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INFORMATION SYSTEMS AS COALGEBRAS

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Information Systems as Coalgebras

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Abstract

In this paper we show that the category Cinf of continuous information systems introduced in [13] can be constructed from the category Rel of sets and relations in a systematic way: we prove that Cinf is the category of coalgebras of the lower powerdomain comonad on the Karoubi envelope of Rel. Informally, this means that the category of continuous dcpo's is in proportion to the Karoubi envelope of Rel, like the category of sets and functions is in proportion to Rel.

1 Introduction

The original notion of an *information system* was introduced in [12], where it was shown that information systems provide concrete representations of Scott domains. Since, other types of information systems have been introduced to provide concrete representations of other classes of domains (see for example [2, 5, 7, 8, 13]).

In [7, 8] we showed that various categories of information systems can be systematically generated from the category Rel of sets and relations (and also from associated categories of coherence spaces and qualitative domains) by means of category-theoretically constructions such as the Karoubi envelope and the Kleisli category construction. This allowed us to give simple definitions of operations on information systems by first defining them on Rel and then considering their extension to the generated categories of information systems. However, the largest category of information systems that could be obtained in this way from (an

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analogue of) Rel was the category of information systems corresponding to the continuous Scott domains.

In this paper we extend these results by showing that the category Cinf of continuous information systems (introduced in [13]), which is equivalent to the category Con of *all* continuous dcpo's, can also be obtained from Rel in a systematical way as follows. First we take the Karoubi envelope $\mathcal{K}(\mathsf{Rel})$ of Rel, and we notice that the finite powerset constructor on Rel can be lifted to a comonad structure ! on $\mathcal{K}(\mathsf{Rel})$ (corresponding to the lower powerdomain constructor). Next we consider the category of coalgebras of !, which turns out to be isomorphic to the category Cinf. As a corollary it follows that Con is equivalent to the category of coalgebras of !.

If we recall that the category Set of sets and functions is isomorphic to the coalgebra category of the powerset comonad on Rel, then the above result can be summarized informally as follows:

$$\frac{\mathsf{Con}}{\mathcal{K}(\mathsf{Rel})} = \frac{\mathsf{Set}}{\mathsf{Rel}}$$

That is, the category Con is in proportion to $\mathcal{K}(\mathsf{Rel})$ like Set is in proportion to Rel: just like Set is the category of coalgebras of a comonad on Rel, the category Con is equivalent to the category of coalgebras of a (related) comonad on $\mathcal{K}(\mathsf{Rel})$.

An alternative way to view the result proved in this paper is the following one. In [8] we showed that $\mathcal{K}(\mathsf{Rel})$ is a model of Linear Logic. The *Girard translation* [3] of intuitionistic logic into linear logic corresponds category-theoretically to the construction of the category of *free* coalgebras with respect to !. Hence, the result proved in this paper informally means that the application of an *extension* of the Girard translation (viz., the construction of the category of *all* coalgebras) to $\mathcal{K}(\mathsf{Rel})$ yields the category of continuous dcpo's.

The rest of this paper is organized as follows. In section 2 we recall the elementary theory about continuous information systems and comonads. In section 3 we introduce the finite powerset comonad on $\mathcal{K}(\mathsf{Rel})$. In section 4 we prove our main result (corollary 10) that the category Cinf of information systems is the category of coalgebras of the finite powerset comonad on $\mathcal{K}(\mathsf{Rel})$. In section 5 we show, as an example of the use of corollary 10, how operators on information systems can be defined in a simple way on Rel, and then lifted to Cinf. We will see that this method of definition is possible for an operator if it *commutes* with the powerdomain constructor. Finally, in section 6, we make some concluding remarks.

2 Preliminaries

In this section we recall some elementary theory about continuous information systems [13] and comonads [11].

2.1 Continuous information systems

A *continuous information system* (in the sense of [13]) is a tuple $\langle A, >_A \rangle$, where A is a set and $>_A \subseteq A \times A$ satisfies

$$\exists b(a >_A b >_A c) \Leftrightarrow a >_A c$$

We will write $>$ for $>_A$ when A is clear from the context. Note that each *preorder* (that is, a set ordered by a reflexive transitive relation) is also a continuous information system.

