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HOW TO BROADEN YOUR HORIZON

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How to broaden your horizon*

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Abstract

We generalize the proofs in Meyer and Ono(1992) of the finite model property for **BCK** and **BCIW** to an infinite collection of extensions of **BCI** with a 'knotted' rule of contraction or expansion. As a corollary we get that **BCI** has the finite model property (with respect to the class of models under consideration) only if it is equal to an intersection of these extensions.

1 Introduction

Linear implicational logic or **BCI** is the fragment of intuitionistic implicational logic obtained by deleting the structural rules of weakening and contraction from the formulation of intuitionistic implicational logic as a sequent-calculus. So we have the identity axiom:

$$[\mathsf{ax}] \qquad A \Rightarrow A,$$

the logical rules:

$$[\supset R] \quad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow AB} \qquad\qquad [\supset L] \quad \frac{\Gamma_1 \Rightarrow A \quad \Gamma_2, B \Rightarrow C}{\Gamma_1, \Gamma_2, AB \Rightarrow C}$$

and the cut-rule:

$$[\mathsf{cut}] \quad \frac{\Gamma_1 \Rightarrow A \quad \Gamma_2, A \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow B}.$$

(Here Γ, Γ_i denote *multisets* (so *exchange* is implicit), while AB abbreviates $A \supset B$.)

*EXERCISE: Explain the title.

An equivalent Hilbert-type formulation is given by the axiom-schemes

[**B**]	(BC)((AB)(AC))
[**C**]	(A(BC))(B(AC))
[**I**]	AA

and *modus ponens* as a single rule of inference.

If we extend **BCI** with the structural rule of *weakening*

$$[\mathsf{weak}] \qquad \frac{\Gamma \Rightarrow B}{\Gamma, A \Rightarrow B}$$

(for the Hilbert-system with the axiom-scheme [**K**] : $A(BA)$) we obtain the implicational fragment of *affine* logic, a.k.a.[1] **BCK**.

If we extend **BCI** with the structural rule of *contraction*

$$[\mathsf{cont}] \qquad \frac{\Gamma, A, A \Rightarrow B}{\Gamma, A \Rightarrow B}$$

(for the Hilbert-system with the axiom-scheme [**W**] : $(A(AB))(AB)$) we get the implicational fragment of the *relevant* logic $\mathcal{R}$, a.k.a. **BCIW**.

Let us write A^n to denote n copies of some formula[2] A. We then can define for any pair of natural numbers (n, k) a structural rule as follows:

$$[n \rightsquigarrow k] \qquad \frac{\Gamma, A^n \Rightarrow B}{\Gamma, A^k \Rightarrow B}.$$

We will denote the logical system thus obtained by $\mathbf{BCI}^n_k$. Note that $\mathbf{BCI}^n_k$ can equivalently be obtained by extending the Hilbert-system for **BCI** with an axiom-scheme $[n \rightsquigarrow k]$ defined as

$$(\underbrace{X \supset (X \supset (\ldots (X}_{n} \supset A)\ldots))) \supset (\underbrace{X \supset (X \supset \ldots (X}_{k} \supset A)\ldots)),$$

(which for $n = 0$ should be read as $A \supset (X \supset (X \supset \ldots (X \supset A)\ldots))$, for $k = 0$ as $(X \supset (X \supset \ldots (X \supset A)\ldots)) \supset A$.)

Clearly $\mathbf{BCI}^k_k$ is just **BCI**, while $\mathbf{BCI}^n_0$ for $n \neq 0$ is weird, to say the least: in $\mathbf{BCI}^1_0$ any formula ϕ is derivable; also, for all $n > 0$ the instance $(p^n \supset p) \supset p$ of the axiom-scheme $[n \rightsquigarrow 0]$ is not valid classically. On the other hand, for all pairs (n, k) such that $k = 0 \rightarrow n = 0$ what we have is an *extension* of **BCI** that is a fragment of intuitionistic implicational logic. In fact, for $n < k$ each logic

[1] I.e. *also known as*

[2] By 'formula' we will of course always mean an *implicational* formula.

$\mathbf{BCI}_k^n$ is a fragment of $\mathbf{BCI}_1^0$ (which is just **BCK**), while for $n > k > 0$ clearly $\mathbf{BCI}_k^n$ is a fragment of $\mathbf{BCI}_1^2$ (which is just **BCIW**).

