Institute for Logic, Language and Computation

UNDECIDABILITY OF MODAL AND INTERMEDIATE FIRST-ORDER LOGICS WITH TWO INDIVIDUAL VARIABLES

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0 Introduction

The interest in fragments of predicate logics is motivated by the well-known fact that full classical predicate calculus is undecidable (Church 1936). So it is desirable to find decidable fragments which are in some sense "maximal", ie which become undecidable if they are "slightly" extended. Or, alternatively, we can look for "minimal" undecidable fragments and try to identify the vague boundary between decidability and undecidability. A great deal of work in this area concerning mainly classical logic has been done since the thirties. We will not give a complete review of decidability and undecidability results in classical logic, referring the reader to existing monographs (Suranyi 1959; Lewis 1979; Dreben, Goldfarb 1979); a short summary can also be found in the well-known book (Church 1956). Let us recall only several facts. Herein we will consider only logics without functional symbols, constants and equality.

(C1) The fragment of the classical logic with only monadic predicate letters is decidable (Behmann 1922).

(C2) The fragment of the classical logic with a single binary predicate letter is undecidable (this is a consequence of (Gödel 1933)).

(C3) The fragment of the classical logic with a single individual variable is decidable; in fact it is equivalent to Lewis' **S5** (Wajsberg 1933).

(C4) The fragment of the classical logic with two individual variables is decidable (Segerberg 1973 contains a proof using modal logic; Scott 1962 and Mortimer 1975 give traditional proofs).

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(C5) The fragment of theclassical logic with three individual variables and binary predicate letters is undecidable (Surańyi 1943).
In fact this paper considers formulas of the following type

$$\forall x \exists y \varphi(x, y) \supset \forall x \exists y \forall z \psi(x, y, z),$$

φ, ψ being quantifier-free, and the set of binary predicate letters which can appear in φ or ψ, being fixed and finite.

(C6) The fragment of the classical logic with three individual variables, a single binary and a (fixed) finite number of monadic predicate letters, is undecidable (Surańyi 1959).
In fact, this result deals only with formulas of the type

$$\forall x \forall y \exists z \varphi \wedge \forall x \forall y \forall z \psi \quad (\varphi, \psi \text{ being quantifier-free}).$$

(C7) The fragment of the classical logic with formulas of the type $\forall x \exists y \forall z$ $\varphi(x, y, z)$, with a quantifier-free φ, which can contain a single binary and monadic predicate letters from a (fixed) finite set, is undecidable (Kahr, Moore, Wang 1962; see also Lewis, 1979).

Now let us cite some results in non-classical predicate logic

(N1) The fragment of any sublogic of **QS5** (Quantified S5) which is conservative over the classical logic, with a single monadic predicate letter is undecidable (Kripke 1962).

(N2) The fragment of **QH** (intuitionistic predicate logic) with a single monadic letter is undecidable (Maslov, Mints, Orevkov 1965; Gabbay 1981; Kripke 1965 proves undecidability for a fragment with two monadic letters).

(N3) The fragment of **QH** with a single individual variable is decidable.
This result was proved syntactically in (Mints 1968) and semantically in (Ono 1977), (Fischer-Servi 1978).

(N4) The fragment of **QS4** with a single individual variable is decidable (Fischer-Servi 1978).

(N5) The fragment of **QS5** with a single individual variable is decidable.

This logic is easily embedded into Segerberg's 2-dimensional modal logic (Segerberg 1973), and the result comes as a consequence of (C4).

(N6) The fragment of any intermediate logic with three individual variables and binary predicate letters is undecidable.

This is a consequence of (C5) obtained by Gödel double-negation translation of classical formulas into intuitionisitic formulas. Of course (C6) and (C7) also can be extended to intermediate logics. As for modal predicate logics with three individual variables, their undecidability immediately follows from their conservativity over the classical predicate logic.

(N7) The intermediate logic of "constant domains" with a single individual variable is decidable (Ono 1977).

(N8) One-variable fragments of quantified modal logics with constant domains were considered in Shehtman 1987 (decidability results are stated for quantified **K**, **S4**, **T** and some other modal systems).

(N9) The one-variable fragment of quantified provability logic for Peano arithmetic is decidable (Artemov, Dzhaparidze, 1990), whereas the full logic itself is not recursively enumerable (Vardanyan, 1986).

Summing up, we see that non-classical predicate logics with one individual variable are usually decidable, whereas three variables usually bring undecidability. So the case of two-variable logics is the most interesting, and it is analyzed in the present paper. We show that undecidability is also very frequent here. Informally, the basic idea underlying the proofs is in viewing a 3-variable classical model as a "3-dimensional cube", while a 2-variable Kripke model is a family of "2-dimensional cubes" parameterized by possible worlds, ie it is also a "3-dimensional structure". That is why sometimes we can embed a 3-cube into a 2-variable Kripke model.

1 Embedding QC-3 into QS5-2

Let us fix a classical first-order language L_0 containing:

- a single dyadic predicate letter R;
- a finite set of monadic predicate letters: $M_k (k \leq K)$;
- three individual variables: x, y, z;
- propositional connectives: $\vee, \neg$;
- the quantifier $\exists$.

Let L_1 be a modal first-order language containing:

1. monadic predicate letters: P_1, P_2 , Q;
2. two individual variables: x, y;
3. propositional letters: $P_3, N_k (k \leq K)$;
4. propositional connectives: $\vee, \neg$;
5. the quantifier: $\exists$.
6. modal connective: $\Box$

Other logical symbols ($\wedge, \equiv, \supset, \forall$, etc) will be used as standard abbreviations.

Let also L_{1C} be the corresponding classical language (i.e. L_1 without $\Box$). We define a translation $\varphi \mapsto \varphi'$ of L_0-formulas to L_1-formulas according to the following inductive definition:

1. $R(z,t)' = P_1(t), R(t,z)' = P_2(t)$ if $t \in \{x,y\}$;
2. $R(z,z)' = P_3$;
3. $R(t,r)' = \Box(Q(t) \supset P_1(r))$ if $t, r \in \{x,y\}$;
4. $M_k(t)' = \Box(Q(t) \supset N_k)$ if $t \in \{x,y\}$
5. $M_k(z)' = N_k$;
6. $(\neg\varphi)' = \neg\varphi'$;
7. $(\varphi \vee \psi)' = \varphi' \vee \psi'$;
8. $(\exists t\varphi)' = \exists t\varphi'$ if $t \in \{x,y\}$;
9. $(\exists z\varphi)' = \Diamond\varphi'$.

We will use some particular L_1-formulas

$$\begin{array}{ll} \alpha_1 & = \forall x \Diamond Q(x); \\ \alpha_{2k} & = \forall x \forall y (\Box(Q(x) \supset P_1(y)) \equiv \Box(Q(y) \supset P_2(x))); \\ \alpha_{3k} & = \forall x (\Box(Q(x) \supset P_1(x)) \equiv \Box(Q(x) \supset P_3)); \\ \alpha_{4,E} & = \forall x \forall y (\Diamond(Q(x) \wedge E) \supset \Box(Q(x) \supset E)) \text{ (for any atomic } E). \end{array}$$

Let α be the conjunction of all these formulas.

A **QS5-frame** is a pair (W, D) of two non-empty sets (elements of W are *worlds*, and those of D are *individuals*). A (Kripke) L_1-*model* over (W, D) is a family of classical L_{1C}-models in the domain D parameterized by $W : M = (M_w)_{w \in W}$.

Let φ be an L_0-formula, $a, b, c \in D$. Sometimes φ will be written as $\varphi(x,y,z)$, then $\varphi(a,b,c)$ denotes the result of substituting free occurrences of x, y, z in φ by a, b, c respectively; also the notations $\varphi(a,y,z), \varphi(a,y,c)$ etc will be used. $\varphi(a,b,c)$ is called a *D-valued formula.*

The same agreements concern L_1-formulas.

$M, s \models A(a,b)$ denotes that the D-valued L_1-formula $A(a,b)$ is true in the world w of L_1-model M, while $\mu \models \varphi(a,b,c)$ denotes that the D-valued L_0-formula $\varphi(a,b,c)$ is true in a classical model μ. $w \models A(a,b)$ is used instead of $M, w \models A(a,b)$ if M can be easily restored from context. $M \models A(a,b)$ denotes that $A(a,b)$ is true in every world of M.

Lemma 1.1 *Let μ be a classical L_0 model with the domain D , M be an L_1-model over $(W, D), u \in W$ so that for any $a, b \in D, k \leq K$:*

(i) $M, u \models \alpha$;

(ii) $M, u \models \Box(Q(a) \supset P_1(b)) \Leftrightarrow \mu \models R(a,b)$;

(iii) $M, u \models \Box(Q(a) \supset N_k) \Leftrightarrow \mu \models M_k(a)$.

Then

1. $M, u \models \Diamond(Q(c) \wedge A(a,b)) \equiv \Box(Q(c) \supset A(a,b))$ *for any* $a, b, c \in D$ *and any* L_1*-formula* $A(x,y)$;
2. $\mu \models \varphi(a,b,c) \Leftrightarrow M, u \models \Diamond(Q(c) \wedge \varphi'(a,b))$ *for any* $a, b, c \in D$ *and any* L_0*-formula* $\varphi(x,y,z)$.