Let A, B be continuous information systems. An *approximable mapping* from A to B is a relation $R \subseteq A \times B$ satisfying the following three requirements:

1. $\exists a'(a > a') \Rightarrow \exists b(aRb)$
2. $\exists a', b'(a > a'Rb' > b) \Leftrightarrow aRb$
3. $a > a'$ & $a'Rb_1$ & $a'Rb_2 \Rightarrow \exists b(aRb$ & $b > b_1, b_2)$

Let Cinf denote the category with continuous information systems as objects and approximable mappings as arrows. Composition in Cinf is just ordinary relation composition. The identity id_A on a continuous information system A is the relation $>_A: A \to A$. Let Ainf denote the full subcategory of Cinf with as objects preorders.

In [13] it is shown that Cinf is equivalent to the category of continuous dcpo's and continuous functions. Recall that a subset S of a poset is *directed* iff each finite subset of S has an upperbound in S. A *directed complete poset (dcpo)* is a poset in which each directed subset S has a least upperbound (denoted by $\bigvee S$). For a dcpo D, the *way-below relation* $\ll$ on D is defined as follows

$$x \ll y \Leftrightarrow \forall \text{ directed } S \subseteq D(y \leq \bigvee S \Rightarrow \exists y' \in S(x \leq y'))$$

A dcpo D is *continuous* iff there exists a subset $B_D \subseteq D$ (called a *basis* of D) such that

1. $B_D(x) = \{x' \in B_D | x' \ll x\}$ is directed
2. $\bigvee B_D(x) = x$

A function $f : D \to E$ between dcpo's D, E is *continuous* iff it preserves least upperbounds of directed sets. Let Con denote the category of continuous dcpo's and continuous functions.

The equivalence between Cinf and Con can be described briefly as follows. An *ideal* of an information system A is a subset $x \subseteq A$ such that

1. $x \neq \emptyset$
2. $\exists b' \in x(b' > b) \Leftrightarrow b \in x$
3. $b_1, b_2 \in x \Rightarrow \exists b \in x(b > b_1, b_2)$

Each information system A corresponds to the continuous dcpo of ideals of A ordered by subset inclusion. The other way round, for a continuous dcpo D, we have a continuous information system $\langle B_D, \gg \rangle$. For more details we refer the reader to [13].

The subcategory Ainf of Cinf also corresponds to a category of domains. An element x of a dcpo D is *compact* iff $x \ll x$. A dcpo D is *algebraic* iff it is continuous with a basis of compact elements. Let Alg denote the full subcategory of Con with as objects the algebraic dcpo's. The equivalence Cinf $\simeq$ Con can be restricted to an equivalence between Ainf and Alg (or see [5] for a direct proof of this last equivalence).

2.2 Comonads and coalgebras

Let C be a category. A *comonad* on C is a tuple $\langle !, \eta, \mu \rangle$, where $! : \mathsf{C} \to \mathsf{C}$ is a functor and $\eta :! \to Id_{\mathsf{C}}$, $\mu :! \to !!$ are natural transformations satisfying

1. $\eta_{!A} \circ \mu_A = !(\eta_A) \circ \mu_A = id_{!A}$
2. $\mu_{!A} \circ \mu_A = !(\mu_A) \circ \mu_A$

Example 1 *Let* Rel *denote the category with sets as objects and relations $R \subseteq A \times B$ as arrows $R : A \to B$. Composition in* Rel *is defined by $a(S \circ R)c \Leftrightarrow \exists b(aRbSc)$ and the identity by $a \; id_A \; a' \Leftrightarrow a = a'$.*

The powerset constructor on sets gives rise to a comonad on Rel *as follows. The functor $\mathcal{P}$:* Rel $\to$ Rel *is defined on objects A as the powerset $\mathcal{P}(A)$, and if $R : A \to B$ is a relation, then $\mathcal{P}(R) : \mathcal{P}(A) \to \mathcal{P}(B) : x\mathcal{P}(R)y \Leftrightarrow y = \{b|\exists a \in x(aRb)\}$. The natural transformation $\eta : \mathcal{P} \to Id$ has components $\eta_A : \mathcal{P}(A) \to A : x\eta_A a \Leftrightarrow a \in x$. The natural transformation $\mu : \mathcal{P} \to \mathcal{P}\mathcal{P}$ has components $\mu_A : \mathcal{P}(A) \to \mathcal{P}\mathcal{P}(A) : x\mu_A S \Leftrightarrow S = \{x\}$.*

Given a comonad $! : \mathsf{C} \to \mathsf{C}$, a *coalgebra* of this comonad is an arrow $\alpha : A \to !A$ in C satisfying

1. $\eta_A \circ \alpha = id_A$

2. $!(\alpha) \circ \alpha = \mu_A \circ \alpha$

Let $\alpha : A \rightarrow !A$, $\beta : B \rightarrow !B$ be coalgebras. A *coalgebra morphism* from α to β is an arrow $\phi : A \rightarrow B$ in C satisfying

$$!(\phi) \circ \alpha = \beta \circ \phi$$

Let $c\mathcal{A}(\mathsf{C}, !)$ denote the category of coalgebras (with respect to !) and coalgebra morphisms.