In Meyer and Ono(1992) it is shown that **BCK** and **BCIW** have the *finite model property* with respect to a certain class of ordered monoids. The purpose of this note is to show that Meyer and Ono's proof of the finite model property for **BCK** uniformly generalizes to a proof for $\mathbf{BCI}_k^n$, for all $n < k$, and that their proof of the finite model property for **BCIW** uniformly generalizes to a proof for $\mathbf{BCI}_k^n$, for all $n > k$. I.e. we will show that for a suitable class of structures, *if* a formula ϕ is valid on all its *finite* members, then it is derivable in $\mathbf{BCI}_k^n$. Crucial in this generalization is the construction of *finite* so-called $\mathbf{BCI}_k^n$-monoids in section 3.

All these logics in fact are *proper* extensions of **BCI** (it is easy to show that $\mathbf{BCI} \vdash \Rightarrow [n \rightsquigarrow k]$ iff $n = k$) and it seems not too farfetched to conjecture that **BCI** is their intersection. In the final section we observe that it is an easy corollary of our proof of the finite model property for all the extensions that in fact this is equivalent to the statement that **BCI** *itself* has the finite model property (with respect to this class of models).

2 $\mathbf{BCI}_k^n$: structures, validity and completeness

We begin by introducing the notion of $\mathbf{BCI}_k^n$-monoid, being an obvious generalization of the **BCI**-, **BCK**-, and **BCIW**-structures of Meyer and Ono(1992).

2.1. DEFINITION. A **BCI**-*structure* is a quadruple $(\mathbf{M}, \cdot, 1, \leq)$, with $\mathbf{M}$ a set and $1 \in \mathbf{M}$, such that $\cdot$ is a binary operation on $\mathbf{M}$ that is monotonous with respect to the binary relation $\leq$ on $\mathbf{M}$ (i.e. for all $x, y, x \in \mathbf{M}$ we have that $x \leq y$ implies $x \cdot z \leq y \cdot z$) and $(\mathbf{M}, \cdot, 1)$ a commutative monoid with unity 1.

A $\mathbf{BCI}_k^n$-*monoid* is a **BCI**-structure of which we moreover demand that, for all $x \in \mathbf{M}$, $x^n \leq x^k$ (taking x^0 to denote the unity 1). □

Note that we do not ask anything special of the relation $\leq$. However, we observe the following.

2.2. LEMMA. *A finite* **BCI**-*structure* $(\mathcal{X}, \leq)$ *is both a* $\mathbf{BCI}_k^n$-*monoid and a* $\mathbf{BCI}_n^k$-*monoid for some* $(n \neq k)$ *if and only if* $\leq$ *is reflexive.*

PROOF: If $\mathcal{X}$ is finite then there are $n \neq k$ such that $x^n = x^k$, for all x. By reflexivity then both $x^n \leq x^k$ and $x^k \leq x^n$, for all x. For the converse, observe that all $\mathbf{BCI}_k^n$-monoids are reflexive. □

We define *validity* on $\mathbf{BCI}_k^n$-monoids just as validity on **BCI**-structures: a *valuation* $\models$ on a $\mathbf{BCI}^n_k$-monoid is a relation between elements of $\mathbf{M}$ and propositional variables satisfying the monotonicity condition

$$[\mathsf{mon}] \qquad x \models p \quad \text{and} \quad x \leq y \quad \text{implies} \quad y \models p.$$

Each valuation has a canonical extension to a relation between elements of $\mathbf{M}$ and implicational formulas by $x \models AB$ iff $\forall y. y \models A \Rightarrow xy \models B$.

A formula ϕ is said to be *valid* on a given $\mathbf{BCI}^n_k$-monoid $\mathcal{X}$ (written as $\mathcal{X} \models \phi$) if $1 \models \phi$ for all valuations $\models$.