Proof.

1. It is easily seen that $u \models \Box(Q(c) \supset A(a,b)) \supset \Diamond(Q(c) \wedge A(a,b))$. Indeed, from (i) we obtain $u \models \alpha_1$, and hence $u \models \Diamond Q(c)$. Then we notice that a formula

$$\Diamond X \supset (\Box(X \supset Y) \supset \Diamond(X \wedge Y))$$

is true in every Kripke model and apply modus ponens in the world u. The converse

(*) $u \models \Diamond(Q(c) \wedge A(a,b)) \supset \Box(Q(c) \supset A(a,b))$ is proved inductively.
If $A = P_i(x)$ (or $P_i(y)$), $i = 1,2$ then $A(a,b) = P_i(a)$ (respectively $P_i(b)$), and (*) follows from $\alpha_{4,P_i(y)}$.
If A is a propositional letter then $A(a,b) = A$, and we apply $\alpha_{4,A}$.
If $A = A_1 \vee A_2$, with A_1, A_2 satisfying (*), we have:

$$u \models (\Diamond(Q(c) \wedge A_1(a,b)) \vee \Diamond(Q(c) \wedge A_2(a,b))) \supset$$
$$(\Box(Q(c) \supset A_1(a,b)) \vee \Box(Q(c) \supset A_2(a,b))).$$

Using laws of minimal modal logic **K** we obtain (*) for A.
If $A = \neg B$, with B satisfying (*):

$$u \models \Diamond(Q(c) \wedge B(a,b)) \supset \Box(Q(c) \supset B(a,b)),$$

then by a contraposition we easily obtain (*) for A.
If $A = \exists x B(x,y)$ then $A(a,b) = \exists x B(x,b)$. Assume that $u \models \Diamond(Q(c) \wedge \exists x B(x,b))$, and let us show that $u \models \Box(Q(c) \supset \exists x B(x,b))$ whenever B satisfies (*), ie that $v \models Q(c) \supset \exists x B(x,b)$ for any $v \in W$.
By our assumption, $u \models \Diamond(Q(c) \wedge B(a,b))$ for some $a \in D$ and so $u \models \Box(Q(c) \supset B(a,b))$ by (*). Hence $v \models Q(x) \supset B(a,b), v \models Q(c) \supset \exists x B(x,b)$.
The case $A = \exists y B(x,y)$ is proved similarly.
If $A = \Box B(x,y)$ then (*) holds in any case. For, the following formulas are valid in every **QS5**-frame:

$$\Diamond(Q(c) \wedge \Box B(a,b)) \supset \Diamond\Box B(a,b),$$
$$\Diamond\Box B(a,b) \supset \Box B(a,b),$$
$$\Box B(a,b) \supset \Box(Q(c) \supset \Box B(a,b)).$$

2. By an induction over the construction of φ.
If $\varphi = R(z,x)$ then $\varphi' = P_1(x), \varphi(a,b,c) = R(c,a), \varphi'(a,b) = P_1(a)$. By the condition (ii),

$$\mu \models R(c,a) \Leftrightarrow M,u \models \Box(Q(c) \supset P_1(a)),$$

and

$$M,u \models \Box(Q(c) \supset P_1(a)) \Leftrightarrow M,u \models \Diamond(Q(c) \wedge P_1(a)),$$

by (1).
If $\varphi = R(x,z)$ then $\varphi' = P_2(x)$, and the proof is almost the same.
Now the cases: $\varphi = R_k(z,y), \varphi = R_k(y,z)$ must also be clear.
If $\varphi = R(z,z)$ then $\varphi' = P_3, \varphi(a,b,c) = R(c,c), \varphi'(a,b) = P_3$.
We have:

$$\mu \models R(c,c) \Leftrightarrow M,u \models \Box(Q(c) \supset P_1(c)) \Leftrightarrow M,u \models \Box(Q(c) \supset P_3)$$
$$\Leftrightarrow M,u \models \Diamond(Q(c) \wedge P_3),$$

by (ii), and (1).
If $\varphi = R(x,y)$ then $\varphi' = \Box(Q(x) \supset P_1(y)), \varphi(a,b,c) = R(a,b), \varphi'(a,b) = \Box(Q(a) \supset P_1(b))$. By (ii) we have:

$$\mu \models \varphi(a,b,c) \Leftrightarrow M,u \models \Box(Q(a) \supset P_1(b)).$$

On the other hand,

$$M,u \models \Diamond Q(c) \text{ (by } \alpha_1\text{)},$$
$$M,u \models \Diamond Q(c) \supset (\Box(Q(a) \supset P_1(b)) \equiv \Diamond(Q(c) \wedge \Box(Q(a) \supset P_1(b))))$$

according to the laws of **S5**. Thus φ satisfies (2).
Similar reasonings can be used for $\varphi = R(x,x), R(y,x), R(y,y), M_k(x), M_k(y)$.
If $\varphi = M_k(z)$ then $\varphi'(a,b) = N_k, \varphi(a,b,c) = M_k(c)$, and we have:

$$\mu \models M_k(c) \Leftrightarrow M,u \models \Box(Q(c) \supset N_k)$$

according to (iii).
If $\varphi = \psi \vee \zeta$ and ψ, ζ satisfy (2) then

$$\mu \models \varphi(a,b,c) \Leftrightarrow \mu \models \psi(a,b,c) \vee \mu \models \zeta(a,b,c) \Leftrightarrow$$
$$\Leftrightarrow M,u, \models \Diamond(Q(c) \wedge \psi'(a,b)) \vee M,u \models \Diamond(Q(c) \wedge \zeta'(a,b)) \Leftrightarrow$$
$$\Leftrightarrow M,u \models \Diamond(Q(c) \wedge \psi'(a,b)) \vee \Diamond(Q(c) \wedge \zeta'(a,b)).$$

The latter formula is obviously equivalent in **QS5** to $\Diamond(Q(c) \wedge \varphi'(a,b))$.
If $\varphi = \neg\psi$ and ψ satisfies (2) then

$$\mu \models \varphi(a,b,c) \Leftrightarrow \mu \not\models \psi(a,b,c) \Leftrightarrow M,u \models \neg\Diamond(Q(c) \wedge \psi'(a,b)) \Leftrightarrow$$
$$M,u \models \Box(Q(c) \supset \varphi'(a,b)).$$

But the latter formula is equivalent in u to $\Diamond(Q(c) \wedge \varphi'(a,b))$, due to (1). If $\varphi = \exists x\psi(x,y,z)$, then $\varphi(a,b,c) = \exists x\psi(x,b,c), \varphi'(a,b) = \exists x\psi'(x,b)$, $\mu \models \varphi(a,b,c) \Leftrightarrow$ for some d $\mu \models \psi(d,b,c) \Leftrightarrow$ for some d $M,u \models \Diamond(Q(c) \wedge \psi'(d,b))$ (if ψ satisfies (2)) $\Leftrightarrow M,u \models \Diamond(Q(c) \wedge \exists x\psi'(x,b))$ (the latter equivalence holds in any Kripke model over a **QS5**-frame).
The case $\varphi = \exists y\psi(x,y,z)$ is considered analogously.
If $\varphi = \exists z\psi(x,y,z)$ then $\varphi(a,b,c) = \exists z\psi(a,b,z), \varphi'(a,b) = \Diamond\psi'(a,b)$. Supposing ψ to satisfy (2) we have:

$$\mu \models \varphi(a,b,c) \Leftrightarrow \text{ for some } d, \mu \models \psi(a,b,d) \Leftrightarrow$$
$$\text{for some } d, M,u \models \Diamond(Q(d) \wedge \psi'(a,b)).$$

Hence $\mu \models \varphi(a,b,c) \Rightarrow M,u \models \Diamond\psi'(a,b) \Rightarrow M,u \models \Box\Diamond\psi'(a,b)$ (because M is an **S5**-model). But $M,u \models \Diamond Q(c)$ due to α_1; therefore $\mu \models \varphi(a,b,c) \Rightarrow M,u \models \Diamond(Q(c) \wedge \Diamond\psi'(a,b))$. The converse implication follows immediately from the above equivalence.

■

Lemma 1.2 *Let φ be an L_0-formula without free occurrences of z. Then* **QS5** $\vdash \Box\varphi' \equiv \varphi', \Diamond\varphi' \equiv \varphi'$.