Example 2 *Let* Set *denote the category of sets and functions. It is easy to show that the coalgebra category* $c\mathcal{A}(\mathsf{Rel}, \mathcal{P})$ *with respect to* $\mathcal{P}$ *is isomorphic to* Set.

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3 The Finite Powerset Constructor as Comonad

As we saw in example 1, the powerset constructor $\mathcal{P}$ can be extended to a comonad on Rel. In [8, 9] we showed that something similar can be done with the *finite* powerset constructor $\mathcal{P}_f$. However, $\mathcal{P}_f$ can only be extended to a *semi*-functor on Rel, that is, a functor which need not preserve identities [6]. Therefore, $\mathcal{P}_f$ gives only rise to a *semi*-comonad structure on Rel (for precise definitions see [8, 9]). In this paper we are not interested in this *semi*-comonad structure, but in the comonad generated by it, which we turn to now.

The canonical way of transforming semi-comonads into comonads is by means of the Karoubi envelope construction. For a category C, the *Karoubi envelope* $\mathcal{K}(\mathsf{C})$ of C is the category with as objects the *idempotent* arrows of C, that is, the arrows $f : A \rightarrow A$ such that $f \circ f = f$. An arrow ϕ of $\mathcal{K}(\mathsf{C})$ between objects $f : A \rightarrow A$ and $g : B \rightarrow B$ is an arrow $\phi : A \rightarrow B$ of C such that $g \circ \phi \circ f = \phi$.

Example 3 *The category* $\mathcal{K}(\mathsf{Rel})$ *has as objects relations* $R \subseteq A \times A$ *satisfying* $R \circ R = R$. *It follows that the objects of* $\mathcal{K}(\mathsf{Rel})$ *are exactly the continuous information systems. The arrows* $R : A \rightarrow B$ *of* $K(\mathsf{Rel})$ *are relations* $R \subseteq A \times B$ *satisfying*

$$\exists a', b'(a >_A a'Rb' >_B b) \Leftrightarrow aRb$$

Note that, although the objects of $\mathcal{K}(\mathsf{Rel})$ *are continuous information systems, the arrows need not be approximable mappings (they are called* lower approximable semimappings *in [13]).*

Let $\mathcal{K}_c(\mathsf{Rel})$ *denote the full subcategory of* $\mathcal{K}(\mathsf{Rel})$ *with as objects preorders.*

A *semi*-functor $F : \mathsf{C} \to \mathsf{D}$ can be extended to a functor $\mathcal{K}(F) : \mathcal{K}(\mathsf{C}) \to \mathcal{K}(\mathsf{D})$ by $\mathcal{K}(F)(f) = F(f)$ and $\mathcal{K}(F)(\phi) = F(\phi)$. More general, *semi*-comonads on a category C give rise to comonads on the Karoubi envelope $\mathcal{K}(\mathsf{C})$. We give an explicit description of the comonad on $\mathcal{K}(\mathsf{Rel})$ generated by the finite powerset semi-comonad on Rel.

Definition 4 *The comonad $\langle !, \eta, \mu\rangle$ on $\mathcal{K}(\mathsf{Rel})$ is defined as follows:*

- *The functor $! : \mathcal{K}(\mathsf{Rel}) \to \mathcal{K}(\mathsf{Rel})$ assigns to objects A the finite powerset $\mathcal{P}_f(A)$ ordered by $X >_{!A} Y \Leftrightarrow \forall b \in Y \exists a \in X(a >_A b)$. If $R : A \to B$ is an arrow in $K(\mathsf{Rel})$, then $!R :!A \to !B : X!RY \Leftrightarrow \forall b \in Y \exists a \in X(aRb)$.*
- *The natural transformation $\eta :! \to Id$ has components $\eta_A :!A \to A : X\eta_A a \Leftrightarrow \exists a' \in X(a' > a)$.*
- *The natural transformation $\mu :! \to !!$ has components $\mu_A :!A \to !!A : X\mu_A\alpha \Leftrightarrow \forall a' \in \bigcup \alpha \exists a \in X(a > a')$.*

The comonad ! corresponds to the *lower* or *Hoare powerdomain* constructor on the objects of $\mathcal{K}(\mathsf{Rel})$ (considered as domains).