With respect to validity on $\mathbf{BCI}$-structures we observe:

2.3. LEMMA. *A formula ϕ is valid on all (finite)* $\mathbf{BCI}$*-structures iff it is valid on all (finite)* reflexive *(even: all* discrete*)* $\mathbf{BCI}$*-structures.*

PROOF: Suppose $(\mathcal{X}, \leq) \not\models \phi$ for some valuation $\models$. Define $\models^*$ on $(\mathcal{X}, =)$ by $x \models^* p$ iff $x \models p$, for propositional variables p. Then obviously $x \models^* \psi$ iff $x \models \psi$ for *any* formula ψ. Therefore $1 \not\models^* \phi$, so ϕ is not valid on a reflexive (more so, a *discrete*) $\mathbf{BCI}$-structure. $\square$

2.4. REMARK. It seems that in general the condition $x^n \leq x^k$ does *not* ensure validity of $[n \rightsquigarrow k]$, so in fact we can not guarantee *soundness* of *all* logics $\mathbf{BCI}^n_k$ with respect to validity in $\mathbf{BCI}^n_k$-monoids. The reader might want to verify however that soundness *does* hold in case $k = 1$.

2.5. DEFINITION. Let Γ, Δ be multisets of formulas. We say that $\Gamma \mid^n_k \Delta$ iff $\Delta \Rightarrow x$ is obtainable from $\Gamma \Rightarrow x$ by means of some (maybe no) applications of rule $[n \rightsquigarrow k]$ (x being an arbitrary formula). $\square$

So $\mid^n_k$ is a partial order on the collection of multisets of formulas. Note that for $k = n$ we have $\Gamma \mid^n_k \Delta$ iff $\Gamma = \Delta$. Also obviously $\Gamma \mid^n_k \Delta$ iff $\Delta \mid^k_n \Gamma$. So Γ and Δ are $\mid^n_k$-comparable if and only if they are $\mid^k_n$-comparable.

2.6. PROPOSITION. $\mathbf{BCI}^n_k$*-logic is* complete *with respect to the class of all* $\mathbf{BCI}^n_k$*-monoids, for any n, k.*

PROOF: (This is standard, and as in Meyer and Ono(1992). We give an outline of the proof, for we will need some of the notions and details later on.)

Suppose $\mathbf{BCI}^n_k \not\vdash A$. Let $S = \{S_1, \ldots S_m\}$ be the set of all subformulas of A. Let Σ denote the collection of all finite multisets with elements in S.

(Note that each element can be unambiguously represented by a vector $\langle a_1, \ldots, a_m \rangle$ with entries in the set $\mathbb{N}$ of natural numbers, and a_i denoting the number of occurences of S_i. Clearly, $(\Sigma, \cup, \emptyset, \mid^n_k)$ is a $\mathbf{BCI}^n_k$-monoid, with as isomorphic representation the set of all m-dimensional vectors with entries in

$\mathbb{N}$, pointwise addition as operation, $\mathbf{0} := \langle 0, \ldots, 0\rangle$ as unity and the obvious interpretation of the relation $|^n_k$.)

Then, for propositional variables p appearing in A, define

$$\Gamma \models p \quad \text{iff} \quad \mathbf{BCI}^n_k \vdash \Gamma \Rightarrow p.$$

One then shows by induction on the complexity of $S_i \in S$, that for all subformulas S_i of A we have that

$$\Gamma \models S_i \quad \text{iff} \quad \mathbf{BCI}^n_k \vdash \Gamma \Rightarrow S_i.$$

(If $\Gamma \models B \supset C$, note that by induction hypothesis $\{B\} \models B$, so $\Gamma, B \models C$, a.w.a.d.[3] by induction hypothesis. Conversely, if $\mathbf{BCI}^n_k \vdash \Gamma \Rightarrow B \supset C$ and $\Delta \models B$, then $\mathbf{BCI}^n_k \vdash \Delta \Rightarrow B$ by inductive hypothesis. So $\mathbf{BCI}^n_k \vdash \Gamma, \Delta \Rightarrow C$ by an application of cut, and $\Gamma, \Delta \models C$ by inductive hypothesis.)

We conclude that $\emptyset \not\models A$. $\square$

3 Finite $\mathbf{BCI}^n_k$-monoids

We are going to construct finite $\mathbf{BCI}^n_k$-monoids on intitial segments of the natural numbers, for any $n \neq k$.

Let $mod(a, b)$ denote the remainder of a on division by b. For any pair of natural numbers $R \geq 0$ and $r \geq 1$ define an operation $[R, \cdot]_r$ on natural numbers as follows:

$$[R, a]_r = \begin{cases} a - R & \text{if } a < R \\ mod(a - R, r) & \text{otherwise.} \end{cases}$$

It is easy to see that then $R + [R, a]_r$ maps the natural numbers onto the initial segment $\{0, 1, \ldots, R+r-1\}$; moreover, it is the identity on this segment, as for $a \leq R + r - 1$ we have that $[R, a]_r = a - R$.