Proof. The first equivalence is easily proved by an induction, using the following **QS5**-theorems:

$$\Box(\Box A \vee \Box B) \equiv \Box A \vee \Box B, \Box\neg\Box A \equiv \neg\Box A, \Box\exists t\Box A \equiv \exists t\Box A.$$

Then we have: $\Box(\neg\varphi') \equiv (\neg\varphi)'$, and thus $\Diamond\varphi' \equiv \varphi'$. ■

Theorem 1.3 *For any closed L_0-formula φ,*

$$\models \varphi \Leftrightarrow \textbf{QS5-2} \vdash \alpha \supset \varphi'.$$

($\models$ means classical validity, **QS5-2** *is the L_1-fragment of the pure quantificational version of* **S5**. [1]*)*

Proof. [$\Rightarrow$] Assume that **QS5-2** $\not\vdash \alpha \supset \varphi'$. By Kripke's completeness theorem (Kripke 1959), there exists a **QS5**-frame (W,D) and an L_1-model M over (W,D) such that

$$M,u \models \alpha \wedge \neg\varphi'$$

for some $u \in W$. Then $M,u \models \Box\neg\varphi'$, by Lemma 1.2. On the other hand, $M,u \models \alpha_1$ implies $M,u \models \Diamond Q(c)$ for any c. Therefore

$$M,u \models \Diamond(Q(c) \wedge \neg\varphi')$$

[1] We leave aside a question of how to axiomatise **QS5-2**; eg is it true that we can take just standard axioms and rules of **QS5** and restrict them to the language L_1?

and $\varphi' = \varphi'(a, b)$ for any $a, b \in D$ since φ (and φ') is closed.

Let μ be a classical L_0-model with the domain D satisfying (ii) (such a model obviously exists). By Lemma 1.1, we obtain

$$\mu \models \neg\varphi(a, b, c)(= \neg\varphi),$$

and thus $\not\models \varphi$.

[$\Leftarrow$] Assume that $\not\models \varphi$; then $\mu \not\models \varphi$ for some L_0-model μ. Let D be the domain of μ, and consider the **QS5**-frame (D, D). We can define an L_1-model M over (D, D) such that for any $a, b \in D$, $k \leq K$:

$$\begin{aligned}
&M, a \models Q(b) \Leftrightarrow b = a;\\
&M, a \models P_3 \Leftrightarrow \mu \models R(a, a);\\
&M, a \models P_1(b) \Leftrightarrow \mu \models R(a, b);\\
&M, a \models P_2(b) \Leftrightarrow \mu \models R(b, a);\\
&M, a \models N_k \Leftrightarrow \mu \models M_k(a).
\end{aligned}$$

Let us show that $M, u \models \alpha$ for any $u \in D$:

1. Since $b \models Q(b)$ for any $b \in D$, we have $u \models \alpha_1$.
2. For any $b, c \in D$:

   $$\begin{aligned}
   &u \models \Box(Q(b) \supset P_1(c)) \Leftrightarrow b \models P_1(c) \Leftrightarrow \mu \models R(b, c) \Leftrightarrow\\
   &c \models P_2(b) \Leftrightarrow u \models \Box(Q(c) \supset P_2(b)).
   \end{aligned}$$

   Hence $u \models \alpha_2$.
3. For any $b \in D$:

   $$\begin{aligned}
   &u \models \Box(Q(b) \supset P_1(b)) \Leftrightarrow b \models P_1(b) \Leftrightarrow \mu \models R(b, b) \Leftrightarrow\\
   &b \models P_3 \Leftrightarrow u \models \Box(Q(b) \supset P_3),
   \end{aligned}$$

   and so $u \models \alpha_3$.
4. For any $b, c \in D$:

   $$u \models \Diamond(Q(b) \wedge P_i(c)) \Leftrightarrow b \models P_i(c) \Leftrightarrow a \models \Box(Q(b) \supset P_i(c)),$$

   and thus $u \models \alpha_{4P_i(y)}$.
   If E is a propositional letter then $u \models \Diamond(Q(b) \wedge E) \Leftrightarrow b \models E \Leftrightarrow u \models \Box(Q(b) \supset E)$, and thus $u \models \alpha_{4,E}$ in this case as well.
   The conditions (ii), (iii) from Lemma 1.1 are satisfied by μ, M and any $u \in D$:

   $$u \models \Box(Q(a) \supset P_1(b)) \Leftrightarrow a \models P_1(b) \Leftrightarrow \mu \models R(a, b),$$

   $$u \models \Box(Q(a) \supset N_k) \Leftrightarrow a \models N_k \Leftrightarrow \mu \models M_k(a),$$

   and thus by Lemma 1.1, $\mu \models \neg\varphi(= \neg\varphi(a, b, c)$ with a, b, c arbitrary) implies

   $$M, u \models \Diamond(Q(c) \wedge \neg\varphi'(a, b))$$

   (and $\varphi'(a, b) = \varphi'$); hence $M, u \models \Diamond\neg\varphi'$, and $M, u \models \neg\varphi'$, by Lemma 1.2. Consequently $M, u \not\models \alpha \supset \varphi'$, and thus **QS5-2** $\not\models \alpha \supset \varphi'$.

■

Corollary 1.4 **QS5-2** *is undecidable.*

Proof. The fragment **QC-3** of classical predicate calculus in the language L_0 is undecidable (if K is sufficiently large) as we have noted in the introduction (Surańyi 1943). Theorem 1.3, (together with Gödel completeness theorem) shows that **QC-3** is reducible to **QS5-2**. ■

QS5-2 can be also defined in purely classical terms. Viz, consider the classical first-order language $L_1^\wedge$ containing

1. binary predicate letters: Q, P_1, P_2;
2. monadic predicate letters: $P_3 N_k (k \leq K)$;
3. individual variables: x, y, z;
4. propositional connectives: $\vee, \neg$;
5. the quantifier $\exists$.

Atomic formulas allowed in $L_1^\wedge$ are only of the types: $S(x,z), S(y,z)$ (S being a binary predicate letter), $N_k(z), P_3(z)$. Non-atomic formulas are built by standard rules.

There is an evident translation $A \mapsto A^\wedge$ of L_1 formulas to $L_1^\wedge$-formulas.

$$\begin{array}{l} S(x)^\wedge = S(x,z), S(y)^\wedge = S(y,z) \quad (\text{if } S = Q, P_1, P_2); \\ P_3^\wedge = P_3(z); N_k^\wedge = N_k(z); \\ (A \vee B))^\wedge = A^\wedge \vee B^\wedge; (\neg A)^\wedge = \neg A^\wedge; \\ (\exists x A)^\wedge = \exists x A^\wedge; (\exists y)^\wedge = \exists y A^\wedge; \\ (\Box A)^\wedge = \forall z A^\wedge. \end{array}$$

It is clear that every $L_1^\wedge$-formula is equivalent (classically) to some $A^\wedge$.

Lemma 1.5 *Let μ be a classical $L_1^\wedge$-model with the domain D, and let M be an L_1-model over (D,D) such that*

(**) $\quad M, c \vDash A(a,b) \Leftrightarrow \mu \vDash A^\wedge(a,b,c)$ *for any $a,b,c \in D$ and any atomic A.*

Then (**) is true for any formula A.

Proof. The proof is inductive and quite easy. ■

Corollary 1.6 *Classical validity of $L_1^\wedge$-formulas is undecidable.*

Proof. By lemma 1.5, for any closed L_1-formula A,

$$(D,D) \vDash A \Leftrightarrow D \vDash A^\wedge$$

(ie validity of A in a **QS5**-frame (D,D) is equivalent to classical validity of $A^\wedge$ in D).

But the set of all L_1-formulas which are valid in every (D,D) is undecidable (cf the proof of Theorem 1.3). Hence the result follows. ■

Remark

In fact **QS5** is complete wrt the frames of the type (D, D). This follows from Kripke completeness theorem (Kripke 1959) and the following observation (which is an analogue of "p-morphism lemma").

Let $(W, D), (W', D')$ be **QS5**-frames, $f : W \to W', g : D \to D'$ be onto maps. Let M, M' be Kripke models over $(W, D), (W', D')$ respectively, such that

$$(***) \qquad M, u \models A(a_1, \dots, a_n) \Leftrightarrow M', f(u) \models A(g(a_1), \dots g(a_n))$$

for any atomic (n-place) formula A. Then $(***)$ holds for any formula $A(x_1, \dots, x_n)$. This claim is proved inductively, and it implies that any formula refutable in some (W', D') is refutable in some (D, D).

To obtain our further undecidability results, we have to specify Theorem 1.3 for $\forall\exists\forall$-formulas. Call an L_1-formula *simple* if it has no occurrences of $\Box$ and quantifiers. A formula of the form $\forall x \exists y (\bigvee_{i=1}^{n} \Box A_i)$, with $A_1, \dots, A_n$ simple, is called *quasisimple.*

Lemma 1.7 *Let φ be a closed L_0-formula of $\forall\exists\forall$-type. Then there is a quasisimple L_1-formula φ^* such that*

$$\mathbf{QS5} \vdash \alpha \supset (\varphi' \equiv \varphi^*),$$

and this φ^* can be found effectively.

Proof. φ is equivalent to $\forall x \exists y \forall z \psi$, with ψ in a disjunctive normal form:

$$\psi = \bigvee_a (\bigwedge_b p_{ab} \wedge \bigwedge_c \neg q_{ac})$$

(p_{ab}, q_{ac} are atomic). Then

$$\mathbf{QS5} \vdash \psi' \equiv \bigvee_a (\bigwedge_b p'_{ab} \wedge \bigwedge_c \neg q'_{ac}).$$

Some of p'_{ab}, q'_{ac} are classical, and we combine them together in every disjunct. All the others are of the form $R(t, r)' = \Box(Q(t) \supset P_1(r))$ or $M_k(t)' = \Box(Q(t) \supset N_k)$, with $t, r \in \{x, y\}$. Thus, we have:

$$\mathbf{QS5} \vdash \psi' \equiv \bigvee_a (X_a \wedge \bigwedge_d \Box Y_{ad} \wedge \bigwedge_e \neg\Box Z_{ae}),$$

for some simple X_a, Y_{ad}, Z_{ae}.