4 Information Systems as Coalgebras

In this section we prove our main result (corollary 10) that the category Cinf of continuous information systems is isomorphic to the category of coalgebras of the comonad ! on $\mathcal{K}(\mathsf{Rel})$. First we show that each continuous information system gives rise to a coalgebra of !.

Definition 5 *Let A be a continuous information system. The relation $\alpha_A \subseteq A \times \mathcal{P}_f(A)$ is defined by*

$$a\alpha_A X \Leftrightarrow \exists a'(a > a' > X)$$

where $a' > X$ denotes that a' is an upperbound of X.

Lemma 6 *For a continuous information system A, the relation α_A is an arrow $A \to !A$ in $\mathcal{K}(\mathsf{Rel})$.*

Proof: We show that

$$\exists b, Y(a > b\alpha_A Y > X) \Leftrightarrow a\alpha_A X$$

- ($\Rightarrow$): Suppose that $a > b\alpha_A Y > X$, then there exists b' such that $a > b > b' > Y > X$. Hence there exists b' such that $a > b' > X$ and $a\alpha_A X$ follows.

- ($\Leftarrow$): Suppose that $a\alpha_A X$. By definition of α_A, we have $\exists a'(a > a' > X)$ and hence, by the properties of the $>$-relation, there exists b, c such that $a > b > c > a' > X$. It follows that $a > b > c > \{a'\} > X$, and hence $\exists b, Y(a > b\alpha_A Y > X)$.

■

Theorem 7 *For a continuous information system A, the arrow $\alpha_A : A \to !A$ is a coalgebra with respect to $!$.*

Proof: By lemma 6, the relation α_A is an arrow $A \to !A$ in $\mathcal{K}(\mathsf{Rel})$. We first show that $\eta \circ \alpha_A = id$, that is, we show that

$$\exists X(a\alpha_A X \;\&\; \exists b \in X(b > c)) \Leftrightarrow a > c$$

- ($\Rightarrow$): Suppose that $a\alpha_A X$ & $\exists b \in X(b > c)$. Then there exists a' such that $a > a' > X$, and hence $a > a' > b > c$. It follows that $a > c$.
- ($\Leftarrow$): Suppose that $a > c$. By the properties of $>$ there exists a', b such that $a > a' > b > c$. Hence $a > a' > \{b\}$ and $\exists b \in \{b\}(b > c)$. It follows that there exists X such that $a\alpha_A X$ & $\exists b \in X(b > c)$.

Next we show that $\mu_A \circ \alpha_A = !(\alpha_A) \circ \alpha_A$, that is, we show that

$$\exists X(a\alpha_A X\mu_A S) \Leftrightarrow \exists X(a\alpha_A X!(\alpha_A)S)$$

Starting from the left-hand side we have

$$\begin{aligned}
&\Leftrightarrow \exists X(\exists a'(a > a' > X) \;\&\; X > \bigcup S) \\
&\Leftrightarrow \exists a'(a > a' > \bigcup S) \\
&\Leftrightarrow \exists X(\exists a'(a > a' > X) \;\&\; \forall Y \in S \exists b \in X \exists b'(b > b' > Y)) \\
&\Leftrightarrow \exists X(\exists a'(a > a' > X) \;\&\; \forall Y \in S \exists b \in X(b\alpha_A Y)) \\
&\Leftrightarrow \exists X(a\alpha_A X!(\alpha_A)S)
\end{aligned}$$

which is the right-hand side of the above equivalence. ■

The following theorem shows that coalgebra morphisms between coalgebras of the form α_A coincide with approximable mappings.