We define on $\{0, 1, \ldots, R + r - 1\}$ an operation $\oplus_r$ by:

$$a \oplus_r b = R + [R, a + b]_r.$$

3.1. PROPOSITION. $(\{0, 1, \ldots, R+r-1\}, \oplus_r, 0)$ *is a commutative monoid with unity* 0.

PROOF: Commutativity is clear because of commutativity of $+$; neutrality of 0 follows from the remarks above. So it remains to show associativity of $\oplus_r$.

[3] I.e. *and we are done*

By definition $(a \oplus_r b) \oplus_r c = R + [R, (a \oplus_r b) + c]_r$. First suppose that $a + b + c < R$. Then obviously $a + b < R$, so $(a \oplus_r b) \oplus_r c = R + [R, a + b + c]_r$. Otherwise $a+b+c \geq R$. If $a+b < R$ we again have $(a \oplus_r b) \oplus_r c = R + [R, a+b+c]_r$. So let us suppose that $a + b \geq R$. Then

$$\begin{aligned}
(a \oplus_r b) \oplus_r c &= R + [R, R + [R, a + b]_r + c]_r \\
&= R + mod(mod(a + b - R, r) + c, r) \\
&= R + mod(a + b + c - R, r) \\
&= R + [R, a + b + c]_r
\end{aligned}$$

So $(a \oplus_r b) \oplus_r c = R + [R, a + b + c]_r$ in all cases. Similarly one shows that $a \oplus_r b(\oplus_r c) = R + [R, a + b + c]_r$, a.w.a.d. □

The following two lemma's give some properties of the operation $\oplus_r$.

3.2. LEMMA. *Let $a, b, h \in \{0, 1, \ldots, R + r - 1\}$. Then (i) $(a + b) - (a \oplus_r b) \equiv 0 \bmod r$; (ii) if $h \leq (a \oplus_r b)$ and $(a+b) - h \equiv 0 \bmod r$, also $(a \oplus_r b) - h \equiv 0 \bmod r$.*

PROOF: For (i), either $a \oplus_r b = a + b$, or $a \oplus_r b = R + mod(a + b - R, r)$. For (ii), just note that $(a \oplus_r b) - h = (a + b) - h - ((a + b) - (a \oplus_r b))$ and use (i). □

Let $\oplus_r^n a$ stand for $\underbrace{a \oplus_r a \oplus_r \ldots \oplus_r a}_{n}$.

3.3. LEMMA. *For all $x, a \in \{0, 1, \ldots, R + r - 1\}$ we have $(\oplus_r^r a) \oplus_r x \geq x$; moreover $((\oplus_r^r a) \oplus_r x) - x \equiv 0 \bmod r$.*

PROOF: By the above $\oplus_r^r a$ equals either ra or $R + mod(ra - R, r)$ which in turn equals $R + mod(-R, r)$. In the first case we obtain $x \oplus_r ra$, which either equals $x + ra \geq x$, or $R + mod(x + ra - R, r) = R + mod(x - R, r)$.

In the second case we obtain

$$\begin{aligned}
x \oplus_r (R + mod(-R, r) &= R + mod(x + R + mod(-R, r) - R, r) \\
&= R + mod(x + mod(-R, r), r) \\
&= R + mod(x - R, r).
\end{aligned}$$

It therefore suffices to show that for any $x \in \{0, 1, \ldots, R+r-1\}$ we have that $x \leq R + mod(x - R, r)$. If $x < R$ this is obvious. Otherwise $R \leq x \leq R + r - 1$ and $R + mod(x - R, r) = R + x - R = x$.

The second claims follows by noting that $mod(x - R, r) = r - mod(R - x, r)$, so $R + mod(x - R, r) - x = (R - x) - mod(R - x, r) + r \equiv 0 \bmod r$. □

We will write $\mathcal{M}$ for $(\{0,1,\ldots,R+r-1\},\oplus_r,0)$, suppose $r = | \, n-k \, |$ for some $n \neq k$ and consider the elements of $\mathcal{M}$ as representing the number of occurrences of some formula ϕ. Let Γ,Δ range over multisets having ϕ as their sole element (when non-empty). Γ,Δ then are uniquely represented by their cardinality and we have a relation $|^n_k$ between natural numbers by defining $x \, |^n_k \, y$ iff Γ,Δ have x,y elements and $\Gamma \, |^n_k \, \Delta$ in the sense of definition 2.5.