But (cf. Lemma 1.1)

$$
\begin{array}{ll}
\mathbf{QS5} \vdash & \alpha \supset .\Box(Q(t) \supset P_1(r)) \equiv \Diamond(Q(t) \wedge P_1(r)), \\
& \alpha \supset .\Box(Q(t) \supset N_k) \equiv \Diamond(Q(t) \wedge N_k),
\end{array}
$$

and hence

$$
\begin{array}{ll}
\mathbf{QS5} \vdash & \alpha \supset .\neg\Box(Q(t) \supset P_1(r)) \equiv \Box(Q(t) \supset \neg P_1(r)), \\
& \alpha \supset .\neg\Box(Q(t) \supset N_k) \equiv \Box(Q(t) \supset \neg N_k).
\end{array}
$$

Therefore, assuming α in **QS5** we obtain:

$$
\begin{array}{lll}
\psi' & \equiv \bigvee_a (X_a \wedge \bigwedge_f \Box W_{af}) & \text{(for some simple } W_{af}) \\
& \equiv \bigvee_a (X_a \wedge \Box W_a) & (\text{if } W_a = \bigwedge_f W_{af}) \\
& \equiv \bigwedge_g (U_g \vee \bigvee_h \Box V_{gh}) & \text{(for some simple } U_g, V_{gh}) \\
\Box\psi' & \equiv \bigwedge_g \Box(U_g \vee \bigvee_h \Box V_{gh}) & \\
& \equiv \bigwedge_g (\Box U_g \vee \bigvee_h \Box V_{gh}) & \text{(according to } \mathbf{QS5}) \\
& \equiv \bigvee_i \Box A_i & \text{(for some simple } A_i),
\end{array}
$$

and we can take $\varphi^* = \forall x \exists y (\bigvee_i \Box A_i)$. ■

Theorem 1.8 *The set* $\{A \mid A$ *is quasisimple,* $\mathbf{QS5} \vdash \alpha \supset A\}$ *is undecidable.*

Proof. By Kahr-Moore-Wang's theorem (Kahr, Moore, Wang 1962), the classical validity of L_0-formulas of $\forall\exists\forall$-type (for sufficiently large K) is undecidable. By Lemma 1.7 and Theorem 1.3,

$$\models \varphi \Leftrightarrow \mathbf{QS5\text{-}2} \vdash \alpha \supset \varphi^*$$

for any formula φ of this kind. Hence the result follows. ■

2 Undecidable Intermediate Logics

Now let us consider the intuitionistic first-order language L_2 having the same predicate and propositional letters as L_1, plus a new propositional letter S.

L_2 also has two individual variables: x, y. Intuitionistic propositional connectives are denoted thus: $\neg, \vee, \wedge, \rightarrow$; quantifiers: $\forall, \exists$.

Every quasisimple L_1-formula

$$A = \forall x \exists y \bigvee_i \Box A_i$$

is translated to L_2:

$$A^- = \exists x \forall y \bigwedge_i ((S \rightarrow A_i) \rightarrow S).$$

The following intuitionistic formulas will serve for the same purpose as α_i in Section 1:

$X_{0,E} = \forall x \forall y(S \rightarrow .E \vee \neg E)$ (for any atomic E)
$X_1 = \exists x(Q(x) \vee \neg Q(x)) \rightarrow S$,
$X_2 = \forall x \forall y((S \vee (Q(x) \rightarrow P_1(y))) \leftrightarrow (S \vee (Q(y) \rightarrow P_2(x))))$,
$X_3 = \forall x((S \vee (Q(x) \rightarrow P_1(x))) \leftrightarrow (S \vee (Q(x) \rightarrow P_3)))$,
$X_{4,E} = \forall x \forall y((Q(x) \rightarrow E) \vee (Q(x) \rightarrow \neg E))$ (for any atomic E).
Let X be the conjunction of all these formulas.

Recall that an *intuitionistic Kripke frame* is a triple $(W, \leq, D)$ in which $(W, \leq)$ is a poset, D is a function sending elements of W to non-empty sets, such that $D(u) \subseteq D(v)$ whenever $u \leq v$. An *intuitionistic Kripke model* (or an *L_2-model* in our case) over $(W, \leq D)$ is obtained by assigning a truth value in every world v to every $D(v)$-valued atomic formula, so that a formula which is true in some v, is true in any $u \geq v$. As usual, $M, w \models A(a_1, a_2)$ (or sometimes $w \models A(a_1, a_2)$) denotes that a $D(w)$-valued formula $A(a_1, a_2)$ is true in a world w of a model M; we suppose the reader to be familiar with the corresponding definition. (In particular:

$$w \models \forall x A(x, a) \Leftrightarrow \forall u \geq w \forall d \in D(u) \;\; u \models A(d, a);$$
$$w \models \exists x A(x, a) \Leftrightarrow \exists d \in D(w) \;\; w \models A(d, a).)$$

An intuitionistic formula A is *true* in M if its universal closure is true in every world of M; A is *valid* in a frame F (notation: $F \models A$) if A is true in every Kripke model over F. $\mathbf{L}(F) = \{A \mid F \models A\}$ is called the *logic of the Kripke frame* F, $\mathbf{L}(\mathbf{F}) - \mathbf{2}$ is its fragment in our language L_2. **QH-2** is the L_2-fragment of the intuitionistic predicate logic.

An L_2-formula is called *simple* if it is a simple L_1-formula (ie. if it is built from atomic formulas using only $\vee$ and $\neg$).

Lemma 2.1 *Let M_2 be an L_2-model over $(W, \leq, D)$ such that $M_2, u \models X_{0,E}$ for any atomic E. Then $M_2, u \models S \rightarrow .A \vee \neg A$ for any simple $D(u)$-valued formula A.*

Proof. By an induction over the length of A. The base is provided by $X_{0,E}$.

If $A = B \vee C$ and $u \models S \rightarrow .B \vee \neg B, S \rightarrow .C \vee \neg C$ then $u \models S \rightarrow (B \vee \neg B) \wedge (C \vee \neg C)$. But $(B \vee \neg B) \wedge (C \vee \neg C)$ implies $(B \vee C) \vee (\neg B \wedge \neg C)$ in the intuitionistic logic, and the latter formula is equivalent to $(A \vee \neg A)$.

If $A = \neg B$ and $u \models S \rightarrow .B \vee \neg B$, then obviously $u \models S \rightarrow .\neg A \vee A$ (since B implies $\neg A$). ∎

Lemma 2.2 *Let M_2 be an L_2-model over $(U, \leq, D)$, A be a quasisimple L_1-formula, $u \in U, M_2, u \models X \wedge A^-, M_2, u \not\models S$. Let also $D_0 = D(u), W = \{v \mid u \leq v \& M_2, v \models S\}$.*

Consider an L_1-model M_1 over (W, D_0) such that

$$(*) \qquad M_1, v \models B \Leftrightarrow M_2, v \models B$$

for any $v \in W$ and any atomic D_0-valued B.

Then $M_1, v \models \alpha \wedge \neg A$ for any $v \in W$.

Proof. At first we observe that (*) holds also for any simple D_0-valued B. This is proved by an induction; the only non-trivial case is: $B = \neg C$. So suppose C satisfies (*). Then

$$M_1, v \models B \Leftrightarrow M_2, v \not\models C.$$

But $M_2, v \models C \vee \neg C$ for any $v \in W$, due to Lemma 2.1. Hence

$$M_2, v \not\models C \Leftrightarrow M_2, \models \neg C \quad (= B),$$

and therefore B satisfies (*).

If $A = \forall x \exists y \bigvee_i \Box A_i$, then by the definition of A^- we have:

$$M_2, u \models \exists x \forall y \bigwedge_i ((S \to A_i) \to S),$$

and hence

$$\exists a \in D_0 \forall b \in D_0 M_2, u \models \bigwedge_i ((S \to A_i(a,b)) \to S);$$
$$\exists a \in D_0 \forall b \in D_0 \forall i M_2, u \not\models S \to A_i(a,b)$$

since $M_2, u \not\models S$. But then

$$\exists a \in D_0 \forall b \in D_0 \forall i \exists v \in W M_2, v \not\models A_i(a,b).$$

Now applying (*) to $A_i(a,b)$ we obtain:

$$\exists a \in D_0 \forall b \in D_0 \forall_i \exists v \in W M_1, v \models \neg A_i(a,b),$$

which is equivalent to

$$M_1, v \models \exists x \forall y \bigwedge_i \Diamond \neg A_i$$

(for any $v \in W$). Thus $M_1, v \models \neg A$.