Theorem 8 *Let A, B be continuous information systems. A relation $R \subseteq A \times B$ is a coalgebra morphism $\alpha_A \to \alpha_B$ iff it is an approximable mapping $A \to B$.*

Proof: Suppose that $R : \alpha_A \to \alpha_B$ is a coalgebra morphism. Because R is an arrow in $\mathcal{K}(\mathsf{Rel})$, it satisfies requirement (2) in the definition of approximable mapping. To see that it also satisfies the requirements (1) and (3), consider the coalgebra morphism equation $!(R) \circ \alpha_A = \alpha_B \circ R$, that is

$$\exists X(a\alpha_A X!(R)Y) \Leftrightarrow \exists b(aRb\alpha_B Y)$$

By writing out the definitions of $\alpha_A, \alpha_B, !(R)$, this is equivalent to the following expression

$$\exists X(\exists a''(a > a'' > X) \ \&\ \forall b' \in Y \exists a' \in X(a'Rb')) \Leftrightarrow \exists b(aRb \ \&\ \exists b''(b > b'' > Y))$$

By some calculus this can be further simplified to

$$\exists a' \forall b' \in Y(a > a'Rb') \Leftrightarrow \exists b(aRb > Y)$$

It is easy to see that this is equivalent to the conditions (1) and (3) in the definition of approximable mapping. ■

Finally, we show that *each* coalgebra α of ! is of the form α_A.

Theorem 9 *If* $\alpha : A \to !A$ *is a coalgebra, then* $\alpha = \alpha_A$.

Proof: Suppose that α is a coalgebra, then it satisfies $\eta_A \circ \alpha = id_A$, that is, the following equivalence (*) holds:

$$\exists X(a\alpha X \ \&\ \exists b \in X(b > c)) \Leftrightarrow a > c$$

We will show that α satisfies

$$a\alpha X \Leftrightarrow \exists b(a > b > X)$$

- ($\Rightarrow$): Suppose that $a\alpha X$. We first show that $a > X$. As α is an arrow in $\mathcal{K}(\mathsf{Rel})$, from $a\alpha X$ it follows that there exists X' such that $a\alpha X' > X$. Hence for all $c \in X$ we have $a\alpha X'$ & $\exists b \in X'(b > c)$. Applying (*) yields $\forall c \in X(a > c)$, that is, $a > X$. Next we have to show that there exists b such that $a > b > X$. But this easily follows from the fact that α is an arrow in $\mathcal{K}(\mathsf{Rel})$: $a\alpha X$ implies $\exists b(a > b\alpha X)$.

- ($\Leftarrow$): Suppose that $\exists b(a > b > X)$. To show that $a\alpha X$, we use the following equivalence which is implied by (*):

$$a\alpha\{b\} \Leftrightarrow a > b$$

  Hence from $\exists b(a > b > X)$ follows $\exists b(a\alpha\{b\} > X)$. Because α is an arrow in $\mathcal{K}(\mathsf{Rel})$, this implies $a\alpha X$.

■

From theorems 7 and 9 it follows that there is a one-one correspondence between coalgebras of ! and continuous information systems. But then, by theorem 8, we also have that coalgebra morphisms correspond to approximable mappings, and hence the following corollary holds.

Corollary 10 *The category* Cinf *of continuous information systems is isomorphic to the category* $c\mathcal{A}(\mathcal{K}(\mathsf{Rel}), !)$ *of coalgebras of* !.

Corollary 11 *The category* Con *of continuous dcpo's is equivalent to the category* $c\mathcal{A}(\mathcal{K}(\mathsf{Rel}), !)$ *of coalgebras of* !.

It is not difficult to see that the subcategory Ainf of Cinf corresponds to the coalgebra category $c\mathcal{A}(\mathcal{K}_c(\mathsf{Rel}), !)$. Hence the following corollary holds.

Corollary 12 *The category* Alg *of algebraic dcpo's is equivalent to the category* $c\mathcal{A}(\mathcal{K}_c(\mathsf{Rel}), !)$ *of coalgebras of* !.

5 Operators on Information Systems

In this section, as an example of the use of corollary 10, we show how operators on continuous information systems can be defined in a simple way on Rel, and then extended to Cinf.

In general, let C be a category with a comonad $\langle !, \eta, \mu \rangle$. Let $F : \mathsf{C} \to \mathsf{C}$ be a functor which *commutes* with ! in the sense that there is a natural transformation $c : F! \to !F$ satisfying

1. $\eta_{FA} \circ c_A = F(\eta_A)$
2. $\mu_{FA} \circ c_A = !(c_A) \circ c_{!A} \circ F(\mu_A)$

It is easy to see that F can be extended to a functor $F' : c\mathcal{A}(\mathsf{C}, !) \to c\mathcal{A}(\mathsf{C}, !)$ by defining