So the following hold:

- if $k > n$ then $x \, |^n_k \, y$ iff $(x = y$ or $n \leq x \leq y$ and $y - x \equiv 0 \bmod (k-n))$;
- if $k < n$ then $x \, |^n_k \, y$ iff $y \, |^k_n \, x$ iff $(x = y$ or $k \leq y \leq x$ and $x - y \equiv 0 \bmod (n-k))$.

One easily verifies that for any $n \neq k$ any subset of $\mathbb{N}$ without $|^n_k$-comparable elements is *finite*. Moreover, for $k < n$, there obviously are no infinite ascending chains, while for $n < k$ there are no infinite descending chains. (I.e. for $n < k$ the relation $|^n_k$ is a *well-quasi-ordering* on the set of natural numbers.)

3.4. PROPOSITION. *If* $\min(n,k) \leq R$, *then* $(\mathcal{M}, |^n_k)$ *is a finite* $\mathbf{BCI}^n_k$*-monoid.*

PROOF: We show monotonicity of $|^n_k$ with respect to $\oplus_r$. Suppose $x \, |^n_k \, y$.

If $k > n$ and $x = y$, then $x \oplus_r z = y \oplus_r z$. So let us assume that $n \leq x \leq y$ and $(y - x) \equiv 0 \bmod r$. Then $y = x + ar$ for some natural number a. Now

$$x \oplus_r z = R + [R, x+z]_r \qquad \text{and}$$

$$y \oplus_r z = R + [R, y+z]_r = R + [R, x+ar+z]_r.$$

If $y + z < R$ then $x \oplus_r z = x + z$, $y \oplus_r z = y + z$ a.w.a.d., as obviously $x+z \, |^n_k \, y+z$. Otherwise $y \oplus_r z = R + mod(x+ar+z-R, r) = R + mod(x+z-R, r)$. If $x + z \geq R$, then in fact $y \oplus_r z = x \oplus_r z$ a.w.a.d. Otherwise $x \oplus_r z = x + z$. Then $n \leq x \oplus_r z < y \oplus_r z$ and moreover $(y \oplus_r z) - (x \oplus_r z) = R + mod(x+z-R, r) - (x+z) = R - (x+z) - mod(R-(x+z), r) + r \equiv 0 \bmod r$.

So $x \oplus_r z \, |^n_k \, y \oplus_r z$ in all cases, q.e.d.

If $k < n$ we reason by duality, as in that case

$$x \, |^n_k \, y \quad \Leftrightarrow \quad y \, |^k_n \, x \quad \Rightarrow \quad y \oplus_r z \, |^k_n \, x \oplus_r z \quad \Leftrightarrow \quad x \oplus_r z \, |^n_k \, y \oplus_r z.$$

To complete the proof we need $(\oplus^n_r x) \, |^n_k \, (\oplus^k_r x)$ for all $x \in \mathcal{M}$. Let us assume that $k > n$. Then $(\oplus^k_r x) = (\oplus^n_r x) \oplus_r (\oplus^r_r x)$. Put $y := (\oplus^n_r x)$. We will show that $y \, |^n_k \, (\oplus^r_r x) \oplus_r y$. If $x = 0$ we have equality. Otherwise $y = nx \geq n$ or $y = R + mod(nx - R, r) \geq R \geq n$ by assumption. S.w.a.d.[4] by lemma 3.3.

For $n > k$ we use once more duality. (Here we need the assumption that $R \geq k$). $\square$

[4] I.e. *So we are done*

3.5. REMARK. Though the condition "$\min(n,k) \leq R$" is sufficient, it is *not* necessary. E.g. for $k > n$ we may have $n > R$, while nevertheless $(\mathcal{M}, |^n_k)$ is a finite $\mathbf{BCI}^n_k$- monoid. This can be the case if $mod(nx - R, r) > 0$, for all $nx \geq R$.