Let us show that $M_1, v \models \alpha_1$. This is equivalent to

$$\forall a \in D_0 \exists w \in W M_2, w \models Q(a).$$

But the latter follows from the conditions of 2.2. Indeed, $M_2, u \models X_1$ implies $M_2, u \models \neg Q(a) \to S$, and thus $M_2, u \not\models \neg Q(a)$ (since $M_2, u \not\models S$). Then $M_2, w \models Q(a)$ for some $w \geq u$. But $M_2, u \models X_1$ also implies $M_2, u \models Q(a) \to S$, and thus $M_2, w \models S$ i.e. $w \in W$.

Now consider α_2.

$$M_1, v \models \alpha_2 \Leftrightarrow \forall a, b \in D_0 (M_1, v \models \Box(Q(a) \supset P_1(b)) \Leftrightarrow$$
$$\Leftrightarrow M_1, v \models \Box(Q(b) \supset P_2(a))).$$

Both directions in the last '⇔' are proved analogously, and let us show '⇒'. Suppose $M_1, v \models \Box(Q(a) \supset P_1(b))$. Then $M_2, u \models Q(a) \to P_1(b)$; indeed, $M_2, w \models Q(a) \& w \geq u$ implies $w \in W$ (as we have already seen), and hence $M_1, w \models Q(a)$; $M_1, w \models P_1(b); M_2, w \models P_1(b)$.

From $M_2, u \models X_2$ we have:

$$M_2, u \models Q(a) \to P_1(b). \to .S \vee (Q(b) \to P_2(a)),$$

and therefore (since $M_2, u \not\models S$):

$$M_2, u \models Q(b) \to P_2(a).$$

Now $M_1, v \models \Box(Q(b) \supset P_2(a))$ follows easily:

$$M_1, w \models Q(b) \Rightarrow M_2, w \models Q(b) \Rightarrow M_2, w \models P(a) \Rightarrow M_1, w \models P_2(a).$$

$M_1, v \models \alpha_3$ is proved likewise, using X_3.

Let us consider $\alpha_{4,P_i(y)}$. Suppose $a, b \in D_0$,

$$M_1, w \models Q(a) \wedge P_i(b)$$

for some $w \in W$. Then

$$M_2, w \models Q(a) \wedge P_i(b),$$

and thus

$$M_2, u \not\models Q(a) \to \neg P_i(b).$$

But $M_2, u \models X_{4,P_i(y)}$ yields:

$$M_2, u \models (Q(a) \to \neg P_i(b)) \vee (Q(a) \to P_i(b)),$$

and hence

$$M_2, u \models Q(a) \to P_i(b).$$

As we have seen before, this implies

$$M_1, v \models \Box(Q(a) \supset P_i(b)).$$

Therefore $M_1, v \models \alpha_{4,P_i(y)}$.

All the other formulas $\alpha_{4,E}$ are checked in the same way. ■

Now consider the intuitionistic Kripke frame $F_2 = (W_2, \preccurlyeq, D)$, with a constant domain $\mathbb{N}$ (the set of all positive integers), in which $\phi_2 = (W_2, \preccurlyeq)$ is the ω-branched tree of height 2, see Figure 1:

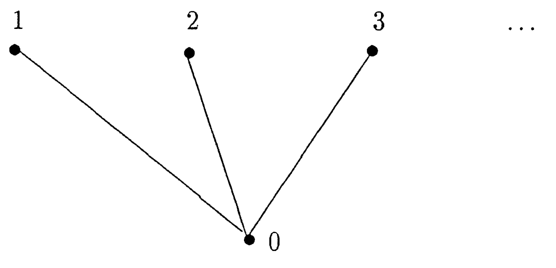


Figure 1

Lemma 2.3 *Let A be a quasisimple L_1-formula, M_1 be a L_1-model over* $(\mathbb{N}, \mathbb{N})$ *such that $M_1, v \models \alpha \wedge \neg A$ for some v. Consider an L_2-model M_2 over F_2 such that*

$$M_2, w \models E \Leftrightarrow w \neq 0 \& M_1, w \models E$$

for any atomic $\mathbb{N}$-valued $E, E \neq S$, and

$$M_2, w \models S \Leftrightarrow w \neq 0.$$

Then $M_2, 0 \models X \wedge A^-$.

Proof. At first we observe that

$$(**) \qquad M_2, w \models B \Leftrightarrow M_1, w \models B$$

for any simple $\mathbb{N}$-valued B and any $w \neq 0$. This is proved like (*) in the previous lemma. (Note that $M_2, w \models C \vee \neg C$ since w is maximal in F_2.)

Let $A = \forall x \exists y \bigvee_i \Box A_i$. Then $M_1, v \models \neg A$ implies

$$\exists a \in \mathbb{N} \forall b \in \mathbb{N} \forall i \exists w \in \mathbb{N} M_1, w \models \neg A_i(a, b),$$

and from (**) we have;

$$\exists a \in \mathbb{N} \forall b \in \mathbb{N} \forall i M_2, 0 \not\models S \rightarrow A_i(a, b).$$

Since $M_2, w \models S$ for any $w \neq 0$, we obtain:

$$\exists a \in \mathbb{N} \forall b \in \mathbb{N} \forall i M_2, 0 \models (S \rightarrow A_i(a, b)) \rightarrow S.$$

Therefore

$$M_2, w \models \exists x \forall y \bigwedge_i (S \rightarrow A_i(a, b). \rightarrow S) \quad (= A^-).$$

Now consider $X_{0,E}$. We have to show

$$M_2, 0 \models S \rightarrow .E(n) \vee \neg E(n)$$

for any $n \in \mathbb{N}$, that is

$$M_2, w \models E(n) \vee \neg E(n)$$

for any $w \neq 0$. But this follows from the maximality of w in ϕ_2.

The formula X_1 is equivalent to

$$\forall x (Q(x) \rightarrow S) \wedge \forall x (\neg Q(x) \rightarrow S).$$

$0 \models \forall x (Q(x) \rightarrow S)$ is obvious from the definition. On the other hand, $M_1, v \models \alpha_1$, and thus

$$\forall a \in \mathbb{N} \exists w \in \mathbb{N} M_1, w \models Q(a),$$

that is

$$\forall a \in \mathbb{N} \exists w \in \mathbb{N} M_2, w \models Q(a).$$

Hence

$$\forall a \in \mathbb{N}\ 0 \not\models \neg Q(a),$$

and this implies

$$0 \models \forall x(\neg Q(x) \to S).$$

The formula X_2 is equivalent to

$$\forall x \forall y(Q(x) \to P_1(y). \to .S \vee (Q(y) \to P_2(x))) \wedge$$
$$\forall x \forall y(Q(y) \to P_2(x). \to .S \vee (Q(x) \to P_1(y))).$$

Let us consider the first conjunct; we have to show that

$$0 \models Q(a) \to P_1(b). \to .S \vee (Q(b) \to P_2(a)).$$

This is equivalent to

$$0 \models Q(a) \to P_1(b) \Rightarrow 0 \models Q(b) \to P_2(a).$$

So suppose

$$0 \models Q(a) \to P_1(b).$$

Then

$$M_1, w \models Q(a) \supset P_1(b)$$

for any $w \neq 0$, and thus

$$M_1, v \models \Box(Q(a) \supset P_1(b)).$$

But α_2 provides then:

$$M_1, v \models \Box(Q(b) \supset P_2(a)),$$

and therefore (since $0 \not\models Q(b)$):

$$M_2, 0 \models Q(b) \to P_2(a).$$

X_3 is checked by the same argument, using α_3.

For $X_{4,E}$ we have to show:

$$0 \models (Q(a) \to E(b)) \vee (Q(a) \to \neg E(b))$$

for any atomic $E(x)$. So suppose

$$0 \not\models Q(a) \to \neg E(b).$$

Then

$$M_2, w \models Q(a) \wedge E(b)$$

for some $w \neq 0$, and hence

$$M_1, v \vDash \Diamond(Q(a) \wedge E(b)).$$

Applying α_4 we conclude:

$$M_1, v \vDash \Box(Q(a) \supset E(b)),$$

and this yields:

$$M_2, 0 \vDash Q(a) \to E(b).$$

Therefore $0 \vDash X_{4,E}$. ■

Theorem 2.4 *Let Σ be a set of intuitionistic formulas such that* $\mathbf{QH\text{-}2} \subseteq \Sigma \subseteq \mathbf{L}(F_2) - 2$. *Then*

1. *for any quasisimple formula A,*

$$\mathbf{QS5} \vdash \alpha \supset A \Leftrightarrow (X \wedge A^- \to S) \in \Sigma;$$

2. *Σ is undecidable.*

Proof. To prove (1), it is sufficient to show that
(1.1) $\mathbf{QS5} \vdash \alpha \supset A \Rightarrow \mathbf{QH} \vdash X \wedge A^- S$,
and
(1.2) $F_2 \vDash X \wedge A^- \to S \Rightarrow \mathbf{QS5} \vdash \alpha \supset A$

For (1.1), assume that $\mathbf{QH} \nvdash X \wedge A^- \to S$. By Kripke's completeness theorem (Kripke 1965), we have then:

$$M_2, t \nvDash X \wedge A^- \to S$$

for some Kripke model M_2 over a Kripke frame $(U, \leq, D)$ and for some $t \in U$. Thus there exists $u \geq t$ such that $u \vDash X \wedge A^-; u \nvDash S$.