- $F'(\alpha : A \to !A) = c_A \circ F(A)$
- $F'(\phi) = F(\phi)$

In this way various operators can be defined on continuous information systems. The simplest example of such an operator is ! itself, which commutes with ! by the identity. As a further example, we consider the lifting operation (that is, the addition of a new bottom element to a poset) more closely. First define the operator $L : \mathsf{Rel} \to \mathsf{Rel}$ on objects and on arrows by

- $LA = A \uplus \{\bot\}$
- $aL(R)b \Leftrightarrow (b = \bot \vee aRb)$

where $\uplus$ denotes disjoint union. Note that L does not preserve identities (although it does preserve composition), and hence is only a *semi*-functor. However, L can be extended to a *functor* $\mathcal{K}(L) : \mathcal{K}(\mathsf{Rel}) \to \mathcal{K}(\mathsf{Rel})$ in a canonical way (see section 3). In particular, on objects $A \in \mathcal{K}(\mathsf{Rel})$ we have

- $\mathcal{K}(L)(A) = A \uplus \{\bot\}$
- $a >_{\mathcal{K}(L)(A)} a' \Leftrightarrow (a' = \bot \vee a >_A a')$

Next define a natural transformation $c : \mathcal{K}(L)! \to !\mathcal{K}(L)$ by

$$Xc_AX' \Leftrightarrow \forall a' \in X'(a' = \bot \vee \exists a \in X(a > a'))$$

(In fact, c can be obtained from a corresponding natural transformation $L\mathcal{P}_f \to \mathcal{P}_fL$ on Rel via the Karoubi envelope construction). As it is easy to see that c satisfies the above requirements, $\mathcal{K}(L)$ can be extended to a functor $\mathcal{K}(L)' : c\mathcal{A}(\mathcal{K}(\mathsf{Rel}), !) \to c\mathcal{A}(\mathcal{K}(\mathsf{Rel}), !)$ on the category of continuous information systems. In particular, it turns out that $\mathcal{K}(L)' = \mathcal{K}(L)$.

The categorical products of $\mathcal{K}(\mathsf{Rel})$ can also be transfered to Cinf as follows. In general, if a category C equipped with a comonad ! has a tensor product $\otimes$, then this tensor product lifts to the category $c\mathcal{A}(\mathsf{C}, !)$ of coalgebras (see for example [1]). Moreover, if C has products that satisfy $!(A \times B) \cong !A \otimes !B$, then the lifted tensor product is actually the product in $c\mathcal{A}(\mathsf{C}, !)$. In our case, the category $\mathcal{K}(\mathsf{Rel})$ has a tensor product defined by

$$A \otimes B = \{\langle a, b\rangle | a \in A, b \in B\}$$

ordered pointwise, and it has a categorical product

$$A \times B = \{\langle a, 0\rangle | a \in A\} \cup \{\langle b, 1\rangle | b \in B\}$$

ordered componentwise. As it is also easy to see that the above mentioned isomorphism is satisfied, it follows that $\otimes$ is the product in Cinf.

6 Conclusion

We conclude this paper with a few final remarks.

- If we replace the category Rel by an associated category and follow the procedure outlined in this paper, then we get extended categories of information systems. For example, if we replace Rel by the category of coherence spaces and linear maps (see [8]), then the resulting coalgebras are equipped with a symmetric binary relation $\sim$ that satisfies

$$a > a' \ \& \ b > b' \ \& \ a \sim b \Rightarrow a' \sim b'$$

This gives rise to continuous dcpo's with additional structure.

- In section 3 we remarked that the comonad ! is generated from the semi-comonad $\mathcal{P}_f$ on Rel. A result corresponding to corollary 10 can be proved directly for the semi-comonad $\mathcal{P}_f$ as follows. First we define, for an arbitrary semi-comonad $T : \mathsf{C} \to \mathsf{C}$, the category $c\mathcal{A}_s(\mathsf{C}, T)$ of semi-coalgebras with respect to T. It then turns out that, in general, we have $c\mathcal{A}_s(\mathsf{C}, T) = c\mathcal{A}(\mathcal{K}(\mathsf{C}), \mathcal{K}(T))$. Hence by corollary 10 it follows that $c\mathcal{A}_s(\mathsf{Rel}, \mathcal{P}_f)$ is isomorphic to the category Cinf of information systems, and equivalent to the category Con of continous dcpo's. Informally, this result can be summarized as follows:

$$\frac{\mathsf{Con}}{\mathsf{Rel}} =_s \frac{\mathsf{Set}}{\mathsf{Rel}}$$

That is, modulo semi-notions, Con stands to Rel like Set stands to Rel.

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