4 The finite model property

After having shown how one may obtain finite $\mathbf{BCI}^n_k$-monoids, we can continue exploiting Meyer and Ono(1992), in order to generalize the proofs there given of the finite model property for **BCK** (which for us is nothing but $\mathbf{BCI}^0_1$) and **BCIW** (which is $\mathbf{BCI}^2_1$) to a proof of the finite model property for $\mathbf{BCI}^n_k$ for all $n < k$, respectively to a proof of the finite model property for $\mathbf{BCI}^n_k$ for all $n > k > 0$.

As a starter, we will generalize their notion of *ϕ-critical* formula.

4.1. DEFINITION. Let ϕ be a formula, Γ a multiset of formulas.

- if $n < k$ we say that Γ is *(ϕ, n, k)-critical* iff
  - (i) $\mathbf{BCI}^n_k \vdash \Gamma \Rightarrow \phi$;
  - (ii) If $\Delta \;|^n_k\; \Gamma$ and $\Delta \neq \Gamma$, then $\mathbf{BCI}^n_k \not\vdash \Delta \Rightarrow \phi$.
- if $k < n$ we say that Γ is *(ϕ, k, n)-critical* iff
  - (i) $\mathbf{BCI}^n_k \not\vdash \Gamma \Rightarrow \phi$;
  - (ii) If $\Gamma \;|^n_k\; \Delta$ and $\Delta \neq \Gamma$, then $\mathbf{BCI}^n_k \vdash \Delta \Rightarrow \phi$. □

4.2. PROPOSITION. *Let Σ be the collection of all finite multisets with elements from some finite set of formulas S. Then the set*

$$\mathcal{R}(\phi) := \{\Gamma \in \Sigma \mid \Gamma \text{ is } (\phi, n, k)\text{–critical}\}$$

is finite, *for any $n \neq k$.*

PROOF: If Γ, Δ are distinct elements in $\mathcal{R}(\phi)$, then they are neither $|^n_k$- nor $|^k_n$-comparable. The claim then follows from the fact that for $n \neq k$ either the relation $|^n_k$ or the relation $|^k_n$ is a well-quasi-ordering, as it is equivalent to the pointwise lifting of the 'singleton-order' to $|S|$-tuples. See e.g. Rosenstein(1982). □

We can now more or less copy the finite model property proofs in Meyer and Ono(1992), with just some additional arguments using properties of $\oplus_r$. To keep this note self-contained, we will nevertheless work through the details.

Suppose $\mathbf{BCI}_k^n \not\vdash\Rightarrow A$. Let S, Σ be as in the proof of 2.6. Then $\bigcup_{\phi \in S} \mathcal{R}(\phi)$ is *finite*, by proposition 4.2.

First we assume that $n < k$.

Let $R := max\{n, max\{a_i \mid \langle a_1, \ldots, a_m \rangle \in \bigcup \mathcal{R}(\phi)\}\}$, where $\langle a_1, \ldots, a_m \rangle$ is the vector representing the multiset $\{S_1^{a_1}, \ldots, S_m^{a_m}\}$. Let $r = k - n$, and take the set $\mathcal{V}$ of m-dimensional vectors with entries $\leq R + r - 1$. Define an operation $\oplus_r$ on $\mathcal{V}$ by $\langle a_1, \ldots, a_m \rangle \oplus_r \langle b_1, \ldots, b_m \rangle = \langle a_1 \oplus_r b_1, \ldots, a_m \oplus_r b_m \rangle$, and a relation $|_k^n$ by $\langle a_1, \ldots, a_m \rangle \, |_k^n \, \langle b_1, \ldots, b_m \rangle$ iff $\forall i. a_i \, |_k^n \, b_i$.

(In what follows we will write $\mathbf{a}$ for $\langle a_1, \ldots, a_m \rangle$, etc.)

By the results of the previous section it is clear that $(\mathcal{V}, \oplus_r, \mathbf{0}, |_k^n)$ is a finite $\mathbf{BCI}_k^n$-monoid. Take the refuting valuation $\models$ used in the proof of 2.6. Define a valuation $\models^*$ on $(\mathcal{V}, |_k^n)$ by $\mathbf{a} \models^* p$ iff $\mathbf{a} \models p$, for any variable p in A. (Observe that $\models^*$ satisfies [mon], because $\models$ satisfies it.)

Then we apply induction on the complexity of $S_i \in S$ in order to show that for *all* subformulas S_i of A we have that $\mathbf{a} \models^* S_i$ iff $\mathbf{a} \models S_i$.