Let M_1 be an L_1-model constructed as in Lemma 2.2. Applying this lemma, we obtain $M_1, v \vDash \alpha \wedge \neg A$, and therefore $\mathbf{QS5} \nvdash \alpha \supset A$.

For (1.2), assume that $\mathbf{QS5} \nvdash \alpha \supset A$. Then by Kripke's completenss theorem (Kripke, 1959), A is refuted in some $\mathbf{QS5}$-frame $(W_0, \mathbb{N})$ with the domain $\mathbb{N}$ and countable or finite W_0. The remark at the end of Section 1 shows that A can also be refuted in the frame $(\mathbb{N}, \mathbb{N})$, ie there exists a model M_1 over $(\mathbb{N}, \mathbb{N})$ such that $M_1, v \nvDash A$ for some v. Then we construct a model M_2 over F_2 according to Lemma 2.3. By this lemma, we obtain: $M_2, 0 \vDash X \wedge A^-$; and $M_2, 0 \nvDash S$ by the construction.

Thus $X \wedge A^- \to S$ is refuted in F_2.
(2) follows from (1) and Theorem 1.8. ■

Since for a propositional formula $A, (W, \leq, D) \models A$ does not depend on D, we can denote this by $(W, \leq) \models A$ (and say that A is *valid* in the *propositional Kripke frame* $(W, \leq)$). Let

$\mathbf{L}(W, \leq) = \{A \mid A$ is a propositional intuitionistic formula $\&(W, \leq) \models A\}$.

For a propositional intermediate logic **L** , **QL** denotes the predicate intermediate logic (in the language with a countable set of individual variables) obtained from Heyting predicate calculus by adjoining all substitution instances of theorems of **L** as new axioms; **QL-2** denotes the fragment of **QL** in the language L_2.

Corollary 2.5 *For any propositional intermediate logic* **L**, *if* $\mathbf{L} \subseteq \mathbf{L}(\phi_2)$ *then* **QL-2** *is undecidable.*

Proof. Obviously **QH-2** $\subseteq$ **QL-2**. On the other hand, $\mathbf{L} \subseteq \mathbf{L}(\phi_2)$ implies **QL** $\subseteq \mathbf{L}(F_2)$ because $\mathbf{L}(F_2)$ is closed under substitution and under the intuitionistic inference rules. Hence **QL-2** $\subseteq \mathbf{L}(F_2) - 2$ and we can apply Theorem 2.4. ■

Theorem 2.4 provides many other examples of predicate intermediate logics with undecidable L_2-fragments; we can add to **QH-2** any new axioms which are valid in F_2 (e.g. $\forall x \neg\neg P(x) \to \neg\neg\forall x P(x), \forall x(P(x) \vee q) \to .\forall x P(x) \vee q$, etc.).

To describe another family of undecidable intermediate logics, consider the language L_3 obtained from L_2 by adding a new propositional letter S_0. Simple and quasisimple L_1-formulas will be translated to L_3 as follows:

- $B^0 = B$ if B is atomic;
- $(\neg B)^0 = B^0 \to S_0$;
- $(B \vee C)^0 = B^0 \vee C^0$;
- $(\forall x \exists y \bigvee_i \Box A_i)^0 = \exists x \forall y \bigwedge_i ((S \to .A_i^0 \vee S_0) \to S)$.
  Let also
  $Y_{0,E} = \forall x \forall y (S \to .E \vee (E \to S_0))$ (if E is atomic);
  $Y_1 = \exists x (Q(x) \vee (Q(x) \to S_0)) \to S$;
  $Y_2 = \forall x \forall y (S \vee (Q(x) \to .P_1(y) \vee S_0). \leftrightarrow .S \vee (Q(y) \to .P_2(x) \vee S_0))$;
  $Y_3 = \forall x (S \vee (Q(x) \to .P_1(x) \vee S_0). \leftrightarrow .S \vee (Q(x) \to P_3))$;
  $Y_{4,E} = \forall x \forall y ((Q(x) \to .E \vee S_0) \vee (Q(x) \wedge E \to S_0))$ (for E atomic).
  Y denotes the conjunction of all these formulas.

Now let us prove the analogues of Lemmas 2.1 and 2.2.

Lemma 2.6 *Suppose* M *is an* L_3*-model over* $(W, \leq, D)$ *such that* $M, u \models Y_{0,E}$ *for any atomic* E. *Then*

$$M, u \models S \to .A^0 \vee (\neg A)^0$$

for any simple $D(u)$*-valued formula* A.

Proof. It is completely analogous to 2.1. Intuitionistic tautologies

$$(\neg B)^0 \wedge (\neg C)^0 \rightarrow (\neg(B \vee C))^0,$$
$$B^0 \rightarrow (\neg\neg B)^0$$

are used to analyze the cases $A = B \vee C, A = \neg B$. ■

Lemma 2.7 *Let M_3 be an L_3-model over $(U, \leq, D)$, A be a quasisimple L_1-formula, $u \in U, M_3, u \models Y \wedge A^0$, $M_3, u \not\models S$. Let also*

$$D_0 = D(u), W = \{v \mid u \leq v \& M_3, v \models S \& M_3, v \not\models S_0\}.$$

Consider an L_1-model M_1 over (W, D_0) such that

$$(*) \quad M_1, v \models B \Leftrightarrow M_3, v \models B^0$$

for any $v \in W$ and any atomic D_0-valued B.
Then $M_1, v \models \alpha \wedge \neg A$ for any $v \in W$.

Proof. As in 2.2, at first we prove that (*) is true for any simple D_0-valued B.
Consider the case $B = \neg C$. Then $B^0 = C^0 \rightarrow S_0$, and

$$M_1, v \models B \Leftrightarrow M_1, v \not\models C \Leftrightarrow M_3, v \not\models C^0 \Leftrightarrow M_3, v \models C^0 \rightarrow S^0(= B^0).$$

In the latter equivalence, '$\Rightarrow$' follows from 2.6 (since $M_2, v \models S$); '$\Leftarrow$' is true since $M_3, v \not\models S_0$.
Next consider $A = \forall x \exists y \bigvee_i \Box A_i$. We have:

$$M_3, u \models A^0(= \exists x \forall y \bigwedge_i ((S \rightarrow .A_i^0 \vee S_0) \rightarrow S)),$$

and thus

$$\exists a \in D_0 \forall b \in D_0 \forall i M_3, u \models (S \rightarrow .A_i^0(a, b) \vee S_0) \rightarrow S;$$
$$\exists a \in D_0 \forall b \in D_0 \forall i M_3, u \not\models S \rightarrow .A_i^0(a, b) \vee S_0$$

since $u \not\models S$. But then

$$\exists a \in D_0 \forall b \in D_0 \forall i \exists v \in W M_3, v \not\models A_i^0(a, b),$$

and by applying (*) to $A_i(a, b)$ we obtain:

$$M_1, v \models \exists x \forall y \bigwedge_i \Diamond \neg A_i$$

i.e. $M_1, v \models \neg A$.
Now consider α_1. From $u \models Y_1, u \not\models S$ we have (for any $a \in D_0$):

$$u \not\models Q(a) \rightarrow S_0,$$

and thus

$$M_2, v \models Q(a), M_2, v \not\models S_0$$

for some $v \geq u$.

On the other hand, $u \models Y_1$ implies

$$u \models Q(a) \rightarrow S\ .$$

Therefore $v \in W$, and $M_1, v \models Q(a)$ by (*). Hence

$$\forall a \in D_0 \exists v \in W M_1, v \models Q(a),$$

i.e. $M_1 \models \alpha_1$.

For the proof of $M_1 \models \alpha_2$ we will show only

$$M_1, v \models \Box(Q(a \supset P_1(b)) \Rightarrow M_1, v \models \Box(Q(b) \supset P_2(a)).$$

Suppose

$$M_1, v \models \Box(Q(a) \supset P_1(b)).$$

Then

$$M_2, u \models Q(a) \rightarrow .P_1(b) \vee S_0$$

(since $u \models Q(a) \rightarrow S$ as we have seen, it follows that for any $v \geq u$,

$$M_2, v \models Q(a) \& M_2, v \not\models S_0$$

implies $v \in W$, and hence

$$M_1, v \models Q(a); M_1, v \models P_1(b); M_2, v \models P_1(b)).$$

Applying Y_2, we conclude:

$$u \models Q(b) \rightarrow .P_2(a) \vee S_0,$$

and hence one can easily get

$$M_1 \models \Box(Q(b) \supset P_2(a)).$$

We skip the proof of $M_1 \models \alpha_3$, and consider $\alpha_{4,P_i(y)}$.

