sieger van Denneheuvel, Karen Kwast — Weak Equivalence
CT-91-05 Sieger van Denneheuvel, Karen Kwast — Weak Equivalence for Constraint Sets
CT-91-06 Edith Spaan — Census Techniques on Relativized Space Classes
CT-91-07 Karen L. Kwast — The Incomplete Database
CT-91-08 Kees Doets — Levationis Laus
CT-91-09 Ming Li, Paul M.B. Vitányi — Combinatorial Properties of Finite Sequences with high Kolmogorov Complexity
CT-91-10 John Tromp, Paul Vitányi — A Randomized Algorithm for Two-Process Wait-Free Test-and-Set
CT-91-11 Lane A. Hemachandra, Edith Spaan — Quasi-Injective Reductions
CT-91-12 Krzysztof R. Apt, Dino Pedreschi — Reasoning about Termination of Prolog Programs

Computational Linguistics

CL-91-01 J.C. Scholtes — Kohonen Feature Maps in Natural Language Processing
CL-91-02 J.C. Scholtes — Neural Nets and their Relevance for Information Retrieval
CL-91-03 Hub Prüst, Remko Scha, Martin van den Berg — A Formal Discourse Grammar tackling Verb Phrase Anaphora

Other Prepublications

X-91-01 Alexander Chagrov, Michael Zakharyaschev — The Disjunction Property of Intermediate Propositional Logics
X-91-02 Alexander Chagrov, Michael Zakharyaschev — On the Undecidability of the Disjunction Property of Intermediate Propositional Logics
X-91-03 V. Yu. Shavrukov — Subalgebras of Diagonalizable Algebras of Theories containing Arithmetic
X-91-04 K.N. Ignatiev — Partial Conservativity and Modal Logics
X-91-05 Johan van Benthem — Temporal Logic
X-91-06 — Annual Report 1990
X-91-07 A.S. Troelstra — Lectures on Linear Logic, Errata and Supplement
X-91-08 Giorgie Dzhaparidze — Logic of Tolerance
X-91-09 L.D. Beklemishev — On Bimodal Provability Logics for Π_1-axiomatized Extensions of Arithmetical Theories
X-91-10 Michiel van Lambalgen — Independence, Randomness and the Axiom of Choice
X-91-11 Michael Zakharyaschev — Canonical Formulas for K4. Part I: Basic Results
X-91-12 Herman Hendriks — Flexibele Categoriale Syntaxis en Semantiek: de proefschriften van Frans Zwarts en Michael Moortgat
X-91-13 Max I. Kanovich — The Multiplicative Fragment of Linear Logic is NP-Complete
X-91-14 Max I. Kanovich — The Horn Fragment of Linear Logic is NP-Complete
X-91-15 V. Yu. Shavrukov — Subalgebras of Diagonalizable Algebras of Theories containing Arithmetic, revised version
X-91-16 V.G. Kanovei — Undecidable Hypotheses in Edward Nelson's Internal Set Theory
X-91-17 Michiel van Lambalgen — Independence, Randomness and the Axiom of Choice, Revised Version
X-91-18 Giovanna Cepparello — New Semantics for Predicate Modal Logic: an Analysis from a standard point of view
X-91-19 Papers presented at the Provability Interpretability Arithmetic Conference, 24-31 Aug. 1991, Dept. of Phil., Utrecht University

1992

Annual Report 1991

Logic, Semantics and Philosophy of Langauge

LP-92-01 Víctor Sánchez Valencia — Lambek Grammar: an Information-based Categorial Grammar
LP-92-02 Patrick Blackburn — Modal Logic and Attribute Value Structures
LP-92-03 Szabolcs Mikulás — The Completeness of the Lambek Calculus with respect to Relational Semantics
LP-92-04 Paul Dekker — An Update Semantics for Dynamic Predicate Logic
LP-92-05 David I. Beaver — The Kinematics of Presupposition
LP-92-06 Patrick Blackburn, Edith Spaan — A Modal Perspective on the Computational Complexity of Attribute Value Grammar
LP-92-07 Jeroen Groenendijk, Martin Stokhof — A Note on Interrogatives and Adverbs of Quantification
LP-92-08 Maarten de Rijke — A System of Dynamic Modal Logic
LP-92-09 Johan van Benthem — Quantifiers in the world of Types

Mathematical Logic and Foundations

ML-92-01 A.S. Troelstra — Comparing the theory of Representations and Constructive Mathematics
ML-92-02 Dmitrij P. Skvortsov, Valentin B. Shehtman — Maximal Kripke-type Semantics for Modal and Superintuitionistic Predicate Logics
ML-92-03 Zoran Marković — On the Structure of Kripke Models of Heyting Arithmetic
ML-92-04 Dimiter Vakarelov — A Modal Theory of Arrows, Arrow Logics I
ML-92-05 Domenico Zambella — Shavrukov's Theorem on the Subalgebras of Diagonalizable Algebras for Theories containing $I\Delta_0$ + EXP

Compution and Complexity Theory

CT-92-01 Erik de Haas, Peter van Emde Boas — Object Oriented Application Flow Graphs and their Semantics
CT-92-02 Karen L. Kwast, Sieger van Denneheuvel — Weak Equivalence: Theory and Applications

Other prepublications

X-92-01 Heinrich Wansing — The Logic of Information Structures
X-92-02 Konstantin N. Ignatiev — The Closed Fragment of Dzhaparidze's Polymodal Logic and the Logic of Σ_1-conservativity
X-92-03 Willem Groeneveld — Dynamic Semantics and Circular Propositions, revised version