Let $S_i = B \supset C$. Suppose $\mathbf{a} \models B \supset C$ and $\mathbf{b} \models^* B$. Then by inductive hypothesis $\mathbf{b} \models B$, and therefore $\mathbf{a} + \mathbf{b} \models C$. Let $\mathbf{c}$ be (C, n, k)-critical and such that $\mathbf{c} \, |_k^n \, \mathbf{a} + \mathbf{b}$. (Such $\mathbf{c}$ has to exists as there are but finitely many vectors $\mathbf{x}$ with the property that $\mathbf{x} \, |_k^n \, \mathbf{a} + \mathbf{b}$.)

CLAIM: $c_i \, |_k^n \, (a_i \oplus_r b_i)$. Indeed, as $c_i \, |_k^n \, a_i + b_i$, $c_i < n$ would imply that $c_i = a_i + b_i = a_i \oplus_r b_i$; otherwise, by criticality, $n \leq c_i \leq R$; if $a_i + b_i \leq R + r - 1$, then $a_i \oplus_r b_i = a_i + b_i$; otherwise $c_i \leq a_i \oplus_r b_i$ (because $R + r - 1 \geq a_i \oplus_r b_i \geq R$.) The second part of lemma 3.2 then gives $(a_i \oplus_r b_i) - c_i \equiv 0 \bmod r$, a.w.a.d.

As $\mathbf{c} \models C$, by inductive hypothesis $\mathbf{c} \models^* C$, and thus $\mathbf{a} \oplus_r \mathbf{b} \models^* C$ by the above claim and [mon]. We thus showed that $\mathbf{a} \models^* B \supset C$.

Conversely, let $\mathbf{a} \models^* B \supset C$ and $\mathbf{b} \models B$. Take a (B, n, k)-critical $\mathbf{c}$ such that $\mathbf{c} \, |_k^n \, \mathbf{b}$.

As $\forall i. c_i \leq R + r - 1$ we have $\mathbf{c} \models^* B$, so $\mathbf{a} \oplus_r \mathbf{c} \models^* C$. The inductive hypothesis then tells us that $\mathbf{a} \oplus_r \mathbf{c} \models C$.

But $a_i \oplus_r c_i \, |_k^n \, a_i + b_i$: for $c_i \, |_k^n \, b_i$ implies $a_i \oplus_r c_i \, |_k^n \, a_i \oplus_r b_i$ (by monotonicity of $|_k^n$ with respect to $\oplus_r$); also $a_i \oplus_r b_i \, |_k^n \, a_i + b_i$, and the result follows by transitivity of $|_k^n$. Therefore $\mathbf{a} + \mathbf{b} \models C$ by [mon], and we showed $\mathbf{a} \models B \supset C$.

But then $(\mathcal{V}, |_k^n)$ (for $n < k$) is a refuting *finite* structure for A.

We proceed to the case that $n > k$.

We now put $R := max\{k, max\{a_i \mid \langle a_1, \ldots, a_m \rangle \in \bigcup \mathcal{R}(\phi)\}\}$ and similar to the previous case we take the finite $\mathbf{BCI}_k^n$-monoid $(\mathcal{V}, \oplus_r, \mathbf{0}, |_k^n)$, with $r = n - k$ and the refuting valuation $\models$ used in the proof of 2.6. Again we define a valuation $\models^*$ on $(\mathcal{V}, |_k^n)$ by $\mathbf{a} \models^* p$ iff $\mathbf{a} \models p$, for any variable p in A.

By induction on the complexity of $S_i \in S$ we show that $\mathbf{a} \models^* S_i$ iff $\mathbf{a} \models S_i$ for *all* subformulas S_i of A.

Let $S_i = B \supset C$, $\mathbf{a} \models B \supset C$ and $\mathbf{b} \models^* B$. By inductive hypothesis $\mathbf{b} \models B$, so $\mathbf{a}+\mathbf{b} \models C$. As $\mathbf{a} \oplus_r \mathbf{b} \mid_n^k \mathbf{a}+\mathbf{b}$, we have $\mathbf{a}+\mathbf{b} \mid_k^n \mathbf{a} \oplus_r \mathbf{b}$, and therefore ([mon]) $\mathbf{a} \oplus_r \mathbf{b} \models C$. By inductive hypothesis $\mathbf{a} \oplus_r \mathbf{b} \models^* C$, so $\mathbf{a} \models^* B \supset C$.