Suppose $a, b \in D_0, w \in W$,

$$M_1, w \models Q(a) \wedge P_i(b).$$

Then

$$M_2, w \models Q(a) \wedge P_i(b), M_2, w \not\models S_0,$$

and thus

$$M_2, u \not\models Q(a) \wedge P_i(b) \rightarrow S_0.$$

Using $Y_{4,P_i(y)}$ we have then

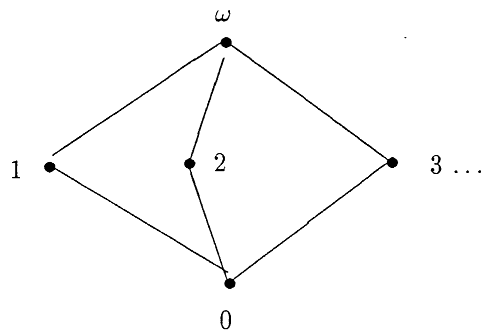
$$M_2, u \models Q(a) \to .P_i(b) \vee S_0,$$

and this implies

$$M_1 \models \Box(Q(a) \supset P_i(b)).$$

Therefore $M_1 \models \alpha_{4,P_i(y)}$. ■

Now consider the frame $F_3 = (W_3, \leq, D)$ with the constant domain $\mathbb{N}$ in which $W_3 = \omega + 1 = \{0, 1, \ldots, \omega\}$ and $\phi_3 = (W_3, \leq)$ as in Figure 2:



(i.e. ϕ_3 is ϕ_2 with the element ω added at the top.)
Figure 2

Lemma 2.8 *Let A be a quasisimple L_1-formula, M_1 be an L_1-model over $(\mathbb{N}, \mathbb{N})$ such that $M_1, v \models \alpha \wedge \neg A$ for some v. Consider an L_3-model M_3 over F_3 such that*

$$M_3, w \models E \Leftrightarrow (w \neq 0, \omega \& M_1, w \models E) \vee w = \omega$$

for any atomic $\mathbb{N}$-valued $E, E \neq S, S_0$;

$$M_3, w \models S \Leftrightarrow w \neq 0;$$
$$M_3, w \models S_0 \Leftrightarrow w = \omega.$$

Then $M_3, 0 \models Y \wedge A^0$.

Proof. Similarly to 2.3. At first we observe that

$$(**) \qquad M_3, w \models B^0 \Leftrightarrow M_1, w \models B$$

for any simple $\mathbb{N}$-vlaued $B, w \neq 0, \omega$. As usual, this is easily proved by an induction; note that

$$M_3, w \models B^0 \vee (B^0 \to S_0),$$

due to the structure of ϕ_3.

Let $A = \forall x \exists y \bigvee_i \Box A_i$. Then $M_1, v \models \alpha \wedge \neg A$ implies

$$\exists a \in \mathbb{N} \forall b \in \mathbb{N} \forall i \exists w \in \mathbb{N} M_1, w \not\models A_i(a,b),$$

and from (**) we obtain:

$$\exists a \in \mathbb{N} \forall b \in \mathbb{N} \forall i M_3, 0 \not\models A \rightarrow .S_0 \vee A_i^0(a,b),$$

and hence

$$M_3, 0 \models \exists x \forall y \bigwedge_i ((A \rightarrow .S_0 \vee A_i^0) \rightarrow S) \quad (= A^0).$$

$0 \models Y_{0,E}$ follows from

$$0 \models S \rightarrow .E(n) \vee (E(n) \rightarrow S_0).$$

Y_1 is equivalent to

$$(\exists x Q(x) \rightarrow S) \wedge (\exists x (Q(x) \rightarrow S_0) \rightarrow S).$$

$0 \models \exists x Q(x) \rightarrow S$ is obvious, and the second conjunct is true because $M_1 \models \forall x \Diamond Q(x)$ (due to α_1).

Y_2 is also equivalent to a conjunction of two formulas. Consider the first conjunct, that is

$$Y_2' = \forall x \forall y ((Q(x) \rightarrow .P_1(y) \vee S_0) \rightarrow .(Q(y) \rightarrow .P_2(x) \vee S_0) \vee S).$$

If

$$0 \models Q(a) \rightarrow .P_1(b) \vee S_0$$

then

$$M_2, w \models Q(a) \rightarrow P_1(b) \text{ for any } w \neq 0, \omega;$$

and thus

$$M_1 \models \Box(Q(a) \supset P_1(b)).$$

By α_2 we get

$$M_1 \models \Box(Q(b) \supset P_2(a))$$

i.e.

$$M_2, w \models Q(b) \rightarrow P_2(a)$$

for any $w \neq 0, \omega$. Then

$$0 \models Q(b) \rightarrow .P_2(a) \vee S_0.$$

Therefore we obtain $0 \models Y_2'$. Y_3 is considered analogously. The proof of $0 \models Y_{4,E}$ is left to the reader. ■

Theorem 2.9 *Let Σ be a set of intuitionistic formulas such that* $\mathbf{QH\text{-}2} \subseteq \Sigma \subseteq \mathbf{L}(F_3) - 2$.[2] *Then*

(1) for any quasisimple formula A,

$$\mathbf{QS5} \vdash \alpha \supset A \Leftrightarrow (Y \wedge A^0 \rightarrow S) \in \Sigma;$$

(2) Σ is undecidable.

Proof. To prove (1), it is sufficient to verify
(1.1) $\mathbf{QS5} \vdash \alpha \supset A \Rightarrow \mathbf{QH} \vdash Y \wedge A^0 \rightarrow S$,
and
(1.2) $F_3 \models Y \wedge A^0 \rightarrow S \Rightarrow \mathbf{QS5} \vdash \alpha \supset A$.

For (1.1), assume that $\mathbf{QH} \nvdash Y \wedge A^0 \rightarrow S$. By Kripke's completeness thoerem we have:

$$M_3, t \not\models Y \wedge A^0 \rightarrow S$$

for some Kriple model M_3 over a Kripke frame $(U, \leq, D)$ and for some $t \in U$. Then there exists $u \geq t$ such that $u \models Y \wedge A^0$ and $u \not\models S$. Consider an L_1-model M_1 constructed as in 2.7. Applying 2.7 we obtain:

$$M_1 \models \alpha \wedge \neg A.$$

Therefore $\mathbf{QS5} \nvdash \alpha \supset A$.

For (1.2), assume that $\mathbf{QS5} \nvdash \alpha \supset A$. Then $M_1, n_0 \not\models A$ for some model M_1 over $(\mathbb{N}, \mathbb{N})$ and for some $n_0 \in \mathbb{N}$ (cf the proof of Theorem 2.4). Take the model M_3 over F_3 constructed in 2.8. Then 2.8 yields

$$M_3, 0 \models Y \wedge A^0,$$

and $M_3, 0 \not\models S$ by the definition. Thus F_3 refutes $Y \wedge A^0 \rightarrow S$.

(2) follows from (1) and 1.8. ∎

Corollary 2.10 *For any propositional intermediate logic* $\mathbf{L}$, *if* $\mathbf{L} \subseteq \mathbf{L}(\phi_3)$ *then* $\mathbf{QL\text{-}2}$ *is undecidable.*

Proof. Immediately from 2.9. (Cf the proof of Corollary 2.5). ∎

Theorem 2.9 allows to prove the undecidability of two-variable fragments for some other intermediate logics (e.g. for those having additional axioms

$$\neg\neg\exists x P(x) \rightarrow \exists x \neg\neg P(x), \neg q \rightarrow \exists x P(x). \rightarrow \exists x(\neg q \rightarrow P(x))$$

etc.).

Recall that an intermediate propositional logic is called *tabular* if it can be presented as $\mathbf{L}(\Theta)$ for some finite propositional Kripke frame Θ. A maximal

[2]The notation '-2' in this theorem (as well as in 2.10) refers to the language L_3.

non-tabular logic is called *pretabular.* A nice theorem (Maksimova, 1972) states that there are exactly three pretabular intermediate logics. Two of them are just $\mathbf{L}(\phi_2)$ and $\mathbf{L}(\phi_3)$, and the third one is the well-known logic $\mathbf{LC} = \mathbf{H} + (p \to q) \vee (q \to p)$

Since every non-tabular logic is included into one of these three, a proof of an analogue of Corollaries 2.5 and 2.10 for sublogics of **LC** would provide the undecidability of **QL-2** for any non-tabular **L**. However, today we have no such proof. So for many intermediate logics **L** the decidability of **QL-2** remains unclear (e.g. for logics of a finite width).

On the other hand, D. P. Skvortsov recently informed us that he can prove decidability of **QL-2** for any tabular **L** in the first-order language without equality and functional symbols.

3 Undecidable Modal Logics

For the modal case, the method used in the previous section becomes much simpler. Let us consider first-order normal modal logics in the language L_1 (the same as in Section 1).

Recall that a *(modal)Kripke frame* is a triple (W, ρ, D) in which $W \neq \varnothing, \rho \subseteq W \times W, D$ is a function sending elements of W to sets such that $D(u) \subseteq D(v)$ whenever $u\rho v$. The notations: $M, w \models A(a_1, a_2), F \models A, \mathbf{L}(F)$ are used analogously to Sections 1, 2. **QK** denotes the minimal normal modal predicate logic.

Lemma 3.1 *Let M be a Kripke model over a frame (U, ρ, D), A be a quasisimple L_1-formula, $u \in U, D_0 = D(u)$,*

$$W = \{v \in U \mid u\rho v\} \neq \varnothing; M, u \models \alpha \wedge \neg A.$$

Consider a **QS5**-*model M_1 over (W, D_0) such that*

$$(*) \qquad M_1, v \models B \Leftrightarrow M, v \models B$$

for any $v \in W$ and any atomic D_0-valued B.

Then $M_1 \models \alpha \wedge \neg A$.

Proof. Suppose $A = \forall x \exists y \bigvee_i \Box A_i$. Then

$$M, u \models \neg A \Rightarrow \exists a \in D_0 \forall b \in D_0 \forall i \exists v \in W M, v \models \neg A_i(a, b)$$
$$\Rightarrow \exists a \in D_0 \forall b \in D_0 \forall i \exists v \in W M_1, v \models \neg A_i(a, b)$$

(because (*) holds also for any simple D_0-valued formula, as it is easily seen)

$$\Rightarrow M_1 \models \exists x \forall y \bigwedge_i \Diamond \neg A_i \Rightarrow M_1 \models \neg A.$$

Also we have:

$M, u \models \alpha_1 \Rightarrow \forall a \in D_0 M, u \models \Diamond Q(a) \Rightarrow \forall a \in D_0 \exists v \in W M_1, v \models Q(a) \Rightarrow M_1 \models \alpha_1.$

Let us show that

$$M_1 \models \forall x \forall y (\Box(Q(x) \supset P_1(y)) \equiv \Box(Q(y) \supset P_2(x))).$$

Indeed,

$$\begin{aligned}
&M_1 \models \Box(Q(a) \supset P_1(b)) \Leftrightarrow \forall v \in W M_1, v \models Q(a) \supset P_1(b) \\
&\Leftrightarrow M, u \models \Box(Q(a) \supset P_1(b)) \quad (\text{since } W = \rho(u)) \\
&\Leftrightarrow M, u \models \Box(Q(b) \supset P_2(a)) \quad (\text{since } M, u \models \alpha_2) \\
&\Leftrightarrow \forall v \in W M, v \models Q(b) \supset P_2(a) \\
&\Leftrightarrow \forall v \in W M_1, v \models Q(b) \supset P_2(a) \quad (\text{by } (*)) \\
&\Leftrightarrow M_1 \models \Box(Q(b) \supset P_2(a)).
\end{aligned}$$

α_3 is considered analogously.

Consider $\alpha_{4,E}$:

$$\begin{aligned}
&M_1 \models \Diamond(Q(a) \wedge E(b)) \Rightarrow \exists v \in W M_1, v \models Q(a) \wedge E(b) \\
&\Rightarrow \exists v \in W M, v \models Q(a) \wedge E(b) \quad (\text{by } (*)) \\
&\Rightarrow M, u \models \Diamond(Q(a) \wedge E(b)) \\
&\Rightarrow M, u \models \Box(Q(a) \supset E(b)) \quad (\text{by } \alpha_{4,E}) \\
&\Rightarrow M_1 \models \Box(Q(a) \supset E(b)) \quad (\text{by } (*) \text{ and since } W = \rho(u)).
\end{aligned}$$

■

Lemma 3.2 *Let M_1 be a* **QS5**-*model over $(\mathbb{N}, \mathbb{N})$, A be a quasisimple formula such that $M_1, v \models \alpha \wedge \neg A$ for some v. Let $F = (W, \rho, D)$ be a Kripke frame with the constant domain $\mathbb{N}$ such that $\rho(u)$ is infinite for some $u \in W$ (so we can assume that $\mathbb{N} \subseteq \rho(u)$).*

Consider a Kripke model M over F such that for any $v \in W$ and any atomic $\mathbb{N}$-valued B:

$$\begin{aligned}
&(*) \qquad M, v \models B \Leftrightarrow M_1, v \models B \quad \text{if } v \in \mathbb{N}, \\
&(**) \qquad M, v \not\models B \quad \text{if } v \notin \mathbb{N}
\end{aligned}$$

Then $M, \mathbf{u} \models \alpha \wedge \neg A$.

Proof. It is almost the same as for 3.1. We have:

$$\begin{aligned}
&M_1, u \models \neg A \Rightarrow \exists a \in \mathbb{N} \forall b \in \mathbb{N} \forall i \exists v \in \mathbb{N} M_1, v \models \neg A_i(a, b) \\
&\Rightarrow \exists a \in \mathbb{N} \forall b \in \mathbb{N} \forall i \exists v \in \mathbb{N} \ M, v \models \neg A_i(a, b)
\end{aligned}$$

(because (*) holds for simple formulas too)

$$\Rightarrow M, u \models \exists x \forall y \bigwedge_i \Diamond \neg A_i \quad (\equiv \neg A).$$

Also

$$M_1, u \models \alpha_1 \Rightarrow \forall a M_1, u \models \Diamond Q(a) \Rightarrow \forall a \exists v \in \mathbb{N} M_1, v \models Q(a)$$
$$\Rightarrow \forall a M, u \models \Diamond Q(a) \Rightarrow M, u \models \alpha_1.$$

Consider α_2:

$$M, u \models \Box(Q(a) \supset P_1(b)) \Rightarrow \forall v \in \mathbb{N} M, v \models Q(a) \supset P_1(b)$$
$$\Rightarrow \forall v \in \mathbb{N} M_1, v \models Q(a) \supset P_1(b) \quad \text{(by (*))}$$
$$\Rightarrow \forall v \in \mathbb{N} M_1, v \models Q(b) \supset P_2(a) \quad \text{(by } \alpha_2)$$
$$\Rightarrow M, u \models \Box(Q(b) \supset P_2(a))$$

because $M, v \models Q(b)$ only if $v \in \mathbb{N}$.

The converse, and α_3 are verified analogously. Let us check α_4:

$$M, u \models \Diamond(Q(a) \wedge E(b)) \Rightarrow \exists v \in \mathbb{N} M, v \models Q(a) \wedge E(b)$$
$$(v \in \mathbb{N} \text{ because otherwise } M, v \not\models Q(a))$$
$$\Rightarrow M_1 \models \Diamond(Q(a) \wedge E(b))$$
$$\Rightarrow M_1 \models \Box(Q(a) \supset E(b)) \quad \text{(by } \alpha_4)$$
$$\Rightarrow M, u \models \Box(Q(a) \supset E(b))$$

again because $Q(a)$ is false outside $\mathbb{N}$. ■

Theorem 3.3 *Let Σ be a set of L_1-formulas such that*

$$\mathbf{QK\text{-}2} \subseteq \Sigma \subseteq \mathbf{L}(F) - 2$$

for some Kripke frame $F = (W, \rho, D)$ with the constant domain $\mathbb{N}$ such that $\rho(u)$ is infinite for some $u \in W$.

Then

(1) for any quasisimple A,

$$\mathbf{QS5} \models \alpha \supset A \Leftrightarrow (\alpha \supset A) \in \Sigma;$$

(2) Σ is undecidable.

Proof.

$$(1.1) \quad \mathbf{QS5} \vdash \alpha \supset A \Rightarrow \mathbf{QK} \vdash \alpha \supset A.$$

Indeed, assume that $\mathbf{QK} \not\vdash \alpha \supset A$. Then by the completeness of QK (Gabbay, 1976),

$$M, u \models \alpha \wedge \neg A$$

for some world u in some Kripke model M. We can apply 3.1 (note that $\rho(u) \neq \varnothing$ since $M, u \models \Diamond Q(a)$ for some a, thanks to α_1) and construct a QS5-model M_1 such that

$$M_1 \models \alpha \wedge \neg A$$

Hence $\mathbf{QS5} \not\vdash \alpha \supset A$.

$$(1.2) \quad F \models \alpha \supset A \Rightarrow \mathbf{QS5} \vdash \alpha \supset A.$$

Indeed, suppose **QS5**$\not\vdash \alpha \supset A$. By the completeness of **QS5** we have a **QS5**-model M_1 such that $M_1, v \models \alpha \wedge \neg A$ for some v. Then we can construct a model M over F according to 3.2, such that $M, u \models \alpha \wedge \neg A$. Hence $F \not\models \alpha \supset A$. ■

Corollary 3.4 *Let* **L** *be a non-tabular normal modal propositional logic containing* **S4**. *Then* **QL-2** *is undecidable.*

Proof. By Ehsakia-Meskhi's result (Ehsakia, Meskhi 1977), every non-tabular normal extension of **S4** is contained in $\mathbf{L}(\phi_2)$ (figure 1) or in $\mathbf{L}(\phi_3)$ (figure 2), or in **S5** (which is the logic of an infinite cluster), or in the logic of the frame

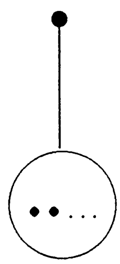


Figure 3

(is an infinite cluster plus the top element). The **QL** is valid in one of the corresponding predicate frames with the domain $\mathbb{N}$, and Theorem 3.3 can be applied. ■

Of course Theorem 3.3 is useful for many other systems, e.g. **QL-2** is undecidable for **L**= **K4**, **GL** (Gödel-Löb logic), **T**, **B** et al. Corresponding logics with the Barcan schema

$$\forall x \Box A \supset \Box \forall x A$$

are also seen to be undecidable.

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stempel:

(Dit gedeelte niet beschrijven)

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