Conversely, suppose $\mathbf{a} \not\models B \supset C$. This means we have $\mathbf{b} \in \mathcal{V}$ such that $\mathbf{b} \models B$ and $\mathbf{a}+\mathbf{b} \not\models C$. There is a (C, k, n)-critical $\mathbf{c}$ such that $\mathbf{a}+\mathbf{b} \mid_k^n \mathbf{c}$, i.e. $\mathbf{c} \mid_n^k \mathbf{a}+\mathbf{b}$. As before we conclude $\mathbf{c} \mid_n^k \mathbf{a} \oplus_r \mathbf{b}$, so $\mathbf{a} \oplus_r \mathbf{b} \mid_k^n \mathbf{c}$, and $\mathbf{a} \oplus_r \mathbf{b} \not\models C$. Therefore $\mathbf{a} \oplus_r \mathbf{b} \not\models^* C$ by inductive hypothesis.

Now say $\mathbf{b} = \langle b_1, \ldots, b_m \rangle$. We define $\mathbf{b}' = \langle b'_1, \ldots, b'_m \rangle$, where $b'_i = R + [R, b_i]_r$. Then $\mathbf{b}' \in \mathcal{V}$ and $\mathbf{b} \mid_k^n \mathbf{b}'$ (for if $b_i \leq R + r - 1$, then $b'_i = b_i$; otherwise $b'_i = R + mod(b_i - R, r)$, so $k \leq b'_i < b_i$ and $b_i - b'_i = b_i - R - mod(b_i - R, r) \equiv 0 \bmod r$). Moreover

$$\begin{aligned} \mathbf{a} \oplus_r \mathbf{b}' &= R + [R, a_i + b'_i]_r \\ &= R + [R, a_i + R + [R, b_i]_r]_r \\ &= R + [R, a_i + b_i]_r \\ &= \mathbf{a} \oplus_r \mathbf{b}. \end{aligned}$$

So $\mathbf{b}' \models B$ and by inductive hypothesis $\mathbf{b}' \models^* B$. However $\mathbf{a} \oplus_r \mathbf{b}' \not\models^* C$. So $\mathbf{a} \not\models^* B \supset C$.

Thus we found also for $k < n$ that $(\mathcal{V}, \mid_k^n)$ is a refuting *finite* structure for A and we have shown:

4.3. THEOREM. (Finite model property for $\mathbf{BCI}_k^n$) *$\mathbf{BCI}_k^n$ is complete with respect to the collection of all finite $\mathbf{BCI}_k^n$-monoids, for all $n \neq k$.* □

4.4. PROPOSITION. *$\mathbf{BCI}_k^n$ is complete with respect to the collection of all* finite *$\mathbf{BCI}$-structures, i.e. if ϕ is valid on all finite $\mathbf{BCI}$-structures then it is derivable in $\mathbf{BCI}_k^n$, for any $n \neq k$.*

PROOF: If ϕ is valid on all finite reflexive $\mathbf{BCI}$-structures, it is valid on all finite $\mathbf{BCI}_k^n$-monoids. □

4.5. COROLLARY. *$\mathbf{BCI}$ has the finite model property only if it is the same as $\bigcap_{\mathcal{J}} \mathbf{BCI}_k^n$, for some $\mathcal{J} \subseteq \{(n, k) \mid n \neq k\}$.* □

4.6. REMARK. The argument given follows a quite general pattern, that might be instructive to sketch.

Let $\mathcal{L}$ be some logic, and suppose we have a collection of logics $L_i \supset \mathcal{L}$. Let moreover classes of structures $\mathcal{M}(\mathcal{L})$ for $\mathcal{L}$ and $\mathcal{M}(L_i)$ for L_i be given, such that $\mathcal{M}(L_i) \subset \mathcal{M}(\mathcal{L})$ and each *finite* $\mathcal{L}$-structure is an L_i-structure for some i. Then one easily shows:

1. $\mathcal{L}$ has fmp[5] $\quad\Rightarrow\quad \bigcap_i L_i = \mathcal{L}$;

2. Each L_i has fmp and $\bigcap_i L_i = \mathcal{L} \quad\Rightarrow\quad \mathcal{L}$ has fmp,

provided (for 1) that L_i is *sound*, (for 2) that L_i is *complete* with respect to the given collection of L_i-structures.

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[5]I.e. *finite model property*

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